A Complete Proof System for 1-Free Regular Expressions in Bisimulation Semantics

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Abstract Robin Milner proposed an axiomatization for the regular expressions in bisimulation semantics in 1984, and posed the question whether it is complete. We consider a subclass of regular expressions by excluding the 1 and changing from the unary to the binary Kleene star. We establish completeness of Milner's axiomatization for this subclass.

1 Introduction

Regular expressions, introduced by Stephen Kleene [14], are widely studied in formal language theory, notably for string searching [19]. They are constructed from constants 0 (no strings), 1 (the empty string), and a (a single letter) from some alphabet; binary operators $+$ and $\cdot$ (union and concatenation); and the unary Kleene star $*$ (zero or more iterations). Aanderaa [1] and Salomaa [17] gave complete axiomatizations for the regular expressions in the trace semantics of formal languages, including a fixed-point rule with a non-empty-word property as side condition.

Robin Milner [15] was the first to study regular expressions in bisimulation semantics [16], where he called them star expressions. Here the interpretation of 0 is deadlock, 1 is (successful) termination, a is an atomic action, and $+$ and $\cdot$ are alternative and sequential composition of two processes, respectively. Milner adapted the aforementioned axiomatizations from formal languages to obtain a sound axiomatization for this setting, and posed the (still open) question whether this axiomatization is complete, meaning that if the process graphs of two star expressions are bisimilar, then they can be proven equal. That Milner's axiomatization contains a fixed-point rule is inevitable [18]. Bergstra, Bethke, and Ponse [4] studied star expressions without 0 and 1, replacing the unary by the binary Kleene star $^{\circledast}$, which represents an iteration of the first argument, followed by the execution of the second argument. They obtained an axiomatization by basically omitting the axioms for 0 and 1 as well as the fixed-point rule from Milner's axiomatization, and adding Troeger's axiom [20]. This axiomatization was proven complete in [10,8]. A sound and complete axiomatization for star expressions without 0 and the unary Kleene star, but with 1 and a unary perpetual loop operator $*0$, was presented in [7,9].

In contrast to the formal languages setting, not all finite-state process graphs can be expressed by a star expression modulo bisimilarity. Milner posed in [15] a second question, namely, to characterize which finite-state process graphs can be expressed. This was settled in [3] by the notion of a well-behaved specification.

In this paper we prove completeness of Milner's axiomatization (tailored to the adapted setting) for star expressions with 0, but without 1 and with the binary Kleene star. While earlier completeness proofs focus on terms, we follow in the footsteps of Milner and focus on their process graphs. A key idea is to determine loops in graphs associated to star expressions. By a loop we mean a subgraph generated by a set of entry transitions from a vertex v such that (1) each infinite path eventually returns to v, and (2) termination in the loop is not permitted. A graph is said to satisfy LLEE (Layered Loop Existence and Elimination) if repeatedly eliminating the entry transitions of a loop leads to a graph without infinite paths. LLEE offers a generalization (and more elegant definition) of the notion of a well-behaved specification. Our completeness proof roughly works as follows (see also Sect. 5). Let star expressions e_1 and e_2 give rise to bisimilar graphs g_1 and g_2. We show that graphs of star expressions satisfy LLEE. We moreover prove that LLEE is preserved under bisimulation collapse. And we construct for each graph that satisfies LLEE a star expression that corresponds to this graph, modulo bisimilarity. In particular such a star expression f can be constructed for the bisimulation collapse of g_1 and g_2. We show that both e_1 and e_2 can be proven equal to f, by a pull-back over the functional bisimulations, from the bisimulation collapse back to g_1 and g_2. This yields the desired completeness result.

This proof strategy is partly inspired by the term graph representation of cyclic λ-terms [12], in order to define and implement maximal sharing in the λ-calculus with `letrec` [13] (see also [11]). We moreover exploited ideas from [7,9] on minimizing the process graph (associated to the term) in the left-hand side of a binary Kleene star modulo bisimilarity. Interestingly, our framework includes the star expression that at the end of [9] is mentioned as problematic for a completeness proof.

The completeness result for star expressions with 0 but without 1 and with the binary Kleene star is interesting in its own right. We are hopeful that the current proof approach can be extended to the full class of star expressions, which would provide an affirmative answer to Milner's long-standing open question. However, we found that serious technical obstacles concerning 1-transitions lie ahead. In particular, in the setting with 1, LLEE is not always preserved under bisimulation collapse.

A version of this paper with an appendix containing omitted proofs, and an argument illustration is available at `https://git.io/Je4X9` on GitHub.

2 Preliminaries

Given a set A of *actions*, the *star expressions* are generated by the grammar

$$e ::= 0 \mid a \mid (e_1 + e_2) \mid (e_1 \cdot e_2) \mid (e_1{}^{\circledast} e_2) \qquad (\text{for } a \in A).$$

0 represents deadlock (i.e., does not perform any action), a an atomic action, $+$ alternative and $\cdot$ sequential composition, and $^{\circledast}$ the binary Kleene star. Note that the empty word 1 is missing from the syntax. $\sum_{i=1}^{k} e_i$ is defined recursively as 0 if $k = 0$, e_1 if $k = 1$, and $(\sum_{i=1}^{k-1} e_i) + e_k$ if $k > 1$.

A (finite, sink-termination) *chart* $\mathcal{C}$ consists of finite sets V of *vertices* and $T \subseteq V \times A \times V$ of *transitions*, a special vertex $\surd \in V$ with no outgoing transitions, which indicates termination, and a *start vertex* $v_s \in V \backslash \{\surd\}$. The chart associated to a star expression is defined by the following rules, where e ranges over star expressions, ξ over star expressions plus $\surd$, and a over A:

$$\frac{}{a \xrightarrow{a} \surd} \qquad \frac{e_i \xrightarrow{a} \xi \quad (i = 1, 2)}{e_1 + e_2 \xrightarrow{a} \xi} \qquad \frac{e_1 \xrightarrow{a} e_1'}{e_1 \cdot e_2 \xrightarrow{a} e_1' \cdot e_2} \qquad \frac{e_1 \xrightarrow{a} \surd}{e_1 \cdot e_2 \xrightarrow{a} e_2}$$

$$\frac{e_1 \xrightarrow{a} e_1'}{e_1{}^{\circledast}e_2 \xrightarrow{a} e_1' \cdot (e_1{}^{\circledast}e_2)} \qquad \frac{e_1 \xrightarrow{a} \surd}{e_1{}^{\circledast}e_2 \xrightarrow{a} e_1{}^{\circledast}e_2} \qquad \frac{e_2 \xrightarrow{a} \xi}{e_1{}^{\circledast}e_2 \xrightarrow{a} \xi}$$

The charts associated to star expressions are finite-state (by [2, Thm. 3.4]). By the absence of 1, termination is only possible at $\surd$.

A *bisimulation* between charts $\mathcal{C}_1$ and $\mathcal{C}_2$ is a symmetric binary relation B between vertices of $\mathcal{C}_1$ and of $\mathcal{C}_2$ that relates the start vertices, such that if $v_1 \, B \, v_2$, then: (1, progress): for every transition $v_1 \xrightarrow{a} v_1'$ in $\mathcal{C}_1$ there is a transition $v_2 \xrightarrow{a} v_2'$ in $\mathcal{C}_2$ with $v_1' \, B \, v_2'$, and (2, termination): if $v_1 = \surd$ then $v_2 = \surd$. If there is a bisimulation between $\mathcal{C}_1$ and $\mathcal{C}_2$, we write $\mathcal{C}_1 \leftrightarrow \mathcal{C}_2$ and say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *bisimilar*.

The axiomatization BBP of the above class of star expressions has the axioms:

$$\begin{array}{lrcl@{\qquad}lrcl}
\text{(A1)} & x + y & = & y + x & \text{(A6)} & x + 0 & = & 0 \\
\text{(A2)} & (x + y) + z & = & x + (y + z) & \text{(A7)} & 0 \cdot x & = & 0 \\
\text{(A3)} & x + x & = & x & \text{(BKS1)} & x \cdot (x^{\circledast}y) + y & = & x^{\circledast}y \\
\text{(A4)} & (x + y) \cdot z & = & x \cdot z + y \cdot z & \text{(BKS2)} & (x^{\circledast}y) \cdot z & = & x^{\circledast}(y \cdot z) \\
\text{(A5)} & (x \cdot y) \cdot z & = & x \cdot (y \cdot z) & & & &
\end{array}$$

together with the inference rules of equational logic, and the additional rule:

$$(\text{RSP}^{\circledast}) \quad \frac{e \;=\; (f \cdot e) + g}{e \;=\; f^{\circledast}g}$$

The axiomatization BBP is clearly sound modulo $\leftrightarrow$. We will prove its completeness modulo $\leftrightarrow$. Derivable equality from BBP is denoted by $=_{\mathsf{BBP}}$.

3 Layered loop existence and elimination

To define the key notion called LLEE, which holds for all charts associated to star expressions, we first define the notion of a 'loop'. A loop captures a subchart, within the chart of a star expression e, that corresponds to the behavior generated by an innermost subterm $f_1{}^{\circledast}f_2$ of e by the iteration of f_1. A *path* from a vertex v_1 is a (finite or infinite) sequence of transitions $v_1 \xrightarrow{a_1} v_2 \xrightarrow{a_2} \cdots$ in a chart.

Definition 3.1. *A chart is a* loop chart *if:*

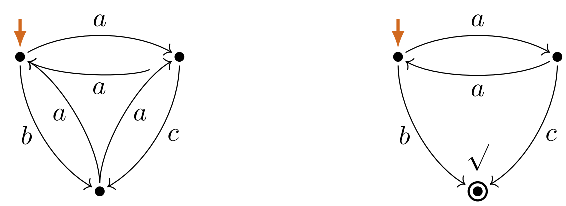
(L1) *There is an infinite path from* v_s*.*

(L2) *Every infinite path from* v_s *returns to* v_s *after a positive number of transitions.*

(L3) *It does not contain the vertex* $\surd$*.*

Given a chart, we repeatedly select a vertex v and a subset S of its outgoing transitions such that the subchart reachable by paths starting with a transition in S, with v as start vertex, forms a loop chart, and remove the transitions in S from the original chart. If this repeated procedure ultimately leads to a chart without infinite paths, then the original chart is bisimilar to the chart of a star expression.

Example 3.2. The two charts below show why (L2) and (L3) are necessary to rule out charts that are not expressible by a star expression modulo $\leftrightarroweq$. The left top arrow marks the start vertex. The first example, which violates (L2), was (as we will see correctly) conjectured not to be expressible (modulo $\leftrightarroweq$) in [15]. The second example, a double-exit structure with a termination vertex at the bottom, which violates (L3), was shown not to be expressible in [5]. We note that there is no loop subchart at any of the vertices in the two examples.



We now introduce the machinery to label entry transitions of loop charts within a chart of a star expression. The labels are layered: in case a Kleene star is within an argument of another Kleene star, the labels corresponding to the inner Kleene star are supposed to be lower than those corresponding to the outer Kleene star.

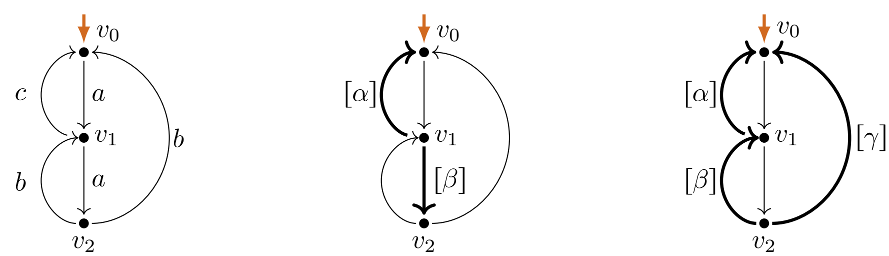
We assume a set Γ of *loop names*, referred to by α, β, γ, and a *loop-layer function* $|\cdot| : \Gamma \to \mathbb{N}^{>0}$. A *loop-labeling* $\hat{\mathcal{C}}$ of a chart $\mathcal{C}$ gives some of its transitions, called *loop-entry transitions*, a subscript $[\alpha]$ with $\alpha \in \Gamma$. All other transitions are called *branch transitions* and carry the subscript 'br'. Let $LI(\hat{\mathcal{C}})$ denote the set of pairs $\langle v, \alpha \rangle \in V \times \Gamma$ such that an $\to_{[\alpha]}$ transition departs from v in $\hat{\mathcal{C}}$. If $\langle v, \alpha \rangle \in LI(\hat{\mathcal{C}})$, then $\mathcal{C}_{\hat{\mathcal{C}}}(v, \alpha)$ is the subchart that consists of the vertices and transitions reachable by paths that start with an $\to_{[\alpha]}$ transition from start vertex v followed by only branch transitions and that halt immediately if v is revisited.

Definition 3.3. *Loop-labeling $\hat{\mathcal{C}}$ is a* LLEE-witness *for chart $\mathcal{C}$ if:*

(W1) *There is no infinite sequence of branch transitions from v_s.*

(W2) *For all $\langle v, \alpha \rangle \in LI(\hat{\mathcal{C}})$, (a) $\mathcal{C}_{\hat{\mathcal{C}}}(v, \alpha)$ is a loop chart, and (b) from all vertices $w \neq v$ of $\mathcal{C}_{\hat{\mathcal{C}}}(v, \alpha)$, in $\mathcal{C}$, no transition $\to_{[\beta]}$ with $|\beta| \geqslant |\alpha|$ departs.*

If such a LLEE-witness exists, $\mathcal{C}$ (and $\hat{\mathcal{C}}$) is called a LLEE-chart.



$\mathcal{C}$ $\qquad$ $\hat{\mathcal{C}}_1$ $\qquad$ $\hat{\mathcal{C}}_2$

Example 3.4. The charts above will serve as running example. (We take the liberty to omit branch labels in pictures.) The chart $\mathcal{C}$ at the left is the bisimulation collapse of the chart of the star expression $(a \cdot ((a \cdot (b + b \cdot a))^{\circledast} c))^{\circledast} 0$, which in [9] is considered problematic. Note that in the latter chart, the start vertex and the vertex reached after the second b in the inner Kleene star are bisimilar; in $\mathcal{C}$ these vertices have been collapsed to v_0. The challenge is to reconstruct a star expression from $\mathcal{C}$. It has two LLEE-charts, $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$; in $\hat{\mathcal{C}}_2$, $|\alpha| < |\beta|$, and $|\alpha| < |\gamma|$.

We refine the rules on page 3 by adding labels l to transitions. In the rules below, a branch label is provided to transitions that cannot return to the star expression in their left-hand side. In particular, the fifth and sixth rule make a distinction on whether e_1 is *normed*, meaning it has a path to $\surd$, as only in the normed case $e_1{}^{\circledast}e_2$ can return to itself. Loop-entry transitions are provided with a loop label based on the *star height* $|e|_{\circledast}$ of their left-hand side e, denoting the maximum number of nestings of Kleene stars; it is defined by $|0|_{\circledast} = |a|_{\circledast} = 0$, $|f + g|_{\circledast} = |f \cdot g|_{\circledast} = \max\{|f|_{\circledast}, |g|_{\circledast}\}$, and $|f^{\circledast}g|_{\circledast} = \max\{|f|_{\circledast} + 1, |g|_{\circledast}\}$.

$$\frac{}{a \xrightarrow{a}_{\mathrm{br}} \surd} \qquad \frac{e_i \xrightarrow{a}_{l} \xi}{e_1 + e_2 \xrightarrow{a}_{\mathrm{br}} \xi}\, i \in \{1,2\} \qquad \frac{e_1 \xrightarrow{a}_{l} e_1'}{e_1 \cdot e_2 \xrightarrow{a}_{l} e_1' \cdot e_2} \qquad \frac{e_1 \xrightarrow{a}_{\mathrm{br}} \surd}{e_1 \cdot e_2 \xrightarrow{a}_{\mathrm{br}} e_2}$$

$$\frac{e_1 \xrightarrow{a}_{l} e_1'}{e_1{}^{\circledast}e_2 \xrightarrow{a}_{[|e_1|_{\circledast}+1]} e_1' \cdot (e_1{}^{\circledast}e_2)}\ \text{if } e_1 \text{ is normed} \qquad \frac{e_1 \xrightarrow{a}_{l} e_1'}{e_1{}^{\circledast}e_2 \xrightarrow{a}_{\mathrm{br}} e_1' \cdot (e_1{}^{\circledast}e_2)}\ \text{if } e_1 \text{ is not normed}$$

$$\frac{e_1 \xrightarrow{a}_{\mathrm{br}} \surd}{e_1{}^{\circledast}e_2 \xrightarrow{a}_{[|e_1|_{\circledast}+1]} e_1{}^{\circledast}e_2} \qquad \frac{e_2 \xrightarrow{a}_{l} \xi}{e_1{}^{\circledast}e_2 \xrightarrow{a}_{\mathrm{br}} \xi}$$

Proposition 3.5. *For each star expression e, the loop-labeling defined by the rules above is a LLEE-witness of the chart defined by the rules on p.3 with start vertex e.*

4 Extraction of star expressions from LLEE-charts

Definition 4.1. *A* provable solution *of a chart $\mathcal{C}$ is a function s that maps the vertices in $V\backslash\{\surd\}$ to star expressions such that, for all $v \in V\backslash\{\surd\}$,*

$$s(v) \ =_{\mathsf{BBP}} \ \Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot s(w_j)\Big)$$

where $\{v \xrightarrow{a_i} \surd \mid i = 1, \ldots, m\} \cup \{v \xrightarrow{b_j} w_j \mid j = 1, \ldots, n \wedge w_j \neq \surd\}$ is the set of transitions from v in $\mathcal{C}$. We call $s(v_{\mathrm{s}})$ the principal value *of s.*

For the chart defined by the rules on page 3, with as start vertex some star expression e, mapping each vertex f to itself yields a provable solution with principal value e.

The following proposition states that provable solutions of charts can be transferred backwards via functional bisimulations. Owing to the definition of bisimulation, its proof boils down to an easy application of axioms (A1,2,3).

Proposition 4.2. *Let $\phi : V_1 \to V_2$ be a bisimulation between $\mathcal{C}_1$ and $\mathcal{C}_2$. If s is a provable solution of $\mathcal{C}_2$, then $s \circ \phi$ (with domain $V_1 \backslash \{\surd\}$) is a provable solution of $\mathcal{C}_1$, with the same principal value as s.*

We construct a provable solution $s_{\hat{\mathcal{C}}}$ with respect to any given LLEE-chart $\hat{\mathcal{C}}$. Consider a vertex $w \neq \surd$ with loop-entry transitions $\{w \xrightarrow{a_i}_{[\alpha_i]} w \mid i = 1, \ldots, m\} \cup \{w \xrightarrow{b_j}_{[\beta_j]} w_j \mid j = 1, \ldots, n \wedge w_j \neq w\}$ and branch transitions $\{w \xrightarrow{c_i}_{\mathrm{br}} \surd \mid i = 1, \ldots, p\} \cup \{w \xrightarrow{d_j}_{\mathrm{br}} u_j \mid j = 1, \ldots, q \wedge u_j \neq \surd\}$. We define $s_{\hat{\mathcal{C}}}$ by:

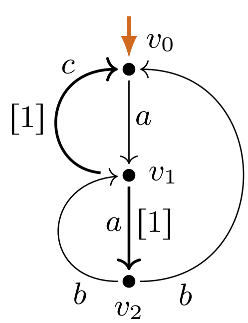
$$s_{\hat{\mathcal{C}}}(w) \;:=\; \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_i \cdot t_{\hat{\mathcal{C}}}(w_j, w)\Big)\Big)^{\circledast}\Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big)$$

using induction on the length $\|w\|_{\mathrm{br}}$ of a longest path of branch transitions from w, where $t_{\hat{\mathcal{C}}}(w, v)$ builds a star expression from w inside a loop that started at v, meaning the construction completes when v is reached. Consider distinct vertices $w, v \neq \surd$ where w is in a loop from v. Let w have loop-entry transitions $\{w \xrightarrow{a_i}_{[\alpha_i]} w \mid i = 1, \ldots, m\} \cup \{w \xrightarrow{b_j}_{[\beta_j]} w_j \mid j = 1, \ldots, n \wedge w_j \neq w\}$ and branch steps $\{w \xrightarrow{c_i}_{\mathrm{br}} v \mid i = 1, \ldots, p\} \cup \{w \xrightarrow{d_j}_{\mathrm{br}} u_j \mid j = 1, \ldots, q \wedge u_j \neq v\}$. We define, by induction on the maximal loop level of a loop at v and with a subinduction on $\|w\|_{\mathrm{br}}$:

$$t_{\hat{\mathcal{C}}}(w, v) \;:=\; \Big(\Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_i \cdot t_{\hat{\mathcal{C}}}(w_j, w)\Big)\Big)^{\circledast}\Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot t_{\hat{\mathcal{C}}}(u_j, v)\Big)\Big)\Big)$$

Proposition 4.3. *For any LLEE-witness $\hat{\mathcal{C}}$, $s_{\hat{\mathcal{C}}}$ is a provable solution of $\mathcal{C}$.*

Example 4.4. We exemplify the construction of a provable solution by means of the LLEE-witness $\hat{\mathcal{C}}_1$ from Ex. 3.4, with $\alpha = \beta = 1$.



$\hat{\mathcal{C}}_1$

$$\begin{aligned}
s_{\hat{\mathcal{C}}_1}(v_0) \;&\equiv\; 0^{\circledast}(a \cdot s_{\hat{\mathcal{C}}_1}(v_1)) \\
&=_{\mathsf{BBP}}\; a \cdot (c \cdot a + a \cdot (b + b \cdot a))^{\circledast}0 \\
s_{\hat{\mathcal{C}}_1}(v_1) \;&\equiv\; (c \cdot t_{\hat{\mathcal{C}}_1}(v_0, v_1) + a \cdot t_{\hat{\mathcal{C}}_1}(v_2, v_1))^{\circledast}0 \\
&=_{\mathsf{BBP}}\; (c \cdot a + a \cdot (b + b \cdot a))^{\circledast}0 \\
t_{\hat{\mathcal{C}}_1}(v_0, v_1) \;&\equiv\; 0^{\circledast}a \\
&=_{\mathsf{BBP}}\; a \\
t_{\hat{\mathcal{C}}_1}(v_2, v_1) \;&\equiv\; 0^{\circledast}(b + b \cdot t_{\hat{\mathcal{C}}_1}(v_0, v_1)) \\
&=_{\mathsf{BBP}}\; b + b \cdot a \\
s_{\hat{\mathcal{C}}_1}(v_2) \;&\equiv\; 0^{\circledast}(b \cdot s_{\hat{\mathcal{C}}_1}(v_1) + b \cdot s_{\hat{\mathcal{C}}_1}(v_0)) \\
&=_{\mathsf{BBP}}\; (b + b \cdot a) \cdot s_{\hat{\mathcal{C}}_1}(v_1) \\
&=_{\mathsf{BBP}}\; (b + b \cdot a) \cdot ((c \cdot a + a \cdot (b + b \cdot a))^{\circledast}0)
\end{aligned}$$

Proposition 4.5. *For all provable solutions s_1 and s_2 of a LLEE-chart $\mathcal{C}$ we have $s_1(w) =_{\mathsf{BBP}} s_2(w)$ for all vertices $w \neq \surd$.*

Proof. Pick a LLEE-witness $\hat{\mathcal{C}}$. It suffices to show, for each provable solution s of $\mathcal{C}$, that $s(w) =_{\mathsf{BBP}} s_{\hat{\mathcal{C}}}(w)$ for all $w \neq \surd$.

The following fact can be proved with induction on the measure used in the definition of $t_{\hat{\mathcal{C}}}$, i.e., the maximal loop level at v, with a subinduction on $\|w\|_{\mathrm{br}}$:

$$v \curvearrowright w \quad \Longrightarrow \quad s(w) =_{\mathsf{BBP}} t_{\hat{\mathcal{C}}}(w, v) \cdot s(v) \tag{4.1}$$

for all vertices v, w. The proof of (4.1), omitted here, strongly resembles the ensuing proof that $s(w) =_{\mathsf{BBP}} s_{\hat{\mathcal{C}}}(w)$ for all $w \neq \surd$. The derivation below is based on the set representation of transitions from w in $\hat{\mathcal{C}}$ as formulated in the definition of $s_{\hat{\mathcal{C}}}(w)$. The first derivation step uses that s is a provable solution of $\mathcal{C}$ and axioms (A1,2,3), the second step (4.1) as $w \curvearrowright w_j$ for $j = 1, \ldots, m$, and the third step axioms (A4,5).

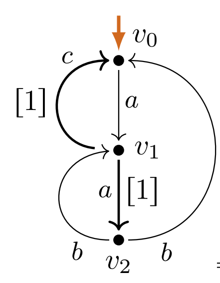
$$\begin{aligned}
s(w) &=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i \cdot s(w) \Big) + \Big(\sum_{j=1}^{n} b_j \cdot s(w_j) \Big) \Big) + \Big(\Big(\sum_{i=1}^{p} c_i \Big) + \Big(\sum_{j=1}^{q} d_j \cdot s(u_j) \Big) \Big) \\
&=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i \cdot s(w) \Big) + \Big(\sum_{j=1}^{n} b_j \cdot \big(t_{\hat{\mathcal{C}}}(w_j, w) \cdot s(w) \big) \Big) \Big) + \Big(\Big(\sum_{i=1}^{p} c_i \Big) + \Big(\sum_{j=1}^{q} d_j \cdot s(u_j) \Big) \Big) \\
&=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i \Big) + \Big(\sum_{j=1}^{n} \big(b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w) \big) \Big) \Big) \cdot s(w) + \Big(\Big(\sum_{i=1}^{p} c_i \Big) + \Big(\sum_{j=1}^{q} d_j \cdot s(u_j) \Big) \Big)
\end{aligned}$$

In view of this derived equality for $s(w)$, the rule $\mathrm{RSP}^{\circledast}$ can be applied to obtain:

$$s(w) =_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i \Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w) \Big) \Big)^{\circledast} \Big(\Big(\sum_{i=1}^{p} c_i \Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j) \Big) \Big) \equiv s_{\hat{\mathcal{C}}}(w)$$

The last step uses the definition of $s_{\hat{\mathcal{C}}}(w)$. □

Example 4.6. Below it is shown that any provable solution s of the chart $\mathcal{C}$ from Ex. 3.4 is provably equal to the solution $s_{\hat{\mathcal{C}}_1}$ extracted in Ex. 4.4.



$\hat{\mathcal{C}}_1$

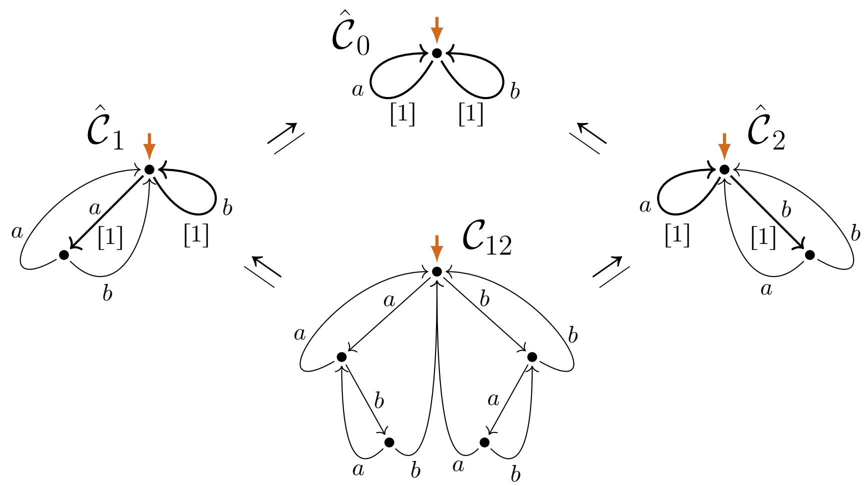
$$\begin{aligned}
s(v_0) &=_{\mathsf{BBP}}^{(\mathrm{sol})} a \cdot s(v_1) \quad (^{(\mathrm{sol})} \text{ means use of ‘provable solution’}) \\
s(v_1) &=_{\mathsf{BBP}}^{(\mathrm{sol})} c \cdot s(v_0) + a \cdot s(v_2) \\
&=_{\mathsf{BBP}}^{(\mathrm{sol})} c \cdot (a \cdot s(v_1)) + a \cdot (b \cdot s(v_1) + b \cdot s(v_0)) \\
&=_{\mathsf{BBP}}^{(\mathrm{sol})} c \cdot (a \cdot s(v_1)) + a \cdot (b \cdot s(v_1) + b \cdot (a \cdot s(v_1))) \\
&=_{\mathsf{BBP}} (c \cdot a + a \cdot (b + b \cdot a)) \cdot s(v_1) + 0 \\
&\Downarrow \text{ applying } \mathrm{RSP}^{\circledast} \\
s(v_1) &=_{\mathsf{BBP}} (c \cdot a + a \cdot (b + b \cdot a))^{\circledast} 0 \\
\Rightarrow \quad s(v_1) &=_{\mathsf{BBP}} s_{\hat{\mathcal{C}}_1}(v_1) \quad (\text{see Ex. 4.4}) \\
\Rightarrow \quad s(v_0) &=_{\mathsf{BBP}}^{(\mathrm{sol})} a \cdot s(v_1) =_{\mathsf{BBP}} a \cdot s_{\hat{\mathcal{C}}_1}(v_1) =_{\mathsf{BBP}}^{(\mathrm{sol})} s_{\hat{\mathcal{C}}_1}(v_0) \\
\Rightarrow \quad s(v_2) &=_{\mathsf{BBP}}^{(\mathrm{sol})} b \cdot s(v_1) + b \cdot s(v_0) \\
&=_{\mathsf{BBP}} b \cdot s_{\hat{\mathcal{C}}_1}(v_1) + b \cdot s_{\hat{\mathcal{C}}_1}(v_0) =_{\mathsf{BBP}}^{(\mathrm{sol})} s_{\hat{\mathcal{C}}_1}(v_2)
\end{aligned}$$

5 Intermezzo: Structure of the completeness proof

Our next goal is to prove the heart of the matter: that the bisimulation collapse of a LLEE-chart is again a LLEE-chart. Then completeness of BBP can be argued as follows. Given two bisimilar star expressions e_1 and e_2, generate their charts $\mathcal{C}_1$ and $\mathcal{C}_2$, which are LLEE-charts according to Prop. 3.5. As remarked below Def. 4.1, e_1 and e_2 are principal values of provable solutions of $\mathcal{C}_1$ and $\mathcal{C}_2$. These charts have the same bisimulation collapse $\mathcal{C}$, which is again a LLEE-chart. Build a provable solution s of $\mathcal{C}$, according to Prop. 4.3. Transfer s backwards over the functional bisimulations to provable solutions s_1 and s_2 of $\mathcal{C}_1$ and $\mathcal{C}_2$, according to Prop. 4.2. By construction, s_1 and s_2 have the same principal value e as s. Finally, e_1 and e_2 are both provably equal to e, according to Prop. 4.5. Hence, $e_1 =_{\mathsf{BBP}} e =_{\mathsf{BBP}} e_2$.

We note that in his completeness proof for regular expressions in formal language theory, Salomaa [17] had to move "upwards" from two equivalent regular expressions to a larger regular expression that can be functionally collapsed onto both of them. By contrast, our proof approach forces us "downwards" to the bisimulation collapse, as in the opposite direction the LLEE-witness may be lost.

Example 5.1. The picture below illustrates why Salomaa's proof strategy to link two charts of regular expressions via their least common unfolding does not work for us. For the bisimilar LLEE-charts $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$, which correspond to the star expressions $(a \cdot (a+b) + b)^{\circledast}0$ and $(b \cdot (a+b) + a)^{\circledast}0$, their least common unfolding $\mathcal{C}_{12}$ is not a LLEE-chart. Namely, it has an infinite path but no loop subchart, as from each of its five vertices an infinite path starts that does not return to this vertex. But the common bisimulation collapse $\hat{\mathcal{C}}_0$ (of $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$), which corresponds to the star expression $(a+b)^{\circledast}0$, is a LLEE-chart. ($\rightleftharpoons$ denotes functional bisimilarity.)



6 Notations and lemmas regarding loops

We introduce notations and lemmas on transitions and loops that are used in Sect. 7. The definitions are with regard to a chart with a loop labeling.

For a binary relation R, let R^+ and R^* be its transitive and transitive-reflexive closures. $u \to_l v$ denotes there is a transition $u \xrightarrow{a}_l v$ for an $a \in A$, and in proofs (but not pictures) $u \to v$ denotes that $u \to_l v$ for some label l. Let $u \xrightarrow[\not\mathbf{t}(w)]{}_l v$ denote that $u \to_l v$ and $v \neq w$; likewise, $u \xrightarrow[\not\mathbf{t}(w)]{} v$ denotes that $u \xrightarrow[\not\mathbf{t}(w)]{}_l v$ for some label l. We write $u \curvearrowright v$, and say that u *descends into a loop to* v, if there is a path $u \xrightarrow[\not\mathbf{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}^*_{\mathrm{br}} v$. We write $v \circlearrowright u$, and say that v *loops back to* u, if $u \curvearrowright v \to^+_{\mathrm{br}} u$. Let $\mathsf{scc}(u)$ denote the strongly connected component (scc) to which u belongs.

Lemma 6.1. *In a LLEE-chart, if* $\mathsf{scc}(u) = \mathsf{scc}(v)$*, then* $u \curvearrowright^* v$ *implies* $v \circlearrowright^* u$*.*

Proof. We prove that $u \curvearrowright^n v$ implies $v \circlearrowright^n u$ for all $n \geqslant 0$, by induction on n. The base case $n = 0$ is trivial, as then $u = v$. If $n > 0$, $u \curvearrowright^{n-1} u' \curvearrowright v$ for some u'. Clearly $\mathsf{scc}(u) = \mathsf{scc}(u') = \mathsf{scc}(v)$. By induction, $u' \circlearrowright^{n-1} u$. Since $u' \curvearrowright v$, there is an acyclic path $u' \to_{[\alpha]} \cdot \to^*_{\mathrm{br}} v$. And since $\mathsf{scc}(u') = \mathsf{scc}(v)$, there is an acyclic path $v \to^*_{\mathrm{br}} \cdot \to_{[\beta_1]} \cdot \to^*_{\mathrm{br}} \cdot \cdots \cdot \to^*_{\mathrm{br}} \cdot \to_{[\beta_k]} \cdot \to^*_{\mathrm{br}} u'$. By (W2)(b), $|\alpha| > |\beta_1| > \cdots > |\beta_k| > |\alpha|$. This means $k = 0$, so $v \to^*_{\mathrm{br}} u'$. This implies $v \circlearrowright u'$ and hence $v \circlearrowright^n u$. ◻

Lemma 6.2. *In a LLEE-chart,* $\mathsf{scc}(u) = \mathsf{scc}(v)$ *if and only if* $u \circlearrowright^* w$ *and* $v \circlearrowright^* w$ *for some* w*.*

Proof. If $u = v$, the lemma is trivial. Let $u \neq v$. Then they are on a cycle, which, since there is no branch cycle, contains a loop-entry transition from some w. Without loss of generality, suppose $w \neq u$. Then $w \curvearrowright^+ u$, so by Lem. 6.1, $u \circlearrowright^+ w$. If $w = v$ we have $v \circlearrowright^* w$, and if $w \neq v$ we can argue in the same fashion that $v \circlearrowright^+ w$. ◻

$\circlearrowright^*$ is a partial order for LLEE-charts. Namely, by definition it is transitive–reflexive. Moreover, it is anti-symmetric, because $u \circlearrowright^+ v$ and $v \circlearrowright^+ u$ would induce a branch cycle from u to v and back, which cannot exist in a LLEE-chart.

Lemma 6.3. *In a LLEE-chart,* $\circlearrowright^*$ *has the least-upper-bound property: if a non-empty set of vertices has an upper bound, then it has a least upper bound.*

Proof. Since the chart is finite, it suffices to show that for each vertex v the set of vertices x with $v \circlearrowright^* x$ is totally ordered with regard to $\circlearrowright^*$. Let $v \circlearrowright^+ u_1$ and $v \circlearrowright^+ u_2$ with $u_1 \neq u_2$. There is a path $u_1 \xrightarrow[\not\mathbf{t}(u_1)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u_1)]{}^* v \to^+_{\mathrm{br}} u_2 \xrightarrow[\not\mathbf{t}(u_2)]{}_{[\beta]} \cdot \xrightarrow[\not\mathbf{t}(u_2)]{}^* v \to^+_{\mathrm{br}} u_1$. Without loss of generality, suppose $|\beta| \geqslant |\alpha|$. Then layeredness implies that each path $v \to^+_{\mathrm{br}} u_2$ must visit u_1, so $v \xrightarrow[\not\mathbf{t}(u_2)]{}^+_{\mathrm{br}} u_1 \to^+_{\mathrm{br}} u_2$. Hence there is a path $u_2 \xrightarrow[\not\mathbf{t}(u_2)]{}_{[\beta]} \cdot \xrightarrow[\not\mathbf{t}(u_2)]{}^* v \xrightarrow[\not\mathbf{t}(u_2)]{}^+_{\mathrm{br}} u_1 \to^+_{\mathrm{br}} u_2$, which implies $u_1 \circlearrowright^+ u_2$. ◻

We write $v \;{}_d\!\circlearrowright\; u$, and say that v *directly loops back to* u, if $v \circlearrowright u$ and for all w with $v \circlearrowright w$ either $w = u$ or $u \circlearrowright w$.

Lemma 6.4. *In a LLEE-chart, if* $v_1 \;{}_d\!\circlearrowright\; u$ *and* $v_2 \;{}_d\!\circlearrowright\; u$ *for distinct* v_1, v_2*, then there does not exist a vertex* w *such that both* $w \circlearrowright^* v_1$ *and* $w \circlearrowright^* v_2$*.*

Proof. $\neg(v_2 \circlearrowright^+ v_1)$ and $\neg(v_1 \circlearrowright^+ v_2)$, for else the definition of ${}_d\!\circlearrowright$ would imply $u \circlearrowright^* v_1$ or $u \circlearrowright^* v_2$, and so $v_1 \circlearrowright^+ v_1$ or $v_2 \circlearrowright^+ v_2$, contradicting irreflexivity of $\circlearrowright^+$. In the proof of Lem. 6.3 we saw that for each w, $\{x \mid w \circlearrowright^* x\}$ is totally ordered with regard to $\circlearrowright^*$, which implies such sets cannot contain both v_1 and v_2. ◻

We write $u \rightarrow_{lb} v$, called a *loop-back transition*, if $u \rightarrow_{\mathrm{br}} v$ and $\mathsf{scc}(u) = \mathsf{scc}(v)$. The *loops-back-to norm* $\|u\|_{lb}^{\min}$ is the length of a longest chain of loop-back transitions from u. In a LLEE-chart this is well-defined, owing to the absence of branch cycles. We note that $\|u\|_{lb}^{\min} = 0$ if and only if u does not loop back, denoted by $\neg(u \circlearrowleft)$.

7 Preservation of LLEE under bisimulation collapse

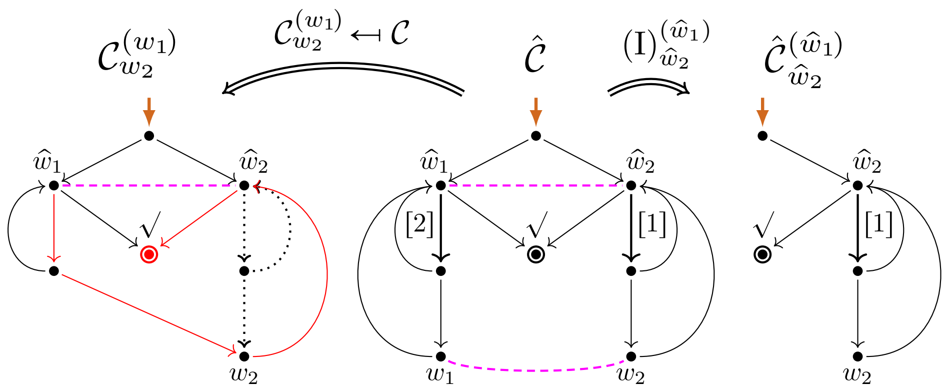
The bisimulation collapse of a LLEE-chart will be constructed in a step-wise fashion, collapsing one pair of bisimilar vertices w_1 and w_2 at a time, whereby the incoming transitions of w_1 are redirected to w_2. The crux is to take care (and prove) that the resulting chart still has a LLEE-witness.

Definition 7.1. *Given a chart $\mathcal{C}$ and vertices w_1, w_2, the* connect-w_1-through-to-w_2 *chart $\mathcal{C}_{w_2}^{(w_1)}$ is obtained by eliminating w_1 and redirecting its incoming transitions to w_2. If w_1 is the start vertex of $\mathcal{C}$, then w_2 becomes the start vertex of $\mathcal{C}_{w_2}^{(w_1)}$.*

Given a loop-labeling $\hat{\mathcal{C}}$, loop-entry and branch labels of transitions to w_1 in $\hat{\mathcal{C}}$ are inherited by their redirections to w_2 in $\mathcal{C}_{w_2}^{(w_1)}$, except if a redirection coincides with a transition to w_2 in $\mathcal{C}$, in which case the label of the existing transition has precedence over the redirected one. The resulting loop-labeling is denoted by $\hat{\mathcal{C}}_{w_2}^{(w_1)}$.

It is not hard to see that if $w_1 \leftrightarrows w_2$, then $\mathcal{C}_{w_2}^{(w_1)} \leftrightarrows \mathcal{C}$. However, the LLEE property may be lost in the transition from $\mathcal{C}$ to $\mathcal{C}_{w_2}^{(w_1)}$.

Example 7.2. Depicted below in the middle is a LLEE-chart $\hat{\mathcal{C}}$. The unspecified action labels are assumed to facilitate that w_1 and w_2 are bisimilar, and so $\widehat{w}_1$ and $\widehat{w}_2$ are bisimilar. Bisimilarity is indicated by the dashed lines. The chart $\mathcal{C}_{w_2}^{(w_1)}$, drawn at the left, is not a LLEE-chart. Namely, three transitions can be removed, which are drawn as dotted arrows: the downwards transition from $\widehat{w}_2$, which induces a loop, and two further transitions that get unreachable. After these removals, the remaining chart still has an infinite path, but it does not contain a loop subchart, because each infinite path can reach termination $\surd$ without returning to its source. An example of this is the red path from $\widehat{w}_1$ via w_2 and $\widehat{w}_2$ to $\surd$. In $\hat{\mathcal{C}}$, the bisimilar pair w_1, w_2 progresses to the bisimilar pair $\widehat{w}_1, \widehat{w}_2$. The connect-$\widehat{w}_1$-through-to-$\widehat{w}_2$ chart at the right is a LLEE-chart, as witnessed by the loop-labeling $\hat{\mathcal{C}}_{\widehat{w}_2}^{(\widehat{w}_1)}$.



Together with three further examples (Ex. 7.7, 7.8, and 7.9), the example above illustrates that the bisimilar pair of vertices must be selected carefully, to safeguard that the connect-through construction preserves LLEE. The proposition below expresses that a bisimilar pair of vertices can always be selected in one of three mutually exclusive categories. Subsequently three LLEE-preserving transformations I, II, and III are defined for each of these categories. In the first category, no path exists from w_2 to w_1; moreover, w_1 does not loop back. In the second, w_2 loops back to w_1. In the third, w_1 and w_2 loop back to the same vertex v; moreover, w_1 directly loops back to v, and no branch path exists from w_2 to w_1.

In the proof of the proposition below, from a given pair of distinct bisimilar vertices we repeatedly progress via transitions, at one side picking loop-back transitions, over pairs of distinct bisimilar vertices, until one of three conditions is met.

Proposition 7.3. *Consider a LLEE-chart $\hat{\mathcal{C}}$ that is not a bisimulation collapse. There are bisimilar vertices w_1, w_2 such that either (C1) $\neg(w_1 \circlearrowright) \wedge \neg(w_2 \rightarrow^* w_1)$, or (C2) $w_2 \circlearrowright^+ w_1$, or (C3) $\exists v \in V\big(w_1 \,{}_d\!\circlearrowright\, v \;\wedge\; w_2 \circlearrowright^+ v\big) \wedge \neg(w_2 \rightarrow^*_{\mathrm{br}} w_1)$.*

Proof. Pick distinct bisimilar vertices u_1, u_2. First regard the case $\mathsf{scc}(u_1) \neq \mathsf{scc}(u_2)$. Without loss of generality, let $\neg(u_2 \rightarrow^* u_1)$. We progress to a pair of vertices where (C1) holds, using induction on $\|u_1\|^{\min}_{lb}$. In the base case, $\|u_1\|^{\min}_{lb} = 0$ implies $\neg(u_1 \circlearrowright)$, so we can define $w_1 = u_1$ and $w_2 = u_2$ and are done. In the induction step, $\|u_1\|^{\min}_{lb} > 0$ implies $u_1 \rightarrow_{lb} u_1'$ and $\|u_1'\|^{\min}_{lb} < \|u_1\|^{\min}_{lb}$ for some u_1'. Since $u_1 \mathrel{\underline{\leftrightarrow}} u_2$, we have $u_2 \rightarrow u_2'$ and $u_1' \mathrel{\underline{\leftrightarrow}} u_2'$ for some u_2'. Since $u_1 \rightarrow_{lb} u_1'$, by definition, u_1 and u_1' are in the same scc. Hence $u_1' \rightarrow^* u_1$. This implies $\neg(u_2' \rightarrow^* u_1')$, for else $u_2 \rightarrow u_2' \rightarrow^* u_1' \rightarrow^* u_1$, which contradicts the assumption $\neg(u_2 \rightarrow^* u_1)$. Since $u_1' \mathrel{\underline{\leftrightarrow}} u_2'$ and $\neg(u_2' \rightarrow^* u_1')$ and $\|u_1'\|^{\min}_{lb} < \|u_1\|^{\min}_{lb}$, by induction there exists a bisimilar pair w_1, w_2 for which (C1) holds.

Now let $\mathsf{scc}(u_1) = \mathsf{scc}(u_2)$. Then by Lem. 6.2, $u_1 \circlearrowright^* v$ and $u_2 \circlearrowright^* v$ for some v. By Lem. 6.3 we pick v as the least upper bound of u_1, u_2 with regard to $\circlearrowright^*$. If $u_1 = v$, then $u_2 \circlearrowright^+ u_1$, so (C2) holds for $w_1 = u_1$ and $w_2 = u_2$. If $u_2 = v$, then likewise (C2) holds for $w_1 = u_2$ and $w_2 = u_1$. Now let $u_1, u_2 \neq v$. Since v is the least upper bound, $u_1 \circlearrowright^* v_1 \,{}_d\!\circlearrowright\, v \,{}_d\!\circlearrowleft\, v_2 \circlearrowleft^* u_2$ for distinct $v_1, v_2 \in V$. There cannot be a cycle of branch transitions, so $\neg(v_2 \rightarrow^*_{\mathrm{br}} v_1)$ or $\neg(v_1 \rightarrow^*_{\mathrm{br}} v_2)$. By symmetry it suffices to consider $\neg(v_2 \rightarrow^*_{\mathrm{br}} v_1)$. Summarizing, $u_1 \circlearrowright^* v_1 \,{}_d\!\circlearrowright\, v \,{}_d\!\circlearrowleft\, v_2 \circlearrowleft^* u_2 \wedge \neg(v_1 \leftarrow^*_{\mathrm{br}} v_2)$. For this situation we use induction on $\|u_1\|^{\min}_{lb}$. If $u_1 = v_1$, then $u_1 \,{}_d\!\circlearrowright\, v$; taking $w_1 = u_1$ and $w_2 = u_2$, (C3) holds. So we can assume $u_1 \circlearrowright^+ v_1 \,{}_d\!\circlearrowright\, v$. Pick a transition $u_1 \rightarrow_{lb} u_1'$ with $\|u_1'\|^{\min}_{lb} < \|u_1\|^{\min}_{lb}$; by definition, $\mathsf{scc}(u_1') = \mathsf{scc}(u_1)$. Since $u_1 \mathrel{\underline{\leftrightarrow}} u_2$, there is a transition $u_2 \rightarrow u_2'$ with $u_1' \mathrel{\underline{\leftrightarrow}} u_2'$ for some u_2'. If $\mathsf{scc}(u_1') \neq \mathsf{scc}(u_2')$, then as before we can find bisimilar w_1, w_2 for which (C1) holds. Now let $\mathsf{scc}(u_1') = \mathsf{scc}(u_2')$, so u_1, u_2, u_1', u_2' are in the same scc. Since $u_1 \circlearrowright^+ v_1$ and $u_1 \rightarrow u_1'$, either $u_1' = v_1$ or $v_1 \rightarrowtail^+ u_1'$. Moreover, $\mathsf{scc}(u_1') = \mathsf{scc}(u_1) = \mathsf{scc}(v_1)$, so by Lem. 6.1, $u_1' \circlearrowright^* v_1$. Since $u_2 \circlearrowright^* v_2$, we can distinguish two cases (for illustrations, see version with appendix).

Case 1: $u_2 \circlearrowright^+ v_2$. Since $u_2 \rightarrow u_2'$, either $u_2' = v_2$ or $v_2 \rightarrowtail^+ u_2'$. Moreover, $\mathsf{scc}(u_2') = \mathsf{scc}(u_2) = \mathsf{scc}(v_2)$, so by Lem. 6.1, $u_2' \circlearrowright^* v_2$. Hence, $u_1' \circlearrowright^* v_1 \,{}_d\!\circlearrowright\, v \,{}_d\!\circlearrowleft\, v_2 \circlearrowleft^* u_2' \wedge \neg(v_1 \leftarrow^*_{\mathrm{br}} v_2)$, and $\|u_1'\|^{\min}_{lb} < \|u_1\|^{\min}_{lb}$. We apply the induction hypothesis to obtain a bisimilar pair w_1, w_2 for which (C1), (C2), or (C3) holds.

Case 2: $u_2 = v_2$. We distinguish two cases.

Case 2.1: $u_2 \to_{[\alpha]} u_2'$. Then either $u_2' = u_2$ or $u_2 \rightharpoonup^{+} u_2'$. Moreover, $\mathsf{scc}(u_2') = \mathsf{scc}(u_2)$, so by Lem. 6.1, $u_2' \circlearrowright^* u_2$, and hence $u_2' \circlearrowright^* v_2$. Thus we have obtained $u_1' \circlearrowright^* v_1 \;{}_{d}{\circlearrowright}\; v \;{}_{d}{\circlearrowleft}\; v_2 \circlearrowleft^* u_2' \wedge \neg(v_1 \leftarrow_{\mathrm{br}}^{*} v_2)$. Due to $\|u_1'\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$, we can apply the induction hypothesis again.

Case 2.2: $u_2 \to_{\mathrm{br}} u_2'$. Then $\neg(v_2 \to_{\mathrm{br}}^{*} v_1)$ together with $v_2 = u_2 \to_{\mathrm{br}} u_2'$ and $u_1' \to_{\mathrm{br}}^{*} v_1$ (because $u_1' \circlearrowright^* v_1$) imply $u_1' \neq u_2'$. We distinguish two cases.

Case 2.2.1: $u_2' = v$. Then $u_1' \circlearrowright^* v_1 \;{}_{d}{\circlearrowright}\; v = u_2'$, i.e., $u_1' \circlearrowright^{+} u_2'$, so we are done, because (C2) holds for $w_1 = u_2'$ and $w_2 = u_1'$.

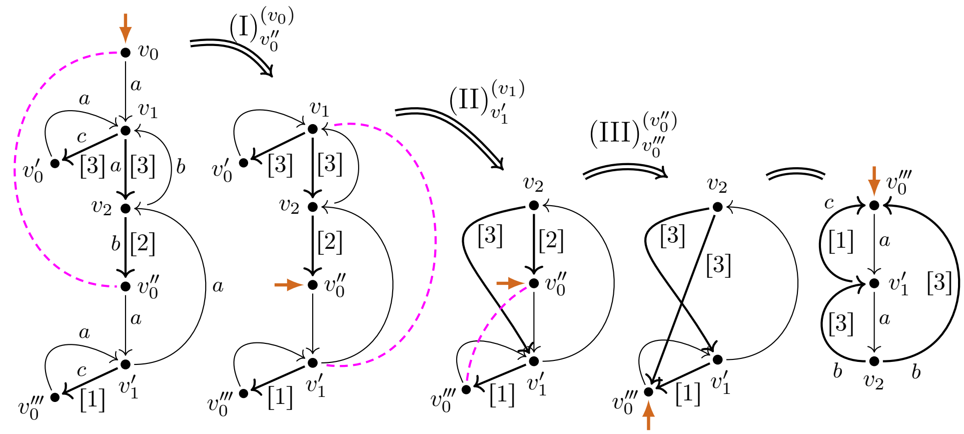
Case 2.2.2: $u_2' \neq v$. By Lem. 6.1, $u_2' \circlearrowright^{+} v$. Hence, $u_2' \circlearrowright^* v_2' \;{}_{d}{\circlearrowright}\; v$ for some v_2'. Since $v_2 = u_2 \to_{\mathrm{br}} u_2' \circlearrowright^* v_2'$ and $\neg(v_2 \to_{\mathrm{br}}^{*} v_1)$, it follows that $\neg(v_2' \to_{\mathrm{br}}^{*} v_1)$. So $u_1' \circlearrowright^* v_1 \;{}_{d}{\circlearrowright}\; v \;{}_{d}{\circlearrowleft}\; v_2' \circlearrowleft^* u_2' \wedge \neg(v_1 \leftarrow_{\mathrm{br}}^{*} v_2')$. Due to $\|u_1'\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$, we can apply the induction hypothesis again. ◻

Given, in a LLEE-chart $\hat{\mathcal{C}}$, bisimilar vertices w_1, w_2 that satisfy condition (C1), (C2), or (C3) of Prop. 7.3, we eliminate w_1 by constructing the connect-w_1-through-to-w_2 chart $\mathcal{C}_{w_2}^{(w_1)}$ (see Def. 7.1). We number the transformations associated to (C1), (C2), and (C3) as I, II, and III, respectively. In each transformation an adaptation of labels of transitions is performed, to avoid violations of LLEE-witness properties in the result. In transformations I and III the adaptation is performed before connecting w_1 through to w_2, and is needed to guarantee that layeredness is preserved; in transformation II it is performed right after eliminating w_1, and avoids the creation of branch cycles. The adaptations for the three transformations are:

- L_{I} Let $m = \max\{\, |\beta| \,:\, \text{there is a path } w_2 \to^* \cdot \to_{[\beta]} \text{ in } \hat{\mathcal{C}}\,\}$. In loop-entry transitions $u \to_{[\alpha]} v$ for which there is a path $v \to^* w_1$ in $\mathcal{C}$, replace α by an α' with $|\alpha'| = |\alpha| + m$. This increases the labels of loop-entry transitions that descend to w_1 in $\hat{\mathcal{C}}$ to a higher level than the loop labels reachable from w_2.
- L_{II} Since $w_2 \circlearrowright^{+} w_1$, there exists a $\widehat{w}_2$ with $w_2 \circlearrowright^* \widehat{w}_2 \;{}_{d}{\circlearrowright}\; w_1$. Let γ be a loop name of maximum loop level among the loop-entries at w_1 in $\hat{\mathcal{C}}$. (Note that since $w_2 \circlearrowright^{+} w_1$, there is at least one such transition.) Turn the branch transitions from $\widehat{w}_2$ into loop-entry transitions with loop label γ.
- L_{III} Let γ be a loop label of maximum level among the loop-entry transitions at v in $\hat{\mathcal{C}}$. (Note that since $w_1 \circlearrowright v$, there is at least one such transition.) Turn the loop labels of the loop-entry transitions from v into γ.

Each of these transformations ends with a clean-up step: if the loop-entry transitions from a vertex with the same loop label no longer induce an infinite path (due to the removal of w_1), then these transitions are changed into branch transitions.

Example 7.4. Below, the LLEE-chart at the left is in three transformation steps reduced at the right to our running example, i.e., the LLEE-witness $\hat{\mathcal{C}}_2$ of the bisimulation collapse of the chart of $(a \cdot ((a \cdot (b \cdot a + b))^{\circledast} c))^{\circledast} 0$ from Ex. 3.4. Dashed lines are between bisimilar vertices. In step one, a transformation I, the start state v_0 is connected through to the bisimilar vertex v_0'', whereby v_0'' becomes the start vertex; note that there is no path from v_0'' to v_0, and v_0 does not loop back. In step two, a transformation II, v_1 is connected through to the bisimilar vertex v_1'; note that $v_1' \curvearrowleft^{+} v_1$. In step three, a transformation III, the start vertex v_0'' is connected through to the bisimilar vertex v_0''', whereby v_0''' becomes the start vertex; note that $v_0'' \;{}_{d}\!\curvearrowleft\; v_2$ and $v_0''' \curvearrowleft^{+} v_2$ and there is no branch path from v_0''' to v_0''; by the loop level adaptation, all loop entries from v_2 get level 3. The final step is an isomorphic deformation. Only the charts at the left and right depict actions on transitions.



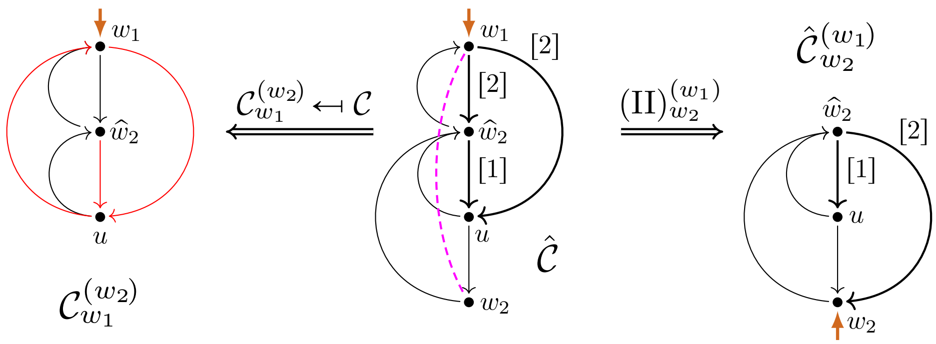
Proposition 7.5. *Let $\mathcal{C}$ be a LLEE-chart. If a pair w_1, w_2 of bisimilar vertices satisfies (C1), (C2), or (C3), then $\mathcal{C}^{(w_1)}_{w_2}$ (see Def. 7.1) is a LLEE-chart.*

As background for the proof of this proposition, we first give examples why conditions (C1), (C2), and (C3) cannot be readily relaxed or changed. For convenience, the pictures in these examples neglect action labels on transitions.

Example 7.6. To show that in (C1) it is crucial that w_1 does not loop back, we refer back to the LLEE-witness $\hat{\mathcal{C}}$ in Ex. 7.2. There $\neg(w_2 \to^* w_1)$, but (C1) is not satisfied by the pair w_1, w_2 because $w_1 \curvearrowleft \widehat{w}_1$. Since in $\hat{\mathcal{C}}$ the levels of loop-entry transitions that descend to w_1 are higher than the loop levels that descend from w_2, the preprocessing step of transformation I is void. We observed that the chart $\mathcal{C}^{(w_1)}_{w_2}$ at the left in Ex. 7.2 has no LLEE-witness. The bisimilar pair w_1, w_2 in $\hat{\mathcal{C}}$ progresses to the bisimilar pair $\widehat{w}_1, \widehat{w}_2$, for which (C1) holds. Since $\hat{\mathcal{C}}^{(\widehat{w}_1)}_{\widehat{w}_2}$ at the right of Ex. 7.2 is obtained by applying transformation I to this pair, it is guaranteed to be a LEE-witness; this will be argued in the proof of Prop. 7.5.

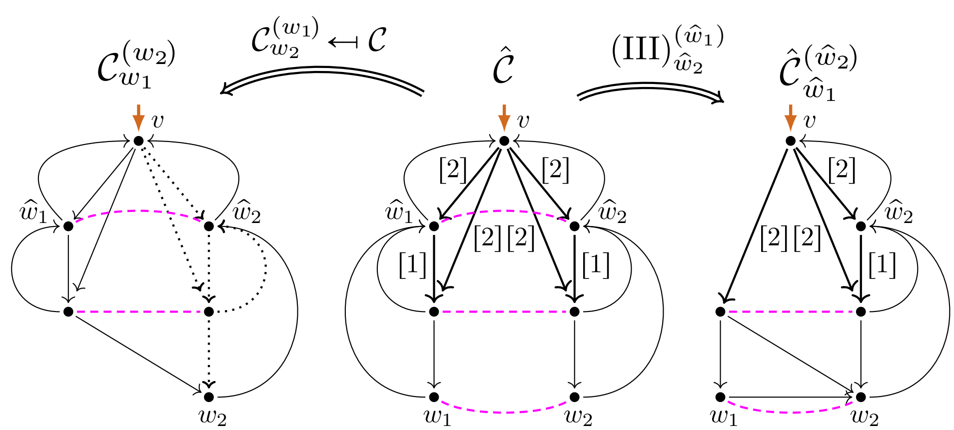
To avoid the creation of branch cycles in transformation II, it would seem expedient to connect transitions to w_2 through to w_1, since (C2), $w_2 \curvearrowleft^{+} w_1$, rules out the existence of a path $w_1 \to^{+}_{\mathrm{br}} w_2$ in $\hat{\mathcal{C}}$. (Instead, transitions to w_1 are connected through to w_2, and resulting branch cycles are eliminated by turning the branch transitions at $\widehat{w}_2$ into loop-entry transitions.) However, connecting transitions to w_2 through to w_1 may produce a chart for which no LLEE-witness exists.

Example 7.7. For the LLEE-chart $\hat{\mathcal{C}}$ below in the middle, the chart $\mathcal{C}^{(w_2)}_{w_1}$ at the left does not have a LLEE-witness: it has no loop subchart, because from each of its three vertices an infinite path starts that does not return to this vertex. From $\widehat{w}_2$ this path, drawn in red, cycles between u and w_1. Transformation II applied to the pair w_1, w_2 (instead of w_2, w_1) in $\hat{\mathcal{C}}$ yields the loop-labeling $\hat{\mathcal{C}}^{(w_1)}_{w_2}$ with additionally $\widehat{w}_2 \to_{\mathrm{br}} w_2$ turned into $\widehat{w}_2 \to_{[2]} w_2$. Since the pair w_1, w_2 satisfies (C2), the proof of Prop. 7.5 guarantees that this loop-labeling, drawn at the right, is a LLEE-witness.



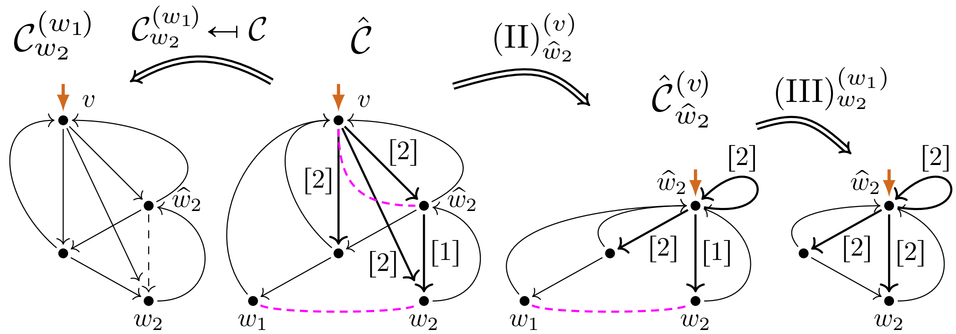
The following example shows that for transformation III it is essential to select a bisimilar pair w_1, w_2 where w_1 *directly* loops back to v.

Example 7.8. In the LLEE-chart $\hat{\mathcal{C}}$ below in the middle, $w_1, w_2 \curvearrowright^{+} v$ and there is no branch path from w_2 to w_1, but (C3) does not hold for the pair w_1, w_2 because $\neg(w_1 \;{}_d\!\curvearrowright\; v)$. All loop-entry transitions from v have the same loop label, so the preprocessing step of transformation III is void. The chart $\mathcal{C}^{(w_1)}_{w_2}$ at the left does not have a LLEE-witness. Namely, the transition from $\widehat{w}_2$ can be declared a loop-entry transition, and after its removal also two transitions from v can be declared loop-entry transitions, leading to the removal of the five transitions that are depicted as dotted arrows. The remaining chart (of solid arrows) however has no further loop subchart, because from each of its vertices an infinite path starts that does not return to this vertex. The bisimilar pair w_1, w_2 progresses to the bisimilar pair $\widehat{w}_1, \widehat{w}_2$ in $\hat{\mathcal{C}}$, for which (C3) holds because $\widehat{w}_1 \;{}_d\!\curvearrowright\; v \curvearrowleft \widehat{w}_2$ and $\neg(\widehat{w}_2 \to^{*}_{\mathrm{br}} \widehat{w}_1)$. Transformation III applied to this pair yields the chart $\hat{\mathcal{C}}^{(\widehat{w}_1)}_{\widehat{w}_2}$ at the right. In the proof of Prop. 7.5 it is argued that this is guaranteed to be a LLEE-chart. The remaining two bisimilar pairs can be eliminated by one or two further applications of transformation III.



The following example shows (C3) cannot be weakened by dropping $\neg(w_2 \to^{*}_{\mathrm{br}} w_1)$.

Example 7.9. For the LLEE-chart $\hat{\mathcal{C}}$ below in the middle, $w_1 \;{}_{d}\!\curvearrowleft\; v \circlearrowleft^{+} w_2$, but there is a branch path from w_2 to w_1. The chart $\mathcal{C}^{(w_1)}_{w_2}$ at the left does not have a LLEE-witness, because from each of its vertices an infinite path starts that does not return to it. The bisimilar pair w_1, w_2 in $\hat{\mathcal{C}}$ progresses to the bisimilar pair $v, \widehat{w}_2$, to which transformation II is applicable because (C2) holds: $\widehat{w}_2 \curvearrowleft v$. In the resulting LLEE-chart $\hat{\mathcal{C}}^{(v)}_{\widehat{w}_2}$, second to the right, (C3) holds for the pair w_1, w_2 because $w_1 \;{}_{d}\!\curvearrowleft\; \widehat{w}_2 \circlearrowleft w_2$ and $\neg(w_2 \rightarrow^{*}_{\mathrm{br}} w_1)$. Applying transformation III to this pair results in the LLEE-chart at the right.



Proof (of Prop. 7.5). Let $\hat{\mathcal{C}}$ be a LLEE-chart. For vertices w_1, w_2 such that (C1), (C2), or (C3) holds, transformation I, II, or III, respectively, produces a loop-labeling $\hat{\mathcal{C}}^{(w_1)}_{w_2}$. We prove for transformations I and III that this is a LLEE-witness.

We first argue it suffices to show that each of the transformations produces, before the final clean-up step, a loop-labeling that satisfies the LLEE-conditions with the exception of possible violations of the loop property (L1) in (W2)(a). The reason is that violations of (L1) can be removed from a loop-labeling while preserving the other LLEE-witness conditions. To show this, suppose (L1) is violated in some $\mathcal{C}_{\hat{\mathcal{C}}}(u,\alpha)$. Then $u \rightarrow_{[\alpha]}$ but $\neg(u \rightarrow_{[\alpha]} \cdot \rightarrow^{*}_{\mathrm{br}} u)$. Let $\hat{\mathcal{C}}_1$ be the result of removing this violation by changing the α-loop-entry transitions from u into branch transitions. No new violation of (L1) is introduced in $\hat{\mathcal{C}}_1$. (W1) and (W2)(a), (L2), are preserved in $\hat{\mathcal{C}}_1$ because an introduced infinite branch path in $\hat{\mathcal{C}}_1$ would be a branch cycle that stems from a path $u \rightarrow_{[\alpha]} u' \rightarrow^{*}_{\mathrm{br}} u$ in $\hat{\mathcal{C}}$. (W2)(b) might only be violated by a path $w \xrightarrow[\not\leftarrow(w)]{}_{[\beta]} \cdot \xrightarrow[\not\leftarrow(w)]{}^{*}_{\mathrm{br}} u \xrightarrow[\not\leftarrow(w,u)]{}_{\mathrm{br}} u' \xrightarrow[\not\leftarrow(w,u)]{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\gamma]}$ with $|\beta| \leqslant |\gamma|$ in $\hat{\mathcal{C}}_1$ where $u \rightarrow_{\mathrm{br}} u'$ stems from $u \rightarrow_{[\alpha]} u'$ in $\hat{\mathcal{C}}$; then $|\beta| > |\alpha| > |\gamma|$ by layeredness of $\hat{\mathcal{C}}$; so (W2)(b) is preserved. Analogously we find that also (W2)(a), (L3), is preserved because $\surd$ is not contained in $\mathcal{C}_{\hat{\mathcal{C}}}(u,\alpha)$, as $\hat{\mathcal{C}}$ is a LLEE-witness.

To show the correctness of transformation I, consider vertices w_1 and w_2 such that (C1) holds. We show that the result $\hat{\mathcal{C}}^{(w_1)}_{w_2}$ of transformation I before the clean-up step satisfies the LLEE-witness properties, except for possible violations of (L1).

To verify (W1) and part (L2) of (W2)(a), it suffices to show that $\hat{\mathcal{C}}^{(w_1)}_{w_2}$ does not contain branch cycles. The original loop-labeling $\hat{\mathcal{C}}$ is a LLEE-witness, so it does not contain branch cycles. Since the level adaptation step does not turn loop-entry steps into branch steps, branch cycles could only arise in the step connecting w_1 through to w_2. Suppose such a branch cycle arises. Then there must be a transition $u \rightarrow_{\mathrm{br}} w_1$ in $\hat{\mathcal{C}}$ (which is redirected to w_2 in $\hat{\mathcal{C}}^{(w_1)}_{w_2}$) and a path $w_2 \rightarrow^{*}_{\mathrm{br}} u$ in $\hat{\mathcal{C}}$. But then $w_2 \to_{\mathrm{br}}^{*} u \to_{\mathrm{br}} w_1$ in $\mathcal{C}$, which contradicts (C1) that there is no path from w_2 to w_1. We conclude that (W1) and part (L2) of (W2)(a) hold for $\hat{\mathcal{C}}_{w_2}^{(w_1)}$.

Now we verify part (L3) of (W2)(a) in $\hat{\mathcal{C}}_{w_2}^{(w_1)}$. Consider a path $u \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{[\alpha]} \cdot \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{\mathrm{br}}^{*} w_1$ in $\hat{\mathcal{C}}$. Then $u \neq w_1$, and $u \rightsquigarrow w_1$. It suffices to show that then $\neg(w_2 \to^{+} \surd)$ in $\mathcal{C}$. Suppose toward a contradiction w_2 is normed. Since $w_1 \leftrightharpoons_{\mathcal{C}} w_2$, then $w_1 \to^{+} \surd$ in $\mathcal{C}$. In view of the path $u \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{[\alpha]} \cdot \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{\mathrm{br}}^{*} w_1$ in $\hat{\mathcal{C}}$, by (L3), all paths $w_1 \to^{+} \surd$ in $\mathcal{C}$ must visit u. Since $u \rightsquigarrow w_1$, u and w_1 are in the same scc. So by $w_1 \neq u$, and Lem. 6.1, $w_1 \circlearrowright^{+} u$. This contradicts condition (C1) that w_1 does not loop back.

Finally we show that (W2)(b) is preserved in $\hat{\mathcal{C}}_{w_2}^{(w_1)}$ by both the level adaptation and the connect-through step. First, since in the level adaptation step all adapted loop labels are increased with the same value m, a violation of (W2)(b) would arise by a path $u \to_{[\alpha]} \cdot \to_{\mathrm{br}}^{*} \cdot \to_{[\beta]} v$ in $\hat{\mathcal{C}}$ where the loop label β is increased while α is not. But such a path cannot exist. Since β is increased, there is a path $v \to^{*} w_1$ in $\mathcal{C}$. But then there is a path $u \to_{[\alpha]} \cdot \to^{+} v \to^{*} w_1$ in $\hat{\mathcal{C}}$, which implies that also α is increased in the level adaptation step. Second, a violation of (W2)(b) in the connect-through step would arise from paths $u \to_{[\alpha]} \cdot \to_{\mathrm{br}}^{*} w_1$ and $w_2 \to_{\mathrm{br}}^{*} \cdot \to_{[\beta]}$ in $\hat{\mathcal{C}}'$ with $|\alpha| \leqslant |\beta|$. However, in view of the path $u \to_{[\alpha]} \cdot \to^{*} w_1$, the loop label α was increased with m in the level adaptation step . On the other hand, in view of (C1) that there is no path from w_2 to w_1 in $\mathcal{C}$, w_1 is unreachable at the end of the path $w_2 \to^{*} \cdot \to_{[\beta]}$. Hence this loop label β was not increased in the level adaptation step. So it is guaranteed that for such a pair of paths in $\hat{\mathcal{C}}_{w_2}^{(w_1)}$ always $|\alpha| > |\beta|$.

We conclude that the result of transformation I is again a LLEE-witness.

To show the correctness of transformation III, consider vertices w_1 and w_2 such that (C3) holds. Let v be such that $w_1 \;{}_{d}{\circlearrowright}\; v \circlearrowleft^{+} w_2$. We show that its intermediate result $\hat{\mathcal{C}}_{w_2}^{(w_1)}$ before the clean-up step satisfies the LLEE-witness properties, except for possible violations of (L1).

First we show that (W2)(b) is preserved by both the level adaptation and the connect-through step. A violation arising by the first step, i.e., in $\hat{\mathcal{C}}'$, would involve a path $u \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{[\alpha]} \cdot \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{\mathrm{br}}^{*} v \to_{[\beta]}$ in $\hat{\mathcal{C}}$ where β is increased to a loop label γ of maximum level among all loop-entries at v. But in this way no violation can arise, since there was already a path $u \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{[\alpha]} \cdot \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{\mathrm{br}}^{*} v \to_{[\gamma]}$ in $\hat{\mathcal{C}}$, so $|\alpha| > |\gamma| \geqslant |\beta|$.

Now we exclude violations of (W2)(b) in the connect-through step, by showing that in $\hat{\mathcal{C}}_{w_2}^{(w_1)}$, $|\alpha| > |\beta|$ for all newly created paths $u \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{[\alpha]} \cdot \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{\mathrm{br}}^{*} \cdot \to_{[\beta]}$ with $u \neq w_1$ that stem from paths $u \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{[\alpha]} \cdot \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{\mathrm{br}}^{*} w_1$ and $w_2 \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{\mathrm{br}}^{*} \cdot \to_{[\beta]}$ in $\hat{\mathcal{C}}'$. As $w_2 \circlearrowright^{+} v$, there is a path $v \xrightarrow[\cancel{\mathbf{t}}(v)]{}_{[\gamma]} \cdot \xrightarrow[\cancel{\mathbf{t}}(v)]{}_{\mathrm{br}}^{*} w_2$ in $\hat{\mathcal{C}}'$. We distinguish two cases.

Case 1: $u = v$. Then, by the level adaptation step, $\alpha = \gamma$. Since $u = v$, there is a path $v \xrightarrow[\cancel{\mathbf{t}}(v)]{}_{[\gamma]} \cdot \xrightarrow[\cancel{\mathbf{t}}(v)]{}_{\mathrm{br}}^{*} w_2 \xrightarrow[\cancel{\mathbf{t}}(v)]{}_{\mathrm{br}}^{*} \cdot \to_{[\beta]}$ in $\hat{\mathcal{C}}'$. By (W2)(b) for $\hat{\mathcal{C}}'$, $|\gamma| > |\beta|$.

Case 2: $u \neq v$. Since $w_1 \;{}_{d}{\circlearrowright}\; v$, there is a path $w_1 \to_{\mathrm{br}}^{+} v$ in $\hat{\mathcal{C}}$ and thus in $\hat{\mathcal{C}}'$. Suppose, toward a contradiction, that this path visits u. Then $u \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{[\alpha]} \cdot \xrightarrow[\cancel{\mathbf{t}}(u)]{}_{\mathrm{br}}^{*} w_1 \to_{\mathrm{br}}^{+} u$, so $w_1 \circlearrowright u$ in $\hat{\mathcal{C}}'$ and thus in $\hat{\mathcal{C}}$. Then $w_1 \;{}_{d}{\circlearrowright}\; v$ and $u \neq v$ imply $v \circlearrowright u$, which together with $u \to_{\mathrm{br}}^{+} v$ yields a branch cycle between u and v in $\hat{\mathcal{C}}$. This contradicts that (W1) holds in $\hat{\mathcal{C}}$. Therefore $w_1 \xrightarrow[\not\leftarrow(u)]{}{}_{\mathrm{br}}^{+} v$ in $\hat{\mathcal{C}}'$. We consider two cases.

Case 2.1: $w_2 \xrightarrow[\not\leftarrow(u)]{}{}_{\mathrm{br}}^{*} \cdot \to_{[\beta]}$ visits v, so $v \xrightarrow[\not\leftarrow(u)]{}{}_{\mathrm{br}}^{*} \cdot \to_{[\beta]}$ in $\hat{\mathcal{C}}'$. Then $u \xrightarrow[\not\leftarrow(u)]{}{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(u)]{}{}_{\mathrm{br}}^{*} w_1 \xrightarrow[\not\leftarrow(u)]{}{}_{\mathrm{br}}^{+} v \xrightarrow[\not\leftarrow(u)]{}{}_{\mathrm{br}}^{*} \cdot \to_{[\beta]}$ in $\hat{\mathcal{C}}'$. By (W2)(b) for $\hat{\mathcal{C}}'$, $|\alpha| > |\beta|$.

Case 2.2: $w_2 \xrightarrow[\not\leftarrow(u)]{}{}_{\mathrm{br}}^{*} \cdot \to_{[\beta]}$ does not visit v. Then since $w_2 \circlearrowright^{+} v$ implies $v \curvearrowright^{+} w_2$, there is a path $v \xrightarrow[\not\leftarrow(v)]{}{}_{[\gamma]} \cdot \xrightarrow[\not\leftarrow(v)]{}{}_{\mathrm{br}}^{*} x_k \xrightarrow[\not\leftarrow(x_k)]{}{}_{[\delta_k]} \cdot \xrightarrow[\not\leftarrow(x_k)]{}{}_{\mathrm{br}}^{*} \cdots x_1 \xrightarrow[\not\leftarrow(x_1)]{}{}_{[\delta_1]} \cdot \xrightarrow[\not\leftarrow(x_1)]{}{}_{\mathrm{br}}^{*} w_2 \xrightarrow[\not\leftarrow(v)]{}{}^{*} \cdot \to_{[\beta]}$ in $\hat{\mathcal{C}}'$, for some $k \geqslant 0$. Since also $u \xrightarrow[\not\leftarrow(u)]{}{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(u)]{}{}_{\mathrm{br}}^{*} w_1 \xrightarrow[\not\leftarrow(u)]{}{}_{\mathrm{br}}^{+} v \to_{[\gamma]}$ in $\hat{\mathcal{C}}'$, by (W2)(b), $|\alpha| > |\gamma| > |\delta_k| > \cdots > |\delta_1| > |\beta|$. So $|\alpha| > |\beta|$.

To verify (W1) together with part (L2) of (W2)(a) for $\hat{\mathcal{C}}_{w_2}^{(w_1)}$, it suffices to show that $\hat{\mathcal{C}}_{w_2}^{(w_1)}$ does not contain branch cycles. This can be verified analogously as for transformation I. That is, under the assumption of a branch cycle we can construct a path $w_2 \to_{\mathrm{br}}^{+} w_1$ in $\hat{\mathcal{C}}$, which contradicts (C3) (as it contradicted (C1)).

To show part (L3) of (W2)(a) for $\hat{\mathcal{C}}_{w_2}^{(w_1)}$, we can use part of the argumentation employed above for proving (W2)(b). It was demonstrated in particular that for every descends-in-loop-to path $u \xrightarrow[\not\leftarrow(u)]{}{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(u)]{}{}_{\mathrm{br}}^{*} x$ in $\hat{\mathcal{C}}''$ there is a descends-in-loop-to path $\tilde{u} \xrightarrow[\not\leftarrow(\tilde{u})]{}{}_{[\gamma']} \cdot \xrightarrow[\not\leftarrow(\tilde{u})]{}{}_{\mathrm{br}}^{*} x$ with the same target x in $\hat{\mathcal{C}}$. This entails that if a descends-in-loop-to path in $\hat{\mathcal{C}}''$ had $\surd$ as target, then there were a descends-in-loop-to path in $\hat{\mathcal{C}}$ with $\surd$ as target, contradicting (L3) for the LLEE-witness $\hat{\mathcal{C}}$. Hence $\hat{\mathcal{C}}''$ must satisfy part (L3) of (W2)(a).

We conclude that the result of transformation III is again a LLEE-witness. $\square$

Corollary 7.10. *The bisimulation collapse of a LLEE-chart is a LLEE-chart.*

Corollary 7.11. *If a chart is expressible by a star expression without 1 modulo bisimilarity, then its collapse is a LLEE-chart.*

Proof. Let $\mathcal{C}$ be a chart with $\mathcal{C} \leftrightarroweq \mathcal{C}(e)$, where $\mathcal{C}(e)$ is the chart associated to a star expression e. Then $\mathcal{C}(e)$ is a LLEE-chart by Prop. 3.5, and so by Cor. 7.10 its collapse is a LLEE-chart. The collapse of $\mathcal{C}$ coincides with the collapse of $\mathcal{C}(e)$. $\square$

It follows that the two charts in Ex. 3.2 are not expressible (without 1) modulo $\leftrightarroweq$.

8 The completeness result

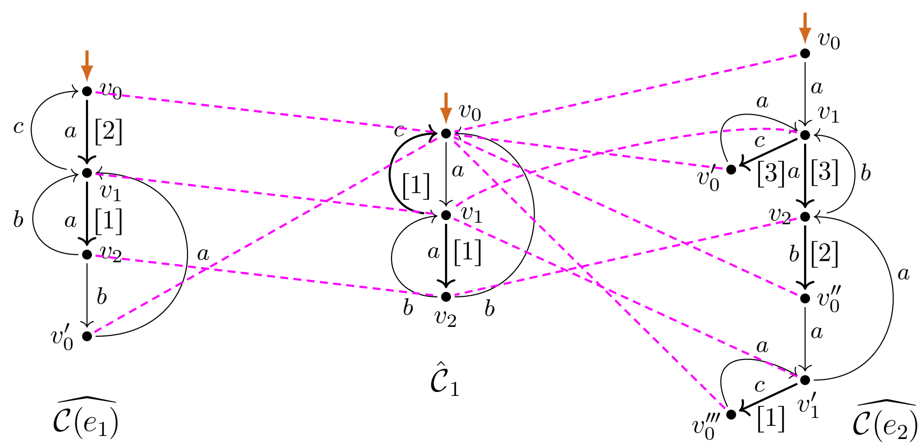
That bisimulation collapse preserves LLEE was the last building block in the proof of the desired completeness result. The proof steps of Thm. 8.1 were already explained in Sect. 5, in order to shed light on the definitions and results in Sect. 3 and Sect. 4, and to motivate the technical developments in Sect. 7.

Theorem 8.1. *The axiomatization* BBP *is complete with respect to the bisimulation semantics for star expressions without* 1 *and with the binary Kleene star* $^{\circledast}$.

We consider two instances of the completeness proof.

Example 8.2. The bisimilar LLEE-charts $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ in Ex. 5.1 have principal solutions $(a \cdot (a + b) + b)^{\circledast}0$ and $(b \cdot (a + b) + a)^{\circledast}0$. Their common bisimulation collapse $\hat{\mathcal{C}}_0$ has principal solution $(a + b)^{\circledast}0$. By Prop. 4.2 and Prop. 4.5, $(a \cdot (a + b) + b)^{\circledast}0$ $=_{\mathsf{BBP}}$ $(a + b)^{\circledast}0$ $=_{\mathsf{BBP}}$ $(b \cdot (a + b) + a)^{\circledast}0$.

Example 8.3. The star expressions $(a \cdot ((a \cdot (b + b \cdot a))^{\circledast}c))^{\circledast}0$, denoted by e_1, and $a \cdot ((c \cdot a + a \cdot (b \cdot a \cdot ((c \cdot a)^{\circledast}a))^{\circledast}b)^{\circledast}0)$, denoted by e_2, are bisimilar. As remarked earlier, in [9] e_1 is mentioned as problematic for a completeness proof, because the bisimulation collapse of its chart does not directly correspond to a star expression. Below at the left and right LLEE-witnesses of the charts $\mathcal{C}(e_1)$ and $\mathcal{C}(e_2)$ are drawn. They have provable solutions with principal values e_1 and e_2. The middle is a LLEE-witness of their bisimulation collapse, the running example $\hat{\mathcal{C}}_1$. The dashed lines connect the vertices in $\widehat{\mathcal{C}(e_1)}$ and $\widehat{\mathcal{C}(e_2)}$ to their bisimilar counterparts in $\hat{\mathcal{C}}_1$. We saw earlier that $\hat{\mathcal{C}}_1$ has a provable solution with principal value $s_{\hat{\mathcal{C}}_1}(v_0) = a \cdot ((c \cdot a + a \cdot (b + b \cdot a))^{\circledast}0)$. By Prop. 4.2 and Prop. 4.5, $e_1 =_{\mathsf{BBP}} s_{\hat{\mathcal{C}}_1}(v_0) =_{\mathsf{BBP}} e_2$.



9 Conclusion

We have shown that Milner's axiomatization, tailored to star expressions without 1 and with $^{\circledast}$, is complete in bisimulation semantics. At the core of the proof is the novel notion LLEE, which also characterizes precisely the process graphs that can be expressed by star expressions without 1 and with $^{\circledast}$: their bisimulation collapse is a LLEE-chart. The main goal for the future is to extend the completeness result to the full class of star expressions, i.e., to include the constant 1.

The completeness result from [10,8] for star expressions without 0 and 1 and with $^{\circledast}$ was extended in [6] to a setting with 1 (but not 0) and *, where an extended version of the non-empty-word property is disallowed for terms directly under a *. We could extend our completeness result to star expressions with 0, 1, and *, but with a syntactic restriction on terms directly under a *, by rewriting each term in this class to a star expression with only 'harmless' occurrences of 1. With this approach one could also obtain the result in [6] directly from the result in [10,8].

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S Supplements (Appendix)

S.1 For Section 3: Layered loop existence and elimination

Proposition (= Prop. 3.5). *For each star expression e, the loop-labeling defined by the rules on page 5 is a LLEE-witness of the chart defined by the rules on page 3 with start vertex e.*

Proof. To verify (W1) it suffices to show there are no infinite branch paths from any star expression e (this is also a preparation for (W2)(a), part (L2)). We prove, by induction on the syntactic structure of e, the stronger statement that if $e \to^{+} f$, then there does not exist an infinite branch path from f. The base cases, in which e is of the form a or 0, are trivial. Suppose $e \equiv e_1 + e_2$. Then $e_i \to^{+} f$ for some $i \in \{1, 2\}$. So by induction, f does not exhibit an infinite branch path. Suppose $e \equiv e_1 \cdot e_2$. Then $e \to^{+} f$ means either $e_1 \to^{+} f_1$ and $f \equiv f_1 \cdot e_2$, or $e_2 \to^{*} f$. In the first case, by induction, f_1 and e_2 do not exhibit infinite branch paths. This induces that $f_1 \cdot e_2$ does not exhibit an infinite branch path. In the second case, by induction, f does not exhibit an infinite branch path. Suppose $e \equiv e_1{}^{\circledast}e_2$. Then $e \to^{+} f$ means (A) $f \equiv e_1{}^{\circledast}e_2$, or (B) $e_1 \to^{+} f_1$ and $f \equiv f_1 \cdot (e_1{}^{\circledast}e_2)$, or (C) $e_2 \to^{+} f$. In case (A), each branch path from f starts with either $f \to_{\mathrm{br}} e_1' \cdot (e_1{}^{\circledast}e_2)$ where $e_1 \to e_1'$ and e_1' is not normed, or $f \to_{\mathrm{br}} e_2'$ where $e_2 \to e_2'$. In the first case, by induction, e_1' does not exhibit an infinite branch path, so since e_1' is not normed, $e_1' \cdot (e_1{}^{\circledast}e_2)$ does not exhibit an infinite branch path. In the second case, by induction, e_2' does not exhibit an infinite branch path. In case (B), since by induction f_1 and by case (A) $e_1{}^{\circledast}e_2$ do not exhibit infinite branch paths, $f_1 \cdot (e_1{}^{\circledast}e_2)$ does not exhibit an infinite branch path. In case (C), by induction, f does not exhibit an infinite branch path.

We verify (W2). From the rules on page 5 it follows that if e has a loop-entry transition, then $e \equiv ((\cdots((e_1{}^{\circledast}e_2) \cdot f_1)\cdots) \cdot f_n)$ for some $n \geqslant 0$ and e_1 normed. Let $\hat{\mathcal{C}}$ denote the loop-labeling defined by the rules on page 5. We prove (W2) for a subchart $\mathcal{C}_{\hat{\mathcal{C}}}(e, \alpha)$ of $\hat{\mathcal{C}}$. We first consider the case $n = 0$, and then generalize it.

Let $e \equiv e_1{}^{\circledast}e_2$ with e_1 normed, and $\alpha = |e_1|_{\circledast} + 1$. Either $e \to_{[\alpha]} e$ or $e \to_{[\alpha]} e_1' \cdot e$ for some normed e_1' with $e_1 \to e_1'$. In the first case (L1) is clearly satisfied; we focus on the second case. It can be argued, by induction on syntactic structure, that every normed star expression has a branch path to $\surd$. Then so does e_1'. This means $e_1' \cdot e$ has a branch path to e. Hence (L1) holds. For the remainder of (W2) it suffices to consider loop-entry transitions $e \to_{[\alpha]} e_1'' \cdot e$ where $e_1 \to e_1''$. Since we showed above there are no branch cycles, every branch path from e_1'' eventually leads to deadlock or $\surd$; in the first case the corresponding branch path of $e_1'' \cdot e$ also deadlocks, and in the second case it returns to e. Hence (L2) holds. Since $e_1'' \cdot e$ cannot reach $\surd$ without returning to e, (L3) holds. It can be shown, by induction on derivation depth, that $f \to f'$ implies $|f|_{\circledast} \geqslant |f'|_{\circledast}$, and clearly $f \to_{[\beta]}$ implies $|\beta| \leqslant |f|_{\circledast}$. So if $e_1'' \to^{*} \cdot \to_{[\beta]}$, then $|\beta| \leqslant |e_1''|_{\circledast} \leqslant |e_1|_{\circledast}$. Hence, if $e_1'' \cdot e \xrightarrow[\not{t}(e_1{}^{\circledast}e_2)]{}{}^{*}_{\mathrm{br}} \cdot \to_{[\beta]}$, then $|\beta| < |e_1|_{\circledast} + 1 = |\alpha|$. So (W2)(*b*) holds.

Now consider $e \equiv ((\cdots((e_1{}^{\circledast}e_2) \cdot f_1)\cdots) \cdot f_n)$ for $n > 0$, with e_1 normed. Again $\alpha = |e_1|_{\circledast} + 1$. The subchart $\mathcal{C}_{\hat{\mathcal{C}}}(e, \alpha)$ basically coincides with $\mathcal{C}_{\hat{\mathcal{C}}}(e_1{}^{\circledast}e_2, \alpha)$, except that the star expressions in the first chart are post-fixed with $f_1, \ldots, f_n$; its transitions are derived by n additional applications of the first rule for concatenation on page 5, to affix these expressions. This chart isomorphism between $\mathcal{C}_{\hat{\mathcal{C}}}(e_1{}^{\circledast}e_2, \alpha)$

and $\mathcal{C}_{\hat{\mathcal{C}}}(e, \alpha)$ preserves action labels as well as the loop-labeling, because the first rule for concatenation preserves these labels. We showed that $\mathcal{C}_{\hat{\mathcal{C}}}(e_1 {}^{\circledast} e_2, \alpha)$ satisfies (W2), so the same holds for $\mathcal{C}_{\hat{\mathcal{C}}}(e, \alpha)$. ◻

S.2 For Section 4: Extraction of star expressions from LLEE-charts

Proposition (= Prop. 4.2). *Let $\phi : V_1 \to V_2$ be a bisimulation between $\mathcal{C}_1$ and $\mathcal{C}_2$. If s is a provable solution of $\mathcal{C}_2$, then $s \circ \phi$ (with domain $V_1 \backslash \{\surd\}$) is a provable solution of $\mathcal{C}_1$. Note that it has the same principal value as s.*

Proof. Let $v \in V_1 \backslash \{\surd\}$. Since ϕ is a functional bisimulation between $\mathcal{C}_1$ and $\mathcal{C}_2$, the forth and back conditions for the graph of ϕ as a bisimulation hold for the pair $\langle v, \phi(v) \rangle$ of vertices. This make it possible to bring the sets of transitions $T_1(v)$ from v in $\mathcal{C}_1$, and $T_2(\phi(v))$ from $\phi(v)$ in $\mathcal{C}_2$ into a 1–1 correspondence such that ϕ again relates their targets:

$$T_1(v) = \big\{ v \xrightarrow{a_i} \surd \;\big|\; i = 1, \ldots, m \big\} \cup \big\{ v \xrightarrow{b_j} v'_{j1} \;\big|\; j = 1, \ldots, n \big\}, \tag{S.1}$$

$$T_2(\phi(v)) = \big\{ \phi(v) \xrightarrow{a_i} \surd \;\big|\; i = 1, \ldots, m \big\} \cup \big\{ \phi(v) \xrightarrow{b_j} v'_{j2} \;\big|\; j = 1, \ldots, n \big\}, \tag{S.2}$$

$$\phi(v'_{j1}) = v'_{j2}, \quad \text{for all } j \in \{1, \ldots, n\}, \tag{S.3}$$

with $n, m \in \mathbb{N}$, and vertices $v'_{j1} \in V_1$, and $v'_{j2} \in V_2$, for $j \in \{1, \ldots, n\}$. (Note that the same transition may be listed multiple times in the set $T_2(\phi(v))$.) On this basis we can argue as follows.

$$(s \circ \phi)(v) \;\equiv\; s(\phi(v)) \;=_{\mathsf{BBP}}\; \Big(\sum_{i=1}^{m} a_i \Big) + \Big(\sum_{j=1}^{n} b_j \cdot s(v'_{j2}) \Big)$$

(since s is a provable solution of $\mathcal{C}_2$, using (S.2) and axioms (A1), (A2), (A3))

$$\equiv \quad \Big(\sum_{i=1}^{m} a_i \Big) + \Big(\sum_{j=1}^{n} b_j \cdot (s \circ \phi)(v'_{j1}) \Big)$$

(using (S.3) and $(s \circ \phi)(v'_{j1}) \equiv s(\phi(v'_{j1}))$)

This shows, in view of (S.1), that $s \circ \phi$ satisfies the condition for a provable solution at v. Now as $v \in V_1 \backslash \{\surd\}$ was arbitrary, $s \circ \phi$ (with domain $V_1 \backslash \{\surd\}$) is a provable solution of $\mathcal{C}_1$. Since furthermore the functional bisimulation ϕ must relate the start vertices of $\mathcal{C}_1$ and $\mathcal{C}_2$, the principal value of $s \circ \phi$ coincides with that of s. ◻

Proposition (= Prop. 4.3). *For any LLEE-witness $\hat{\mathcal{C}}$, $s_{\hat{\mathcal{C}}}$ is a provable solution of $\mathcal{C}$.*

Proof. We first show an auxiliary result about the connection between the extracted solution $s_{\hat{\mathcal{C}}}$ and the relative extracted solution $t_{\hat{\mathcal{C}}}$. For all vertices v, w:

$$v \curvearrowright w \;\;\Longrightarrow\;\; s_{\hat{\mathcal{C}}}(w) \;=_{\mathsf{BBP}}\; t_{\hat{\mathcal{C}}}(w, v) \cdot s_{\hat{\mathcal{C}}}(v)\,. \tag{S.4}$$

Note that if $v \curvearrowright w$, then $v \neq \surd$, and also $w \neq \surd$, because then w is in the body of a loop at v, and therefore cannot be $\surd$.

We proceed by complete induction (without explicit treatment of the base case) on the length $\|w\|_{\mathrm{br}}$ of a longest path of branch transitions from w. For performing the induction step, we consider arbitrary $v, w \neq \surd$ with $v \curvearrowright w$. We assume a representation of the set $\hat{T}(w)$ of transitions from w in $\hat{\mathcal{C}}$:

$$\hat{T}(w) = \{w \xrightarrow{a_i}_{[\alpha_i]} w \mid i = 1, \ldots, m\} \cup \{w \xrightarrow{b_j}_{[\beta_j]} w_j \mid w_j \neq w,\ j = 1, \ldots, n\} \\ \cup \{w \xrightarrow{c_i}_{\mathrm{br}} v \mid i = 1, \ldots, p\} \cup \{w \xrightarrow{d_j}_{\mathrm{br}} u_j \mid u_j \neq \surd,\ j = 1, \ldots, q\} \quad \text{(S.5)}$$

that partitions $\hat{T}(w)$ into loop-entry transitions to w and to other targets $w_1, \ldots, w_n$, and branch transitions to v and to other targets $u_1, \ldots, u_q$. Since w is contained in a loop at v, none of these targets can be $\surd$. In order to show provable equality at the right-hand side of (S.4), we argue as follows:

$$\begin{aligned}
s_{\hat{\mathcal{C}}}(w) \equiv\ & \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w)\Big)\Big)^{\circledast}\Big(0 + \Big(\Big(\sum_{i=1}^{p} c_i \cdot s_{\hat{\mathcal{C}}}(v)\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big)\Big) \\
& \quad \text{(by the definition of } s_{\hat{\mathcal{C}}}(w)\text{, based on the representation (S.5),} \\
& \quad \text{using that none of the target vertices is } \surd) \\
=_{\mathsf{BBP}}\ & \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w)\Big)\Big)^{\circledast}\Big(\Big(\sum_{i=1}^{p} c_i \cdot s_{\hat{\mathcal{C}}}(v)\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big) \\
& \quad \text{(using axiom (A6))} \\
=_{\mathsf{BBP}}\ & \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w)\Big)\Big)^{\circledast}\Big(\Big(\sum_{i=1}^{p} c_i \cdot s_{\hat{\mathcal{C}}}(v)\Big) + \Big(\sum_{j=1}^{q} d_j \cdot \big(t_{\hat{\mathcal{C}}}(u_j, v) \cdot s_{\hat{\mathcal{C}}}(v)\big)\Big)\Big) \\
& \quad \text{(by the induction hypothesis, using that } \|u_j\|_{\mathrm{br}} < \|w\|_{\mathrm{br}} \\
& \quad \text{because } w \to_{\mathrm{br}} u_j \text{ for } j = 1, \ldots, q\text{, see (S.5))} \\
=_{\mathsf{BBP}}\ & \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w)\Big)\Big)^{\circledast}\Big(\Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot t_{\hat{\mathcal{C}}}(u_j, v)\Big)\Big) \cdot s_{\hat{\mathcal{C}}}(v)\Big) \\
& \quad \text{(using axioms (A5), (A4))} \\
=_{\mathsf{BBP}}\ & \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w)\Big)\Big)^{\circledast}\Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot t_{\hat{\mathcal{C}}}(u_j, v)\Big)\Big) \cdot s_{\hat{\mathcal{C}}}(v) \\
& \quad \text{(using axiom (BKS2))} \\
\equiv\ & t_{\hat{\mathcal{C}}}(w, v) \cdot s_{\hat{\mathcal{C}}}(v) \\
& \quad \text{(by the definition of } t_{\hat{\mathcal{C}}}(w, v)\text{, based on the representation (S.5))}
\end{aligned}$$

This chain of provable equalities demonstrates (S.4).

We now prove that $s_{\hat{\mathcal{C}}}$ is a provable solution of the chart $\mathcal{C}$. Let $w \neq \surd$. We show that $s_{\hat{\mathcal{C}}}(w)$ satisfies the defining equation of $s_{\hat{\mathcal{C}}}$ to be a provable solution of $\mathcal{C}$ at w.

We consider a representation of the set $\hat{T}(w)$ of transitions from w in $\hat{\mathcal{C}}$ as follows:

$$\hat{T}(w) = \{w \xrightarrow{a_i}_{[\alpha_i]} w \mid i = 1, \ldots, m\} \cup \{w \xrightarrow{b_j}_{[\beta_j]} w_j \mid w_j \neq w,\ j = 1, \ldots, n\} \\ \cup \{w \xrightarrow{c_i}_{\mathrm{br}} \surd \mid i = 1, \ldots, p\} \cup \{w \xrightarrow{d_j}_{\mathrm{br}} u_j \mid u_j \neq \surd,\ j = 1, \ldots, q\} \quad \text{(S.6)}$$

that partitions $\hat{T}(w)$ into loop-entry transitions to w and to other targets $w_1, \ldots, w_n$, and branch transitions to $\surd$ and to other targets $u_1, \ldots, u_q$. We argue as follows:

$$\begin{aligned}
s_{\hat{\mathcal{C}}}(w) \;\equiv\;\; & \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w)\Big)\Big)^{\circledast}\Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big) \\
& \text{(by the definition of } s_{\hat{\mathcal{C}}}\text{, in view of (S.6))} \\
=_{\mathsf{BBP}}\; & \Big(\Big(\sum_{i=1}^{m} a_i\Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w)\Big)\Big) \cdot s_{\hat{\mathcal{C}}}(w) + \Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big) \\
& \text{(using axiom (BKS1) and the defining equality in the first step)} \\
=_{\mathsf{BBP}}\; & \Big(\Big(\sum_{i=1}^{m} a_i \cdot s_{\hat{\mathcal{C}}}(w)\Big) + \Big(\sum_{j=1}^{n} b_j \cdot (t_{\hat{\mathcal{C}}}(w_j, w) \cdot s_{\hat{\mathcal{C}}}(w))\Big)\Big) + \Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big) \\
& \text{(using axioms (A4), (A5))} \\
=_{\mathsf{BBP}}\; & \Big(\Big(\sum_{i=1}^{m} a_i \cdot s_{\hat{\mathcal{C}}}(w)\Big) + \Big(\sum_{j=1}^{n} b_j \cdot s_{\hat{\mathcal{C}}}(w_j)\Big)\Big) + \Big(\Big(\sum_{i=1}^{p} c_i\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big) \\
& \text{(using (S.4), in view of } w \curvearrowright w_j \text{ for } j = 1, \ldots, n\text{)} \\
=_{\mathsf{BBP}}\; & \Big(\sum_{i=1}^{p} c_i\Big) + \Big(\Big(\Big(\sum_{i=1}^{m} a_i \cdot s_{\hat{\mathcal{C}}}(w)\Big) + \Big(\sum_{j=1}^{n} b_j \cdot s_{\hat{\mathcal{C}}}(w_j)\Big)\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s_{\hat{\mathcal{C}}}(u_j)\Big)\Big) \\
& \text{(using axioms (A1), (A2))}
\end{aligned}$$

This chain of equalities demonstrates that $s_{\hat{\mathcal{C}}}(w)$ is a provable solution of $\mathcal{C}$ at w, in view of (S.6). As $w \neq \surd$ is arbitrary, $s_{\hat{\mathcal{C}}}$ is indeed a provable solution of $\mathcal{C}$. ◻

Proof (Supplement for the proof of Prop. 4.5). Let $\hat{\mathcal{C}}$ be a LLEE-witness of $\mathcal{C}$. We prove the auxiliary result (4.1) about the connection of s with the relative extracted solution $t_{\hat{\mathcal{C}}}$, which was employed in the proof of Prop. 4.5:

$$v \curvearrowright w \quad\Longrightarrow\quad s(w) =_{\mathsf{BBP}} t_{\hat{\mathcal{C}}}(w, v) \cdot s(v) \tag{S.7}$$

for all vertices v and w. Note again that if $v \curvearrowright w$, then $v \neq \surd$, and also $w \neq \surd$, because w is in the body of a loop at v, and therefore cannot be $\surd$.

We proceed by complete induction (without explicit treatment of the base case) on the same measure as used in the definition of the relative extraction function $t_{\hat{\mathcal{C}}}$, namely, induction on the maximal loop level of a loop at v, with a subinduction on $\|w\|_{\mathrm{br}}$. For performing the induction step, consider vertices v, w with $v \curvearrowright w$. As in the proof of Prop. 4.3 we assume the representation (S.5) of the set $\hat{T}(w)$ of transitions from w in $\hat{\mathcal{C}}$, which partitions $\hat{T}(w)$ into loop-entry transitions to w and to other targets $w_1, \ldots, w_n$, and branch transitions to v and to other targets $u_1, \ldots, u_q$. Since w is contained in a loop at v, none of these targets can be $\surd$. We now argue as follows:

$$\begin{aligned}
s(w) \;=_{\mathsf{BBP}}\; & 0 + \Big(\Big(\sum_{i=1}^{m} a_i \cdot s(w)\Big) + \Big(\Big(\sum_{j=1}^{n} b_j \cdot s(w_j)\Big) + \Big(\sum_{i=1}^{p} c_i \cdot s(v)\Big) + \Big(\sum_{j=1}^{q} d_j \cdot s(u_j)\Big)\Big)\Big) \\
& \text{(since } s \text{ is a provable solution of } \mathcal{C} \text{ at } w\text{, using (S.5))}
\end{aligned}$$

$$
\begin{aligned}
&=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i \cdot s(w) \Big) + \Big(\sum_{j=1}^{n} b_j \cdot s(w_j) \Big) \Big) + \Big(\Big(\sum_{i=1}^{p} c_i \cdot s(v) \Big) + \Big(\sum_{j=1}^{q} d_j \cdot s(u_j) \Big) \Big) \\
&\qquad \text{(using axioms (A6), (A2))} \\
&=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i \cdot s(w) \Big) + \Big(\sum_{j=1}^{n} b_j \cdot \big(t_{\hat{\mathcal{C}}}(w_j, w) \cdot s(w) \big) \Big) \Big) + \Big(\Big(\sum_{i=1}^{p} c_i \cdot s(v) \Big) + \Big(\sum_{j=1}^{q} d_j \cdot \big(t_{\hat{\mathcal{C}}}(u_j, v) \cdot s(v) \big) \Big) \Big) \\
&\qquad \text{(using the induction hypothesis, which is applicable because} \\
&\qquad\; \text{the maximal loop level at } w \text{ is smaller than that at } v \text{ due to } v \rightharpoonup w, \\
&\qquad\; \text{and also } \|u_j\|_{\mathrm{br}} < \|w\|_{\mathrm{br}} \text{ due to } w \rightarrow_{\mathrm{br}} u_j \text{ for } j = 1, \ldots, q, \text{ see (S.5))} \\
&=_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i \Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w) \Big) \Big) \cdot s(w) + \Big(\Big(\sum_{i=1}^{p} c_i \Big) + \Big(\sum_{j=1}^{q} d_j \cdot t_{\hat{\mathcal{C}}}(u_j, v) \Big) \Big) \cdot s(v) \\
&\qquad \text{(using axioms (A5), (A4))}
\end{aligned}
$$

This chain of equalities justifies:

$$
s(w) =_{\mathsf{BBP}} \Big(\Big(\sum_{i=1}^{m} a_i \Big) + \Big(\sum_{j=1}^{n} b_j \cdot t_{\hat{\mathcal{C}}}(w_j, w) \Big) \Big) \cdot s(w) + \Big(\Big(\sum_{i=1}^{p} c_i \Big) + \Big(\sum_{j=1}^{q} d_j \cdot t_{\hat{\mathcal{C}}}(u_j, v) \Big) \Big) \cdot s(v)
$$

To this equality we can apply the rule $\mathrm{RSP}^{\circledast}$:

$$
\begin{aligned}
s(w) &=_{\mathsf{BBP}} \Big(\Big(\Big(\sum_{i=1}^{m} a_i \Big) + \Big(\sum_{j=1}^{n} b_i \cdot t_{\hat{\mathcal{C}}}(w_j, w) \Big) \Big)^{\circledast} \Big(\Big(\sum_{i=1}^{p} c_i \Big) + \Big(\sum_{j=1}^{q} d_j \cdot t_{\hat{\mathcal{C}}}(u_j, v) \Big) \Big) \Big) \cdot s(v) \\
&\qquad \text{(by applying rule } \mathrm{RSP}^{\circledast}\text{)} \\
&\equiv \quad t_{\hat{\mathcal{C}}}(w, v) \cdot s(v) \,,
\end{aligned}
$$

The last step uses the definition of $t_{\hat{\mathcal{C}}}(w, v)$, based on representation (S.5) of $\widehat{T}(w)$. In this way we have carried out the induction step. We conclude that (4.1) holds for all vertices v and w of $\mathcal{C}$. ◻

S.3 For Section 7: Preservation of LLEE under collapse

Proposition (= Prop. 7.3). *In a LLEE-chart $\hat{\mathcal{C}}$ that is not a bisimulation collapse there are bisimilar vertices w_1, w_2 such that either (C1) $\neg(w_1 \circlearrowright) \wedge \neg(w_2 \rightarrow^{*} w_1)$, or (C2) $w_2 \circlearrowright^{+} w_1$, or (C3) $\exists v \in V\big(w_1 \,{}_{d}{\circlearrowright}\, v \wedge w_2 \circlearrowright^{+} v\big) \wedge \neg(w_2 \rightarrow^{*}_{\mathrm{br}} w_1)$.*

Proof (Supplementary illustrations for the proof of Prop. 7.3 on pages 11–12). The proof started from a pair u_1, u_2 of distinct bisimilar vertices. By progression of u_1, u_2 by means of bisimilarity via other pairs of distinct bisimilar vertices we constructed two bisimilar vertices w_1, w_2 such that either of the conditions (C1), (C2), or (C3) holds. Note that each of (C1), (C2), and (C3) implies that w_1 and w_2 are distinct.

In the case $\mathsf{scc}(u_1) = \mathsf{scc}(u_2)$ we reached the situation:

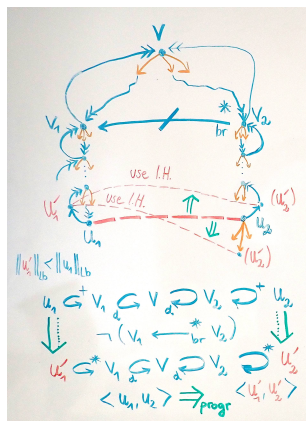
$$
u_1 \circlearrowright^{*} v_1 \,{}_{d}{\circlearrowright}\, v \,{}_{d}{\circlearrowleft}\, v_2 \circlearrowleft^{*} u_2 \wedge \neg\big(v_1 \leftarrow^{*}_{\mathrm{br}} v_2\big) \,. \tag{S.8}
$$

Note that also (S.8) implies that u_1 and u_2 are distinct. Now for pairs of vertices u_1 and u_2 such that (S.8) holds for some vertices v_1, v_2, and v, we used induction on $\|u_1\|_{lb}^{\min}$ in order show that u_1 and u_2 progress via bisimilarity to vertices w_1 and w_2 such that one of the conditions (C1), (C2), or (C3) holds.

In order to carry out the induction step we used a case distinction. Below we repeat the arguments, and supplement them with pictures illustrations.

Case 1: $u_2 \circlearrowright^{+} v_2$.

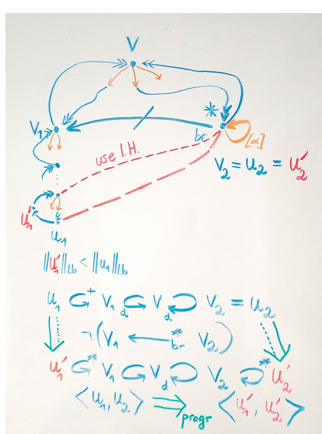
Since $u_2 \to u_2'$, either $u_2' = v_2$ or $v_2 \rightharpoonup^{+} u_2'$. Moreover, $\mathsf{scc}(u_2') = \mathsf{scc}(u_2) = \mathsf{scc}(v_2)$, so by Lem. 6.1, $u_2' \circlearrowright^{*} v_2$. Hence, $u_1' \circlearrowright^{*} v_1 \;{}_{d}{\circlearrowright}\; v \;{}_{d}{\circlearrowleft}\; v_2 \circlearrowleft^{*} u_2' \wedge \neg(v_1 \leftarrow_{\mathrm{br}}^{*} v_2)$, and $\|u_1'\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$. We apply the induction hypothesis to obtain a bisimilar pair w_1, w_2 for which (C1), (C2), or (C3) holds.

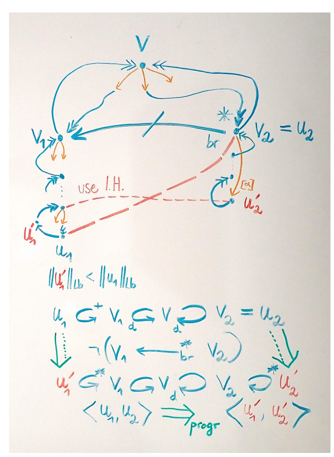


Case 2: $u_2 = v_2$.

Case 2.1: $u_2 \to_{[\alpha]} u_2'$.

$u_2 \to_{[\alpha]} u_2'$. Then either $u_2' = u_2$ or $u_2 \rightharpoonup u_2'$. Moreover, $\mathsf{scc}(u_2') = \mathsf{scc}(u_2)$, so by Lem. 6.1, $u_2' \circlearrowright^{*} u_2$, and hence $u_2' \circlearrowright^{*} v_2$. Thus we have obtained $u_1' \circlearrowright^{*} v_1 \;{}_{d}{\circlearrowright}\; v \;{}_{d}{\circlearrowleft}\; v_2 \circlearrowleft^{*} u_2' \wedge \neg(v_1 \leftarrow_{\mathrm{br}}^{*} v_2)$. Due to $\|u_1'\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$, we can apply the induction hypothesis again.



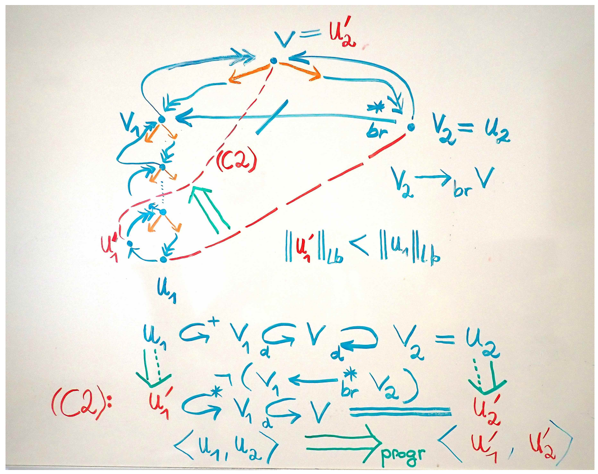


Case 2.2: $u_2 \to_{\mathrm{br}} u_2'$.

Then $\neg(v_2 \to_{\mathrm{br}}^{*} v_1)$ together with $v_2 = u_2 \to_{\mathrm{br}} u_2'$ and $u_1' \to_{\mathrm{br}}^{*} v_1$ (because $u_1' \circlearrowleft^{*} v_1$) imply $u_1' \neq u_2'$. We distinguish two cases.

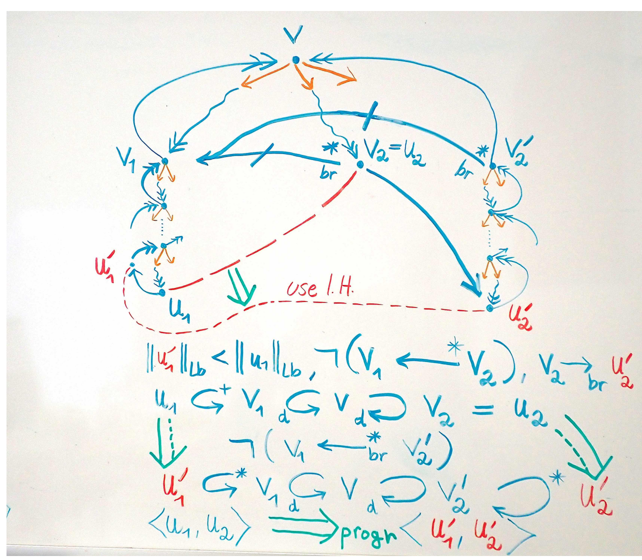
Case 2.2.1: $u_2' = v$.

Then $u_1' \circlearrowleft^{*} v_1 \;{}_{d}\!\circlearrowleft v = u_2'$, i.e., $u_1' \circlearrowleft^{+} u_2'$, so we are done, because (C2) holds for $w_1 = u_2'$ and $w_2 = u_1'$.



Case 2.2.2: $u_2' \neq v$.

By Lem. 6.1, $u_2' \circlearrowleft^{+} v$. Hence, $u_2' \circlearrowleft^{*} v_2' \;{}_{d}\!\circlearrowleft v$ for some v_2'. Since $v_2 = u_2 \to_{\mathrm{br}} u_2' \circlearrowleft^{*} v_2'$ and $\neg(v_2 \to_{\mathrm{br}}^{*} v_1)$, it follows that $\neg(v_2' \to_{\mathrm{br}}^{*} v_1)$. So $u_1' \circlearrowleft^{*} v_1 \;{}_{d}\!\circlearrowleft v \;{}_{d}\!\circlearrowright v_2' \circlearrowright^{*} u_2' \wedge \neg(v_1 \leftarrow_{\mathrm{br}}^{*} v_2')$. Due to $\|u_1'\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$, we can apply the induction hypothesis again.



$\square$

We write $v \,{}^{\alpha}\!\curvearrowright\, w$, and say that v *descends into loop α to w*, if there is a path of the form $v \xrightarrow[\not\leftarrow(v)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(v)]{}^{*}_{\mathrm{br}} w$. Note that $v \curvearrowright w$ (as defined on page 9) holds if and only if there is a loop name α such that $v \,{}^{\alpha}\!\curvearrowright\, w$.

Lemma S.1. *In a chart with a* LLEE*-witness, if $v \,{}^{\alpha}\!\curvearrowright \cdot \curvearrowright^{*} \cdot \to_{[\beta]}$, then $|\alpha| > |\beta|$.*

Proof. By induction on the number n of $\curvearrowright$-steps in a path $v \,{}^{\alpha}\!\curvearrowright \cdot \curvearrowright^{n} \cdot \to_{[\beta]}$. If $n = 0$, then from $v \,{}^{\alpha}\!\curvearrowright \cdot \to_{[\beta]}$ we get $|\alpha| > |\beta|$ by means of the LLEE-witness condition (W2)(b). If $n > 0$, then the path $v \,{}^{\alpha}\!\curvearrowright \cdot \curvearrowright^{n} \cdot \to_{[\beta]}$ is of the form $v \,{}^{\alpha}\!\curvearrowright \cdot \curvearrowright^{n-1} \cdot {}^{\gamma}\!\curvearrowright \cdot \to_{[\beta]}$ for some loop name γ. This path contains an initial segment $v \,{}^{\alpha}\!\curvearrowright \cdot \curvearrowright^{n-1} \cdot \to_{[\gamma]}$. Then $|\alpha| > |\gamma|$ follows by the induction hypothesis. From the part ${}^{\gamma}\!\curvearrowright \cdot \to_{[\beta]}$ of this path we get $|\gamma| > |\beta|$ by LLEE-witness condition (W2)(b). So we conclude that $|\alpha| > |\beta|$ holds. ◻

Lemma S.2. *In a LLEE-chart, for every path $v \xrightarrow[\not\leftarrow(v)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(v)]{}^{*} w$ there is an acyclic path of the form $v \,{}^{\alpha}\!\curvearrowright \cdot \curvearrowright^{*} w$.*

Proof. Let π be a path from v to w that starts with a loop-entry step with loop name α such that all targets of transitions in π avoid v. By removing cycles we obtain an acyclic path π' from v to w that starts with an α-loop-entry step whose target is not v. We can write π' as a sequence of loop-entry and branch steps of the form $v \xrightarrow[\not\leftarrow(v)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(v)]{}^{*}_{\mathrm{br}} u_1 \xrightarrow[\not\leftarrow(u_0)]{}_{[\alpha_0]} \cdot \xrightarrow[\not\leftarrow(u_0)]{}^{*}_{\mathrm{br}} \cdots u_{n-2} \xrightarrow[\not\leftarrow(u_{n-2})]{}_{[\alpha_{n-2}]} \cdot \xrightarrow[\not\leftarrow(u_{n-2})]{}^{*}_{\mathrm{br}} u_{n-1} \xrightarrow[\not\leftarrow(u_{n-1})]{}_{[\alpha_{n-1}]} \cdot \xrightarrow[\not\leftarrow(u_{n-1})]{}^{*}_{\mathrm{br}} w$ for some $n \geqslant 1$, where the target-avoidance parts are due to acyclicity of π'. Hence π' is of the form $v \,{}^{\alpha}\!\curvearrowright \cdot {}^{\alpha_1}\!\curvearrowright \cdots {}^{\alpha_{n-2}}\!\curvearrowright \cdot {}^{\alpha_{n-1}}\!\curvearrowright w$, and therefore of the form $v \,{}^{\alpha}\!\curvearrowright \cdot \curvearrowright^{*} w$. ◻

The following lemma was also used implicitly in the proof of Lem. 6.3.

Lemma S.3. *In a LLEE-chart, if $v \xrightarrow[\not\leftarrow(v)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(v)]{}^{*} \cdot \to_{[\beta]}$, then $|\alpha| > |\beta|$.*

Proof. This is a direct consequence of Lem. S.2 and Lem. S.1. ◻

Lemma S.4. *In a LLEE-chart, if $u \circlearrowleft^{*} v \circlearrowleft^{*} w$, then each path $u \to^{*}_{\mathrm{br}} w$ visits v.*

Proof. Let $v \neq u, w$, as else the lemma trivially holds. Since $u \circlearrowleft^{+} v \circlearrowleft^{+} w$, there is a path $w \xrightarrow[\not\leftarrow(w)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(w)]{}^{*} v \xrightarrow[\not\leftarrow(v)]{}_{[\beta]} \cdot \xrightarrow[\not\leftarrow(v)]{}^{*} u$. By layeredness, $|\alpha| > |\beta|$. A path $u \xrightarrow[\not\leftarrow(v)]{}^{*}_{\mathrm{br}} w$ would yield $v \xrightarrow[\not\leftarrow(v)]{}_{[\beta]} \cdot \xrightarrow[\not\leftarrow(v)]{}^{*} u \xrightarrow[\not\leftarrow(v)]{}^{*}_{\mathrm{br}} w \to_{[\alpha]}$. Then layeredness would require $|\beta| > |\alpha|$, which cannot be the case. ◻

Proposition (= Prop. 7.5). *Let $\mathcal{C}$ be a LLEE-chart. If a pair w_1, w_2 of bisimilar vertices satisfies (C1), (C2), or (C3), then $\mathcal{C}^{(w_1)}_{w_2}$ (see Def. 7.1) is a LLEE-chart.*

Proof (Supplement for the proof of Prop. 7.5). We argue the correctness of transformation II. Consider vertices w_1, w_2 such that (C2) holds, that is, $w_2 \circlearrowleft^{+} w_1$. Let $\widehat{w}_2$ be the ${}_{d}\!\circlearrowleft$-predecessor of w_1 in the ${}_{d}\!\circlearrowleft$-chain from w_2 to w_1, i.e., $w_2 \circlearrowleft^{*} \widehat{w}_2 \;{}_{d}\!\circlearrowleft w_1$.

As for the transformations I and III it suffices to show, in view of the alleviation of the proof obligation at the start of the proof on page 15, that the intermediate result $\hat{\mathcal{C}}''$ of transformation II before the clean-up step satisfies the LLEE-witness properties, except for possible violations of (L1). By the definition of transformation II, $\hat{\mathcal{C}}''$ results by performing the adaptation step L_{II} to the chart $\hat{\mathcal{C}}' := \hat{\mathcal{C}}^{(w_1)}_{w_2}$ that arises from $\hat{\mathcal{C}}$ by connecting w_1 through to w_2.

To prove that (W1), and the part concerning (L2) for (W2)(a) is satisfied for $\hat{\mathcal{C}}''$, it suffices to show that the transformed chart does not contain a cycle of branch transitions. At first, the step of connecting w_1 through to w_2 in $\hat{\mathcal{C}}$ may introduce a branch cycle in $\hat{\mathcal{C}}' = \hat{\mathcal{C}}^{(w_1)}_{w_2}$. But every such cycle is removed in the subsequent level adaptation step L_{II}. Namely, each branch cycle introduced in $\hat{\mathcal{C}}'$ must stem from a transition $u \rightarrow_{\mathrm{br}} w_1$ (which is redirected to w_2 in $\hat{\mathcal{C}}'$) and a path $w_2 \rightarrow^*_{\mathrm{br}} u$ in $\hat{\mathcal{C}}$, for some $u \neq w_1$. Since $w_2 \curvearrowleft^* \widehat{w}_2 \,{}_d\!\curvearrowleft w_1$, by Lem. S.4, the path $w_2 \rightarrow^*_{\mathrm{br}} u \rightarrow_{\mathrm{br}} w_1$ in $\hat{\mathcal{C}}$ must visit $\widehat{w}_2$. Since all branch transitions from $\widehat{w}_2$ are turned into loop-entry transitions in step L_{II}, the branch cycle $w_2 \rightarrow^*_{\mathrm{br}} u \rightarrow_{\mathrm{br}} w_2$ in $\hat{\mathcal{C}}'$ that was introduced in the connect-through step, is after step L_{II} no longer a branch cycle in $\hat{\mathcal{C}}''$.

Now we prove that (W2)(b) is preserved by the two steps from $\hat{\mathcal{C}}$ via $\mathcal{C}' = \hat{\mathcal{C}}^{(w_1)}_{w_2}$ to $\hat{\mathcal{C}}''$. Every path $u \xrightarrow[\not\mathbf{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}''$ with $u \neq w_1, w_2$ arises by a, possibly empty, combination of the following three kinds of modifications in the first two transformation steps:

(i) A transition to w_1 was redirected to w_2 in the connect-through step.

(ii) The loop-entry transition at the beginning of the path is from $\widehat{w}_2$ and was a branch transition before step L_{II}, meaning that $u = \widehat{w}_2$ and $\alpha = \gamma$. (Recall that γ is a loop name of maximum loop level among the loop-entries at w_1 in $\hat{\mathcal{C}}$.)

(iii) The loop-entry transition at the end of the path is from $\widehat{w}_2$ and was a branch transition before step L_{II}, meaning that $\beta = \gamma$.

This gives $2^3 = 8$ possibilities. Of these, three possibilities are void: if all three adaptations are not the case, the path is already present in $\hat{\mathcal{C}}$, and so $|\alpha| > |\beta|$ is guaranteed; (ii) and (iii) together cannot hold, because then the path would return to $u = \widehat{w}_2$, which it cannot, because all of its steps avoid u as target. We now show that in the remaining five cases always $|\alpha| > |\beta|$. Since $w_2 \curvearrowleft^+ w_1$, there is a path $w_1 \xrightarrow[\not\mathbf{t}(w_1)]{}_{[\delta]} \cdot \xrightarrow[\not\mathbf{t}(w_1)]{}^{*}_{\mathrm{br}} w_2$ in $\hat{\mathcal{C}}$. By definition of γ, $|\gamma| \geqslant |\delta|$.

A Let only (i) hold: there are paths $u \xrightarrow[\not\mathbf{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}^{*}_{\mathrm{br}} w_1$ and $w_2 \xrightarrow[\not\mathbf{t}(u)]{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$ (which do not visit $\widehat{w}_2$). Then there is a path $u \xrightarrow[\not\mathbf{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}^{*}_{\mathrm{br}} w_1 \rightarrow_{[\gamma]}$ in $\hat{\mathcal{C}}$, so $|\alpha| > |\gamma|$. We distinguish two cases.

Case 1: The path $w_2 \xrightarrow[\not\mathbf{t}(u)]{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ visits w_1. Then there is a path $w_1 \xrightarrow[\not\mathbf{t}(u)]{}^{*} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$. So $u \xrightarrow[\not\mathbf{t}(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\mathbf{t}(u)]{}^{*} w_1 \xrightarrow[\not\mathbf{t}(u)]{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$. So by (W2)(b), $|\alpha| > |\beta|$.

Case 2: The path $w_2 \xrightarrow[\not\mathbf{t}(u)]{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ does not visit w_1. Then there is a path $w_1 \xrightarrow[\not\mathbf{t}(w_1)]{}_{[\delta]} \cdot \xrightarrow[\not\mathbf{t}(w_1)]{}^{*}_{\mathrm{br}} w_2 \xrightarrow[\not\mathbf{t}(w_1)]{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$, so $|\delta| > |\beta|$. Hence $|\alpha| > |\gamma| \geqslant |\delta| > |\beta|$.

B Let only (ii) hold. Then $u = \widehat{w}_2$, $\alpha = \gamma$, and there is a path $\widehat{w}_2 \xrightarrow[\not\leftarrow(\widehat{w}_2,\, w_1)]{}{}^{+}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$. As $\widehat{w}_2 \;{}_{d}\!\circlearrowleft w_1$, there is a path $w_1 \xrightarrow[\not\leftarrow(w_1)]{}_{[\delta]} \cdot \xrightarrow[\not\leftarrow(w_1)]{}{}^{*}_{\mathrm{br}} \widehat{w}_2$ in $\hat{\mathcal{C}}$. Hence $w_1 \xrightarrow[\not\leftarrow(w_1)]{}_{[\delta]} \cdot \xrightarrow[\not\leftarrow(w_1)]{}{}^{*}_{\mathrm{br}} \widehat{w}_2 \xrightarrow[\not\leftarrow(w_1)]{}{}^{+}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$, so $|\delta| > |\beta|$. Hence $|\alpha| = |\gamma| \geqslant |\delta| > |\beta|$.

C Let only (iii) hold. Then $\beta = \gamma$, and $u \xrightarrow[\not\leftarrow(u,\, w_1)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(u,\, w_1)]{}{}^{*}_{\mathrm{br}} \widehat{w}_2$ with $u \neq w_1$ is a path in in $\hat{\mathcal{C}}$. Since $\widehat{w}_2 \;{}_{d}\!\circlearrowleft w_1$ and $u \neq w_1$, it follows that $\neg(\widehat{w}_2 \;{}_{d}\!\circlearrowleft u)$. So in view of the path $u \xrightarrow[\not\leftarrow(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(u)]{}{}^{*}_{\mathrm{br}} \widehat{w}_2$, there is no path $\widehat{w}_2 \rightarrow^{*}_{\mathrm{br}} u$ in $\hat{\mathcal{C}}$. Since $\widehat{w}_2 \;{}_{d}\!\circlearrowleft w_1$, there is a path $\widehat{w}_2 \rightarrow^{*}_{\mathrm{br}} w_1$ in $\hat{\mathcal{C}}$, which by the previous observation is of the form $\widehat{w}_2 \xrightarrow[\not\leftarrow(u)]{}{}^{*}_{\mathrm{br}} w_1$. Hence there is a path $u \xrightarrow[\not\leftarrow(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(u)]{}{}^{*}_{\mathrm{br}} \widehat{w}_2 \xrightarrow[\not\leftarrow(u)]{}{}^{*}_{\mathrm{br}} w_1 \rightarrow_{[\gamma]}$ in $\hat{\mathcal{C}}$, so $|\alpha| > |\gamma| = |\beta|$.

D Let only (i) and (ii) hold, meaning $u = \widehat{w}_2$, $\alpha = \gamma$, and there are paths $\widehat{w}_2 \xrightarrow[\not\leftarrow(\widehat{w}_2)]{}{}^{+} w_1$ and $w_2 \xrightarrow[\not\leftarrow(\widehat{w}_2)]{}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$. Since $w_2 \circlearrowleft^{*} \widehat{w}_2 \;{}_{d}\!\circlearrowleft^{+} w_1$, and $u = \widehat{w}_2$ implies $w_2 \neq \widehat{w}_2$, by Lem. S.4, the path $w_2 \xrightarrow[\not\leftarrow(\widehat{w}_2)]{}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ cannot visit w_1. Hence $w_1 \xrightarrow[\not\leftarrow(w_1)]{}_{[\delta]} \cdot \xrightarrow[\not\leftarrow(w_1)]{}{}^{*}_{\mathrm{br}} w_2 \xrightarrow[\not\leftarrow(w_1)]{}{}^{*}_{\mathrm{br}} \cdot \rightarrow_{[\beta]}$ in $\hat{\mathcal{C}}$. So $|\delta| > |\beta|$. Hence $|\alpha| = |\gamma| \geqslant |\delta| > |\beta|$.

E Let only (i) and (iii) hold. Then $\beta = \gamma$, and $u \xrightarrow[\not\leftarrow(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(u)]{}{}^{*}_{\mathrm{br}} w_1$ and $w_2 \xrightarrow[\not\leftarrow(u)]{}{}^{*}_{\mathrm{br}} \widehat{w}_2$ are paths in $\hat{\mathcal{C}}$. Since $u \xrightarrow[\not\leftarrow(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(u)]{}{}^{*}_{\mathrm{br}} w_1 \rightarrow_{[\gamma]}$ in $\hat{\mathcal{C}}$, $|\alpha| > |\gamma| = |\beta|$.

We conclude that in all five cases, $\hat{\mathcal{C}}''$ satisfies (W2)(b).

Finally we argue that part (L3) of (W2)(a) holds for $\hat{\mathcal{C}}''$, i.e., there are no descends-in-loop-to paths of the form $u \xrightarrow[\not\leftarrow(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(u)]{}{}^{*}_{\mathrm{br}} \surd$ in $\hat{\mathcal{C}}''$. We can use part of the argumentation employed for demonstrating (W2)(b) above. It was demonstrated in particular that for every descends-in-loop-to path $u \xrightarrow[\not\leftarrow(u)]{}_{[\alpha]} \cdot \xrightarrow[\not\leftarrow(u)]{}{}^{*}_{\mathrm{br}} x$ in $\hat{\mathcal{C}}''$, there is a descends-in-loop-to path $\tilde{u} \xrightarrow[\not\leftarrow(\tilde{u})]{}_{[\gamma]} \cdot \xrightarrow[\not\leftarrow(\tilde{u})]{}{}^{*}_{\mathrm{br}} x$ with the same target x in $\hat{\mathcal{C}}$. From this it follows that if a descends-in-loop-to path in $\hat{\mathcal{C}}''$ had $\surd$ as target, then there were a descends-in-loop-to path already in $\hat{\mathcal{C}}$ that had $\surd$ as target, violating (L3) for the LLEE-chart $\hat{\mathcal{C}}$. Hence $\hat{\mathcal{C}}''$ must satisfy (L3).

We conclude that the result of transformation II is a LLEE-chart. $\square$