# A Complete Proof System for 1-Free Regular Expressions in Bisimulation Semantics

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**Abstract** Robin Milner in 1984 proposed an axiomatization for the regular expressions in bisimulation semantics, and posed the question of whether it is complete. We consider a subclass of regular expressions by excluding the 1 and changing from the unary to the binary Kleene star. We establish completeness of Milner's axiomatization for this subclass.

## 1 Introduction

Regular expressions, introduced by Stephen Kleene [14], are widely studied in formal language theory, notably for string searching [19]. They are constructed from constants 0 (no strings), 1 (the empty string), and $a$ (a single letter) from some alphabet; binary operators + and $\cdot$, (union and concatenation); and the unary Kleene star * (zero or more iterations). Aanderaa [1] and Salomaa [17] gave complete axiomatizations for the regular expressions in the trace semantics of formal languages, including a fixed-point rule with a non-empty-word property as side condition.

Robin Milner [15] was the first to study regular expressions in bisimulation semantics [16], where he called them star expressions. Here the interpretation of 0 is deadlock, 1 is (successful) termination, $a$ is an atomic action, and + and $\cdot$ are alternative and sequential composition of two processes, respectively. Milner adapted the aforementioned axiomatizations from formal languages to obtain a sound axiomatization for this setting, and posed the (still open) question whether this axiomatization is complete, meaning that if the process graphs of two star expressions are bisimilar, then they can be proven equal. That Milner's axiomatization contains a fixed-point rule is inevitable [18]. Bergstra, Bethke, and Ponse [4] studied star expressions without 0 and 1, replacing the unary by the binary Kleene star $^{\circledast}$, which represents an iteration of the first argument, followed by the execution of the second argument. They obtained an axiomatization by basically omitting the axioms for 0 and 1 as well as the fixed-point rule from Milner's axiomatization, and adding Troeger's axiom [20]. This axiomatization was proven complete in [10,8]. A sound and complete axiomatization for star expressions without 0 and the unary Kleene star, but with 1 and a unary perpetual loop operator $^{*}0$, was presented in [7,9].

In contrast to the formal languages setting, not all finite-state process graphs can be expressed by a star expression modulo bisimilarity. Milner posed in [15] a second question, to characterize which finite-state process graphs can be expressed. This was settled in [3] by the notion of a well-behaved specification.

In this paper we prove completeness of Milner's axiomatization (tailored to the adapted setting) for star expressions with 0, but without 1 and with the binary Kleene star. While the aforementioned completeness proofs all focus on terms, we follow in the footsteps of Milner and focus on their process graphs. A key idea is to determine loops in process graphs associated to star expressions. By a loop we mean a subgraph generated by a set of entry transitions from a vertex $v$ such that (1) each infinite path starting with an entry transition eventually returns to $v$, and (2) termination in the loop is not permitted. A process graph is said to satisfy LLEE (Layered Loop Existence and Elimination) if repeatedly eliminating the entry transitions of a loop leads to a graph without infinite paths. LLEE offers a generalization (and more elegant definition) of the notion of a well-behaved specification. Our completeness proof roughly works as follows. Let star expressions $e_1$ and $e_2$ give rise to bisimilar process graphs $g_1$ and $g_2$. We show that they satisfy LLEE. We prove that LLEE is preserved under bisimulation collapse. We construct for each graph that satisfies LLEE a star expression that corresponds to this graph, modulo bisimilarity. In particular such a star expression $f$ can be constructed for the bisimulation collapse of $g_1$ and $g_2$. Finally we show that $e_1$ and $e_2$ can be proven equal to $f$, which yields the desired completeness result.

We were inspired by the term graph representation of cyclic $\lambda$-terms [12], in order to define and implement maximal sharing in the $\lambda$-calculus with `letrec` [13] (see also [11]). We also exploited ideas from [7,9] on minimizing the process graph (associated to the term) in the left-hand side of a binary Kleene star modulo bisimilarity. Interestingly, our framework includes the pair of bisimilar star expressions that at the end of [9] is mentioned as problematic for a completeness proof.

The completeness result for star expressions with 0 but without 1 and with the binary Kleene star is interesting in its own right. We are hopeful that the current proof approach can be extended to the full class of star expressions, which would provide an affirmative answer to Milner's long-standing open question. However, we found that serious technical obstacles concerning 1-transitions lie ahead. In particular, in the setting with 1, LLEE is not always preserved under bisimulation collapse.

A version of this paper with supplements containing the proofs of statements that are only well-motivated here, and a laborious, but conceptually repetitive part of the LLEE-preservation proof is available via `https://git.io/Je4X9` on GitHub.

## 2 Preliminaries

Given $A$ a set of *actions*, the *star expressions* are generated by the grammar

$$e ::= 0 \mid a \mid (e_1 + e_2) \mid (e_1 \cdot e_2) \mid (e_1{}^{\circledast} e_2) \qquad (\text{for } a \in A).$$

0 represents deadlock (i.e., does not perform any action), $a$ an atomic action, $+$ alternative and $\cdot$ sequential composition, and $^{\circledast}$ the binary Kleene star. Note that the empty word 1 is missing from the syntax.

A (finite sink-termination) *chart* $\mathcal{C}$ consists of finite sets $V$ of *vertices*, $A$ of *actions*, and $T \subseteq V \times A \times V$ of *transitions*, a special vertex $\surd \in V$ with no outgoing transitions, which indicates termination, and a *start vertex* $v_s \in V \backslash \{\surd\}$. The chart associated to a star expression is defined by the following rules, where $e$ ranges over star expressions, $\xi$ over star expressions plus $\surd$, and $a$ over $A$:

$$\frac{}{a \xrightarrow{a} \surd} \qquad \frac{e_i \xrightarrow{a} \xi \quad (i=1,2)}{e_1 + e_2 \xrightarrow{a} \xi} \qquad \frac{e_1 \xrightarrow{a} e_1'}{e_1 \cdot e_2 \xrightarrow{a} e_1' \cdot e_2} \qquad \frac{e_1 \xrightarrow{a} \surd}{e_1 \cdot e_2 \xrightarrow{a} e_2}$$

$$\frac{e_1 \xrightarrow{a} e_1'}{e_1{}^{\circledast}e_2 \xrightarrow{a} e_1' \cdot (e_1{}^{\circledast}e_2)} \qquad \frac{e_1 \xrightarrow{a} \surd}{e_1{}^{\circledast}e_2 \xrightarrow{a} e_1{}^{\circledast}e_2} \qquad \frac{e_2 \xrightarrow{a} \xi}{e_1{}^{\circledast}e_2 \xrightarrow{a} \xi}$$

The charts associated to star expressions are finite-state (by [2, Thm. 3.4]). By the absence of 1, termination is only possible at $\surd$.

A *bisimulation* between charts $\mathcal{C}_1$ and $\mathcal{C}_2$ is a symmetric binary relation $B$ between vertices of $\mathcal{C}_1$ and of $\mathcal{C}_2$ that relates the start vertices, such that if $v_1 \, B \, v_2$, then: (1, progress): for every transition $v_1 \xrightarrow{a} v_1'$ in $\mathcal{C}_1$ there is a transition $v_2 \xrightarrow{a} v_2'$ in $\mathcal{C}_2$ with $v_1' \, B \, v_2'$, and (2, termination): if $v_1 = \surd$ then $v_2 = \surd$. If there is a bisimulation between $\mathcal{C}_1$ and $\mathcal{C}_2$, we write $\mathcal{C}_1 \leftrightarrow \mathcal{C}_2$ and say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *bisimilar*.

The axiomatization $\mathsf{BBP}$ of the above class of star expressions has the axioms:

$$\begin{array}{llcllllcl}
\text{(A1)} & x+y & = & y+x & & \text{(A6)} & x+0 & = & 0 \\
\text{(A2)} & (x+y)+z & = & x+(y+z) & & \text{(A7)} & 0\cdot x & = & 0 \\
\text{(A3)} & x+x & = & x & & \text{(BKS1)} & x\cdot(x^{\circledast}y)+y & = & x^{\circledast}y \\
\text{(A4)} & (x+y)\cdot z & = & x\cdot z + y\cdot z & & \text{(BKS2)} & (x^{\circledast}y)\cdot z & = & x^{\circledast}(y\cdot z) \\
\text{(A5)} & (x\cdot y)\cdot z & = & x\cdot(y\cdot z) & & & & &
\end{array}$$

together with the inference rules of equational logic, and the additonal rule:

$$(\mathrm{RSP}^{\circledast}) \quad \frac{e \;=\; (f\cdot e) + g}{e \;=\; f^{\circledast}g}$$

The axiomatization $\mathsf{BBP}$ is clearly sound modulo $\leftrightarrow$. We will prove its completeness modulo $\leftrightarrow$. Derivable equality from $\mathsf{BBP}$ is denoted by $=_{\mathsf{BBP}}$.

## 3 Layered loop existence and elimination

To define the key notion called LLEE that holds for all charts associated to the star expressions, we first define the notion of a 'loop'. It captures a subchart within the chart of a star expression $e$ that corresponds to the behavior of the iteration of $f_1$ in an innermost subterm $f_1{}^{\circledast}f_2$ of $e$. A *path* from a vertex $v_1$ is a (finite or infinite) sequence of transitions $v_1 \xrightarrow{a_1} v_2 \xrightarrow{a_2} \cdots$ in a chart.

**Definition 3.1.** *A chart is a* loop chart *if:*

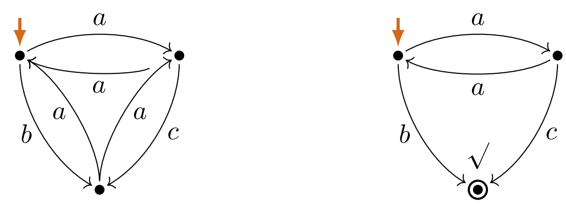
(L1) *There is an infinite path from* $v_s$.

(L2) *Every infinite path from* $v_s$ *returns to* $v_s$ *after a positive number of transitions.*

(L3) *It does not contain the vertex* $\surd$.

Given a chart, repeatedly select a vertex $v$ and a subset $S$ of its outgoing transitions such that the subchart reachable by paths starting with a transition in $S$, with $v$ as start vertex, forms a loop chart, and remove the transitions in $S$ from the original chart. If this repeated procedure ultimately leads to a chart without infinite paths, then the original chart is bisimilar to the chart of a star expression.

*Example 3.2.* The two charts below show why (L2) and (L3) are necessary to rule out charts that are not expressible by a star expression modulo $\leftrightarrows$. The left top arrow marks the start vertex. The first example, which violates (L2), was (as we will see correctly) conjectured not to be expressible (modulo $\leftrightarrows$) in [15]. The second example, a double-exit structure with a termination vertex at the bottom, which violates (L3), was shown not to be expressible in [5]. We note that there is no loop subchart at any of the vertices in the two examples.




We now introduce the machinery to label entry transitions of loop charts within a chart of a star expression. The labels are layered: in case a Kleene star is within an argument of another Kleene star, the labels corresponding to the inner Kleene star are supposed to be lower than those corresponding to the outer Kleene star.

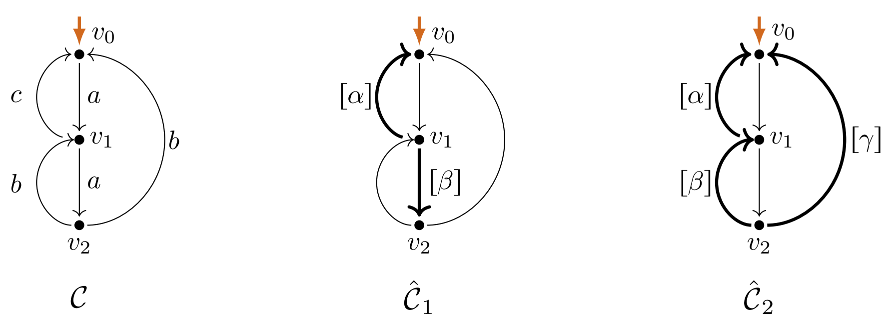
We assume a set $\Gamma$ of *loop names*, referred to by $\alpha$, $\beta$, $\gamma$, and a *loop-layer function* $|\cdot| : \Gamma \to \mathbb{N}^{>0}$. A *loop-labeling* $\hat{\mathcal{C}}$ of a chart $\mathcal{C}$ gives some of its transitions, called *loop-entry transitions*, a subscript $[\alpha]$ with $\alpha \in \Gamma$. All other transitions are called *branch transitions* and carry the subscript 'br'. Let $LI(\hat{\mathcal{C}})$ denote the set of pairs $\langle v, \alpha \rangle \in V \times \Gamma$ such that an $\rightarrow_{[\alpha]}$ transition departs from $v$ in $\hat{\mathcal{C}}$. If $\langle v, \alpha \rangle \in LI(\hat{\mathcal{C}})$, then $\mathcal{C}_{\hat{\mathcal{C}}}(v, \alpha)$ is the subchart that consists of the vertices and transitions reachable by paths that start with an $\rightarrow_{[\alpha]}$ transition from start vertex $v$ followed by only branch transitions and that halt immediately if $v$ is revisited.

**Definition 3.3.** *Loop-labeling $\hat{\mathcal{C}}$ is a* LLEE-witness *for chart $\mathcal{C}$ if:*

(W1) *There is no infinite sequence of branch transitions from $v_s$.*

(W2) *For all $\langle v, \alpha \rangle \in LI(\hat{\mathcal{C}})$, (a) $\mathcal{C}_{\hat{\mathcal{C}}}(v, \alpha)$ is a loop chart, and (b) from all vertices $w \neq v$ of $\mathcal{C}_{\hat{\mathcal{C}}}(v, \alpha)$, in $\mathcal{C}$, no transition $\rightarrow_{[\beta]}$ with $|\beta| \geqslant |\alpha|$ departs.*

*If such a LLEE-witness exists, $\mathcal{C}$ (and $\hat{\mathcal{C}}$) is called a* LLEE-chart.




*Example 3.4.* The charts above will serve as running example. (We take the liberty to omit branch labels in pictures.) $\mathcal{C}$ is the bisimulation collapse of the chart of the

star expression $(a \cdot ((a \cdot (b + b \cdot a))^{\circledast} c))^{\circledast} 0$, which in [9] is mentioned as problematic. Note that in the latter chart, the start vertex and the vertex reached after the second $b$ in the star expression are bisimilar; in $\mathcal{C}$ these vertices are collapsed to $v_1$. The challenge is to reconstruct a star expression from $\mathcal{C}$. Here $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ are the two possible, generically labeled, LLEE-witnesses; in $\hat{\mathcal{C}}_2$, $|\alpha| < |\beta|$, and $|\alpha| < |\gamma|$.

We refine the rules on page 3 by adding labels $l$ to transitions. In the rules below, a branch label is provided to transitions that cannot return to the star expression in their left-hand side. In particular, the third and fourth rule make a distinction on whether $e_1$ is *normed*, meaning it has a path to $\surd$, as only in the normed case $e_1{}^{\circledast}e_2$ can return to itself. Loop-entry transitions are provided with a loop label based on the *star height* $|e|_{\circledast}$ of their left-hand side $e$, denoting the maximum number of nestings of Kleene stars; it is defined by $|0|_{\circledast} = |a|_{\circledast} = 0$, $|f + g|_{\circledast} = |f \cdot g|_{\circledast} = \max\{|f|_{\circledast}, |g|_{\circledast}\}$, and $|f^{\circledast}g|_{\circledast} = \max\{|f|_{\circledast} + 1, |g|_{\circledast}\}$.

$$\frac{}{a \xrightarrow{a}_{\mathrm{br}} \surd} \qquad \frac{e_i \xrightarrow{a}_{l} \xi}{e_1 + e_2 \xrightarrow{a}_{\mathrm{br}} \xi}\; i \in \{1,2\} \qquad \frac{e_1 \xrightarrow{a}_{l} e_1'}{e_1 \cdot e_2 \xrightarrow{a}_{l} e_1' \cdot e_2} \qquad \frac{e_1 \xrightarrow{a}_{\mathrm{br}} \surd}{e_1 \cdot e_2 \xrightarrow{a}_{\mathrm{br}} e_2}$$

$$\frac{e_1 \xrightarrow{a}_{l} e_1'}{e_1{}^{\circledast}e_2 \xrightarrow{a}_{[|e_1|_{\circledast}+1]} e_1' \cdot (e_1{}^{\circledast}e_2)}\; \text{if } e_1 \text{ is normed} \qquad \frac{e_1 \xrightarrow{a}_{l} e_1'}{e_1{}^{\circledast}e_2 \xrightarrow{a}_{\mathrm{br}} e_1' \cdot (e_1{}^{\circledast}e_2)}\; \text{if } e_1 \text{ is not normed}$$

$$\frac{e_1 \xrightarrow{a}_{\mathrm{br}} \surd}{e_1{}^{\circledast}e_2 \xrightarrow{a}_{[|e_1|_{\circledast}+1]} e_1{}^{\circledast}e_2} \qquad \frac{e_2 \xrightarrow{a}_{l} \xi}{e_1{}^{\circledast}e_2 \xrightarrow{a}_{\mathrm{br}} \xi}$$

**Proposition 3.5.** *For each star expression $e$, the loop-labeling defined by the rules above is a LLEE-witness of the chart defined by the rules on p.3 with start vertex $e$.*

## 4 Extraction of star expressions from LLEE-charts

**Definition 4.1.** *A* provable solution *of a chart $\mathcal{C} = \langle V, A, T, v_{\mathrm{s}} \rangle$ is a function $s$ that maps the vertices in $V \backslash \{\surd\}$ to star expressions such that, for all $v \in V \backslash \{\surd\}$,*

$$s(v) \;=_{\mathsf{BBP}}\; a_1 + \cdots + a_m + b_1 \cdot s(w_1) + \cdots + b_n \cdot s(w_n)$$

*where $\{v \xrightarrow{a_i} \surd \mid i = 1, \ldots, m\} \cup \{v \xrightarrow{b_j} w_j \mid j = 1, \ldots, n \wedge w_j \neq \surd\}$ is the set of transitions from $v$ in $\mathcal{C}$. We call $s(v_{\mathrm{s}})$ the* principal value *of $s$.*

For the chart defined by the rules on page 3, with as start vertex some star expression $e$, mapping each vertex $f$ to itself yields a provable solution with principal value $e$.

The following proposition states that provable solutions of charts can be transferred backwards via functional bisimulations. Owing to the definition of bisimulation, its proof boils down to an easy application of axioms (A1,2,3).

**Proposition 4.2.** *Let $\phi : V_1 \to V_2$ be a bisimulation between $\mathcal{C}_1$ and $\mathcal{C}_2$. If $s$ is a provable solution of $\mathcal{C}_2$, then $s \circ \phi$ (with domain $V_1 \backslash \{\surd\}$) is a provable solution of $\mathcal{C}_1$. Note that it has the same principal value as $s$.*

We construct a provable solution $s_{\hat{\mathcal{C}}}$ with respect to any given LLEE-chart $\hat{\mathcal{C}}$. Consider a vertex $w \neq \surd$ with loop-entry transitions $\{w \xrightarrow{a_i}_{[\alpha_i]} w \mid i = 1, \ldots, m\} \cup \{w \xrightarrow{b_j}_{[\beta_j]} w_j \mid j = 1, \ldots, n \wedge w_j \neq w\}$ and branch transitions $\{w \xrightarrow{c_i}_{\mathrm{br}} \surd \mid i = 1, \ldots, p\} \cup \{w \xrightarrow{d_j}_{\mathrm{br}} u_j \mid j = 1, \ldots, q \wedge u_j \neq \surd\}$. We define $s_{\hat{\mathcal{C}}}$ by:

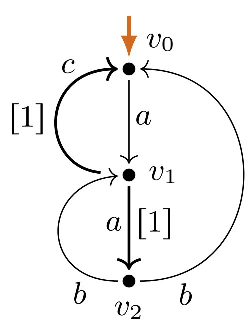
$$\begin{aligned} s_{\hat{\mathcal{C}}}(w) \;:=\; & (a_1 + \cdots + a_m + b_1 \cdot t_{\hat{\mathcal{C}}}(w_1, w) + \cdots + b_n \cdot t_{\hat{\mathcal{C}}}(w_n, w))^{\circledast} \\ & \qquad\qquad (c_1 + \cdots + c_p + d_1 \cdot s_{\hat{\mathcal{C}}}(u_1) + \cdots + d_q \cdot s_{\hat{\mathcal{C}}}(u_q)) \end{aligned}$$

by induction on the length $\|w\|_{\mathrm{br}}$ of a longest path of branch transitions from $w$, where $t_{\hat{\mathcal{C}}}(w, v)$ builds a star expression from $w$ inside a loop that started at $v$, meaning the construction completes when $v$ is reached. Consider distinct vertices $w, v \neq \surd$ where $w$ is in a loop from $v$. Let $w$ have loop-entry transitions $\{w \xrightarrow{a_i}_{[\alpha_i]} w \mid i = 1, \ldots, m\} \cup \{w \xrightarrow{b_j}_{[\beta_j]} w_j \mid j = 1, \ldots, n \wedge w_j \neq w\}$ and branch steps $\{w \xrightarrow{c_i}_{\mathrm{br}} v \mid i = 1, \ldots, p\} \cup \{w \xrightarrow{d_j}_{\mathrm{br}} u_j \mid j = 1, \ldots, q \wedge u_j \neq v\}$. We define, by induction on the maximal loop level of a loop at $v$ and with a subinduction on $\|w\|_{\mathrm{br}}$:

$$\begin{aligned} t_{\hat{\mathcal{C}}}(w, v) \;:=\; & (a_1 + \cdots + a_m + b_1 \cdot t_{\hat{\mathcal{C}}}(w_1, w) + \cdots + b_n \cdot t_{\hat{\mathcal{C}}}(w_n, w)))^{\circledast} \\ & \qquad (c_1 + \cdots + c_p + d_1 \cdot t_{\hat{\mathcal{C}}}(u_1, w) + \cdots + d_q \cdot t_{\hat{\mathcal{C}}}(u_q, v))) \end{aligned}$$

**Proposition 4.3.** *For any LLEE-witness $\hat{\mathcal{C}}$, $s_{\hat{\mathcal{C}}}$ is a provable solution of $\mathcal{C}$.*

*Example 4.4.* We exemplify the construction of a provable solution by means of the LLEE-witness $\hat{\mathcal{C}}_1$ of the running example, with $\alpha = \beta = 1$.



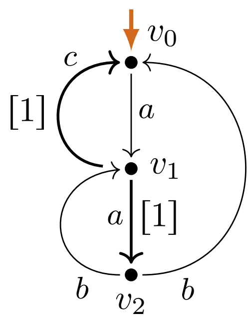

$\hat{\mathcal{C}}_1$

$$\begin{aligned}
s_{\hat{\mathcal{C}}_1}(v_0) &\equiv 0^{\circledast}(a \cdot s_{\hat{\mathcal{C}}_1}(v_1)) \\
&=_{\mathsf{BBP}} a \cdot (c \cdot a + a \cdot (b + b \cdot a))^{\circledast} 0 \\
s_{\hat{\mathcal{C}}_1}(v_1) &\equiv (c \cdot t_{\hat{\mathcal{C}}_1}(v_0, v_1) + a \cdot t_{\hat{\mathcal{C}}_1}(v_2, v_1))^{\circledast} 0 \\
&=_{\mathsf{BBP}} (c \cdot a + a \cdot (b + b \cdot a))^{\circledast} 0 \\
t_{\hat{\mathcal{C}}_1}(v_0, v_1) &\equiv 0^{\circledast} a \\
&=_{\mathsf{BBP}} a \\
t_{\hat{\mathcal{C}}_1}(v_2, v_1) &\equiv 0^{\circledast}(b + b \cdot t_{\hat{\mathcal{C}}_1}(v_0, v_1)) \\
&=_{\mathsf{BBP}} b + b \cdot a \\
s_{\hat{\mathcal{C}}_1}(v_2) &\equiv 0^{\circledast}(b \cdot s_{\hat{\mathcal{C}}_1}(v_1) + b \cdot s_{\hat{\mathcal{C}}_1}(v_0)) \\
&=_{\mathsf{BBP}} (b + b \cdot a) \cdot s_{\hat{\mathcal{C}}_1}(v_1) \\
&=_{\mathsf{BBP}} (b + b \cdot a) \cdot ((c \cdot a + a \cdot (b + b \cdot a))^{\circledast} 0)
\end{aligned}$$

Given a LLEE-witness $\hat{\mathcal{C}}$, it can be shown that each provable solution of the corresponding chart $\mathcal{C}$ is provably equal to $s_{\hat{\mathcal{C}}}$, which implies the following result.

**Proposition 4.5.** *Given provable solutions $s_1$ and $s_2$ of a LLEE-chart, always $s_1(v) =_{\mathsf{BBP}} s_2(v)$ for all vertices $v \neq \surd$.*

*Example 4.6.* Below an example is given in which a provable solution $s$ of the LLEE-witness $\hat{\mathcal{C}}_1$ is shown to be provably equal to the solution $s_{\hat{\mathcal{C}}_1}$ extracted in Ex. 4.4.




$\hat{\mathcal{C}}_1$

$$s(v_0) =^{(\mathrm{sol})}_{\mathsf{BBP}} a \cdot s(v_1) \quad (^{(\mathrm{sol})} \text{ means use of ‘provable solution’})$$

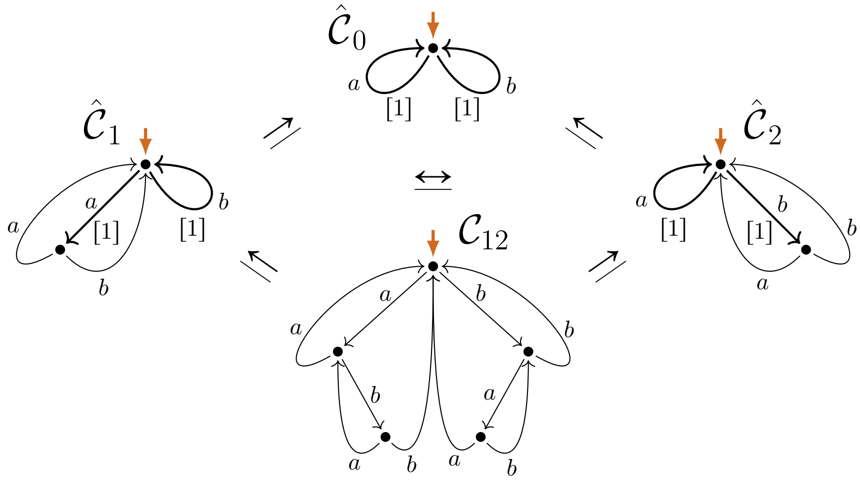
$$\begin{aligned}
s(v_1) &=^{(\mathrm{sol})}_{\mathsf{BBP}} c \cdot s(v_0) + a \cdot s(v_2) \\
&=^{(\mathrm{sol})}_{\mathsf{BBP}} c \cdot (a \cdot s(v_1)) + a \cdot (b \cdot s(v_1) + b \cdot s(v_0)) \\
&=^{(\mathrm{sol})}_{\mathsf{BBP}} c \cdot (a \cdot s(v_1)) + a \cdot (b \cdot s(v_1) + b \cdot (a \cdot s(v_1))) \\
&=_{\mathsf{BBP}} (c \cdot a + a \cdot (b + b \cdot a)) \cdot s(v_1) + 0 \\
&\Downarrow \text{ applying RSP}^{\circledast} \\
s(v_1) &=_{\mathsf{BBP}} (c \cdot a + a \cdot (b + b \cdot a))^{\circledast} 0 \\
\Rightarrow \quad s(v_1) &=_{\mathsf{BBP}} s_{\hat{\mathcal{C}}_1}(v_1) \quad \text{(see extraction example)} \\
\Rightarrow \quad s(v_0) &=^{(\mathrm{sol})}_{\mathsf{BBP}} a \cdot s(v_1) =_{\mathsf{BBP}} a \cdot s_{\hat{\mathcal{C}}_1}(v_1) =^{(\mathrm{sol})}_{\mathsf{BBP}} s_{\hat{\mathcal{C}}_1}(v_0) \\
\Rightarrow \quad s(v_2) &=^{(\mathrm{sol})}_{\mathsf{BBP}} b \cdot s(v_1) + b \cdot s(v_0) \\
&=_{\mathsf{BBP}} b \cdot s_{\hat{\mathcal{C}}_1}(v_1) + b \cdot s_{\hat{\mathcal{C}}_1}(v_0) \\
&=^{(\mathrm{sol})}_{\mathsf{BBP}} s_{\hat{\mathcal{C}}_1}(v_2)
\end{aligned}$$

## 5 Intermezzo: Structure of the completeness proof

Our next goal is to prove the heart of the matter: that the bisimulation collapse of a LLEE-chart is again a LLEE-chart. Then completeness of $\mathsf{BBP}$ can be argued as follows. Given two bisimilar star expressions $e_1$ and $e_2$, generate their charts $\mathcal{C}_1$ and $\mathcal{C}_2$, which are LLEE-charts according to Prop. 3.5. As remarked below Def. 4.1, $e_1$ and $e_2$ are principal values of provable solutions of $\mathcal{C}_1$ and $\mathcal{C}_2$. These charts have the same bisimulation collapse $\mathcal{C}$, which is again a LLEE-chart. Build a provable solution $s$ of $\mathcal{C}$, according to Prop. 4.3. Transfer $s$ backwards over the bisimulation collapses to provable solutions $s_1$ and $s_2$ of $\mathcal{C}_1$ and $\mathcal{C}_2$, according to Prop. 4.2. By construction, $s_1$ and $s_2$ have the same principal value $e$ as $s$. Finally, $e_1$ and $e_2$ are provably equal to $e$, according to Prop. 4.5. Hence, $e_1 =_{\mathsf{BBP}} e =_{\mathsf{BBP}} e_2$.

We note that in his completeness proof for regular expressions in formal language theory, Salomaa [17] has to move "upwards" from two equivalent regular expressions to a larger regular expression that can be functionally collapsed onto both of them. By contrast, our proof approach forces us "downwards" to the bisimulation collapse, as in the opposite direction the LLEE-witness may be lost.

*Example 5.1.* The example below illustrates why Salomaa's proof strategy to link two charts of regular expressions via their least common unfolding does not work for us. For the bisimilar LLEE-charts $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$, which correspond to the star expressions $(a \cdot (a + b) + b)^{\circledast}0$ and $(b \cdot (a + b) + a)^{\circledast}0$, their least common unfolding $\mathcal{C}_{12}$ is not a LLEE-chart. But their common bisimulation collapse $\hat{\mathcal{C}}_0$, which corresponds to the star expression $(a + b)^{\circledast}0$, is a LLEE-chart. ($\rightrightarrows$ denotes functional bisimilarity.)




## 6 Notations and lemmas regarding loops

We introduce notations and lemmas on transitions and loops that are used in Sect. 7. The definitions are with regard to a chart with a loop labeling. For a binary relation $R$, we write $R^+$ and $R^*$ for its transitive and transitive-reflexive closures.

$u \to_l v$ denotes there is a transition $u \xrightarrow{a}_l v$ for some $a \in A$, and in proofs (but not in pictures) $u \to v$ denotes that $u \to_l v$ for some label $l$. Let $u \xrightarrow[\not{\mathsf{t}}(w)]{}_l v$ denote that $u \to_l v$ and $v \neq w$; likewise, $u \xrightarrow[\not{\mathsf{t}}(w)]{} v$ denotes that $u \xrightarrow[\not{\mathsf{t}}(w)]{}_l v$ for some label $l$.

We write $u \curvearrowright v$, and say that $u$ *descends into a loop to* $v$, if there is a path $u \xrightarrow[\not{\mathsf{t}}(u)]{}_{[\alpha]} \xrightarrow[\not{\mathsf{t}}(u)]{}^*_{\mathrm{br}} v$. We write $v \circlearrowright u$, and say that $v$ *loops back to* $u$, if $u \curvearrowright v \to^+_{\mathrm{br}} u$. Let $\mathsf{scc}(u)$ denote the strongly connected component (scc) to which $u$ belongs.

**Lemma 6.1.** *In a LLEE-chart, if* $\mathsf{scc}(u) = \mathsf{scc}(v)$*, then* $u \curvearrowright^* v$ *implies* $v \circlearrowright^* u$*.*

*Proof.* We prove that $u \curvearrowright^n v$ implies $v \circlearrowright^n u$ for all $n \geqslant 0$, by induction on $n$. The base case $n = 0$ is trivial, as then $u = v$. If $n > 0$, $u \curvearrowright^{n-1} u' \curvearrowright v$ for some $u'$. Clearly $\mathsf{scc}(u) = \mathsf{scc}(u') = \mathsf{scc}(v)$. By induction, $u' \circlearrowright^{n-1} u$. Since $u' \curvearrowright v$, there is an acyclic path $u' \to_{[\alpha]} \cdot \to^*_{\mathrm{br}} v$. And since $\mathsf{scc}(u') = \mathsf{scc}(v)$, there is an acyclic path $v \to^*_{\mathrm{br}} \cdot \to_{[\beta_1]} \cdot \to^*_{\mathrm{br}} \cdots \cdot \to^*_{\mathrm{br}} \cdot \to_{[\beta_k]} \cdot \to^*_{\mathrm{br}} u'$. By (W2)(b), $|\alpha| > |\beta_1| > \cdots > |\beta_k| > |\alpha|$. This means $k = 0$, so $v \to^*_{\mathrm{br}} u'$. This implies $v \circlearrowright u'$ and hence $v \circlearrowright^n u$. $\square$

**Lemma 6.2.** *In a LLEE-chart,* $\mathsf{scc}(u) = \mathsf{scc}(v)$ *if and only if* $u \circlearrowright^* w$ *and* $v \circlearrowright^* w$ *for some* $w$*.*

*Proof.* If $u = v$, the lemma is trivial. Let $u \neq v$. Then they are on a cycle, which, since there is no branch cycle, contains a loop-entry transition from some $w$. Without loss of generality assume $w \neq u$. Then $w \curvearrowright^+ u$, so by Lem. 6.1, $u \circlearrowright^+ w$. If $w = v$ we have $v \circlearrowright^* w$, and if $w \neq v$ we can argue in the same fashion that $v \circlearrowright^+ w$. $\square$

We argue that $\circlearrowright^*$ is a partial order for LLEE-charts. By definition it is transitive–reflexive. Moreover, it is anti-symmetric, because $u \circlearrowright^+ v$ and $v \circlearrowright^+ u$ would induce a branch cycle from $u$ to $v$ and back, which cannot exist in a LLEE-chart.

**Lemma 6.3.** *In a LLEE-chart, $\circlearrowleft^*$ has the least-upper-bound property: if a non-empty set of vertices has an upper bound, then it has a least upper bound.*

*Proof.* Since the chart is finite, it suffices to show that for each vertex $v$ the set of vertices $x$ with $v \circlearrowleft^* x$ is totally ordered with regard to $\circlearrowleft^*$. Let $v \circlearrowleft^+ u_1$ and $v \circlearrowleft^+ u_2$ with $u_1 \neq u_2$. There is a path $u_1 \xrightarrow[\not\leftarrow(u_1)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(u_1)]{}^{*} v \to_{\text{br}}^{+} u_2 \xrightarrow[\not\leftarrow(u_2)]{}_{[\beta]} \cdot \xrightarrow[\not\leftarrow(u_2)]{}^{*} v \to_{\text{br}}^{+} u_1$. Without loss of generality, suppose $|\beta| \geqslant |\alpha|$. Then layeredness implies that the part $v \to_{\text{br}}^{+} u_2$ must visit $u_1$, so $v \xrightarrow[\not\leftarrow(u_2)]{}_{\text{br}}^{+} u_1 \to_{\text{br}}^{+} u_2$. Hence there is a path $u_2 \xrightarrow[\not\leftarrow(u_2)]{}_{[\beta]} \cdot \xrightarrow[\not\leftarrow(u_2)]{}^{*} v \xrightarrow[\not\leftarrow(u_2)]{}_{\text{br}}^{+} u_1 \to_{\text{br}}^{+} u_2$, which implies $u_1 \circlearrowleft^+ u_2$. □

We write $v \,{}_{d}\!\circlearrowleft u$, and say that $v$ *directly loops back to* $u$, if $v \circlearrowleft u$ and for all $w$ with $v \circlearrowleft w$ either $w = u$ or $u \circlearrowleft w$.

**Lemma 6.4.** *In a LLEE-chart, if $v_1 \,{}_{d}\!\circlearrowleft u$ and $v_2 \,{}_{d}\!\circlearrowleft u$ for distinct $v_1, v_2$, then there does not exist a vertex $w$ such that both $w \circlearrowleft^* v_1$ and $w \circlearrowleft^* v_2$.*

*Proof.* $\neg(v_1 \circlearrowleft^+ v_2)$ and $\neg(v_2 \circlearrowleft^+ v_1)$, for else the definition of ${}_{d}\!\circlearrowleft$ would imply $u \circlearrowleft^* v_2$ or $u \circlearrowleft^* v_1$, and so $v_2 \circlearrowleft^+ v_2$ or $v_1 \circlearrowleft^+ v_1$, contradicting irreflexivity of $\circlearrowleft^+$. In the proof of Lem. 6.3 we saw that for each $w$, $\{x \mid w \circlearrowleft^* x\}$ is totally ordered with regard to $\circlearrowleft^*$, which implies such sets cannot contain both $v_1$ and $v_2$. □

We write $u \to_{lb} v$, called a *loop-back transition*, if $u \to_{\text{br}} v$ and $\mathsf{scc}(u) = \mathsf{scc}(v)$. The *loops-back-to norm* $\|u\|_{lb}^{\min}$ is the length of a longest chain of loop-back transitions from $u$. In a LLEE-chart this is well-defined, owing to the absence of branch cycles. We note that $\|u\|_{lb}^{\min} = 0$ if and only if $\neg(u \circlearrowleft)$.

## 7 Preservation of LLEE under bisimulation collapse

The bisimulation collapse of a LLEE-chart will be constructed in a step-wise fashion, collapsing one pair of bisimilar vertices $w_1$ and $w_2$ at a time, whereby the incoming transitions of $w_1$ are redirected to $w_2$. The crux is to take care (and prove) that LLEE is preserved, that is, that the resulting chart still has a LLEE-witness.

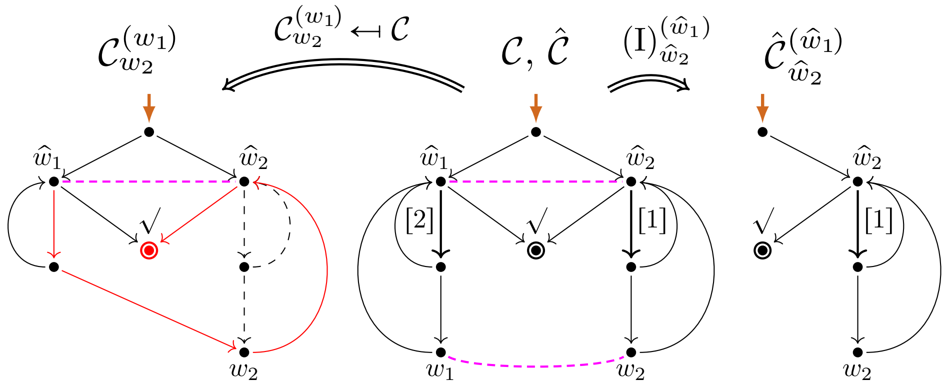
**Definition 7.1.** *Given a chart $\mathcal{C}$ and vertices $w_1, w_2$, the* connect-$w_1$-through-to-$w_2$ chart *$\mathcal{C}_{w_2}^{(w_1)}$ is obtained by eliminating $w_1$ and redirecting its incoming transitions to $w_2$. If $w_1$ is the start vertex of $\mathcal{C}$, then $w_2$ becomes the start vertex of $\mathcal{C}_{w_2}^{(w_1)}$.*

*Given a loop-labeling $\hat{\mathcal{C}}$, loop-entry and branch labels of transitions to $w_1$ in $\hat{\mathcal{C}}$ are inherited by their redirections to $w_2$ in $\mathcal{C}_{w_2}^{(w_1)}$, except if a redirection coincides with a transition to $w_2$ in $\mathcal{C}$, in which case the label of the existing transition has precedence over the redirected one. The resulting loop-labeling is denoted by $\hat{\mathcal{C}}_{w_2}^{(w_1)}$.*

It is not hard to see that if $w_1 \leftrightarrows w_2$, then $\mathcal{C}_{w_2}^{(w_1)} \leftrightarrows \mathcal{C}$.

While the process semantics of a chart is preserved from $\mathcal{C}$ to $\mathcal{C}_{w_2}^{(w_1)}$ for bisimilar vertices $w_1$ and $w_2$, this does not hold for the property LLEE.

*Example 7.2.* Depicted below is a LLEE-chart $\mathcal{C}$ in which the action labels are unspecified, but assumed to facilitate that $w_1$ and $w_2$ are bisimilar, and then so are $\widehat{w}_1$ and $\widehat{w}_2$. Bisimilarity is indicated by the magenta dashed links. Then $\mathcal{C}^{(w_1)}_{w_2}$ as drawn on the left is not a LLEE-chart: after removing the dashed downwards transition from $\widehat{w}_2$, which induces a loop, and two other dashed transitions that get unreachable, the remaining chart still has an infinite path, but it does not contain a loop subchart. Namely, each infinite path can reach termination $\surd$ without returning to its source. An example of this is the red path from $\widehat{w}_1$ to $\surd$. However, the pair $\langle w_1, w_2\rangle$ progresses to $\langle \widehat{w}_1, \widehat{w}_2\rangle$ under bisimilarity. Then the connect-$\widehat{w}_1$-through-to-$\widehat{w}_2$ chart on the right is a LLEE-chart, as is witnessed by the loop-labeling $\hat{\mathcal{C}}^{(\widehat{w}_1)}_{\widehat{w}_2}$:




Together with two further examples on pages 13 and 14, the example above illustrates that the bisimilar pair of vertices must be selected carefully, to safeguard that the connect-through construction preserves LLEE. The proposition below expresses that a bisimilar pair of vertices can always be selected in one of three mutually exclusive categories. Subsequently three LLEE-preserving transformation I, II, and III are defined for each of these categories. (Transformation I already appeared in the example above.) Only in the first category are $w_1$ and $w_2$ not in the same scc: there is no path from $w_2$ to $w_1$; moreover, $w_1$ does not loop back. In the second, $w_2$ loops back to $w_1$. In the third, $w_1$ and $w_2$ loop back to the same vertex $v$; moreover, $w_1$ directly loops back to $v$, and there is no branch path from $w_2$ to $w_1$.

In the proof of the proposition below, from a given pair of distinct bisimilar vertices we repeatedly progress via transitions, at one side picking loop-back transitions, over pairs of distinct bisimilar vertices, until one of three conditions is met.

**Proposition 7.3.** *Consider a LLEE-chart $\hat{\mathcal{C}}$ that is not a bisimulation collapse. There are bisimilar vertices $w_1, w_2$ such that either (C1) $\neg(w_1 \circlearrowleft) \wedge \neg(w_2 \to^* w_1)$, or (C2) $w_2 \circlearrowleft^+ w_1$, or (C3) $\exists v \in V\big(w_1 \,{}_d{\circlearrowleft}\, v \wedge w_2 \circlearrowleft^+ v\big) \wedge \neg(w_2 \to^*_{\mathrm{br}} w_1)$.*

*Proof.* Pick distinct bisimilar vertices $u_1, u_2$. First consider the case $\mathsf{scc}(u_1) \neq \mathsf{scc}(u_2)$. Without loss of generality, assume $\neg(u_2 \to^* u_1)$. We show by induction on $\|u_1\|^{\min}_{lb}$ that (1) holds. In the base case, $\|u_1\|^{\min}_{lb} = 0$ implies $\neg(u_1 \circlearrowleft)$, so we can define $w_1 = u_1$ and $w_2 = u_2$ and are done. In the induction step, $\|u_1\|^{\min}_{lb} > 0$ implies $u_1 \to_{lb} u_1'$ and $\|u_1'\|^{\min}_{lb} < \|u_1\|^{\min}_{lb}$ for some $u_1'$. Since $u_1 \mathrel{\underline{\leftrightarrow}} u_2$, we have $u_2 \to u_2'$ and $u_1' \mathrel{\underline{\leftrightarrow}} u_2'$ for some $u_2'$. Suppose, toward a contradiction, that $u_2' \to^* u_1'$. Since

$u_1 \to_{lb} u_1'$, by definition, $u_1$ and $u_1'$ are in the same scc. Hence $u_1' \to^* u_1$, and so $u_2 \to u_2' \to^* u_1' \to^* u_1$. This however contradicts the assumption $\neg(u_2 \to^* u_1)$. So we conclude $\neg(u_2' \to^* u_1')$. Since $u_1' \,\underline{\leftrightarrow}\, u_2'$ and $\neg(u_2' \to^* u_1')$ and $\|u_1'\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$, by induction there exist bisimilar $w_1, w_2$ for which (1) holds.

Now let $\mathsf{scc}(u_1) = \mathsf{scc}(u_2)$. Then by Lem. 6.2 $u_1 \curvearrowleft^* v$ and $u_2 \curvearrowleft^* v$ for some $v$. By Lem. 6.3 we can pick $v$ as the least upper bound of $u_1, u_2$ with regard to $\curvearrowleft^*$. If $u_1$ (or $u_2$) $= v$, then $u_2$ (or $u_1$) $\curvearrowleft^+ u_1$, so (2) holds for $w_1 = u_1$ (or $u_2$) and $w_2 = u_2$ (or $u_1$). Now let $u_1, u_2 \neq v$. Since $v$ is the least upper bound, $u_1 \curvearrowleft^* v_1 \;{}_{d}\!\curvearrowleft v$ and $u_2 \curvearrowleft^* v_2 \;{}_{d}\!\curvearrowleft v$ for distinct $v_1, v_2 \in V$. There cannot be a cycle of branch transitions, so $\neg(v_2 \to_{\mathrm{br}}^* v_1)$ or $\neg(v_1 \to_{\mathrm{br}}^* v_2)$. By symmetry it suffices to consider only $\neg(v_2 \to_{\mathrm{br}}^* v_1)$. We apply induction on $\|u_1\|_{lb}^{\min}$. If $u_1 = v_1$, then $u_1 \;{}_{d}\!\curvearrowleft v$; taking $w_1 = u_1$ and $w_2 = u_2$, (3) holds. So we can assume $u_1 \curvearrowleft^+ v_1 \;{}_{d}\!\curvearrowleft v$. Pick a transition $u_1 \to_{lb} u_1'$ with $\|u_1'\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$; by definition, $\mathsf{scc}(u_1') = \mathsf{scc}(u_1)$. Since $u_1 \,\underline{\leftrightarrow}\, u_2$, there is a transition $u_2 \to u_2'$ with $u_1' \,\underline{\leftrightarrow}\, u_2'$ for some $u_2'$. If $\mathsf{scc}(u_1') \neq \mathsf{scc}(u_2')$, then as before we can find bisimilar $w_1, w_2$ for which (1) holds. Now let $\mathsf{scc}(u_1') = \mathsf{scc}(u_2')$. Since $u_1 \curvearrowleft^+ v_1$ and $u_1 \to u_1'$, either $u_1' = v_1$ or $v_1 \rightsquigarrow^+ u_1'$. Moreover, $\mathsf{scc}(u_1') = \mathsf{scc}(u_1) = \mathsf{scc}(v_1)$, so by Lem. 6.1, $u_1' \curvearrowleft^* v_1$. We distinguish two cases.

*Case 1:* $u_2 \curvearrowleft^+ v_2$. Since $u_2 \to u_2'$, either $u_2' = v_2$ or $v_2 \rightsquigarrow^+ u_2'$. Moreover, $\mathsf{scc}(u_2') = \mathsf{scc}(u_1') = \mathsf{scc}(u_1) = \mathsf{scc}(u_2) = \mathsf{scc}(v_2)$, so by Lem. 6.1, $u_2' \curvearrowleft^* v_2$. Since $v_1, v_2 \;{}_{d}\!\curvearrowleft v$, $u_1' \curvearrowleft^* v_1$, $u_2' \curvearrowleft^* v_2$, and $v_1 \neq v_2$, by Lem. 6.4, $u_1' \neq u_2'$. Note that $u_i' \curvearrowleft^* v_i \;{}_{d}\!\curvearrowleft v$ implies $u_i' \neq v$ for $i = 1, 2$. Since $\|u_1'\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$, and moreover $u_1', u_2' \neq v$, $\mathsf{scc}(u_1') = \mathsf{scc}(u_2')$, $u_i' \curvearrowleft^* v_i \;{}_{d}\!\curvearrowleft v$ for $i = 1, 2$, and $\neg(v_2 \to_{\mathrm{br}}^* v_1)$, by induction there is a bisimilar pair $w_1, w_2$ for which (1), (2) or (3) holds.

*Case 2:* $u_2 = v_2$. We distinguish two cases.

*Case 2.1:* $u_2 \to_{[\alpha]} u_2'$. Then either $u_2' = u_2$ or $u_2 \rightsquigarrow^+ u_2'$. Moreover, $\mathsf{scc}(u_2') = \mathsf{scc}(u_2)$, so by Lem. 6.1, $u_2' \curvearrowleft^* u_2$, and hence $u_2' \curvearrowleft^* v_2$. Now this case can be dealt with in exactly the same way as case 1.

*Case 2.2:* $u_2 \to_{\mathrm{br}} u_2'$. Then $\neg(v_2 \to_{\mathrm{br}}^* v_1)$ together with $v_2 = u_2 \to_{\mathrm{br}} u_2'$ and $u_1' \curvearrowleft^* v_1$ imply $u_1' \neq u_2'$. We distinguish two cases.

*Case 2.2.1:* $u_2' = v$. Then $u_1' \curvearrowleft^* v_1 \;{}_{d}\!\curvearrowleft v = u_2'$, i.e., $u_1' \curvearrowleft^+ u_2'$, so we are done, because (2) holds for $w_1 = u_2'$ and $w_2 = u_1'$.

*Case 2.2.2:* $u_2' \neq v$. By Lem. 6.1, $u_2' \curvearrowleft^+ v$. Hence, $u_2' \curvearrowleft^* v_3 \;{}_{d}\!\curvearrowleft v$ for some $v_3$. Since $v_2 = u_2 \to_{\mathrm{br}} u_2' \curvearrowleft^* v_3$ and $\neg(v_2 \to_{\mathrm{br}}^* v_1)$, it follows that $\neg(v_3 \to_{\mathrm{br}}^* v_1)$. Since $\|u_1'\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$, and moreover $u_1', u_2' \neq v$, $\mathsf{scc}(u_1') = \mathsf{scc}(u_2')$, $u_1' \curvearrowleft^* v_1 \;{}_{d}\!\curvearrowleft v$, $u_2' \curvearrowleft^* v_3 \;{}_{d}\!\curvearrowleft v$, and $\neg(v_3 \to_{\mathrm{br}}^* v_1)$, by induction there is a bisimilar pair $w_1, w_2$ such that (1), (2) or (3) holds. □

Given, in a LLEE-chart $\hat{\mathcal{C}}$, bisimilar vertices $w_1, w_2$ that satisfy condition (1), (2) or (3) of Prop. 7.3, we eliminate $w_1$ by constructing the connect-$w_1$-through-to-$w_2$ chart $\mathcal{C}_{w_2}^{(w_1)}$ (see Def. 7.1). We number the transformations associated to (1), (2) and (3) as I, II and III, respectively. In each transformation an adaptation of labels of transitions is performed, to avoid violations of LLEE-witness properties in the result. In transformations I and III the adaptation is performed before eliminating $w_1$, and is needed to guarantee that layeredness is preserved; in transformation II it is performed right after eliminating $w_1$, and avoids the creation of branch cycles. The adaptations for the three transformations are:

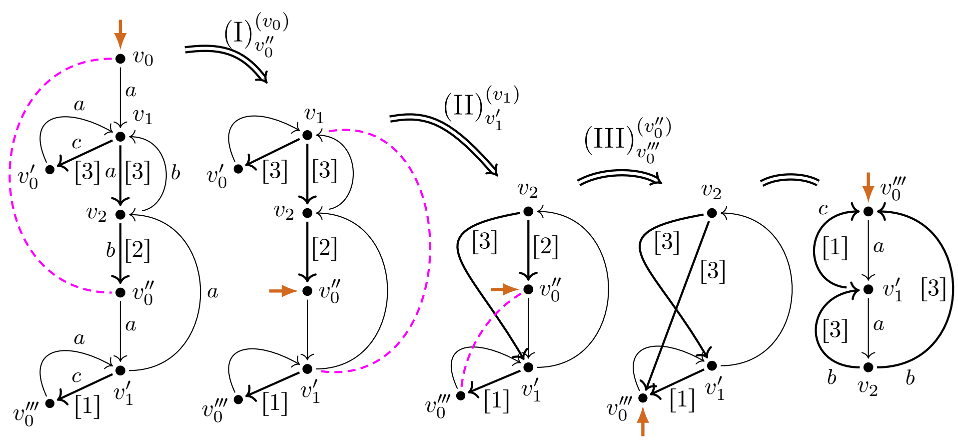
$L_{\mathrm{I}}$ Let $m = \max\{\,|\beta| : \text{there is a path } w_2 \to^* \cdot \to_{[\beta]} \text{ in } \hat{\mathcal{C}}\,\}$. In loop-entry transitions $u \to_{[\alpha]} v$ for which there is a path $v \to^* w_1$ in $\mathcal{C}$, replace $\alpha$ by an $\alpha'$ with $|\alpha'| = |\alpha| + m$. This ensures that the loop labels of loop-entry transitions that descend to $w_1$ are of a higher level than of the loop labels reachable from $w_2$.

$L_{\mathrm{II}}$ Since $w_2 \curvearrowleft^+ w_1$, there exists a $\widehat{w}_2$ with $w_2 \curvearrowleft^* \widehat{w}_2 \; {}_d\!\curvearrowleft \; w_1$. Let $\gamma$ be a loop name of maximum loop level among the loop-entries at $w_1$ in $\hat{\mathcal{C}}$. (Note that since $w_2 \curvearrowleft^+ w_1$, there is at least one such transition.) Turn the branch transitions from $\widehat{w}_2$ into loop-entry transitions with loop label $\gamma$.

$L_{\mathrm{III}}$ Let $\gamma$ be a loop label of maximum level among the loop-entry transitions at $v$ in $\hat{\mathcal{C}}$. (Note that since $w_1 \curvearrowleft v$, there is at least one such transition.) Turn the loop labels of the loop-entry transitions from $v$ into $\gamma$.

Each of these transformations ends with a clean-up step: if the loop-entry transitions from a vertex with the same loop label do not longer induce an infinite path (due to the removal of $w_1$), then they are changed into branch transitions.

*Example 7.4.* Below, the LLEE-chart at the left is in three transformation steps reduced to our running example at the right, the LLEE-witness $\hat{\mathcal{C}}_2$ of the bisimulation collapse of the chart of $(a \cdot ((a \cdot (b \cdot a + b))^{\circledast} c))^{\circledast} 0$. Dashed lines connect bisimilar vertices. In the first step, a transformation I, the start state $v_0$ is connected through to the bisimilar vertex $v_0''$, whereby $v_0''$ becomes the start vertex; note that $v_0$ and $v_0''$ are not in the same scc and $v_0$ does not loop back. In the second step, a transformation II, $v_1$ is connected through to the bisimilar vertex $v_1'$; note that $v_1' \curvearrowleft^+ v_1$. In the third step, a transformation III, the start vertex $v_0''$ is connected through to the bisimilar vertex $v_0'''$, whereby $v_0'''$ becomes the start vertex; note that $v_0'' \; {}_d\!\curvearrowleft \; v_2$ and $v_0''' \curvearrowleft^+ v_2$ and there is no branch path from $v_0'''$ to $v_0''$; by the loop level adaptation, all loop entries from $v_2$ get level 3. The final step is an isomorphic deformation. Only the charts at the left and right depict actions on transitions.



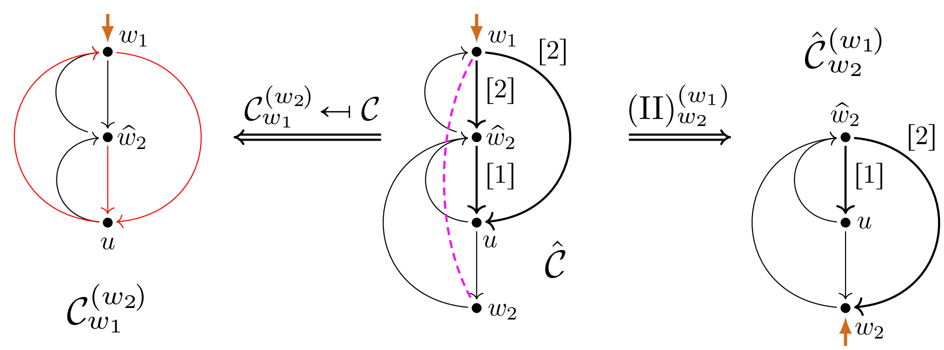

**Proposition 7.5.** *Let $\mathcal{C}$ be a LLEE-chart. If a pair $w_1, w_2$ of distinct, bisimilar vertices satisfies (C1), (C2), or (C3), then $\mathcal{C}^{(w_1)}_{w_2}$ (see Def. 7.1) is a LLEE-chart.*

As background for the proof of this proposition, we give first examples why conditions (C1), (C2), an (C3) cannot be readily relaxed or changed. For convenience, the pictures in these examples neglect action labels on transitions.

For showing that in condition (C1) it is crucial that $w_1$ does not loop back, we refer back to the illustration on page 10. In the LLEE-witness $\hat{\mathcal{C}}$ there, $\neg(w_2 \rightarrow^* w_1)$, but condition (C1) is not satisfied by $w_1, w_2$ because $w_1 \circlearrowleft \widehat{w}_1$. Since in $\hat{\mathcal{C}}$ the loop levels of loop-entry transitions that descend to $w_1$ are higher than the loop levels that descend from $w_2$, the preprocessing step of transformation I is void. The chart $\mathcal{C}^{(w_1)}_{w_2}$ at the left has no LLEE-witness. For the progressed bisimilar pair $\widehat{w}_1, \widehat{w}_2$, condition (C1) is satisfied. Since $\hat{\mathcal{C}}^{(\widehat{w}_1)}_{\widehat{w}_2}$ on the right is obtained by applying transformation I to this pair, it as a LEE-witness by the proof of Prop. 7.5.

To avoid the creation of branch cycles in transformation II, it would seem expedient to connect transitions to $w_2$ through to $w_1$, since condition (C2), $w_2 \circlearrowleft^+ w_1$, rules out the existence of a path $w_1 \rightarrow^+_{\mathrm{br}} w_2$ in $\hat{\mathcal{C}}$. (Instead, we connect transitions to $w_1$ through to $w_2$, and resulting branch cycles are eliminated by turning the branch transitions at $\widehat{w}_2$ into loop-entry transitions.) However, connecting transitions to $w_2$ through to $w_1$ may produce a chart for which no LLEE-witness exists.

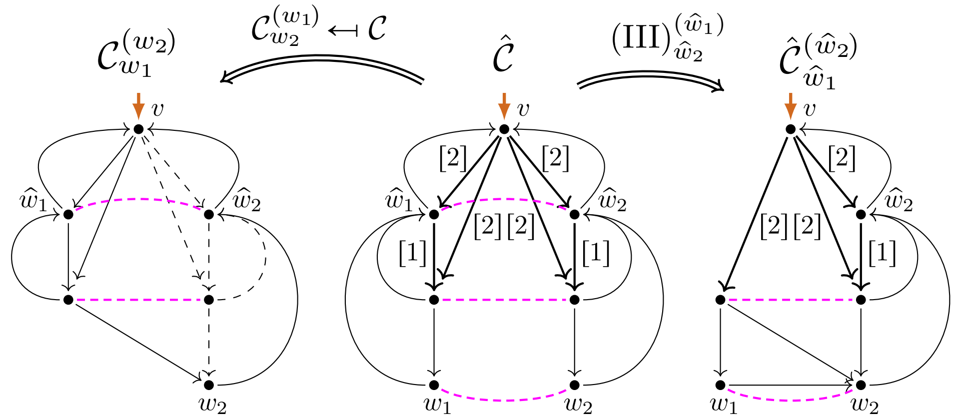
*Example 7.6.* For the LLEE-chart $\hat{\mathcal{C}}$ in the middle, the chart $\mathcal{C}^{(w_2)}_{w_1}$ at the left does not have a LLEE-witness. Namely, it has an infinite path but no loop subchart, as from each of its three vertices an infinite path starts that does not return to this vertex. For $\widehat{w}_2$ this path is drawn in red. Transformation II applied to $\hat{\mathcal{C}}$ yields the loop-labeling $\hat{\mathcal{C}}^{(w_1)}_{w_2}$ with additionally $\widehat{w}_2 \rightarrow_{\mathrm{br}} w_2$ turned into $\widehat{w}_2 \rightarrow_{[2]} w_2$. The proof of Prop. 7.5 guarantees that this loop-labeling, drawn on the right, is a LLEE-witness.




The following example shows that for transformation III it is essential to select a bisimilar pair $w_1, w_2$ where $w_1$ *directly* loops back to $v$.

*Example 7.7.* Consider the picture below. In the LLEE-chart $\hat{\mathcal{C}}$ in the middle, $w_1, w_2 \circlearrowleft^+ v$ and there is no branch path from $w_2$ to $w_1$, but condition (C3) does not hold for $w_1, w_2$ because $\neg(w_1 \;{}_d\!\circlearrowleft v)$. Note that all loop-entry transitions from $v$ have the same loop label, so the preprocessing step of transformation III is void. The chart $\mathcal{C}^{(w_1)}_{w_2}$ drawn at the left does not have a LLEE-witness. The transition from $\widehat{w}_2$ can be declared a loop-entry transition, and after its removal also two transitions from $v$ can be declared loop-entry transitions, leading to the removal of the five transitions that are depicted as dashed arrows. The remaining chart (of solid arrows) however still has an infinite path but no further loop subcharts. Namely, from each of its vertices an infinite path starts that does not return to this vertex. For the bisimilar pair $\widehat{w}_1, \widehat{w}_2$ in $\hat{\mathcal{C}}$, (C3) is satisfied. Transformation III applied to this pair in $\hat{\mathcal{C}}$ yields the chart $\hat{\mathcal{C}}^{(w_1)}_{w_2}$ drawn at the right, which according to Prop.

7.5 is a LLEE-chart. The remaining two bisimilar pairs can be eliminated by one or two further applications of transformation III.




*Proof (of Prop. 7.5).* Let $\hat{\mathcal{C}}$ be a LLEE-witness of a LLEE-chart $\mathcal{C}$. For vertices $w_1$, $w_2$ such that (C1), (C2), or (C3) holds, transformation I, II, or III, respectively, as defined above produces a loop-labeling $\hat{\mathcal{C}}^{(w_1)}_{w_2}$. We show that it is a LLEE-witness.

We first argue that it suffices to show that each of the transformations I, II, and III produces, before the final clean-up step, a loop-labeling that satisfies the LLEE-conditions with the exception of possible violations of the loop property (L1) in (W2)(a). The reason is that violations of (L1) can be removed from a loop-labeling while preserving the other LLEE-witness conditions. For showing this, suppose (L1) is violated in some $\mathcal{C}_{\hat{\mathcal{C}}}(u, \alpha)$. Then $u \to_{[\alpha]}$ but $\neg(u \to_{[\alpha]} \cdot \to^{*}_{\mathrm{br}} u)$. Let $\hat{\mathcal{C}}_1$ be the result of removing this violation by changing the $\alpha$-loop-entry transitions from $u$ into branch transitions. No new violation of (L1) is introduced in $\hat{\mathcal{C}}_1$. (W1) and (W2)(a), (L2), are preserved in $\hat{\mathcal{C}}_1$ because an introduced infinite branch step path in $\hat{\mathcal{C}}_1$ would be a branch step cycle that stems from a path $u \to_{[\alpha]} u' \to^{*}_{\mathrm{br}} u$ in $\hat{\mathcal{C}}$. (W2)(b) might only be violated by a path $w \xrightarrow[\not t(w)]{}_{[\beta]} \cdot \xrightarrow[\not t(w)]{}^{*}_{\mathrm{br}} u \xrightarrow[\not t(w,u)]{}_{\mathrm{br}} u' \xrightarrow[\not t(w,u)]{}^{*}_{\mathrm{br}} \cdot \to_{[\gamma]}$ with $|\beta| \leqslant |\gamma|$ in $\hat{\mathcal{C}}_1$ where $u \to_{\mathrm{br}} u'$ stems from $u \to_{[\alpha]} u'$ in $\hat{\mathcal{C}}$; then $|\beta| > |\alpha| > |\gamma|$ by layeredness of $\hat{\mathcal{C}}$; so (W2)(b) is preserved. Analogously we find that also (W2)(a), (L3), is preserved because $\surd$ was not contained in $\mathcal{C}_{\hat{\mathcal{C}}}(u, \alpha)$ as $\hat{\mathcal{C}}$ is a LLEE-witness.

To show correctness of transformation I, consider vertices $w_1$ and $w_2$ such that (C1) holds. We show that the result $\hat{\mathcal{C}}^{(w_1)}_{w_2}$ of transformation I before the clean-up step satisfies the LLEE-witness properties, except for possible violations of (L1).

To verify (W1) and the (L2) part of (W2)(a), it suffices to show that $\hat{\mathcal{C}}^{(w_1)}_{w_2}$ does not contain branch cycles. The original loop-labeling $\hat{\mathcal{C}}$ is a LLEE-witness, so it does not contain branch cycles. Since the level adaptation step does not turn loop-entry steps into branch steps, branch cycles could only arise in the step connecting $w_1$ through to $w_2$. Suppose that such a branch cycle arises. Then there must be a transition $u \to_{\mathrm{br}} w_1$ in $\hat{\mathcal{C}}$ (which is redirected to $w_2$ in $\hat{\mathcal{C}}^{(w_1)}_{w_2}$), and a path $w_2 \to^{*}_{\mathrm{br}} u$ in $\hat{\mathcal{C}}$. But then $w_2 \to^{*}_{\mathrm{br}} u \to_{\mathrm{br}} w_1$ in $\mathcal{C}$, which contradicts (C1) that there is no path from $w_2$ to $w_1$. We conclude that (W1) and part (L2) of (W2)(a) hold for $\hat{\mathcal{C}}^{(w_1)}_{w_2}$.

Now we verify the (L3) part of (W2)(a) on $\hat{\mathcal{C}}^{(w_1)}_{w_2}$, that $\surd$ is not reachable by descending via a loop-entry step without returning. Only the redirection step could introduce a violation, due to paths $u \xrightarrow[\not\mathrm{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathrm{t}(u)]{}^{*}_{\mathrm{br}} w_1$ and $w_2 \xrightarrow[\not\mathrm{t}(u)]{}^{+}_{\mathrm{br}} \surd$ in $\hat{\mathcal{C}}$ for some vertex $u \neq w_1$. But since the first of these paths implies $\curvearrowright w_1$, such a combination of paths is excluded by (C1).

Finally we show that (W2)(b) is preserved in $\hat{\mathcal{C}}^{(w_1)}_{w_2}$ by both the level adaptation and the connect-through step. First, since in the level adaptation step all adapted loop labels are increased with the same value $m$, a violation of (W2)(b) would arise by a path $u \to_{[\alpha]} \cdot \to^{*}_{\mathrm{br}} \cdot \to_{[\beta]} v$ in $\hat{\mathcal{C}}$ where the loop label $\beta$ is increased while $\alpha$ is not. But such a path cannot exist. Since $\beta$ is increased, there is a path $v \to^{*} w_1$ in $\mathcal{C}$. But then there is a path $u \to_{[\alpha]} \cdot \to^{+} v \to^{*} w_1$ in $\hat{\mathcal{C}}$, which implies that also $\alpha$ is increased in the level adaptation step. Second, a violation of (W2)(b) in the connect-through step would arise from paths $u \to_{[\alpha]} \cdot \to^{*}_{\mathrm{br}} w_1$ and $w_2 \to^{*}_{\mathrm{br}} \cdot \to_{[\beta]}$ in $\hat{\mathcal{C}}'$ with $|\alpha| \leqslant |\beta|$. However, in view of the path $u \to_{[\alpha]} \cdot \to^{*} w_1$, the loop label $\alpha$ was increased with $m$ in the level adaptation step . On the other hand, in view of (C1) that there is no path from $w_2$ to $w_1$ in $\mathcal{C}$, $w_1$ is unreachable at the end of the path $w_2 \to^{*} \cdot \to_{[\beta]}$. Hence this loop label $\beta$ was not increased in the level adaptation step. So it is guaranteed that for such a pair of paths in $\hat{\mathcal{C}}^{(w_1)}_{w_2}$ always $|\alpha| > |\beta|$.

We conclude that the result of transformation I is again a LLEE-witness.

In order to show correctness of transformation III, we consider vertices $w_1$ and $w_2$ such that (C3) holds. Let $v$ be such that $w_1 \;{}_{d}\!\circlearrowright v \circlearrowleft^{+} w_2$. We show that its intermediate result $\hat{\mathcal{C}}^{(w_1)}_{w_2}$ before the clean-up step satisfies the LLEE-witness properties, except for possible violations of (L1).

First we show that (W2)(b) is preserved by both the level adaptation and the connect-through step. A violation arising by the first step, i.e., in $\hat{\mathcal{C}}'$, would involve a path $u \xrightarrow[\not\mathrm{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathrm{t}(u)]{}^{*}_{\mathrm{br}} v \to_{[\beta]}$ in $\hat{\mathcal{C}}$ where $\beta$ is increased to a loop label $\gamma$ of maximum level among all loop-entries at $v$. But in this way no violation can arise, since there was already a path $u \xrightarrow[\not\mathrm{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathrm{t}(u)]{}^{*}_{\mathrm{br}} v \to_{[\gamma]}$ in $\hat{\mathcal{C}}$, so $|\alpha| > |\gamma| \geqslant |\beta|$.

Now we exclude violations of (W2)(b) in the connect-through step, by showing that in $\hat{\mathcal{C}}^{(w_1)}_{w_2}$, $|\alpha| > |\beta|$ for all newly created paths $u \xrightarrow[\not\mathrm{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathrm{t}(u)]{}^{*}_{\mathrm{br}} \cdot \to_{[\beta]}$ with $u \neq w_1$ that stem from paths $u \xrightarrow[\not\mathrm{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathrm{t}(u)]{}^{*}_{\mathrm{br}} w_1$ and $w_2 \xrightarrow[\not\mathrm{t}(u)]{}^{*}_{\mathrm{br}} \cdot \to_{[\beta]}$ in $\hat{\mathcal{C}}'$. As $w_2 \circlearrowright^{+} v$, there is a path $v \xrightarrow[\not\mathrm{t}(v)]{}_{[\gamma]} \cdot \xrightarrow[\not\mathrm{t}(v)]{}^{*}_{\mathrm{br}} w_2$ in $\hat{\mathcal{C}}'$. We distinguish two cases.

Case 1: $u = v$. Then, by the level adaptation step, $\alpha = \gamma$. Since $u = v$, there is a path $v \xrightarrow[\not\mathrm{t}(v)]{}_{[\gamma]} \cdot \xrightarrow[\not\mathrm{t}(v)]{}^{*}_{\mathrm{br}} w_2 \xrightarrow[\not\mathrm{t}(v)]{}^{*}_{\mathrm{br}} \cdot \to_{[\beta]}$ in $\hat{\mathcal{C}}'$. By (W2)(b) for $\hat{\mathcal{C}}'$, $|\gamma| > |\beta|$.

Case 2: $u \neq v$. Since $w_1 \;{}_{d}\!\circlearrowright v$, there is a path $w_1 \to^{+}_{\mathrm{br}} v$ in $\hat{\mathcal{C}}$ and thus in $\hat{\mathcal{C}}'$. Suppose toward a contradiction that this path visits $u$. Then $u \xrightarrow[\not\mathrm{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathrm{t}(u)]{}^{*}_{\mathrm{br}} w_1 \to^{+}_{\mathrm{br}} u$, so $w_1 \circlearrowright u$ in $\hat{\mathcal{C}}'$ and thus in $\hat{\mathcal{C}}$. Then $w_1 \;{}_{d}\!\circlearrowright v$ and $u \neq v$ imply $v \circlearrowright u$, which together with $u \to^{+}_{\mathrm{br}} v$ yields a branch cycle between $u$ and $v$ in $\hat{\mathcal{C}}$. This contradicts that (W1) holds in $\hat{\mathcal{C}}$. Therefore $w_1 \xrightarrow[\not\mathrm{t}(u)]{}^{+}_{\mathrm{br}} v$ in $\hat{\mathcal{C}}'$. We consider two cases.

Case 2.1: The path $w_2 \underset{\not\leftarrow(u)}{\longrightarrow}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ visits $v$, so $v \underset{\not\leftarrow(u)}{\longrightarrow}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}'$. Then $u \underset{\not\leftarrow(u)}{\longrightarrow}{}_{[\alpha]} \cdot \underset{\not\leftarrow(u)}{\longrightarrow}{}^{*}_{\mathrm{br}}\, w_1 \underset{\not\leftarrow(u)}{\longrightarrow}{}^{+}_{\mathrm{br}}\, v \underset{\not\leftarrow(u)}{\longrightarrow}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}'$. By (W2)(b) for $\hat{\mathcal{C}}'$, $|\alpha| > |\beta|$.

Case 2.2: The path $w_2 \underset{\not\leftarrow(u)}{\longrightarrow}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ does not visit $v$. Then there is a path $v \underset{\not\leftarrow(v)}{\longrightarrow}{}_{[\gamma]} \cdot \underset{\not\leftarrow(v)}{\longrightarrow}{}^{*}_{\mathrm{br}}\, w_2 \underset{\not\leftarrow(v)}{\longrightarrow}{}^{*} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}'$, so $|\gamma| > |\beta|$. And since there is a path $u \underset{\not\leftarrow(u)}{\longrightarrow}{}_{[\alpha]} \cdot \underset{\not\leftarrow(u)}{\longrightarrow}{}^{*}_{\mathrm{br}}\, w_1 \underset{\not\leftarrow(u)}{\longrightarrow}{}^{+}_{\mathrm{br}}\, v \rightarrow_{[\gamma]}$ in $\hat{\mathcal{C}}'$, by (W2)(b) for $\hat{\mathcal{C}}'$, $|\alpha| > |\gamma|$. So $|\alpha| > |\beta|$.

To verify (W1) together with part (L2) of (W2)(a) for $\hat{\mathcal{C}}^{(w_1)}_{w_2}$, it suffices to show that $\hat{\mathcal{C}}^{(w_1)}_{w_2}$ does not contain branch cycles. This can be verified analogously as for transformation I. That is, under the assumption of a branch cycle we can construct a path $w_2 \rightarrow^{+}_{\mathrm{br}} w_1$ in $\hat{\mathcal{C}}$, which contradicts (C3) (as it contradicted (C1)).

Part (L3) of (W2)(a) in $\hat{\mathcal{C}}^{(w_1)}_{w_2}$ can be shown similarly as we did for (W2)(b) above. We can use the analysis of newly created descends-in-loop-to paths developed there to see that violations of (L3) would arise from violations of (L3) in $\hat{\mathcal{C}}$.

We conclude that the result of transformation III is again a LLEE-witness. □

**Corollary 7.8.** *The bisimulation collapse of a LLEE-chart is a LLEE-chart.*

**Corollary 7.9.** *If a chart is expressible by a star expression modulo bisimilarity, then its collapse is a LLEE-chart.*

*Proof.* Let $\mathcal{C}$ be a chart with $\mathcal{C} \leftrightarrows \mathcal{C}(e)$, where $\mathcal{C}(e)$ is the chart associated to a star expression $e$. Then $\mathcal{C}(e)$ is a LLEE-chart by Prop. 3.5, and so by Cor. 7.8 its collapse is a LLEE-chart. The collapse of $\mathcal{C}$ coincides with the collapse of $\mathcal{C}(e)$. □

It follows that the two charts in Ex. 3.2 are indeed not expressible modulo $\leftrightarrows$.

## 8 The completeness result

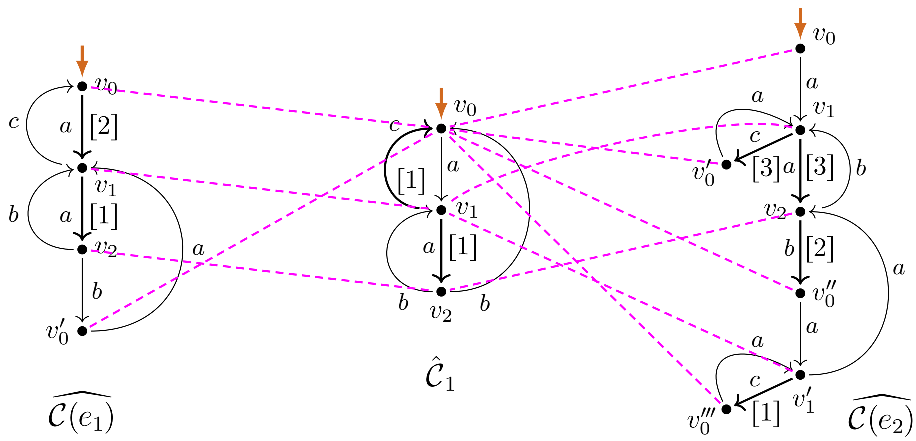
That bisimulation collapse preserves LLEE was the last building block in the proof of the desired completeness result. The proof steps of Thm. 8.1 were already explained in Sect. 5, in order to shed light on the definitions and results in Sect. 3 and Sect. 4, and to motivate the technical developments in Sect. 7.

**Theorem 8.1.** *The axiomatization* BBP *is complete with respect to the bisimulation semantics for star expressions without* 1 *and with the binary Kleene star* $^{\circledast}$.

We consider two instances of the completeness proof.

*Example 8.2.* Bisimilar LLEE-charts $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ in the picture in Sect. 5 have principal solutions $(a \cdot (a+b) + b)^{\circledast} 0$ and $(b \cdot (a+b) + a)^{\circledast} 0$. Their common bisimulation collapse $\hat{\mathcal{C}}_0$ has principal solution $(a+b)^{\circledast} 0$. By Prop. 4.2 and Prop. 4.5, $(a \cdot (a+b) + b)^{\circledast} 0 =_{\mathsf{BBP}} (a+b)^{\circledast} 0 =_{\mathsf{BBP}} (b \cdot (a+b) + a)^{\circledast} 0$.

*Example 8.3.* The bisimilar star expressions $(a \cdot ((a \cdot (b + b \cdot a))^{\circledast} c))^{\circledast} 0$, denoted by $e_1$, and $a \cdot ((c \cdot a + a \cdot (b \cdot a \cdot ((c \cdot a)^{\circledast} a))^{\circledast} b)^{\circledast} 0)$, denoted by $e_2$, As remarked earlier, in [9] $e_1$ is mentioned as problematic for a completeness proof, because the bisimulation collapse of its chart does not directly correspond to a star expression. Below at the left and right LLEE-witnesses of the charts $\mathcal{C}(e_1)$ and $\mathcal{C}(e_2)$ are drawn. They have provable solutions with principal values $e_1$ and $e_2$. The middle is a LLEE-witness of their bisimulation collapse, the running example $\hat{\mathcal{C}}_1$. The dashed lines connect the vertices in $\widehat{\mathcal{C}(e_1)}$ and $\widehat{\mathcal{C}(e_2)}$ to their bisimilar counterparts in $\hat{\mathcal{C}}_1$. We saw earlier that $\hat{\mathcal{C}}_1$ has a provable solution with principal value $s_{\hat{\mathcal{C}}_1}(v_0) = a \cdot ((c \cdot a + a \cdot (b + b \cdot a))^{\circledast} 0)$. By Prop. 4.2 and Prop. 4.5, $e_1 =_{\mathsf{BBP}} s_{\hat{\mathcal{C}}_1}(v_0) =_{\mathsf{BBP}} e_2$.




## 9 Conclusion

We have shown that Milner's axiomatization, tailored to star expressions without 1 and with $^{\circledast}$, is complete in bisimulation semantics. At the core of the proof is the novel notion LLEE, which also characterizes precisely the process graphs that can be expressed by star expressions without 1 and with $^{\circledast}$: their bisimulation collapse is a LLEE-chart. The main goal for the future is to extend the completeness result to the full class of star expressions, i.e., to include the constant 1.

The completeness result from [10,8] for star expressions without 0 and 1 and with $^{\circledast}$ was extended in [6] to a setting with 1 (but not 0) and *, where an extended version of the non-empty-word property is disallowed for terms directly under a *. We could extend our completeness result to star expressions with 0, 1, and *, but with a syntactic restriction on terms directly under a *, by rewriting each term in this class to a star expression with only 'harmless' occurrences of 1. With this approach one could also obtain the result in [6] directly from the result in [10,8].

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## S Supplements (Appendix)

### S.1 For Section 3: layered loop existence and elimination

**Proposition (= Prop. 3.5).** *For each star expression $e$, the loop-labeling defined by the rules on page 5 is a LLEE-witness of the chart defined by the rules on page 3 with start vertex $e$.*

*Proof.* To verify (W1) it suffices to show that there are no infinite branch paths from any star expression $e$ (this is also a preparation for (W2)(a), part (L2)). We prove, by induction on the syntactic structure of $e$, the stronger statement that if $e \to^{+} f$, then there does not exist an infinite branch path from $f$. The base cases, in which $e$ is of the form $a$ or $0$, are trivial. Suppose $e \equiv e_1 + e_2$. Then $e_i \to f$ for some $i \in \{1, 2\}$. So by induction, $f$ does not exhibit an infinite branch path. Suppose $e \equiv e_1 \cdot e_2$. Then $e \to^{+} f$ means either $e_1 \to^{+} f_1$ and $f \equiv f_1 \cdot e_2$, or $e_2 \to^{*} f$. In the first case, by induction, $f_1$ and $e_2$ do not exhibit an infinite branch path. This implies that $f_1 \cdot e_2$ does not exhibit an infinite branch path. In the second case, by induction $f$ does not exhibit an infinite branch path. Suppose $e \equiv e_1{}^{\circledast} e_2$. Then $e \to^{+} f$ means (A) $f \equiv e_1{}^{\circledast} e_2$, or (B) $e_1 \to^{+} f_1$ and $f \equiv f_1 \cdot (e_1{}^{\circledast} e_2)$, or (C) $e_2 \to^{+} f$. In case (A), each branch path from $f$ starts with either $f \to_{\mathrm{br}} e_1' \cdot (e_1{}^{\circledast} e_2)$ where $e_1 \to e_1'$ and $e_1'$ is not normed, or $f \to_{\mathrm{br}} e_2'$ where $e_2 \to e_2'$. In the first case, since by induction $e_1'$ does not exhibit an infinite branch path and $e_1'$ is not normed, it follows that $e_1' \cdot (e_1{}^{\circledast} e_2)$ does not exhibit an infinite branch path. In the second case, by induction $e_2'$ does not exhibit an infinite branch path. In case (B), since by induction $f_1$ does not exhibit an infinite branch path and by case (A) $e_1{}^{\circledast} e_2$ does not exhibit an infinite branch path, it follows that $f_1 \cdot (e_1{}^{\circledast} e_2)$ does not exhibit an infinite branch path. In case (C), by induction $f$ does not exhibit an infinite branch path.

To verify (W2), we denote $\hat{\mathcal{C}}$ we labeling that is defined on the set of all star expressions by the rules on page 5, and consider subcharts $\mathcal{C}_{\hat{\mathcal{C}}}(e, \alpha)$ that consist of the vertices and transitions reachable after loop-entry transitions $e \to_{[\alpha]} e'$ of $\hat{\mathcal{C}}$ before returning to $e$. From the rules on page 5 it follows that if $e$ enables a loop-entry transition, then $e \equiv ((\cdots((e_1{}^{\circledast} e_2) \cdot f_1) \cdots) \cdot f_n)$ for some $n \in \mathbb{N}$ ($n = 0$ is permitted) and $e_1$ normed. We first consider the case $n = 0$, and then generalize it.

Suppose that $e \equiv e_1{}^{\circledast} e_2$ with $e_1$ normed. Let $\alpha = |e_1|_{\circledast} + 1$. Either $e \to_{[\alpha]} e$ or $e \to_{[\alpha]} e_1' \cdot e$ for some normed $e_1'$ with $e_1 \to e_1'$. In the first case (L1) is satisfied; we focus on the second case. It can be argued, by induction on syntactic structure, that every normed star expression has a branch path to $\surd$. Then so does $e_1'$. This means $e_1' \cdot e$ has a branch path to $e$. Hence (L1) holds for $\mathcal{C}_{\hat{\mathcal{C}}}(e, \alpha)$. For the remainder of (W2) it suffices to consider loop-entry transitions $e \to_{[\alpha]} e_1'' \cdot e$ where $e_1 \to e_1''$. Since we showed above there are no branch cycles, every branch path from $e_1''$ eventually leads to deadlock or to $\surd$; in the former case the corresponding branch path of $e_1'' \cdot e$ also deadlocks, and in the latter case it returns to $e$. Hence (L2) holds as well for $\mathcal{C}_{\hat{\mathcal{C}}}(e, \alpha)$. Since $e_1'' \cdot e$ cannot reach $\surd$ without returning to $e$, (L3) holds. It can be shown, by induction on derivation depth, that $f \to f'$ implies $|f|_{\circledast} \geqslant |f'|_{\circledast}$, and clearly $f \to_{[\beta]}$ implies $|\beta| \leqslant |f|_{\circledast}$. So if $e_1'' \to^{*} \cdot \to_{[\beta]}$, then $|\beta| \leqslant |e_1''|_{\circledast} \leqslant |e_1|_{\circledast}$. Hence, if $e_1'' \cdot e \xrightarrow[\not{t}(e_1{}^{\circledast} e_2)]{}{}^{*}_{\mathrm{br}} \cdot \to_{[\beta]}$, then $|\beta| < |e_1|_{\circledast} + 1 = |\alpha|$. So (W2)(*b*) holds.

Finally, consider $e \equiv ((\cdots((e_1{}^{\circledast} e_2) \cdot f_1) \cdots) \cdot f_n)$ for $n > 0$, and where $e_1$ is normed. Let again $\alpha = |e_1|_{\circledast} + 1$. Then, due to the product rules, the subchart

$\mathcal{C}_{\hat{\mathcal{C}}}(e, \alpha)$ is the result of successively post-fixing the expressions $f_1, \ldots, f_n$ in each of the expressions in the vertices of $\mathcal{C}_{\hat{\mathcal{C}}}(e_1{}^{\circledast}e_2, \alpha)$. The reason is that $\mathcal{C}_{\hat{\mathcal{C}}}(e_1{}^{\circledast}e_2, \alpha)$ has as vertices $e_1{}^{\circledast}e_2$, and all expressions $g \cdot (e_1{}^{\circledast}e_2)$ where $g$ is an iterated derivative of $e_1$, and that similarly $\mathcal{C}_{\hat{\mathcal{C}}}(e_1{}^{\circledast}e_2, \alpha)$ consists of $e$, and all expressions $((\cdots((g \cdot (e_1{}^{\circledast}e_2)) \cdot f_1) \cdots) \cdot f_n)$ where $g$ is an iterated derivative of $e_1$. The bijective correspondence function thus introduced from $\mathcal{C}_{\hat{\mathcal{C}}}(e_1{}^{\circledast}e_2, \alpha)$ to $\mathcal{C}_{\hat{\mathcal{C}}}(e, \alpha)$ is a chart isomorphism that preserves and reflects action labels, and also marking labels in the underlying loop labeling. This follows from the fact that rules for concatenation on page 5 preserve, and reflect the marking and action labels of a transition. As a consequence $\mathcal{C}_{\hat{\mathcal{C}}}(e_1{}^{\circledast}e_2, \alpha)$ is a loop chart if and only if $\mathcal{C}_{\hat{\mathcal{C}}}(e, \alpha)$ is a loop chart. Also, $\mathcal{C}_{\hat{\mathcal{C}}}(e_1{}^{\circledast}e_2, \alpha)$ satisfies the layeredness condition (W2)(b) with respect to $\hat{\mathcal{C}}$ if and only if so does $\mathcal{C}_{\hat{\mathcal{C}}}(e, \alpha)$. Now since above we have recognized $\mathcal{C}_{\hat{\mathcal{C}}}(e_1{}^{\circledast}e_2, \alpha)$ as a loop chart that satisfies the layeredness condition (W2)(b), the same holds also for $\mathcal{C}_{\hat{\mathcal{C}}}(e, \alpha)$.

□

### S.2 For Section 4: Extraction of star expressions from LLEE-charts

**Proposition (= Prop. 4.2).** *Let $\phi : V_1 \to V_2$ be a bisimulation between $\mathcal{C}_1$ and $\mathcal{C}_2$. If $s$ is a provable solution of $\mathcal{C}_2$, then $s \circ \phi$ (with domain $V_1 \backslash \{\surd\}$) is a provable solution of $\mathcal{C}_1$. Note that it has the same principal value as $s$.*

*Proof.* Let $v \in V_1 \backslash \{\surd\}$. Since $\phi$ is a functional bisimulation between $\mathcal{C}_1$ and $\mathcal{C}_2$, the forth and back conditions for the graph of $\phi$ as a bisimulation hold for the pair $\langle v, \phi(v) \rangle$ of vertices. This make it possible to bring the sets of transitions $T_1(v)$ from $v$ in $\mathcal{C}_1$, and $T_2(\phi(v))$ from $\phi(v)$ in $\mathcal{C}_2$ into a 1–1 correspondence such that $\phi$ again relates their targets:

$$T_1(v) = \{v \xrightarrow{a_i} \surd \mid i = 1, \ldots, m\} \cup \{v \xrightarrow{b_j} v'_{j1} \mid j = 1, \ldots, n\}, \tag{S.1}$$

$$T_2(\phi(v)) = \{\phi(v) \xrightarrow{a_i} \surd \mid i = 1, \ldots, m\} \cup \{\phi(v) \xrightarrow{b_j} v'_{j2} \mid j = 1, \ldots, n\}, \tag{S.2}$$

$$\phi(v'_{j1}) = v'_{j2}, \quad \text{for all } j \in \{1, \ldots, n\}, \tag{S.3}$$

with $n, m \in \mathbb{N}$, and vertices $v'_{j1} \in V_1$, and $v'_{j2} \in V_2$, for $j \in \{1, \ldots, n\}$. (Note that the same transition may be listed multiple times in the set $T_2(\phi(v))$.) On this basis we can argue as follows, where $\sum_{i=1}^{k} e_i$ is defined recursively as 0 if $k = 0$, as $e_1$ if $k = 1$, and as $(\sum_{i=1}^{k-1} e_i) + e_k$ if $k > 1$:

$$(s \circ \phi)(v) \;\equiv\; s(\phi(v)) \;=_{\mathsf{BBP}}\; \Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot s(v'_{j2})\Big)$$

(since $s$ is a provable solution of $\mathcal{C}_2$, using (S.2) and axioms (A1), (A2), (A3))

$$\equiv \quad \Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot (s \circ \phi)(v'_{j1})\Big)$$

(using (S.3) and $(s \circ \phi)(v'_{j1}) \equiv s(\phi(v'_{j1}))$)

This shows, in view of (S.1), that $s \circ \phi$ satisfies the condition for a provable solution at $v$. Now as $v \in V_1 \backslash \{\surd\}$ was arbitrary, $s \circ \phi$ (with domain $V_1 \backslash \{\surd\}$) is a provable solution of $\mathcal{C}_1$. Since furthermore the functional bisimulation $\phi$ must relate the start vertices of $\mathcal{C}_1$ and $\mathcal{C}_2$, the principal value of $s \circ \phi$ coincides with that of $s$. $\square$

**Proposition (= Prop. 4.3).** *For every LLEE-witness $\hat{\mathcal{C}}$, $s_{\hat{\mathcal{C}}}$ is a provable solution of the chart $\mathcal{C}$ underlying $\hat{\mathcal{C}}$.*

*Proof.* We first show an auxiliary result about the connection between the extracted solution $s_{\hat{\mathcal{C}}}$ and the relative extracted solution $t_{\hat{\mathcal{C}}}$. For all vertices $v, w$:

$$v \curvearrowright w \implies s_{\hat{\mathcal{C}}}(w) =_{\mathsf{BBP}} t_{\hat{\mathcal{C}}}(w, v) \cdot s_{\hat{\mathcal{C}}}(v)\,. \tag{S.4}$$

Note that if $v \curvearrowright w$, then $v \neq \surd$, and also $w \neq \surd$, because then $w$ is in the body of a loop at $v$, and therefore cannot be $\surd$.

We proceed by complete induction (without explicit treatment of the base case) on the length $\|w\|_{\mathrm{br}}$ of a longest path of branch transitions from $w$. For performing the induction step, we consider arbitrary $v, w \neq \surd$ with $v \curvearrowright w$. We assume a representation of the set $\hat{T}(w)$ of transitions from $w$ in $\hat{\mathcal{C}}$:

$$\begin{aligned}\hat{T}(w) = {} & \big\{ w \xrightarrow{a_i}_{[\alpha_i]} w \;\big|\; i = 1, \dots, m \big\} \cup \big\{ w \xrightarrow{b_j}_{[\beta_j]} w_j \;\big|\; w_j \neq w,\ j = 1, \dots, n \big\} \\ & \cup \big\{ w \xrightarrow{c_i}_{\mathrm{br}} v \;\big|\; i = 1, \dots, p \big\} \cup \big\{ w \xrightarrow{d_j}_{\mathrm{br}} u_j \;\big|\; u_j \neq \surd,\ j = 1, \dots, q \big\}\end{aligned} \tag{S.5}$$

that partitions $\hat{T}(w)$ into loop-entry transitions to $w$ and to other targets $w_1, \dots, w_n$, and branch transitions to $v$ and to other targets $u_1, \dots, u_q$. Since $w$ is contained in a loop at $v$, none of these target can be $\surd$. Then in order to show the provable equality at the right-hand side of (S.4) we argue as follows:

$$s_{\hat{\mathcal{C}}}(w) \equiv \Big( \Big( \sum_{i=1}^{m} a_i \Big) + \Big( \sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w) \Big) \Big)^{\circledast} \Big( 0 + \Big( \Big( \sum_{i=1}^{p} c_i \cdot s_{\hat{\mathcal{C}}}(v) \Big) + \Big( \sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j) \Big) \Big) \Big)$$

(by the definition of $s_{\hat{\mathcal{C}}}(w)$ based on the representation (S.5), using that none of the target vertices is the terminating sink $\surd$, because $w$ is in the body of a loop starting at $v$.

$$=_{\mathsf{BBP}} \Big( \Big( \sum_{i=1}^{m} a_i \Big) + \Big( \sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w) \Big) \Big)^{\circledast} \Big( \Big( \sum_{i=1}^{p} c_i \cdot s_{\hat{\mathcal{C}}}(v) \Big) + \Big( \sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j) \Big) \Big)$$

(using axiom (A6))

$$=_{\mathsf{BBP}} \Big( \Big( \sum_{i=1}^{m} a_i \Big) + \Big( \sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w) \Big) \Big)^{\circledast} \Big( \Big( \sum_{i=1}^{p} c_i \cdot s_{\hat{\mathcal{C}}}(v) \Big) + \Big( \sum_{j=1}^{q} d_j \cdot \big( t_{\hat{\mathcal{C}}}(u_j, v) \cdot s_{\hat{\mathcal{C}}}(v) \big) \Big) \Big)$$

(by the induction hypothesis, using that $\|u_j\|_{\mathrm{br}} < \|w\|_{\mathrm{br}}$ because $w \to_{\mathrm{br}} u_j$, see (S.5), for $j \in \{1, \dots, q\}$)

$$=_{\mathsf{BBP}} \Big( \Big( \sum_{i=1}^{m} a_i \Big) + \Big( \sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w) \Big) \Big)^{\circledast} \Big( \Big( \Big( \sum_{i=1}^{p} c_i \Big) + \Big( \sum_{j=1}^{q} d_j \cdot t_{\hat{\mathcal{C}}}(u_j, v) \Big) \Big) \cdot s_{\hat{\mathcal{C}}}(v) \Big)$$

(using axioms (A5), (A4))

$$=_{\mathsf{BBP}} \Big( \Big( \sum_{i=1}^{m} a_i \Big) + \Big( \sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w) \Big) \Big)^{\circledast} \Big( \Big( \sum_{i=1}^{p} c_i \Big) + \Big( \sum_{j=1}^{q} d_j \cdot t_{\hat{\mathcal{C}}}(u_j, v) \Big) \Big) \cdot s_{\hat{\mathcal{C}}}(v)$$

(using axiom (BKS2))

$$\equiv \quad t_{\hat{\mathcal{C}}}(w,v) \cdot s_{\hat{\mathcal{C}}}(v)$$
(by the definition of $t_{\hat{\mathcal{C}}}(w,v)$ based on the representation (S.5))

This chain of provable equalities demonstrates the provable equality at the right-hand side of (S.4). We have performed the induction step. So we have shown (S.4).

In order to prove that $s_{\hat{\mathcal{C}}}$ is a provable solution of the chart $\mathcal{C}$ underlying $\hat{\mathcal{C}}$, we let $w \neq \surd$. We show that $s_{\hat{\mathcal{C}}}(w)$ satisfies the defining equation of $s_{\hat{\mathcal{C}}}$ to be a provable solution of $\mathcal{C}$ at $w$.

We consider a representation of the set $\hat{T}(w)$ of transitions from $w$ in $\hat{\mathcal{C}}$ as follows:

$$\begin{aligned}\hat{T}(w) = \{w \xrightarrow{a_i}_{[\alpha_i]} w \mid i = 1,\ldots,m\} &\cup \{w \xrightarrow{b_j}_{[\beta_j]} w_j \mid w_j \neq w,\ j = 1,\ldots,n\} \\ \cup \{w \xrightarrow{c_i}_{\text{br}} \surd \mid i = 1,\ldots,p\} &\cup \{w \xrightarrow{d_j}_{\text{br}} u_j \mid u_j \neq \surd,\ j = 1,\ldots,q\} \end{aligned} \tag{S.6}$$

that partitions $\hat{T}(w)$ into loop-entry transitions to $w$ and to other targets $w_1,\ldots,w_n$, and branch transitions to $\surd$ and to other targets $u_1,\ldots,u_q$. Now we argue as follows:

$$s_{\hat{\mathcal{C}}}(w) \equiv \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j,w)\Big)\Big)^{\circledast}\Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big)$$
(by the definition of $s_{\hat{\mathcal{C}}}$ in view of the representation (S.6))

$$=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j,w)\Big)\Big) \cdot s_{\hat{\mathcal{C}}}(w) + \Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big)$$
(using axiom (BKS1) and the defining equality in the first step)

$$=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i \cdot s_{\hat{\mathcal{C}}}(w)\Big) + \Big(\sum_{j=1}^{n} b_j \cdot (t_{\hat{\mathcal{C}}}(w_j,w) \cdot s_{\hat{\mathcal{C}}}(w))\Big)\Big) + \Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big)$$
(using axioms (A4), (A5))

$$=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i \cdot s_{\hat{\mathcal{C}}}(w)\Big) + \Big(\sum_{j=1}^{n} b_j \cdot s_{\hat{\mathcal{C}}}(w_j)\Big)\Big) + \Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big)$$
(using (S.4) in view of $w \curvearrowright w_j$ for $j \in \{1,\ldots,n\}$)

$$=_{\mathsf{BBP}} \Big(\sum_{i=1}^{p} c_i\Big) + \Big(\Big(\Big(\sum_{i=1}^{m} a_i \cdot s_{\hat{\mathcal{C}}}(w)\Big) + \Big(\sum_{j=1}^{n} b_j \cdot s_{\hat{\mathcal{C}}}(w_j)\Big)\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big)$$
(using axioms (A1), (A2))

This chain of equalities demonstrates the provable equality for $s_{\hat{\mathcal{C}}}(w)$ to be a provable solution of $\mathcal{C}$ at $w$ in view of the representation (S.6) of $\hat{T}(w)$. As the vertex $w \neq \surd$ was arbitrary for this argument, $s_{\hat{\mathcal{C}}}$ is indeed a provable solution of $\mathcal{C}$. ◻

**Proposition (= Prop. 4.5).** *Let $\mathcal{C}$ be a LLEE-chart. Then all provable solutions $s_1$ and $s_2$ of $\mathcal{C}$ are provably equal, that is, $s_1(v) =_{\mathsf{BBP}} s_2(v)$ for all vertices $v \neq \surd$.*

*Proof.* Let $\hat{\mathcal{C}}$ be a LLEE-witness of $\mathcal{C}$. It suffices to show that every provable solution $s$ is provably equal to the extracted solution $s_{\hat{\mathcal{C}}}$ of $\mathcal{C}$ at all vertices $\neq \surd$. To show this, we will again need, as in the proof of Prop. 4.3, an auxiliary result about the connection of $s$ with the relative extracted solution $t_{\hat{\mathcal{C}}}$:

$$v \curvearrowright w \quad \Longrightarrow \quad s(w) =_{\mathsf{BBP}} t_{\hat{\mathcal{C}}}(w,v) \cdot s(v) \tag{S.7}$$

for all vertices $v$ and $w$. Note again that if $v \curvearrowright w$, then $v \neq \surd$, and also $w \neq \surd$, because $w$ is in the body of a loop at $v$, and therefore cannot be $\surd$.

In order to show (S.7) for every provable solution of $\mathcal{C}$, we let $s$ be an arbitrary provable solution of $\mathcal{C}$. In order to show (S.7) we proceed by complete induction (without explicit treatment of the base case) on the same measure as used in the definition of the relative extraction function $t_{\hat{\mathcal{C}}}$, namely, induction on the maximal loop level of a loop at $v$ with a subinduction on $\|w\|_{\mathrm{br}}$. For performing the induction step, we consider vertices $v$, $w$ with $v \curvearrowright w$. As in the proof of Prop. 4.3 we assume a representation (S.5) of the set $\hat{T}(w)$ of transitions from $w$ in $\hat{\mathcal{C}}$ that partitions $\hat{T}(w)$ into loop-entry transitions to $w$ and to other targets $w_1, \ldots, w_n$, and branch transitions to $v$ and to other targets $u_1, \ldots, u_q$. Since $w$ is contained in a loop at $v$, none of these target can be $\surd$. With these representations we now argue as follows:

$$\begin{aligned}
s(w) &=_{\mathsf{BBP}} 0 + \Big( \Big( \sum_{i=1}^{m} a_i \cdot s(w) \Big) + \Big( \Big( \Big( \sum_{j=1}^{n} b_j \cdot s(w_j) \Big) + \Big( \sum_{i=1}^{p} c_i \cdot s(v) \Big) \Big) + \Big( \sum_{j=1}^{q} d_j \cdot s(u_j) \Big) \Big) \Big) \\
&\qquad \text{(since $s$ is a provable solution of $\mathcal{C}$ at $w$,} \\
&\qquad \text{ using list representation (S.5) of $\widehat{T}(w)$, parsing it} \\
&\qquad \text{ as a representation of $\widehat{T}(w)$ in Def. 4.1 without $\surd$)} \\
&=_{\mathsf{BBP}} \Big( \Big( \sum_{i=1}^{m} a_i \cdot s(w) \Big) + \Big( \sum_{j=1}^{n} b_j \cdot s(w_j) \Big) \Big) + \Big( \Big( \sum_{i=1}^{p} c_i \cdot s(v) \Big) + \Big( \sum_{j=1}^{q} d_j \cdot s(u_j) \Big) \Big) \\
&\qquad \text{(using axioms (A6), A2))} \\
&=_{\mathsf{BBP}} \Big( \Big( \sum_{i=1}^{m} a_i \cdot s(w) \Big) + \Big( \sum_{j=1}^{n} b_j \cdot \big( t_{\hat{\mathcal{C}}}(w_j,w) \cdot s(w) \big) \Big) \Big) + \Big( \Big( \sum_{i=1}^{p} c_i \cdot s(w) \Big) + \Big( \sum_{j=1}^{q} d_j \cdot \big( t_{\hat{\mathcal{C}}}(u_j,v) \cdot s(v) \big) \Big) \Big) \\
&\qquad \text{(using the induction hypothesis, which is applicable, because} \\
&\qquad \text{ the maximal loop level at $w$ is smaller than that at $v$ due to $v \curvearrowright w$,} \\
&\qquad \text{ and also $\|u_j\|_{\mathrm{br}} < \|w\|_{\mathrm{br}}$ due to $w \to_{\mathrm{br}} u_j$ for $j \in \{1,\ldots,q\}$, see (S.5))} \\
&=_{\mathsf{BBP}} \Big( \Big( \sum_{i=1}^{m} a_i \Big) + \Big( \sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j,w) \Big) \Big) \cdot s(w) + \Big( \Big( \sum_{i=1}^{p} c_i \Big) + \Big( \sum_{j=1}^{q} d_j \cdot t_{\hat{\mathcal{C}}}(u_j,v) \Big) \Big) \cdot s(v) \\
&\qquad \text{(using axioms (A5), (A4))}
\end{aligned}$$

This chain of equalities justifies:

$$s(w) =_{\mathsf{BBP}} \Big( \Big( \sum_{i=1}^{m} a_i \Big) + \Big( \sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j,w) \Big) \Big) \cdot s(w) + \Big( \Big( \sum_{i=1}^{p} c_i \Big) + \Big( \sum_{j=1}^{q} d_j \cdot t_{\hat{\mathcal{C}}}(u_j,v) \Big) \Big) \cdot s(v)$$

To this equality we can apply the rule RSP$^{\circledast}$:

$$s(w) =_{\mathsf{BBP}} \Big(\Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_i \cdot t_{\hat{\mathcal{C}}}(w_j, w)\Big)\Big)^{\circledast}\Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot t_{\hat{\mathcal{C}}}(u_j, v)\Big)\Big)\Big) \cdot s(v)$$

(by applying rule RSP$^{\circledast}$)

$$\equiv \quad t_{\hat{\mathcal{C}}}(w, v) \cdot s(w)\,,$$

where in the last step we have used the definition of $t_{\hat{\mathcal{C}}}(w, v)$ in view of our assumption that it is based on the list representation (S.5) of $\widehat{T}(w)$. In this way we have carried out the induction step.

As $s$ was an arbitrary solution of $\mathcal{C}$, we can conclude that (S.7) holds for all vertices $v$ and $w$ of $\mathcal{C}$.

Now we turn to showing the remaining proof obligation: that every provable solution of $\mathcal{C}$ is provably equal to the solution $s_{\hat{\mathcal{C}}}$ that is extracted from the LLEE-witness $\hat{\mathcal{C}}$. For this we let $s$ be an arbitrary provable solution of $\mathcal{C}$. We also consider an arbitrary vertex $w \neq \surd$, and show that $s(w) =_{\mathsf{BBP}} s_{\hat{\mathcal{C}}}(w)$. For this, we consider, as in the proof of Prop. 4.3 a representation (S.6) of the set $\widehat{T}(w)$ of transitions from $w$ in $\hat{\mathcal{C}}$. With this representation we argue as follows:

$$s(w) =_{\mathsf{BBP}} \Big(\sum_{i=1}^{p} c_i\Big) + \Big(\Big(\Big(\sum_{i=1}^{m} a_i \cdot s(w)\Big) + \Big(\sum_{j=1}^{n} b_j \cdot s(w_j)\Big)\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s(u_j)\Big)\Big)$$

(since $s$ is a provable solution of $\mathcal{C}$ at $w$,
using list representation (S.6) of $\widehat{T}(w)$,
and using axioms (A2) for rearranging sum expressions)

$$=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i \cdot s(w)\Big) + \Big(\sum_{j=1}^{n} b_j \cdot s(w_j)\Big)\Big) + \Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s(u_j)\Big)\Big)$$

(using axioms (A2), (A3))

$$=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i \cdot s(w)\Big) + \Big(\sum_{j=1}^{n} b_j \cdot \big(t_{\hat{\mathcal{C}}}(w_j, w) \cdot s(w)\big)\Big)\Big) + \Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s(u_j)\Big)\Big)$$

(using (S.7) due to $w \curvearrowright w_j$, for all $j \in \{1, \ldots, m\}$,
as a consequence of the representation (S.6) of $\widehat{T}(w)$)

$$=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} \big(b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w)\big)\Big)\Big) \cdot s(w) + \Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s(u_j)\Big)\Big)$$

(using axioms (A5), (A4))

This chain of equalities justifies:

$$s(w) =_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} \big(b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w)\big)\Big)\Big) \cdot s(w) + \Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s(u_j)\Big)\Big)$$

To this equality we can apply the rule RSP$^{\circledast}$ in order to obtain, and reason further:

$$\begin{aligned} s(w) &=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w)\Big)\Big)^{\circledast}\Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big) \\ &\equiv \quad s_{\hat{\mathcal{C}}}(w)\,, \end{aligned}$$

where in the last step we have used the definition of $s_{\hat{\mathcal{C}}}(w)$ in view of our assumption that it was based on the list representation (S.6) of $\widehat{T}(w)$. In this way we have proved that $s(w) =_{\mathsf{BBP}} s_{\hat{\mathcal{C}}}(w)$ holds.

Since $w$ was arbitrary above, we have shown that the assumed solution $s$ of $\mathcal{C}$ is provably equal to the solution $s_{\mathcal{C}}$ that is extracted from the LLEE-witness $\hat{\mathcal{C}}$ of $\mathcal{C}$.

As this was the statement that we had to show for arbitrary solutions $s$ of $\mathcal{C}$, we have proved the proposition. $\square$

## S.3 For Section 7: Preservation of LLEE under collapse

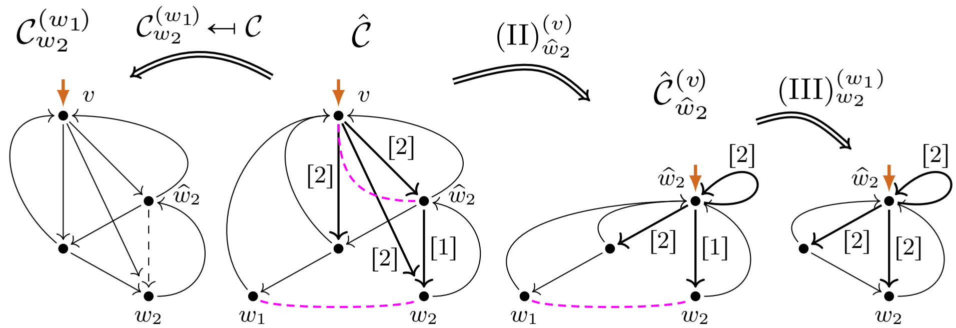


**Figure 1.** The step to the left is a counterexample that shows that the condition (C3) in Prop. 7.5 cannot be weakened by dropping the conjunct $\neg(w_2 \rightarrow^*_{\mathrm{br}} w_1)$. For the LLEE-witness $\hat{\mathcal{C}}$ it holds that $w_1 \;{}_d\!\circlearrowright v \circlearrowleft^+ w_2$, but there is a branch path from $w_2$ to $w_1$. The connect-$w_1$-through-to-$w_2$ chart at the left does not have a LLEE-witness, because from each of its vertices an infinite path starts that does not return to it. Note that the bisimilar pair $w_1, w_2$ in $\hat{\mathcal{C}}$ progresses to the bisimilar pair $v, \hat{w}_2$, to which transformation II is applicable because $\hat{w}_2 \circlearrowright v$. This results in the LLEE-witness $\hat{\mathcal{C}}^{(v)}_{\hat{w}_2}$, second to the right. Subsequently transformation III is applicable to the bisimilar pair $w_1, w_2$ because $w_1 \;{}_d\!\circlearrowright \hat{w}_2 \circlearrowleft w_2$. This second step results in the LLEE-witness at the right.

**Lemma S.1.** *If $u \circlearrowright^* v \circlearrowright^* w$ holds in a LLEE-witness, then every branch step path $u \rightarrow^*_{\mathrm{br}} w$ from $u$ to $w$ visits $v$.*

*Proof.* It suffices to show the following stronger statement for all $n \in \mathbb{N}$. Namely, that if $w_0 \circlearrowright w_1 \circlearrowright \cdots \circlearrowright w_{n-1} \circlearrowright w_n$ holds, then every branch step path $w_0 \rightarrow^*_{\mathrm{br}} w_1$ from $w_0$ to $w_n$ channels through $w_1, \ldots, w_{n-1}$ in that order. We proceed by induction on $n$. In the base case $n = 0$, and also in the case $n = 1$, the conclusion of the statement is satisfied for trivial reasons. So now suppose that $n \geqslant 2$. Then we first notice that it holds $w_n \;{}^{\alpha_n}\!\!\curvearrowright\; w_{n-1} \;{}^{\alpha_{n-1}}\!\!\curvearrowright\; \cdots \;{}^{\alpha_2}\!\!\curvearrowright\; w_1 \;{}^{\alpha_1}\!\!\curvearrowright\; w_0$ with loop names $\alpha_1, \ldots, \alpha_n$ such that $|\alpha_n| > |\alpha_{n-1}| > \cdots > |\alpha_1| > |\alpha_0|$. Let $\pi : w_0 \rightarrow^*_{\mathrm{br}} w_n$ be a branch step path from $w_0$ to $w_n$. We first show that $\pi$ does not channel through $w_1$. Toward a contradiction, assume that that is not the case. Let $n_0 > 1$ be minimal such that $\pi$ channels through $w_{n_0}$. Then there is a path $w_0 \xrightarrow[\not{t}(w_1)]{}{}^*_{\mathrm{br}} w_{n_0}$ Due to $w_0 \;{}^{\alpha_1}\!\circlearrowright w_1$, it holds that $w_1 \;{}^{\alpha_1}\!\!\curvearrowright\; w_0 \xrightarrow[\not{t}(w_1)]{}{}^*_{\mathrm{br}} w_{n_0}$. This implies $w_1 \;{}^{\alpha_1}\!\!\curvearrowright\; w_{n_0} \;{}^{\alpha_{n_0}}\!\!\curvearrowright$, which due to $|\alpha_{n_0}| > |\alpha_1|$ contradicts (W2)(b) for the underlying LLEE-witness. Therefore $\pi$ must channel through $w_1$. Then $\pi$ is the concatentation $\pi_0 \cdot \pi_1$ of branch step paths $\pi_0$ from $w_0$ to $w_1$, and $\pi_1$ from $w_1$ to $w_n$. By applying the induction hypothesis to $\pi_1$ we get that $\pi_1$ channels through $w_2, \ldots, w_{n-1}$ in that order. Now $\pi_0$ cannot also channel through one of $w_2, \ldots, w_{n-1}$, because otherwise $\pi$ would contain a branch step cycle. So we conclude that $\pi$ channels through $w_1, \ldots, w_{n-1}$ in that order. $\square$

*Proof (Supplement for the proof of Prop. 7.5).* Here we give the detailed argument for the correctness of transformation II. For this, we consider vertices $w_1$ and $w_2$ such that (C2) holds, that is, $w_2 \curvearrowleft^{+} w_1$. We denote by $\widehat{w}_2$ the ${}_{d}{\curvearrowleft}$-predecessor of $w_1$ in the ${}_{d}{\curvearrowleft}$-chain from $w_1$ to $w_2$, for which it holds $w_2 \curvearrowleft^{*} \widehat{w}_2 \;{}_{d}{\curvearrowleft}\; w_1$.

As for the transformations I and III it suffices to show, in view of the alleviation of the proof obligation at the start of the proof on page 14, that the intermediate result $\hat{\mathcal{C}}''$ of transformation II before the clean-up step satisfies the LLEE-witness properties, except for possible violations of (L1). By the definition of transformation II, $\hat{\mathcal{C}}''$ results by performing the adaptation step $L_{\mathrm{II}}$ to the chart $\hat{\mathcal{C}}' := \hat{\mathcal{C}}^{(w_1)}_{w_2}$ that arises from $\hat{\mathcal{C}}$ by connecting $w_1$ through to $w_2$.

To prove that (W1), and the part concerning (L2) for (W2)(a) is satisfied for $\hat{\mathcal{C}}''$, it suffices to show that the transformed chart does not contain a cycle of branch transitions. At first, the step of connecting $w_1$ through to $w_2$ in $\hat{\mathcal{C}}$ may introduce a branch cycle in $\hat{\mathcal{C}}' = \hat{\mathcal{C}}^{(w_1)}_{w_2}$. But every such cycle is then immediately removed in the level adaptation step $L_{\mathrm{II}}$. Namely, each branch cycle introduced in $\hat{\mathcal{C}}'$ must stem from paths $u \to_{\mathrm{br}} w_1$ in $\hat{\mathcal{C}}$ (a branch transition in that is redirected to $w_1$ in $\hat{\mathcal{C}}'$) and $w_2 \to^{*}_{\mathrm{br}} u$ in $\hat{\mathcal{C}}$, for some vertex $u$. Since we have $w_2 \curvearrowleft^{*} \widehat{w}_2 \;{}_{d}{\curvearrowleft}\; w_1$, it follows by Lemma S.1 that the branch step path $w_2 \to^{*}_{\mathrm{br}} u \to_{\mathrm{br}} w_1$ in $\hat{\mathcal{C}}$ must visit $\widehat{w}_2$ before arriving at $w_1$ (since $\widehat{w}_2 \neq w_1$ due to $\widehat{w}_1 \;{}_{d}{\curvearrowleft}\; w_1$ and $\curvearrowleft$ is irreflexive). Due to the fact that all branch transitions from $\widehat{w}_2$ are turned into loop-entry transitions in step $L_{\mathrm{II}}$, the branch step cycle $w_2 \to^{*}_{\mathrm{br}} u \to_{\mathrm{br}} w_2$ in $\hat{\mathcal{C}}'$ that was introduced in the connect-through step is then no longer a branch step cycle after step $L_{\mathrm{II}}$ in $\hat{\mathcal{C}}''$.

Next we prove that (W2)(b) is preserved by the two steps from $\hat{\mathcal{C}}$ via $\mathcal{C}' = \hat{\mathcal{C}}^{(w_1)}_{w_2}$ to $\hat{\mathcal{C}}''$. Every path $u \xrightarrow[\nleftarrow(u)]{}_{[\alpha]} \cdot \xrightarrow[\nleftarrow(u)]{}^{*}_{\mathrm{br}} \cdot \to_{[\beta]}$ in $\hat{\mathcal{C}}''$ with $u \neq w_1, w_2$ arises by a, possibly empty, combination of the following three kinds of modifications in the first two transformation steps:

(i) A transition to $w_1$ was redirected to $w_2$ in the connect-through step.
(ii) The loop-entry transition at the beginning of the path is from $\widehat{w}_2$ and was a branch transition before step $L_{\mathrm{II}}$, meaning that $u = \widehat{w}_2$ and $\alpha = \gamma$.
(iii) The loop-entry transition at the end of the path is from $\widehat{w}_2$ and was a branch transition before step $L_{\mathrm{II}}$, meaning that $\beta = \gamma$.

This gives $2^3 = 8$ possibilities. Of these, three possibilities are void: if all three adaptations are not the case, the path is already present in $\hat{\mathcal{C}}$, and so $|\alpha| > |\beta|$ is guaranteed; and if both (ii) and (iii) hold, the path would return to $u = \widehat{w}_2$, which it cannot, because all of its step avoid $u$ as target. We now show that in the remaining five cases always $|\alpha| > |\beta|$. Since $w_2 \curvearrowleft^{+} w_1$, there is a path $w_1 \xrightarrow[\nleftarrow(w_1)]{}_{[\delta]} \cdot \xrightarrow[\nleftarrow(w_1)]{}^{*}_{\mathrm{br}} w_2$ in $\hat{\mathcal{C}}$. By definition of $\gamma$, $|\gamma| \geqslant |\delta|$.

A Suppose that only (i) holds, meaning there are paths $u \xrightarrow[\nleftarrow(u)]{}_{[\alpha]} \cdot \xrightarrow[\nleftarrow(u)]{}^{*}_{\mathrm{br}} w_1$ and $w_2 \xrightarrow[\nleftarrow(u)]{}^{*}_{\mathrm{br}} \cdot \to_{[\beta]}$ in $\hat{\mathcal{C}}$. Then there is a path $u \xrightarrow[\nleftarrow(u)]{}_{[\alpha]} \cdot \xrightarrow[\nleftarrow(u)]{}^{*}_{\mathrm{br}} w_1 \to_{[\gamma]}$ in $\hat{\mathcal{C}}$, so $|\alpha| > |\gamma|$. We distinguish two cases.

Case 1: The path $w_2 \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ visits $w_1$, so there is a path $w_1 \xrightarrow[\not\mathbf{t}(u)]{}{}^{*} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$. Then there is a path $u \xrightarrow[\not\mathbf{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}\ w_1 \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$. Then by (W2)(b) for $\hat{\mathcal{C}}$, $|\alpha| > |\beta|$.

Case 2: The path $w_2 \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ does not visit $w_1$. Then there is a path $w_1 \xrightarrow[\not\mathbf{t}(w_1)]{}_{[\delta]} \cdot \xrightarrow[\not\mathbf{t}(w_1)]{}{}^{*}_{\mathrm{br}}\ w_2 \xrightarrow[\not\mathbf{t}(w_1)]{}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$, and hence by (W2)(b) for $\hat{\mathcal{C}}$, $|\delta| > |\beta|$. Hence $|\alpha| > |\gamma| \geqslant |\delta| > |\beta|$.

B Suppose that only (ii) holds. Then $u = \widehat{w}_2$, $\alpha = \gamma$ hold, and there must be a path $\widehat{w}_2 \xrightarrow[\not\mathbf{t}(\widehat{w}_2,\, w_1)]{}{}^{+}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$. As $\widehat{w}_2 \mathrel{{}_{d}{\curvearrowleft}} w_1$, there is a path $w_1 \xrightarrow[\not\mathbf{t}(w_1)]{}_{[\delta]} \cdot \xrightarrow[\not\mathbf{t}(w_1)]{}{}^{*}_{\mathrm{br}}\ \widehat{w}_2$ in $\hat{\mathcal{C}}$. Hence there is also a path $w_1 \xrightarrow[\not\mathbf{t}(w_1)]{}_{[\delta]} \cdot \xrightarrow[\not\mathbf{t}(w_1)]{}{}^{*}_{\mathrm{br}}\ \widehat{w}_2\ \xrightarrow[\not\mathbf{t}(w_1)]{}{}^{+}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$, so $|\delta| > |\beta|$. Hence $|\alpha| = |\gamma| \geqslant |\delta| > |\beta|$.

C Suppose that only (iii) holds. Then $\beta = \gamma$, and $u \xrightarrow[\not\mathbf{t}(u,\, w_1)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u,\, w_1)]{}{}^{*}_{\mathrm{br}}\ \widehat{w}_2$ with $u \neq w_1$ is a path in in $\hat{\mathcal{C}}$. Since $\widehat{w}_2 \mathrel{{}_{d}{\curvearrowleft}} w_1$ and $u \neq w_1$, it follows that $\neg(\widehat{w}_2 \mathrel{{}_{d}{\curvearrowleft}} u)$. So in view of the path $u \xrightarrow[\not\mathbf{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}}\ \widehat{w}_2$, there is no path $\widehat{w}_2 \rightarrow^{*}_{\mathrm{br}} u$ in $\hat{\mathcal{C}}$. Since $\widehat{w}_2 \mathrel{{}_{d}{\curvearrowleft}} w_1$, there is a path $\widehat{w}_2 \rightarrow^{*}_{\mathrm{br}} w_1$ in $\hat{\mathcal{C}}$, which by the previous observation is of the form $\widehat{w}_2 \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}}\ w_1$. Hence there is a path $u \xrightarrow[\not\mathbf{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}}\ \widehat{w}_2 \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}}\ w_1 \rightarrow_{[\gamma]}$ in $\hat{\mathcal{C}}$, so $|\alpha| > |\gamma| = |\beta|$.

D Suppose only (i) and (ii) hold, meaning $u = \widehat{w}_2$, $\alpha = \gamma$, and there are paths $\widehat{w}_2 \xrightarrow[\not\mathbf{t}(\widehat{w}_2)]{}{}^{+}\ w_1$ and $w_2 \xrightarrow[\not\mathbf{t}(\widehat{w}_2)]{}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$. Since we have $w_2 \curvearrowleft^{*} \widehat{w}_2 \mathrel{{}_{d}{\curvearrowleft}^{+}} w_1$, and $u = \widehat{w}_2$ implies $w_2 \neq \widehat{w}_2$, so by Lemma S.1 the path $w_2 \xrightarrow[\not\mathbf{t}(\widehat{w}_2)]{}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ cannot visit $w_1$. Hence there is a path $w_1 \xrightarrow[\not\mathbf{t}(w_1)]{}_{[\delta]} \cdot \xrightarrow[\not\mathbf{t}(w_1)]{}{}^{*}_{\mathrm{br}}\ w_2 \xrightarrow[\not\mathbf{t}(w_1)]{}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$. By (W2)(b) for $\hat{\mathcal{C}}$, $|\delta| > |\beta|$. Hence $|\alpha| = |\gamma| \geqslant |\delta| > |\beta|$.

E Suppose only (i) and (iii) hold. Then $\beta = \gamma$, and $u \xrightarrow[\not\mathbf{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}}\ w_1$ as well as $w_2 \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}}\ \widehat{w}_2$ are paths in $\hat{\mathcal{C}}$. Since there is a path $u \xrightarrow[\not\mathbf{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}}\ w_1 \rightarrow_{[\gamma]}$ in $\hat{\mathcal{C}}$, By (W2)(b) for $\hat{\mathcal{C}}$, $|\alpha| > |\gamma| = |\beta|$.

We conclude that $\hat{\mathcal{C}}''$ satisfies (W2)(b).

Finally we argue for the part (L3) of (W2)(a). We have to show that there are no descends-in-loop-to paths of the form $u \xrightarrow[\not\mathbf{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}} \surd$ in $\hat{\mathcal{C}}''$. For showing this, we can use the argumentation that we employed for demonstrating (W2)(b) above. Namely, we have demonstrated in particular that for every descends-in-loop-to path $u \xrightarrow[\not\mathbf{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}}\ x$ in $\hat{\mathcal{C}}''$ there is a descends-in-loop-to path $\tilde{u} \xrightarrow[\not\mathbf{t}(u)]{}_{[\gamma]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}{}^{*}_{\mathrm{br}}\ x$ with the same target $x$ in $\hat{\mathcal{C}}''$. From this it follows that if a descends-in-loop-to path in $\hat{\mathcal{C}}''$ had $\surd$ as a target, there there were a descends-in-loop-to path already in $\hat{\mathcal{C}}$ that had $\surd$ as a target, contradicting (L3) in (W2)(a) for the LLEE-witness $\hat{\mathcal{C}}$. We conclude that $\hat{\mathcal{C}}''$ satisfies part (L3) of (W2)(a).

We conclude that the result of transformation II is a LLEE-witness. $\square$