Lecture 4: Fixed-Parameter Intractability

(A Short Introduction to Parameterized Complexity)

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June 14, 2024

Course overview

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results		**Algorithmic Meta-Theorems**		
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	GDA	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	GDA	GDA
Algorithmic Techniques		*Formal-Method & Algorithmic Techniques*		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width			**FPT-Intractability Classes & Hierarchies**
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Overview

- simple graphs (correction)
- Motivation for fixed-parameter intractability
- Fixed parameter reductions
- The classes para-NP and XP
- The class W[P]
- Logic preliminaries (continued)
- W-hierarchy
  - definitions
    - with Boolean circuits
    - as parameterized weighted Fagin definability problems
- A-hierarchy
  - definition as parameterized model-checking problems
- picture overview of these classes

Expressing graph properties by MSO formulas (5)

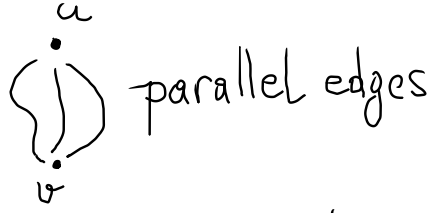
Exercise

Express by a MSO(τ_{HG}) formula *conn*(X) with one free unary predicate variable X over $\tau_{\text{HG}} = \{\mathit{VERT}/_1, \mathit{EDGE}/_1, \mathit{INC}/_2\}$, the vocabulary for graphs, that for all hypergraphs $\mathcal{G} = \langle V, E \rangle$:

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \models \mathit{hamiltonian} \iff \text{there is a Hamiltonian path in } \mathcal{G}.$$

Note:

- This property is not expressible by a (single) MSO(τ_{G}) formula.
- Other properties that are not MSO(τ_{G}) expressible:
  - balanced bipartite graphs
  - existence of a perfect matching
  - simple graphs (graphs with no parallel edges)
  - existence of spanning trees with maximum degree 3



the edge predicate $E(\cdot,\cdot)$ alone can obviously not express absence of parallel edges

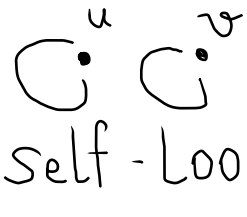
$MSO(\mathcal{T}_G)$ formula

MSO_1 formula

$\forall x \neg E(x,x)$

$\forall_{(vert)} x \neg E(x,x)$

expresses self-loop-freeness



self-loops

$\mathcal{A}_{\mathcal{T}_G}(G) \models \forall x \neg E(x,x)$

$\Updownarrow$

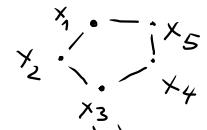
G contains no self-loops

Example suggestions

Examples

1. Find a first-order logic formula over τ_{G} that expresses that a graph has a cycle of length precisely k.
2. Find an MSO_1 or $\mathrm{MSO}(\tau_{\mathsf{G}})$ formula that expresses that a graph has a dominating set of $\leq k$ elements.
3. Find an MSO_2 or $\mathrm{MSO}(\tau_{\mathsf{HG}})$ formula $\mathit{feedback}(S)$ that expresses that $S \subseteq V$ is a feedback vertex set.
4. $(^\star)$ Find an MSO_1 or $\mathrm{MSO}(\tau_{\mathsf{G}})$ formula that expresses that a graph is connected.
5. $(^\star)$ Find an MSO_2 or $\mathrm{MSO}(\tau_{\mathsf{HG}})$ formula $\mathit{path}(x, y, Z)$ that expresses that Z is a set of edges that forms a path from x to y.

1. First-order logic τ_G-formula "graph has cycle of length $= k$"



$$\varphi_{k\text{-cycle}} := \exists x_1 \dots \exists x_k \left(\left(\bigwedge_{1 \le i < j \le k} \neg x_i = x_j \right) \wedge \left(\bigwedge_{1 \le i \le k-1} E(x_i, x_{i+1}) \right) \wedge E(x_k, x_1) \right)$$

for $G = \langle V, E \rangle$ and $k \ge 3$: $\mathcal{A}_{\tau_G}(G) \models \varphi_{k\text{-cycle}} \iff G$ has a k-cycle

$|\varphi_{k\text{-cycle}}| \in O(k^2)$

---

2. $MSO(\tau_G)$ or MSO_1 formula: "graph G has a dominating set of $\le k$ elements"

$G = \langle V, E \rangle$. $S \subseteq V$ is dominating set $:\iff \forall v \in V [v \notin S \to \exists e = \{v, u\} \in E (u \in S)]$

$$\varphi_{\text{domset}}(X) := \forall x \left(\neg X(x) \to \exists y \left(E(x,y) \wedge X(y) \right) \right) \quad \in \text{For}(MSO(\tau_G))$$

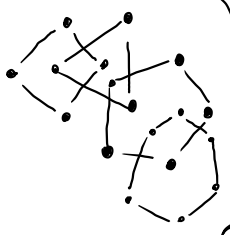
$$\forall_{(\text{vert})} x \left(\neg X(x) \to \exists_{(\text{vert})} y \left(E(x,y) \wedge X(y) \right) \right) \in \text{For}(MSO_1)$$

$$\varphi_{\text{domset} \le k}(X) := \varphi_{\text{domset}}(X) \wedge \exists_{(\text{vert})} x_1 \dots \exists_{(\text{vert})} x_k \left(\forall_{(\text{vert})} y \left(X(y) \iff \bigvee_{i=1}^{k} y = x_i \right) \right)$$

$$\varphi_{\exists \text{domset} \le k} := \exists_{(\text{Vert})} X \; \varphi_{\text{domset} \le k}(X)$$

Then $\mathcal{A}(G) \models \varphi_{\exists \text{domset} \le k} \iff G$ contains a dominating set of $\le k$ elements

3. MSO(τ_{HG})-formula for "S is a feedback-vertex set"



$\varphi_{fb}(X)$ $\Leftrightarrow$ contains a vertex of every cycle $\Leftrightarrow$ contains a vertex of every disjoint cycle collection

$$\text{disjoint-cycle-collection}(Y) :=$$
$$:= \forall e\,(Y(e) \rightarrow \text{EDGE}(e))$$
$$\wedge\ \forall v\,(\text{VERT}(v) \wedge \exists e\,(Y(e) \wedge \text{INC}(v,e)) \rightarrow \exists^{=2} e\,(Y(e) \wedge \text{INC}(v,e)))$$

Y is a set of edges such that every vertex that is incident to one of the edges in Y is incident to precisely 2 edges in Y

X is a set of vertices that has a non-empty intersection with every dcc Y

$$\varphi_{fb}(X) := \forall x\,(X(x) \rightarrow \text{VERT}(x)) \wedge$$
$$\forall C\,(\text{disjoint-cycle-collection}(C) \rightarrow \exists x\,(X(x) \wedge \exists e\,(C(e) \wedge \text{INC}(e))))$$

Then: for all finite graphs $G = \langle V, E \rangle$:

$$\mathcal{A}_{\tau_{HG}}(G) \models \varphi(S) \quad \Leftrightarrow \quad S \text{ is a feedback vertex set of } G.$$

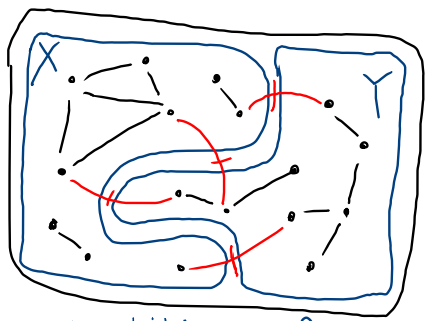
4. Find an MSO_1 or $MSO(\tau_{HG})$ formula "G is connected"

φ_{conn}

$$\varphi_{conn} := \neg \exists_{(vert)} X \exists_{(vert)} Y \; Partition(X, Y)$$

$$Partition(X, Y) := \forall_{(vert)} x \Big((X(x) \vee Y(x)) \wedge \neg (X(x) \wedge Y(x)) \wedge$$
$$\wedge (\exists_{(vert)} x \; X(x)) \wedge (\exists_{(vert)} x \; Y(x)) \wedge$$
$$\wedge \neg \exists_{(vert)} x \exists_{(vert)} y \; (X(x) \wedge Y(y) \wedge E(x,y)) \Big)$$

G



partition of G

into non-empty subsets X and Y of vertices
that are not connected by any edge

5. Find an MSO_2 / $MSO(\mathcal{T}_{HG})$ formula $path(x, y, Z)$ such that

for every finite graph $G = \langle V, E \rangle$
for every $Z \subseteq E$ and $x, y \in V$:

$$\mathcal{A}_{\mathcal{T}_{HG}}(G) \models path(\bar{x}, \bar{y}, \bar{Z}) \iff$$

$x \neq y$ and Z is a path (of edges) in G from x to y

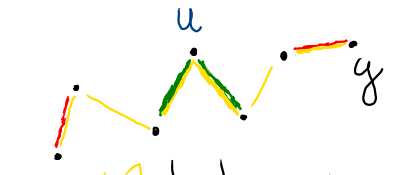
$$path(x, y, Z) := \neg x = y \wedge VERT(x) \wedge VERT(y) \wedge \forall e\,(Z(e) \rightarrow EDGE(e))$$

$$\wedge\ \exists^{=1} e\,(Z(e) \wedge INC(x, e))$$
$$\wedge\ \exists^{=1} e\,(Z(e) \wedge INC(y, e))$$

} x and y are incident to precisely 1 edge of Z

$$\wedge\ \forall u\,(VERT(u) \wedge \exists e\,(Z(e) \wedge INC(u, e)) \wedge \neg u = x \wedge \neg u = y \rightarrow \exists^{=2} e\,(Z(e) \wedge INC(u, e)))$$

every vertex on an edge in Z except x and y is incident to precisely 2 edges of Z

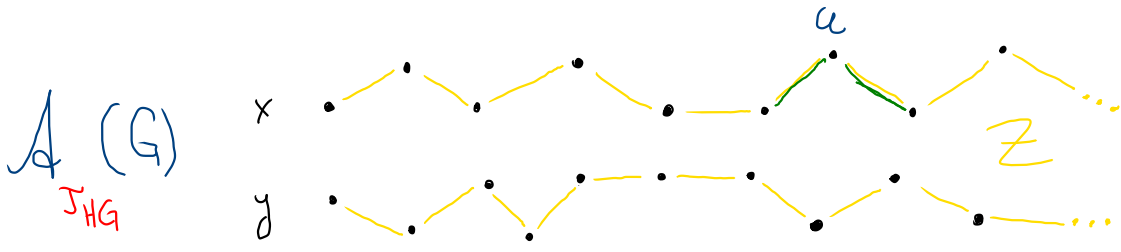


edgeset Z between x and y of a path

x and y are incident to precisely 1 edge of Z

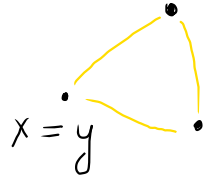
every internal vertex u of the edge set Z is incident to precisely 2 edges of Z

Note: for infinite graphs G this formula is not correct



while $\mathcal{A}_{\mathcal{T}_{HG}}(G) \models path(x,y,z)$ holds (and all vertices u at internal edges have precisely 2 incident edges), it may be impossible to reach y from x in finitely many steps ($\Rightarrow z$ is no path)

Further more: if $x=y$ were permitted, cyclic paths would have to be included



$\Rightarrow$ slight change of the formula $path(x,y,z)$

(See next page)

$$\text{path}(x, y, Z) :=$$

$$\text{VERT}(x) \wedge \text{VERT}(y) \wedge \forall e \left(Z(e) \rightarrow \text{EDGE}(e) \right) \wedge$$

$$\wedge \forall u \left(\text{VERT}(u) \wedge \neg u = x \wedge \neg u = y \wedge \exists e \left(Z(e) \wedge \text{INC}(u, e) \right) \right.$$
$$\left. \rightarrow \exists^{=2} e \left(Z(x) \wedge \text{INC}(u, e) \right) \right)$$

every vertex u on an edge of Z is incident to precisely 2 edges of Z

$$\wedge \left(x = y \rightarrow \exists^{=2} e \left(Z(e) \wedge \text{INC}(x, e) \right) \right)$$

if $x = y$ then both are also on precisely 2 edges of Z

$$\wedge \left(\neg x = y \rightarrow \exists^{=1} e \left(Z(e) \wedge \text{INC}(x, e) \right) \right.$$
$$\left. \wedge \exists^{=1} e \left(Z(e) \wedge \text{INC}(y, e) \right) \right)$$

if $x \neq y$ both are separately on precisely 1 edge of Z

Then:

$$\mathcal{A}_{S_G}(G) \models \text{path}(x, y, Z) \iff \begin{cases} Z \text{ is a set of edges on} \\ \text{a path in } G \text{ from } x \text{ to } y \end{cases}$$

Two classical problems

QUERIES

Instance: a relational database D, a conjunctive query α.
Compute: answer to query α from database D.

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LTL-MODEL-CHECKING

Instance: a Kripke structure (state space) $\mathcal{K}$, an LTL formula φ
Parameter: size $|\varphi|$ of formula φ
Question: Does $\mathcal{K} \models \varphi$ hold?

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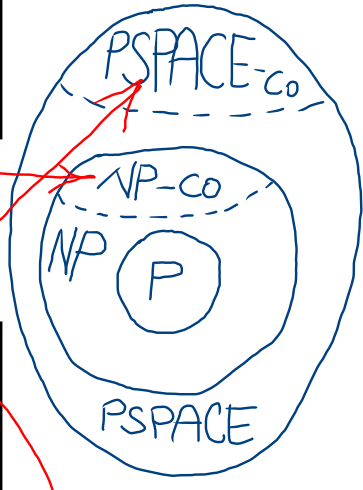
- LTL-MODEL-CHECKING $\in$ PSPACE-complete.



Comparing their parameterizations

QUERIES

Instance: a relational database D, a conjunctive query α.
Parameter: size $k = |\alpha|$ of query α
Compute: answer to query α from database D.

- QUERIES $\in$ NP-complete.

LTL-MODEL-CHECKING

Instance: a Kripke structure (state space) $\mathcal{K}$, an LTL formula φ
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- LTL-MODEL-CHECKING $\in$ PSPACE-complete,

Comparing their parameterizations

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Instance: a relational database D, a conjunctive query α.
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- QUERIES $\in$ NP-complete.
- QUERIES $\in O(n^k)$ for $n = \|D\|$, which does not give an FPT result.

LTL-MODEL-CHECKING

Instance: a Kripke structure (state space) $\mathcal{K}$, an LTL formula φ
Parameter: size $k = |\varphi|$ of formula φ
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- LTL-MODEL-CHECKING $\in$ PSPACE-complete,

Comparing their parameterizations

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LTL-MODEL-CHECKING

Instance: a Kripke structure (state space) $\mathcal{K}$, an LTL formula φ
Parameter: size $k = |\varphi|$ of formula φ
Question: Does $\mathcal{K} \models \varphi$ hold?

- LTL-MODEL-CHECKING $\in$ PSPACE-complete,
- LTL-MODEL-CHECKING $\in O(k \cdot 2^{2k} \cdot n) \in$ FPT for $n = \|\mathcal{K}\|$.

Fixed-parameter intractability

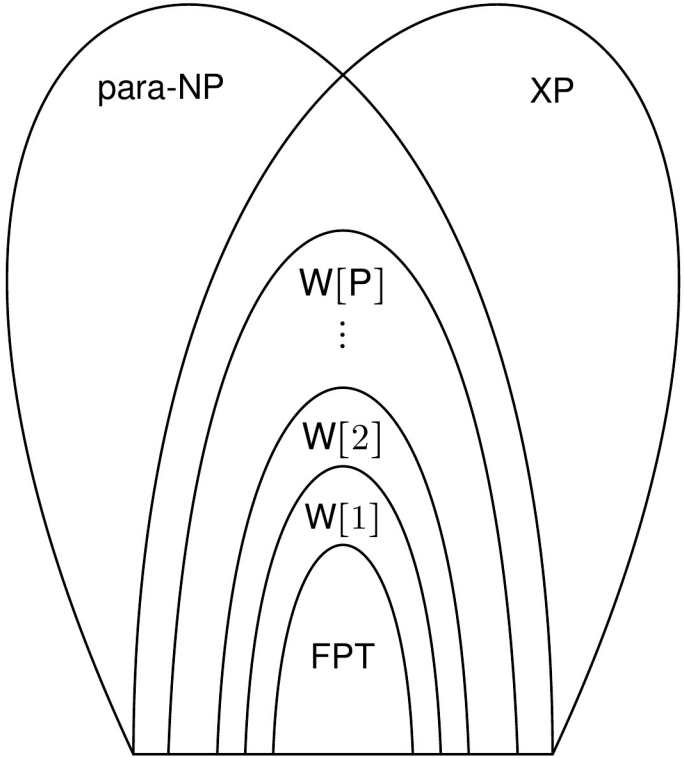
'The purpose [...] is to give evidence that certain problems are not fixed-parameter tractable (just as the main purpose of the theory of NP*-completeness is to give evidence that certain problems are not polynomial time computable.)*

In classical theory, the notion of NP*-completeness is central to a nice, simple, and far-reaching theory for intractable problems.*

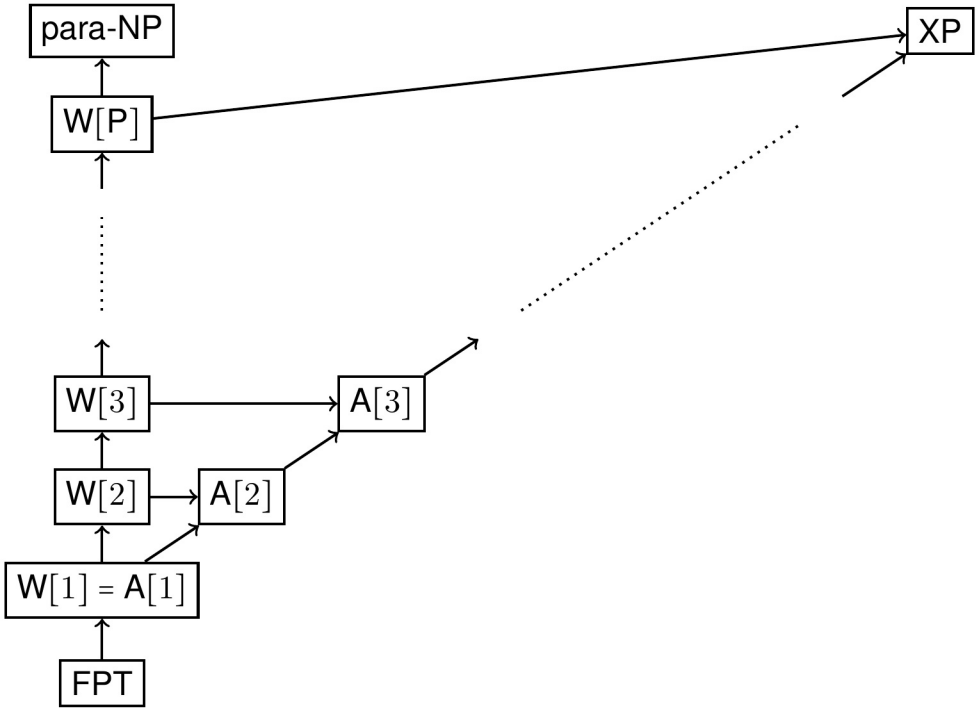
Unfortunately, the world of parameterized intractability is more complex: There is a big variety of seemingly different classes of intractable parameterized problems.'

(Flum, Grohe [2])

W-Hierarchy versus para-NP and XP



W-Hierarchy and A-Hierarchy versus para-NP and XP



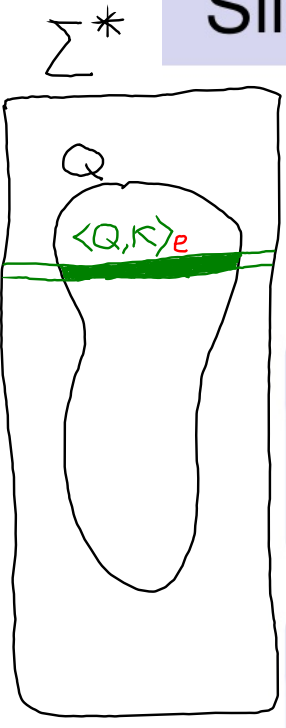
Fixed-Parameter tractable

Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *fixed-parameter tractable* (is in FPT) if:

$$\exists f : \mathbb{N} \to \mathbb{N} \text{ computable } \exists p \in \mathbb{N}[X] \text{ polynomial}$$
$$\exists \mathbb{A} \text{ algorithm, takes inputs in } \Sigma^*$$
$$\forall x \in \Sigma^* \big[\, \mathbb{A} \text{ decides whether } x \in Q \text{ holds in time } \leq f(\kappa(x)) \cdot p(|x|) \,\big]$$

Slices of parameterized problems



The ℓ-*th slice*, for $\ell \in \mathbb{N}$, of a parameterized problem $\langle Q, \kappa \rangle$ is:

$$\langle Q, \kappa \rangle_\ell := \{x \in Q \mid \kappa(x) = \ell\} \ .$$

Proposition (slices of FPT problems are in PTIME)

Let $\langle Q, \kappa \rangle$ be a parameterized problem, and $\ell \in \mathbb{N}$.
If $\langle Q, \kappa \rangle \in$ FPT, then $\langle Q, \kappa \rangle_\ell \in$ PTIME.

Proof

Let ℓ be fixed. Then for all $x \in \Sigma^*$:
Decide $x \in Q$, $\kappa(x) = \ell$ in time $\leq f(\kappa(x)) \cdot p(|x|) = f(\ell) \cdot p(|x|) \in$ PTIME.

A problem not in FPT / XP

The *ℓ-th slice*, for $\ell \in \mathbb{N}$, of a parameterized problem $\langle Q, \kappa \rangle$ is:

$$\langle Q, \kappa \rangle_\ell := \{x \in Q \mid \kappa(x) = \ell\} \ .$$

XP

Proposition:

Slices of FPT problems are in PTIME

If $\langle Q, \kappa \rangle \in$ FPT, then $\langle Q, \kappa \rangle_\ell \in$ PTIME.

$\in$ XP

p-COLORABILITY
Instance: A graph $\mathcal{G}$, and $\ell \in \mathbb{N}$.
Parameter: ℓ.
Problem: Decide whether $\mathcal{G}$ is ℓ-colorable.

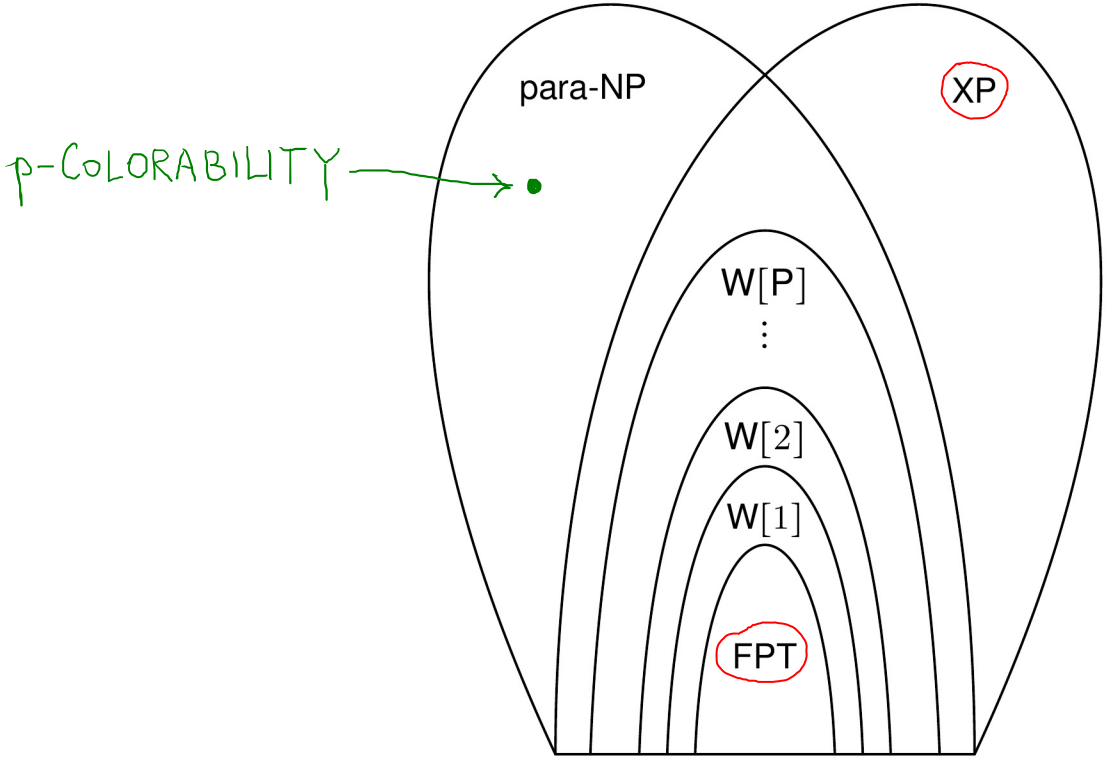
$\notin$ XP

Consequence: p-COLORABILITY $\notin$ FPT (unless P = NP).

It is well-known: 3-COLORABILITY $\in$ NP-complete. Now since 3-COLORABILITY is the third slice of p-COLORABILITY, the proposition entails p-COLORABILITY $\notin$ FPT unless P = NP.

$\notin$ XP

W-Hierarchy versus para-NP and XP



Polynomial reductions / hardness / completeness

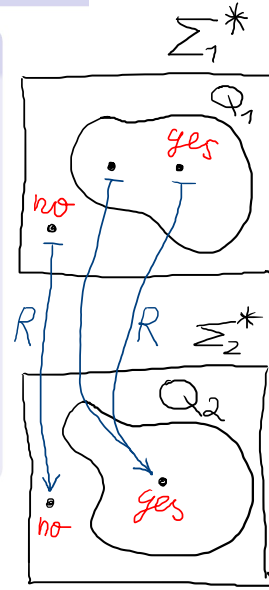
Definition

Let $\langle Q_1, \Sigma_1 \rangle$, $\langle Q_2, \Sigma_2 \rangle$ be classical problems.
An *polynomial-time reduction* from $\langle Q_1, \Sigma_1 \rangle$ to $\langle Q_2, \Sigma_2 \rangle$ is a mapping $R : \Sigma_1^* \to \Sigma_2^*$:

R1. $(x \in Q_1 \iff R(x) \in Q_2)$ for all $x \in \Sigma_1^*$.

R2. R is computable by a polynomial-time algorithm: there is a polynomial $p(X)$ such that R is computable in time $p(|x|)$.

$\langle Q_1, \Sigma_1 \rangle \leq_{\text{pol}} \langle Q_2, \Sigma_2 \rangle :=$
there is a polynomial-time reduction from $\langle Q_1, \Sigma_1 \rangle$ to $\langle Q_2, \Sigma_2 \rangle$.



Polynomial reductions / hardness / completeness

Definition (Richard Karp)

Let $\langle Q_1, \Sigma_1 \rangle$, $\langle Q_2, \Sigma_2 \rangle$ be classical problems.
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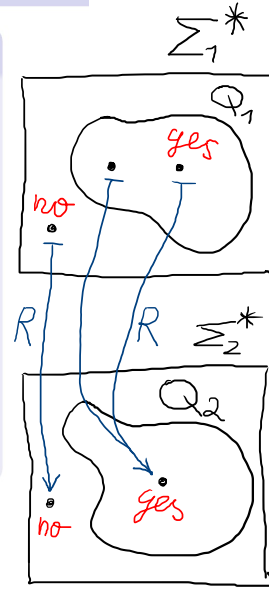
R1. $(x \in Q_1 \iff R(x) \in Q_2)$ for all $x \in \Sigma_1^*$.

R2. R is computable by a polynomial-time algorithm: there is a polynomial $p(X)$ such that R is computable in time $p(|x|)$.

$\langle Q_1, \Sigma_1 \rangle \leq_{\text{pol}} \langle Q_2, \Sigma_2 \rangle :=$
there is a polynomial-time reduction from $\langle Q_1, \Sigma_1 \rangle$ to $\langle Q_2, \Sigma_2 \rangle$.

(polynomial many-to-one reduction)



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An *polynomial-time reduction* from $\langle Q_1, \Sigma_1 \rangle$ to $\langle Q_2, \Sigma_2 \rangle$ is a mapping $R : \Sigma_1^* \to \Sigma_2^*$:

R1. $(x \in Q_1 \iff R(x) \in Q_2)$ for all $x \in \Sigma_1^*$.

R2. R is computable by a polynomial-time algorithm: there is a polynomial $p(X)$ such that R is computable in time $p(|x|)$.

$\langle Q_1, \Sigma_1 \rangle \leq_{\text{pol}} \langle Q_2, \Sigma_2 \rangle :=$
there is a polynomial-time reduction from $\langle Q_1, \Sigma_1 \rangle$ to $\langle Q_2, \Sigma_2 \rangle$.

Proposition

If $\langle Q_1, \Sigma_1 \rangle \leq_{\text{pol}} \langle Q_2, \Sigma_2 \rangle$, then: $\langle Q_1, \Sigma_1 \rangle \in \mathsf{P} \iff \langle Q_2, \Sigma_2 \rangle \in \mathsf{P}$.
$\langle Q_1, \Sigma_1 \rangle \notin \mathsf{P} \implies \langle Q_2, \Sigma_2 \rangle \notin \mathsf{P}$.

Proposition.

if $\langle Q_1, \Sigma_1 \rangle \leq_p \langle Q_2, \Sigma_2 \rangle$

then $Q_1 \in P \Leftarrow Q_2 \in P$

Proof Idea.

Construct alg. A_1

$\Sigma_1^* \ni x \xrightarrow{R} R(x) \in \Sigma_2^*$

A_R

A_2 polynomial decision alg. for Q_2

$time_{A_2}(x) \leq p_2(|x|)$

$R(x) \notin Q_2$ $\Leftrightarrow$ (R1) $x \notin Q_1$

$R(x) \in Q_2$ $\Leftrightarrow$ (R1) $x \in Q_1$

$$\begin{aligned} time\ A_R(x) &= time\ A_R(x) + time\ A(R(x)) \\ &\leq p_R(|x|) + p_2(|R(x)|) \\ &\leq p_R(|x|) + p_2(|x| + p_R(|x|)) \\ &\leq \tilde{p}(|x|) \end{aligned}$$

$|R(x)| \leq |x| + p_R(|x|)$

for polynomial $\tilde{p}(n) := p_R(n) + p_2(n + p_R(n))$

Polynomial reductions / hardness / completeness

Definition

Let $\langle Q_1, \Sigma_1 \rangle$, $\langle Q_2, \Sigma_2 \rangle$ be classical problems.
An *polynomial-time reduction* from $\langle Q_1, \Sigma_1 \rangle$ to $\langle Q_2, \Sigma_2 \rangle$ is a mapping $R : \Sigma_1^* \to \Sigma_2^*$:

R1. $(x \in Q_1 \iff R(x) \in Q_2)$ for all $x \in \Sigma_1^*$.

R2. R is computable by a polynomial-time algorithm: there is a polynomial $p(X)$ such that R is computable in time $p(|x|)$.

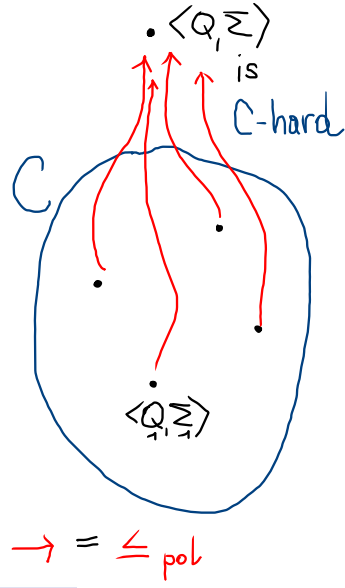
$\langle Q_1, \Sigma_1 \rangle \leq_{\text{pol}} \langle Q_2, \Sigma_2 \rangle :=$
there is a polynomial-time reduction from $\langle Q_1, \Sigma_1 \rangle$ to $\langle Q_2, \Sigma_2 \rangle$.

Proposition

If $\langle Q_1, \Sigma_1 \rangle \leq_{\text{pol}} \langle Q_2, \Sigma_2 \rangle$, then: $\langle Q_1, \Sigma_1 \rangle \in \mathsf{P} \iff \langle Q_2, \Sigma_2 \rangle \in \mathsf{P}$.
$\langle Q_1, \Sigma_1 \rangle \notin \mathsf{P} \implies \langle Q_2, \Sigma_2 \rangle \notin \mathsf{P}$.

Let C be class of classical problems.

- $\langle Q, \Sigma \rangle$ is C-hard: if, for all $\langle Q', \Sigma' \rangle \in \mathsf{C}$, $\langle Q', \Sigma' \rangle \leq_{\text{pol}} \langle Q, \Sigma \rangle$.



Polynomial reductions / hardness / completeness

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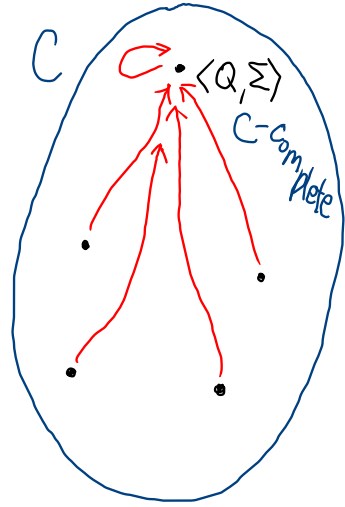
- $\langle Q, \Sigma \rangle$ is C-complete: if $\langle Q, \Sigma \rangle$ is C-hard, and $\langle Q, \Sigma \rangle \in \mathsf{C}$.



Polynomial reductions / hardness / completeness

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$\langle Q_1, \Sigma_1 \rangle \leq_{\text{pol}} \langle Q_2, \Sigma_2 \rangle :=$
there is an polynomial-time reduction from $\langle Q, \kappa \rangle$ to $\langle Q', \kappa' \rangle$.

Proposition

If $\langle Q_1, \Sigma_1 \rangle \leq_{\text{pol}} \langle Q_2, \Sigma_2 \rangle$, then: $\langle Q_1, \Sigma_1 \rangle \in \mathsf{P} \iff \langle Q_2, \Sigma_2 \rangle \in \mathsf{P}$.
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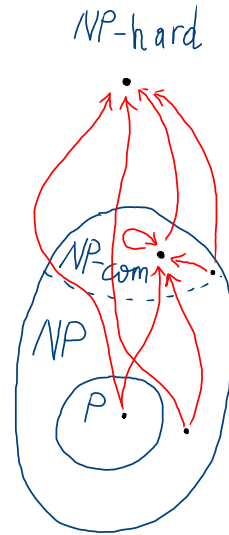

Fixed-parameter tractable reductions

Definition

Let $\langle Q_1, \Sigma_1, \kappa \rangle$, $\langle Q_2, \Sigma_2, \kappa_2 \rangle$ be parameterized problems.
An *fpt-reduction* from $\langle Q_1, \kappa_1 \rangle$ to $\langle Q_2, \kappa_2 \rangle$ is a mapping $R : \Sigma_1^* \to (\Sigma_2)^*$:

R1. $(x \in Q_1 \iff R(x) \in Q_2)$ for all $x \in \Sigma_1^*$.

R2. R is computable by a fpt-algorithm (with respect to κ): there are f computable and $p(X)$ polynomial such that R is computable in time $f(\kappa_1(x)) \cdot p(|x|)$.

Fixed-parameter tractable reductions

Definition

Let $\langle Q_1, \Sigma_1, \kappa\rangle$, $\langle Q_2, \Sigma_2, \kappa_2\rangle$ be parameterized problems.
An *fpt-reduction* from $\langle Q_1, \kappa_1\rangle$ to $\langle Q_2, \kappa_2\rangle$ is a mapping $R : \Sigma_1^* \to (\Sigma_2)^*$:

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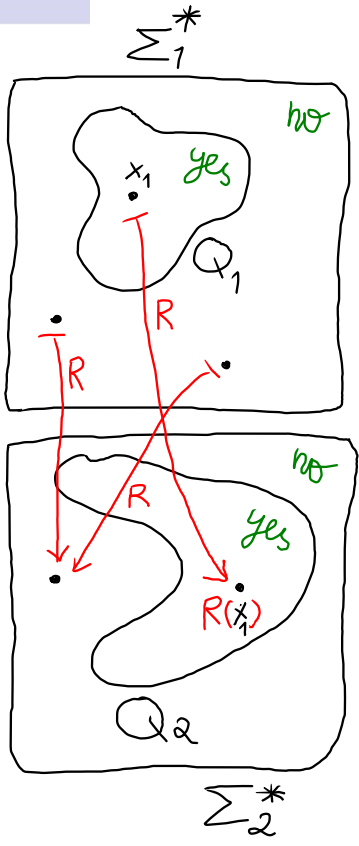
R2. R is computable by a fpt-algorithm (with respect to κ): there are f computable and $p(X)$ polynomial such that R is computable in time $f(\kappa_1(x)) \cdot p(|x|)$.

R3. $\kappa_2(R(x)) \leq g(\kappa_1(x))$ for all $x \in \Sigma_1^*$, for some computable function $g : \mathbb{N} \to \mathbb{N}$.

$\langle Q_1, \kappa_1\rangle \leq_{\text{fpt}} \langle Q_2, \kappa_2\rangle :=$ there is an fpt-red. from $\langle Q_1, \kappa_1\rangle$ to $\langle Q_2, \kappa_2\rangle$.



$\kappa_2(R(x)) \leq g(\kappa_1(x))$

Fixed-parameter tractable reductions

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R3. $\kappa_2(R(x)) \leq g(\kappa_1(x))$ for all $x \in \Sigma_1^*$, for some computable function $g : \mathbb{N} \to \mathbb{N}$.

$\langle Q_1, \kappa_1\rangle \leq_{\text{fpt}} \langle Q_2, \kappa_2\rangle$:= there is an fpt-red. from $\langle Q_1, \kappa_1\rangle$ to $\langle Q_2, \kappa_2\rangle$.

Proposition

If $\langle Q_1, \kappa_1\rangle \leq_{\text{fpt}} \langle Q_2, \kappa_2\rangle$, then: $\langle Q_1, \kappa_1\rangle \in \text{FPT} \iff \langle Q_2, \kappa_2\rangle \in \text{FPT}$.
$\langle Q_1, \kappa_1\rangle \notin \text{FPT} \implies \langle Q_2, \kappa_2\rangle \notin \text{FPT}$.

$\langle Q_1, \kappa_1\rangle, \langle Q_2, \kappa_2\rangle$ parameterized problems

Def. $\langle Q_1,\kappa_1\rangle \leq_{fpt} \langle Q_2,\kappa_2\rangle \iff \exists R: \Sigma_1^* \to \Sigma_2^*$: (R1) $\forall x \in \Sigma^*\ (x\in Q_1 \iff R(x)\in Q_2)$,
(R2) computable in $f(\kappa_1(x))\cdot p(|x|)$,
(R3) $\kappa_2(R(x)) \leq g(\kappa_1(x))$.

Prop. $\langle Q_1,\kappa_1\rangle \leq_{fpt} \langle Q_2,\kappa_2\rangle \Rightarrow \left(\langle Q_1,\kappa_1\rangle \in FPT \Leftarrow \langle Q_2,\kappa_2\rangle \in FPT\right)$ assuming $\langle Q_1,\Sigma_1\rangle \leq_{fpt} \langle Q_2,\Sigma_2\rangle$.
↳ (R1), (R2), (R3)

Proof

$\langle Q_1,\kappa_1\rangle \in FPT \Leftarrow \langle Q_2,\kappa_2\rangle \in FPT$

A_1 (x): $\Sigma_1^* \ni x \xrightarrow{R} \boxed{A_R} \to R(x)\in\Sigma_2^* \to \boxed{A_2}$

$\text{time}_{A_R}(x) \leq f(\kappa_1(x))\cdot p(|x|)$ (R2)

$R(x)\in Q_2$?
- yes: $R(x)\in Q_2$ (R1) $x\in Q_1$
- no: $R(x)\notin Q_2$ (R1) $x\notin Q_1$

$\text{time}_{A_1}(x) = \text{time}_{A_R}(x) + \text{time}_{A_2}(R(x))$

$\leq f(\kappa_1(x))\cdot p(|x|) + f_2(g(\kappa_1(x)))\cdot \underbrace{p_2(|x|+f(\kappa_1(x))\cdot p(|x|))}$

$= \tilde{f}(\kappa_1(x))\cdot\tilde{p}(|x|)$ for appropriate $\tilde{f}: \mathbb{N}\to\mathbb{N}$ and polynomial $\tilde{p}(x)$

$\leq f(\kappa_1(x))\cdot p(|x|) + (f_2\circ g)(\kappa_1(x))\cdot \tilde{f}(\kappa_1(x))\cdot\tilde{p}(|x|)$

$\leq \widetilde{\tilde{f}}(\kappa_1(x))\cdot\widetilde{\tilde{p}}(|x|) \in FPT$ for appropriate $\widetilde{\tilde{f}}: \mathbb{N}\to\mathbb{N}$ and polynomial $\widetilde{\tilde{p}}$.

$\text{time}_{A_2}(R(x))$
$\leq f_2(\underbrace{\kappa_2(R(x))}_{\leq g(\kappa_1(x))\ (R3)})\cdot p_2(\underbrace{|R(x)|}_{\leq |x|\ (R2)})$
$+\text{time}_{A_R}(x)$
$\leq f_2(g(\kappa_1(x)))\cdot p_2(|x|+f(\kappa_1(x))\cdot p(|x|))$ (R2)

Comparing parameterizations (revisited)

Definition (computably bounded below)

Let $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ parameterizations.

- $\kappa_1 \geq \kappa_2 \;:\Longleftrightarrow\; \exists g : \mathbb{N} \to \mathbb{N}$ computable $\forall x \in \Sigma^* \big[\, g(\kappa_1(x)) \geq \kappa_2(x) \,\big]$.
- $\kappa_1 \approx \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \geq \kappa_2 \;\wedge\; \kappa_2 \geq \kappa_1$.
- $\kappa_1 > \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \geq \kappa_2 \;\wedge\; \neg\big(\kappa_2 \geq \kappa_1\big)$.

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with $\kappa_1 \geq \kappa_2$:

$$\langle Q, \kappa_1 \rangle \in \mathsf{FPT} \iff \langle Q, \kappa_2 \rangle \in \mathsf{FPT}\,,$$
$$\langle Q, \kappa_1 \rangle \notin \mathsf{FPT} \implies \langle Q, \kappa_2 \rangle \notin \mathsf{FPT}\,.$$

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- $\kappa_1 \geq \kappa_2 \iff \exists g : \mathbb{N} \to \mathbb{N}$ computable $\forall x \in \Sigma^* \big[\, g(\kappa_1(x)) \geq \kappa_2(x) \,\big]$.
- $\kappa_1 \approx \kappa_2 \iff \kappa_1 \geq \kappa_2 \;\wedge\; \kappa_2 \geq \kappa_1$.
- $\kappa_1 > \kappa_2 \iff \kappa_1 \geq \kappa_2 \;\wedge\; \neg\big(\kappa_2 \geq \kappa_1\big)$.

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$$\langle Q, \kappa_1 \rangle \notin \mathsf{FPT} \implies \langle Q, \kappa_2 \rangle \notin \mathsf{FPT}.$$

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with $Q \subseteq \Sigma^*$:

$$\kappa_1 \geq \kappa_2 \iff \langle Q, \kappa_1 \rangle \leq_{\mathsf{fpt}} \langle Q, \kappa_2 \rangle \text{ via } R : \Sigma^* \to \Sigma^*,\ x \mapsto x.$$

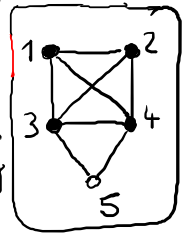
Fixed-parameter tractable reductions

Examples

- p-CLIQUE $\equiv_{\text{fpt}}$ p-INDEPENDENT-SET.
- p-DOMINATING-SET $\equiv_{\text{fpt}}$ p-HITTING-SET.

p-CLIQUE $\equiv_{fpt}$ p-INDEPENDENT-SET

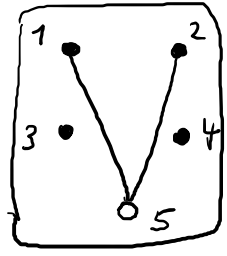
$\langle V, E \rangle = G$



$S = \{1,2,3,4\}$

k-clique

$\overline{G}$



$S = \{1,2,3,4\}$

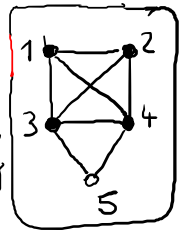
k-independent set

$\overline{G} = \langle V, \{\{u,v\} / u \neq v, \{u,v\} \notin E\}$

complement graph

In general: $S \subseteq V$ k-clique in G $\Leftrightarrow$ $S \subseteq V$ is k-element independent set in $\overline{G}$

p-CLIQUE $\equiv_{fpt}$ p-INDEPENDENT-SET



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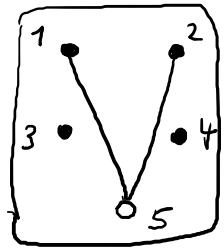


$\overline{G}$

$S = \{1,2,3,4\}$

k-independent set

$\overline{G} = \langle V, \{\{u,v\} / u \neq v, \{u,v\} \notin E\}$

complement graph

(R1)

In general: $S \subseteq V$ k-clique in G $\Leftrightarrow$ $S \subseteq V$ is k-element independent set in $\overline{G}$

$R: \langle G, k \rangle \xmapsto{R} \langle \overline{G}, k \rangle$ is $\leq_{fpt}$-reduction

complement graph

(R1) ✓

(R2) computable by poly. algor. (surely by cubratic)

(R3) $\kappa_2(R(\langle G,k\rangle)) = \kappa_2(\langle \overline{G}, k\rangle) = k$

$= \kappa_1(\langle G,k\rangle) = id(\kappa_1(\langle G,k\rangle))$

Fixed-parameter tractable reductions

Examples

- p-CLIQUE $\equiv_{\text{fpt}}$ p-INDEPENDENT-SET.
- p-DOMINATING-SET $\equiv_{\text{fpt}}$ p-HITTING-SET.

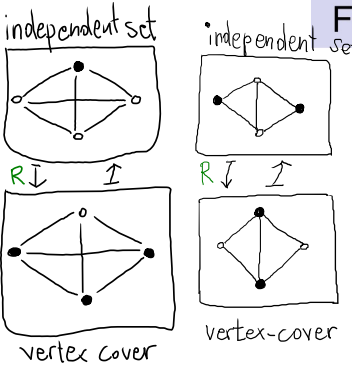
Non-Example

- For graphs $\mathcal{G} = \langle V, E \rangle$, and sets $X \subseteq V$:

  X is independent set of $\mathcal{G}$ $\iff$ $V \setminus X$ is a vertex cover of $\mathcal{G}$

  yields a polynomial reduction between p-INDEPENDENT-SET and p-VERTEX-COVER, but does not yield an fpt-reduction.

Fixed-parameter tractable reductions

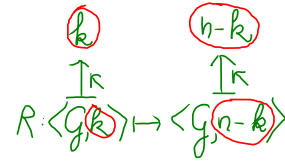


Examples

- p-CLIQUE $\equiv_{\text{fpt}}$ p-INDEPENDENT-SET.
- p-DOMINATING-SET $\equiv_{\text{fpt}}$ p-HITTING-SET.

Non-Example

- For graphs $\mathcal{G} = \langle V, E \rangle$, and sets $X \subseteq V$:
  X is independent set of $\mathcal{G}$ (of size k) $\iff$ $V \setminus X$ is a vertex cover of $\mathcal{G}$ (of size $n-k$)
  yields a polynomial reduction between p-INDEPENDENT-SET and p-VERTEX-COVER, but does not yield an fpt-reduction.



(R1) ✓
(R2) ✓
(R3) ✗

$\forall x \in \Sigma^*_1 \left(\kappa_2(R(x)) \leq g(\kappa_1(x)) \right)$
$\forall n \forall k \left(n-k \leq g(k) \right)$ ✗

$\kappa_2(R(x)) \leq g(\kappa_1(x))$
↓
for some g

p-INDEPENDENT-SET
Instance: graph G, $k \in \mathbb{N}$
Parameter: $k \in \mathbb{N}$
Question: Does G have an independent set of size $\geq k$?

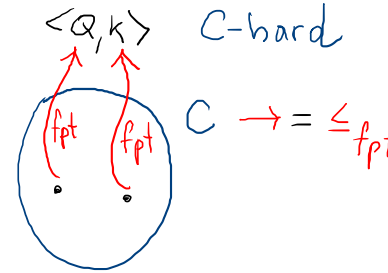
p-VERTEX-COVER
Instance: graph G, $k \in \mathbb{N}$
Parameter: $k \in \mathbb{N}$
Question: Does G have a vertex cover of size $\leq k$?

Fpt-reduction closure / hardness / reducibility

Let C be a class of parameterized problems.

We define for all parameterized problems $\langle Q, \kappa \rangle$:

- $\langle Q, \kappa \rangle$ is C-*hard under fpt-reductions*
  if every problem in C is fpt-reducible to $\langle Q, \kappa \rangle$



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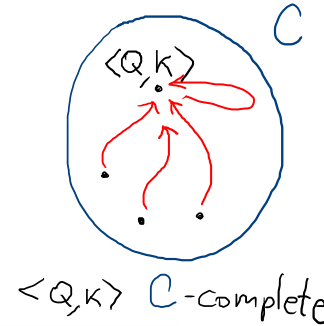
  if every problem in C is fpt-reducible to $\langle Q, \kappa \rangle$
- $\langle Q, \kappa \rangle$ is C-*complete under fpt-reductions*
  if $\langle Q, \kappa \rangle \in$ C and $\langle Q, \kappa \rangle$ is C-hard under fpt-reductions,



Fpt-reduction closure / hardness / reducibility

Let C be a class of parameterized problems.

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- $\left[\langle Q, \kappa \rangle\right]^{\text{fpt}} := \left\{\langle Q', \kappa' \rangle \mid \langle Q', \kappa' \rangle \leq_{\text{fpt}} \langle Q, \kappa \rangle\right\}$.
- $\left[\mathsf{C}\right]^{\text{fpt}} := \bigcup_{\langle Q, \kappa \rangle \in \mathsf{C}} \left[\langle Q, \kappa \rangle\right]^{\text{fpt}}$
  is the *closure of* C *under fpt-reductions.*
- $\langle Q, \kappa \rangle$ is C*-hard under fpt-reductions*
  if every problem in C is fpt-reducible to $\langle Q, \kappa \rangle$
- $\langle Q, \kappa \rangle$ is C*-complete under fpt-reductions*
  if $\langle Q, \kappa \rangle \in$ C and $\langle Q, \kappa \rangle$ is C-hard under fpt-reductions,

Fpt-reduction closure / hardness / reducibility

Let C be a class of parameterized problems.

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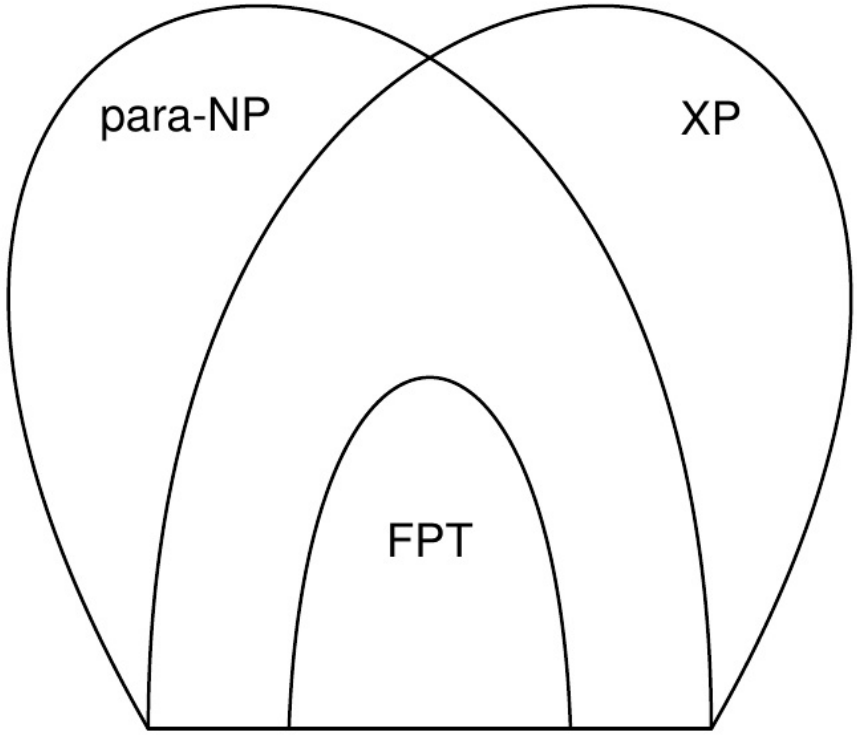
- $\left[\mathsf{C}\right]^{\text{fpt}} := \bigcup_{\langle Q, \kappa \rangle \in \mathsf{C}} \left[\langle Q, \kappa \rangle\right]^{\text{fpt}}$
  is the *closure of* C *under fpt-reductions.*
- $\langle Q, \kappa \rangle$ is C*-hard under fpt-reductions*
  if every problem in C is fpt-reducible to $\langle Q, \kappa \rangle$
  that is: $\mathsf{C} \subseteq \left[\langle Q, \kappa \rangle\right]^{\text{fpt}}$, and hence $\left[\mathsf{C}\right]^{\text{fpt}} \subseteq \left[\langle Q, \kappa \rangle\right]^{\text{fpt}}$.
- $\langle Q, \kappa \rangle$ is C*-complete under fpt-reductions*
  if $\langle Q, \kappa \rangle \in$ C and $\langle Q, \kappa \rangle$ is C-hard under fpt-reductions,
  and then: $\left[\mathsf{C}\right]^{\text{fpt}} = \left[\langle Q, \kappa \rangle\right]^{\text{fpt}}$.

para-NP and XP



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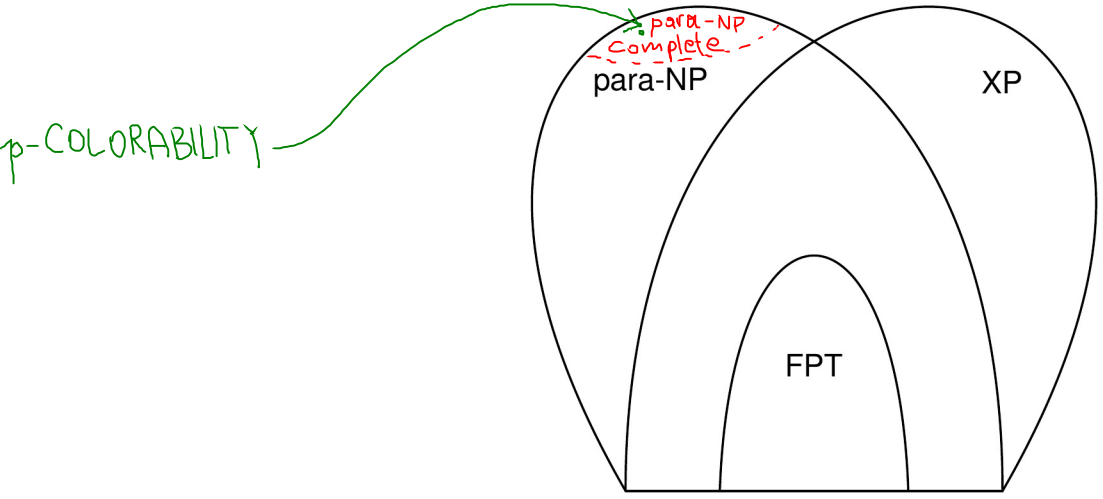
para-NP and XP



XP (slicewise polynomial problems)

Recall: slices of FPT-problems are in PTIME. This suggests a class:

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equivalently, if in addition to computable $f : \mathbb{N} \to \mathbb{N}$ there are polynomials $p_k \in \mathbb{N}[X]$ for all $k \in \mathbb{N}$ such that:

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FPT $\subsetneqq$ XP .

Model checking

The *model checking problem* for a class Φ of first-order formulas:

MC(Φ)

Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.

Problem: Decide whether $\mathcal{A} \models \varphi$ (that is, $\varphi(\mathcal{A}) \neq \emptyset$).

Theorem

MC(FO) *can be solved in time* $O(|\varphi| \cdot |A|^w \cdot w)$*, where* w *is the width of the input formula* φ *(max. no. of free variables in a subformula of* φ*).*

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The *parameterized model checking problem* for a class Φ of formulas:

p-MC(Φ).
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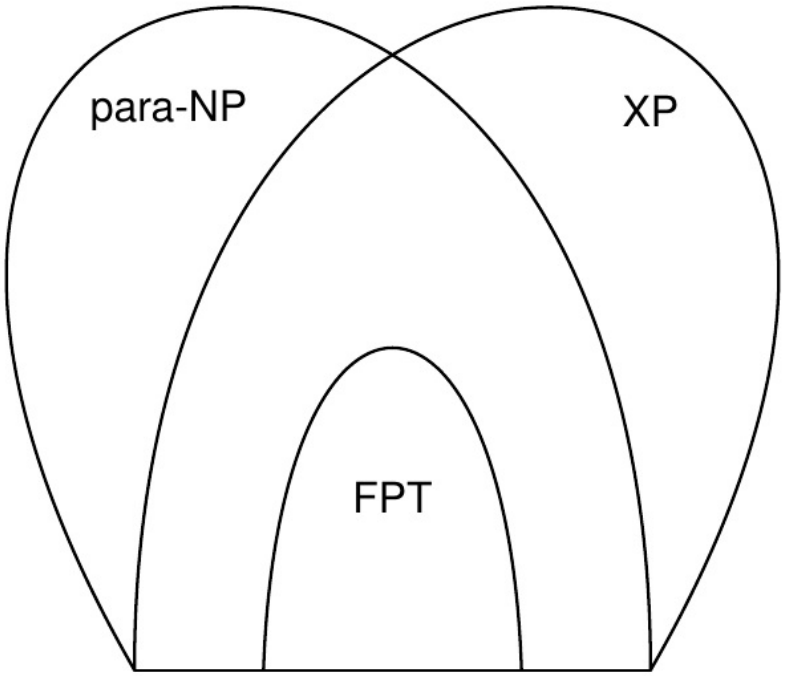
Parameter: $|\varphi|$.
Problem: Decide whether $\mathcal{A} \models \varphi$.

Theorem

p-MC(Φ) $\in$ XP.

FPT versus para-NP and XP



Proposition

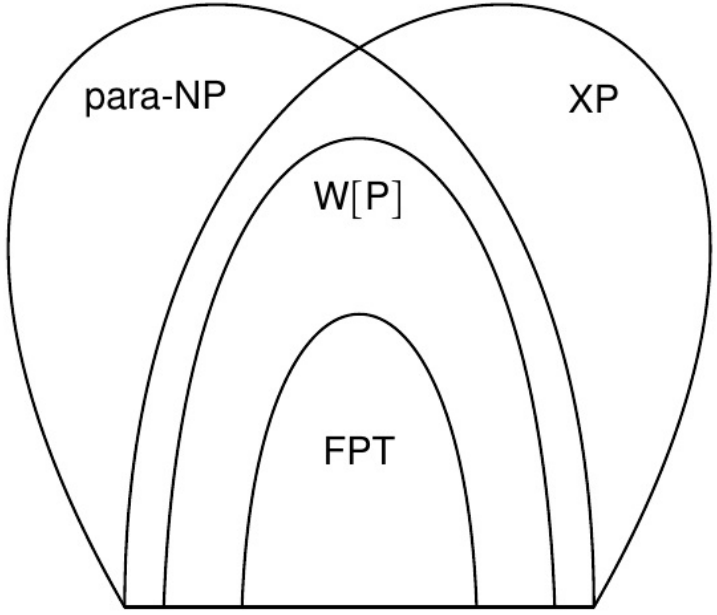
- FPT $\subseteq$ para-NP, and:
  FPT $=$ para-NP if and only if PTIME $=$ NP.
- para-NP $\not\subseteq$ XP if PTIME $\neq$ NP.
- FPT $\subsetneq$ XP.

W[P]

'There is no definite single class that can be viewed as "the parameterized NP*". Rather, there is a whole hierarchy of classes playing this role.*

The class W[P] *can be placed on top of this hierarchy. It is one of the most important parameterized complexity classes.'*

(Flum, Grohe [2])



W[P] and limited non-determinism

$\langle Q, \Sigma \rangle \in \mathsf{NP}[f]$ means:

$\iff$ $\exists p(X)$ polynomial $\exists \mathbb{M}$ non-deterministic Turingmachine

$\big(\forall x \in \Sigma^* \big((x \in Q \iff \mathbb{M}$ accepts $x)$

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Fact

$$\mathsf{NP}[\log n] = \mathsf{P}\,, \qquad \mathsf{NP}[n^{O(1)}] = \mathsf{NP}\,.$$

W[P]

Definition

- Let Σ be an alphabet, and $\kappa : \Sigma^* \to \mathbb{N}$ a parameterization.

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- W[P] contains all problems $\langle Q, \kappa \rangle$ that can be decided by a κ-restricted nondeterministic Turing machine.

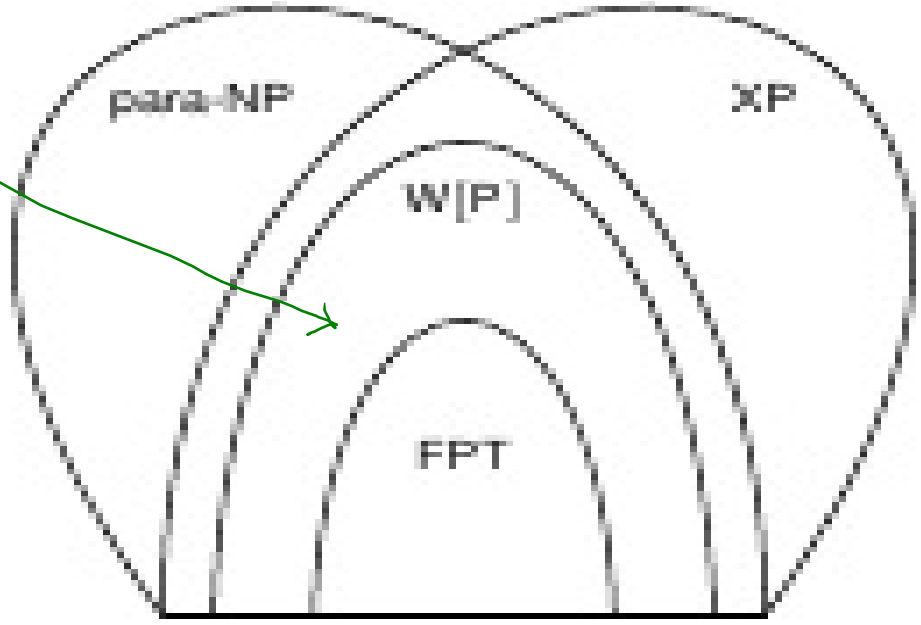
W[P] (properties)

Theorems

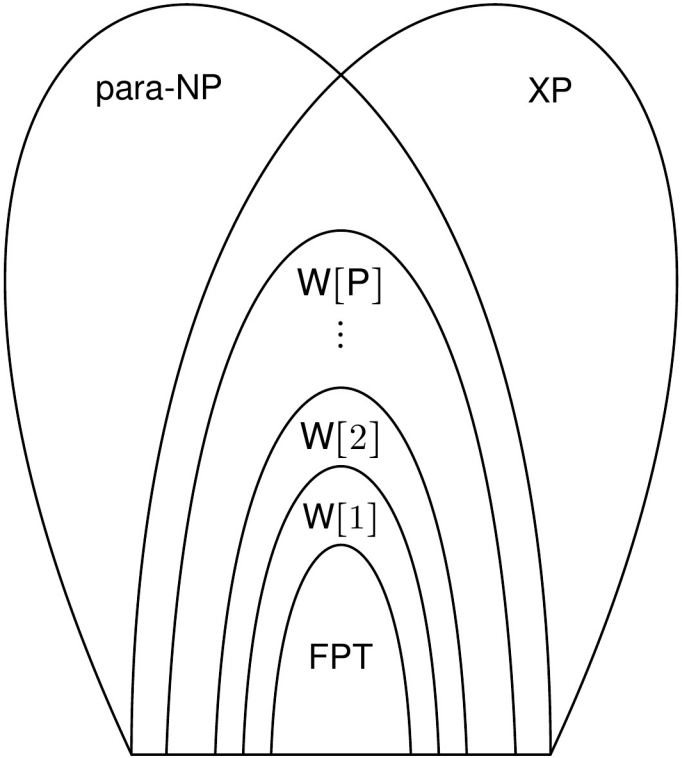
T1. FPT $\subseteq$ W[P] $\subseteq$ XP $\cap$ para-NP

T2. W[P] is closed under fpt-reductions.

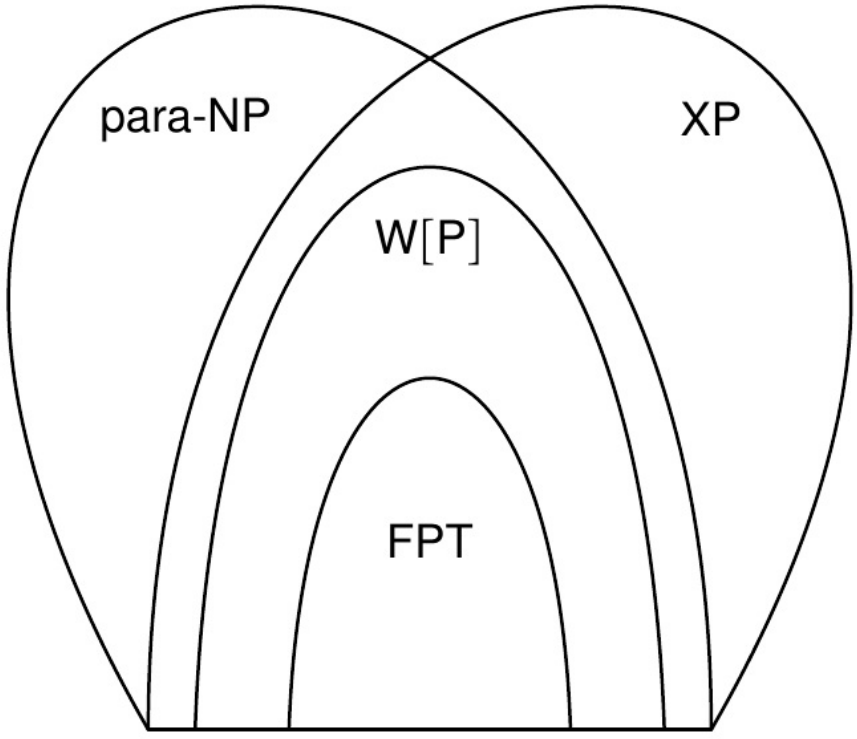
T3. p-CLIQUE, p-INDEPENDENT-SET, p-DOMINATING-SET, and p-HITTING-SET are in W[P].



W-Hierarchy



FPT and W[P] versus para-NP and XP



Proposition

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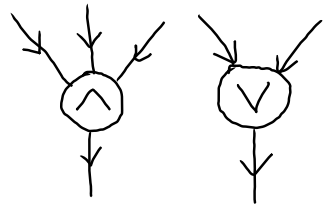
The W-hierarchy – Boolean circuits

A *(Boolean) circuit* is a DAG in which nodes are labeled:



- nodes of in-degree > 1 as *and-node* or as *or-node*,
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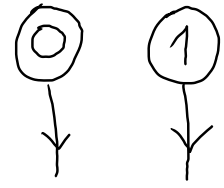

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A circuit $\mathcal{C}$ with n input nodes defines a function $\mathcal{C}(\cdot) : \{0,1\}^n \to \{0,1\}$ (a Boolean function) in the natural way.

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We say that $\mathcal{C}$ is *k-satisfiable* if $\mathcal{C}$ is satisfied by a tuple of weight k.

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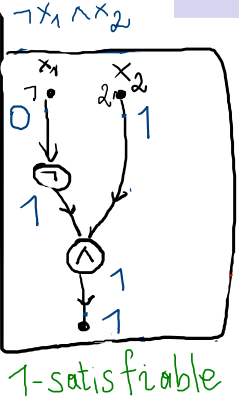

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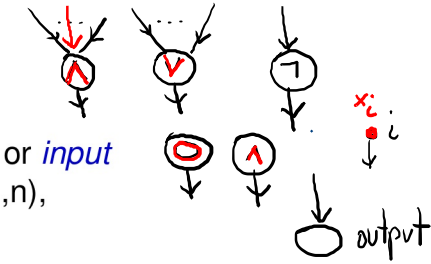

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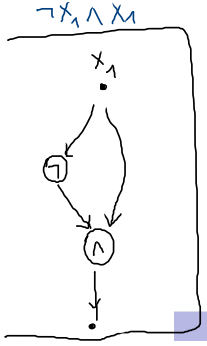

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W[P] complete problems

p-WSAT(CIRC)
Instance: A circuit $\mathcal{C}$ and $k \in \mathbb{N}$
Parameter: k.
Problem: Decide whether $\mathcal{C}$ is k-satisfiable.

Theorem

p-WSAT(CIRC) *is* W[P]*-complete under fpt-reductions.*

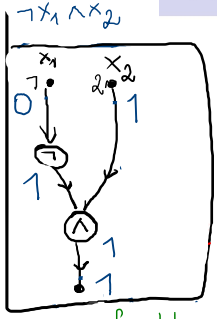
Definition

The depth of the circuit is the max. length of a path from an input node to the output node. Small nodes have indegree at most 2 while large nodes have indegree > 2. The weft of a circuit is the max. number of large nodes on a path from an input node to the output node. We denote by $\text{CIRC}_{t,d}$ the class of circuits with weft $\leq t$ and depth $\leq d$.

Application

p-DOMINATING-SET $\in$ W[P], since it reduces to p-WSAT($\text{CIRC}_{2,3}$).

The W-hierarchy – Boolean circuits



1-satisfiable

depth 3
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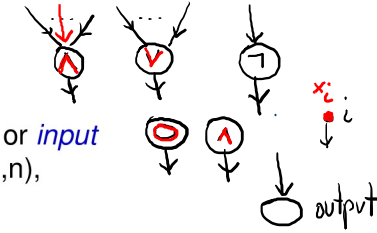
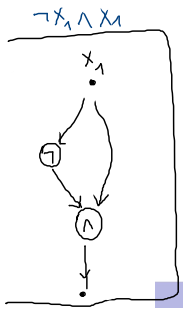
A *(Boolean) circuit* is a DAG in which nodes are labeled:

- nodes of in-degree > 1 as *and-node* or as *or-node*,
- nodes of in-degree $= 1$ as *negation nodes*,
- nodes of in-degree $= 0$ as *Boolean constants* 0 or 1, or *input node* (we assume input nodes to be numbered 1,...,n),
- one node of out-degree 0 is labeled as *output node*.



A circuit $\mathcal{C}$ with n input nodes defines a function $\mathcal{C}(\cdot) : \{0,1\}^n \to \{0,1\}$ (a Boolean function) in the natural way.

- If $\mathcal{C}(x) = 1$, for $x \in \{0,1\}^n$, we say that x *satisfies* $\mathcal{C}$.
- The *weight* of a tuple $x = \langle x_1, \ldots, x_n \rangle \in \{0,1\}^*$ is $\sum_{i=1}^{n} x_i$.



depth 3
weft 0

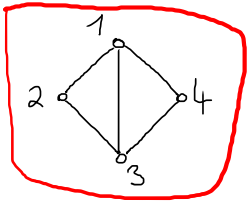
not satisfiable

Definition

We say that $\mathcal{C}$ is *k-satisfiable* if $\mathcal{C}$ is satisfied by a tuple of weight k.

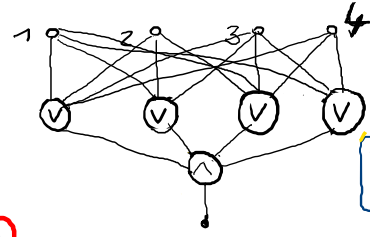
p DOMINATING SET

$\forall x \in V \; \exists y \in V \; (E(x,y) \wedge S(y))$



G → $e(G)$

$e(G) \in CIRC_3(2)$



weft($e(G)$) = 2

depth($e(G)$) = 3

every vertex should get an input from itself or from a neighbour

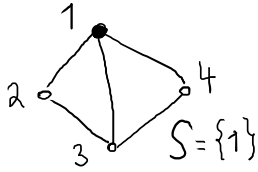
G has dom. set S of k elements $\iff$ $e(G)$ is satisfied by tuple x_S of weight k (R_1)

$\Downarrow$

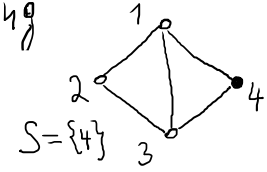
p-DOMINATING-SET $\leq_{fpt}$ p-WSAT($CIRC_{2,3}$)

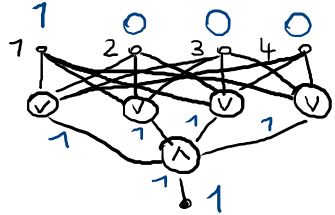
$\Rightarrow$ p-DOM.SET $\in W[2]$

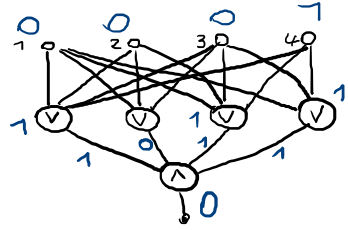
dominating set



no dominating set







W-hierarchy

Definition (with Boolean circuits)

(W-hierarchy) For $t \geq 1$, a parameterized problem $\langle Q, \kappa \rangle$ belongs to the class W$[t]$ if there is a parameterized reduction from $\langle Q, \kappa \rangle$ to p-WSAT(CIRC$_{t,d}$) (with parameter t) for some $d \geq 1$.

FPT $\subseteq$ W$[1] \subseteq$ W$[2] \ldots$

- p-CLIQUE, p-INDEPENDENT-SET are W$[1]$-Complete.
- p-DOMINATING-SET, p-HITTING-SET are W$[2]$-Complete.

Hypothesis: W$[1] \neq$ FPT

Proposition

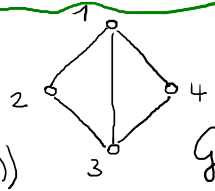
Later

This definition of the W-hierarchy is equivalent to the one here [redacted]. [redacted] It holds, for all $t \geq 1$:

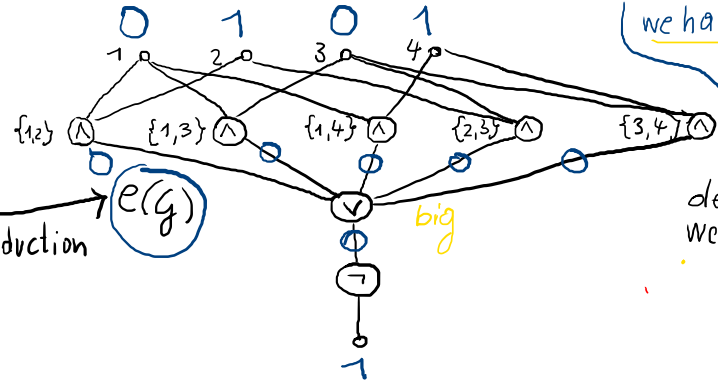
$$\text{W}[t] = \left[\{p\text{-WSAT}(\text{CIRC}_{t,d}) \mid d \geq 1\}\right]^{\text{fpt}}.$$

p-INDEPENDENT-SET

$\forall x \forall y (E(x,y) \rightarrow \neg S(x) \wedge S(y))$



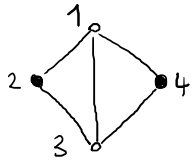
G → reduction → $e(G)$

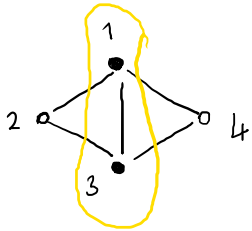


we have to make sure that no edge gets input from both of its vertices

depth = 4
weft = 1

$S = \{2,4\}$ independent set





$S = \{1,3\}$ not an independent set

Hence:

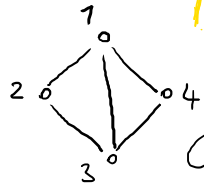
p-INDEPENDENT-SET $\leq_{fpt}$ p-WSAT(CIRC(1,4)) $\Rightarrow$ p-IND.SET $\in$ W[1]

G has a k-element independent set $\Leftrightarrow$ $e(G)$ is k-satisfiable

p-VERTEX-COVER

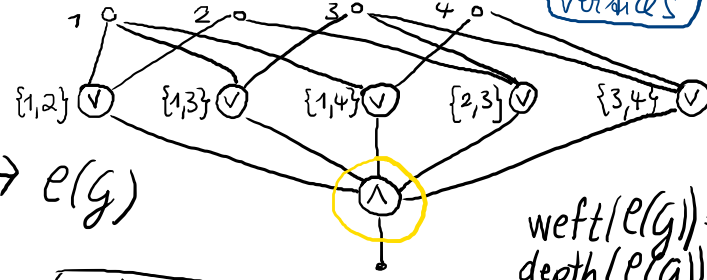
$\Downarrow$

$\forall x \forall y (\in(x,y) \rightarrow S(x) \vee S(y))$



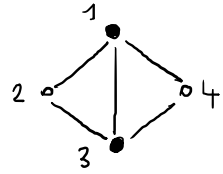
$G \hookrightarrow e(G)$

every edge should receive an input from one of its vertices

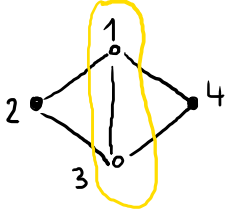


$weft(e(G)) = 1$

$depth(e(G)) = 3$



$S = \{1,3\}$ vertex-cover



$S = \{2,4\}$ not a vertex cover.

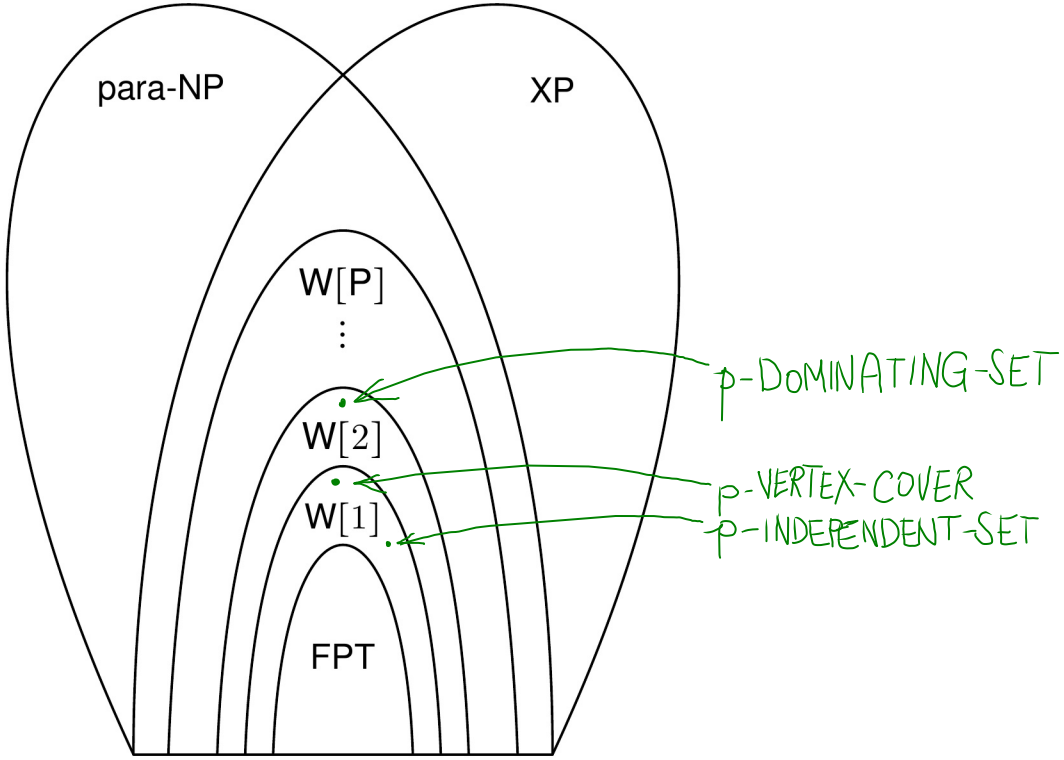
Hence:

p-VERTEX-COVER $\leq_{fpt}$ p-WSAT(CIRC$_{\neg,3}$)

G is a k-element vertex cover $\Leftrightarrow$ $e(G)$ is k-satisfiable

$\Rightarrow$ p-VERTEX-COVER $\in$ W[1]

W-Hierarchy


para-NP
XP
W[P]
W[2]
W[1]
FPT
p-DOMINATING-SET
p-VERTEX-COVER
p-INDEPENDENT-SET

Limited non-determinism (classically)

$\langle Q, \Sigma \rangle \in \mathsf{NP}[f]$ means:

$\iff$ $\exists p(X)$ polynomial $\exists \mathbb{M}$ non-deterministic Turingmachine

$\big(\forall x \in \Sigma^* \big((x \in Q \iff \mathbb{M} \text{ accepts } x)$

$\wedge$ on input x, $\mathbb{M}$ halts in $\leq p(|x|)$ steps, of which

at most $\leq f(|x|)$ are non-deterministic$\big)$

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$\mathsf{NP}[\mathcal{F}] := \bigcup_{f \in \mathcal{F}} \mathsf{NP}[f]$ for class of functions $\mathcal{F}$.

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Fact

$$\mathsf{NP}[\log n] = \mathsf{P}\,, \qquad \mathsf{NP}[n^{O(1)}] = \mathsf{NP}\,.$$

Theorem (Cai, Chen, 1997)

The following are equivalent:

(i) $\mathsf{FPT} = \mathsf{W}[\mathsf{P}]$.

(ii) *There is a computable, nondecreasing, unbounded function* $\iota : \mathbb{N} \to \mathbb{N}$ *such that* $\mathsf{P} = \mathsf{NP}[\iota(n) \cdot \log n]$.

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Theorem (Cai, Chen, 1997)

The following are equivalent:

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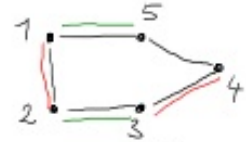
(ii) *There is a computable, nondecreasing, unbounded function* $\iota : \mathbb{N} \to \mathbb{N}$ *such that* $\mathsf{P} \subsetneq \mathsf{NP}[\iota(n) \cdot \log n]$.

"Showing $\mathsf{FPT} \subsetneq \mathsf{W}[\mathsf{P}]$ is probably HARDER than showing $\mathsf{P} \subsetneq \mathsf{NP}$"

Why is the theory of W[P]/W/A-hardness important?

- Prevents from wasting hours tackling a problem which is fundamentally difficult;
- Finding results on a problem is always a ping-pong game between trying to design a hardness/FPT result;
- There is a hierarchy on parameters and it is worth knowing which is the smallest one such that the problem remains FPT;
- There is a hierarchy on complexity classes and it is worth noting to which extent a problem is hard.

Maximum 2-edge colorable subgraphs



p^*-*tw*-max-2-edge-colorable-subgraph
Instance: A graph $\mathcal{G} = \langle V, E \rangle$.
Parameter: $tw(\mathcal{G})$.
Compute: Maximum number of edges in a 2-edge colored subgraph of G.

$S \subseteq E$ is 2-colorable

⇕

Example [AA & Vahan Mkrtchyan]

p^*-*tw*-max-2-edge-colorable-subgraph $\in$ FPT.

We find an MSO_2-formula $\varphi_{2ec}(S)$ such that $(A_{TH_G} \models \varphi(S))$

$\varphi_{2ec}(X) := \forall e\,(X(e) \rightarrow EDGE(e))$
$\wedge\ \exists C_1 \exists C_2 [(\text{edge-set-partition}(C_1, C_2, X)$
$\wedge\ \forall e_1 \forall e_2\,(EDGE(e_1) \wedge EDGE(e_2) \wedge \neg e_1 = e_2$
$\wedge\ \exists v\,(INC(v, e_1), INC(v, e_2))$
$\rightarrow \neg(C_1(e_1) \wedge C_1(e_2))$
$\wedge \neg(C_2(e_1) \wedge C_2(e_2))]$

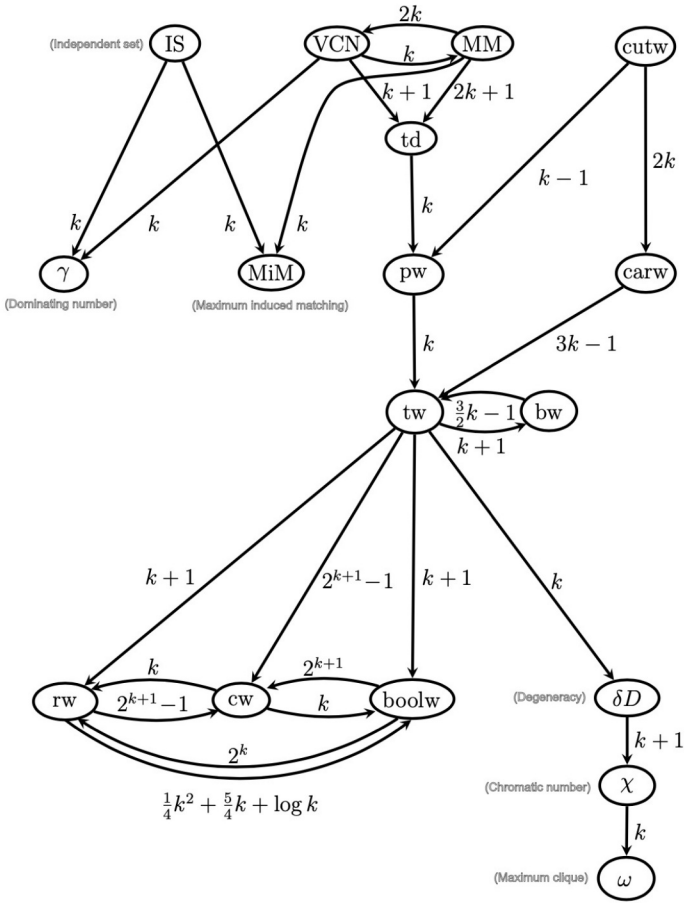
edge-set-partition $(C_1, C_2, X) :=$
$\forall x\,(X(x) \rightarrow EDGE(x))$
$\wedge\ \forall x\,(EDGE(x) \rightarrow (X(x) \leftrightarrow C_1(x) \vee C_2(x))$
$\wedge \neg(C_1(x) \wedge C_2(x)))$

Computably boundedness between notions of width

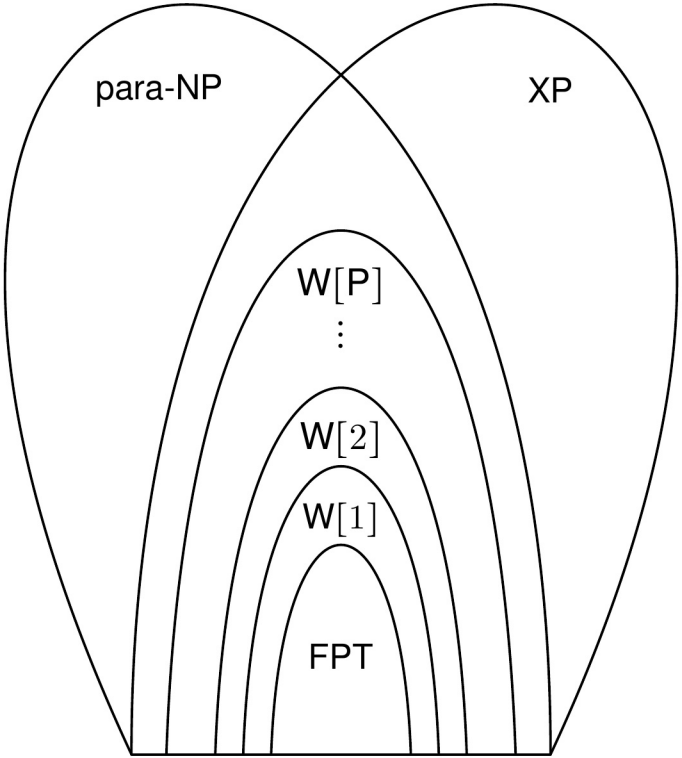
(from Sasák, [6])

$g(wd_1) \geq wd_2 \;:\Leftrightarrow\; wd_1 \xrightarrow{g} wd_2$

- FPT-results transfer upwards
  (and conversely to $\xrightarrow{g}$)
- ($\notin$ FPT)-results transfer downwards
  (and along $\xrightarrow{g}$)



W-Hierarchy



Logic preliminaries (continued)

- *atomic formulas/atoms*: a formula $x = y$ or $Rx_1 \dots x_n$
- *literal*: an atom or a negated atom
- *quantifier-free formula*: a formula without quantifiers
- formula in *negation-normal form*:
  negations only occur in front of atoms
- formula in *prenex normal form*: formula of the form
  $Q_1 x_1 \dots Q_k x_k \, \psi$, where ψ is quantifier-free
  and $Q_1, \dots, Q_k \in \{\exists, \forall\}$

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  $Q_1 x_1 \dots Q_k x_k \, \psi$, where ψ is quantifier-free
  and $Q_1, \dots, Q_k \in \{\exists, \forall\}$
- Σ_0 and Π_0 : the class of quantifier-free formulas
- Σ_{t+1} : class of all formulas $\exists x_1 \dots \exists x_k \varphi$ where $\varphi \in \Pi_t$
- Π_{t+1} : class of all formulas $\forall x_1 \dots \forall x_k \varphi$ where $\varphi \in \Sigma_t$

Weighted Fagin definability

Let $\varphi(X)$ be a f-o formula with a free relation variable X with arity s.
Let τ be a vocabulary for φ, plus a relation symbol R of arity s.

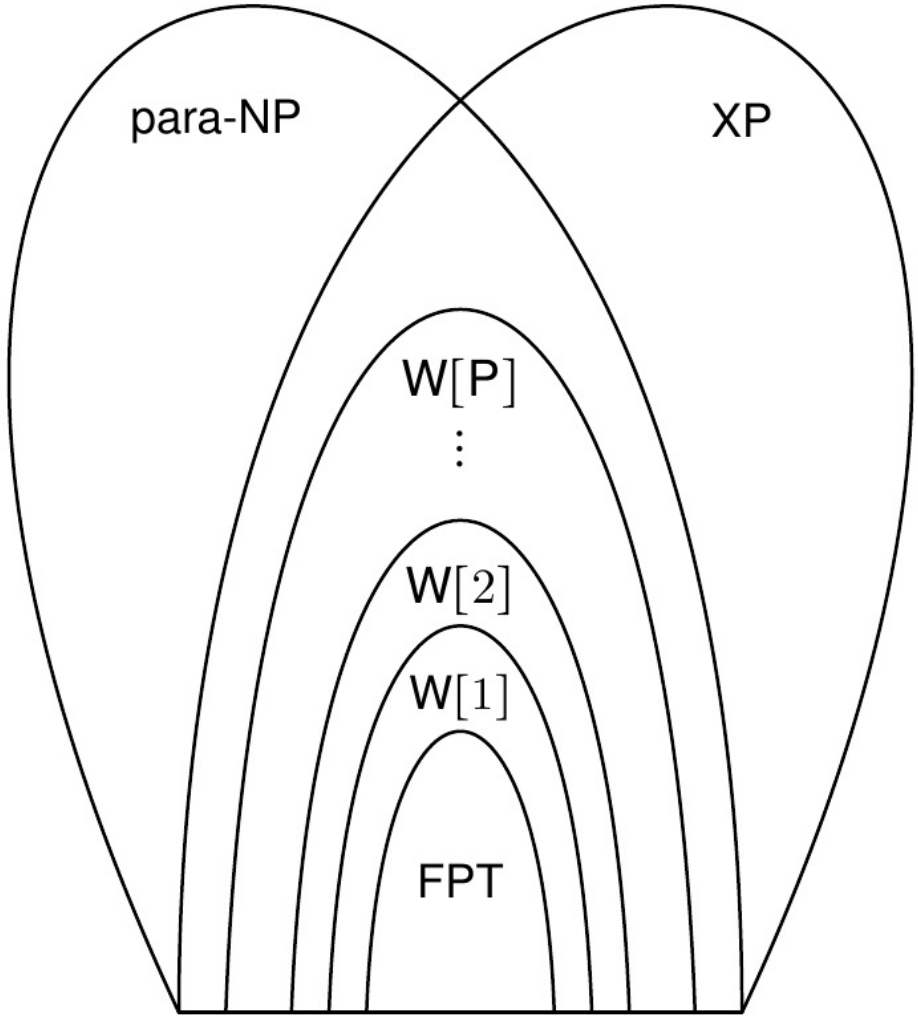
A *solution for φ in a τ-structure $\mathcal{A}$* is a relation $S \subseteq A^s$ such that $\mathcal{A} \models \varphi(\overline{S})$.

The *weighted Fagin definability problem* for $\varphi(X)$ is:

WD_φ
Instance: A structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Problem: Decide whether there is a solution $S \subseteq A^s$ for φ of cardinality $|S| = k$.

WD_Φ: the class of all problems WD_φ with $\varphi \in \Phi$, where Φ is a class of first-order formulas with free relation variable X.

W-Hierarchy



W-Hierarchy

p-WD$_\varphi$ (φ a fo-formula with free relation variable X of arity s)

Instance: A structure $\mathcal{A}$ and $k \in \mathbb{N}$.

Parameter: k.

Problem: Is there a relation $S \subseteq A^s$ of cardinality $|S| = k$ with $\mathcal{A} \models \varphi(S)$.

p-WD-Φ: the class of all problems p-WD-φ with $\varphi \in \Phi$, Φ is a class of first-order formulas.

Definition (Downey–Fellows, 1995)

$W[t] := \left[p\text{-WD-}\Pi_t\right]^{\text{fpt}}$, for $t \geq 1$, form the *W-hierarchy*.

W-Hierarchy

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Examples

- p-Clique $\in$ W$[1]$.
- p-Dominating-Set $\in$ W$[2]$.
- p-Hitting-Set $\in$ W$[2]$.

W-Hierarchy

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Examples

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W-Hierarchy

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Examples

- p-DOMINATING-SET $\in$ W$[2]$.

W-Hierarchy

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Instance: A structure $\mathcal{A}$ and $k \in \mathbb{N}$.

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Examples

- p-HITTING-SET $\in$ W[2].

W-hierarchy

Definition

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Proposition

This definition of the W-hierarchy is equivalent to the one here before. That is, it holds, for all $t \geq 1$:

$$\text{W}[t] = \left[\{p\text{-WSAT}(\text{CIRC}_{t,d}) \mid d \geq 1\}\right]^{\text{fpt}}.$$

W-Hierarchy (properties)

Immediate from definition follows: $\left[p\text{-WD-FO}\right]^{\text{fpt}} = \bigcup_{i=1}^{\infty} \mathsf{W}[i]$.

Theorems

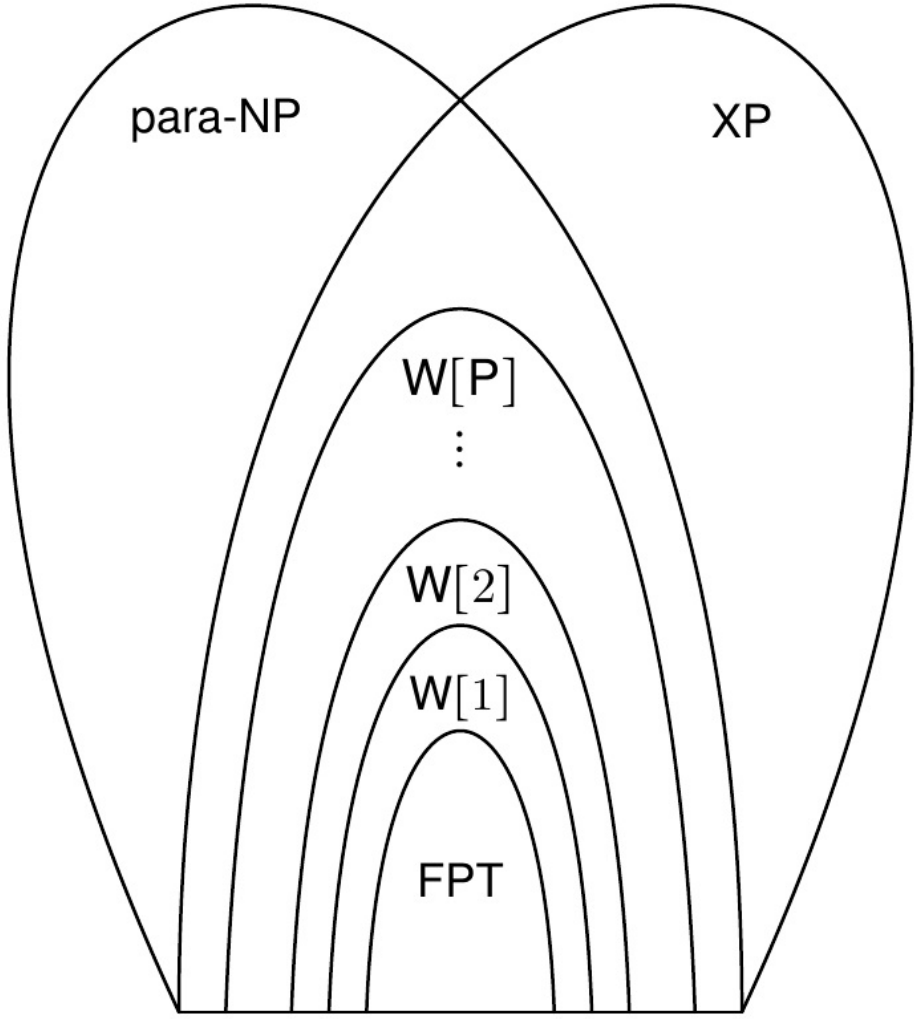
T1. p-WD-FO $\subseteq$ W[P], and hence W$[t] \subseteq$ W[P] for all $t \geq 1$.

T2. p-WD-$\Sigma_1 \subseteq$ FPT.

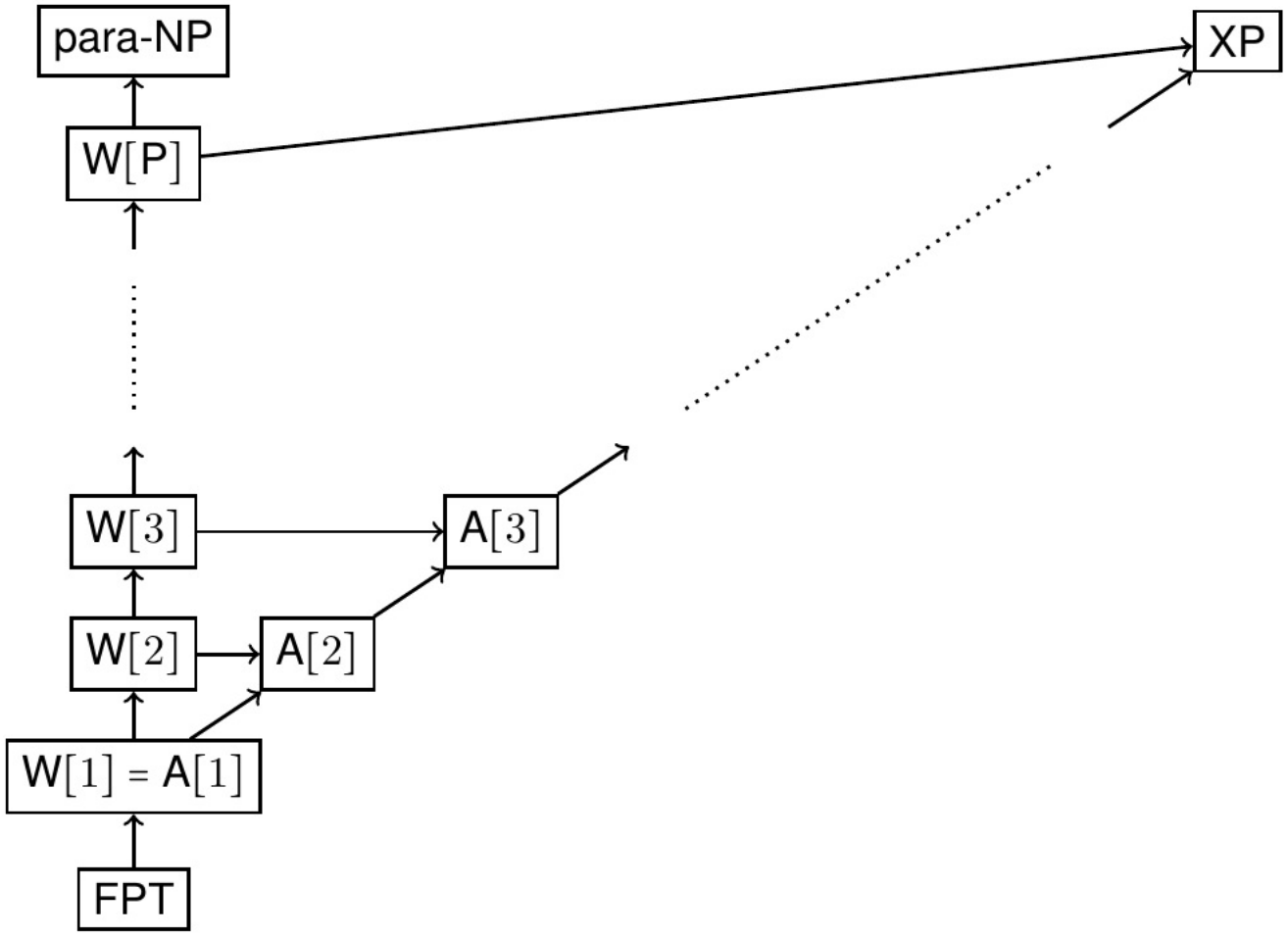
T3. p-WD-$\Sigma_{t+1} \subseteq p$-WD-Π_t, for all $t \geq 1$.

T4. $\mathsf{W}[t] = \left[p\text{-WD-}\Sigma_{t+1}\right]^{\text{fpt}}$ for all $t \geq 1$.

W-Hierarchy versus para-NP and XP



A-Hierarchy



A-Hierarchy (definition and examples 1,2)

The parameterized model checking problem for a class Φ of formulas:

p-MC(Φ)
Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.
Parameter: $|\varphi|$.
Problem: Decide whether $\varphi(\mathcal{A}) \neq \varnothing$.

Definition (Flum, Grohe, 2001)

$A[t] := \left[p\text{-MC}(\Sigma_t)\right]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

A-Hierarchy (definition and examples 1,2)

The parameterized model checking problem for a class Φ of formulas:

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$A[t] := \left[p\text{-MC}(\Sigma_t)\right]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

- p-CLIQUE $\in$ A$[1]$.
- p-DOMINATING-SET $\in$ A$[2]$.

A-Hierarchy (definition and examples 3,4)

The parameterized model checking problem for a class Φ of formulas:

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Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.
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Definition (Flum, Grohe, 2001)

$A[t] := \left[p\text{-MC}(\Sigma_t)\right]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

- p-HITTING-SET $\in$ A$[2]$.
- p-SUBGRAPH-ISOMORPHISM $\in$ A$[1]$.

A-Hierarchy (example 5)

The parameterized model checking problem for a class Φ of formulas:

p-MC(Φ)
Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.
Parameter: $|\varphi|$.
Problem: Decide whether $\varphi(\mathcal{A}) \neq \varnothing$.

Definition (Flum, Grohe, 2001)

$A[t] := \left[p\text{-MC}(\Sigma_t)\right]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

- p-SUBGRAPH-ISOMORPHISM $\in$ A$[1]$.

p-SUBGRAPH-ISOMORPHISM
Instance: Graphs $\mathcal{G}$ and $\mathcal{H}$.
Parameter: The number of vertices of $\mathcal{H}$.
Problem: Does $\mathcal{G}$ have a subgraph isomorphic to $\mathcal{H}$.

A-Hierarchy (example 6)

The parameterized model checking problem for a class Φ of formulas:

p-MC(Φ)
Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.
Parameter: $|\varphi|$.
Problem: Decide whether $\varphi(\mathcal{A}) \neq \varnothing$.

Definition (Flum, Grohe, 2001)

$A[t] := \left[p\text{-MC}(\Sigma_t)\right]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

- p-VERTEX-DELETION $\in$ A$[2]$.

p-VERTEX-DELETION
Instance: Graphs $\mathcal{G}$ and $\mathcal{H}$, and $k \in \mathbb{N}$.
Parameter: $k + \ell$, where ℓ the number of vertices of $\mathcal{H}$.
Problem: Is it possible to delete at most k vertices from $\mathcal{G}$ such that the resulting graph has no subgraph isomorphic to $\mathcal{H}$?

A-Hierarchy (example 7)

The parameterized model checking problem for a class Φ of formulas:

p-MC(Φ)
Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.
Parameter: $|\varphi|$.
Problem: Decide whether $\varphi(\mathcal{A}) \neq \varnothing$.

Definition (Flum, Grohe, 2001)

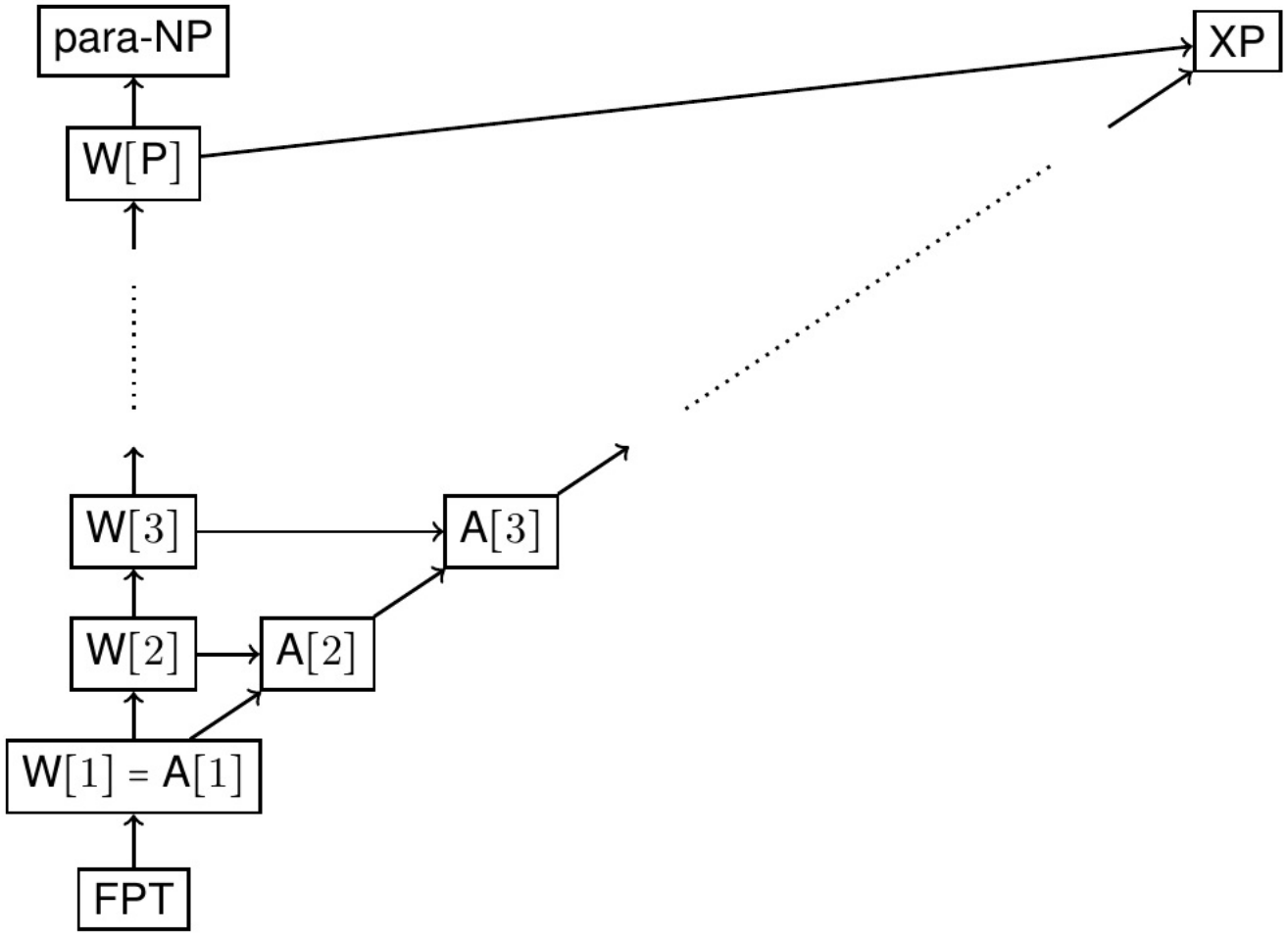
$A[t] := \left[p\text{-MC}(\Sigma_t)\right]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

- p-CLIQUE-DOMINATING-SET $\in$ A$[2]$.

p-CLIQUE-DOMINATING-SET
Instance: Graphs $\mathcal{G}$, and $k, \ell \in \mathbb{N}$.
Parameter: $k + \ell$, where ℓ the number of vertices of $\mathcal{H}$.
Problem: Decide whether $\mathcal{G}$ contains a set of k vertices from $\mathcal{G}$ that dominates every clique of ℓ elements.

W-Hierarchy and A-Hierarchy versus para-NP and XP



A-Hierarchy (properties)

Theorems

T1. $\mathsf{A}[1] \subseteq \mathsf{W}[\mathsf{P}]$.

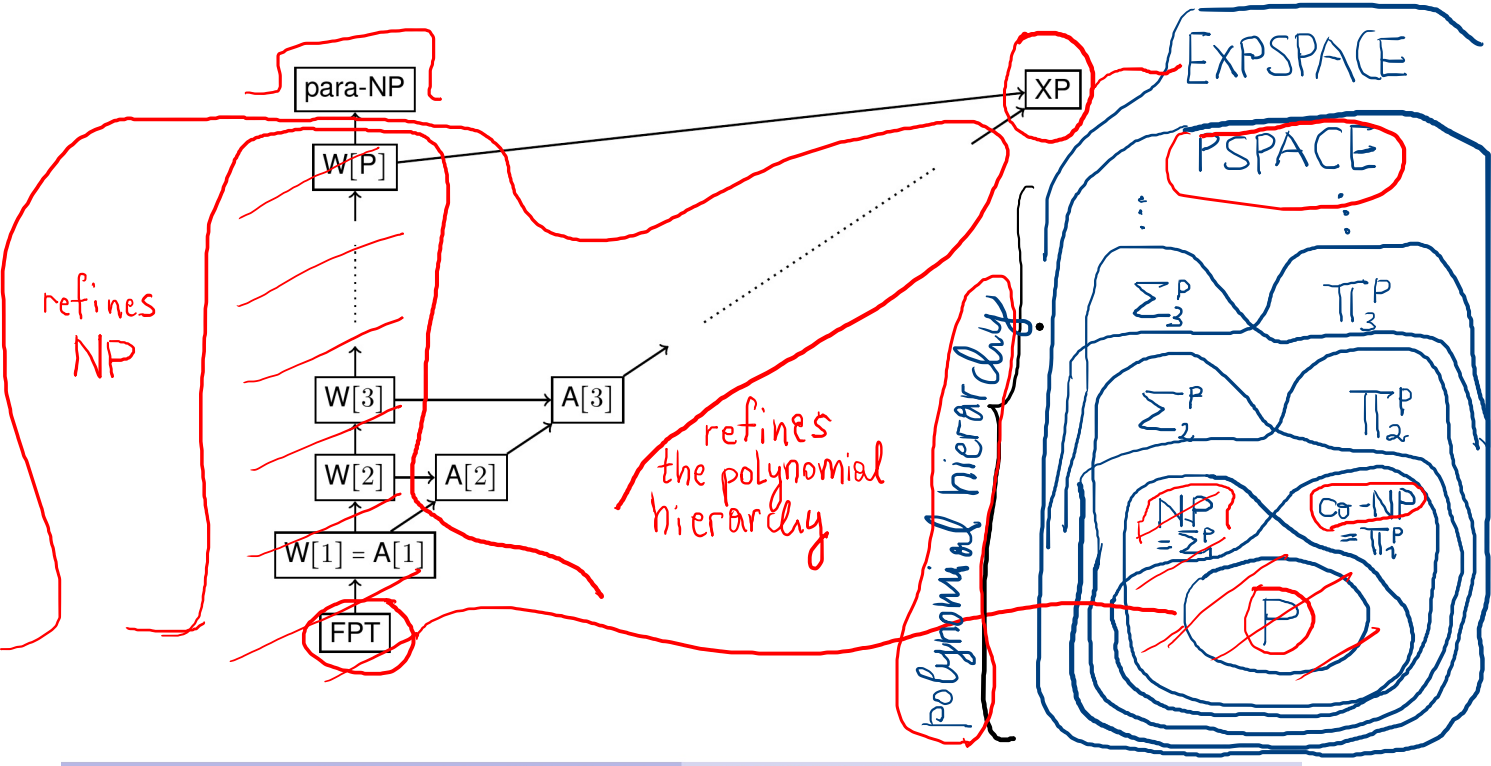
T2. $\mathsf{W}[t] \subseteq \mathsf{A}[t]$, for all $t \in \mathbb{N}$.

- Unlikely: $\mathsf{A}[t] \subseteq \mathsf{W}[t]$, for $t > 1$.

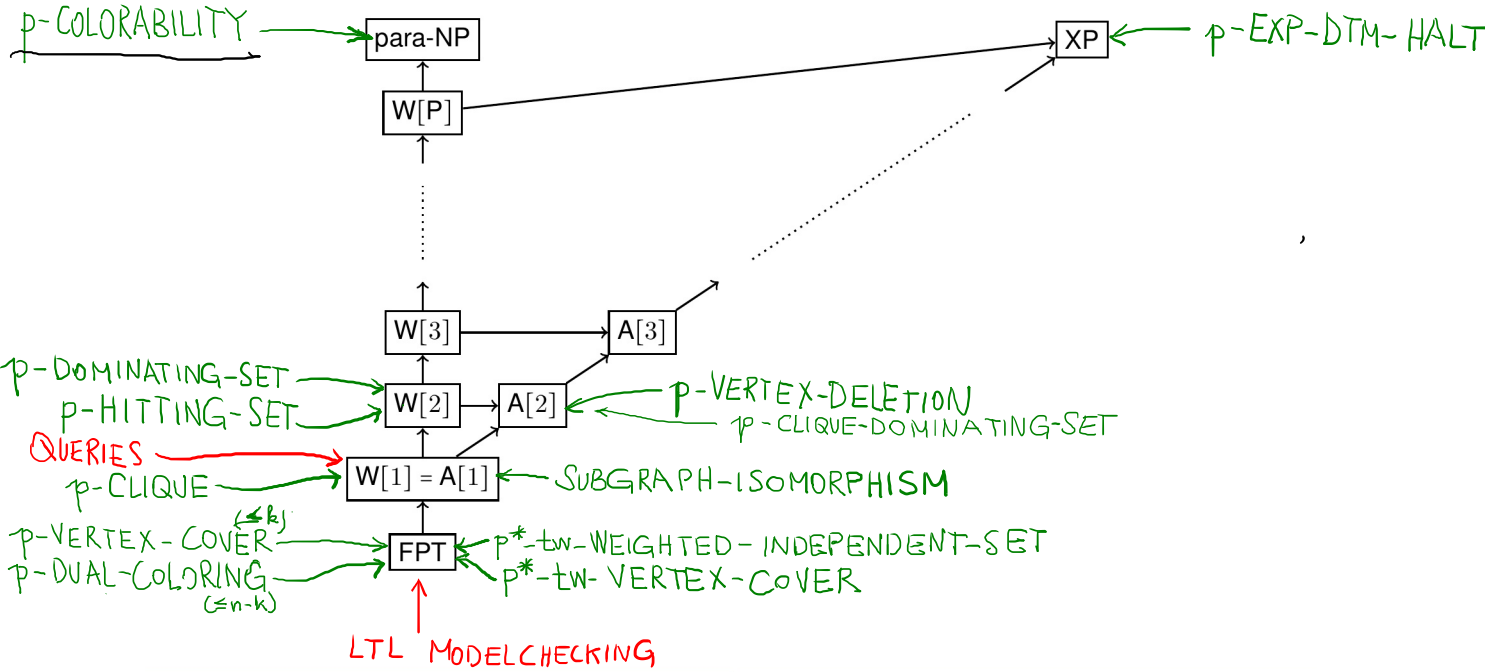
  Reason:
  - the A-hierarchy are parameterizations of problems that are complete for the levels of the polynomial hierarchy
  - the W-hierarchy is a refinement of NP in parameterized complexity
- Unlikely: $\left[p\text{-MC(FO)}\right]^{\text{fpt}} = \bigcup_{i=1}^{\infty} \mathsf{A}[i]$,

  contrasting with: $\left[p\text{-WD-FO}\right]^{\text{fpt}} = \bigcup_{i=1}^{\infty} \mathsf{W}[i]$.

A-Hierarchy


classical hierarchy
para-NP
W[P]
XP
refines NP
W[3]
A[3]
W[2]
A[2]
W[1] = A[1]
FPT
refines the polynomial hierarchy
polynomial hierarchy
EXPSPACE
PSPACE
Σ_3^P
Π_3^P
Σ_2^P
Π_2^P
NP = Σ_1^P
Co-NP = Π_1^P
P

W-Hierarchy and A-Hierarchy versus para-NP and XP



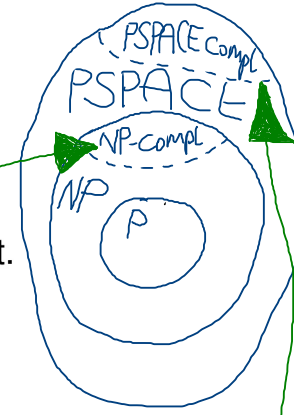
Why is the theory of W[P]/W/A-hardness important?

- Prevents from wasting hours tackling a problem which is fundamentally difficult;
- Finding results on a problem is always a ping-pong game between trying to design a hardness/FPT result;
- There is a hierarchy on parameters and it is worth knowing which is the smallest one such that the problem remains FPT;
- There is a hierarchy on complexity classes and it is worth noting to which extent a problem is hard.

Revisiting the two problems at start today

QUERIES
Instance: a relational database D, a conjunctive query α.
Parameter: size $k = |\alpha|$ of query α
Compute: answer to query α from database D.

- QUERIES $\in$ NP-complete.
- QUERIES $\in O(n^k)$ for $n = \|D\|$, which does not give an FPT result.



LTL-MODEL-CHECKING
Instance: a Kripke structure (state space) $\mathcal{K}$, an LTL formula φ
Parameter: size $k = |\varphi|$ of formula φ
Question: Does $\mathcal{K} \models \varphi$ hold?

- LTL-MODEL-CHECKING $\in$ PSPACE-complete,
- LTL-MODEL-CHECKING $\in O(k \cdot 2^{2k} \cdot n) \in$ FPT for $n = \|\mathcal{K}\|$.

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Compute: answer to query α from database D.

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LTL-MODEL-CHECKING

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Parameter: size $k = |\varphi|$ of formula φ
Question: Does $\mathcal{K} \vDash \varphi$ hold?

- LTL-MODEL-CHECKING $\in$ PSPACE-complete,
- LTL-MODEL-CHECKING $\in O(k \cdot 2^{2^k} \cdot n) \in$ FPT for $n = \|\mathcal{K}\|$.

Revisiting the two problems at start today

QUERIES

Instance: a relational database D, a conjunctive query α.
Parameter: size $k = |\alpha|$ of query α
Compute: answer to query α from database D.

- QUERIES $\in$ NP-complete.
- QUERIES $\in O(n^k)$ for $n = \|D\|$, which does not give an FPT result.
- QUERIES $\in$ W[1] (= strong evidence for it likely not to be in FPT).

LTL-MODEL-CHECKING

Instance: a Kripke structure (state space) $\mathcal{K}$, an LTL formula φ
Parameter: size $k = |\varphi|$ of formula φ
Question: Does $\mathcal{K} \models \varphi$ hold?

- LTL-MODEL-CHECKING $\in$ PSPACE-complete,
- LTL-MODEL-CHECKING $\in O(k \cdot 2^{2^k} \cdot n) \in$ FPT for $n = \|\mathcal{K}\|$.

Summary

- Motivation for fixed-parameter intractability
- Fixed parameter reductions
- The classes para-NP and XP
- The class W[P]
- Logic preliminaries (continued)
- W-hierarchy
  - definitions
    - with Boolean circuits
    - as parameterized weighted Fagin definability problems
- A-hierarchy
  - definition as parameterized model-checking problems
- picture overview of these classes

Course overview

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results		**Algorithmic Meta-Theorems**		
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	GDA	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	GDA	GDA
Algorithmic Techniques		*Formal-Method & Algorithmic Techniques*		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width			**FPT-Intractability Classes & Hierarchies**
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Example suggestions

Examples

1. FPT results transfer backwards over fpt-reductions:
   If $\langle Q_1, \kappa_1 \rangle \leq_{\text{fpt}} \langle Q_2, \kappa_2 \rangle$, then $Q_2 \in \text{FPT}$ implies $Q_1 \in \text{FPT}$.
2. Find the idea for:
   p-DOMINATING-SET $\equiv_{\text{fpt}}$ p-HITTING-SET.
3.

References


Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh.
Parameterized Algorithms.
Springer, 1st edition, 2015.


Jörg Flum and Martin Grohe.
Parameterized Complexity Theory.
Springer, 2006.

Fixed-parameter tractable reductions

Examples

- p-CLIQUE $\equiv_{\text{fpt}}$ p-INDEPENDENT-SET.
- p-DOMINATING-SET $\equiv_{\text{fpt}}$ p-HITTING-SET.

Non-Example

- For graphs $\mathcal{G} = \langle V, E \rangle$, and sets $X \subseteq V$:

  X is independent set of $\mathcal{G}$ $\iff$ $V \setminus X$ is a vertex cover of $\mathcal{G}$

  yields a polynomial reduction between p-INDEPENDENT-SET and p-VERTEX-COVER, but does not yield an fpt-reduction.

p-DOMINATING-SET

Instance: Graph $G=\langle V,E\rangle$, $k\in\mathbb{N}$
Parameter: k
Question: Does G have a dominating set of size $\leq k$

p-HITTING-SET

Instance: Used, $\mathcal{B}\subseteq\mathcal{S}(U)$, $k\in\mathbb{N}$
Parameter: $k\in\mathbb{N}$
Question: Is there a hitting set for $\mathcal{B}$ of size $\leq k$

p-DOMINATING-SET $\equiv_{fpt}$ p-HITTING-SET

$\leq_{fpt}$

$\geq_{fpt}$

p-DOMINATING-SET

Instance: Graph $G=\langle V,E\rangle$, $k\in\mathbb{N}$
Parameter: k
Question: Does G have a dominating set of size $\leq k$

p-HITTING SET

Instance: Used, $\mathcal{B}\subseteq\mathcal{P}(U)$, $k\in\mathbb{N}$
Parameter: $k\in\mathbb{N}$
Question: Is there a hitting set for $\mathcal{A}$ of size $\leq k$

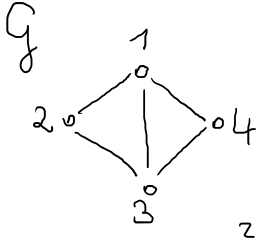
p-DOMINATING-SET $\equiv_{fpt}$ p-HITTING-SET

$\leq_{fpt}$

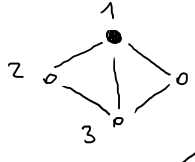
$\geq_{fpt}$

$N_G(v) := \{w \mid \{v,w\}\in E\}$

$\langle G,k\rangle \longmapsto \langle \mathcal{B}_G := \{\{v\}\cup N_G(v) \mid v\in V\}, k\rangle$

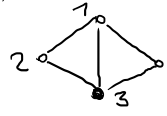


$\mathcal{B}_G := \{\{1,2,3,4\},\{1,2,3\},\{1,3,4\}\}$



$S=\{1\}$ domset $\Rightarrow$ $S=\{1\}$ is hitting set

$S=\{2\}$ not domset $\Rightarrow$ $S=\{2\}$ not a hitting set



$S=\{3\}$ domset $\Leftarrow$ $S=\{3\}$ hitting set

p-DOMINATING-SET

Instance: Graph $G = \langle V, E \rangle$, $k \in \mathbb{N}$
Parameter: k
Question: Does G have a dominating set of size $\leq k$

p-HITTING-SET

Instance: Used, $\mathcal{B} \subseteq \mathcal{P}(U)$, $k \in \mathbb{N}$
Parameter: $k \in \mathbb{N}$
Question: Is there a hitting set for $\mathcal{A}$ of size $\leq k$

p-DOMINATING-SET $\equiv_{fpt}$ p-HITTING-SET

$\equiv_{fpt}$
$\leq_{fpt}$
$\geq_{fpt}$

$G_{\mathcal{B}} := \langle U, E_{\mathcal{B}} \rangle \longleftarrow \langle U, \mathcal{B}, k \rangle$

$E_{\mathcal{B}} = \{\{u,v\} \mid \exists B \in \mathcal{B}(u, v \in B \wedge u \neq v)\}$

$\langle G_{\mathcal{B}} := \langle U \cup \mathcal{B}, E_1 \cup E_2 \rangle, k \rangle \longleftarrow \langle U, \mathcal{B}, k \rangle$

$E_1 := \{\{u, B\} \mid u \in U, u \in B \in \mathcal{B}\}$

$E_2 := \{\{v, w\} \mid v, w \in U, v \neq w\}$