

A Complete Proof System for 1-Free Regular Expressions Modulo Bisimilarity

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Abstract

Robin Milner (1984) gave a sound proof system for bisimilarity of regular expressions interpreted as processes: Basic Process Algebra with unary Kleene star iteration, deadlock 0, successful termination 1, and a fixed-point rule. He asked whether this system is complete. Despite intensive research over the last 35 years, the problem is still open.

This paper gives a partial positive answer to Milner's problem. We prove that the adaptation of Milner's system over the subclass of regular expressions that arises by dropping the constant 1, and by changing to binary Kleene star iteration is complete. The crucial tool we use is a graph structure property that guarantees expressibility of a process graph by a regular expression, and is preserved by going over from a process graph to its bisimulation collapse.

Keywords regular expressions, process algebra, bisimilarity, process graphs, complete proof system

1 Introduction

Regular expressions, introduced by Kleene [14], are widely studied in formal language theory, notably for string searching [23]. They are constructed from constants 0 (no strings), 1 (the empty string), and a (a single letter) from some alphabet; binary operators $+$ and \cdot (union and concatenation); and the unary Kleene star $*$ (zero or more iterations).

Their interpretations are Kleene algebras with as prime example the algebra of regular events, the language semantics of regular expressions, which is closely linked with deterministic finite state automata. Aanderaa [1] and Salomaa [19] gave complete axiomatizations for the language semantics of regular expressions, with a non-algebraic fixed-point rule that has a non-empty-word property as side condition. Krob [16] gave an infinitary, and then Kozen [15] a finitary algebraic axiomatization involving equational implications.

Robin Milner [17] was the first to study regular expressions in bisimulation process semantics [18], where he called them star expressions. Here the interpretation of 0 is deadlock, 1 is (successful) termination, a is an atomic action, and $+$ and \cdot are alternative and sequential composition of two processes, respectively. Milner adapted Salomaa's axiomatization to obtain a sound proof system for this setting, and posed the (still open) question whether this axiomatization is complete, meaning that if the process graphs of two star expressions are bisimilar, then they can be proven equal.

That Milner's axiomatization contains a fixed-point rule is inevitable [20, 21]. Bergstra, Bethke, and Ponse [4] studied star expressions without 0 and 1, replacing the unary by the binary Kleene star \otimes , which represents an iteration of the first argument, possibly eventually followed by the execution of the second argument. They obtained an axiomatization by basically omitting the axioms for 0 and 1 as well as the fixed-point rule from Milner's axiomatization, and adding Troeger's axiom [24]. This axiomatization was proven complete in [8, 10]. A sound and complete axiomatization for star expressions without 0 and the unary Kleene star, but with 1 and a unary perpetual loop operator $*0$, was given in [7, 9].

In contrast to the formal languages setting, not all finite-state process graphs can be expressed by a star expression modulo bisimilarity. Milner posed a second question in [17], namely, to characterize which finite-state process graphs can be expressed. This was shown to be decidable in [3] by defining and using 'well-behaved' specifications.

In this paper we prove completeness of Milner's axiomatization (tailored to the adapted setting) for star expressions with 0, but without 1 and with the binary Kleene star.

While earlier completeness proofs focus on manipulation of terms, we follow in the footsteps of Milner and focus on their process graphs. A key idea is to determine loops in graphs associated to star expressions. By a loop we mean a subgraph generated by a set of entry transitions from a vertex v in which (1) there is an infinite path from v , (2) each infinite path eventually returns to v , and (3) termination is not permitted. A graph is said to satisfy LLEE (Layered Loop Existence and Elimination) if repeatedly eliminating the entry transitions of a loop, and performing garbage collection, leads to a graph without infinite paths. LLEE offers a generalization (and more elegant definition) of the notion of a well-behaved specification.

Our completeness proof roughly works as follows (for more details see Sect. 4). Let star expressions e_1 and e_2 give rise to bisimilar graphs g_1 and g_2 . We show that the process graph interpretations of e_1 and e_2 satisfy LLEE. We moreover prove that LLEE is preserved under bisimulation collapse. And we construct for each graph that satisfies LLEE a star expression that corresponds to this graph, modulo bisimilarity. In particular such a star expression f can be constructed for the bisimulation collapse of g_1 and g_2 . We show that both e_1 and e_2 can be proven equal to f , by a pull-back over the functional bisimulations from the bisimulation collapse back to g_1 and g_2 . This yields the desired completeness result.

In our proof, the minimization of terms (and thereby of the associated process graphs) in the left-hand side of a binary Kleene star modulo bisimilarity is partly inspired by [7, 9]. Interestingly, we will be able to use as running example the process graph interpretation of the star expression that at the end of [9] is mentioned as problematic for a completeness proof. Our crucial use of witnesses for the graph property LLEE borrows from the representation of cyclic λ -terms [12] as structure-constrained term graphs, as used for defining and implementing maximal sharing in the λ -calculus with letrec [13] (see also [11]).

The completeness result for star expressions with 0 but without 1 and with the binary Kleene star settles a natural question. We are also hopeful that the property LLEE provides a strong conceptual tool for approaching Milner's long-standing open question regarding the class of all star expressions. The presence of 1-transitions in graphs presents new challenges, such as that LLEE is not always preserved under bisimulation collapse. In order to be able to still work with this concept, we will need workarounds.

Please see the appendix for details of proofs that have been omitted or that are only sketched.

2 Preliminaries

In this section we define star expressions, their process semantics as 'charts', the proof system BBP for bisimilarity of their chart interpretations, and provable solutions of charts.

Definition 2.1. Given a set A of actions, the set $StExp(A)$ of star expressions over A is generated by the grammar:

$e ::= 0 \mid a \mid (e_1 + e_2) \mid (e_1 \cdot e_2) \mid (e_1^* e_2) \quad (\text{with } a \in A).$

0 represents deadlock (i.e., does not perform any action), a an atomic action, $+$ alternative and \cdot sequential composition, and * the binary Kleene star. Note that 1 (for empty steps) is missing from the syntax. $\sum_{i=1}^k e_i$ is defined recursively as 0 if $k = 0$, e_1 if $k = 1$, and $(\sum_{i=1}^{k-1} e_i) + e_k$ if $k > 1$.

The *star height* $|e|_*$ of a star expression $e \in StExp(A)$ denotes the maximum number of nestings of Kleene stars in e : it is defined by $|0|_* := |a|_* := 0$, $|f + g|_* := |f|_* \vee |g|_*$, $|f \cdot g|_* := \max\{|f|_*, |g|_*\}$, and $|f^* g|_* := \max\{|f|_* + 1, |g|_*\}$.

Definition 2.2. By a (finite sink-termination) *chart* C we understand a 5-tuple $\langle V, \sqrt{}, v_s, A, T \rangle$ where V is a finite set of vertices, $\sqrt{}$ is, in case $\sqrt{} \in V$, a special vertex with no outgoing transitions that indicates termination, $v_s \in V \setminus \{\sqrt{}\}$ is the *start vertex*, A is a set actions, and $T \subseteq V \times A \times V$ the set of *transitions*. Since A can be reconstructed from T , we will frequently keep A implicit, denote a chart as a 4-tuple $\langle V, \sqrt{}, v_s, T \rangle$. A chart is *start-vertex connected* if every vertex is reachable by a path from the start vertex. This property can be achieved by removing unreachable vertices ('garbage collection'). We will assume charts to be start-vertex connected.

In a chart C , let $v \in V$ and $U \subseteq T$ be a set of transitions from v . By the $\langle v, U \rangle$ -generated subchart of C we mean the

chart $C_0 = \langle V_0, \sqrt{}, v, A, T_0 \rangle$ with start vertex v where V_0 is the set of vertices and T_0 the set of transitions that are on paths in C from v that first take a transition in U , and then, until v is reached again, continue with other transitions of C . We use the standard notation $v \xrightarrow{a} v'$ in lieu of $\langle v, a, v' \rangle \in T$.

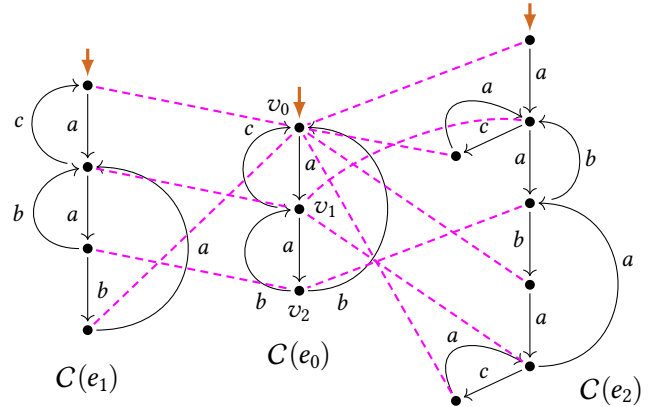
Definition 2.3. Let $C_i = \langle V_i, \sqrt{}, v_{s,i}, T_i \rangle$ for $i \in \{1, 2\}$ be two charts. A *bisimulation* between C_1 and C_2 is a relation $B \subseteq V_1 \times V_2$ with (start) $v_{s,1} B v_{s,2}$ (it relates the start vertices), and such that if $v_1 B v_2$, then: (*forth*) for every transition $v_1 \xrightarrow{a} v'_1$ in C_1 there is a transition $v_2 \xrightarrow{a} v'_2$ in C_2 with $v'_1 B v'_2$, (*back*) for every transition $v_2 \xrightarrow{a} v'_2$ in C_2 there is a transition $v_1 \xrightarrow{a} v'_1$ in C_1 with $v'_1 B v'_2$, and (*termination*) $v_1 = \sqrt{}$ if and only if $v_2 = \sqrt{}$. If there is a bisimulation between C_1 and C_2 , we write $C_1 \Leftrightarrow C_2$ and say that C_1 and C_2 are *bisimilar*. If a bisimulation is the graph of a function, we say that it is a *functional bisimulation*. We write $C_1 \Rightarrow C_2$ if there is a functional bisimulation between C_1 and C_2 .

Definition 2.4. For every star expression $e \in StExp(A)$ the *chart interpretation* $C(e) = \langle V(e), \sqrt{}, e, A, T(e) \rangle$ of e is the chart with start vertex e that is specified by iteration via the following transition rules, which form a transition system specification (TSS), with $e, e_1, e_2, e'_1 \in StExp(A)$, $a \in A$:

$$\begin{array}{c} \frac{}{a \xrightarrow{a} \sqrt{}} \quad \frac{e_i \xrightarrow{a} \xi}{e_1 + e_2 \xrightarrow{a} \xi} \quad (i = 1, 2) \\ \frac{e_1 \xrightarrow{a} e'_1}{e_1 \cdot e_2 \xrightarrow{a} e'_1 \cdot e_2} \quad \frac{e_1 \xrightarrow{a} \sqrt{}}{e_1 \cdot e_2 \xrightarrow{a} e_2} \\ \frac{e_1 \xrightarrow{a} e'_1}{e_1^* e_2 \xrightarrow{a} e'_1 \cdot (e_1^* e_2)} \quad \frac{e_1 \xrightarrow{a} \sqrt{}}{e_1^* e_2 \xrightarrow{a} e_1^* e_2} \quad \frac{e_2 \xrightarrow{a} \xi}{e_1^* e_2 \xrightarrow{a} \xi} \end{array}$$

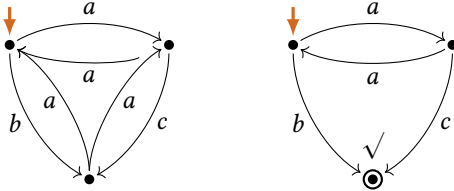
with $\xi \in StExp(A)_{\sqrt{}} := StExp(A) \cup \{\sqrt{}\}$, where $\sqrt{}$ indicates termination. If $e \xrightarrow{a} \xi$ can be proved, ξ is called an *a-derivative*, or just *derivative*, of e . The set $V(e) \subseteq StExp(A)_{\sqrt{}}$ consists of the *iterated derivatives* of e . To see that $C(e)$ is finite, Antimirov's result [2], that a regular expression has only finitely many iterated derivatives, can be adapted.

We say that a star expression $e \in StExp(A)$ is *normed* if there is a path of transitions from e to $\sqrt{}$ in $C(e)$.



Example 2.5. By the rules in Def. 2.4, $e_0 := a \cdot e'_0$ with $e'_0 := (c \cdot a + a \cdot (b + b \cdot a))^{\otimes 0}$ has the chart $C(e_0)$ as above, with $v_0 := e_0$, $v_1 := e'_0$ and $v_2 := (b + b \cdot a) \cdot e'_0$. This chart is the bisimulation collapse of the charts $C(e_1)$ and $C(e_2)$ of star expressions $e_1 := (a \cdot ((a \cdot (b + b \cdot a))^{\otimes c}))^{\otimes 0}$, and $e_2 := a \cdot ((c \cdot a + a \cdot (b \cdot a \cdot ((c \cdot a)^{\otimes a}))^{\otimes b})^{\otimes 0})$. Bisimulations between $C(e_1)$ and $C(e_0)$, and between $C(e_0)$ and $C(e_2)$ are indicated by the broken lines. The chart $C(e_0)$ was considered problematic in [9].

Example 2.6. The left chart below does not admit termination. The right chart is a double-exit graph with the sink termination vertex \checkmark at the bottom.



These charts are not bisimilar to chart interpretations of star expressions. For the left chart this was shown by Milner [17], and for the right chart by Bosscher [5].

Definition 2.7. The proof system BBP or the class of star expressions has the axioms (B1)–(B6), (BKS1), (BKS2), the inference rules of equational logic, and the rule RSP^{\otimes} :

$$\begin{array}{ll}
 \text{(B1)} & x + y = y + x \\
 \text{(B2)} & (x + y) + z = x + (y + z) \\
 \text{(B3)} & x + x = x \\
 \text{(B4)} & (x + y) \cdot z = x \cdot z + y \cdot z \\
 \text{(B5)} & (x \cdot y) \cdot z = x \cdot (y \cdot z) \\
 \text{(B6)} & x + 0 = x \\
 \text{(B7)} & 0 \cdot x = 0 \\
 \text{(BKS1)} & x \cdot (x^{\otimes} y) + y = x^{\otimes} y \\
 \text{(BKS2)} & (x^{\otimes} y) \cdot z = x^{\otimes} (y \cdot z) \\
 \text{(RSP}^{\otimes}) & \frac{x = (y \cdot x) + z}{x = y^{\otimes} z}
 \end{array}$$

By $e_1 =_{\text{BBP}} e_2$ we denote that $e_1 = e_2$ is derivable in BBP.

BBP is a finite ‘implicational’ proof system [22], because unlike in Salomaa’s and Milner’s systems for regular expressions with 1 the fixed-point rule does not require any side-condition to ensure ‘guardedness’.

Definition 2.8. For a chart $C = \langle V, \checkmark, v_s, A, T \rangle$, a *provable solution* of C is a function $s : V \setminus \{\checkmark\} \rightarrow \text{StExp}(A)$ such that:

$$s(v) =_{\text{BBP}} \left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot s(w_j) \right) \quad (\text{for all } v \in V \setminus \{\checkmark\})$$

holds, given that the union of $\{v \xrightarrow{a_i} \checkmark \mid i = 1, \dots, m\}$ and $\{v \xrightarrow{b_j} w_j \mid j = 1, \dots, n, w_j \neq \checkmark\}$ is the set of transitions from v in C . We call $s(v_s)$ the *principal value* of s .

Proposition 2.9. For every $e \in \text{StExp}(A)$, the identity function $\text{id}_{V(e)} : V(e) \rightarrow V(e) \subseteq \text{StExp}(A)$, $e' \mapsto e'$, is a provable solution of the chart interpretation $C(e)$ of e .

Proof (Idea). Each e in $\text{StExp}(A)$ is the BBP-provable sum of expressions a and $a \cdot e'$ over all $a \in A$ for a -derivatives \checkmark and e' , respectively, of e . This ‘fundamental theorem of differential calculus for star expressions’ implies, quite directly, that $\text{id}_{V(e)}$ is a provable solution of $C(e)$. \square

3 Layered loop existence and elimination

As preparation for the definition of the central concept of ‘LLEE-witness’, we start with an informal explanation of the structural chart property ‘LEE’. It is a necessary condition for a chart to be the chart interpretation of a star expression. LEE is defined by a dynamic elimination procedure that analyses the structure of the graph by peeling off ‘loop subcharts’. Such subcharts capture, within the chart interpretation of a star expression e , the behaviour of the iteration of f_1 within innermost subterms $f_1^{\otimes} f_2$ in e . (A weaker form of ‘loop’ by Milner [17], which describes the behavior of general iteration subterms, is not sufficient for our aims.)

Definition 3.1. A chart $\mathcal{L} = \langle V, \checkmark, v_s, T \rangle$ is a *loop chart* if:

- (L1) There is an infinite path from the start vertex v_s .
- (L2) Every infinite path from v_s returns to v_s after a positive number of transitions (and so visits v_s infinitely often).
- (L3) V does not contain the vertex \checkmark .

In such a loop chart we call the transitions from v_s *loop-entry transitions*, and all other transitions *loop-body transitions*.

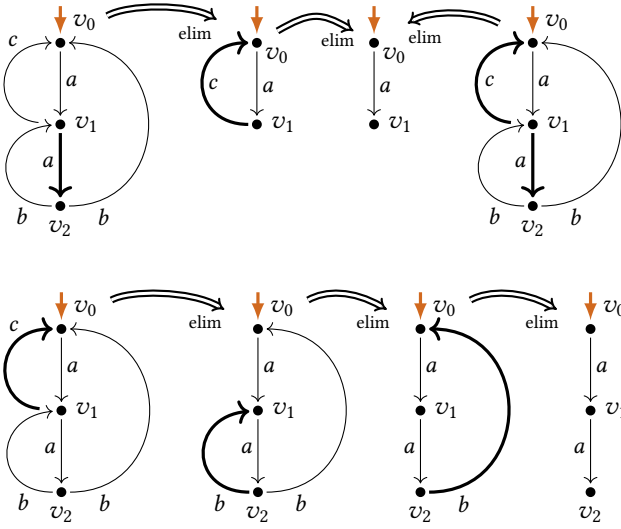
Let C be a chart. A loop chart \mathcal{L} is called a *loop subchart* of C if \mathcal{L} is the $\langle v, U \rangle$ -generated subchart of C for some vertex v of C , and a set U of transitions of C that depart from v (so the transitions in U are the loop-entry transitions of \mathcal{L}).

Note that the two charts in Ex. 2.6 are not loop charts: the left one violates (L2), and the right one violates (L3). Moreover, none of these charts contains a loop subchart. While the chart $C(e_0)$ in Ex. 2.5 is not a loop chart either, as it violates (L2), we will see that it has loop subcharts.

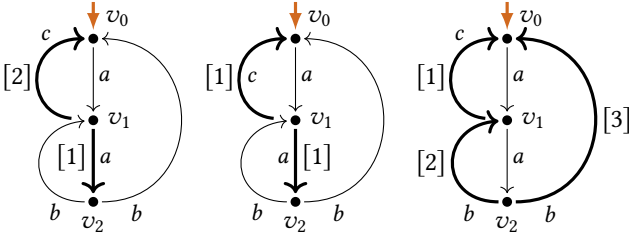
Let \mathcal{L} be a loop subchart of a chart C . Then the result of *eliminating \mathcal{L} from C* arises by removing all loop-entry transitions of \mathcal{L} from C , and then removing all vertices and transitions that get unreachable. We say that a chart C has the *loop existence and elimination property (LEE)* if the process, started on C , of repeated eliminations of loop subcharts results in a chart that does not have an infinite path.

For the charts in Ex. 2.6 the procedure stops immediately, as they do not contain loop subcharts. Since both of them have infinite paths, it follows that they do not satisfy LEE.

We consider three runs of the elimination procedure for the chart $C(e_0)$ in Ex. 2.5. The loop-entry transitions of loop subcharts that are removed in each step are marked in bold.



Each run witnesses that C satisfies LEE. Note that loop elimination does not yield a unique result.¹ Runs can be recorded by attaching, in the original chart, to transitions that get removed in the elimination procedure as marking label the sequence number of the appertaining elimination step. For the three runs of loop elimination above we get the following marking labeled versions of C , respectively:



Since all three runs were successful (as they yield charts without infinite paths), these recordings (marking-labeled charts) can be viewed as ‘LEE-witnesses’. We now will define a concept of a ‘layered LEE-witness’ (LLEE-witness), i.e., a LEE-witness with the added constraint that in the formulated run of the loop elimination procedure it never happens that a loop-entry transition is removed from within the body of a previously removed loop subchart. This refined concept has simpler properties, and it will fit our purpose.

Before introducing ‘LLEE-witnesses’, we first define chart labelings that mark transitions in a chart as ‘(loop-)entry’ and as ‘(loop-)body’ transitions, but without safeguarding that these markings refer to actual loops.

Definition 3.2. Let $C = \langle V, v_s, \sqrt{\cdot}, A, T \rangle$ be a chart. An *entry/body-labeling* $\hat{C} = \langle V, v_s, \sqrt{\cdot}, A \times \mathbb{N}, \hat{T} \rangle$ of C is a chart that arises from C by adding, for each transition $\tau = \langle v_1, a, v_2 \rangle \in T$, to the action label a of τ a *marking label* $\alpha \in \mathbb{N}$, yielding

¹Confluence, and unique normalization, can be shown if a pruning operation is added that permits to drop transitions to deadlocking vertices.

$\hat{\tau} = \langle v_1, \langle a, \alpha \rangle, v_2 \rangle \in \hat{T}$. In such an entry/body-labeling we call transitions with marking label 0 *body transitions*, and transitions with marking labels in \mathbb{N}^+ *entry transitions*.

Let \hat{C} be an entry/body-labeling of C , and let v and w be vertices of C and \hat{C} . We denote by $v \rightarrow_{b_0} w$ that there is a body transition $v \xrightarrow{\langle a, 0 \rangle} w$ in \hat{C} for some $a \in A$, and by $v \rightarrow_{[\alpha]} w$, for $\alpha \in \mathbb{N}^+$ that there is an entry transition $v \xrightarrow{\langle a, \alpha \rangle} w$ in \hat{C} for some $a \in A$. We will use $\alpha, \beta, \gamma, \dots$ for marking labels in \mathbb{N}^+ of entry transitions. By the set $E(\hat{C})$ of *entry transition identifiers* we denote the set of pairs $\langle v, \alpha \rangle \in V \times \mathbb{N}^+$ such that an entry transition $\rightarrow_{[\alpha]}$ departs from v in \hat{C} . For $\langle v, \alpha \rangle \in E(\hat{C})$, we define by $C_{\hat{C}}(v, \alpha)$ the subchart of C with start vertex v_s that consists of the vertices and transitions which occur on paths in C as follows: they start with a $\rightarrow_{[\alpha]}$ entry transition from v , continue with body transitions only, and halt immediately if v is revisited.

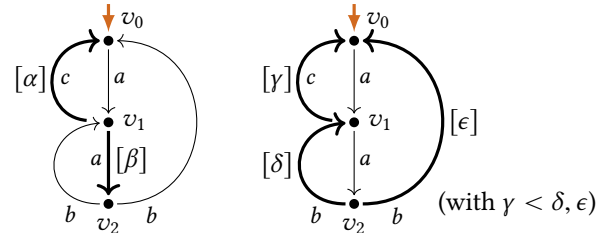
Definition 3.3. A *LLEE-witness* \hat{C} of a chart C is an entry/body-labeling of C that satisfies the following properties:

- (W1) There is no infinite path of \rightarrow_{b_0} transitions from v_s .
- (W2) For all $\langle v, \alpha \rangle \in E(\hat{C})$, (a) $C_{\hat{C}}(v, \alpha)$ is a loop chart, and (b) (*layeredness*) from no vertex $w \neq v$ of $C_{\hat{C}}(v, \alpha)$ there departs in \hat{C} an entry transition $\rightarrow_{[\beta]}$ with $\beta \geq \alpha$.

The stipulation in (W2)(a) justifies to call entry transitions in a LLEE-witness a *loop-entry transition*. For a loop-entry transition $\rightarrow_{[\beta]}$ with $\beta \in \mathbb{N}^+$, we call β its *loop level*.

A chart is a *LLEE-chart* if it has a LLEE-witness.

Example 3.4. The three labelings of the chart $C(e_0)$ in Ex. 2.5 that arose as recordings of runs of the loop elimination procedure can be viewed as entry/body-labelings of that chart. There, and below, we dropped the body labels of transitions, and instead only indicated the entry labels in boldface together with their levels. By checking conditions (W1) and (W2)(a)-(b), it is easy to verify that these entry/body-labelings are LLEE-witnesses. In fact it is not difficult to establish that every LLEE-witness of $C(e_0)$ in Ex. 2.5 is of either of the following two forms, with marking labels $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{N}^+$:



We now argue that LLEE-witnesses guarantee the property LEE. Let \hat{C} be a LLEE-witness of a chart C . Repeatedly pick an entry transition identifier $\langle v, \alpha \rangle$ with $\alpha \in \mathbb{N}^+$ minimal, remove the loop subchart that is generated by loop-entry transitions of level α from v (it is indeed a loop by (W2)(a), and minimality of α and (W2)(b) ensure the absence of departing loop-entry transitions of lower level), and perform

garbage collection. Eventually the part of C that is reachable by body transitions from the start vertex is obtained. This subchart does not have an infinite path due to (W1). Therefore C indeed satisfies LEE, as witnessed by \hat{C} .

The property LEE and the concept of LLEE-witness are closely linked with the process semantics of star expressions. In fact, we now define a labeling of the TSS in Def. 2.4 that permits to define, for every star expression e , an entry/body-labeling of the chart interpretation $C(e)$ of e , which can then be recognized as a LLEE-witness of $C(e)$.

We refine the TSS rules in Def. 2.4 as follows: A body label is added to transitions that cannot return to the star expression in their left-hand side. The rule for transitions into the iteration part e_1 of an iteration $e_1^*e_2$ is split into the cases where e_1 is normed or not. Only in the normed case can $e_1^*e_2$ return to itself, and then a loop-entry transition with the star height $|e_1|_\otimes$ of e_1 as its level is created.

Definition 3.5. For every $e \in \text{StExp}(A)$, we define the entry/body-labeling $\widehat{C}(e)$ of the chart interpretation $C(e)$ of e in analogy with $C(e)$ by using the following transition rules that refine the rules in Def. 2.4 by adding marking labels:

$$\begin{array}{c}
\frac{}{a \xrightarrow{a}_{\text{bo}} \checkmark} \quad \frac{e_i \xrightarrow{a}_l \xi}{e_1 + e_2 \xrightarrow{a}_{\text{bo}} \xi} \quad i \in \{1, 2\} \\
\\
\frac{e_1 \xrightarrow{a}_l e'_1}{e_1 \cdot e_2 \xrightarrow{a}_l e'_1 \cdot e_2} \quad \frac{e_1 \xrightarrow{a}_{\text{bo}} \checkmark}{e_1 \cdot e_2 \xrightarrow{a}_{\text{bo}} e_2} \\
\\
\frac{e_1 \xrightarrow{a}_l e'_1}{e_1^*e_2 \xrightarrow{a}_{[|e_1|_\otimes+1]} e'_1 \cdot (e_1^*e_2)} \text{ if } e_1 \text{ is normed} \\
\\
\frac{e_1 \xrightarrow{a}_l e'_1}{e_1^*e_2 \xrightarrow{a}_{\text{bo}} e'_1 \cdot (e_1^*e_2)} \text{ if } e_1 \text{ is not normed} \\
\\
\frac{e_1 \xrightarrow{a}_{\text{bo}} \checkmark}{e_1^*e_2 \xrightarrow{a}_{[|e_1|_\otimes+1]} e_1^*e_2} \quad \frac{e_2 \xrightarrow{a}_l \xi}{e_1^*e_2 \xrightarrow{a}_{\text{bo}} \xi}
\end{array}$$

for $l \in \{\text{bo}\} \cup \{[\alpha] \mid \alpha \in \mathbb{N}^+\}$, where we employed notation defined in Def. 2.4 for writing marking labels as subscripts.

Example 3.6. In Fig. 1 we depict the entry/body-labelings, as defined in Def. 3.2, for star expressions e_1 , e_0 , and e_2 in Ex. 2.5. It is easy to verify that these labelings are LLEE-witnesses of the charts $C(e_0)$, $C(e_1)$, and $C(e_2)$ in Ex. 2.5, resp..

Proposition 3.7. For every $e \in \text{StExp}(A)$, the entry/body-labeling $\widehat{C}(e)$ of $C(e)$ is a LLEE-witness of $C(e)$.

For a binary relation R , let R^+ and R^* be its transitive and transitive-reflexive closures. $u \rightarrow_l v$ denotes there is a transition $u \xrightarrow{a}_l v$ for an $a \in A$, and in proofs (but not pictures) $u \rightarrow v$ denotes that $u \rightarrow_l v$ for some label l . By $u \xrightarrow{+}_{(w)} v$ we denote that $u \rightarrow_l v$ and $v \neq w$ (this transition avoids

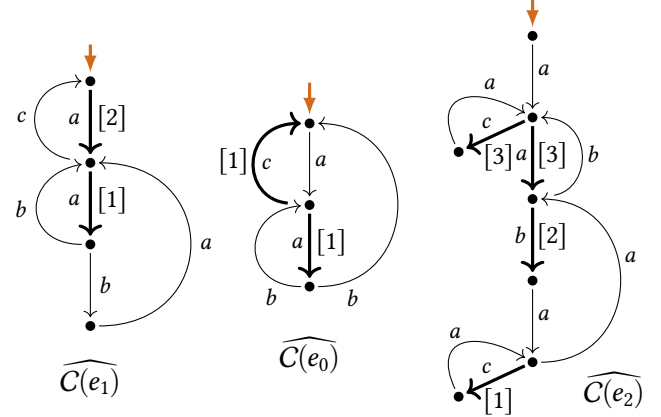


Figure 1. LLEE-witness entry/body-labelings as defined by Def. 3.5 for the chart interpretations of e_0 , e_1 , and e_2 in Ex. 2.5.

target w). Likewise, $u \xrightarrow{+}_{(w)} v$ denotes that $u \xrightarrow{l}_{(w)} v$ for some label l . By $\text{scc}(u)$ we denote the strongly connected component (scc) to which u belongs.

Definition 3.8. Let \hat{C} be a LLEE-witness of chart C . If there is a path $v \xrightarrow{+}_{(v)} [\alpha] \cdot \xrightarrow{+}_{(v)}^*_{\text{bo}} w$, then we write $v \alpha \curvearrowright w$. (Note that $v \alpha \curvearrowright w$ holds if and only if w is a vertex $\neq v$ of the loop chart $C_{\hat{C}}(v, \alpha)$ that is generated by the $\rightarrow_{[\alpha]}$ entry transitions at v in C .) We write $v \curvearrowright w$ and say that v descends in a loop to w if $v \alpha \curvearrowright w$ for some $\alpha \in \mathbb{N}^+$.

We write $w \sqsubset v$ (or $v \sqsupset w$), and say that w loops back to v , if $v \curvearrowright w \rightarrow^+_{\text{bo}} v$. We write $w_d \sqsubset v$ (or $v_d \sqsupset w$), and say that w directly loops back to v , if $w \sqsubset v$ and for all u with $w \sqsubset u$ either $u = v$ or $v \sqsubset u$.

Lemma 3.9. The relations \rightarrow_{bo} , \curvearrowright , \sqsubset , \sqsupset as defined by a LLEE-witness \hat{C} on a chart C satisfy the following properties:

- (i) There are no infinite \rightarrow_{bo} paths (so no \rightarrow_{bo} cycles).
- (ii) If $\text{scc}(u) = \text{scc}(v)$, then $u \curvearrowright^* v$ implies $v \sqsubset^* u$.
- (iii) If $v \curvearrowright w$ and $\neg(w \sqsubset v)$, then w is not normed.
- (iv) $\text{scc}(u) = \text{scc}(v)$ if and only if $u \sqsubset^* w$ and $v \sqsubset^* w$ for some vertex w .
- (v) \sqsubset^* is a partial order with the least-upper-bound property: if a nonempty set of vertices has an upper bound with respect to \sqsubset^* , then it has a least upper bound.
- (vi) If $v_1_d \sqsubset u$ and $v_2_d \sqsubset u$ for distinct v_1, v_2 , then there is no vertex w such that both $w \sqsubset^* v_1$ and $w \sqsubset^* v_2$.

4 The completeness proof, anticipated

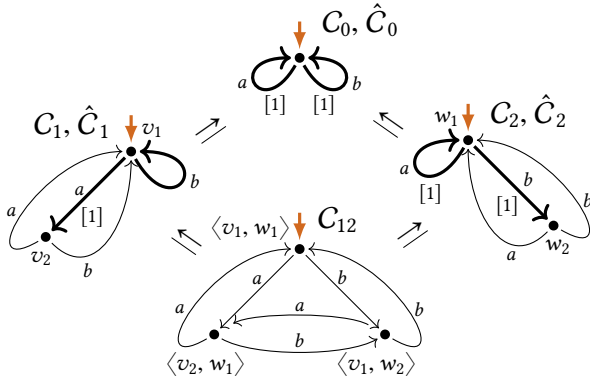
After having introduced LLEE-charts as our crucial auxiliary concept, we now sketch the completeness proof. In doing so we need to anticipate four results that will be developed in the next two sections: **(C)** The bisimulation collapse of a LLEE-chart is again a LLEE-chart. **(E)** From every LLEE-chart a provable solution can be extracted. **(S)** All provable solutions of LLEE-charts are provably equal. **(P)** All provable

solutions can be pulled back from the target to the source chart of a functional bisimulation.

Then completeness of BBP can be argued as follows. Given two bisimilar star expressions e_1 and e_2 , obtain their chart interpretations $C(e_1)$ and $C(e_2)$, which are LLEE-charts due to Prop. 3.7. By Prop. 2.9, e_1 and e_2 are principal values of provable solutions of $C(e_1)$ and $C(e_2)$. These charts have the same bisimulation collapse C . By (C, Thm. 6.9), C is again a LLEE-chart. Use (E, Prop. 5.5) to build a provable solution s of C ; let its principal value be e . Apply (P, Prop. 5.1) to transfer s backwards over the functional bisimulations to obtain provable solutions s_1 and s_2 of $C(e_1)$ and $C(e_2)$, respectively. By construction, s_1 and s_2 have the same principal value e as s . Finally, by using (S, Prop. 5.8), e_1 and e_2 are both provably equal to e . Hence, $e_1 =_{\text{BBP}} e =_{\text{BBP}} e_2$.

In his completeness proof for regular expressions in formal language theory, Salomaa [19] argued ‘upwards’ from two equivalent regular expressions to a larger regular expression that can be homomorphically collapsed onto both of them. In contrast, our proof approach forces us ‘downwards’ to the bisimulation collapse, because in the opposite direction the property of being a LLEE-chart may be lost.

Example 4.1. The picture below highlights why we cannot adopt Salomaa’s proof strategy of linking two language-equivalent regular expressions via the product of the DFAs they represent. The bisimilar LLEE-charts C_1 and C_2 are interpretations of $(a \cdot (a + b) + b)^*0$ and $(b \cdot (a + b) + a)^*0$, respectively (the indicated labelings \hat{C}_1 and \hat{C}_2 are LLEE-witnesses). But their product C_{12} is not a LLEE-chart; it is of the form of one of the not expressible charts from Ex. 2.6. Yet their common bisimulation collapse C_0 , the chart interpretation of $(a + b)^*0$, is a LLEE-chart with LLEE-witness \hat{C}_0 .



In view of $C_1 \approx C_{12} \approx C_2$ this also shows that LLEE-charts are not closed under converse functional bisimilarity \approx .

5 Extraction of star expressions from, and transferral between, LLEE-charts

In this section we develop the results (E), (S), and (P) as mentioned in Sect. 4. We start with the statement (P).

Proposition 5.1. Let $\phi : V_1 \rightarrow V_2$ be a functional bisimulation between charts C_1 and C_2 . If $s_2 : V_2 \setminus \{\sqrt{\cdot}\} \rightarrow \text{StExp}(A)$ is a provable solution of C_2 , then $s_2 \circ \phi : V_1 \setminus \{\sqrt{\cdot}\} \rightarrow \text{StExp}(A)$ is a provable solution of C_1 with the same principal value as s_2 .

Proof (Idea). The bisimulation clauses make it possible to demonstrate the condition for $s_2 \circ \phi$ to be a provable solution of C_1 at w by using the condition for the provable solution s_2 of C_2 at $\phi(w)$, together with the axioms (B1), (B2), (B3). \square

We now turn to proving results (E) and (S) from Sect. 4. We show that from every chart C with LLEE-witness \hat{C} a provable solution $s_{\hat{C}}$ of C can be extracted. Intuitively, the extraction process follows a run of the loop-elimination procedure on C , guided by the LLEE-witness \hat{C} . All loop subcharts that are generated by the loop-entry transitions from a vertex v are removed in a row.² Extraction synthesizes a star expression e_1 whose behavior captures the eliminated loop subcharts of v and their previously eliminated inner loop subcharts, and that will later be part of an iteration expression $e_1^*e_2$ in the solution value at v . This idea motivates an inside-out extraction process that works with partial solutions, and eventually builds up a provable solution of C .

In particular, we inductively define ‘relative extracted solutions’ $t_{\hat{C}}(w, v)$ for vertices v and w where w is in a loop subchart $C_{\hat{C}}(v, \alpha)$ at v , for some $\alpha \in \mathbb{N}$, that is, $v \xrightarrow{\alpha} w$. Hereby $t_{\hat{C}}(w, v)$ captures the part of the behavior in C from w until v is reached. Then we define the from \hat{C} ‘extracted solution’ $s_{\hat{C}}(v)$ at v by using the relative solutions $t_{\hat{C}}(w_j, v)$ for all targets w_j of loop-entry transitions from v to define the iteration part e_1 of the extracted solution $s_{\hat{C}}(v) = e_1^*e_2$ at v . We start with a preparation.

Let \hat{C} be a LLEE-witness, and let v be a vertex of \hat{C} . By the *entry step level* $|v|_{\text{en}}$ of v we mean the maximum loop level of a loop-entry transition in \hat{C} that departs from v , or 0 if no loop-entry transition departs from v . By the *body step norm* $\|v\|_{\text{bo}}$ of v we mean the maximal length of a body transition path in C from v (well-defined by Lem. 3.9, (i)).

Lemma 5.2. For all vertices v, w in a chart C with LLEE-witness \hat{C} it holds (for the concepts as defined with respect to \hat{C}):

- (i) $v \rightarrow_{\text{bo}} w \Rightarrow \|v\|_{\text{bo}} > \|w\|_{\text{bo}}$,
- (ii) $v \rightsquigarrow w \Rightarrow |v|_{\text{en}} > |w|_{\text{en}}$.

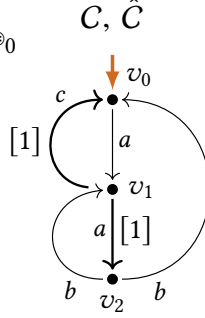
Definition 5.3. Let \hat{C} be a LLEE-witness of a chart C . Then the *relative extraction function* of \hat{C} is defined inductively as:

$$t_{\hat{C}} : \{\langle w, v \rangle \mid v, w \in V \setminus \{\sqrt{\cdot}\}, v \rightsquigarrow w\} \rightarrow \text{StExp}(A),$$

$$t_{\hat{C}}(w, v) := \left(\left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right)^* \right. \\ \left. \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot t_{\hat{C}}(u_j, v) \right) \right) \right),$$

²We repeatedly pick vertices v in the remaining LLEE-witness with entry step level $|v|_{\text{en}}$ (see in the text below) minimal.

$$\begin{aligned}
s_{\hat{C}}(v_0) &:= 0^{\otimes}(a \cdot s_{\hat{C}}(v_1)) \\
&=_{\text{BBP}} a \cdot s_{\hat{C}}(v_1) \\
&=_{\text{BBP}} a \cdot (c \cdot a + a \cdot (b + b \cdot a))^{\otimes 0} \\
s_{\hat{C}}(v_1) &:= (c \cdot t_{\hat{C}}(v_0, v_1) + a \cdot t_{\hat{C}}(v_2, v_1))^{\otimes 0} \\
&=_{\text{BBP}} (c \cdot a + a \cdot (b + b \cdot a))^{\otimes 0} \\
t_{\hat{C}}(v_0, v_1) &:= 0^{\otimes} a \\
&=_{\text{BBP}} a \\
t_{\hat{C}}(v_2, v_1) &:= 0^{\otimes}(b + b \cdot t_{\hat{C}}(v_0, v_1)) \\
&=_{\text{BBP}} b + b \cdot a \\
s_{\hat{C}}(v_2) &:= 0^{\otimes}(b \cdot s_{\hat{C}}(v_1) + b \cdot s_{\hat{C}}(v_0)) \\
&=_{\text{BBP}} b \cdot s_{\hat{C}}(v_1) + b \cdot (a \cdot s_{\hat{C}}(v_1)) \\
&=_{\text{BBP}} (b + b \cdot a) \cdot s_{\hat{C}}(v_1) \\
&=_{\text{BBP}} (b + b \cdot a) \cdot ((c \cdot a + a \cdot (b + b \cdot a))^{\otimes 0})
\end{aligned}$$



$$\begin{aligned}
s(v_0) &=_{\text{BBP}}^{(\text{sol})} a \cdot s(v_1) \quad (\text{sol means use of 'is provable solution'}) \\
s(v_1) &=_{\text{BBP}}^{(\text{sol})} c \cdot s(v_0) + a \cdot s(v_2) \\
&=_{\text{BBP}}^{(\text{sol})} c \cdot (a \cdot s(v_1)) + a \cdot (b \cdot s(v_1) + b \cdot s(v_0)) \\
&=_{\text{BBP}}^{(\text{sol})} c \cdot (a \cdot s(v_1)) + a \cdot (b \cdot s(v_1) + b \cdot (a \cdot s(v_1))) \\
&=_{\text{BBP}} (c \cdot a + a \cdot (b + b \cdot a)) \cdot s(v_1) + 0 \\
&\Downarrow \text{applying RSP}^{\otimes} \\
s(v_1) &=_{\text{BBP}} (c \cdot a + a \cdot (b + b \cdot a))^{\otimes 0} \\
&=_{\text{BBP}} s_{\hat{C}}(v_1) \quad (\text{see in the derivation on the left}) \\
&\Downarrow \\
s(v_0) &=_{\text{BBP}}^{(\text{sol})} a \cdot s(v_1) =_{\text{BBP}} a \cdot s_{\hat{C}}(v_1) =_{\text{BBP}}^{(\text{sol})} s_{\hat{C}}(v_0) \\
&\Downarrow \\
s(v_2) &=_{\text{BBP}}^{(\text{sol})} b \cdot s(v_1) + b \cdot s(v_0) \\
&=_{\text{BBP}} b \cdot s_{\hat{C}}(v_1) + b \cdot s_{\hat{C}}(v_0) =_{\text{BBP}}^{(\text{sol})} s_{\hat{C}}(v_2)
\end{aligned}$$

Figure 2. Left: the process of extracting the provable solution $s_{\hat{C}}$ of a chart C from an LLEE-witness \hat{C} of C as in the middle. Right: steps for showing that an arbitrary provable solution s of C is BBP-provably equal to the extracted solution $s_{\hat{C}}$.

provided that w has loop-entry transitions $\{w \xrightarrow{a_i}_{[\alpha_i]} w \mid i = 1, \dots, m\} \cup \{w \xrightarrow{b_j}_{[\beta_j]} w_j \mid j = 1, \dots, n \wedge w_j \neq w\}$ and body transitions $\{w \xrightarrow{c_i}_{\text{bo}} v \mid i = 1, \dots, p\} \cup \{w \xrightarrow{d_j}_{\text{bo}} u_j \mid j = 1, \dots, q \wedge u_j \neq v\}$. Hereby the induction proceeds on $\langle |v|_{\text{en}}, \|w\|_{\text{bo}} \rangle$ with the lexicographic order $<_{\text{lex}}$ on $\mathbb{N} \times \mathbb{N}$: For $t_{\hat{C}}(w_j, w)$ we have $\langle |w|_{\text{en}}, \|w_j\|_{\text{bo}} \rangle <_{\text{lex}} \langle |v|_{\text{en}}, \|w\|_{\text{bo}} \rangle$ due to $|w_j|_{\text{en}} < |v|_{\text{en}}$, which follows from $v \curvearrowright w$ by Lem. 5.2, (ii). For $t_{\hat{C}}(u_j, v)$ we have $\langle |v|_{\text{en}}, \|u_j\|_{\text{bo}} \rangle <_{\text{lex}} \langle |v|_{\text{en}}, \|w\|_{\text{bo}} \rangle$ due to $\|u_j\|_{\text{bo}} < \|w\|_{\text{bo}}$, which follows from $w \rightarrow_{\text{bo}} u_j$ by Lem 5.2, (i).

The extraction function of \hat{C} is defined by:

$$\begin{aligned}
s_{\hat{C}} : V \setminus \{\sqrt{}\} &\rightarrow \text{StExp}(A), \\
s_{\hat{C}}(w) &:= \left(\left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right)^{\otimes} \right. \\
&\quad \left. \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot s_{\hat{C}}(u_j) \right) \right) \right),
\end{aligned}$$

with induction on $\|w\|_{\text{bo}}$, provided that w has loop-entry transitions $\{w \xrightarrow{a_i}_{[\alpha_i]} w \mid i = 1, \dots, m\} \cup \{w \xrightarrow{b_j}_{[\beta_j]} w_j \mid j = 1, \dots, n \wedge w_j \neq w\}$ and body transitions $\{w \xrightarrow{c_i}_{\text{bo}} \sqrt{} \mid i = 1, \dots, p\} \cup \{w \xrightarrow{d_j}_{\text{bo}} u_j \mid j = 1, \dots, q \wedge u_j \neq \sqrt{}\}$. For $s_{\hat{C}}(u_j)$ the induction hypothesis holds due to $\|u_j\|_{\text{bo}} < \|w\|_{\text{bo}}$, which follows from $w \rightarrow_{\text{bo}} u_j$ by Lem. 5.2, (i).

Lemma 5.4. In a chart C with LLEE-witness \hat{C} , if $v \curvearrowright w$, then $s_{\hat{C}}(w) =_{\text{BBP}} t_{\hat{C}}(w, v) \cdot s_{\hat{C}}(v)$.

Proposition 5.5. For every LLEE-witness \hat{C} of a chart C , the extraction function $s_{\hat{C}}$ is a provable solution of C .

The proof of Lem. 5.4 proceeds by induction on $\|w\|_{\text{bo}}$; no induction is needed for the proof of Prop. 5.5 (cf. appendix).

Example 5.6. Left in Fig. 2 we illustrate the extraction of a provable solution for the LLEE-witness $\hat{C} = \widehat{C}(e_0)$ in Ex. 3.6 of the chart $C = C(e_0)$ in Ex. 2.5. In order to obtain the principal value $s_{\hat{C}}(v_0)$ of the extracted solution $s_{\hat{C}}$, its definition is expanded. It recurs on $s_{\hat{C}}(v_1)$, and then on $t_{\hat{C}}(v_0, v_1)$ and $t_{\hat{C}}(v_2, v_1)$. After computing those star expressions by using the definition of $t_{\hat{C}}$, the principal value can be obtained by substitution. The star expressions $s_{\hat{C}}(v_1)$ and $s_{\hat{C}}(v_2)$ are obtained similarly. For readability we have simplified the arising terms on the way by using the equality $0^{\otimes}x =_{\text{BBP}} x$ (which follows by (B1), (B6), (B7), and (BKS1)).

Lemma 5.7. If $v \curvearrowright w$, then $s(w) =_{\text{BBP}} t_{\hat{C}}(w, v) \cdot s(v)$ for every provable solution s of a chart C with LLEE-witness \hat{C} .

Proposition 5.8. Let s_1 and s_2 be provable solutions of a LLEE-chart. Then $s_1(w) =_{\text{BBP}} s_2(w)$ for all vertices $w \neq \sqrt{}$.

For the proof of this proposition, see Fig. 3. The proof of Lem. 5.7 (see in the appendix) proceeds by the same induction measure as we used for the relative extraction function.

Example 5.9. In the right half of Fig. 2 we prove that an arbitrary provable solution s of LLEE-chart $C = C(e_0)$ in Ex. 2.5 with LLEE-witness $\hat{C} = \widehat{C}(e_0)$ in Ex. 3.6 is provably equal to the extracted solution $s_{\hat{C}}$ of C . Crucially, the defining conditions for s as a provable solution of C are expanded along the loop at v_1 . The loop behavior obtained is the same as that which is used in the definition of $s_{\hat{C}}(v_1)$. By applying the fixed-point rule RSP^{\otimes} we can then deduce BBP-provable equality of $s(v_1)$ and $s_{\hat{C}}(v_1)$. By using the solution conditions for s again, provable equality is then transferred to v_0 and v_1 .

Proof (of Prop. 5.8). Let \hat{C} be a LLEE-witness of a chart C . Let s be a provable solution of C . We have to show that $s(w) =_{\text{BBP}} s_{\hat{C}}(w)$ for all $w \neq \sqrt{}$. For this, let $w \neq \sqrt{}$. The derivation below is based on the set representation of transitions from w in \hat{C} as formulated in the definition of $s_{\hat{C}}(w)$. The first derivation step uses that s is a provable solution of C and axioms (B1), (B2), and (B3), the second step uses Lem. 5.7 in view of $w \rightsquigarrow w_j$ for $j = 1, \dots, n$, and the third step uses axioms (B4) and (B6).

$$\begin{aligned} s(w) &=_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \cdot s(w) \right) + \left(\sum_{j=1}^n b_j \cdot s(w_j) \right) \right) + \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot s(u_j) \right) \right) \\ &=_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \cdot s(w) \right) + \left(\sum_{j=1}^n b_j \cdot (t_{\hat{C}}(w_j, w) \cdot s(w)) \right) \right) + \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot s(u_j) \right) \right) \\ &=_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n (b_j \cdot t_{\hat{C}}(w_j, w)) \right) \right) \cdot s(w) + \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot s(u_j) \right) \right) \end{aligned}$$

In view of this derived provable equality for $s(w)$, we can now apply the rule RSP^{\otimes} in order to obtain:

$$s(w) =_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right)^{\otimes} \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot s_{\hat{C}}(u_j) \right) \right) \equiv s_{\hat{C}}(w)$$

In this last step we have used the definition of $s_{\hat{C}}(w)$. □

Figure 3. Proof of Prop. 5.8.

6 Preservation of LLEE under collapse

In this section we establish the remaining result (C) from Sect. 4 that is crucial for the completeness proof: that the bisimulation collapse of a LLEE-chart is again a LLEE-chart.

This result is achieved by a step-wise construction of a bisimulation collapse. Pairs of bisimilar vertices w_1 and w_2 are collapsed one at a time, whereby the incoming transitions of w_1 are redirected to w_2 . The crux is to take care, and to prove, that the resulting chart has again a LLEE-witness.

Definition 6.1. Let C be a chart, with vertices w_1 and w_2 .

The *connect- w_1 -through-to- w_2 chart* $C_{w_2}^{(w_1)}$ of C is obtained by redirecting all incoming transitions at w_1 over to w_2 , and, if w_1 is the start vertex of C , making w_2 the new start vertex; in this way w_1 gets unreachable, and it is removed with other unreachable vertices to obtain a start-vertex connected chart.

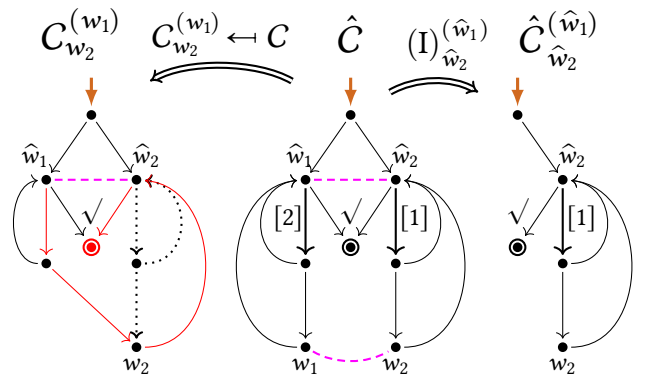
Let \hat{C} be an entry/body-labeling of C . Then we define the entry/body-labeling $\hat{C}_{w_2}^{(w_1)}$ of $C_{w_2}^{(w_1)}$ as follows: every transition in $C_{w_2}^{(w_1)}$ that was already a transition τ in C inherits its marking label from τ in \hat{C} ; and every transition in $C_{w_2}^{(w_1)}$ that arises as the redirection τ_{w_2} to w_2 of a transition τ to w_1 in C such that τ_{w_2} does not coincide with a transition already in C inherits its marking label from τ in \hat{C} .

Lemma 6.2. If $w_1 \rightleftharpoons w_2$ in C , then $C_{w_2}^{(w_1)} \rightleftharpoons C$.

While the connect-through operation of bisimilar vertices in a chart thus results in a bisimilar chart, its application to a LLEE-witness (an entry/body-labeling) does not need to yield a LLEE-witness again: the property LEE may be lost.

Example 6.3. Consider the LLEE-witness \hat{C} in the middle below. The unspecified action labels are assumed to facilitate that w_1 and w_2 are bisimilar. Hence also \hat{w}_1 and \hat{w}_2 are

bisimilar. Bisimilarity is indicated by the broken lines. The connect- w_1 -through-to- w_2 chart on the left is not a LLEE-chart, because it does not satisfy LEE: after the loop subchart induced by the downwards transition from \hat{w}_2 is eliminated, and garbage collection is done, the remaining chart without the dotted transitions still has an infinite path; yet it does not contain another loop subchart, because each infinite path can reach $\sqrt{}$ without returning to its source. An example of this is the red path from \hat{w}_1 via w_2 and \hat{w}_2 to $\sqrt{}$. In \hat{C} , the bisimilar pair w_1, w_2 progresses to the bisimilar pair \hat{w}_1, \hat{w}_2 . The connect- \hat{w}_1 -through-to- \hat{w}_2 chart on the right is a LLEE-chart, as witnessed by the entry/body-labeling $\hat{C}_{\hat{w}_2}^{(\hat{w}_1)}$.



This illustrates that bisimilar pairs of vertices must be selected carefully, to safeguard that the connect-through construction preserves LEE. The proposition below expresses that a pair of distinct bisimilar vertices can always be selected in one of three mutually exclusive categories. Later, three LEE-preserving transformations I, II, and III will be defined for each of these categories.

Proposition 6.4. *If a LLEE-chart C is not a bisimulation collapse, then it contains a pair of bisimilar vertices w_1, w_2 that satisfy, for a LLEE-witness of C , one of the conditions:*

- (C1) $\neg(w_2 \rightarrow^* w_1) \wedge (\neg w_1 \Rightarrow w_2 \text{ is not normed})$,
- (C2) $w_2 \rhd^+ w_1$,
- (C3) $\exists v \in V (w_1 \xrightarrow{d} v \wedge w_2 \rhd^+ v) \wedge \neg(w_2 \rightarrow^*_{bo} w_1)$.

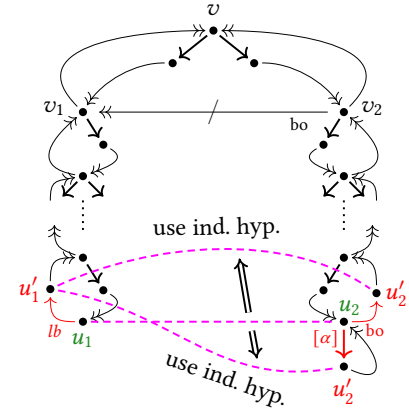
Condition (C1) requires that w_1 and w_2 are in different scc's, as there is no path from w_2 to w_1 . The additional proviso in (C1) constrains the pair in such a way that if both are normed, then w_1 must be outside of all loops (otherwise the connect- w_1 -through-to- w_2 operation does not preserve LLEE-charts, see Ex. 6.3); its asymmetric formulation helps to avoid the assumption of bisimilarity in Prop. 6.8 below. The two other conditions concern the situation that w_1 and w_2 are in the same scc. While in (C2) w_1 and w_2 are comparable (but different) by the loops-back-to relation \rhd^* , they are incomparable in (C3). In the situation that w_1, w_2 loop back to the same vertex v , but w_1 directly loops back to v , (C3) also demands that no body step path exists from w_2 to w_1 (otherwise the connect- w_1 -through-to- w_2 construction does not preserve LLEE-charts, see an example in the appendix).

In the proof of Prop. 6.4 we progress, from a given pair of distinct bisimilar vertices, repeatedly via transitions, at one side picking loop-back transitions, over pairs of distinct bisimilar vertices, until one of the conditions (C1), (C2), (C3) is met. We will use a subset of the body transitions in a LLEE-witness. By a *loop-back transition*, written as $u \rightarrow_{lb} v$, we mean a transition $u \rightarrow_{bo} v$ that stays within an scc, that is, $\text{scc}(u) = \text{scc}(v)$. The *loops-back-to norm* $\|u\|_{lb}^{\min}$ of u is the maximal length of a \rightarrow_{lb} path from u (which is well-defined by Lem. 3.9, (i) and chart finiteness). Note that $\|u\|_{lb}^{\min} = 0$ if and only if u does not loop back (denoted by $\neg(u \rhd)$).

Proof of Prop. 6.4. We pick distinct bisimilar vertices u_1, u_2 . First we consider the case $\text{scc}(u_1) \neq \text{scc}(u_2)$. Without loss of generality, suppose $\neg(u_2 \rightarrow^* u_1)$. We progress to a pair of vertices where (C1) holds, using induction on $\|u_1\|_{lb}^{\min}$. In the base case, $\|u_1\|_{lb}^{\min} = 0$, it suffices to show that it is not possible that both $\neg u_1$ holds and u_2 is normed, because then we can define $w_1 = u_1$ and $w_2 = u_2$, and are done. Therefore suppose, toward a contradiction, that $\neg u_1$ holds and u_2 is normed. Then u_1 is normed, too, since u_1 and u_2 are bisimilar. Also $\neg(u_1 \rhd)$ follows from $\|u_1\|_{lb}^{\min} = 0$, which says that there are no loops-back-to steps from u_1 . So we get that $\neg u_1$, $\neg(u_1 \rhd)$, and u_1 is normed. This contradicts Lemma 3.9, (iii). In the induction step, $\|u_1\|_{lb}^{\min} > 0$ implies $u_1 \rightarrow_{lb} u'_1$ and $\|u'_1\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$ for some u'_1 . Since $u_1 \rightleftharpoons u_2$, we have $u_2 \rightarrow u'_2$ and $u'_1 \rightleftharpoons u'_2$ for some u'_2 . Since $u_1 \rightarrow_{lb} u'_1$, by definition, u_1 and u'_1 are in the same scc. Hence $u'_1 \rightarrow^* u_1$. This implies $\neg(u'_2 \rightarrow^* u'_1)$, for else $u_2 \rightarrow u'_2 \rightarrow^* u'_1 \rightarrow^* u_1$, which contradicts the assumption $\neg(u_2 \rightarrow^* u_1)$. Since $u'_1 \rightleftharpoons u'_2$ and $\neg(u'_2 \rightarrow^* u'_1)$ and $\|u'_1\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$, by induction there exists a bisimilar pair w_1, w_2 for which (C1) holds.

Now let $\text{scc}(u_1) = \text{scc}(u_2)$. Then by Lem. 3.9, (iv), $u_1 \rhd^* v$ and $u_2 \rhd^* v$ for some v . By Lem. 3.9, (v) we pick v as the least upper bound of u_1, u_2 with regard to \rhd^* . If $u_1 = v$, then $u_2 \rhd^+ u_1$, so (C2) holds for $w_1 = u_1$ and $w_2 = u_2$. If $u_2 = v$, then likewise (C2) holds for $w_1 = u_2$ and $w_2 = u_1$. Now let $u_1, u_2 \neq v$. Since v is the least upper bound, $u_1 \rhd^* v_1 \xrightarrow{d} v \xrightarrow{d} v_2 \rhd^* u_2$ for distinct $v_1, v_2 \in V$. There cannot be a cycle of body transitions, so $\neg(v_2 \rightarrow^*_{bo} v_1)$ or $\neg(v_1 \rightarrow^*_{bo} v_2)$. By symmetry it suffices to consider $\neg(v_2 \rightarrow^*_{bo} v_1)$. Summarizing, $u_1 \rhd^* v_1 \xrightarrow{d} v \xrightarrow{d} v_2 \rhd^* u_2$ and $\neg(v_2 \rightarrow^*_{bo} v_1)$. For this situation we use induction on $\|u_1\|_{lb}^{\min}$. If $u_1 = v_1$, then $u_1 \xrightarrow{d} v$; taking $w_1 = u_1$ and $w_2 = u_2$, (C3) holds. So we can assume $u_1 \rhd^+ v_1 \xrightarrow{d} v$. Pick a transition $u_1 \rightarrow_{lb} u'_1$ with $\|u'_1\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$; by definition, $\text{scc}(u'_1) = \text{scc}(u_1)$. Since $u_1 \rightleftharpoons u_2$, there is a transition $u_2 \rightarrow u'_2$ with $u'_1 \rightleftharpoons u'_2$ for some u'_2 . If $\text{scc}(u'_1) \neq \text{scc}(u'_2)$, then as before we can find bisimilar w_1, w_2 for which (C1) holds. Now let $\text{scc}(u'_1) = \text{scc}(u'_2)$, so u_1, u_2, u'_1, u'_2 are in the same scc. Since $u_1 \rhd^+ v_1$ and $u_1 \rightarrow u'_1$, either $u'_1 = v_1$ or $v_1 \rhd^+ u'_1$. Moreover, $\text{scc}(u'_1) = \text{scc}(u_1) = \text{scc}(v_1)$, so by Lem. 3.9, (ii), $u'_1 \rhd^* v_1$. Since $u_2 \rhd^* v_2$, we can distinguish two cases (for illustrations for each of the subcases, see the appendix).

Case 1: $u_2 \rhd^+ v_2$. Since $u_2 \rightarrow u'_2$, either $u'_2 = v_2$ or $v_2 \rhd^+ u'_2$. Moreover, $\text{scc}(u'_2) = \text{scc}(u_2) = \text{scc}(v_2)$, so by Lem. 3.9, (ii), $u'_2 \rhd^* v_2$. Hence, $u'_1 \rhd^* v_1 \xrightarrow{d} v \xrightarrow{d} v_2 \rhd^* u'_2 \wedge \neg(v_2 \rightarrow^*_{bo} v_1)$, and $\|u'_1\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$. We apply the induction hypothesis to obtain a bisimilar pair w_1, w_2 for which (C1), (C2), or (C3) holds. Below we illustrate both of the cases in which $u_2 \rightarrow u'_2$ is a loop-entry transition, or a body transition.



Case 2: $u_2 = v_2$. We distinguish two cases.

Case 2.1: $u_2 \rightarrow_{[a]} u'_2$. Then either $u'_2 = u_2$ or $u_2 \rhd^+ u'_2$. Moreover, $\text{scc}(u'_2) = \text{scc}(u_2)$, so by Lem. 3.9, (ii), $u'_2 \rhd^* u_2$, and hence $u'_2 \rhd^* v_2$. Thus we have obtained $u'_1 \rhd^* v_1 \xrightarrow{d} v \xrightarrow{d} v_2 \rhd^* u'_2 \wedge \neg(v_2 \rightarrow^*_{bo} v_1)$. Due to $\|u'_1\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$, we can apply the induction hypothesis again.

Case 2.2: $u_2 \rightarrow_{\text{bo}} u'_2$. Then $\neg(v_2 \rightarrow_{\text{bo}}^* v_1)$ together with $v_2 = u_2 \rightarrow_{\text{bo}} u'_2$ and $u'_1 \rightarrow_{\text{bo}}^* v_1$ (because $u'_1 \not\rightarrow_{\text{bo}}^* v_1$) imply $u'_1 \neq u'_2$. We distinguish two cases.

Case 2.2.1: $u'_2 = v$. Then $u'_1 \not\rightarrow_{\text{bo}}^* v_1 \not\rightarrow_{\text{bo}}^* v = u'_2$, i.e., $u'_1 \not\rightarrow_{\text{bo}}^* u'_2$, so we are done, because (C2) holds for $w_1 = u'_2$ and $w_2 = u'_1$.

Case 2.2.2: $u'_2 \neq v$. By Lem. 3.9, (ii), $u'_2 \not\rightarrow_{\text{bo}}^+ v$. Hence, $u'_2 \not\rightarrow_{\text{bo}}^* v'_2 \not\rightarrow_{\text{bo}}^* v$ for some v'_2 . Since $v_2 = u_2 \rightarrow_{\text{bo}} u'_2 \not\rightarrow_{\text{bo}}^* v'_2$ and $\neg(v_2 \rightarrow_{\text{bo}}^* v_1)$, it follows that $\neg(v'_2 \rightarrow_{\text{bo}}^* v_1)$. So $u'_1 \not\rightarrow_{\text{bo}}^* v_1 \not\rightarrow_{\text{bo}}^* v \not\rightarrow_{\text{bo}}^* v'_2 \not\rightarrow_{\text{bo}}^* u'_2 \wedge \neg(v_2 \rightarrow_{\text{bo}}^* v_1)$.

Due to $\|u'_1\|_{\text{lb}}^{\min} < \|u_1\|_{\text{lb}}^{\min}$, we can apply the induction hypothesis again.

This exhaustive case analysis concludes the proof. \square

Now we define, for LLEE-witnesses \hat{C} of a LLEE-chart C , and for bisimilar vertices w_1, w_2 in C , in each of the three cases (C1), (C2), or (C3) of Prop. 6.4 a transformation of \hat{C} into an entry/body-labeling of the connect- w_1 -through-to- w_2 chart $C_{w_2}^{(w_1)}$ that can be shown to be a LLEE-witness again. We number the transformations for (C1), (C2), and (C3) as I, II, and III, respectively. Each transformation makes use of the *connect-through construction for entry/body-labelings* as defined in Def. 6.1. Additionally, in each transformation an adaptation of labels of transitions is performed, to avoid violations of LLEE-witness properties. In transformations I and III the adaptation is performed before connecting w_1 through to w_2 , and is needed to guarantee that layeredness is preserved; in transformation II it is performed right after eliminating w_1 , and avoids the creation of body step cycles. The *level adaptations* for the three transformations are:

L_I Let $m = \max\{\beta : \text{there is a path } w_2 \rightarrow^* \cdot \rightarrow_{[\beta]} \text{ in } \hat{C}\}$. In loop-entry transitions $u \rightarrow_{[\alpha]} v$ for which there is a path $v \rightarrow^* w_1$ in C , replace α by an α' with $\alpha' = \alpha + m$. This increases the labels of loop-entry transitions that descend to w_1 in \hat{C} to a higher level than the loop labels reachable from w_2 .

L_{II} Since $w_2 \not\rightarrow_{\text{bo}}^+ w_1$, there exists a \hat{w}_2 with $w_2 \not\rightarrow_{\text{bo}}^* \hat{w}_2 \not\rightarrow_{\text{bo}}^* w_1$. Let γ be the maximum loop level among the loop-entries at w_1 in \hat{C} . (Note that since $w_2 \not\rightarrow_{\text{bo}}^+ w_1$, there is at least one such transition.) Turn the body transitions from \hat{w}_2 into loop-entry transitions with loop label γ .

L_{III} Let γ be a loop label of maximum level among the loop-entry transitions at v in \hat{C} . (Note that since $w_1 \not\rightarrow_{\text{bo}}^+ v$, there is at least one such transition.) Turn the loop labels of the loop-entry transitions from v into γ .

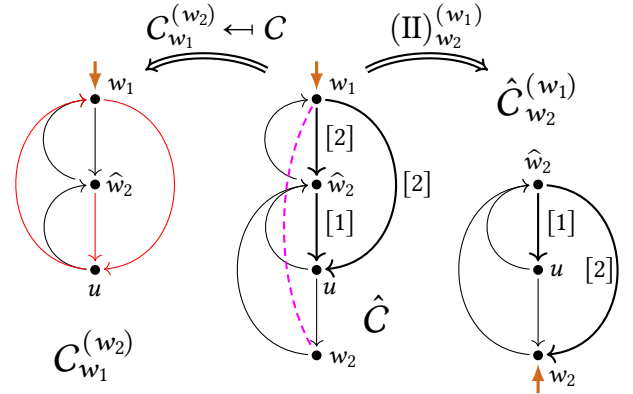
Each of these transformations ends with a *clean-up step*: if the loop-entry transitions from a vertex with the same loop label no longer induce an infinite path (due to the removal of w_1), then they are changed into body transitions.

Example 6.5. The LLEE-witness on the left in Fig. 4 is reduced in three transformation steps to a LLEE-witness of

the chart $C(e_0)$ in Ex. 2.5. Broken lines are between bisimilar vertices. In step one, a transformation I, the start state v_0 is connected through to the bisimilar vertex v_0'' , whereby v_0'' becomes the start vertex; note that there is no path from v_0'' to v_0 , and no vertex descends into a loop to v_0 . In step two, a transformation II, v_1 is connected through to the bisimilar vertex v_1' ; note that $v_1' \not\rightarrow_{\text{bo}}^+ v_1$. In step three, a transformation III, the start vertex v_0'' is connected through to the bisimilar vertex v_0''' , whereby v_0''' becomes the start vertex; note that $v_0'' \not\rightarrow_{\text{bo}}^* v_2$ and $v_0''' \not\rightarrow_{\text{bo}}^+ v_2$ and there is no body step path from v_0''' to v_0'' . By the loop level adaptation L_{III} , all loop entries from v_2 get level 3. The final step is an isomorphic deformation. Only the left and right charts depict actions.

The following examples provide more illustrations of the transformations II and III. Similarly as Ex. 6.3 does so for transformation I and (C1), they also show that the conditions (C2) and (C3) mark rather sharp borders between whether, on a given LLEE-witness, a connect-through operation is possible while preserving LLEE, or not.

Example 6.6. For the LLEE-witness \hat{C} below in the middle, the chart $C_{w_1}^{(w_2)}$ on the left has no LLEE-witness.



It does not satisfy LEE: it has no loop subchart, since from each of its three vertices an infinite path starts that does not return to this vertex; this path from \hat{w}_2 , drawn in red, cycles between u and w_1 . Transformation II applied to the pair w_1, w_2 (instead of w_2, w_1) in \hat{C} yields the entry/body-labeling $\hat{C}_{w_2}^{(w_1)}$ where $\hat{w}_2 \rightarrow_{\text{bo}} w_2$ is turned into $\hat{w}_2 \rightarrow_{[2]} w_2$. As the pair w_1, w_2 satisfies (C2), the proof of Prop. 6.8 ensures that this labeling, drawn on the right, is a LLEE-witness.

Example 6.7. In the LLEE-witness \hat{C} below in the middle, $w_1, w_2 \not\rightarrow_{\text{bo}}^+ v$ and there is no body step path from w_2 to w_1 , but (C3) does not hold for the pair w_1, w_2 due to $\neg(w_1 \not\rightarrow_{\text{bo}}^* v)$. The chart $C_{w_2}^{(w_1)}$ on the left has no LLEE-witness. It does not satisfy LEE: the downwards loop-entry transition from \hat{w}_2 can be eliminated, and then two more arising loop-entry transitions from v ; the remaining chart of solid arrows has no further loop subchart, because from each of its vertices an infinite path starts that does not return to this vertex.

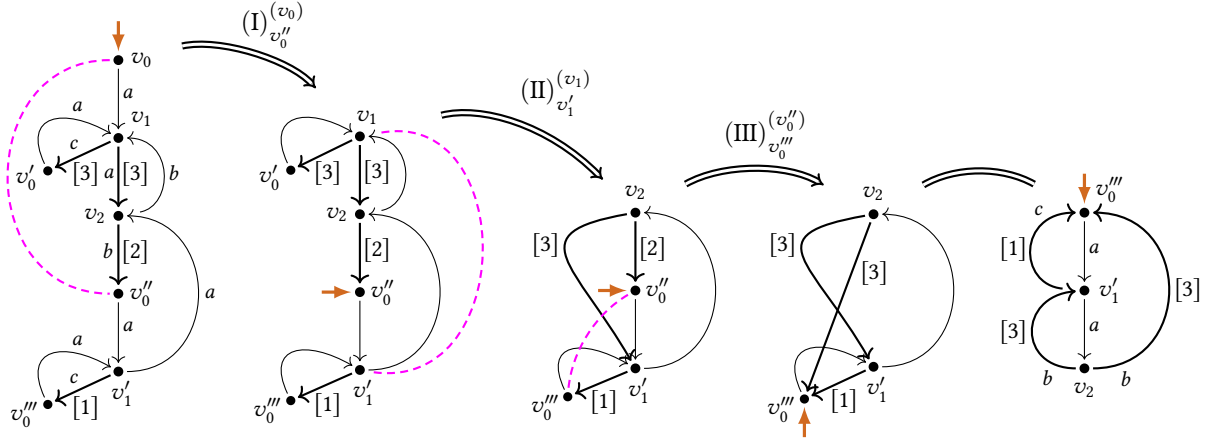
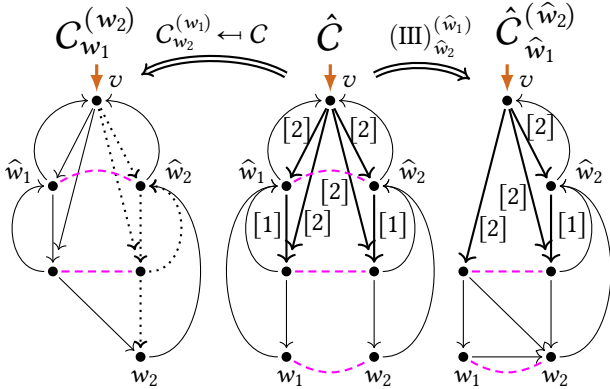


Figure 4. Three connect-through-steps according to the transformations I, II, and III from the LLEE-witness on the left, and a final isomorphic deformation, leading to the LLEE-witness on the right. For clarity, we neglected action labels in the middle.

In \hat{C} , loop-entry transitions from v have the same loop label, so the preprocessing step of transformation III is void. The bisimilar pair w_1, w_2 progresses to the bisimilar pair \hat{w}_1, \hat{w}_2 in \hat{C} , for which (C3) holds because $\hat{w}_1 \hat{a} \hat{G} v \hookrightarrow \hat{w}_2$ and $\neg(\hat{w}_2 \rightarrow_{\text{bo}}^* \hat{w}_1)$. Transformation III applied to this pair yields the labeling $\hat{C}^{(\hat{w}_1)}$ on the right. In the proof of Prop. 6.8 it is argued that this is guaranteed to be a LLEE-witness. The remaining two bisimilar pairs can be eliminated by one or by two further applications of transformation III.



Proposition 6.8. *Let C be a LLEE-chart. If a pair $\langle w_1, w_2 \rangle$ of vertices satisfies (C1), (C2), or (C3) with respect to a LLEE-witness of C , then $C_{w_2}^{(w_1)}$ is a LLEE-chart.*

Proof. Let \hat{C} be a LLEE-witness. For vertices w_1, w_2 such that (C1), (C2), or (C3) holds, transformation I, II, or III, respectively, produces an entry/body-labeling $\hat{C}^{(w_1)}$. We prove for transformation I that this is a LLEE-witness, and refer to the appendix with regard to transformations II, and III.

We first argue it suffices to show that each of the transformations produces, before the final clean-up step, a labeling that satisfies the LLEE-witness conditions, except possible violations of loop property (L1) in (W2)(a). Such violations can

be removed from a loop-labeling while preserving the other LLEE-witness conditions. To show this, suppose (L1) is violated in some $C_{\hat{C}}(u, \alpha)$. Then $u \rightarrow_{[\alpha]}$ but $\neg(u \rightarrow_{[\alpha]} \cdot \rightarrow_{\text{bo}}^* u)$. Let \hat{C}_1 be the result of removing this violation by changing the α -loop-entry transitions from u into body transitions. No new violation of (L1) is introduced in \hat{C}_1 . (W1) and (W2)(a), (L2), are preserved in \hat{C}_1 because an introduced infinite body step path in \hat{C}_1 would be a body step cycle that stems from a path $u \rightarrow_{[\alpha]} u' \rightarrow_{\text{bo}}^* u$ in \hat{C} . (W2)(b) might only be violated by a path $w \xrightarrow{\dagger(w)}_{[\beta]} \cdot \xrightarrow{\dagger(w)}_{\text{bo}}^* u \xrightarrow{\dagger(w, u)}_{\text{bo}} u' \xrightarrow{\dagger(w, u)}_{\text{bo}}^* \cdot \rightarrow_{[\gamma]}$ with $\beta \leq \gamma$ in \hat{C}_1 where $u \rightarrow_{\text{bo}} u'$ stems from $u \rightarrow_{[\alpha]} u'$ in \hat{C} ; then $\beta > \alpha > \gamma$ by layeredness of \hat{C} ; so (W2)(b) is preserved. Analogously we find that also (W2)(a), (L3) is preserved, because \surd is never in $C_{\hat{C}}(u, \alpha)$.

To show the correctness of transformation I, consider vertices w_1 and w_2 with (C1). We show that the result $\hat{C}_{w_2}^{(w_1)}$ of transformation I before the clean-up step satisfies the LLEE-witness properties, except for possible violations of (L1).

To verify (W1) and part (L2) of (W2)(a), it suffices to show that $\hat{C}_{w_2}^{(w_1)}$ does not contain body step cycles. The original loop-labeling \hat{C} is a LLEE-witness, so it does not contain body step cycles. Since the level adaptation step does not turn loop-entry steps into body steps, body step cycles could only arise in the step connecting w_1 through to w_2 . Suppose such a body step cycle arises. Then there must be a transition $u \rightarrow_{\text{bo}} w_1$ in \hat{C} (which is redirected to w_2 in $\hat{C}_{w_2}^{(w_1)}$) and a path $w_2 \rightarrow_{\text{bo}}^* u$ in \hat{C} . But then $w_2 \rightarrow_{\text{bo}}^* u \rightarrow_{\text{bo}} w_1$ in C , which contradicts (C1) that there is no path from w_2 to w_1 . Hence (W1) and part (L2) of (W2)(a) hold for $\hat{C}_{w_2}^{(w_1)}$.

Now we verify part (L3) of (W2)(a) in $\hat{C}_{w_2}^{(w_1)}$. Consider a path $u \xrightarrow{\dagger(u)}_{[\alpha]} \cdot \xrightarrow{\dagger(u)}_{\text{bo}}^* w_1$ in \hat{C} . Then $u \neq w_1$, and $u \hookrightarrow w_1$. It suffices to show that then $\neg(w_2 \rightarrow^+ \surd)$ in C . But this is

guaranteed, because otherwise w_2 were normed, and due to $u \curvearrowright w_1$ we would have a contradiction with condition (C1).

Finally we show that (W2)(b) is preserved in $\hat{C}_{w_2}^{(w_1)}$ by both the level adaptation and the connect-through step. First, since in the level adaptation step all adapted loop labels are increased with the same value m , a violation of (W2)(b) would arise by a path $u \rightarrow_{[\alpha]} \cdot \rightarrow_{bo}^* \cdot \rightarrow_{[\beta]} v$ in \hat{C} where loop label β is increased while α is not. But such a path cannot exist. Since β is increased, there is a path $v \rightarrow^* w_1$ in C . But then there is a path $u \rightarrow_{[\alpha]} \cdot \rightarrow^+ v \rightarrow^* w_1$ in \hat{C} , which implies that also α is increased in the level adaptation step. Second, a violation of (W2)(b) in the connect-through step would arise from paths $u \rightarrow_{[\alpha]} \cdot \rightarrow_{bo}^* w_1$ and $w_2 \rightarrow_{bo}^* \cdot \rightarrow_{[\beta]} v$ in \hat{C}' with $\alpha \leq \beta$. However, in view of the path $u \rightarrow_{[\alpha]} \cdot \rightarrow^* w_1$, the loop label α was increased with m in the level adaptation step. On the other hand, in view of (C1) that there is no path from w_2 to w_1 in C , w_1 is unreachable at the end of the path $w_2 \rightarrow^* \cdot \rightarrow_{[\beta]} v$. Hence this loop label β was not increased in the level adaptation step. So it is guaranteed that for such a pair of paths in $\hat{C}_{w_2}^{(w_1)}$ always $\alpha > \beta$.

We conclude that the result of transformation I is again a LLEE-witness. \square

Theorem 6.9. *The bisimulation collapse of a LLEE-chart is again a LLEE-chart.*

Proof. Given a LLEE-chart C , repeat the following step: based on a LLEE-witness pick, by Prop. 6.4, bisimilar vertices w_1 and w_2 with (C1), (C2), or (C3), and then connect w_1 through to w_2 , obtaining by Prop. 6.8 a LLEE-chart bisimilar to C , due to Lem. 6.2. Hence the bisimulation collapse of C , which is reached eventually, is a LLEE-chart. \square

This theorem can be strengthened: the bisimulation collapse of a LLEE-chart is the chart interpretation of some star expression (which then is a LLEE-chart by Prop. 2.9). This can be proved by showing that, on collapsed LLEE-charts, chart interpretation is the converse of solution extraction.

Corollary 6.10. *If a chart is expressible by a star expression modulo bisimilarity, then its collapse is a LLEE-chart.*

The converse statement holds as well. But this corollary does not hold for star expressions with 1 and unary star: for example, $(a \cdot (1 + c \cdot 0) + b \cdot (1 + d \cdot 0))^*$ has a chart interpretation whose bisimulation collapse does not satisfy LEE.

7 The completeness result, and conclusion

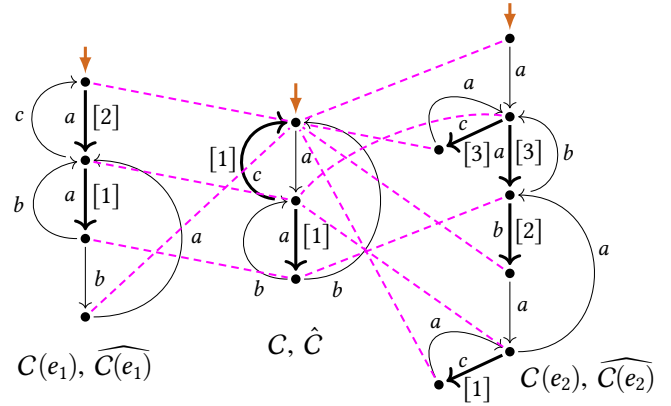
That bisimulation collapse preserves LLEE was the last building block in the proof of the desired completeness result.

Theorem 7.1. *The proof system BBP is complete with respect to the bisimulation semantics of star expressions, that is, with respect to bisimilarity of charts that interpret star expressions without 1 and with binary Kleene star * .*

Proof. The proof steps were already explained in Sect. 4. \square

Example 7.2. The bisimilar LLEE-charts C_1 and C_2 in Ex. 4.1 have $(a \cdot (a + b) + b)^{\otimes} 0$ and $(b \cdot (a + b) + a)^{\otimes} 0$ as their principal solutions. Their bisimulation collapse C_0 has principal solution $(a + b)^{\otimes} 0$. Then $(a \cdot (a + b) + b)^{\otimes} 0 =_{\text{BBP}} (a + b)^{\otimes} 0 =_{\text{BBP}} (b \cdot (a + b) + a)^{\otimes} 0$ by Prop. 5.1, Prop. 5.8.

Example 7.3. Revisiting the star expressions e_1, e_2 in Ex. 2.5 with bisimilar chart interpretations $C(e_1)$ and $C(e_2)$, we can apply our proof in order to show that $e_1 =_{\text{BBP}} e_2$. $C(e_1)$ and $C(e_2)$ have provable solutions with principal values e_1 and e_2 by Prop. 2.9. As $C(e_1)$ and $C(e_2)$ are LLEE-charts by Prop. 3.7 with LLEE-witnesses $\widehat{C}(e_1)$ and $\widehat{C}(e_2)$, their bisimulation collapse C is a LLEE-chart by Thm. 6.9. We take here the more familiar \hat{C} , but could also take the one obtained in Fig. 4. We saw in Fig. 2 that \hat{C} has a provable solution with principal value $s_{\hat{C}}(v_0) = a \cdot ((c \cdot a + a \cdot (b + b \cdot a))^{\otimes} 0)$. Then by Prop. 5.1 and Prop. 5.8 it follows that $e_1 =_{\text{BBP}} s_{\hat{C}}(v_0) =_{\text{BBP}} e_2$.



We have shown that Milner's axiomatization, tailored to star expressions without 1 and with $^{\otimes}$, is complete in bisimulation semantics. At the core of our proof is the graph structure property LLEE, which characterizes the process graphs that can be expressed by star expressions without 1 and with $^{\otimes}$ as charts whose bisimulation collapse is a LLEE-chart.

Completeness of BBP covers completeness of the theory $\text{BPA}_0^{\omega} + \text{RSP}^{\omega}$ of perpetual loop iteration $(\cdot)^{\omega}$ [9] in the sense that the latter result can be shown by our means, or by a faithful interpretation $e^{\omega} \mapsto e^{\otimes} 0$ of $\text{BPA}_0^{\omega} + \text{RSP}^{\omega}$ in BBP.

Completeness of BBP can be extended, also by means of a faithful interpretation, to cover star expressions with 0, 1, and * , but with a syntactic restriction on terms directly under a * : that they can be rewritten to star expressions with only 'harmless' occurrences of 1. This is analogous to the situation that the completeness result from [8, 10] for star expressions without 0 and 1 and with $^{\otimes}$ was extended in [6] to a setting with 1 (but not 0) and * , where a generalized version of the non-empty-word property is disallowed for terms directly under a * . With the interpretation approach, also the result in [6] can be obtained from the one in [8, 10].

The main future goal is to solve Milner's problem entirely by extending our result to the full class of star expressions.

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A Appendix: supplements, more proof details, and omitted proofs

A.1 Proofs in Section 2: Preliminaries

Proposition (= Proposition 2.9). *For every $e \in \text{StExp}(A)$, the identity function $\text{id}_{V(e)} : V(e) \rightarrow V(e) \subseteq \text{StExp}(A)$, $e' \mapsto e'$, is a provable solution of the chart interpretation $C(e)$ of e .*

In the proof of this proposition we will use the following definition concerning ‘action derivatives’, and the subsequent lemma. That statement can be viewed as the ‘fundamental theorem of differential calculus for star expressions’ which says that every star expressions can be reassembled by a form of ‘integration’ from its action derivatives. In this context ‘differentiation’ follows the definition of action derivatives in Definition 2.4 (corresponding to Antimirov’s concept of ‘partial derivative’ in [2]), and ‘integration’ means sum formation over products of pairs $\langle a, \xi \rangle$ for actions a and a -derivatives ξ .

Definition A.1. For star expressions $e \in \text{StExp}(A)$ we define the set $A\partial(e)$ of *action derivatives* of e as follows:

$$A\partial(e) := \{ \langle a, \xi \rangle \mid a \in A, \xi \in \text{StExp}(A)_{\sqrt{}} , e \xrightarrow{a} \xi \}.$$

Lemma A.2. *Every $e \in \text{StExp}(A)$ can be provably reassembled from its action derivatives as:*

$$e =_{\text{BBP}} \left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot e'_j \right), \quad (\text{A.1})$$

$$\text{provided that } A\partial(e) = \{ \langle a_1, \sqrt{} \rangle, \dots, \langle a_m, \sqrt{} \rangle, \langle b_1, e'_1 \rangle, \dots, \langle b_n, e'_n \rangle \}. \quad (\text{A.2})$$

Proof. We proceed by induction on the structure of star expressions in $\text{StExp}(A)$. For performing the induction step, we distinguish the five cases of productions in the grammar in Definition 2.1.

Case 1: $e \equiv 0$.

Then e does not enable any transitions, and hence $A\partial(e) = \emptyset$. We find the provable equality:

$$e \equiv 0 =_{\text{BBP}} 0 + 0 \quad (\text{by axiom (B1) of BBP}).$$

This is of the form as in (A.1) with $m = n = 0$ when we construe $A\partial(e) = \emptyset$ as a list representation of the form (A.2).

Case 2: $e \equiv a$ for some $a \in A$.

Then according to the TSS in Definition 2.4 the expression e enables precisely one transition, an a -transition to $\sqrt{}.$ Hence the set of action derivatives of e consists only of one element:

$$A\partial(e) = \{ \langle a, \sqrt{} \rangle \}. \quad (\text{A.3})$$

We find the provable equality:

$$e =_{\text{BBP}} a + 0 \quad (\text{by axiom (B6) of BBP}).$$

The right-hand side is of the form (A.1) with $m = 1$, $a_1 = a$ and $n = 0$ in relation to (A.3) when we construe $A\partial(e)$ as a list representation of the form (A.2).

Case 3: $e \equiv e_1 + e_2$.

Since every star expression has only finitely many derivatives, each of which is either $\sqrt{}.$ or a star expression, we may assume that the sets of action derivatives of the constituent expressions e_1 and e_2 of $e_1 + e_2$ have list representations:

$$\begin{aligned} A\partial(e_1) &= \{ \langle a_{11}, \sqrt{} \rangle, \dots, \langle a_{m_1}, \sqrt{} \rangle, \langle b_{11}, e'_{11} \rangle, \dots, \langle b_{n_1}, e'_{n_1} \rangle \}, \\ A\partial(e_2) &= \{ \langle a_{12}, \sqrt{} \rangle, \dots, \langle a_{m_2}, \sqrt{} \rangle, \langle b_{12}, e'_{12} \rangle, \dots, \langle b_{n_2}, e'_{n_2} \rangle \}. \end{aligned} \quad (\text{A.4})$$

Then it follows from the form of the TSS rules in Definition 2.4 concerning sums of star expressions that the sets of action derivatives of $e_1 + e_2$ is the union of the sets of action derivatives of e_1 , and of e_2 . By permuting the action derivatives with tick to the front, this union has the list representation:

$$\begin{aligned} A\partial(e_1 + e_2) &= \{ \langle a_{11}, \sqrt{} \rangle, \dots, \langle a_{m_1}, \sqrt{} \rangle, \langle a_{12}, \sqrt{} \rangle, \dots, \langle a_{m_2}, \sqrt{} \rangle, \\ &\quad \langle b_{11}, e'_{11} \rangle, \dots, \langle b_{n_1}, e'_{n_1} \rangle, \langle b_{12}, e'_{12} \rangle, \dots, \langle b_{n_2}, e'_{n_2} \rangle \}. \end{aligned} \quad (\text{A.5})$$

Now we can argue as follows to reassemble $e_1 + e_2$ from its action derivatives:

$$e \equiv e_1 + e_2 =_{\text{BBP}} \left(\left(\sum_{i=1}^{m_1} a_{i1} \right) + \left(\sum_{j=1}^{n_1} b_{j1} \cdot e'_{j1} \right) \right) + \left(\left(\sum_{i=1}^{m_2} a_{i2} \right) + \left(\sum_{j=1}^{n_2} b_{j2} \cdot e'_{j2} \right) \right)$$

$$\begin{aligned}
& \text{(by the induction hypothesis, using representation (A.5))} \\
& =_{\text{BBP}} \left(\left(\left(\sum_{i=1}^{m_1} a_{i1} \right) + \left(\sum_{i=1}^{m_2} a_{i2} \right) \right) + \left(\sum_{j=1}^{n_1} b_{j1} \cdot e'_{j1} \right) \right) + \left(\sum_{j=1}^{n_2} b_{j2} \cdot e'_{j2} \right). \\
& \text{(by axioms (B2) and (B1))}
\end{aligned}$$

Since ACI is a subsystem of BBP, this chain of provably equalities is one in BBP. It demonstrates, together with applications of the axiom (B2) that are needed to bring each of the subexpressions of the two outermost summands into a form with association of summation subterms to the left, that e satisfies (A.1) when we construe $A\partial(e)$ in (A.5) as a list representation of the form (A.2) with $m = m_1 + m_2$ and $n = n_1 + n_2$.

Case 4: $e \equiv e_1 \cdot e_2$.

As argued in the previous case, we may assume that the action derivatives of e_1 are of the form:

$$A\partial(e_1) = \{ \langle a_{11}, \sqrt{} \rangle, \dots, \langle a_{m_1 1}, \sqrt{} \rangle, \langle b_{11}, e'_{11} \rangle, \dots, \langle b_{n_1 1}, e'_{n_1 1} \rangle \}. \quad (\text{A.6})$$

Then it follows from the forms of the two rules in the TSS in Definition 2.4 concerning transitions from expressions with concatenation as their outermost symbol that the set of action derivatives of $e_1 \cdot e_2$ has a list representation of the form:

$$A\partial(e_1 \cdot e_2) = \{ \langle a_{11}, e_2 \rangle, \dots, \langle a_{m_1 1}, e_2 \rangle, \langle b_{11}, e'_{11} \cdot e_2 \rangle, \dots, \langle b_{n_1 1}, e'_{n_1 1} \cdot e_2 \rangle \}. \quad (\text{A.7})$$

Now we can reassemble $e_1 \cdot e_2$ as follows:

$$\begin{aligned}
e & \equiv e_1 \cdot e_2 =_{\text{BBP}} \left(\left(\sum_{i=1}^{m_1} a_{i1} \right) + \left(\sum_{j=1}^{n_1} b_{j1} \cdot e'_{j1} \right) \right) \cdot e_2 && \text{(by the induction hypothesis, using representation (A.6))} \\
& =_{\text{BBP}} \left(\sum_{i=1}^{m_1} a_{i1} \cdot e_2 \right) + \left(\sum_{j=1}^{n_1} (b_{j1} \cdot e'_{j1}) \cdot e_2 \right) && \text{(by axiom (B4))} \\
& =_{\text{BBP}} \left(\sum_{i=1}^{m_1} a_{i1} \cdot e_2 \right) + \left(\sum_{j=1}^{n_1} b_{j1} \cdot (e'_{j1} \cdot e_2) \right) && \text{(by axiom (B5))} \\
& =_{\text{BBP}} 0 + \left(\left(\sum_{i=1}^{m_1} a_{i1} \cdot e_2 \right) + \left(\sum_{j=1}^{n_1} b_{j1} \cdot (e'_{j1} \cdot e_2) \right) \right) && \text{(by axiom (B6))}
\end{aligned}$$

This chain of provable equalities demonstrates, together with applications of the axiom (B2) that are needed to bring each of the subexpressions of the right outermost summands into a form with association of summation subterms to the left, that e satisfies (A.1) when we construe $A\partial(e)$ in (A.7) as a list representation (A.2) with $m = 0$ and $n = m_1 + n_1$.

Case 5: $e \equiv e_1^{\otimes} e_2$.

As in Case 3 we may assume that the sets of action derivatives of the constituent expressions e_1 and e_2 of $e_1 + e_2$ have list representations of the form (A.4). Then it follows from the forms of the three rules in Definition 2.4 concerning transitions from expressions with binary iteration as their outermost symbol, that the set of action derivatives of $e_1^{\otimes} e_2$ has a list representation of the form:

$$\begin{aligned}
A\partial(e_1^{\otimes} e_2) & = \{ \langle a_{11}, e_1^{\otimes} e_2 \rangle, \dots, \langle a_{m_1 1}, e_1^{\otimes} e_2 \rangle, \\
& \quad \langle b_{11}, e'_{11} \cdot (e_1^{\otimes} e_2) \rangle, \dots, \langle b_{n_1 1}, e'_{n_1 1} \cdot (e_1^{\otimes} e_2) \rangle, \\
& \quad \langle a_{12}, \sqrt{} \rangle, \dots, \langle a_{m_2 2}, \sqrt{} \rangle, \langle b_{12}, e'_{12} \rangle, \dots, \langle b_{n_2 2}, e'_{n_2 2} \rangle \}.
\end{aligned}$$

By permuting the action derivatives with tick to the front, this representation can be changed into:

$$A\partial(e_1^{\otimes} e_2) = \left\{ \begin{aligned} & \langle a_{12}, \sqrt{} \rangle, \dots, \langle a_{m_2 2}, \sqrt{} \rangle, \\ & \langle a_{11}, e_1^{\otimes} e_2 \rangle, \dots, \langle a_{m_1 1}, e_1^{\otimes} e_2 \rangle, \\ & \langle b_{11}, e'_{11} \cdot (e_1^{\otimes} e_2) \rangle, \dots, \langle b_{n_1 1}, e'_{n_1 1} \cdot (e_1^{\otimes} e_2) \rangle, \\ & \langle b_{12}, e'_{12} \rangle, \dots, \langle b_{n_2 2}, e'_{n_2 2} \rangle \end{aligned} \right\}. \quad (\text{A.8})$$

Now we argue as follows in order to reassemble $e_1 \otimes e_2$ from its action derivatives in $A\hat{\partial}(e)$:

$$\begin{aligned}
e &\equiv e_1 \otimes e_2 && \text{(assumption in this case)} \\
&=_{\text{BBP}} e_1 \cdot (e_1 \otimes e_2) + e_2 && \text{(by axiom (BKS1))} \\
&=_{\text{BBP}} \left(\sum_{i=1}^{m_1} a_{i1} \right) + \left(\sum_{j=1}^{n_1} b_{j1} \cdot e'_{j1} \right) \cdot (e_1 \otimes e_2) + \left(\left(\sum_{i=1}^{m_2} a_{i2} \right) + \left(\sum_{j=1}^{n_2} b_{j2} \cdot e'_{j2} \right) \right) && \text{(by the induction hypothesis, using representation (A.4))} \\
&=_{\text{BBP}} \left(\left(\sum_{i=1}^{m_1} a_{i1} \cdot (e_1 \otimes e_2) \right) + \left(\sum_{j=1}^{n_1} (b_{j1} \cdot e'_{j1}) \cdot (e_1 \otimes e_2) \right) \right) + \left(\left(\sum_{i=1}^{m_2} a_{i2} \right) + \left(\sum_{j=1}^{n_2} b_{j2} \cdot e'_{j2} \right) \right) && \text{(by axiom (B4))} \\
&=_{\text{BBP}} \left(\left(\sum_{i=1}^{m_1} a_{i1} \cdot (e_1 \otimes e_2) \right) + \left(\sum_{j=1}^{n_1} b_{j1} \cdot (e'_{j1} \cdot (e_1 \otimes e_2)) \right) \right) + \left(\left(\sum_{i=1}^{m_2} a_{i2} \right) + \left(\sum_{j=1}^{n_2} b_{j2} \cdot e'_{j2} \right) \right) && \text{(by axiom (B5))} \\
&=_{\text{ACI}} \left(\sum_{i=1}^{m_2} a_{i2} \right) + \left(\left(\sum_{i=1}^{m_1} a_{i1} \cdot (e_1 \otimes e_2) \right) + \left(\sum_{j=1}^{n_1} b_{j1} \cdot (e'_{j1} \cdot (e_1 \otimes e_2)) \right) + \left(\sum_{j=1}^{n_2} b_{j2} \cdot e'_{j2} \right) \right) && \text{(by axioms (B2) and (B1))}
\end{aligned}$$

This chain of provably equalities demonstrates, together with applications of the axiom (B2) that are needed to bring each of the subexpressions of the right outermost summand into a form with association of summation subterms to the left, that e satisfies (A.1) when we construe $A\hat{\partial}(e)$ in (A.8) as a list representation of the form (A.2) with $m = m_2$ and $n = m_1 + n_1 + n_2$.

In each of these five possible cases concerning the outermost structure of e we have successfully performed the induction step. In this way we have proved the statement of the lemma. \square

Proof of Proposition 2.9. Let $C(e) = \langle V(e), \sqrt{\cdot}, e, A, T(e) \rangle$ be the chart interpretation of a star expression $e \in \text{StExp}(A)$.

Let $f \in V(e) \subseteq \text{StExp}(A)$ be a vertex of $C(e)$. By Lemma A.2 every star expression in $\text{StExp}(A)$ can be reassembled as the BBP-provable sum over products of over its action derivatives $\langle a, \xi \rangle$, that is, over all actions $a \in A$ and a -derivatives ξ of e . In particular, (A.1) guarantees that $\text{id}_{V(e)}(f) = f$ satisfies the condition for $\text{id}_{V(e)}$ to be a provable solution at the vertex f of $C(e)$, relative to a representation (A.2) of the action derivatives of f which corresponds to a representation as assumed in Definition 2.8. Since $f \in V(e)$ was arbitrary in this argument, it follows that $\text{id}_{V(e)}$ is a provable solution of $C(e)$. \square

A.2 Proofs in Section 3: Layered loop existence and elimination

Proposition (= Proposition 3.7). *For every $e \in \text{StExp}(A)$, the entry/body-labeling $\widehat{C}(e)$ of $C(e)$ is a LLEE-witness of $C(e)$.*

Proof. To verify (W1) it suffices to show that there are no infinite body step paths from any star expression e (this is also a preparation for (W2)(a), part (L2)). We prove, by induction on the syntactic structure of e , the stronger statement that if $e \rightarrow^+ f$, then there does not exist an infinite body step path from f . The base cases, in which e is of the form a or 0 , are trivial. Suppose $e \equiv e_1 + e_2$. Then $e_i \rightarrow^+ f$ for some $i \in \{1, 2\}$. So by induction, f does not exhibit an infinite body step path. Suppose $e \equiv e_1 \cdot e_2$. Then $e \rightarrow^+ f$ means either $e_1 \rightarrow^+ f_1$ and $f \equiv f_1 \cdot e_2$, or $e_2 \rightarrow^+ f$. In the first case, by induction, f_1 and e_2 do not exhibit infinite body step paths. This induces that $f_1 \cdot e_2$ does not exhibit an infinite body step path. In the second case, by induction, f does not exhibit an infinite body step path. Suppose $e \equiv e_1 \otimes e_2$. Then $e \rightarrow^+ f$ means (A) $f \equiv e_1 \otimes e_2$, or (B) $e_1 \rightarrow^+ f_1$ and $f \equiv f_1 \cdot (e_1 \otimes e_2)$, or (C) $e_2 \rightarrow^+ f$. In case (A), each body step path from f starts with either $f \rightarrow_{\text{bo}} e'_1 \cdot (e_1 \otimes e_2)$ where $e_1 \rightarrow e'_1$ and e'_1 is not normed, or $f \rightarrow_{\text{bo}} e'_2$ where $e_2 \rightarrow e'_2$. In the first case, by induction, e'_1 does not exhibit an infinite body step path, so since e'_1 is not normed, $e'_1 \cdot (e_1 \otimes e_2)$ does not exhibit an infinite body step path. In the second case, by induction, e'_2 does not exhibit an infinite body step path. In case (B), since by induction f_1 and by case (A) $e_1 \otimes e_2$ do not exhibit infinite body step paths, $f_1 \cdot (e_1 \otimes e_2)$ does not exhibit an infinite body step path. In case (C), by induction, f does not exhibit an infinite body step path.

We verify (W2). From the TSS-rules in Definition 2.4 it follows that if e has a loop-entry transition, then $e \equiv ((\dots((e_1 \otimes e_2) \cdot f_1) \dots) \cdot f_n)$ for some $n \geq 0$ and e_1 normed. Let \hat{C} denote the entry/body-labeling defined by the TSS-rules in Definition 2.4 on the 'free' (= start-vertex free) chart of all star expressions in $\text{StExp}(A)$. We prove (W2) for a subchart $C_{\hat{C}}(e, \alpha)$ of \hat{C} . We first consider the case $n = 0$, and then generalize it.

Let $e \equiv e_1 \otimes e_2$ with e_1 normed, and $\alpha = |e_1|_{\otimes} + 1$. Either $e \rightarrow_{[\alpha]} e$ or $e \rightarrow_{[\alpha]} e'_1 \cdot e$ for some normed e'_1 with $e_1 \rightarrow e'_1$. In the first case (L1) is clearly satisfied; we focus on the second case. It can be argued, by induction on syntactic structure, that every normed star expression has a body step path to $\sqrt{}$. Then so does e'_1 . This means $e'_1 \cdot e$ has a body step path to e . Hence (L1) holds. For the remainder of (W2) it suffices to consider loop-entry transitions $e \rightarrow_{[\alpha]} e''_1 \cdot e$ where $e_1 \rightarrow e''_1$. Since we showed above there are no body step cycles, every body step path from e''_1 eventually leads to deadlock or $\sqrt{}$; in the first case the corresponding body step path of $e''_1 \cdot e$ also deadlocks, and in the second case it returns to e . Hence (L2) holds. Since $e''_1 \cdot e$ cannot reach $\sqrt{}$ without returning to e , (L3) holds. It can be shown, by induction on derivation depth, that $f \rightarrow f'$ implies $|f|_{\otimes} \geq |f'|_{\otimes}$, and clearly $f \rightarrow_{[\beta]}$ implies $\beta \leq |f|_{\otimes}$. So if $e''_1 \rightarrow^* \cdot \rightarrow_{[\beta]}$, then $\beta \leq |e''_1|_{\otimes} \leq |e_1|_{\otimes}$. Hence, if $e''_1 \cdot e \xrightarrow{\star}_{\text{bo}} \cdot \rightarrow_{[\beta]}$, then $\beta < |e_1|_{\otimes} + 1 = \alpha$. So (W2)(b) holds.

Now consider $e \equiv ((\dots((e_1 \otimes e_2) \cdot f_1) \dots) \cdot f_n)$ for $n > 0$, with e_1 normed. Again $\alpha = |e_1|_{\otimes} + 1$. The subchart $C_{\hat{C}}(e, \alpha)$ basically coincides with $C_{\hat{C}}(e_1 \otimes e_2, \alpha)$, except that the star expressions in the first chart are post-fixed with f_1, \dots, f_n ; its transitions are derived by n additional applications of the first rule for concatenation in Definition 2.4, to affix these expressions. This chart isomorphism between $C_{\hat{C}}(e_1 \otimes e_2, \alpha)$ and $C_{\hat{C}}(e, \alpha)$ preserves action labels as well as the loop-labeling, because the first rule for concatenation preserves these labels. We showed that $C_{\hat{C}}(e_1 \otimes e_2, \alpha)$ satisfies (W2), so the same holds for $C_{\hat{C}}(e, \alpha)$. \square

We now turn to the proof of Lemma 3.9, which expresses properties of the body transition relation \rightarrow_{bo} , the descends-in-loop-to relation \curvearrowright , the loops-back-to relation \curvearrowright , and the directly-loops-back-to relation ${}_d\curvearrowright$.

Lemma (= Lemma 3.9). *The relations \rightarrow_{bo} , \curvearrowright , \curvearrowright , ${}_d\curvearrowright$ as defined by a LLEE-witness \hat{C} on a chart C satisfy the following properties:*

- (i) \hat{C} does not have infinite \rightarrow_{bo} paths (so no \rightarrow_{bo} cycles).
- (ii) If $\text{scc}(u) = \text{scc}(v)$, then $u \curvearrowright^* v$ implies $v \curvearrowright^* u$.
- (iii) If $v \curvearrowright w$ and $\neg(w \curvearrowright)$, then w is not normed.
- (iv) $\text{scc}(u) = \text{scc}(v)$ if and only if $u \curvearrowright^* w$ and $v \curvearrowright^* w$ for some vertex w .
- (v) \curvearrowright^* is a partial order that has the least-upper-bound property: if a nonempty set of vertices has an upper bound with respect to \curvearrowright^* , then it has a least upper bound.
- (vi) If $v_1 {}_d\curvearrowright u$ and $v_2 {}_d\curvearrowright u$ for distinct v_1, v_2 , then there is no vertex w such that both $w \curvearrowright^* v_1$ and $w \curvearrowright^* v_2$.

We split the proof into the arguments for the parts (i)–(vi), respectively. In doing so we repeat these statements as individual lemmas, and add a few more on the way.

Lemma A.3. *In a chart with a LLEE-witness, if $v \curvearrowright^{\alpha} \cdot \curvearrowright^* \cdot \rightarrow_{[\beta]}$, then $\alpha > \beta$.*

Proof. By induction on the number n of \curvearrowright -steps in a path $v \curvearrowright^{\alpha} \cdot \curvearrowright^n \cdot \rightarrow_{[\beta]}$. If $n = 0$, then from $v \curvearrowright^{\alpha} \cdot \rightarrow_{[\beta]}$ we get $\alpha > \beta$ by means of the LLEE-witness condition (W2)(b). If $n > 0$, then the path $v \curvearrowright^{\alpha} \cdot \curvearrowright^n \cdot \rightarrow_{[\beta]}$ is of the form $v \curvearrowright^{\alpha} \cdot \curvearrowright^{n-1} \cdot \gamma \curvearrowright \cdot \rightarrow_{[\beta]}$ for some loop name γ . This path contains an initial segment $v \curvearrowright^{\alpha} \cdot \curvearrowright^{n-1} \cdot \rightarrow_{[\gamma]}$. Then $\alpha > \gamma$ follows by the induction hypothesis. From the part $\gamma \curvearrowright \cdot \rightarrow_{[\beta]}$ of this path we get $\gamma > \beta$ by LLEE-witness condition (W2)(b). So we conclude that $\alpha > \beta$ holds. \square

Lemma A.4. *In a chart with a LLEE-witness, if $v \curvearrowright^+ w$, then $w \neq \sqrt{}$.*

Proof. Let \hat{C} be a LLEE-witness of a chart C . It suffices to show that $\curvearrowright w$ implies $w \neq \sqrt{}$. For this, we let v and w be vertices such that $v \curvearrowright w$. Then we can pick $\alpha \in \mathbb{N}^+$ such that $v \curvearrowright^{\alpha} w$. Since this means $v \xrightarrow{\star}_{\text{bo}} \xrightarrow{[\alpha]} \xrightarrow{\star}_{\text{bo}} w$, it follows that $w \in C_{\hat{C}}(v, \alpha)$. Now since $C_{\hat{C}}(v, \alpha)$ is a loop chart by condition (W2)(a) for the LLEE-witness \hat{C} , it follows that $w \neq \sqrt{}$. \square

Lemma A.5. *In a chart with a LLEE-witness (assumed to be start-vertex connected, see Definition 2.2), every vertex is reachable by an acyclic $\rightarrow_{\text{bo}}^* \cdot \curvearrowright^*$ path from the start vertex v_s , that is, $v_s \rightarrow_{\text{bo}}^* \cdot \curvearrowright^* w$ holds for all vertices w .*

Proof. Let π be a path from v_s to w . By removing cycles from π we obtain an acyclic path π' from v_s to w that consists of a sequence of loop-entry and body transitions. Hence π' is of the form $v_s \rightarrow_{\text{bo}}^* w$ or $v_s \rightarrow_{\text{bo}}^* u_0 \xrightarrow{\star}_{\text{bo}} \xrightarrow{[\alpha_0]} \xrightarrow{\star}_{\text{bo}} u_1 \xrightarrow{\star}_{\text{bo}} \dots \xrightarrow{[\alpha_n]} \xrightarrow{\star}_{\text{bo}} u_n \equiv w$ for some $n \in \mathbb{N}$, and $\alpha_0, \dots, \alpha_n \in \mathbb{N}^+$, where the target-avoidance parts are due to acyclicity of π' . Hence π' is of the form $v_s \rightarrow_{\text{bo}}^* u_0 \curvearrowright^{\alpha_0} \cdot \alpha_1 \curvearrowright \dots \alpha_{n-2} \curvearrowright \cdot \alpha_{n-1} \curvearrowright w$, for some $n \in \mathbb{N}$, and $\alpha_0, \dots, \alpha_n \in \mathbb{N}^+$, and therefore of the form $v_s \rightarrow_{\text{bo}}^* \cdot \curvearrowright^* w$. \square

Lemma A.6. *In a chart with a LLEE-witness, for every path $v \xrightarrow{\star}_{\text{bo}} \xrightarrow{[\alpha]} \xrightarrow{\star}_{\text{bo}} w$ there is an acyclic path $v \curvearrowright^{\alpha} \cdot \curvearrowright^* w$.*

Proof. Let π be a path from v to w that starts with a loop-entry step with loop name α such that all targets of transitions in π avoid v . By removing cycles we obtain an acyclic path π' from v to w that starts with an α -loop-entry step whose target is not v . We can write π' as a sequence of loop-entry and body steps of the form $v \xrightarrow{\alpha} [\alpha] \cdot \xrightarrow{*}_{\text{bo}} u_1 \xrightarrow{[u_0]} [\alpha_0] \cdot \xrightarrow{*}_{\text{bo}} u_2 \xrightarrow{[u_1]} [\alpha_1] \cdot \xrightarrow{*}_{\text{bo}} u_3 \xrightarrow{[u_2]} [\alpha_2] \cdot \xrightarrow{*}_{\text{bo}} \dots \xrightarrow{[u_{n-2}]} [\alpha_{n-2}] \cdot \xrightarrow{*}_{\text{bo}} u_{n-1} \xrightarrow{[u_{n-1}]} [\alpha_{n-1}] \cdot \xrightarrow{*}_{\text{bo}} w$ for some $n \geq 1$, where the target-avoidance parts are due to acyclicity of π' . Hence π' is of the form $v \xrightarrow{\alpha} \cdot \xrightarrow{*}_{\text{bo}} w$, and therefore of the form $v \xrightarrow{\alpha} \cdot \xrightarrow{*}_{\text{bo}} w$. \square

The following lemma was also used implicitly in the proof of Lem. 3.9, (v).

Lemma A.7. *In a chart with a LLEE-witness, if $v \xrightarrow{[\alpha]} \cdot \xrightarrow{*}_{\text{bo}} \cdot \rightarrow [\beta]$, then $\alpha > \beta$.*

Proof. This is a direct consequence of Lem. A.6 and Lem. A.3. \square

Lemma A.8. *In a chart with a LLEE-witness, if $u \xrightarrow{*}_{\text{bo}} v \xrightarrow{*}_{\text{bo}} w$, then each path $u \rightarrow^*_{\text{bo}} w$ visits v .*

Proof. Let $v \neq u, w$, as else the lemma trivially holds. Since $u \xrightarrow{*}_{\text{bo}} v \xrightarrow{*}_{\text{bo}} w$, there is a path $w \xrightarrow{[\alpha]} \cdot \xrightarrow{*}_{\text{bo}} v \xrightarrow{[\beta]} \cdot \xrightarrow{*}_{\text{bo}} u$. By layeredness, $\alpha > \beta$. A path $u \xrightarrow{*}_{\text{bo}} w$ would yield $v \xrightarrow{[\beta]} \cdot \xrightarrow{*}_{\text{bo}} u \xrightarrow{*}_{\text{bo}} w \rightarrow [\alpha]$. Then layeredness would require $\beta > \alpha$, which cannot be the case. \square

Lemma (= Lemma 3.9, (i)). *In a chart with a LLEE-witness, there are no infinite \rightarrow_{bo} paths (so no \rightarrow_{bo} cycles).*

Proof. Let C be a chart with LLEE-witness \hat{C} , and with start vertex v_s . Due to Lemma A.5 every vertex of v is reachable by a $\rightarrow^*_{\text{bo}} \cdot \xrightarrow{*}_{\text{bo}}$ path from v_s . In order to show that there are no infinite \rightarrow_{bo} paths in \hat{C} it therefore suffices to show that if $v_s \rightarrow^*_{\text{bo}} \cdot \xrightarrow{*}_{\text{bo}} v$, then there is no infinite \rightarrow_{bo} path from v .

For the base case, $n = 0$, let w be such that $v_s \rightarrow^*_{\text{bo}} w$. Now suppose that there is an infinite $\rightarrow^*_{\text{bo}}$ path from w in \hat{C} . Then due to $v_s \rightarrow^*_{\text{bo}} w$ it follows that there is also an infinite $\rightarrow^*_{\text{bo}}$ path from v_s in \hat{C} . This, however, contradicts with the condition (W1) that the LLEE-witness \hat{C} must satisfy. We conclude that there is no infinite $\rightarrow^*_{\text{bo}}$ path from w in \hat{C} .

For performing the induction step from n to $n + 1$, let w be such that $v_s \rightarrow^*_{\text{bo}} \cdot \xrightarrow{*}_{\text{bo}} w$. Then we can pick w_0 with $v_s \rightarrow^*_{\text{bo}} \cdot \xrightarrow{*}_{\text{bo}} w_0 \xrightarrow{*}_{\text{bo}} w$. It follows that $w_0 \xrightarrow{[\alpha]} \cdot \xrightarrow{*}_{\text{bo}} w$ for some $\alpha \in \mathbb{N}^+$, which we pick accordingly. Now suppose that there is an infinite $\rightarrow^*_{\text{bo}}$ path π from w in \hat{C} . Then it cannot be the case that π avoids w_0 forever, because otherwise it would give rise to an infinite path $w_0 \xrightarrow{[\alpha]} \cdot \xrightarrow{*}_{\text{bo}} w \xrightarrow{[\alpha]} \cdot \xrightarrow{*}_{\text{bo}} w_1 \xrightarrow{[\alpha]} \cdot \xrightarrow{*}_{\text{bo}} w_2 \xrightarrow{[\alpha]} \cdot \xrightarrow{*}_{\text{bo}} \dots$, which is not possible since the condition (W2)(a) for the LLEE-witness C implies that $C_{\hat{C}}(u_0, \alpha)$ is a loop chart. Therefore it follows that π must visit v_0 . But then π also gives rise to an infinite $\rightarrow^*_{\text{bo}}$ path from w_0 . This, however, contradicts the the statement that the induction hypothesis guarantees for w_0 due to $v_s \rightarrow^*_{\text{bo}} \cdot \xrightarrow{*}_{\text{bo}} w_0$, namely that there is no infinite $\rightarrow^*_{\text{bo}}$ path from w_0 . We have reached a contradiction. Therefore we can conclude that there is no infinite $\rightarrow^*_{\text{bo}}$ path π from w in \hat{C} . In this way we have successfully performed the induction step. \square

Lemma (= Lemma 3.9, (ii)). *In a chart with a LLEE-witness, if $\text{scc}(u) = \text{scc}(v)$, then $u \xrightarrow{*}_{\text{bo}} v$ implies $v \xrightarrow{*}_{\text{bo}} u$.*

Proof. We prove that $u \xrightarrow{*}_{\text{bo}} v$ implies $v \xrightarrow{*}_{\text{bo}} u$ for all $n \geq 0$, by induction on n . The base case $n = 0$ is trivial, as then $u = v$. If $n > 0$, $u \xrightarrow{*}_{\text{bo}} u' \xrightarrow{*}_{\text{bo}} v$ for some u' . Clearly $\text{scc}(u) = \text{scc}(u') = \text{scc}(v)$. By induction, $u' \xrightarrow{*}_{\text{bo}} u$. Since $u' \xrightarrow{*}_{\text{bo}} v$, there is an acyclic path $u' \rightarrow [\alpha] \cdot \xrightarrow{*}_{\text{bo}} v$. And since $\text{scc}(u') = \text{scc}(v)$, there is an acyclic path $v \rightarrow [\beta_1] \cdot \xrightarrow{*}_{\text{bo}} \dots \rightarrow [\beta_k] \cdot \xrightarrow{*}_{\text{bo}} u'$. By (W2)(b), $\alpha > \beta_1 > \dots > \beta_k > \alpha$. This means $k = 0$, so $v \rightarrow [\beta] \cdot \xrightarrow{*}_{\text{bo}} u'$. This implies $v \xrightarrow{*}_{\text{bo}} u'$ and hence $v \xrightarrow{*}_{\text{bo}} u$. \square

Lemma (= Lemma 3.9, (iii)). *If, in a chart with a LLEE-witness, $\neg(w \xrightarrow{*}_{\text{bo}})$ and $\neg(w \xrightarrow{*}_{\text{bo}})$, then w is not normed.*

Proof. We argue indirectly by showing that the negation of the implication in the statement of the lemma leads to a contradiction. For this, suppose that $v \xrightarrow{*}_{\text{bo}} w$ and $\neg(w \xrightarrow{*}_{\text{bo}})$ hold for some vertices v and w , and that additionally w is normed. From $v \xrightarrow{*}_{\text{bo}} w$ and $\neg(w \xrightarrow{*}_{\text{bo}})$ we obtain by Lem. 3.9, (ii) that $w \notin \text{scc}(v)$. Since $v \xrightarrow{*}_{\text{bo}} w$ entails $v \rightarrow^*_{\text{bo}} w$ this entails $\neg(w \rightarrow^*_{\text{bo}} v)$. Now since that w is normed means $w \rightarrow^*_{\text{bo}} \checkmark$, we obtain $v \xrightarrow{*}_{\text{bo}} w \rightarrow^*_{\text{bo}} \checkmark$, which means $v \xrightarrow{[\alpha]} \cdot \xrightarrow{*}_{\text{bo}} w \rightarrow^*_{\text{bo}} \checkmark$ for some $\alpha \in \mathbb{N}^+$. Then it follows from Lemma A.6 that $v \xrightarrow{*}_{\text{bo}} \checkmark$. This, however, contradicts, Lemma A.4. \square

Lemma (= Lemma 3.9, (iv)). *In a chart with a LLEE-witness, $\text{scc}(u) = \text{scc}(v)$ if and only if $u \xrightarrow{*}_{\text{bo}} w$ and $v \xrightarrow{*}_{\text{bo}} w$ for some vertex w .*

Proof. The direction from right to left of the lemma trivially holds; we focus on the direction from left to right. Let $\text{scc}(u) = \text{scc}(v)$. The case $u = v$ is trivial. Let $u \neq v$. Then they are on a cycle, which, since there is no body step cycle, contains a loop-entry transition from some w . Without loss of generality, suppose $w \neq u$. Then $w \rhd^+ u$, so by Lemma 3.9, (ii), $u \rhd^+ w$. If $w = v$ we have $v \rhd^* w$, and if $w \neq v$ we can argue in the same fashion that $v \rhd^+ w$. \square

Lemma A.9. *In a chart with a LLEE-witness, \rhd^+ is irreflexive.*

Proof. Let \hat{C} be a LLEE-witness of a LLEE-chart C . Suppose that $w \rhd^+ w$ holds for some vertex w of C and \hat{C} . Then it follows from the definition of \rhd^+ that there is a \rightarrow_{bo} path of non-zero length from w to w itself. But such a \rightarrow_{bo} cycle in \hat{C} is not possible, as it would give rise to an infinite \rightarrow_{bo} path in \hat{C} , contradicting Lemma 3.9, (i). \square

Lemma A.10. *In a chart with a LLEE-witness, \rhd^* is a partial order.*

Proof. By definition, \rhd is transitive-reflexive. Moreover, \rhd is anti-symmetric, because $u \rhd^+ v$ and $v \rhd^+ u$ for $u \neq v$ would imply $u \rhd^+ v$ and $v \rhd^+ u$, in contradiction with irreflexivity of \rhd^+ , see Lemma A.9. \square

Lemma (= Lemma 3.9, (v)). *In a chart with a LLEE-witness, \rhd^* is a partial order that has the least-upper-bound property: if a nonempty set of vertices has an upper bound with respect to \rhd^* , then it has a least upper bound.*

Proof. Let C be a chart with a LLEE-witness \hat{C} . Let the relation \rhd be defined on C according to \hat{C} .

\rhd^* is a partial order by Lemma A.10. Since C as a chart is finite, it suffices to show that for each vertex v the set of vertices x with $v \rhd^* x$ is totally ordered with regard to \rhd^* . Let $v \rhd^+ u_1$ and $v \rhd^+ u_2$ with $u_1 \neq u_2$. There is a path $u_1 \xrightarrow{+} [\alpha] \cdot \xrightarrow{+}^* v \xrightarrow{+}_{\text{bo}} u_2 \xrightarrow{+} [\beta] \cdot \xrightarrow{+}^* v \xrightarrow{+}_{\text{bo}} u_1$. Without loss of generality, suppose $\beta \geq \alpha$. Then layeredness implies that each path $v \xrightarrow{+}_{\text{bo}} u_2$ must visit u_1 , so $v \xrightarrow{+}_{\text{bo}} u_1 \xrightarrow{+}_{\text{bo}} u_2$. Hence there is a path $u_2 \xrightarrow{+} [\beta] \cdot \xrightarrow{+}^* v \xrightarrow{+}_{\text{bo}} u_1 \xrightarrow{+}_{\text{bo}} u_2$, which implies $u_1 \rhd^+ u_2$. \square

Lemma (= Lemma 3.9, (vi)). *In a chart with a LLEE-witness, if $v_1 \rhd_d u$ and $v_2 \rhd_d u$ for distinct v_1, v_2 , then there is no vertex w such that both $w \rhd^* v_1$ and $w \rhd^* v_2$.*

Proof. $\neg(v_2 \rhd^+ v_1)$ and $\neg(v_1 \rhd^+ v_2)$, for else the definition of \rhd_d would imply $u \rhd^* v_1$ or $u \rhd^* v_2$, and so $v_1 \rhd^+ v_1$ or $v_2 \rhd^+ v_2$, contradicting irreflexivity of \rhd^+ , see Lemma A.9. In the proof of Lemma 3.9, (v), we furthermore saw that for each w , $\{x \mid w \rhd^* x\}$ is totally ordered with regard to \rhd^* , which implies that any such sets cannot contain both v_1 and v_2 . \square

A.3 Proofs in Section 5: Extraction of star expressions from, and transferral between, LLEE-charts

Proposition (= Proposition 5.1). *Let $\phi : V_1 \rightarrow V_2$ be a functional bisimulation between charts C_1 and C_2 . Let $s_2 : V_2 \setminus \{\sqrt{\cdot}\} \rightarrow \text{StExp}(A)$ be a provable solution of C_2 . Then $s_2 \circ \phi : V_1 \setminus \{\sqrt{\cdot}\} \rightarrow \text{StExp}(A)$ is a provable solution of C_1 with the same principal value as s_2 .*

Proof. Let s_2 be a provable solution of C_2 . Let $v \in V_1 \setminus \{\sqrt{\cdot}\}$. Since ϕ is a functional bisimulation between C_1 and C_2 , the forth, back, and termination conditions for the graph of ϕ as a bisimulation hold for the pair $\langle v, \phi(v) \rangle$ of vertices. This makes it possible to bring the sets of transitions $T_1(v)$ from v in C_1 , and $T_2(\phi(v))$ from $\phi(v)$ in C_2 into a 1–1 correspondence such that ϕ again relates their targets:

$$T_1(v) = \{v \xrightarrow{a_i} \sqrt{\cdot} \mid i = 1, \dots, m\} \cup \{v \xrightarrow{b_j} v'_{j1} \mid j = 1, \dots, n\}, \quad (\text{A.9})$$

$$T_2(\phi(v)) = \{\phi(v) \xrightarrow{a_i} \sqrt{\cdot} \mid i = 1, \dots, m\} \cup \{\phi(v) \xrightarrow{b_j} v'_{j2} \mid j = 1, \dots, n\}, \quad (\text{A.10})$$

$$\phi(v'_{j1}) = v'_{j2}, \quad \text{for all } j \in \{1, \dots, n\}, \quad (\text{A.11})$$

with $n, m \in \mathbb{N}$, and vertices $v'_{j1} \in V_1 \setminus \{\sqrt{\cdot}\}$, and $v'_{j2} \in V_2 \setminus \{\sqrt{\cdot}\}$, for $j \in \{1, \dots, n\}$. Note that the same transition may be listed multiple times in the set $T_2(\phi(v))$. On this basis we can argue as follows.

$$(s_2 \circ \phi)(v) \equiv s_2(\phi(v)) \stackrel{\text{BBP}}{=} \left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot s_2(v'_{j2}) \right)$$

(since s_2 is a provable solution of C_2 , using (A.10) and axioms (B1), (B2), (B3))

$$\equiv \left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot (s_2 \circ \phi)(v'_{j1}) \right)$$

$$\text{(using (A.11) and } (s_2 \circ \phi)(v'_{j1}) \equiv s_2(\phi(v'_{j1})) \text{)}$$

This shows, in view of (A.9), that $s \circ \phi$ satisfies the condition for a provable solution at v . Now as $v \in V_1 \setminus \{\sqrt{\cdot}\}$ was arbitrary, $s_2 \circ \phi$ (with domain $V_1 \setminus \{\sqrt{\cdot}\}$) is a provable solution of C_1 . Since furthermore the functional bisimulation ϕ must relate the start vertices of C_1 and C_2 , the principal value of $s_2 \circ \phi$ coincides with that of s_2 . \square

Lemma (= Lemma 5.2). *In a chart with a LLEE-witness, for all vertices v, w :*

- (i) $v \rightarrow_{\text{bo}} w \Rightarrow \|v\|_{\text{bo}} > \|w\|_{\text{bo}}$,
- (ii) $v \curvearrowright w \Rightarrow |v|_{\text{en}} > |w|_{\text{en}}$.

Proof. For statement (i) we argue as follows. Recall that the body step norm $\|v\|_{\text{bo}}$ in a LLEE-witness \hat{C} was defined as the maximal length of a body step path from v in \hat{C} . This was well-defined due to Lemma 3.9, (i), and the finiteness of charts. Now suppose that $v \rightarrow_{\text{bo}} w$. Then every body step path from w gives rise to a body step path from v that starts with the transition $v \rightarrow_{\text{bo}} w$. Hence a longest body step path from w of length $\|w\|_{\text{bo}}$ gives rise to a body step path from v of length $\|w\|_{\text{bo}} + 1$. It follows that $\|v\|_{\text{bo}} \geq \|w\|_{\text{bo}} + 1 > \|w\|_{\text{bo}}$, and hence $\|v\|_{\text{bo}} > \|w\|_{\text{bo}}$.

For showing statement (ii), suppose that $v \curvearrowright w$. Then $v \xrightarrow{\alpha} w$ holds for some $\alpha \in \mathbb{N}^+$. Then $|v|_{\text{en}} \geq \alpha$. If there is no loop-entry transition that departs from w , then $|w|_{\text{en}} = 0$ holds, and hence we get $|v|_{\text{en}} \geq \alpha > 0 = |w|_{\text{en}}$. Otherwise we let $\beta \in \mathbb{N}^+$ be the maximal index of a loop-entry transition from w . Then $v \xrightarrow{\alpha} w \rightarrow_{[\beta]}$. By Lemma A.3 it follows that $\alpha > \beta$. Consequently we find $|v|_{\text{en}} \geq \alpha > \beta = |w|_{\text{en}}$. In both cases we have shown $|v|_{\text{en}} > |w|_{\text{en}}$. \square

Lemma (= Lemma 5.4). *For a LLEE-chart C with LLEE-witness \hat{C} the following connection holds between the extracted solution $s_{\hat{C}}$ and the relative extracted solution $t_{\hat{C}}$, for all vertices v, w :*

$$v \curvearrowright w \implies s_{\hat{C}}(w) =_{\text{BBP}} t_{\hat{C}}(w, v) \cdot s_{\hat{C}}(v). \quad (\text{A.12})$$

Note that if $v \curvearrowright w$, then $v \neq \sqrt{\cdot}$, and also $w \neq \sqrt{\cdot}$, because w is in the body of a loop at v , and therefore cannot be $\sqrt{\cdot}$ (see Lem. A.4).

Proof. In order to show (A.12) we proceed by complete induction (without explicit treatment of the base case) on the length $\|w\|_{\text{bo}}$ of a longest body step path from w . For performing the induction step, we consider arbitrary $v, w \neq \sqrt{\cdot}$ with $v \curvearrowright w$. We assume a representation of the set $\hat{T}(w)$ of transitions from w in \hat{C} :

$$\begin{aligned} \hat{T}(w) = & \{w \xrightarrow{a_i}_{[\alpha_i]} w \mid i = 1, \dots, m\} \cup \{w \xrightarrow{b_j}_{[\beta_j]} w_j \mid w_j \neq w, j = 1, \dots, n\} \\ & \cup \{w \xrightarrow{c_i}_{\text{bo}} v \mid i = 1, \dots, p\} \cup \{w \xrightarrow{d_j}_{\text{bo}} u_j \mid u_j \neq v, j = 1, \dots, q\} \end{aligned} \quad (\text{A.13})$$

that partitions $\hat{T}(w)$ into loop-entry transitions to w and to other targets w_1, \dots, w_n , and body transitions to v and to other targets u_1, \dots, u_q . Since w is contained in a loop at v , none of these targets can be $\sqrt{\cdot}$. In order to show provable equality at the right-hand side of (A.12), we argue as follows:

$$\begin{aligned} s_{\hat{C}}(w) & \equiv \left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right)^{\otimes} \left(0 + \left(\sum_{i=1}^p c_i \cdot s_{\hat{C}}(v) \right) + \left(\sum_{j=1}^q d_j \cdot s_{\hat{C}}(u_j) \right) \right) \\ & \quad \text{(by the definition of } s_{\hat{C}}(w), \text{ based on the representation (A.13),} \\ & \quad \text{using that none of the target vertices is } \sqrt{\cdot}) \\ & =_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right)^{\otimes} \left(\left(\sum_{i=1}^p c_i \cdot s_{\hat{C}}(v) \right) + \left(\sum_{j=1}^q d_j \cdot s_{\hat{C}}(u_j) \right) \right) \\ & \quad \text{(using axiom (B6))} \\ & =_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right)^{\otimes} \left(\left(\sum_{i=1}^p c_i \cdot s_{\hat{C}}(v) \right) + \left(\sum_{j=1}^q d_j \cdot (t_{\hat{C}}(u_j, v) \cdot s_{\hat{C}}(v)) \right) \right) \\ & \quad \text{(by the induction hypothesis, using that } v \curvearrowright u_j \text{ and } \|u_j\|_{\text{bo}} < \|w\|_{\text{bo}} \\ & \quad \text{because } w \rightarrow_{\text{bo}} u_j \text{ for } j = 1, \dots, q, \text{ see (A.13))} \\ & =_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right)^{\otimes} \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot t_{\hat{C}}(u_j, v) \right) \right) \cdot s_{\hat{C}}(v) \\ & \quad \text{(using axioms (B5), (B4))} \end{aligned}$$

$$\begin{aligned}
&=_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right)^{\otimes} \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot t_{\hat{C}}(u_j, v) \right) \right) \cdot s_{\hat{C}}(v) \\
&\quad \text{(using axiom (BKS2))} \\
&\equiv t_{\hat{C}}(w, v) \cdot s_{\hat{C}}(v) \\
&\quad \text{(by the definition of } t_{\hat{C}}(w, v), \text{ based on the representation (A.13))}
\end{aligned}$$

This chain of provable equalities demonstrates (A.12). \square

Proposition (= Proposition 5.5). *In a chart C with a LLEE-witness \hat{C} , $s_{\hat{C}}$ is a provable solution of C .*

Proof. We prove that $s_{\hat{C}}$ is a provable solution of the chart C . Let $w \neq \sqrt{}$. We show that $s_{\hat{C}}(w)$ satisfies the defining equation of $s_{\hat{C}}$ to be a provable solution of C at w .

We consider a representation of the set $\hat{T}(w)$ of transitions from w in \hat{C} as follows:

$$\begin{aligned}
\hat{T}(w) = & \{ w \xrightarrow{a_i}_{[\alpha_i]} w \mid i = 1, \dots, m \} \cup \{ w \xrightarrow{b_j}_{[\beta_j]} w_j \mid w_j \neq w, j = 1, \dots, n \} \\
& \cup \{ w \xrightarrow{c_i}_{\text{bo}} \sqrt{} \mid i = 1, \dots, p \} \cup \{ w \xrightarrow{d_j}_{\text{bo}} u_j \mid u_j \neq \sqrt{}, j = 1, \dots, q \}
\end{aligned} \tag{A.14}$$

that partitions $\hat{T}(w)$ into loop-entry transitions to w and to other targets w_1, \dots, w_n , and body transitions to $\sqrt{}$ and to other targets u_1, \dots, u_q . We argue as follows:

$$\begin{aligned}
s_{\hat{C}}(w) &\equiv \left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right)^{\otimes} \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot s_{\hat{C}}(u_j) \right) \right) \\
&\quad \text{(by the definition of } s_{\hat{C}}, \text{ in view of (A.14))} \\
&=_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right) \cdot s_{\hat{C}}(w) + \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot s_{\hat{C}}(u_j) \right) \right) \\
&\quad \text{(using axiom (BKS1) and the defining equality in the first step)} \\
&=_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \cdot s_{\hat{C}}(w) \right) + \left(\sum_{j=1}^n b_j \cdot (t_{\hat{C}}(w_j, w) \cdot s_{\hat{C}}(w)) \right) \right) + \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot s_{\hat{C}}(u_j) \right) \right) \\
&\quad \text{(using axioms (B5), (B4))} \\
&=_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \cdot s_{\hat{C}}(w) \right) + \left(\sum_{j=1}^n b_j \cdot s_{\hat{C}}(w_j) \right) \right) + \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot s_{\hat{C}}(u_j) \right) \right) \\
&\quad \text{(using (A.12), in view of } w \curvearrowright w_j \text{ for } j = 1, \dots, n) \\
&=_{\text{BBP}} \left(\sum_{i=1}^p c_i \right) + \left(\left(\sum_{i=1}^m a_i \cdot s_{\hat{C}}(w) \right) + \left(\sum_{j=1}^n b_j \cdot s_{\hat{C}}(w_j) \right) \right) + \left(\sum_{j=1}^q d_j \cdot s_{\hat{C}}(u_j) \right) \\
&\quad \text{(using axioms (B1), (B2))}
\end{aligned}$$

This chain of provable equalities demonstrates that $s_{\hat{C}}(w)$ is a provable solution of C at w , in view of (A.14). As $w \neq \sqrt{}$ is arbitrary, $s_{\hat{C}}$ is indeed a provable solution of C . \square

Lemma (= Lemma 5.7). *For every provable solution s of a chart C with LLEE-witness \hat{C} , the following connection holds with the relative extraction function $t_{\hat{C}}$ holds, for all vertices v, w :*

$$v \curvearrowright w \implies s(w) =_{\text{BBP}} t_{\hat{C}}(w, v) \cdot s(v) \tag{A.15}$$

Note that if $v \curvearrowright w$, then $v \neq \sqrt{}$, and also $w \neq \sqrt{}$, because w is in the body of a loop at v , and therefore cannot be $\sqrt{}$.

Proof. In order to prove (A.15) we proceed by complete induction on the same measure as used in the definition of the relative extraction function $t_{\hat{C}}$, namely, induction on the maximal loop level of a loop at v , with a subinduction on $\|w\|_{\text{bo}}$. For performing the induction step, consider vertices v, w with $v \curvearrowright w$. As in the proof of Prop. 5.5 we assume the representation (A.13) of the set $\hat{T}(w)$ of transitions from w in \hat{C} , which partitions $\hat{T}(w)$ into loop-entry transitions to w and to other targets w_1, \dots, w_n ,

and body transitions to v and to other targets u_1, \dots, u_q . Since w is contained in a loop at v , none of these targets can be $\sqrt{\cdot}$. We now argue as follows:

$$\begin{aligned}
s(w) &=_{\text{BBP}} 0 + \left(\left(\sum_{i=1}^m a_i \cdot s(w) \right) + \left(\left(\sum_{j=1}^n b_j \cdot s(w_j) \right) + \left(\sum_{i=1}^p c_i \cdot s(v) \right) + \left(\sum_{j=1}^q d_j \cdot s(u_j) \right) \right) \right) \\
&\quad (\text{since } s \text{ is a provable solution of } C \text{ at } w, \text{ using (A.13)}) \\
&=_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \cdot s(w) \right) + \left(\sum_{j=1}^n b_j \cdot s(w_j) \right) \right) + \left(\left(\sum_{i=1}^p c_i \cdot s(v) \right) + \left(\sum_{j=1}^q d_j \cdot s(u_j) \right) \right) \\
&\quad (\text{using axioms (B6), (B2)}) \\
&=_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \cdot s(w) \right) + \left(\sum_{j=1}^n b_j \cdot (t_{\hat{C}}(w_j, w) \cdot s(w)) \right) \right) + \left(\left(\sum_{i=1}^p c_i \cdot s(v) \right) + \left(\sum_{j=1}^q d_j \cdot (t_{\hat{C}}(u_j, v) \cdot s(v)) \right) \right) \\
&\quad (\text{using the induction hypothesis, which is applicable because} \\
&\quad \text{the maximal loop level at } w \text{ is smaller than that at } v \text{ due to } v \curvearrowright w, \text{ and} \\
&\quad v \curvearrowright u_i \text{ and } \|u_j\|_{\text{bo}} < \|w\|_{\text{bo}} \text{ due to } w \rightarrow_{\text{bo}} u_j \text{ for } j = 1, \dots, q, \text{ see (A.13)}) \\
&=_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right) \cdot s(w) + \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot t_{\hat{C}}(u_j, v) \right) \right) \cdot s(v) \\
&\quad (\text{using axioms (B5), (B4)})
\end{aligned}$$

This chain of provable equalities justifies:

$$s(w) =_{\text{BBP}} \left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right) \cdot s(w) + \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot t_{\hat{C}}(u_j, v) \right) \right) \cdot s(v)$$

To this equality we can apply the rule RSP^{\otimes} :

$$\begin{aligned}
s(w) &=_{\text{BBP}} \left(\left(\left(\sum_{i=1}^m a_i \right) + \left(\sum_{j=1}^n b_j \cdot t_{\hat{C}}(w_j, w) \right) \right)^{\otimes} \left(\left(\sum_{i=1}^p c_i \right) + \left(\sum_{j=1}^q d_j \cdot t_{\hat{C}}(u_j, v) \right) \right) \right) \cdot s(v) \\
&\quad (\text{by applying rule } \text{RSP}^{\otimes}) \\
&\equiv t_{\hat{C}}(w, v) \cdot s(v),
\end{aligned}$$

The last step uses the definition of $t_{\hat{C}}(w, v)$, based on representation (A.13) of $\hat{T}(w)$. In this way we have carried out the induction step. We conclude that (A.15) holds for all vertices v and w of C . \square

A.4 Proofs in Section 6: Preservation of LLEE under collapse

Lemma (= Lemma 6.2). *If $w_1 \Leftrightarrow w_2$ in C , then $C_{w_2}^{(w_1)} \Leftrightarrow C$.*

Proof. Let $C = \langle V_1, \sqrt{\cdot}, v_{s,1}, T_1 \rangle$ and $C_{w_2}^{(w_1)} = \langle V_2, \sqrt{\cdot}, v_{s,2}, T_2 \rangle$. Let $B_1 \subseteq V_1 \times V_1$ be the largest bisimulation relation on C . In particular, $\langle w_1, w_2 \rangle \in B_1$. We argue that $B_2 = B_1 \cap (V_1 \times V_2)$ is a bisimulation relation between C and $C_{w_2}^{(w_1)}$. Take any $\langle u, v \rangle \in B_2 \subseteq B_1$.

- (*forth*): Let $u \xrightarrow{a} u' \in T_1$. Then $\langle u, v \rangle \in B_1$ implies there is a $v \xrightarrow{a} v' \in T_1$ with $\langle u', v' \rangle \in B_1$. If $v \xrightarrow{a} v' \in T_2$, then $v' \in V_2$, so $\langle u', v' \rangle \in B_2$ and we are done. If $v \xrightarrow{a} v' \notin T_2$, then $v' = w_1$ and $v \xrightarrow{a} w_2 \in T_2$. Since $\langle u', w_1 \rangle \in B_1$ and $\langle w_1, w_2 \rangle \in B_1$, also $\langle u', w_2 \rangle \in B_1$. Since $w_2 \in V_2$, it follows that $\langle u', w_2 \rangle \in B_2$.
- (*back*): Let $v \xrightarrow{a} v' \in T_2$. If $v \xrightarrow{a} v' \in T_1$, then $\langle u, v \rangle \in B_1$ implies there is a $u \xrightarrow{a} u' \in T_1$ with $\langle u', v' \rangle \in B_1$. Since $v' \in V_2$, also $\langle u', v' \rangle \in B_2$ and we are done. If $v \xrightarrow{a} v' \notin T_1$, then $v' = w_2$ and $v \xrightarrow{a} w_1 \in T_1$. So $\langle u, v \rangle \in B_1$ implies there is a $u \xrightarrow{a} u' \in T_1$ with $\langle u', w_1 \rangle \in B_1$. Since $\langle u', w_1 \rangle \in B_1$ and $\langle w_1, w_2 \rangle \in B_1$, also $\langle u', w_2 \rangle \in B_1$. Since $w_2 \in V_2$, it follows that $\langle u', w_2 \rangle \in B_2$.
- (*termination*): Since $B_2 \subseteq B_1$ clearly $u = \sqrt{\cdot}$ if and only if $v = \sqrt{\cdot}$.

Finally, concerning (*start*): If $v_{s,1} = v_{s,2}$, then trivially $\langle v_{s,1}, v_{s,2} \rangle \in B_2$. If $v_{s,1} \neq v_{s,2}$, then $v_{s,1} = w_1$ and $v_{s,2} = w_2$. Since $\langle w_1, w_2 \rangle \in B_1$ and $w_2 \in V_2$, we have $\langle w_1, w_2 \rangle \in B_2$. \square

Proposition (= Proposition 6.4). *If a LLEE-chart C is not a bisimulation collapse, then it contains a pair of bisimilar vertices w_1, w_2 that satisfy, for a LLEE-witness of C , one of the following conditions:*

(C1) $\neg(w_2 \rightarrow^* w_1) \wedge (\neg w_1 \implies w_2 \text{ is not normed})$,

(C2) $w_2 \hookrightarrow^+ w_1$,

(C3) $\exists v \in V (w_1 \xrightarrow{d} v \wedge w_2 \hookrightarrow^+ v) \wedge \neg(w_2 \rightarrow_{bo}^* w_1)$.

More supplementary illustrations for the proof of Prop. 6.4 on pages 9–10. The proof started from a pair u_1, u_2 of distinct bisimilar vertices. In the case $\text{scc}(u_1) = \text{scc}(u_2)$, we had the following situation:

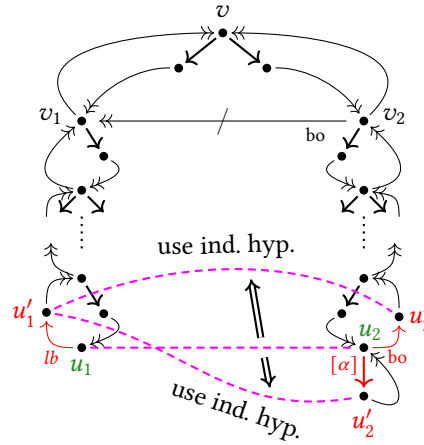
$$u_1 \hookrightarrow^* v_1 \xrightarrow{d} v \xrightarrow{d} v_2 \hookrightarrow^* u_2 \wedge \neg(v_2 \rightarrow_{bo}^* v_1). \quad (\text{A.16})$$

For pairs of vertices u_1 and u_2 such that (A.16) holds, for some v_1, v_2 , and v , we used induction on $\|u_1\|_{lb}^{\min}$ in order to show that u_1 and u_2 progress, via pairs of distinct bisimilar vertices, to bisimilar vertices w_1 and w_2 such that one of the conditions (C1), (C2), or (C3) holds. Note that each of (C1), (C2), and (C3) implies that w_1 and w_2 are distinct.

In order to carry out the induction step we used a case distinction. Below we repeat the arguments, and supplement them with illustrations.

Case 1: $u_2 \hookrightarrow^+ v_2$.

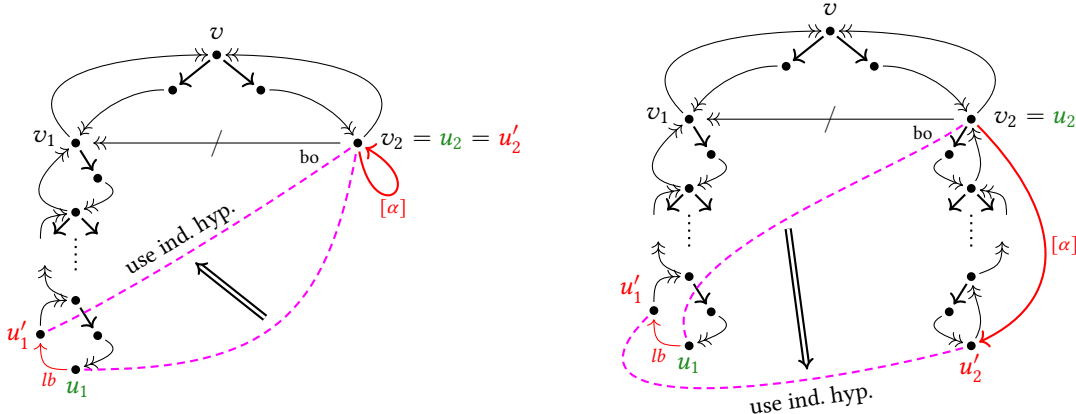
Since $u_2 \rightarrow u'_2$, either $u'_2 = v_2$ or $v_2 \leadsto^+ u'_2$. Moreover, $\text{scc}(u'_2) = \text{scc}(u_2) = \text{scc}(v_2)$, so by Lem. 3.9, (ii), $u'_2 \hookrightarrow^* v_2$. Hence, $u'_1 \hookrightarrow^* v_1 \xrightarrow{d} v \xrightarrow{d} v_2 \hookrightarrow^* u'_2 \wedge \neg(v_2 \rightarrow_{bo}^* v_1)$, and $\|u'_1\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$. We apply the induction hypothesis to obtain a bisimilar pair w_1, w_2 for which (C1), (C2), or (C3) holds. In the illustration below, we drew both of the two cases in which the transition $u_2 \rightarrow u'_2$ is a loop-entry transition, or a body transition, from u_2 .



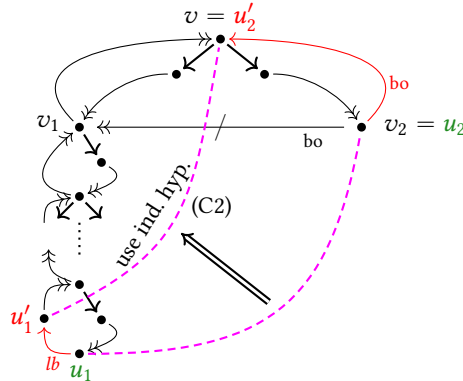
Case 2: $u_2 = v_2$.

Case 2.1: $u_2 \rightarrow_{[\alpha]} u'_2$.

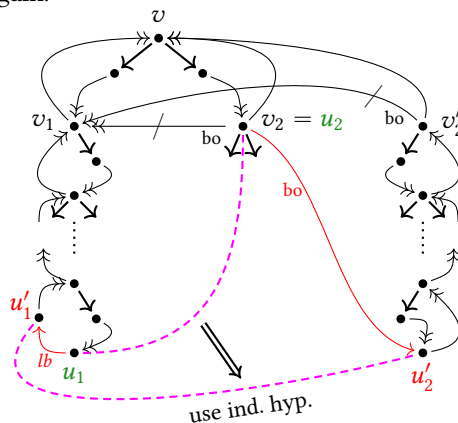
Then either $u'_2 = u_2$ or $u_2 \leadsto^+ u'_2$. Moreover, $\text{scc}(u'_2) = \text{scc}(u_2)$, so by Lem. 3.9, (ii), $u'_2 \hookrightarrow^* u_2$, and hence $u'_2 \hookrightarrow^* v_2$. Thus we have obtained $u'_1 \hookrightarrow^* v_1 \xrightarrow{d} v \xrightarrow{d} v_2 \hookrightarrow^* u'_2 \wedge \neg(v_2 \rightarrow_{bo}^* v_1)$. Due to $\|u'_1\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$, we can apply the induction hypothesis again.



Then $u'_1 \mathcal{G}^* v_1 \mathcal{G} v = u'_2$, i.e., $u'_1 \mathcal{G}^+ u'_2$, so we are done, because (C2) holds for $w_1 = u'_2$ and $w_2 = u'_1$.

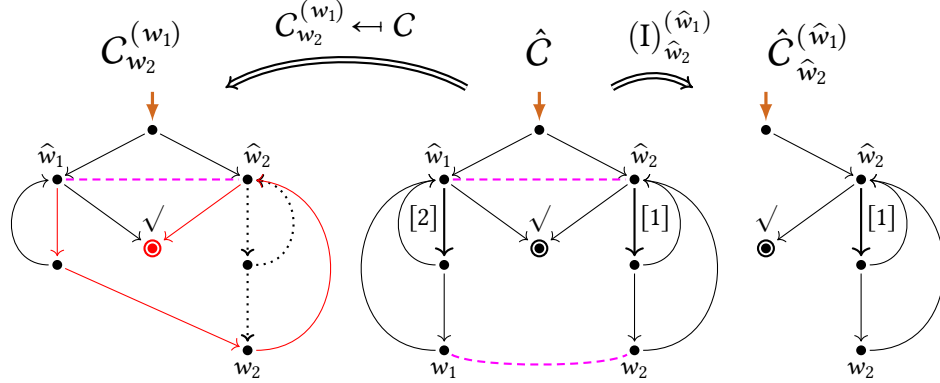


By Lem. 3.9, (ii), $u'_2 \triangleleft^+ v$. Hence, $u'_2 \triangleleft^* v'_2 \triangleleft_d v$ for some v'_2 . Since $v_2 = u_2 \rightarrow_{\text{bo}} u'_2 \triangleleft^* v'_2$ and $\neg(v_2 \rightarrow_{\text{bo}}^* v_1)$, it follows that $\neg(v'_2 \rightarrow_{\text{bo}}^* v_1)$. So $u'_1 \triangleleft^* v_1 \triangleleft_d v \triangleleft_d v'_2 \triangleleft^* u'_2 \wedge \neg(v_2 \rightarrow_{\text{bo}}^* v'_1)$. Due to $\|u'_1\|_{lb}^{\min} < \|u_1\|_{lb}^{\min}$, we can apply the induction hypothesis again.



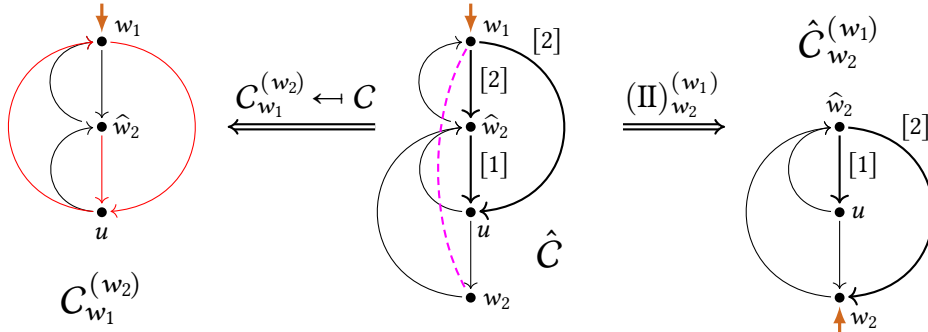
Example A.11 (= Example 6.3). To show that in (C1) it is crucial that w_1 does not loop back, we refer back to the LLEE-witness \hat{C} in Ex. 6.3. There $\neg(w_2 \rightarrow^* w_1)$, but (C1) is not satisfied by the pair w_1, w_2 because $w_1 \hookrightarrow \hat{w}_1$. Since in \hat{C} the levels of loop-entry transitions that descend to w_1 are higher than the loop levels that descend from w_2 , the preprocessing step of transformation I is void. We observed that the connect- w_1 -through-to- w_2 chart $C_{w_2}^{(w_1)}$ on the left in Ex. 6.3 has no LLEE-witness. The bisimilar pair w_1, w_2 in \hat{C} progresses to the bisimilar pair \hat{w}_1, \hat{w}_2 , for which (C1) holds. Since $\hat{C}_{\hat{w}_2}^{(\hat{w}_1)}$ on the right of Ex. 6.3

is obtained by applying transformation I to this pair, it is guaranteed to be a LEE-witness; this will be argued in the proof of Prop. 6.8.



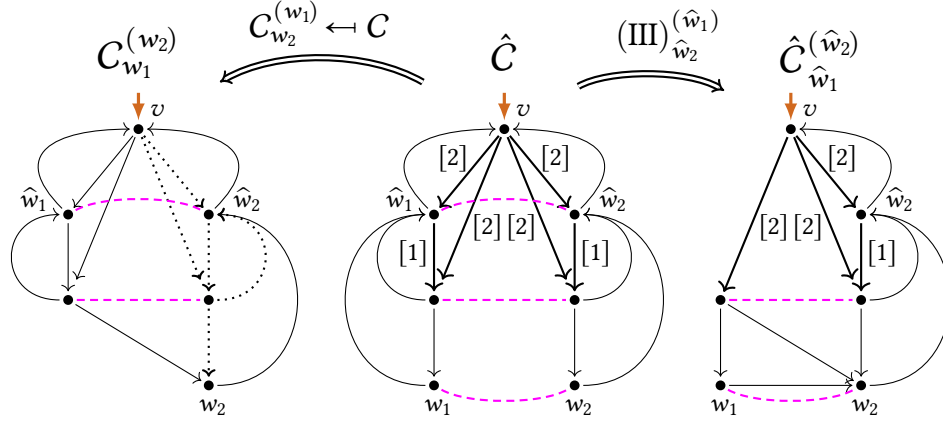
To avoid the creation of body step cycles in transformation II, it would seem expedient to connect transitions to w_2 through to w_1 , since (C2), $w_2 \hookrightarrow^+ w_1$, rules out the existence of a path $w_1 \rightarrow_{\text{bo}}^+ w_2$ in \hat{C} . (Instead, transitions to w_1 are connected through to w_2 , and resulting body step cycles are eliminated by turning the body transitions at \hat{w}_2 into loop-entry transitions.) However, connecting transitions to w_2 through to w_1 may produce a chart for which no LLEE-witness exists.

Example A.12 (= Example 6.6). For the LLEE-chart C with LLEE-witness \hat{C} below in the middle, the connect- w_2 -through-to- w_1 chart $C_{w_1}^{(w_2)}$ on the left does not have a LLEE-witness: it has no loop subchart, because from each of its three vertices an infinite path starts that does not return to this vertex. From \hat{w}_2 this path, drawn in red, cycles between u and w_1 . Transformation II applied to the pair w_1, w_2 (instead of w_2, w_1) in \hat{C} yields the entry/body-labeling $\hat{C}_{w_2}^{(w_1)}$ for the connect- w_1 -through-to- w_2 chart with additionally $\hat{w}_2 \rightarrow_{\text{bo}} w_2$ turned into $\hat{w}_2 \rightarrow_{[2]} w_2$. Since the pair w_1, w_2 satisfies (C2), the proof of Prop. 6.8 guarantees that this entry/body-labeling, drawn on the right, is a LLEE-witness.



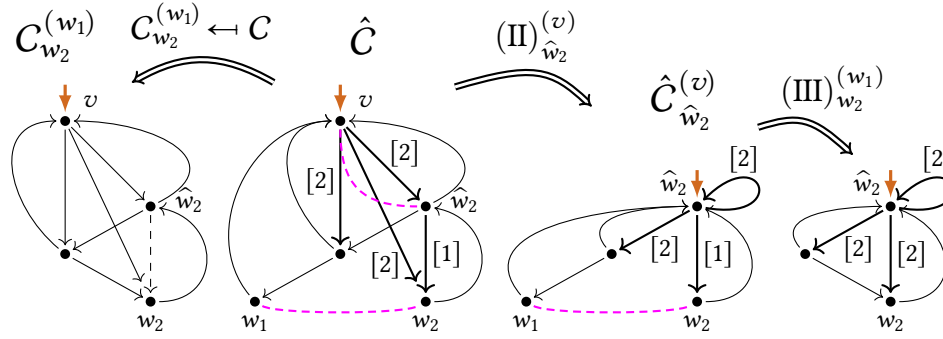
The following example shows that for transformation III it is essential to select a bisimilar pair w_1, w_2 where w_1 *directly* loops back to v .

Example A.13 (= Example 6.7). In the LLEE-witness \hat{C} below in the middle, $w_1, w_2 \hookrightarrow^+ v$, and there is no body step path from w_2 to w_1 , but (C3) does not hold for the pair w_1, w_2 because $\neg(w_1 \hookrightarrow_{\text{d}} v)$. All loop-entry transitions from v have the same loop label, so the preprocessing step of transformation III is void. The connect- w_1 -through-to- w_2 chart $C_{w_2}^{(w_1)}$ on the left does not have a LLEE-witness. Namely, the transition from \hat{w}_2 can be declared a loop-entry transition, and after its removal also two transitions from v can be declared loop-entry transitions, leading to the removal of the five transitions that are depicted as dotted arrows. The remaining chart (of solid arrows) however has no further loop subchart, because from each of its vertices an infinite path starts that does not return to this vertex. The bisimilar pair w_1, w_2 progresses to the bisimilar pair \hat{w}_1, \hat{w}_2 in \hat{C} , for which (C3) holds because $\hat{w}_1 \hookrightarrow_{\text{d}} v \hookrightarrow \hat{w}_2$ and $\neg(\hat{w}_2 \rightarrow_{\text{bo}}^* \hat{w}_1)$. Transformation III applied to this pair yields the entry/body-labeling $\hat{C}_{\hat{w}_2}^{(\hat{w}_1)}$ on the right. In the proof of Prop. 6.8 it is argued that this is guaranteed to be a LLEE-witness. The remaining two bisimilar pairs can be eliminated by one or two further applications of transformation III.



The following example shows (C3) cannot be weakened by dropping $\neg(w_2 \rightarrow_{bo}^* w_1)$.

Example A.14. For the LLEE-witness \hat{C} below in the middle, $w_1 \not\sqsubseteq_d v \sqsubseteq^+ w_2$, but there is a body step path from w_2 to w_1 . The connect- w_1 -through-to- w_2 chart $C_{w_2}^{(w_1)}$ on the left does not have a LLEE-witness, because from each of its vertices an infinite path starts that does not return to it. The bisimilar pair w_1, w_2 in \hat{C} progresses to the bisimilar pair v, \hat{w}_2 , to which transformation II is applicable because (C2) holds: $\hat{w}_2 \sqsubseteq v$. In the resulting LLEE-witness $\hat{C}_{\hat{w}_2}^{(v)}$, second to the right, (C3) holds for the pair w_1, w_2 because $w_1 \sqsubseteq_d \hat{w}_2 \sqsubseteq w_2$ and $\neg(w_2 \rightarrow_{bo}^* w_1)$. Applying transformation III to this pair results in the LLEE-witness on the right.



Supplement for the proof of Proposition 6.8. Let \hat{C} be a LLEE-chart. For vertices w_1, w_2 such that (C1), (C2), or (C3) holds, transformation I, II, or III, respectively, produces an entry/body-labeling $\hat{C}_{w_2}^{(w_1)}$. In the article submission we have proved for transformation I that it is a LLEE-witness. Here we do the same for transformations II and III.

We recall that in the proof in the article submission we have shown that it suffices to show that each of the transformations produces, before the final clean-up step, an entry/body-labeling that satisfies the LLEE-conditions with the exception of possible violations of the loop property (L1) in (W2)(a).

Transformation II: We argue the correctness of transformation II. Consider vertices w_1, w_2 such that (C2) holds, that is, $w_2 \sqsubseteq^+ w_1$. Let \hat{w}_2 be the \sqsubseteq_d -predecessor of w_1 in the \sqsubseteq_d -chain from w_2 to w_1 , i.e., $w_2 \sqsubseteq^* \hat{w}_2 \sqsubseteq_d w_1$.

As for the transformations I and III it suffices to show, in view of the alleviation of the proof obligation at the start of the proof on page 11, that the intermediate result \hat{C}'' of transformation II before the clean-up step satisfies the LLEE-witness properties, except for possible violations of (L1). By the definition of transformation II, \hat{C}'' results by performing the adaptation step L_{II} to the chart $\hat{C}' := \hat{C}_{w_2}^{(w_1)}$ that arises from \hat{C} by connecting w_1 through to w_2 .

To prove that (W1), and the part concerning (L2) for (W2)(a) is satisfied for \hat{C}'' , it suffices to show that the transformed chart does not contain a cycle of body transitions. At first, the step of connecting w_1 through to w_2 in \hat{C} may introduce a body step cycle in $\hat{C}' = \hat{C}_{w_2}^{(w_1)}$. But every such cycle is removed in the subsequent level adaptation step L_{II} . Namely, each body step cycle introduced in \hat{C}' must stem from a transition $u \rightarrow_{bo} w_1$ (which is redirected to w_2 in \hat{C}') and a path $w_2 \rightarrow_{bo}^* u$ in \hat{C} , for some $u \neq w_1$. Since $w_2 \sqsubseteq^* \hat{w}_2 \sqsubseteq_d w_1$, by Lem. A.8, the path $w_2 \rightarrow_{bo}^* u \rightarrow_{bo} w_1$ in \hat{C} must visit \hat{w}_2 . Since all body transitions from \hat{w}_2 are turned into loop-entry transitions in step L_{II} , the body step cycle $w_2 \rightarrow_{bo}^* u \rightarrow_{bo} w_2$ in \hat{C}' that was introduced in the connect-through step, is after step L_{II} no longer a body step cycle in \hat{C}'' .

Now we prove that (W2)(b) is preserved by the two steps from \hat{C} via $C' = \hat{C}_{w_2}^{(w_1)}$ to \hat{C}'' . Every path $u \xrightarrow{\dagger(u)}[\alpha] \cdot \xrightarrow{\dagger(u)}^*_{bo} \cdot \rightarrow[\beta]$ in \hat{C}'' with $u \neq w_1, w_2$ arises by a, possibly empty, combination of the following three kinds of modifications in the first two transformation steps:

- (i) A transition to w_1 was redirected to w_2 in the connect-through step.
- (ii) The loop-entry transition at the beginning of the path is from \hat{w}_2 and was a body transition before step L_{II} , meaning that $u = \hat{w}_2$ and $\alpha = \gamma$. (Recall that γ is a loop name of maximum loop level among the loop-entries at w_1 in \hat{C} .)
- (iii) The loop-entry transition at the end of the path is from \hat{w}_2 and was a body transition before step L_{II} , meaning that $\beta = \gamma$.

This gives $2^3 = 8$ possibilities. Of these, three possibilities are void: if all three adaptations are not the case, the path is already present in \hat{C} , and so $\alpha > \beta$ is guaranteed; (ii) and (iii) together cannot hold, because then the path would return to $u = \hat{w}_2$, which it cannot, because all of its steps avoid u as target. We now show that in the remaining five cases always $\alpha > \beta$. Since $w_2 \sqsubset^+ w_1$, there is a path $w_1 \xrightarrow{\dagger(w_1)}[\delta] \cdot \xrightarrow{\dagger(w_1)}^*_{bo} w_2$ in \hat{C} . By definition of γ , $\gamma \geq \delta$.

A Let only (i) hold: there are paths $u \xrightarrow{\dagger(u)}[\alpha] \cdot \xrightarrow{\dagger(u)}^*_{bo} w_1$ and $w_2 \xrightarrow{\dagger(u)}^*_{bo} \cdot \rightarrow[\beta]$ in \hat{C} (which do not visit \hat{w}_2). Then there is a path $u \xrightarrow{\dagger(u)}[\alpha] \cdot \xrightarrow{\dagger(u)}^*_{bo} w_1 \rightarrow[\gamma]$ in \hat{C} , so $\alpha > \gamma$. We distinguish two cases.

CASE 1: The path $w_2 \xrightarrow{\dagger(u)}^*_{bo} \cdot \rightarrow[\beta]$ visits w_1 . Then there is a path $w_1 \xrightarrow{\dagger(u)}^* \cdot \rightarrow[\beta]$ in \hat{C} . So $u \xrightarrow{\dagger(u)}[\alpha] \cdot \xrightarrow{\dagger(u)}^*_{bo} w_1 \xrightarrow{\dagger(u)}^* \cdot \rightarrow[\beta]$ in \hat{C} . So by (W2)(b), $\alpha > \beta$.

CASE 2: The path $w_2 \xrightarrow{\dagger(u)}^*_{bo} \cdot \rightarrow[\beta]$ does not visit w_1 . Then there is a path $w_1 \xrightarrow{\dagger(w_1)}[\delta] \cdot \xrightarrow{\dagger(w_1)}^*_{bo} w_2 \xrightarrow{\dagger(w_1)}^*_{bo} \cdot \rightarrow[\beta]$ in \hat{C} , so $\delta > \beta$. Hence $\alpha > \gamma \geq \delta > \beta$.

B Let only (ii) hold. Then $u = \hat{w}_2$, $\alpha = \gamma$, and there is a path $\hat{w}_2 \xrightarrow{\dagger(\hat{w}_2, w_1)}^+_{bo} \cdot \rightarrow[\beta]$ in \hat{C} . As $\hat{w}_2 \sqsubset_d w_1$, there is a path $w_1 \xrightarrow{\dagger(w_1)}[\delta] \cdot \xrightarrow{\dagger(w_1)}^*_{bo} \hat{w}_2$ in \hat{C} . Hence $w_1 \xrightarrow{\dagger(w_1)}[\delta] \cdot \xrightarrow{\dagger(w_1)}^*_{bo} \hat{w}_2 \xrightarrow{\dagger(w_1)}^+_{bo} \cdot \rightarrow[\beta]$ in \hat{C} , so $\delta > \beta$. Hence $\alpha = \gamma \geq \delta > \beta$.

C Let only (iii) hold. Then $\beta = \gamma$, and $u \xrightarrow{\dagger(u, w_1)}[\alpha] \cdot \xrightarrow{\dagger(u, w_1)}^*_{bo} \hat{w}_2$ with $u \neq w_1$ is a path in \hat{C} . Since $\hat{w}_2 \sqsubset_d w_1$ and $u \neq w_1$, it follows that $\neg(\hat{w}_2 \sqsubset_d u)$. So in view of the path $u \xrightarrow{\dagger(u)}[\alpha] \cdot \xrightarrow{\dagger(u)}^*_{bo} \hat{w}_2$, there is no path $\hat{w}_2 \rightarrow^*_{bo} u$ in \hat{C} . Since $\hat{w}_2 \sqsubset_d w_1$, there is a path $\hat{w}_2 \rightarrow^*_{bo} w_1$ in \hat{C} , which by the previous observation is of the form $\hat{w}_2 \xrightarrow{\dagger(u)}^*_{bo} w_1$.

Hence there is a path $u \xrightarrow{\dagger(u)}[\alpha] \cdot \xrightarrow{\dagger(u)}^*_{bo} \hat{w}_2 \xrightarrow{\dagger(u)}^*_{bo} w_1 \rightarrow[\gamma]$ in \hat{C} , so $\alpha > \gamma = \beta$.

D Let only (i) and (ii) hold, meaning $u = \hat{w}_2$, $\alpha = \gamma$, and there are paths $\hat{w}_2 \xrightarrow{\dagger(\hat{w}_2)}^+ w_1$ and $w_2 \xrightarrow{\dagger(\hat{w}_2)}^*_{bo} \cdot \rightarrow[\beta]$ in \hat{C} . Since $w_2 \sqsubset^* \hat{w}_2 \sqsubset_d^+ w_1$, and $u = \hat{w}_2$ implies $w_2 \neq \hat{w}_2$, by Lem. A.8, the path $w_2 \xrightarrow{\dagger(\hat{w}_2)}^*_{bo} \cdot \rightarrow[\beta]$ cannot visit w_1 . Hence $w_1 \xrightarrow{\dagger(w_1)}[\delta] \cdot \xrightarrow{\dagger(w_1)}^*_{bo} w_2 \xrightarrow{\dagger(w_1)}^*_{bo} \cdot \rightarrow[\beta]$ in \hat{C} . So $\delta > \beta$. Hence $\alpha = \gamma \geq \delta > \beta$.

E Let only (i) and (iii) hold. Then $\beta = \gamma$, and $u \xrightarrow{\dagger(u)}[\alpha] \cdot \xrightarrow{\dagger(u)}^*_{bo} w_1$ and $w_2 \xrightarrow{\dagger(u)}^*_{bo} \hat{w}_2$ are paths in \hat{C} . Since $u \xrightarrow{\dagger(u)}[\alpha] \cdot \xrightarrow{\dagger(u)}^*_{bo} w_1 \rightarrow[\gamma]$ in \hat{C} , $\alpha > \gamma = \beta$.

We conclude that in all five cases, \hat{C}'' satisfies (W2)(b).

Finally we argue that part (L3) of (W2)(a) holds for \hat{C}'' , i.e., there are no descends-in-loop-to paths of the form $u \xrightarrow{\dagger(u)}[\alpha] \cdot \xrightarrow{\dagger(u)}^*_{bo} \surd$ in \hat{C}'' . We can use part of the argumentation employed for demonstrating (W2)(b) above. It was demonstrated in particular that for every descends-in-loop-to path $u \xrightarrow{\dagger(u)}[\alpha] \cdot \xrightarrow{\dagger(u)}^*_{bo} x$ in \hat{C}'' , there is a descends-in-loop-to path $\tilde{u} \xrightarrow{\dagger(\tilde{u})}[\gamma] \cdot \xrightarrow{\dagger(\tilde{u})}^*_{bo} x$ with the same target x in \hat{C} . From this it follows that if a descends-in-loop-to path in \hat{C}'' had \surd as target, then there were a descends-in-loop-to path already in \hat{C} that had \surd as target, violating (L3) for the LLEE-chart \hat{C} . Hence \hat{C}'' must satisfy (L3).

We conclude that the result of transformation II is a LLEE-chart.

Transformation III: To show the correctness of transformation III, consider vertices w_1 and w_2 such that (C3) holds. Let v be such that $w_1 \xrightarrow{d} v \xrightarrow{+} w_2$. We show that its intermediate result $\hat{C}_{w_2}^{(w_1)}$ before the clean-up step satisfies the LLEE-witness properties, except for possible violations of (L1).

First we show that (W2)(b) is preserved by both the level adaptation and the connect-through step. A violation arising by the first step, i.e., in \hat{C}' , would involve a path $u \xrightarrow{\uparrow(u)}[\alpha] \cdot \xrightarrow{\uparrow(u)}^*_{bo} v \rightarrow[\beta]$ in \hat{C} where β is increased to a loop label γ of maximum level among all loop-entries at v . But in this way no violation can arise, since there was already a path $u \xrightarrow{\uparrow(u)}[\alpha] \cdot \xrightarrow{\uparrow(u)}^*_{bo} v \rightarrow[\gamma]$ in \hat{C} , so $\alpha > \gamma \geq \beta$.

Now we exclude violations of (W2)(b) in the connect-through step, by showing that in $\hat{C}_{w_2}^{(w_1)}$, $\alpha > \beta$ for all newly created paths $u \xrightarrow{\uparrow(u)}[\alpha] \cdot \xrightarrow{\uparrow(u)}^*_{bo} \cdot \rightarrow[\beta]$ with $u \neq w_1$ that stem from paths $u \xrightarrow{\uparrow(u)}[\alpha] \cdot \xrightarrow{\uparrow(u)}^*_{bo} w_1$ and $w_2 \xrightarrow{\uparrow(u)}^*_{bo} \cdot \rightarrow[\beta]$ in \hat{C}' . As $w_2 \xrightarrow{+} v$, there is a path $v \xrightarrow{\uparrow(v)}[\gamma] \cdot \xrightarrow{\uparrow(v)}^*_{bo} w_2$ in \hat{C}' . We distinguish two cases.

CASE 1: $u = v$. Then, by the level adaptation step, $\alpha = \gamma$. Since $u = v$, there is a path $v \xrightarrow{\uparrow(v)}[\gamma] \cdot \xrightarrow{\uparrow(v)}^*_{bo} w_2 \xrightarrow{\uparrow(v)}^*_{bo} \cdot \rightarrow[\beta]$ in \hat{C}' . By (W2)(b) for \hat{C}' , $\gamma > \beta$.

CASE 2: $u \neq v$. Since $w_1 \xrightarrow{d} v$, there is a path $w_1 \rightarrow^+_{bo} v$ in \hat{C} and thus in \hat{C}' . Suppose, toward a contradiction, that this path visits u . Then $u \xrightarrow{\uparrow(u)}[\alpha] \cdot \xrightarrow{\uparrow(u)}^*_{bo} w_1 \rightarrow^+_{bo} u$, so $w_1 \xrightarrow{+} u$ in \hat{C}' and thus in \hat{C} . Then $w_1 \xrightarrow{d} v$ and $u \neq v$ imply $v \xrightarrow{+} u$, which together with $u \rightarrow^+_{bo} v$ yields a body step cycle between u and v in \hat{C} . This contradicts that (W1) holds in \hat{C} . Therefore $w_1 \xrightarrow{\uparrow(u)}^+_{bo} v$ in \hat{C}' . We consider two cases.

CASE 2.1: $w_2 \xrightarrow{\uparrow(u)}^*_{bo} \cdot \rightarrow[\beta]$ visits v , so $v \xrightarrow{\uparrow(v)}^*_{bo} \cdot \rightarrow[\beta]$ in \hat{C}' . Then $u \xrightarrow{\uparrow(u)}[\alpha] \cdot \xrightarrow{\uparrow(u)}^*_{bo} w_1 \xrightarrow{\uparrow(u)}^+_{bo} v \xrightarrow{\uparrow(u)}^*_{bo} \cdot \rightarrow[\beta]$ in \hat{C}' .

By (W2)(b) for \hat{C}' , $\alpha > \beta$.

CASE 2.2: $w_2 \xrightarrow{\uparrow(u)}^*_{bo} \cdot \rightarrow[\beta]$ does not visit v . Then since $w_2 \xrightarrow{+} v$ implies $v \xrightarrow{+} w_2$, there is a path $v \xrightarrow{\uparrow(v)}[\gamma] \cdot \xrightarrow{\uparrow(v)}^*_{bo} x_k \xrightarrow{\uparrow(x_k)}[\delta_k] \cdot \xrightarrow{\uparrow(x_k)}^*_{bo} \dots x_1 \xrightarrow{\uparrow(x_1)}[\delta_1] \cdot \xrightarrow{\uparrow(x_1)}^*_{bo} w_2 \xrightarrow{\uparrow(v)}^*_{bo} \cdot \rightarrow[\beta]$ in \hat{C}' , for some $k \geq 0$. Since also $u \xrightarrow{\uparrow(u)}[\alpha] \cdot \xrightarrow{\uparrow(u)}^*_{bo} w_1 \xrightarrow{\uparrow(u)}^+_{bo} v \rightarrow[\gamma]$ in \hat{C}' , by (W2)(b), $\alpha > \gamma > \delta_k > \dots > \delta_1 > \beta$. So $\alpha > \beta$.

To verify (W1) together with part (L2) of (W2)(a) for $\hat{C}_{w_2}^{(w_1)}$, it suffices to show that $\hat{C}_{w_2}^{(w_1)}$ does not contain body step cycles. This can be verified analogously as for transformation I. That is, under the assumption of a body step cycle we can construct a path $w_2 \rightarrow^+_{bo} w_1$ in \hat{C} , which contradicts (C3) (as it contradicted (C1)).

To show part (L3) of (W2)(a) for $\hat{C}_{w_2}^{(w_1)}$, we can use part of the argumentation employed above for proving (W2)(b). It was demonstrated in particular that for every descends-in-loop-to path $u \xrightarrow{\uparrow(u)}[\alpha] \cdot \xrightarrow{\uparrow(u)}^*_{bo} x$ in \hat{C}'' there is a descends-in-loop-to path $\tilde{u} \xrightarrow{\uparrow(\tilde{u})}[\gamma'] \cdot \xrightarrow{\uparrow(\tilde{u})}^*_{bo} x$ with the same target x in \hat{C} . This entails that if a descends-in-loop-to path in \hat{C}'' had $\sqrt{}$ as target, then there were a descends-in-loop-to path in \hat{C} with $\sqrt{}$ as target, contradicting (L3) for the LLEE-witness \hat{C} . Hence \hat{C}'' must satisfy part (L3) of (W2)(a).

We conclude that the result of transformation III is again a LLEE-witness.

□