

Lecture 3: Algorithmic Meta-Theorems

(A Short Introduction to Parameterized Complexity)

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Ph.D. Program, Advanced Courses Period

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L'Aquila, Italy

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Course overview

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
<i>Algorithmic Techniques</i>		<i>Formal-Method & Algorithmic Techniques</i>		
Introduction & basic FPT results motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	Notions of bounded graph width path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	Algorithmic Meta-Theorems 1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	FPT-Intractability Classes & Hierarchies motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour

Overview

- ▶ Courcelle's theorem
 - ▶ FPT-results by model-checking MSO-formulas
 - ▶ for graphs / structures with bounded tree-width
 - ▶ for maximization problems over graphs of bounded tree-width
 - ▶ for graphs of bounded clique-width
 - ▶ applications to concrete problems

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- ▶ logic preliminaries
 - ▶ first-order logic
 - ▶ expressing graph problems by f-o formulas
 - ▶ monadic second-order logic (MSO)
 - ▶ expressing graph problems by MSO formulas
 - ▶ complexity of evaluation and model checking problems
- ▶ Courcelle's theorem
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 - ▶ applications to concrete problems
- ▶ graph minors
- ▶ meta-theorems for first-order model-checking: an example

Meta-theorems: idea, benefits and limitations

idea:

- ▶ express a problem P by a logical formula φ_P (of 'short' size)
- ▶ use **model checking** of φ_P
on logical structures of **bounded width k** (tree-, clique-width, ...)
 - ▶ is time bounded depending on k , size of φ_P , size of the structure
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- ▶ **without working out the tedious details**
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limitations:

- ▶ algorithms obtained by meta-theorems
cannot be expected to be optimal.
- ▶ a careful analysis of a specific problem at hand
will usually yield more efficient fpt-algorithms

Logical preliminaries

First-order logic (formula example)

$$\begin{aligned}
 \varphi_3 := \quad & \exists x_1 \exists x_2 \exists x_3 \Big(\neg(x_1 = x_2) \wedge \neg E(x_1, x_2) \\
 & \quad \wedge \neg(x_1 = x_3) \wedge \neg E(x_1, x_3) \\
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$$\mathcal{A}(\mathcal{G}) \models \varphi_3 \iff \mathcal{G} \text{ has a 3-element independent set.}$$

$$S \subseteq V \text{ is independent set in } \mathcal{G} = \langle V, E \rangle : \iff \forall e = \{u, v\} \in E \left(\neg(u \in S \wedge v \in S) \right) \\ \iff \forall u, v \in S \left(u \neq v \Rightarrow \{u, v\} \notin E \right)$$

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$$\varphi_k := \exists x_1 \dots \exists x_k \left(\bigwedge_{1 \leq i < j \leq k} (\neg(x_i = x_j) \wedge \neg E(x_i, x_j)) \right)$$

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- ▶ *sentences*: formulas without *free* variables.

First-order logic: semantics (structures)

Definition

Let $\tau = \{R_1, \dots, R_n\}$ be a vocabulary.

A τ -*structure* is a tuple $\mathcal{A} = \langle A; R_1^{\mathcal{A}}, \dots, R_n^{\mathcal{A}} \rangle$ consisting of:

- ▶ the *universe* A ,
- ▶ *interpretations* $R_i^{\mathcal{A}} \subseteq A^{ar(R_i)} = \overbrace{A \times \dots \times A}^{ar(R_i)}$ for each of the relation symbols R_i in τ , where $i \in \{1, \dots, n\}$.

Examples

Let $\tau_G = \{E/2\}$ vocabulary with binary edge relation.

The *standard structure* for a graph $\mathcal{G} = \langle V, E \rangle$:

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Example

Let $\tau_{\text{HG}} = \{VERT/1, EDGE/1, INC/2\}$ vocabulary (for hypergraphs).

The *hypergraph structure* for a graph $\mathcal{G} = \langle V, E \rangle$:

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) := \langle V \cup E; V, E, \{\langle v, e \rangle \mid v \in V, e \in E, v \in e\} \rangle.$$

Interpretation of first-order formulas in structures

Let $\mathcal{A} = \langle A; \{R^{\mathcal{A}}\}_{R \in \tau} \rangle$ be a τ -structure. For a τ -formula $\varphi(x_1, \dots, x_k)$ its *interpretation* $\varphi(\mathcal{A}) \subseteq A^k$ in \mathcal{A} is defined by:

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- ▶ $\mathcal{A} \models \varphi(a_1, \dots, a_k)$ will mean: $\langle a_1, \dots, a_k \rangle \in \varphi(\mathcal{A})$.
- ▶ For a sentence φ , $\mathcal{A} \models \varphi$ will mean $\varphi(\mathcal{A}) \neq \emptyset$ (then $\varphi(\mathcal{A}) = \{ \langle \rangle \}$).

Expressing graph properties by first-order formulas

Exercise

For given formulas $\varphi(x)$ and for all $k \in \mathbb{N}$, $k \geq 1$ define formulas $\exists^{\geq k} x \varphi(x)$, $\exists^{< k} x \varphi(x)$, $\exists^{=k} x \varphi(x)$, such that in a given τ -structure $\mathcal{A} = \langle A; \{R^{\mathcal{A}}\}_{R \in \tau} \rangle$:

$$\mathcal{A} \models \exists^{\geq k} x \varphi(x) \iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| \geq k$$

$$\mathcal{A} \models \exists^{< k} x \varphi(x) \iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| < k$$

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Expressing graph properties by first-order formulas

Exercise

For given formulas $\varphi(x)$ and for all $k \in \mathbb{N}$, $k \geq 1$ define formulas $\exists^{\geq k} x \varphi(x)$, $\exists^{< k} x \varphi(x)$, $\exists^{=k} x \varphi(x)$, such that in a given τ -structure $\mathcal{A} = \langle A; \{R^{\mathcal{A}}\}_{R \in \tau} \rangle$:

$$\mathcal{A} \models \exists^{\geq k} x \varphi(x) \iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| \geq k$$

$$\mathcal{A} \models \exists^{< k} x \varphi(x) \iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| < k$$

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Expressing graph properties by first-order formulas

Exercise

Express by a first-order formula with the vocabulary $\tau_G = \{E/2\}$ for graphs that:

- (i) a graph \mathcal{G} contains a **clique** with precisely k elements,
- (ii) a graph \mathcal{G} has a **dominating set** with less or equal to k elements,
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Recall:

$$\varphi_k := \exists x_1 \dots \exists x_k \left(\bigwedge_{1 \leq i < j \leq k} (\neg(x_i = x_j) \wedge \neg E(x_i, x_j)) \right)$$

$$\mathcal{A}_{\tau_G}(\mathcal{G}) \models \varphi_k \iff \mathcal{G} \text{ has a } k\text{-element independent set.}$$

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Express by a first-order formula with the vocabulary with vocabulary $\tau_{\text{HG}} = \{ \text{VERT}/_1, \text{EDGE}/_1, \text{INC}/_2 \}$ for hypergraphs that:

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Evaluation and model checking (first-order logic)

Let Φ be a class of **first-order** formulas.

The *evaluation problem* for Φ :

EVAL(Φ)

Instance: A structure \mathcal{A} and a formula $\varphi \in \Phi$.

Problem: Compute $\varphi(\mathcal{A})$.

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Width of formula φ : maximal number of free variables in a subformula of φ .

Theorem

EVAL(FO) and **MC(FO)** can be solved in time $O(|\varphi| \cdot |\mathcal{A}|^w \cdot w)$, where w is the width of the input formula φ .

Monadic second-order logic (formula example)

$$\begin{aligned}
 \psi_3 := & \exists C_1 \exists C_2 \exists C_3 \left(\left(\forall x \left(\bigvee_{i=1}^3 C_i(x) \right) \right) \wedge \forall x \left(\bigwedge_{1 \leq i < j \leq 3} \neg (C_i(x) \wedge C_j(x)) \right) \right) \\
 & \wedge \forall x \forall y \left(E(x, y) \rightarrow \bigwedge_{i=1}^3 \neg (C_i(x) \wedge C_i(y)) \right)
 \end{aligned}$$

$$\mathcal{A}(\mathcal{G}) \models \psi_3 \iff \mathcal{G} \text{ has is } 3\text{-colorable.}$$

Monadic second-order logic

- ▶ language based on:
 - ▶ a *vocabulary* $\tau = \{R_1, \dots, R_n\}$ of *predicate symbols* R_i together with arity $ar(R_i) \in \mathbb{N}$
 - ▶ the binary equality predication $=$
 - ▶ first-order variable symbols: $x, y, z, w, x_1, y_1, z_1, w_1, x_2, \dots$
 - ▶ *monadic second-order variable symbols* (symbolizing variables for unary predicate symbols): $X, Y, Z, W, X_1, Y_1, Z_1, W_1, X_1, \dots$
 - ▶ propositional connectives: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
 - ▶ existential quantifier \exists , universal quantifier \forall

Interpretation of MSO-formulas in first-order structures

Let $\mathcal{A} = \langle A; \{R^{\mathcal{A}}\}_{R \in \tau} \rangle$ be a τ -structure.

For a $\text{MSO}(\tau)$ -formula $\varphi(x_1, \dots, x_k, X_1, \dots, X_\ell)$ its *interpretation* $\varphi(\mathcal{A}) \subseteq A^k \times \mathcal{P}(A)^\ell$ in \mathcal{A} is defined by:

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there exists $P_{\ell+1} \in \mathcal{P}(A)$ such that

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will mean: $\langle a_1, \dots, a_k, P_1, \dots, P_\ell \rangle \in \varphi(\mathcal{A})$.

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- ▶ For a sentence φ , $\mathcal{A} \models \varphi$ will mean $\varphi(\mathcal{A}) \neq \emptyset$ (then $\varphi(\mathcal{A}) = \{ \langle \rangle \}$).

Monadic second-order logic (formula example)

$$\begin{aligned}
 \psi_3 := & \exists C_1 \exists C_2 \exists C_3 \left(\left(\forall x \left(\bigvee_{i=1}^3 C_i(x) \right) \right) \wedge \forall x \left(\bigwedge_{1 \leq i < j \leq 3} \neg (C_i(x) \wedge C_j(x)) \right) \right) \\
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$$\begin{aligned}
 \equiv & \exists C_1 \exists C_2 \exists C_3 \left(\forall x (C_1(x) \vee C_2(x) \vee C_3(x)) \right. \\
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 & \wedge \forall x \forall y \left(E(x, y) \rightarrow \neg (C_1(x) \wedge C_1(y)) \right. \\
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Expressing graph properties by MSO formulas (1)

Exercise

Express by a monadic second-order formula $\varphi(X)$ with one free unary predicate variable X over the vocabulary $\tau_G = \{E/2\}$ for graphs that for all graphs $\mathcal{G} = \langle V, E \rangle$:

$$\mathcal{A}_{\tau_G}(\mathcal{G}) \models \varphi(S) \iff S \subseteq V \text{ is an independent set in } \mathcal{G}$$

Recall:

$$\begin{aligned} S \subseteq V \text{ is independent set in } \mathcal{G} &\iff \forall e = \{u, v\} \in E \ (\neg(u \in S \wedge v \in S)) \\ &\iff \forall u, v \in S \ (u \neq v \Rightarrow \{u, v\} \notin E) \end{aligned}$$

Exercise

Express the independent set property by a $\text{MSO}(\tau_{\text{HG}})$ formula ψ with vocabulary $\tau_{\text{HG}} = \{ \text{VERT}/1, \text{EDGE}/1, \text{INC}/2 \}$ for hypergraphs:

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Exercise

Express the independent set property by a MSO(τ_{HG}) formula ψ with vocabulary $\tau_{HG} = \{VERT/1, EDGE/1, INC/2\}$ for hypergraphs:

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Expressing graph properties by MSO formulas (2)

Exercise

Express by a monadic second-order formula $\text{feedback}(X)$ with one free unary predicate variable X over $\tau_{\text{HG}} = \{\text{VERT}/_1, \text{EDGE}/_1, \text{INC}/_2\}$, the vocabulary for graphs, that for all hypergraphs $\mathcal{G} = \langle V, E \rangle$:

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \models \text{feedback}(S) \iff S \subseteq V \text{ is a feedback vertex set}$$

(A set $S \subseteq V$ is a feedback vertex set in \mathcal{G} if S contains a vertex of every cycle of \mathcal{G} .)

Steps:

- ▶ Construct a formula $\text{cycle-family}(X)$ that expresses the property of a set being the union of cycles.
- ▶ Using $\text{cycle-family}(X)$, construct $\text{feedback}(X)$.

MSO for graphs and hypergraphs

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Correspondences

$\text{MSO}(\tau_G)$ corresponds to MSO_1

where 'corresponds to' means: 'is equally expressive as'.

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Note:

$\text{MSO}(\tau_{\text{HG}}) / \text{MSO}_2$ are more expressive than $\text{MSO}(\tau_G) / \text{MSO}_1$.

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Note:

We use MSO for either logic, restrict to $\text{MSO}(\tau_G)$ / MSO_1 if needed.

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Correspondences

$\text{MSO}(\tau_G)$ corresponds to MSO_1
 $\text{MSO}(\tau_{\text{HG}})$ corresponds to MSO_2

where 'corresponds to' means: 'is equally expressive as'.

Note:

$\text{MSO}(\tau_{\text{HG}}) / \text{MSO}_2$ are more expressive than $\text{MSO}(\tau_G) / \text{MSO}_1$.

Expressing graph properties by MSO formulas (5)

Exercise

Express by a $\text{MSO}(\tau_{\text{HG}})$ formula $\text{conn}(X)$ with one free unary predicate variable X over $\tau_{\text{HG}} = \{\text{VERT}/_1, \text{EDGE}/_1, \text{INC}/_2\}$, the vocabulary for graphs, that for all hypergraphs $\mathcal{G} = \langle V, E \rangle$:

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \models \text{hamiltonian} \iff \text{there is a Hamiltonian path in } \mathcal{G}.$$

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$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \models \text{hamiltonian} \iff \text{there is a Hamiltonian path in } \mathcal{G}.$$

Note:

- ▶ This property is not expressible by a (single) $\text{MSO}(\tau_{\text{G}})$ formula.
- ▶ Other properties that are **not** $\text{MSO}(\tau_{\text{G}})$ expressible:
 - ▶ balanced bipartite graphs
 - ▶ existence of a perfect matching
 - ▶ simple graphs (graphs with no parallel edges)
 - ▶ existence of spanning trees with maximum degree 3

Expressing graph properties by MSO formulas (5)

Exercise

$\mathcal{A}_{\text{THG}}(\mathcal{G}) \models \textit{hamiltonian} \iff$ there is a Hamiltonian path in \mathcal{G} .

Expressing graph properties by MSO formulas (5)

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$\mathcal{A}_{\text{THG}}(\mathcal{G}) \models \textit{hamiltonian} \iff$ there is a Hamiltonian path in \mathcal{G} .

Evaluation and model checking (MSO)

The *model checking problem* for MSO-formulas on labeled, ordered unranked trees:

MC(MSO, TREE_{lo})

Instance: A labeled, ordered, unranked Σ -tree \mathcal{T} ,
and a MSO(τ_Σ^u)-formula φ

Problem: Decide whether $\mathcal{T} \models \varphi$.

where for given alphabet Σ , $\tau_\Sigma^u := \{E/2, N/2\} \cup \{P_a/1 \mid a \in \Sigma\}$.

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Theorem

MC(MSO, TREE_{lo}) \in FPT.

More precisely, there is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that MC(MSO, TREE_{lo}) can be decided in time $\leq O(f(|\varphi|) + \|\mathcal{T}\|)$.

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Note that here: $f(k) \geq 2^{\cdot^{\cdot^{\cdot^2}}}$ (a non-elementary function).

Courcelle's Theorem

Courcelle's Theorem for graphs

p^* -*tw*-MC(MSO)

Instance: A graph \mathcal{G} and an $\text{MSO}(\tau_{\text{HG}})$ -sentence φ .

Parameter: $\text{tw}(\mathcal{G}) + |\varphi|$ (where $\text{tw}(\mathcal{G})$ the tree-width of \mathcal{G})

Problem: Decide whether $\mathcal{A}(\mathcal{G}) \models \varphi$.

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Theorem (special case of Courcelle's Theorem)

p^* - tw -MC(MSO) \in FPT. More precisely, the problem is decidable, for some computable and non-decreasing function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ by an algorithm in time:

$$f(k_1, k_2) \cdot n, \quad \text{where } k_1 := tw(\mathcal{A}), k_2 := |\varphi|, n := |V(\mathcal{G})|$$

Courcelle's Theorem: applications (1)

p^* - tw -COLORABILITY \in **FPT**

Instance: A graph \mathcal{G} and $\ell \in \mathbb{N}$.

Parameter: $tw(\mathcal{C})$

Problem: Decide whether is \mathcal{G} ℓ -colorable.

Example

► p^* - tw -3-COLORABILITY \in **FPT**.

Courcelle's Theorem: applications (1)

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Courcelle's Theorem: applications (2)

p^* - tw -HAMILTONICITY

Instance: A graph \mathcal{G}

Parameter: $tw(\mathcal{C})$

Problem: Decide whether \mathcal{G} is a hamiltonian (that is, contains a cyclic path that visits every vertex precisely once).

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Courcelle's Theorem: applications (2)

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Example

p^* - tw -HAMILTONICITY \in FPT.

Tree decompositions, and tree-width for graphs

Definition (recalling tree-width for graphs)

A **tree decomposition** of a graph $\mathcal{G} = \langle V, E \rangle$ is a pair $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ where $\mathcal{T} = \langle T, F \rangle$ a (undirected, unrooted) tree, and $B_t \subseteq V$ for all $t \in T$ such that:

- (T1) $A = \bigcup_{t \in T} B_t$ (every vertex of \mathcal{G} is in some bag).
- (T2) $(\forall \{u, v\} \in E) (\exists t \in T) [\{u, v\} \subseteq B_t]$
(the vertices of **every edge** of \mathcal{G} are realized in some bag).
- (T3) $(\forall v \in V) [\text{subgraph of } \mathcal{T} \text{ defd. by } \{t \in T \mid v \in B_t\} \text{ is connected}]$
(the tree vertices (in \mathcal{T}) whose bags contain some vertex of \mathcal{G} induce a subgraph of \mathcal{T} that is connected).

The **width** of a tree dec. $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ is $\max \{|B_t| - 1 \mid t \in T\}$.

The **tree-width** $tw(\mathcal{A})$ of a τ -structure \mathcal{A} is defined by:

$tw(\mathcal{A})$:= minimal width of a tree decomposition of \mathcal{A} .

Tree decompositions, and tree-width for structures

Definition (extension of tree-width to structures)

A **tree decomposition** of a τ -structure $\mathcal{A} = \langle A; \{R^A\}_{R \in \tau} \rangle$ is a pair $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ where $\mathcal{T} = \langle T, F \rangle$ a (undirected, unrooted) tree, and $B_t \subseteq V$ for all $t \in T$ such that:

- (T1) $A = \bigcup_{t \in T} B_t$ (every element of the universe of \mathcal{A} is in some bag).
- (T2) $(\forall R \in \tau) (\forall \langle a_1, \dots, a_r \rangle \in R^A) (\exists t \in T) [\{a_1, \dots, a_r\} \subseteq B_t]$
(the vertices of every 'hyperedge' in R^A are realized in some bag).
- (T3) $(\forall v \in V) [\text{subgraph of } \mathcal{T} \text{ defd. by } \{t \in T \mid v \in B_t\} \text{ is connected}]$
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$tw(\mathcal{A}) :=$ minimal width of a tree decomposition of \mathcal{A} .

Courcelle's Theorem

p^* -tw-MC(MSO)

Instance: A structure \mathcal{A} and an MSO-sentence φ .

Parameter: $tw(\mathcal{A}) + |\varphi|$.

Problem: Decide whether $\mathcal{A} \models \varphi$.

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Theorem ([Courcelle, 1990])

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Theorem ([Courcelle, 1990])

p^* -tw-MC(MSO) $\in \text{FPT}$. More precisely, the problem is decidable by an algorithm in time:

$f(k_1, k_2) \cdot |\mathcal{A}| + O(\|\mathcal{A}\|)$, where $k_1 := \text{tw}(\mathcal{A})$, and $k_2 := |\varphi|$,
 f computable and non-decreasing

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$f(k_1, k_2) \cdot |A| + O(\|\mathcal{A}\|) \leq f(k_1, k_2) \cdot |A| + c \cdot \|\mathcal{A}\|$ with some $c > 0$

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 $\leq (f(k_1, k_2) + c) \cdot \|\mathcal{A}\|$

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$$\begin{aligned} f(k_1, k_2) \cdot |A| + O(\|\mathcal{A}\|) &\leq f(k_1, k_2) \cdot |A| + c \cdot \|\mathcal{A}\| && \text{with some } c > 0 \\ &\leq (f(k_1, k_2) + c) \cdot \|\mathcal{A}\| \\ &\leq g(k) \cdot (\|\mathcal{A}\| + |\varphi|) && \text{for } g(x) := f(x, x) + c \\ &&& k := k_1 + k_2 \end{aligned}$$

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Vertex Cover (first attempt)

Let $\mathcal{G} = \langle V, E \rangle$ a graph. For all $S \subseteq V$:

S is a **vertex cover** of $\mathcal{G} : \iff \forall e = \{u, v\} \in E (u \in S \vee v \in S)$

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p^* -tw-VERTEX-COVER

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and $\ell \in \mathbb{N}$.

Instance: $tw(\mathcal{G})$.

Problem: Does \mathcal{G} have a vertex cover of size at most ℓ ?

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Courcelle's Theorem: Refinement 1

$p^*\text{-tw-MC}^{\leq}(\text{MSO})$

Instance: A structure \mathcal{A} , an $\varphi(X)$, and $m \in \mathbb{N}$.

Parameter: $\text{tw}(\mathcal{A}) + |\varphi(X)|$.

Problem: Decide whether $\mathcal{A} \models \exists X (\text{card}^{\leq m}(X) \wedge \varphi(X))$.

Refinement 1 of Courcelle's Theorem

$p^*\text{-tw-MC}^{\leq}(\text{MSO}) \in \text{FPT}$. More precisely, the problem is decidable by an algorithm in time:

$$f(k_1, k_2) \cdot |A| + O(\|\mathcal{A}\|), \quad \text{where } k_1 := \text{tw}(\mathcal{A}), \text{ and } k_2 := |\varphi|, \\ f \text{ computable and non-decreasing}$$

Vertex Cover

Let $\mathcal{G} = \langle V, E \rangle$ a graph. For all $S \subseteq V$:

S is a **vertex cover** of $\mathcal{G} : \iff \forall e = \{u, v\} \in E (u \in S \vee v \in S)$

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Example

p^* -tw-VERTEX-COVER \in FPT.

Vertex Cover

Let $\mathcal{G} = \langle V, E \rangle$ a graph. For all $S \subseteq V$:

S is a **vertex cover** of $\mathcal{G} : \iff \forall e = \{u, v\} \in E (u \in S \vee v \in S)$

p^* -tw-VERTEX-COVER

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and $\ell \in \mathbb{N}$.

Instance: $tw(\mathcal{G})$.

Problem: Does \mathcal{G} have a vertex cover of size at most ℓ ?

Example

p^* -tw-VERTEX-COVER \in FPT.

Courcelle's Theorem: Refinement 2

$p^*\text{-tw-MC}^=(\text{MSO})$

Instance: A structure \mathcal{A} , an MSO-sentence $\varphi(X)$, and $m \in \mathbb{N}$.

Parameter: $\text{tw}(\mathcal{A}) + |\varphi(X)|$.

Problem: Decide whether $\mathcal{A} \models \exists X (\text{card}^m(X) \wedge \varphi(X))$.

Refinement 2 of Courcelle's Theorem

$p^*\text{-tw-MC}^=(\text{MSO}) \in \text{FPT}$. More precisely, the problem is decidable by an algorithm in time:

$f(k_1, k_2) \cdot |A|^2 + O(\|\mathcal{A}\|)$, where $k_1 := \text{tw}(\mathcal{A})$, and $k_2 := |\varphi|$,
 f computable and non-decreasing

Courcelle's Theorem Ref. 3: Optimization version

p^* -**tw-opt-MC**(**MSO**)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, an **MSO**-sentence $\varphi(X_1, \dots, X_p)$.

Parameter: $\text{tw}(\mathcal{G}) + |\varphi(X_1, \dots, X_p)|$.

Compute: $\max_{\min} \left\{ \alpha(|X_1|, \dots, |X_p|) \mid \begin{array}{l} X_1, \dots, X_p \subseteq V \cup E \\ \mathcal{A}(\mathcal{G}) \models \varphi(X_1, \dots, X_p) \end{array} \right\}$.

where α is an affine function $\alpha(x_1, \dots, x_p) = a_0 + \sum_{i=1}^p a_i \cdot x_i$.

Optimization version of Courcelle's Theorem

p^* -**tw-opt-MC**(**MSO**) \in **FPT**, and it is decidable by an algorithm in time:

$f(\text{tw}(\mathcal{G}), |\varphi|) \cdot |V|$, where f computable and non-decreasing.

Maximum 2-edge colorable subgraphs

p^* -tw-max-2-edge-colorable-subgraph

Instance: A graph $\mathcal{G} = \langle V, E \rangle$.

Parameter: $tw(\mathcal{G})$.

Compute: Maximum number of edges
in a 2-edge colored subgraph of G .

Example [AA & Vahan Mkrtchyan]

p^* -tw-max-2-edge-colorable-subgraph \in FPT.

Maximum 2-edge colorable subgraphs

p^* -*tw*-max-2-edge-colorable-subgraph

Instance: A graph $\mathcal{G} = \langle V, E \rangle$.

Parameter: $tw(\mathcal{G})$.

Compute: Maximum number of edges
in a 2-edge colored subgraph of \mathcal{G} .

Example [AA & Vahan Mkrtchyan]

p^* -*tw*-max-2-edge-colorable-subgraph \in FPT.

Courcelle's Theorem: applications (3)

p^* - tw -INDEPENDENT-SET

Instance: A graph \mathcal{G} , a number $\ell \in \mathbb{N}$.

Parameter: $tw(\mathcal{G})$

Problem: Decide whether \mathcal{G} has an independent set of ℓ elements.

Example

p^* - tw -INDEPENDENT-SET \in FPT.

Courcelle's Theorem: applications (3)

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Example

p^* - tw -INDEPENDENT-SET \in FPT.

Courcelle's Theorem: applications (4)

p^* - tw -FEEDBACK-VERTEX-SET

Instance: A graph \mathcal{G} and $\ell \in \mathbb{N}$.

Parameter: $tw(\mathcal{C})$

Problem: Decide whether \mathcal{G} has a feedback vertex set of ℓ elements.

Example

p^* - tw -FEEDBACK-VERTEX-SET \in FPT.

Courcelle's Theorem: applications (4)

p^* - tw -FEEDBACK-VERTEX-SET

Instance: A graph \mathcal{G} and $\ell \in \mathbb{N}$.

Parameter: $tw(\mathcal{C})$

Problem: Decide whether \mathcal{G} has a feedback vertex set of ℓ elements.

Example

p^* - tw -FEEDBACK-VERTEX-SET \in FPT.

Courcelle's Theorem: applications (5)

p^* - tw -CROSSING-NUMBER

Instance: A graph \mathcal{G} , and $k \in \mathbb{N}$

Parameter: $tw(\mathcal{G}) + k$

Problem: Decide whether the crossing number of \mathcal{G} is k .

Example

p^* - tw -CROSSING-NUMBER \in FPT.

The *crossing number* is the least number of edge crossings required to draw the graph in the plane such that at each point at most two edges cross.

Courcelle's Theorem: applications (5)

Definition

Let $\mathcal{G}_1 = \langle V_1, E_1 \rangle$ and $\mathcal{G}_2 = \langle V_2, E_2 \rangle$ be graphs.

\mathcal{G}_1 is a *subdivision* of \mathcal{G}_2 if:

- ▶ \mathcal{G}_1 arises by splitting the edges of \mathcal{G}_2 into paths with intermediate vertices.

\mathcal{H} is a *topological subgraph* of \mathcal{G}

if \mathcal{G} has a subgraph that is a subdivision of \mathcal{H} .

Courcelle's Theorem: applications (5)

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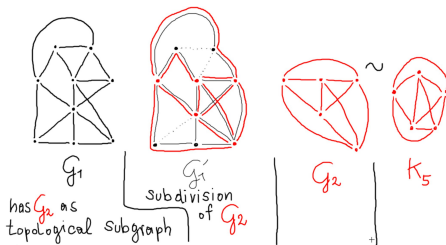
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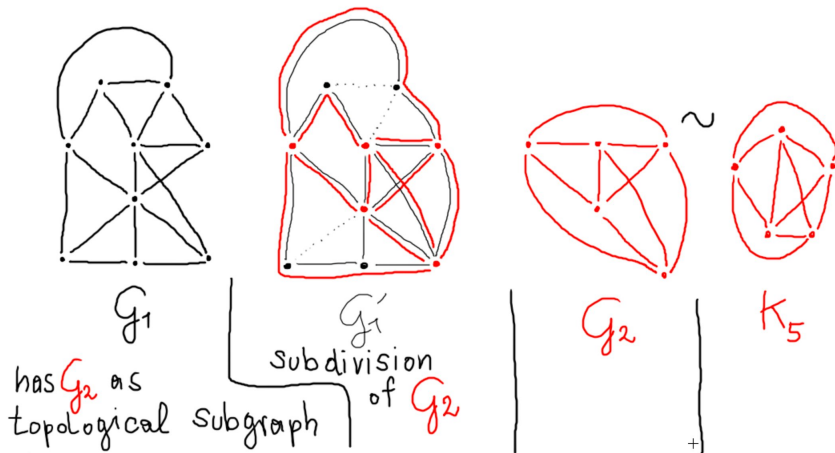
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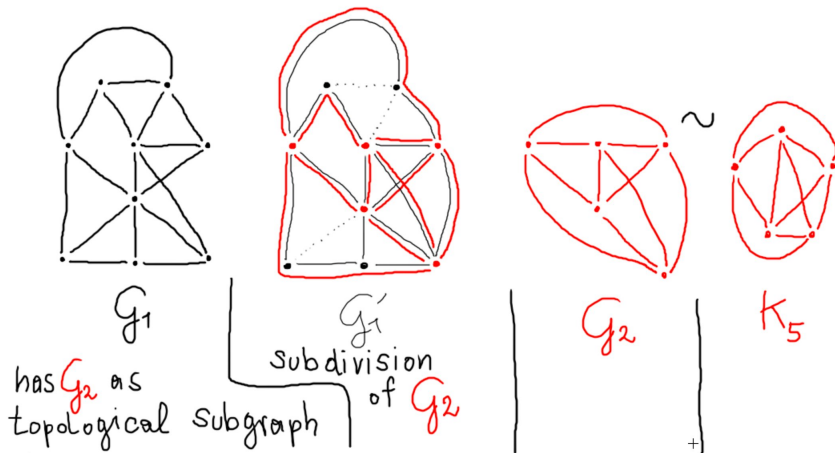
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Courcelle's Theorem: applications (5)



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Theorem (Kuratowski)

A graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as topological subgraph.

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Lemma

There is a $\text{MSO}(\tau_{\text{HG}})$ formula $\text{top-sub}_{\mathcal{H}}$ such that for every graph \mathcal{G} :

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \models \text{top-sub}_{\mathcal{H}} \iff \mathcal{H} \text{ is a topological subgraph of } \mathcal{G}.$$

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Using $\text{MSO}(\tau_{\text{HG}})$ formula $\text{path}(x, y, Z)$ that Z is a path from x to y .

Courcelle's Theorem: applications (5)

Theorem (Kuratowski)

A graph is planar if and only if it contains neither \mathcal{K}_5 nor $\mathcal{K}_{3,3}$ as topological subgraph.

Lemma

There is a $\text{MSO}(\tau_{\text{HG}})$ formula $\text{top-sub}_{\mathcal{H}}$ such that for every graph \mathcal{G} :

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \models \text{top-sub}_{\mathcal{H}} \iff \mathcal{H} \text{ is a topological subgraph of } \mathcal{G}.$$

Using $\text{MSO}(\tau_{\text{HG}})$ formula $\text{path}(x, y, Z)$ that Z is a path from x to y .

Lemma

There is a $\text{MSO}(\tau_{\text{HG}})$ formula cross_k such that for every graph \mathcal{G} :

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \models \text{cross}_k \iff \text{the crossing number of } \mathcal{G} \text{ is at most } k.$$

Proof: By induction, where $\text{cross}_0 := \neg \text{top-sub}_{\mathcal{K}_5} \wedge \neg \text{top-sub}_{\mathcal{K}_{3,3}}$.

Computably boundedness between notions of width

(from Sasák, [Sásak, 2010])

$$g(wd_1) \geq wd_2 : \Leftrightarrow wd_1 \xrightarrow{g} wd_2$$

► FPT-results

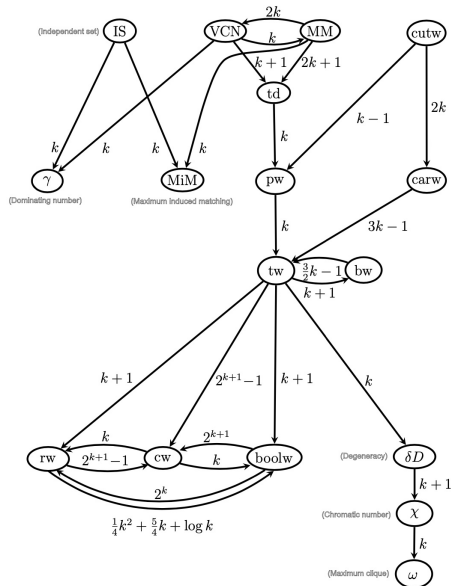
transfer upwards

(and conversely to \xrightarrow{g})

► (\notin FPT)-results

transfer downwards

(and along \xrightarrow{g})



Comparing parameterizations

Definition (computably bounded below)

Let $\kappa_1, \kappa_2 : \Sigma^* \rightarrow \mathbb{N}$ parameterizations.

- ▶ $\kappa_1 \geq \kappa_2 : \iff \exists g : \mathbb{N} \rightarrow \mathbb{N} \text{ computable } \forall x \in \Sigma^* [g(\kappa_1(x)) \geq \kappa_2(x)]$.
- ▶ $\kappa_1 \approx \kappa_2 : \iff \kappa_1 \geq \kappa_2 \wedge \kappa_2 \geq \kappa_1$.
- ▶ $\kappa_1 > \kappa_2 : \iff \kappa_1 \geq \kappa_2 \wedge \neg(\kappa_2 \geq \kappa_1)$.

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \rightarrow \mathbb{N}$ with $\kappa_1 \geq \kappa_2$:

$$\langle Q, \kappa_1 \rangle \in \text{FPT} \iff \langle Q, \kappa_2 \rangle \in \text{FPT}$$

$$\langle Q, \kappa_1 \rangle \notin \text{FPT} \implies \langle Q, \kappa_2 \rangle \notin \text{FPT}$$

Courcelle's Theorem for clique-width

Recall that $\text{MSO}(\tau_G) \sim \text{MSO}_1$ (quantification over sets of vertices, but not sets of edges).

p^* - clw -MC($\text{MSO}(\tau_G)/\text{MSO}_1$)

Instance: A graph \mathcal{G} and an $\text{MSO}(\tau_G)$ -sentence φ .

Parameter: $\text{clw}(\mathcal{G}) + |\varphi|$.

Problem: Decide whether $\mathcal{A}(\mathcal{G}) \models \varphi$.

Theorem ([Courcelle et al., 2000])

p^* - clw -MC($\text{MSO}(\tau_G)/\text{MSO}_1$) $\in \text{FPT}$.

Also, there is a **maximization version** of this theorem.

Courcelle's Theorem for clique-width (example)

Let $\mathcal{G} = \langle V, E \rangle$ a graph. For all $S \subseteq V$:

S is a **vertex cover** of $\mathcal{G} : \iff \forall e = \{u, v\} \in E (u \in S \vee v \in S)$

p^* -**clw**-VERTEX-COVER

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and $\ell \in \mathbb{N}$.

Instance: **clw**(\mathcal{G}).

Problem: Does \mathcal{G} have a vertex cover of size at most ℓ ?

Example

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p^* -**clw**-VERTEX-COVER \in **FPT**.

Application to maximum 2-edge colorable subgraphs?

p^* - clw -max-2-edge-colorable-subgraph

Instance: A graph $\mathcal{G} = \langle V, E \rangle$.

Parameter: $clw(\mathcal{G})$.

Compute: Maximum number of edges
in a 2-edge colored subgraph of G .

Open problem [AA, Vahan Mkrtchyan]

p^* - clw -max-2-edge-colorable-subgraph \in FPT ?

Application to maximum 2-edge colorable subgraphs?

p^* -*clw*-max-2-edge-colorable-subgraph

Instance: A graph $\mathcal{G} = \langle V, E \rangle$.

Parameter: *clw*(\mathcal{G}).

Compute: Maximum number of edges
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Open problem [AA, Vahan Mkrtchyan]

p^* -*clw*-max-2-edge-colorable-subgraph \in FPT ?

We saw that there is a MSO_2 formula $\varphi(X)$ such that:

$$\mathcal{A}_{\text{THG}}(\mathcal{G}) \models \varphi(S) \iff S \subseteq E \text{ is an 2-colorable set of edges in } \mathcal{G}$$

But there seems not to be such an MSO_1 formula.

Courcelle's Theorem for clique-width (non-example)

p^* -*clw*-HAMILTONICITY

Instance: A graph \mathcal{G}

Parameter: *clw*(\mathcal{C})

Problem: Decide whether \mathcal{G} is a hamiltonian (that is, contains a cyclic path that visits every vertex precisely once).

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There is no MSO_1 formula that expresses Hamiltonicity.

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Hence we cannot apply Courcelle's Theorem for clique-width.

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Recall

There is no **MSO**₁ formula that expresses Hamiltonicity.

Hence we cannot apply Courcelle's Theorem for clique-width. Indeed:

Theorems

- (T1) p^* -**clw**-HAMILTONICITY \notin FPT,
since it is not decidable in time $\notin n^{o(\text{clw}(\mathcal{C}))}$ (Fomin et al, 2014).
- (T2) p^* -**clw**-HAMILTONICITY $\in O(n^{o(\text{clw}(\mathcal{C}))})$
(Bergougnoux, Kanté, Kwon, 2020).

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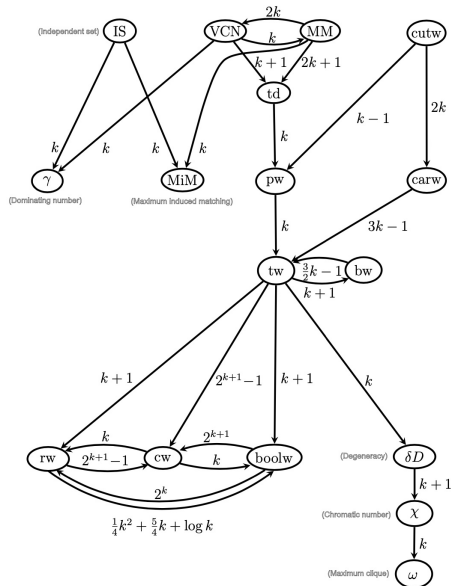
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Graph Minors

Graph minors

Definition

A graph \mathcal{G}_0 is a *minor* of a graph \mathcal{G} if \mathcal{G}_0 is obtained by:

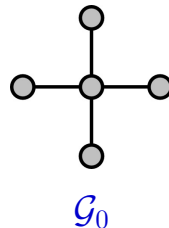
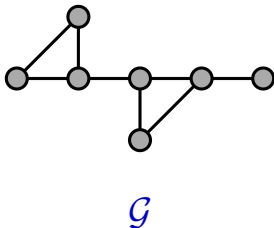
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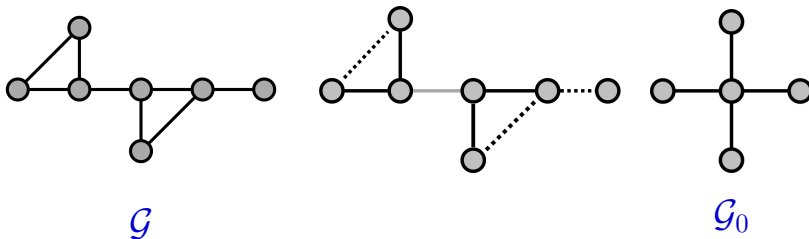


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A class \mathcal{G} is *minor closed* if for every $\mathcal{G} \in \mathcal{G}$ all minors of \mathcal{G} are in \mathcal{G} .

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$$\mathcal{G} := \text{Excl}(\mathcal{H}) := \{\mathcal{G} \mid \mathcal{G} \text{ does not have a minor in } \mathcal{H}\}$$

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Theorem (Graph Minor Theorem (Robertson–Seymour, 1983–2004))

Every class of graphs that is minor closed can be characterized by finitely many excluded minors. That is, for every class \mathcal{G} of minor closed graphs there are graphs $\mathcal{H}_1, \dots, \mathcal{H}_m$ such that:

$$\mathcal{G} = \text{Excl}(\{\mathcal{H}_1, \dots, \mathcal{H}_m\}).$$

Deciding minor closed classes

p -MINOR

Instance: Graphs \mathcal{G} and \mathcal{H} .

Parameter: $\|\mathcal{G}\|$

Problem: Decide whether \mathcal{G} is a minor of \mathcal{H} .

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Corollary

Every minor-closed class of graphs is decidable in cubic time.

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Corollary

Let $\langle Q, \kappa \rangle$ be a parameterized problem on graphs such that for every $k \in \mathbb{N}$, either $\{\mathcal{G} \in Q \mid \kappa(\mathcal{G}) = k\}$ or $\{\mathcal{G} \notin Q \mid \kappa(\mathcal{G}) = k\}$ is minor closed.

Then every slice $\langle Q, \kappa \rangle_k$ is decidable in cubic time. In this case we can say that $\langle Q, \kappa \rangle$ is **nonuniformly fixed-parameter tractable**.

Non-uniformly fixed-parameter tractable

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *fixed-parameter tractable* if:

$\exists f : \mathbb{N} \rightarrow \mathbb{N}$ computable $\exists p \in \mathbb{N}[X]$ polynomial

$\exists \mathbb{A}$ algorithm, takes inputs in Σ^*

$\forall x \in \Sigma^* [\mathbb{A} \text{ decides whether } x \in Q \text{ holds}$
in time $\leq f(\kappa(x)) \cdot p(|x|)]$

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Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *non-uniformly fixed-parameter tractable* (in *nu-FPT*) if:

$\exists f : \mathbb{N} \rightarrow \mathbb{N}$ computable $\exists p \in \mathbb{N}[X]$ polynomial

$\exists \{ \mathbb{A}_k \}_{k \in \mathbb{N}}$ algorithms, takes inputs in Σ^*

$\forall x \in \Sigma^* \left[\mathbb{A}_{\kappa(x)} \text{ decides whether } x \in Q \text{ holds} \right.$
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Using minor-closed classes for FPT results

Corollary

*Let $\langle Q, \kappa \rangle$ be a parameterized problem on graphs such that for every $k \in \mathbb{N}$, either $\{\mathcal{G} \in Q \mid \kappa(\mathcal{G}) = k\}$ or $\{\mathcal{G} \notin Q \mid \kappa(\mathcal{G}) = k\}$ is minor closed. Then $\langle Q, \kappa \rangle$ is **non-uniformly fixed-parameter tractable** (in **nu-FPT**).*

Applications:

- ▶ p -VERTEX-COVER \in **nu-FPT** (p -VERTEX-COVER is minor closed).

Using minor-closed classes for FPT results

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Let $\langle Q, \kappa \rangle$ be a parameterized problem on graphs such that for every $k \in \mathbb{N}$, either $\{\mathcal{G} \in Q \mid \kappa(\mathcal{G}) = k\}$ or $\{\mathcal{G} \notin Q \mid \kappa(\mathcal{G}) = k\}$ is minor closed. Then $\langle Q, \kappa \rangle$ is *non-uniformly fixed-parameter tractable* (in *nu-FPT*).

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- ▶ p -VERTEX-COVER \in *nu-FPT* (p -VERTEX-COVER is minor closed).
- ▶ p -FEEDBACK-VERTEX-SET \in *nu-FPT* (problem is minor closed).

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▶ p -DISJOINT-CYCLES

Instance: A graph \mathcal{G} , and $k \in \mathbb{N}$.

Parameter: k .

Problem: Decide whether \mathcal{G} has k disjoint cycles.

p -DISJOINT-CYCLES \in *nu-FPT*, since the class of graphs that do not have k disjoint cycles is minor closed.

First-Order Meta-Theorem (example)

Seese's theorem

A class \mathcal{G} of graphs has *bounded degree* if there is $d \in \mathbb{N}$ such that $\Delta(\mathcal{G}) \leq d$ for all $\mathcal{G} \in \mathcal{G}$ (where $\Delta(\mathcal{G}) = \max.$ degree of vertex in \mathcal{G}).

p -MC(FO, \mathcal{G})

Instance: A graph $\mathcal{G} \in \mathcal{G}$, and a f-o formula φ over τ_{HG}

Parameter: $|\varphi|$.

Problem: Decide whether $\mathcal{A}(\mathcal{G}) \models \varphi$.

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Theorem ([Seese, 1995])

$p\text{-MC}(\text{FO}, \mathcal{G}) \in \text{FPT}$ for every class \mathcal{G} of bounded degree. This model checking problem can be solved in time $f(|\varphi|) \cdot |\mathcal{G}|$, (linear in $|\mathcal{G}|$).

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Theorem (for comparison, we saw it earlier)

$\text{EVAL}(\text{FO})$ and $\text{MC}(\text{FO})$ can be solved in time $O(|\varphi| \cdot |A|^w \cdot w)$, where w is the width of the input formula φ .

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On the class of graphs of bounded degree:

- ▶ $p\text{-CLIQUE} \in \text{FPT}$ ('is there a clique of size k '?)

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- ▶ 'Does \mathcal{G} contain a cycle of length k ' (parameter k) is in **FPT**.
- ▶ $p\text{-VERTEX-COVER} \in \text{FPT}$ ('vertex cover of size at most k '?)

First-order metatheorems: reference

A good reference for other meta-theorems for first-order logic is:

[[Kreutzer, 2009](#)]: Stephan Kreutzer: *Algorithmic Meta-Theorems*.

Summary

- ▶ Logic preliminaries
 - ▶ first-order logic
 - ▶ expressing graph problems by f-o formulas
 - ▶ monadic second-order logic (MSO)
 - ▶ expressing graph problems by MSO formulas
 - ▶ complexity of evaluation and model checking problems
- ▶ Courcelle's theorem
 - ▶ FPT-results by model-checking MSO-formulas
 - ▶ for graphs with bounded tree-width
 - ▶ for structures with bounded tree-width
 - ▶ for graphs of bounded clique-width
 - ▶ applications to concrete problems
- ▶ graph minors
- ▶ meta-theorems for first-order model-checking: an example

Course overview

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
<i>Algorithmic Techniques</i>		<i>Formal-Method & Algorithmic Techniques</i>		
Introduction & basic FPT results motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	Notions of bounded graph width path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	Algorithmic Meta-Theorems 1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	FPT-Intractability Classes & Hierarchies motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour

Example suggestions

Examples

1. Find a **first-order logic formula** over τ_G that expresses that a graph has a **cycle of length precisely k** .
2. Find an **MSO_1** or **$MSO(\tau_G)$** formula that expresses that a graph has a **dominating set of $\leq k$** elements.
3. Find an **MSO_2** or **$MSO(\tau_{HG})$** formula ***feedback***(S) that expresses that $S \subseteq V$ is a feedback vertex set.
4. (*) Find an **MSO_1** or **$MSO(\tau_G)$** formula that expresses that a graph is **connected**.
5. (*) Find an **MSO_2** or **$MSO(\tau_{HG})$** formula ***path***(x, y, Z) that expresses that Z is a set of edges that forms a path from x to y .

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