From Compactifying Lamba-Letrec Terms to Recognizing Regular-Expression Processes

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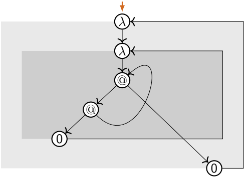

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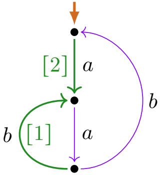
July 2, 2023



Overview

1. Compactifying λ-terms with `letrec`
   - higher-order λ-term graphs

2. Recognizing regular-expression processes
   - LEE-witnesses: graph labelings based on a loop–condition LEE

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   - from terms in the λ-calculus with letrec to:
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     - first-order λ-term graphs
     - λ-NFAs, and λ-DFAs
   - minimization / readback / efficiency / Haskell implementation
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   - Milner's questions, known results
   - structure-constrained process graphs:
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     - preservation under bisimulation collapse
   - readback: from graph labelings to regular expressions

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- Comparison desiderata

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- Comparison results

Comparison original desiderata

λ-calculus with letrec under unfolding semantics

Not available: term graph interpretation that is studied under $\underline{\leftrightarrow}$

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Given: process graph interpretation $P(\cdot)$, studied under $\underline{\leftrightarrow}$

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Desired: reason with graphs that are $P(\cdot)$-expressible modulo $\underline{\leftrightarrow}$ (at least with 'sufficiently many')

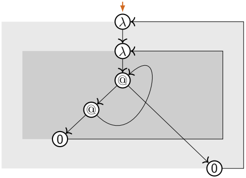
understand incompleteness by a structural graph property

structure constraints (L'Aquila)

Maximal sharing of functional programs

(joint work with Jan Rochel)

$$\begin{tikzcd} L \arrow[r, mapsto, bend left, "\text{interpret}"] & G \arrow[d, Rightarrow, "\text{collapse}"] \\ L_0 & G_0 \arrow[l, mapsto, bend left, "\text{readback}"] \end{tikzcd}$$



Maximal sharing: example (fix)

$$
\begin{tikzcd}
L \arrow[rr, mapsto, bend left, "\text{interpret}"] \arrow[d, mapsto, "\text{unfold}"'] & & G \arrow[dd, Rrightarrow, "\text{collapse}"] \\
M & & \\
L_0 \arrow[u, mapsto, "\text{unfold}"] \arrow[rr, mapsto, bend left, "\text{interpret}"] & & G_0 \arrow[ll, mapsto, bend left, "\text{readback}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\lambda f.\,\text{let } r = f\,(f\;r) \text{ in } r \arrow[d, mapsto, "{[\![\cdot]\!]_{\lambda^\infty}}"'] \arrow[r, mapsto, "{[\![\cdot]\!]_{\mathcal{T}}}"] & \text{(graph 1)} \arrow[d, Rrightarrow] \\
\lambda f.\,f\,(f\,(\ldots)) & \\
\lambda f.\,\text{let } r = f\;r \text{ in } r \arrow[u, mapsto, "{[\![\cdot]\!]_{\lambda^\infty}}"] \arrow[r, mapsto, shift left, "{[\![\cdot]\!]_{\mathcal{T}}}"] & \text{(graph 2)} \arrow[l, mapsto, shift left, "\text{readback}"]
\end{tikzcd}
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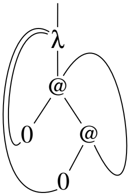

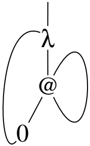

Maximal sharing: the method

$$L \xmapsto{[\![\cdot]\!]_{\mathcal{H}}} \mathcal{G}$$

1. term graph interpretation $[\![\cdot]\!]$.
   of $\boldsymbol{\lambda}_{\mathsf{letrec}}$-term L as:
   a. higher-order term graph
      $\mathcal{G} = [\![L]\!]_{\mathcal{H}}$

Maximal sharing: the method

$$L \xmapsto{[\![\cdot]\!]_{\mathcal{H}}} \mathcal{G} \longmapsto G$$

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Maximal sharing: the method

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\begin{tikzcd}
L \arrow[r, mapsto, "⟦\cdot⟧_{\mathcal{H}}"] \arrow[rr, mapsto, bend left, "⟦\cdot⟧_{\mathcal{T}}"] & \mathcal{G} \arrow[r, mapsto] & G
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2. bisimulation collapse $\Downarrow$
   of f-o term graph G into G_0

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   yielding program $L_0 = \mathsf{rb}(G_0)$.

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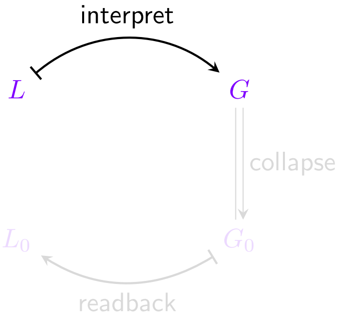
Interpretation



Running example

instead of:

$\lambda f.\, \mathsf{let}\ r = f\,(f\,r)\ \mathsf{in}\ r \quad \longmapsto_{\text{max-sharing}} \quad \lambda f.\, \mathsf{let}\ r = f\,r\ \mathsf{in}\ r$

we use:

$\lambda x.\, \lambda f.\, \mathsf{let}\ r = f\,(f\,r\,x)\,x\ \mathsf{in}\ r \quad \longmapsto_{\text{max-sharing}} \quad \lambda x.\, \lambda f.\, \mathsf{let}\ r = f\,r\,x\ \mathsf{in}\ r$

$L \quad \longmapsto_{\text{max-sharing}} \quad L_0$

Graph interpretation (example 1)

$L_0 = \lambda x.\,\lambda f.\,\mathsf{let}\ r = f\,r\,x\ \mathsf{in}\ r$

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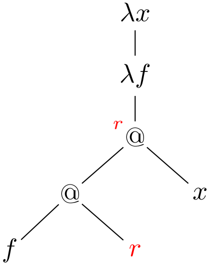
$L_0 = \lambda x.\, \lambda f.\,$ let $r = f\; r\; x$ in r



syntax tree

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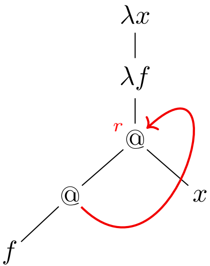

syntax tree (+ recursive backlink)

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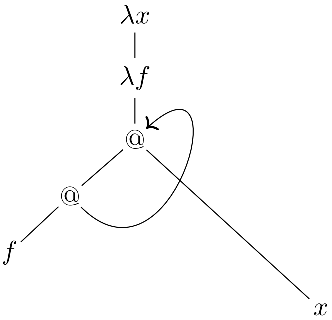

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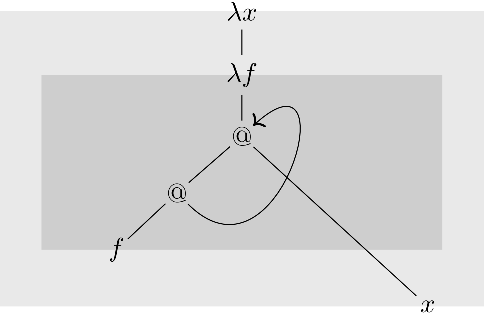

syntax tree (+ recursive backlink, + scopes)

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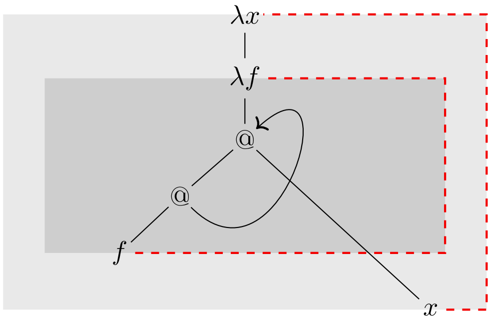

syntax tree (+ recursive backlink, + scopes, + binding links)

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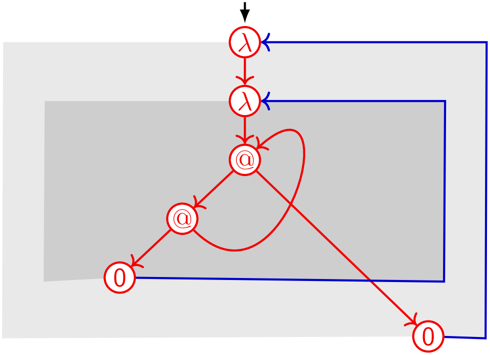

first-order term graph with binding backlinks (+ scope sets)

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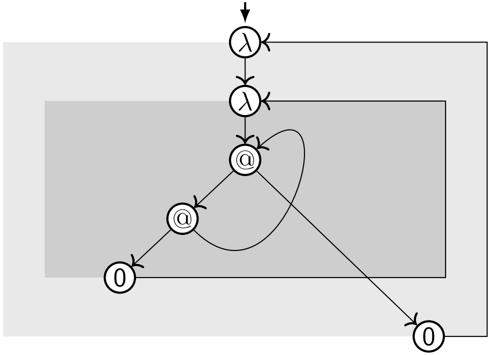

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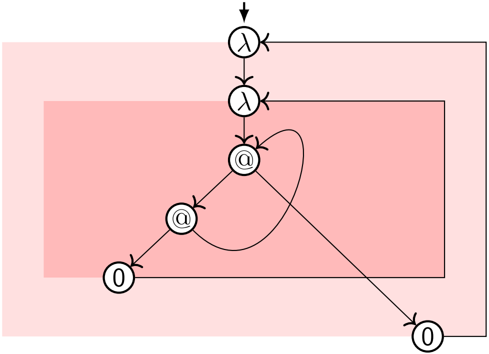

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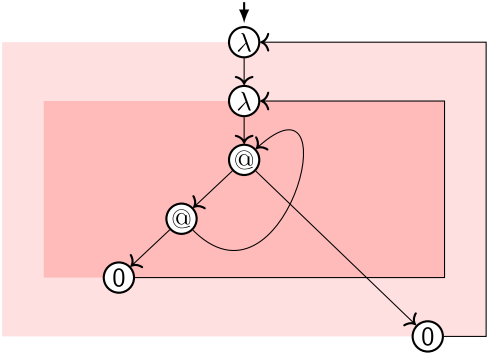

higher-order term graph (with scope sets, Blom [2003])

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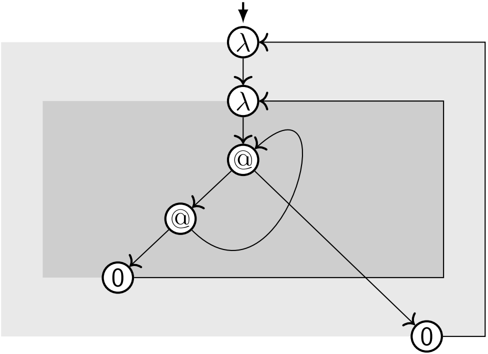

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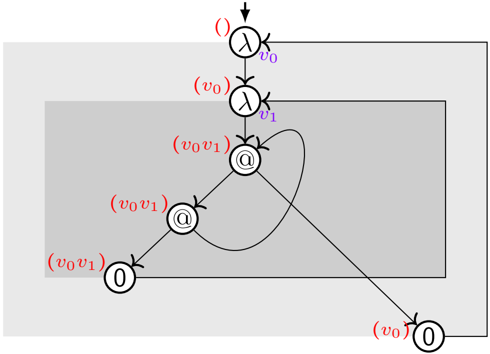

higher-order term graph (with scope sets, + abstraction-prefix function)

Graph interpretation (example 1)

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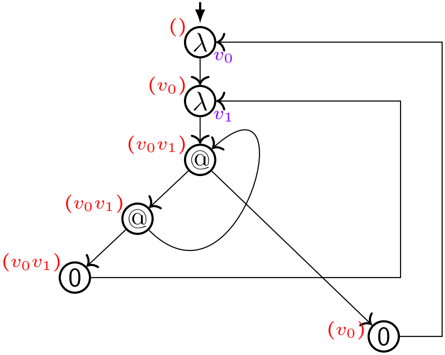

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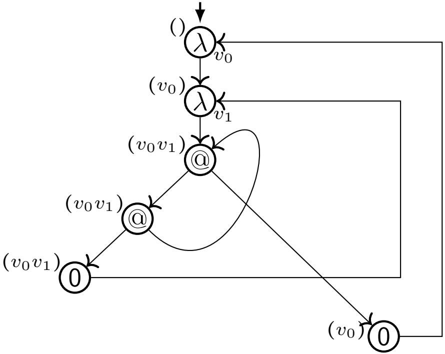
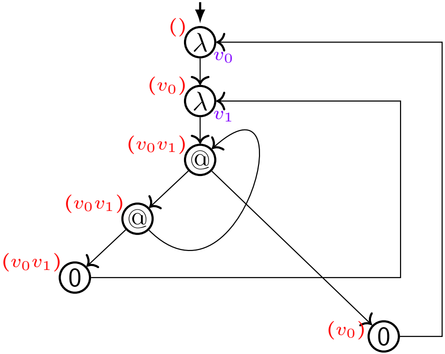


λ-higher-order-term-graph $[\![L_0]\!]_{\mathcal{H}}$

Graph interpretation (example 1)

$L_0 = \lambda x. \lambda f. \mathsf{let}\ r = f\ r\ x\ \mathsf{in}\ r$



first-order term graph (+ abstraction-prefix function)

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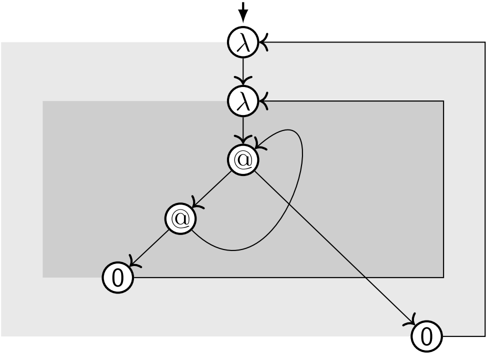

first-order term graph with binding backlinks (+ scope sets)

Graph interpretation (example 1)

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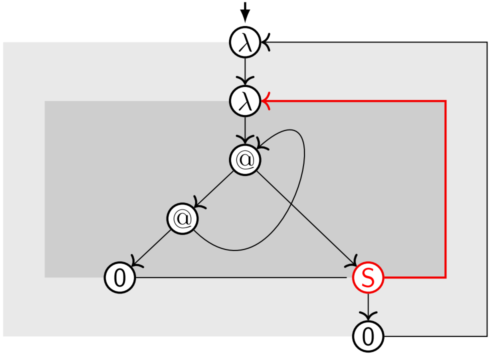

first-order term graph with scope vertices with backlinks (+ scope sets)

Graph interpretation (example 1)

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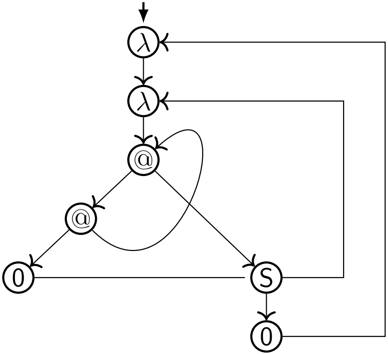

first-order term graph with scope vertices with backlinks

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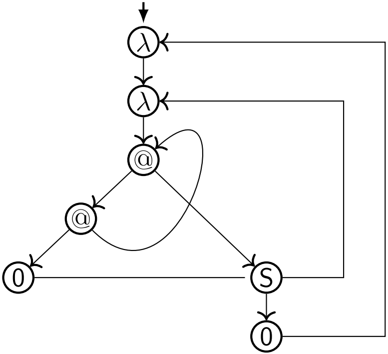

λ-term-graph $[\![L_0]\!]_{\mathcal{T}}$

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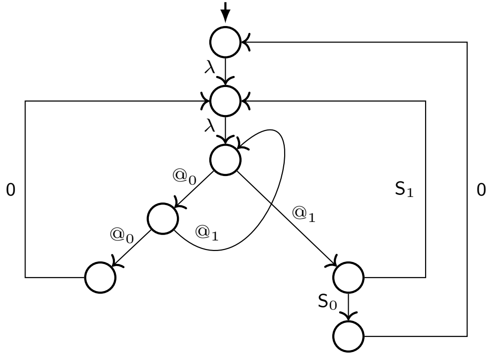

incomplete DFA

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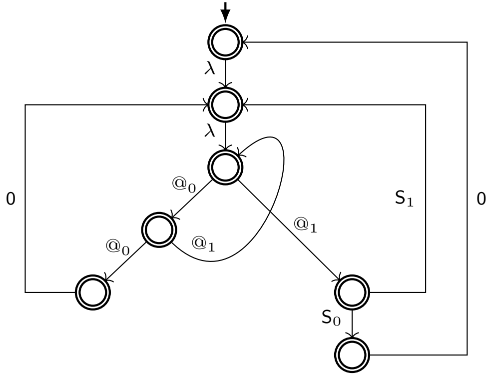

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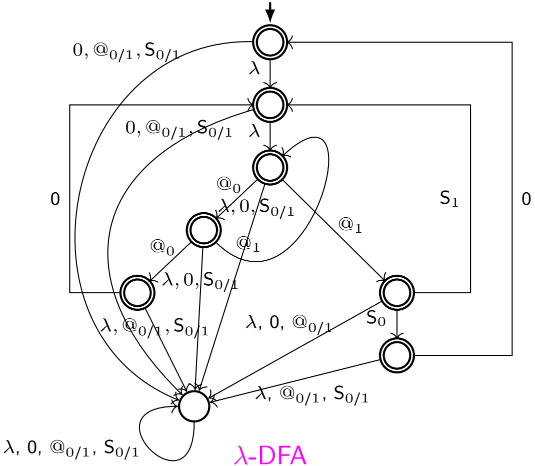

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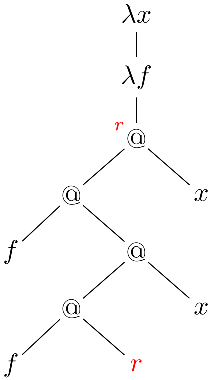

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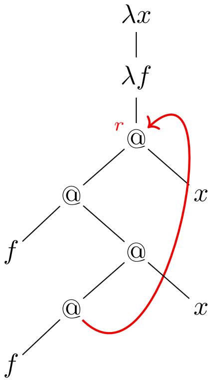

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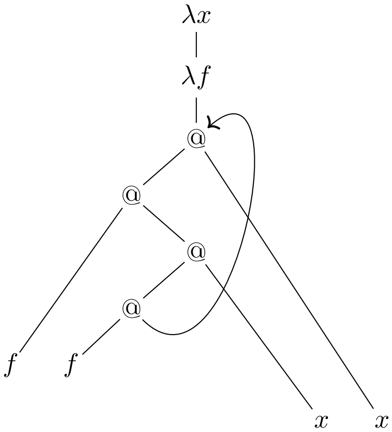

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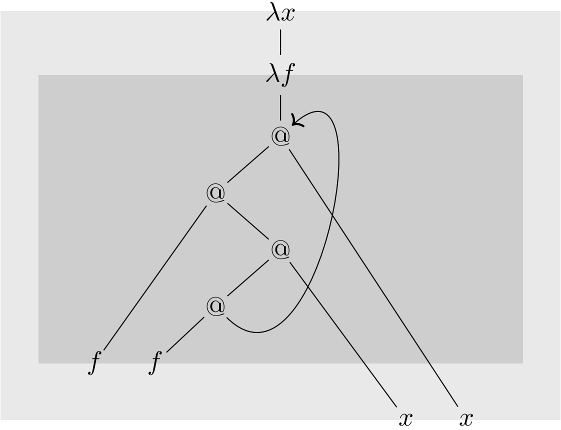

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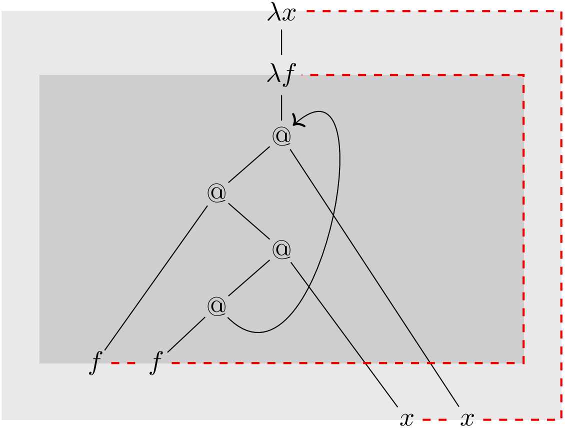
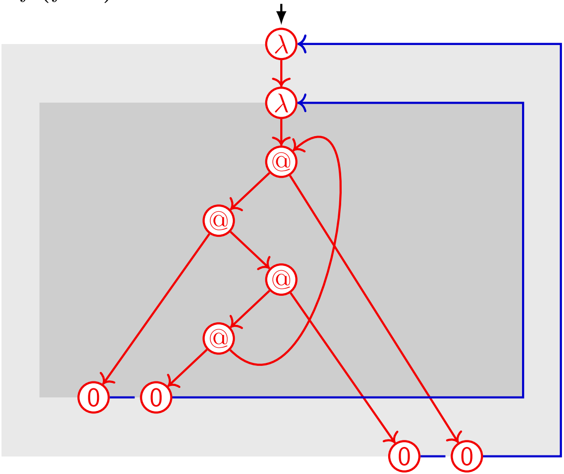


syntax tree (+ recursive backlink, + scopes, + binding links)

Graph interpretation (example 2)

$L = \lambda x. \lambda f.$ let $r = f\,(f\,r\,x)\,x$ in r



first-order term graph with binding backlinks (+ scope sets)

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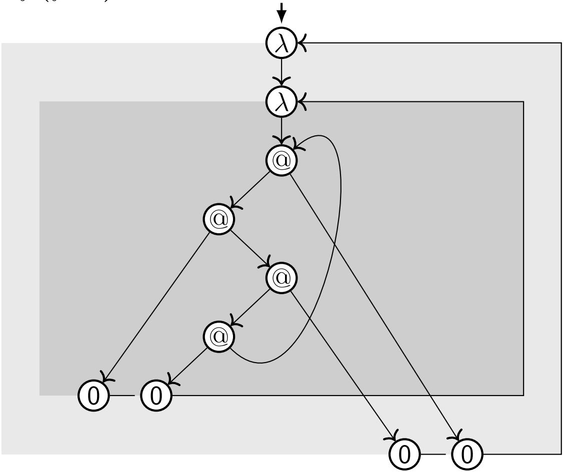

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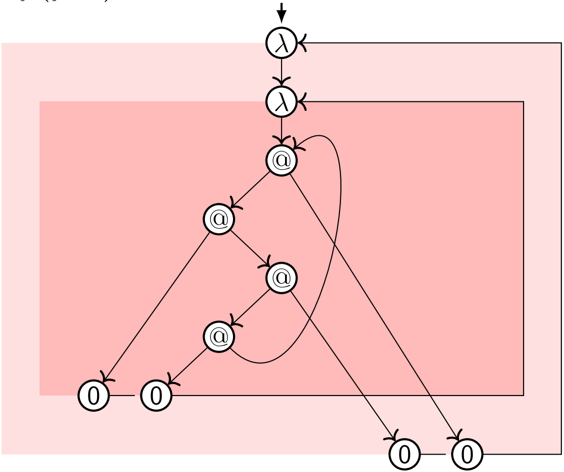

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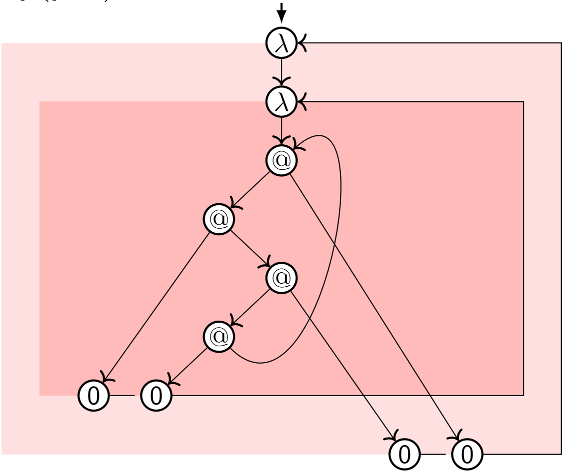

higher-order term graph (with scope sets, Blom [2003])

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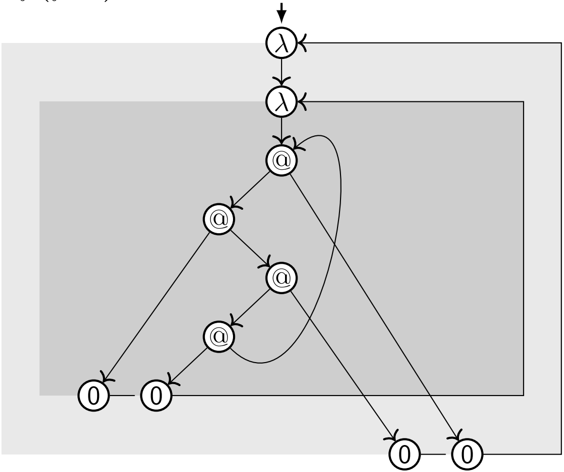

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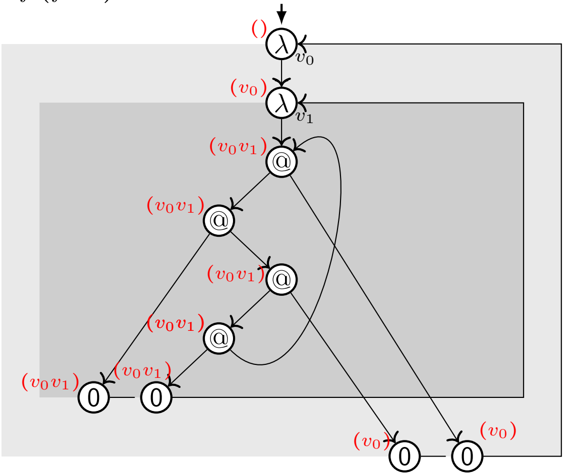
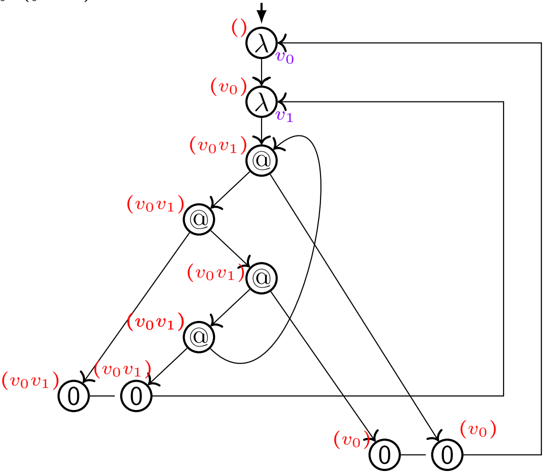


higher-order term graph (with scope sets, + abstraction-prefix function)

Graph interpretation (example 2)

$L = \lambda x.\, \lambda f.\,$ let $r = f\,(f\,r\,x)\,x$ in r



higher-order term graph (with abstraction-prefix function)

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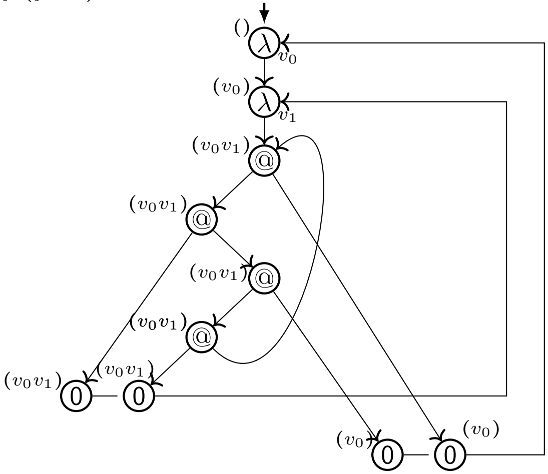

λ-higher-order-term-graph $[\![L]\!]_{\mathcal{H}}$

Graph interpretation (example 2)

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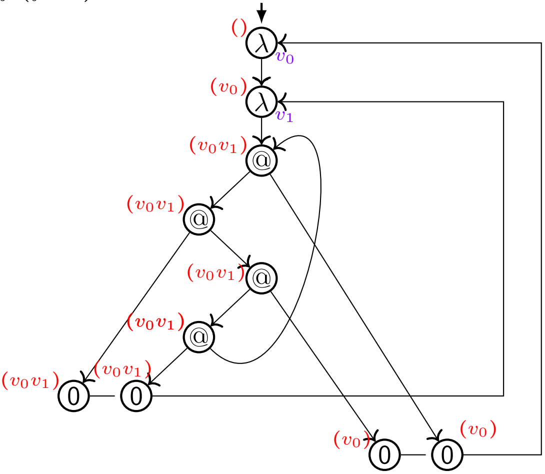

first-order term graph (+ abstraction-prefix function)

Graph interpretation (example 2)

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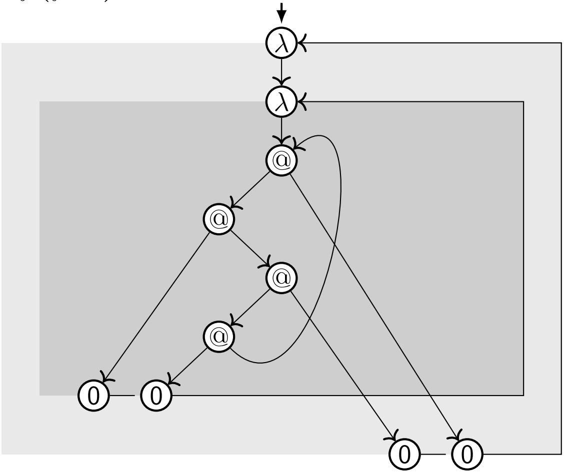

first-order term graph with binding backlinks (+ scope sets)

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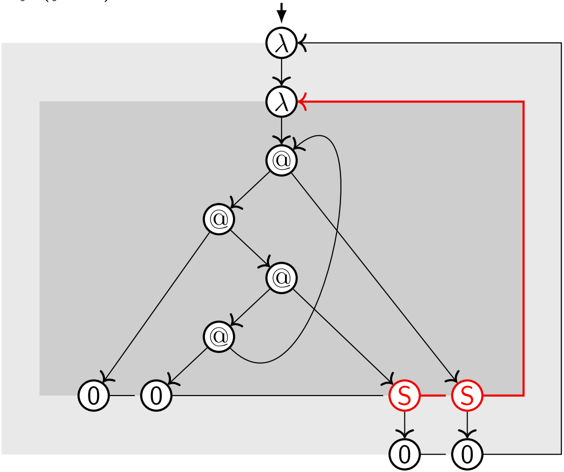

first-order term graph with scope vertices with backlinks (+ scope sets)

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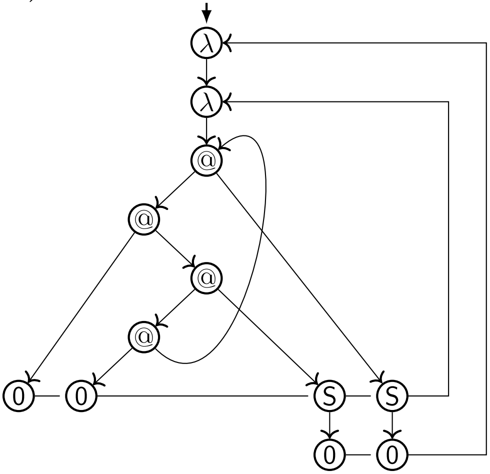
$L = \lambda x. \lambda f.$ let $r = f\,(f\,r\,x)\,x$ in r



first-order term graph with scope vertices with backlinks

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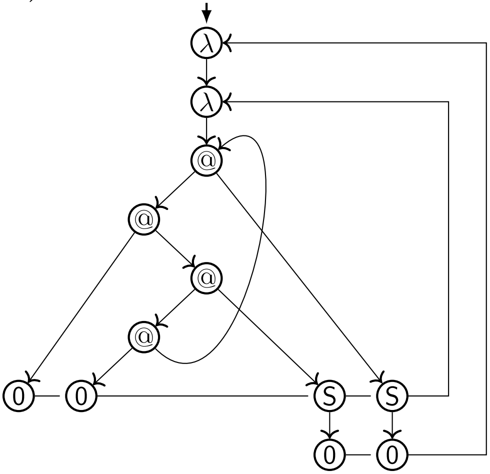

λ-term-graph $[\![L]\!]_{\mathcal{T}}$

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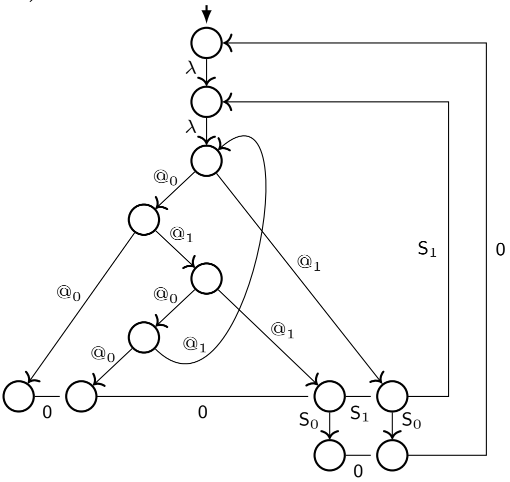

incomplete DFA

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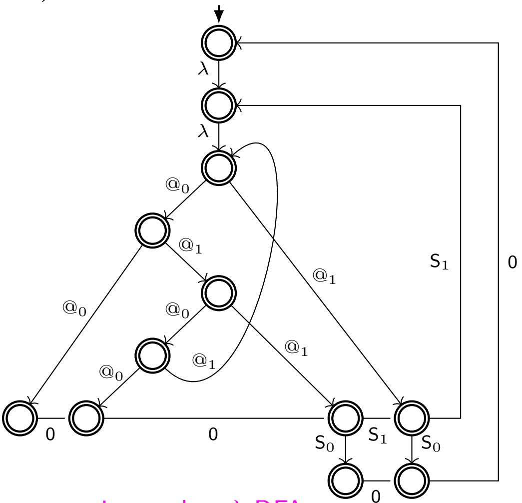

incomplete λ-DFA

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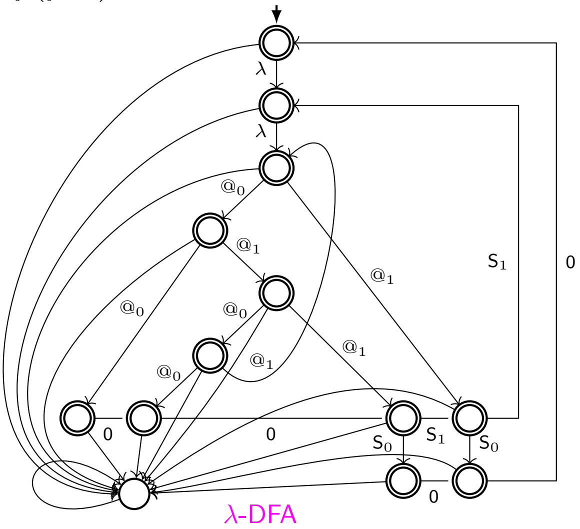
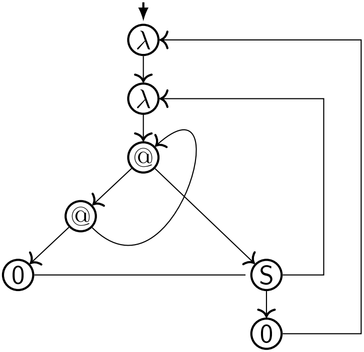
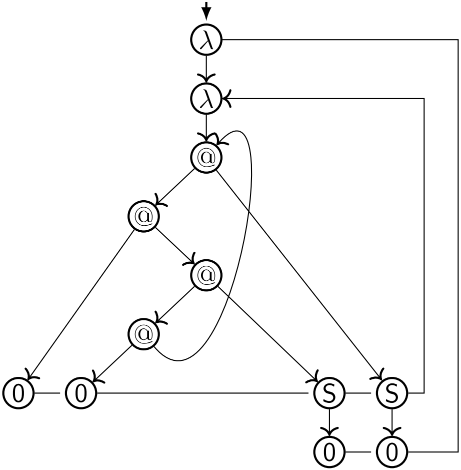


Graph interpretation (examples 1 and 2)



$[\![L_0]\!]_{\mathcal{T}}$



$[\![L]\!]_{\mathcal{T}}$

Interpretation $[\![\cdot]\!]_{\mathcal{T}}$: properties (cont.)

interpretation $\boldsymbol{\lambda}_{\text{letrec}}$-term $L \longmapsto \lambda$-term-graph $[\![L]\!]_{\mathcal{T}}$

- defined by induction on structure of L
- similar analysis as fully-lazy lambda-lifting
- yields eager-scope λ-term-graphs: $\sim$ minimal scopes

Theorem

For $\boldsymbol{\lambda}_{\text{letrec}}$-terms L_1 and L_2 it holds: Equality of infinite unfolding coincides with bisimilarity of λ-term-graph interpretations:

$$[\![L_1]\!]_{\lambda^\infty} = [\![L_2]\!]_{\lambda^\infty} \iff [\![L_1]\!]_{\mathcal{T}} \leftrightarrows [\![L_2]\!]_{\mathcal{T}}$$

Interpretation $[\![\cdot]\!]_{\mathcal{T}}$: properties (cont.)

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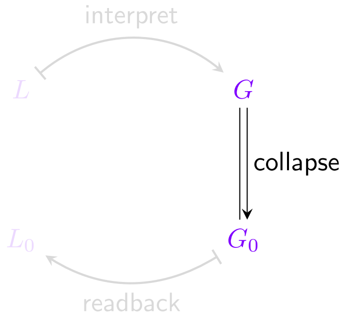
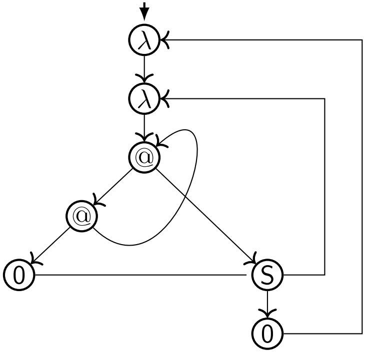
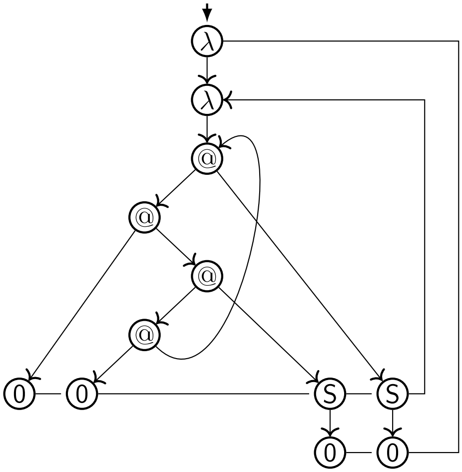
Collapse



Bisimulation check between λ-term-graphs



$[\![L_0]\!]_{\mathcal{T}}$



$[\![L]\!]_{\mathcal{T}}$

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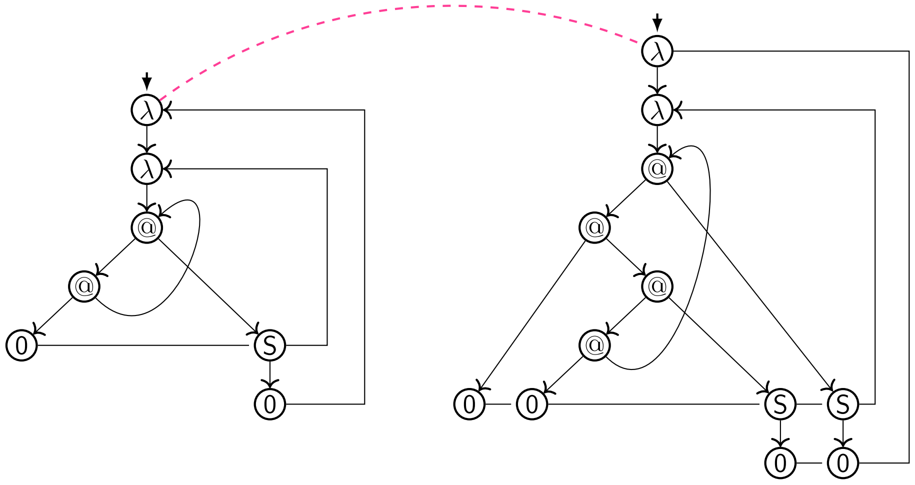

$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

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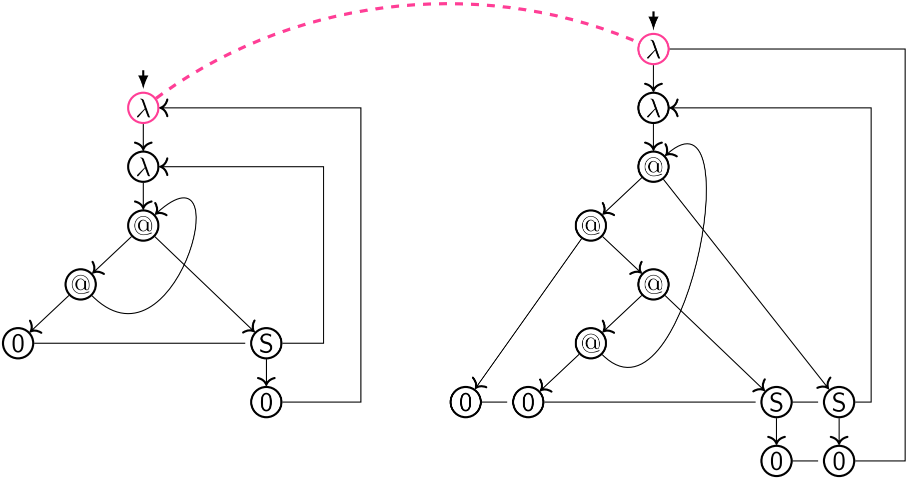

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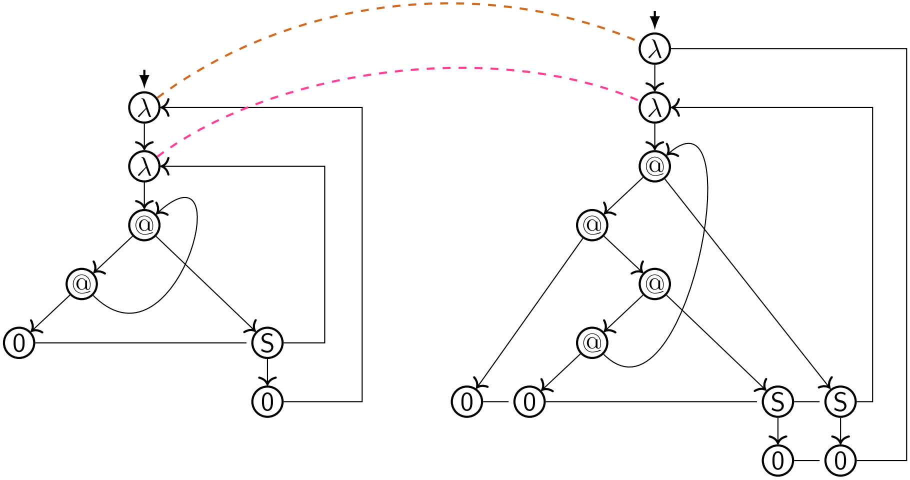

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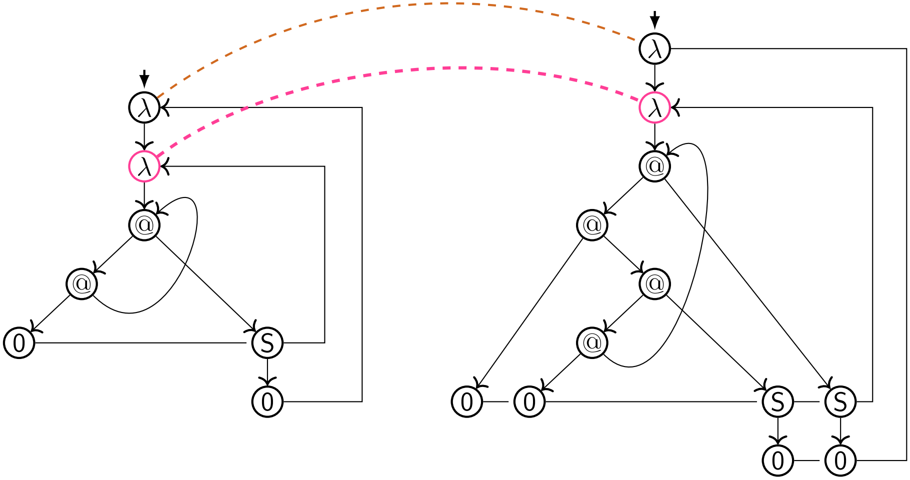

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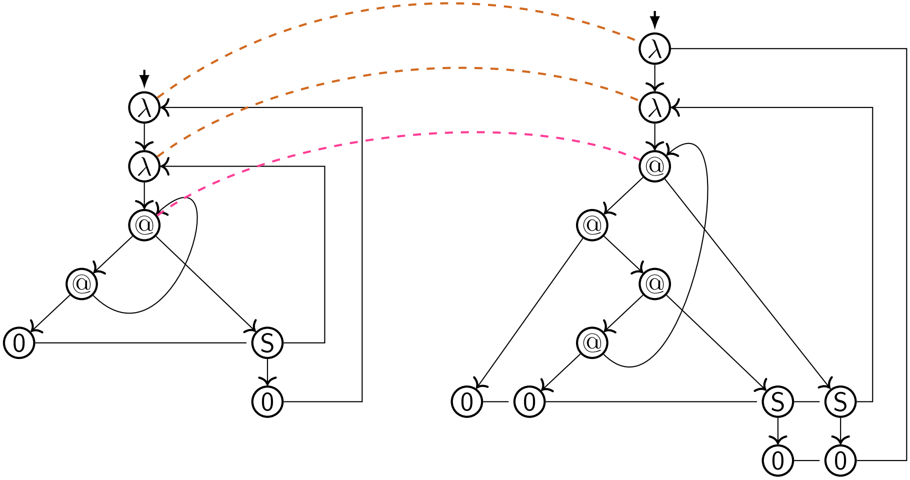

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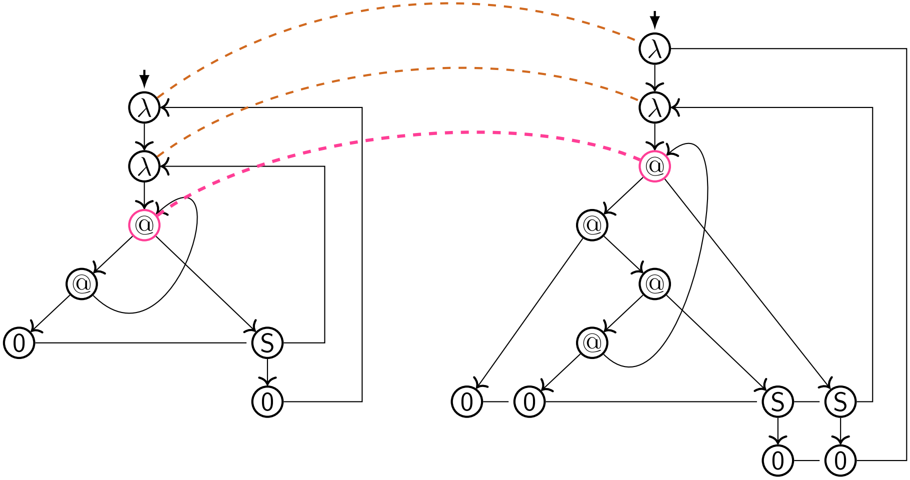

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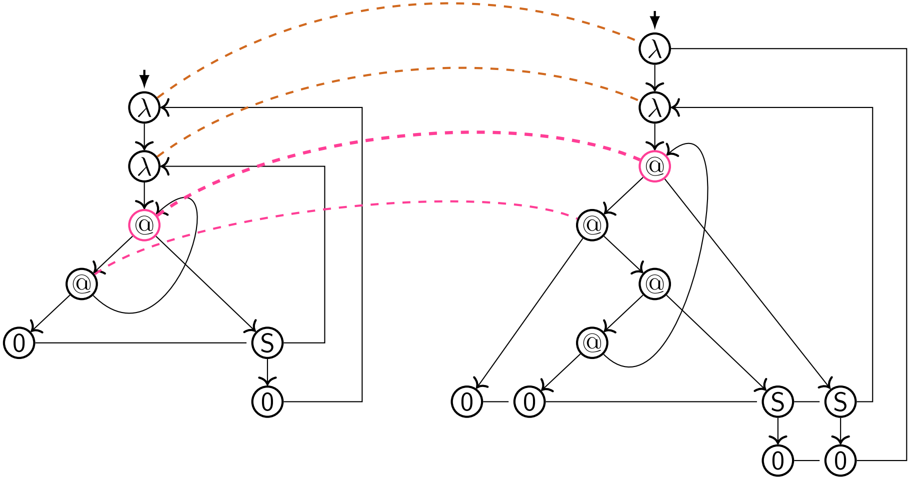

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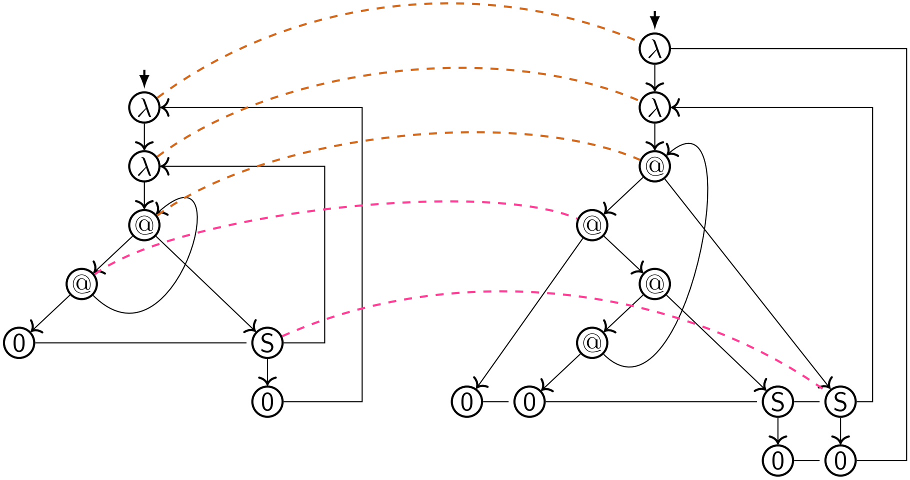

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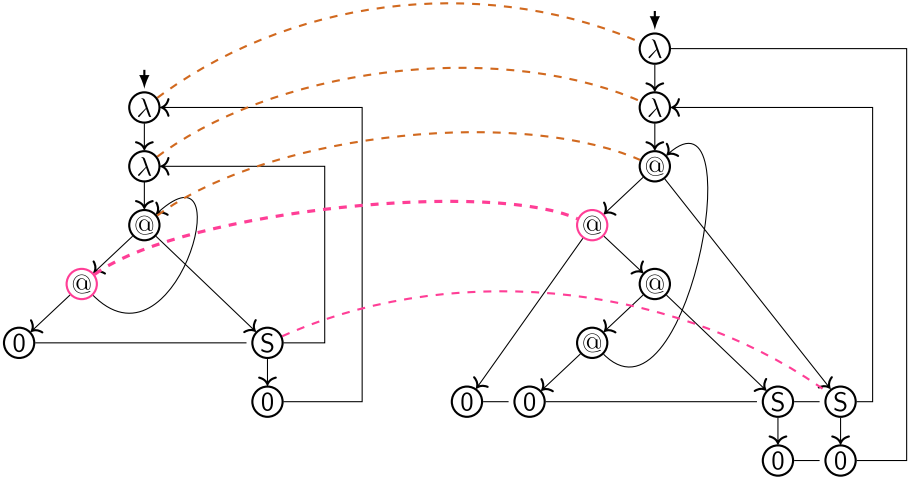

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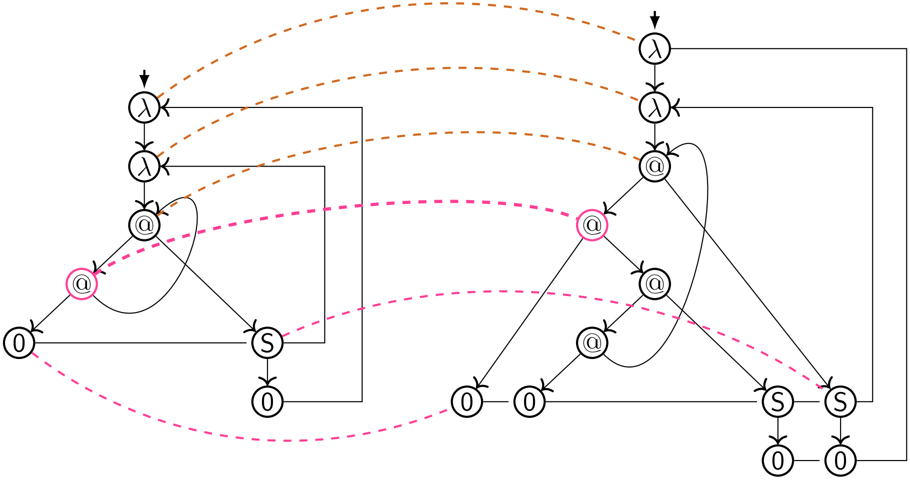

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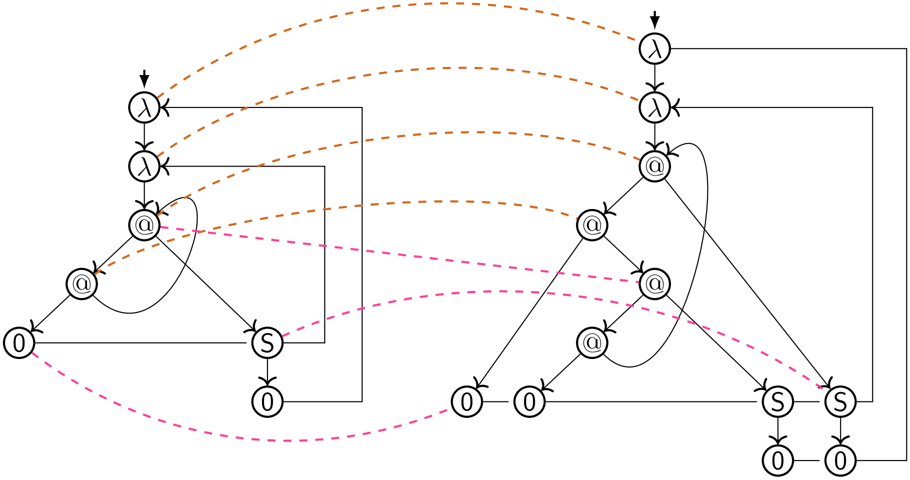

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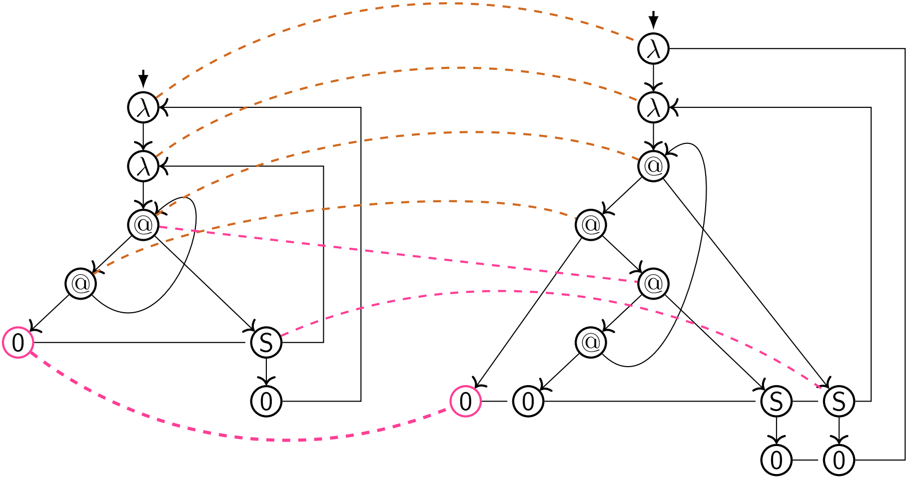

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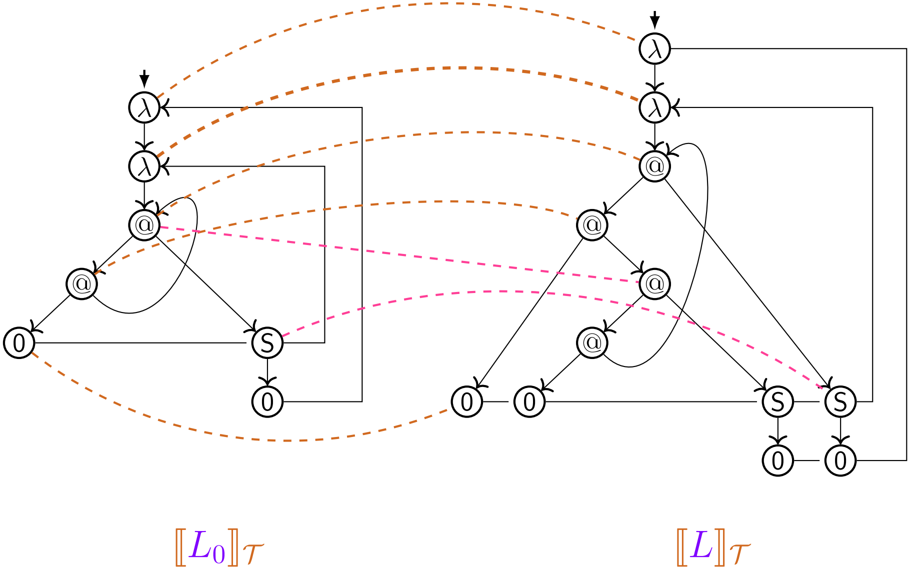

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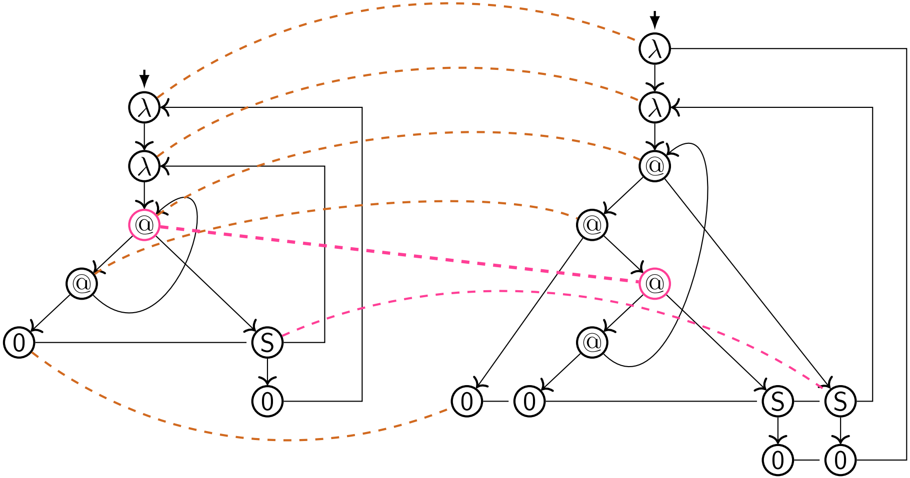

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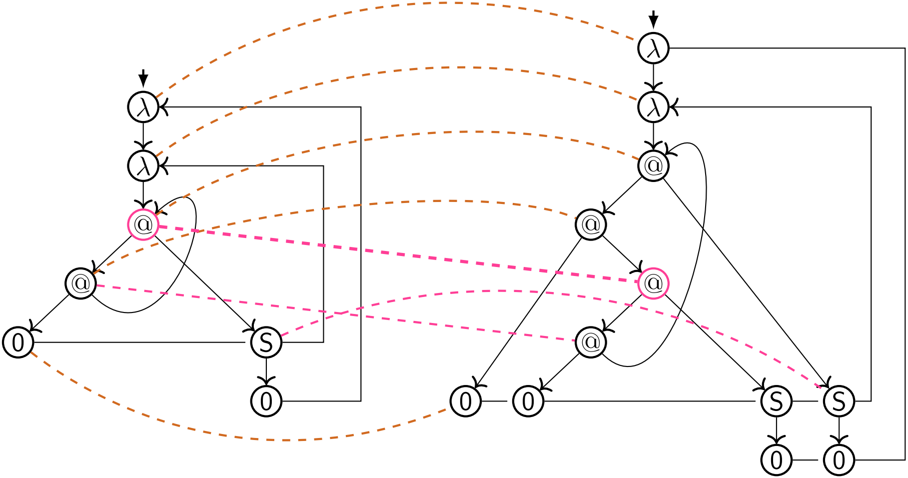

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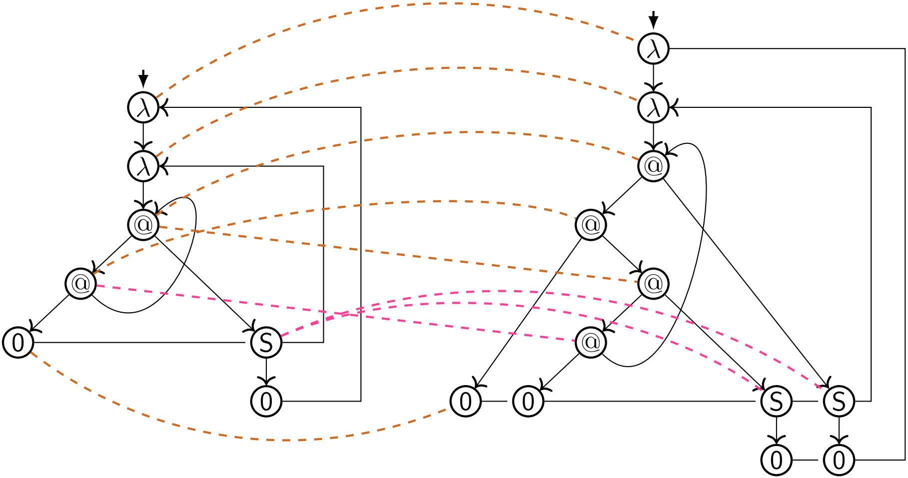

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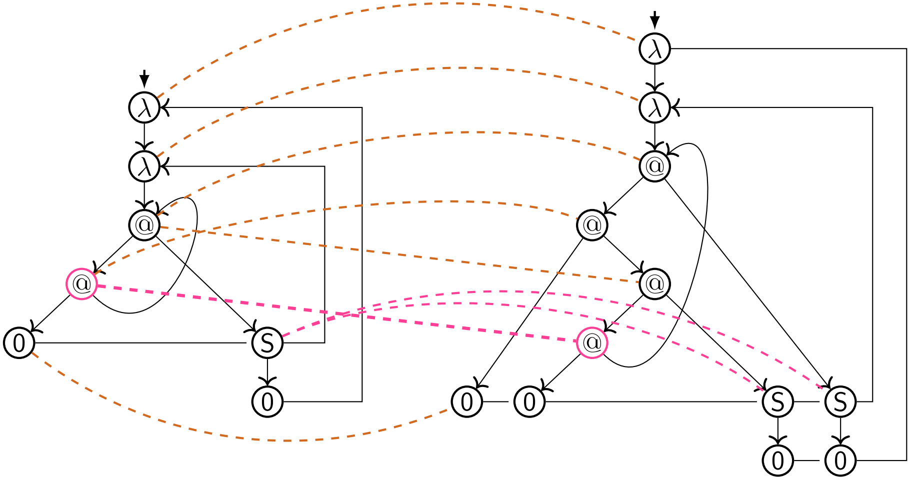

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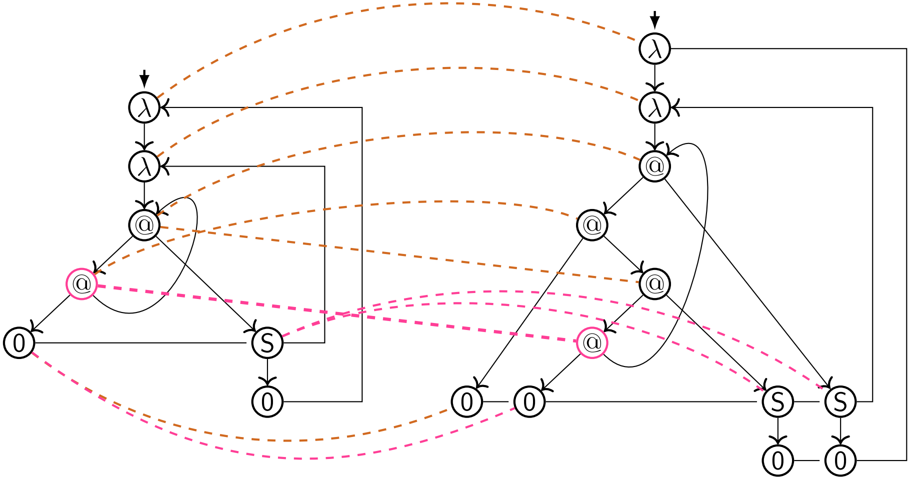

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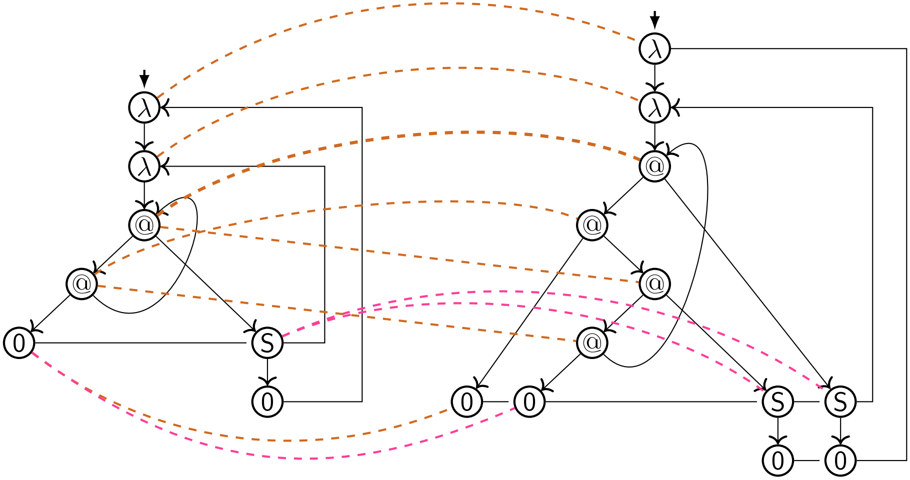

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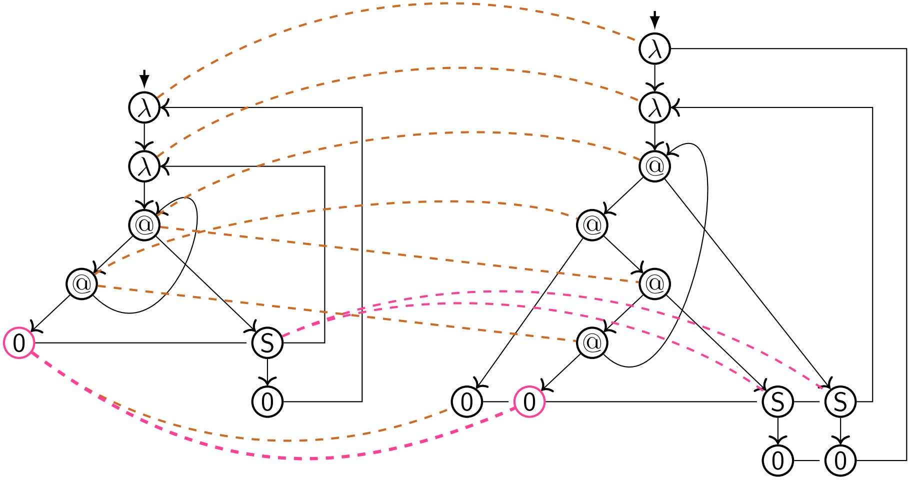

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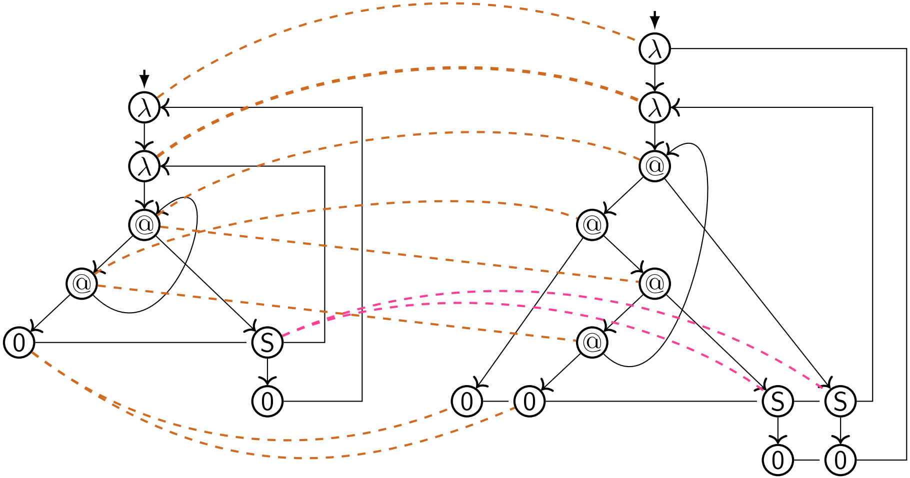

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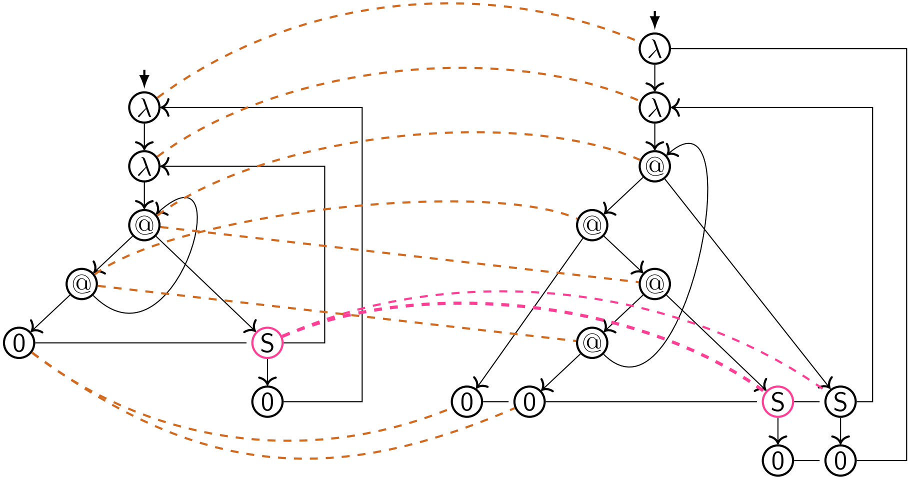

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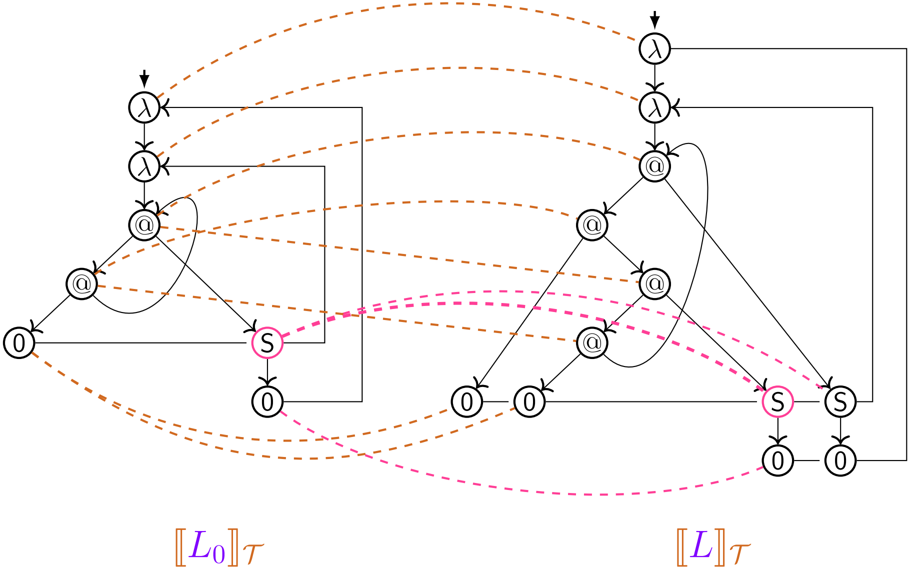

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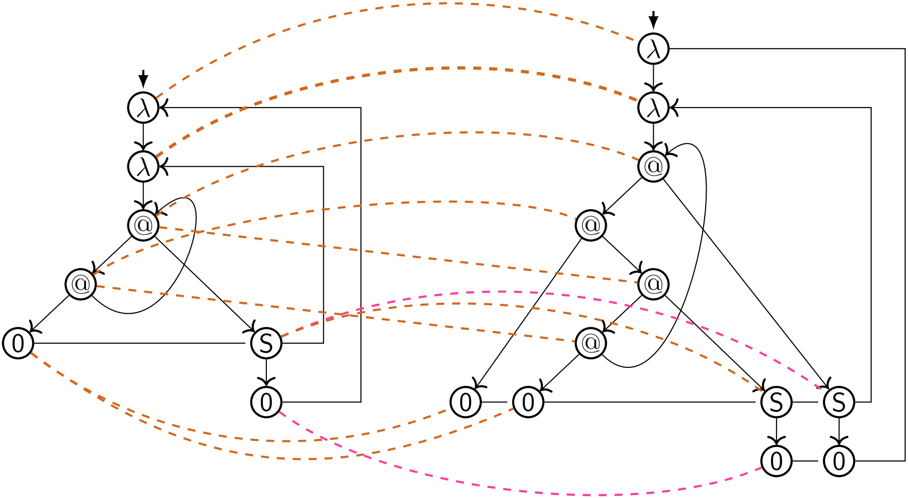

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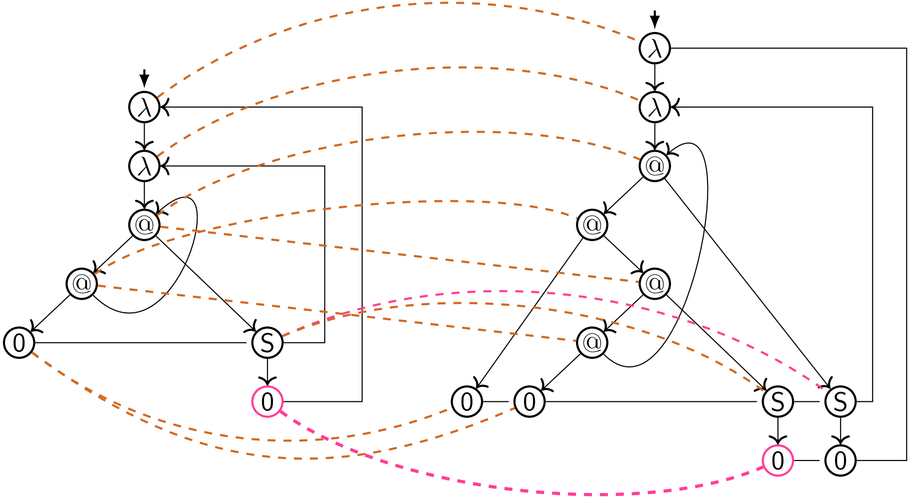

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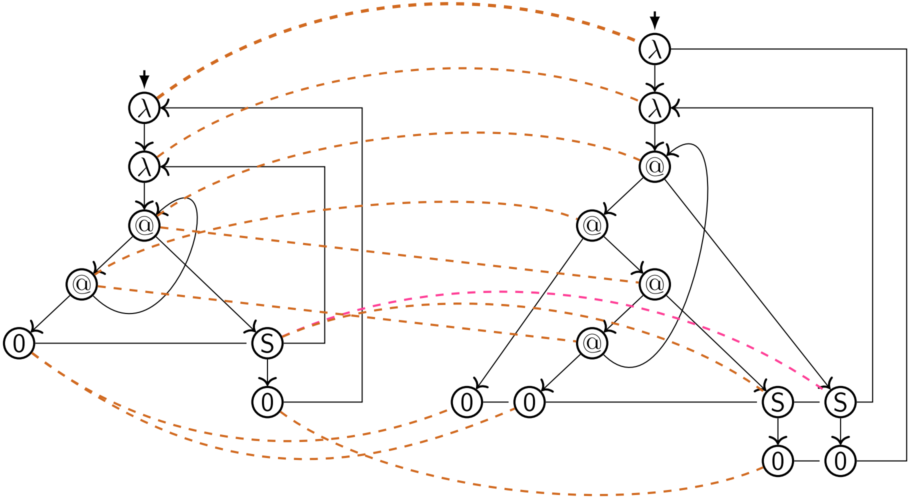

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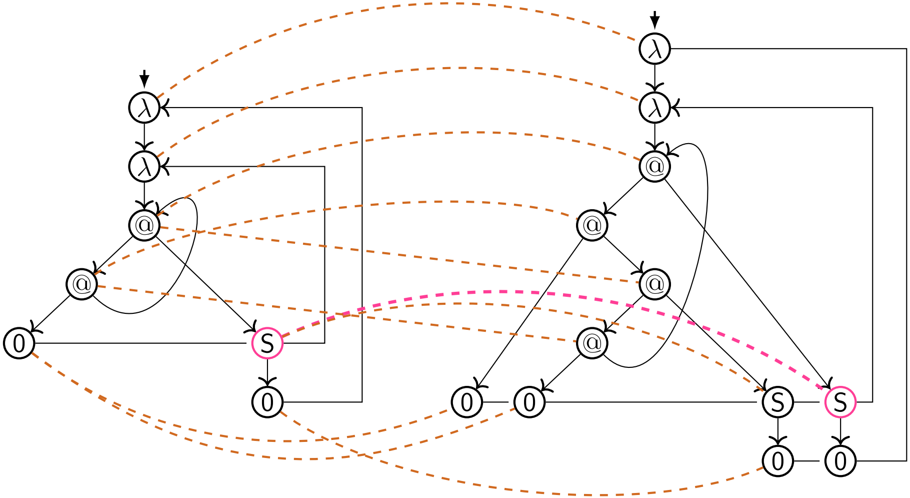

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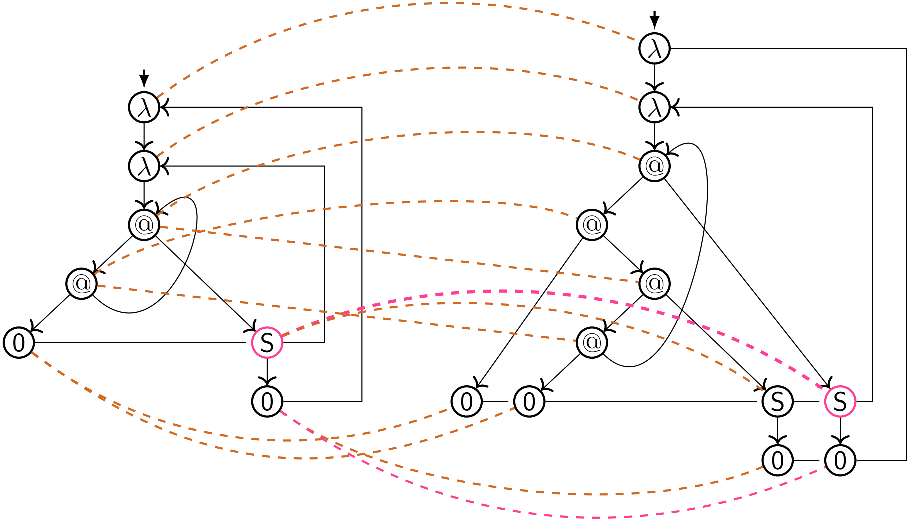

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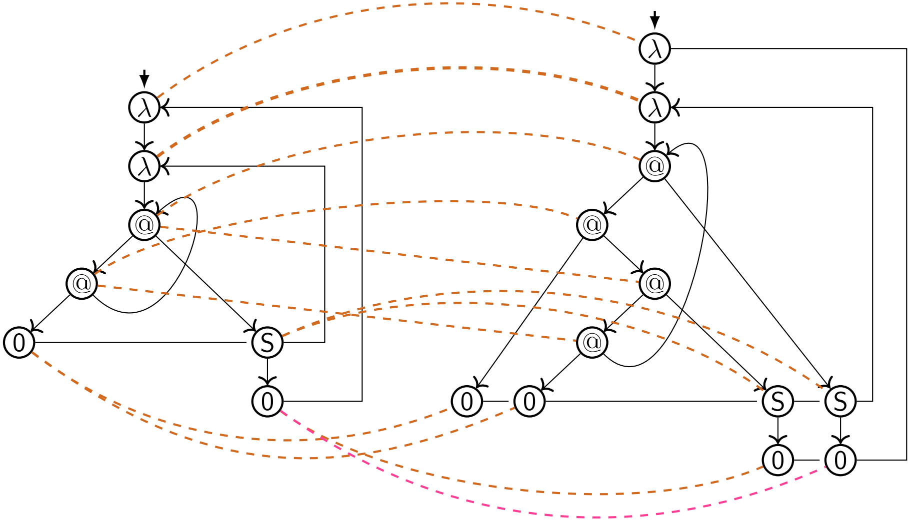

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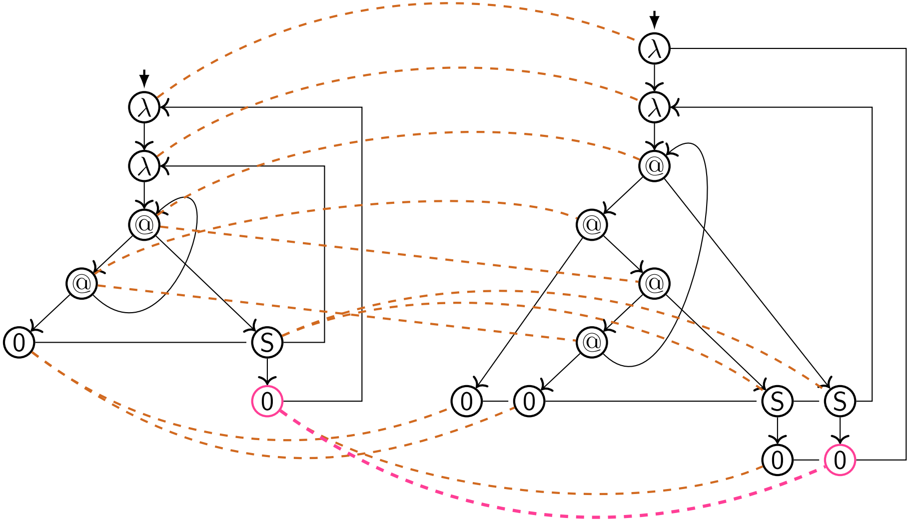

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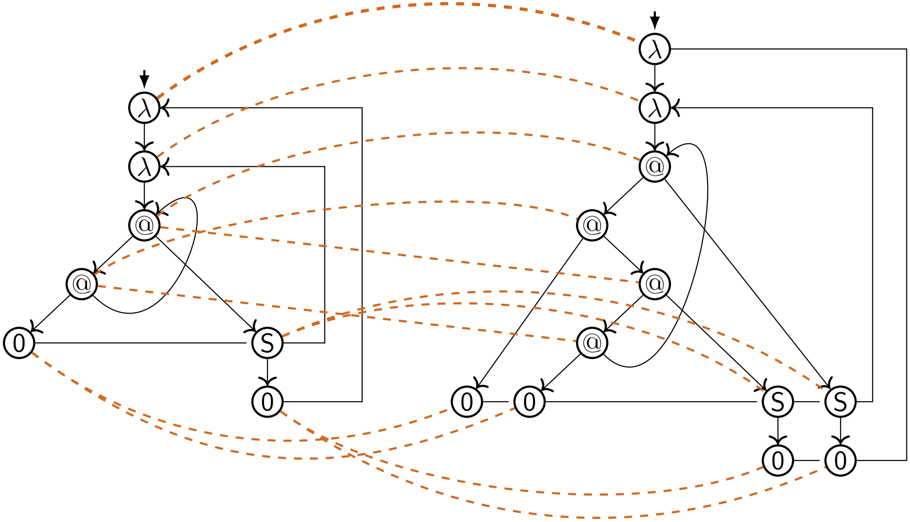
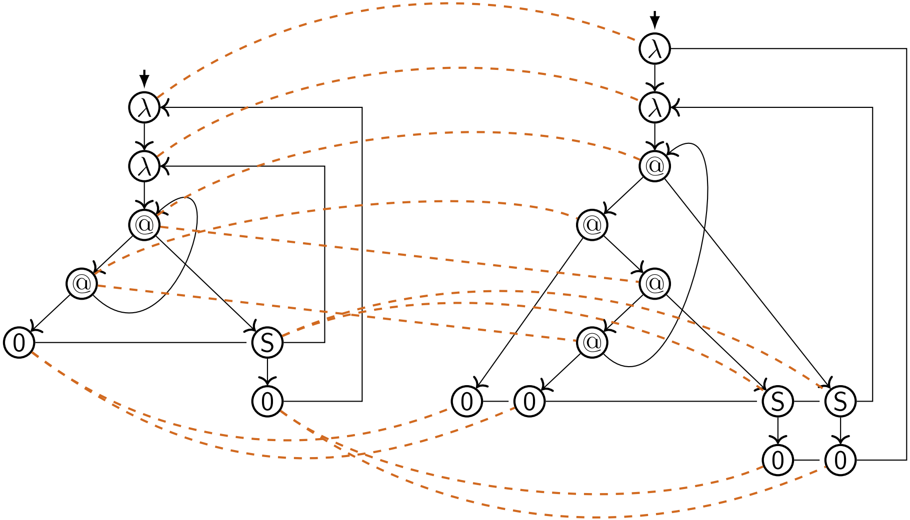
Bisimulation check between λ-term-graphs



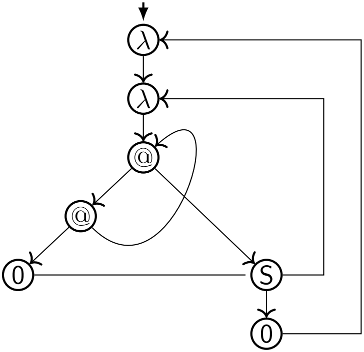
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

bisimulation between λ-term-graphs

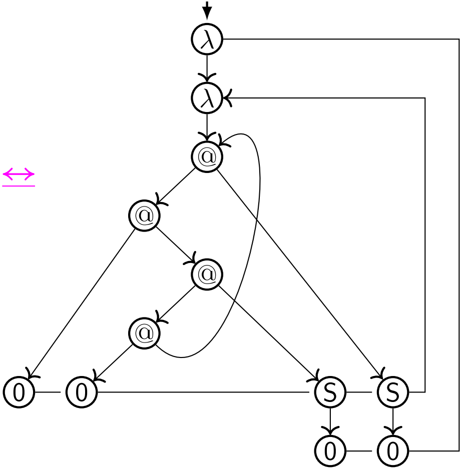


$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

bisimilarity between λ-term-graphs

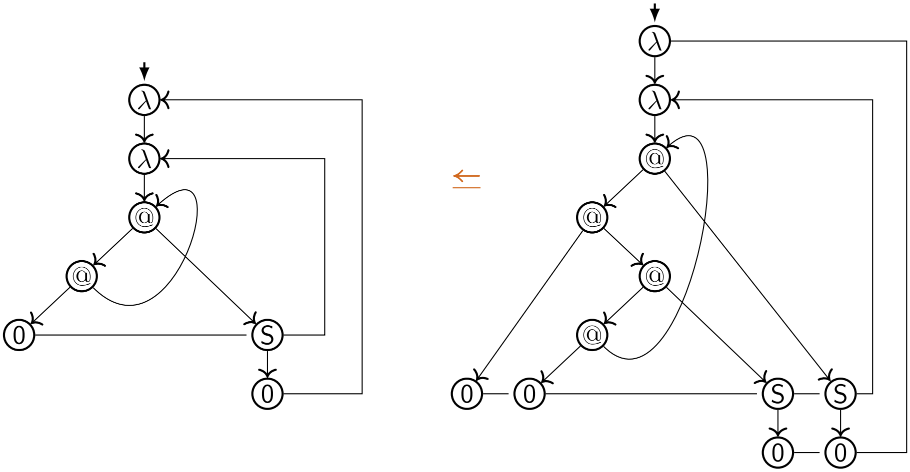


$\underline{\leftrightarrow}$



$[\![L_0]\!]_{\mathcal{T}}$ $\underline{\leftrightarrow}$ $[\![L]\!]_{\mathcal{T}}$

functional bisimilarity and bisimulation collapse



$$[\![L_0]\!]_{\mathcal{T}} \quad \Leftarrow \quad [\![L]\!]_{\mathcal{T}}$$

Bisimulation collapse: property

Theorem

The class of eager-scope λ-term-graphs is closed under functional bisimilarity $\rightarrow$.

$\Longrightarrow$ For a $\boldsymbol{\lambda}_{\mathsf{letrec}}$-term L

the bisimulation collapse of $[\![L]\!]_{\mathcal{T}}$ is again an eager-scope λ-term-graph.

Readback

$$\begin{tikzcd}[column sep=huge, row sep=huge]
L \arrow[r, mapsto, bend left, "\text{interpret}"] & G \arrow[d, Rightarrow, "\text{collapse}"] \\
L_0 & G_0 \arrow[l, mapsto, bend left, "\text{readback}"]
\end{tikzcd}$$

Readback

defined with property:

$$L \xleftarrow[\text{rb}]{} G \quad \text{eager-scope}$$

Readback

defined with property:

$$\begin{tikzcd} L \arrow[r, bend left, mapsto, "{[\![\cdot]\!]_{\mathcal{T}}}"] & G \arrow[l, bend left, mapsto, "\mathrm{rb}"] \end{tikzcd} \quad \text{eager-scope}$$

Readback

defined with property:

$$\begin{tikzcd} L \arrow[r, mapsto, bend left, "[\![\cdot]\!]_{\mathcal{T}}"] & G \arrow[l, mapsto, bend left, "\mathsf{rb}"] \end{tikzcd} \quad \text{eager-scope}$$

Theorem

For all eager-scope λ-term-graphs G:

$$([\![\cdot]\!]_{\mathcal{T}} \circ \mathsf{rb})(G) \simeq G$$

The readback rb is a right-inverse of $[\![\cdot]\!]_{\mathcal{T}}$ modulo isomorphism $\simeq$.

Readback

defined with property:

$$\begin{tikzcd} L \arrow[r, bend left, mapsto, "\llbracket \cdot \rrbracket_{\mathcal{T}}"] & G \arrow[l, bend left, mapsto, "\mathsf{rb}"] \end{tikzcd} \quad \text{eager-scope}$$

Theorem

For all eager-scope λ-term-graphs G:

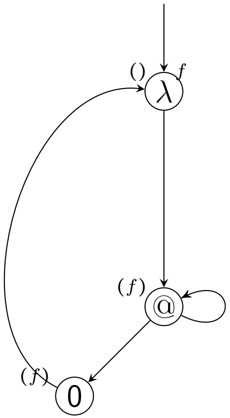
$$(\llbracket \cdot \rrbracket_{\mathcal{T}} \circ \mathsf{rb})(G) \simeq G$$

The readback rb *is a right-inverse of* $\llbracket \cdot \rrbracket_{\mathcal{T}}$ *modulo isomorphism* $\simeq$.

idea:

1. construct a spanning tree T of G
2. using local rules, in a bottom-up traversal of T synthesize $L = \mathsf{rb}(G)$

Readback: example (fix)



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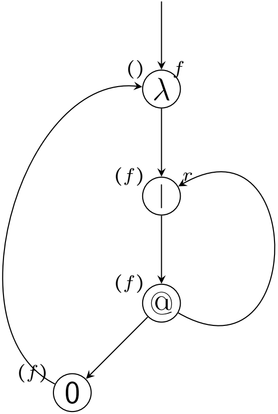

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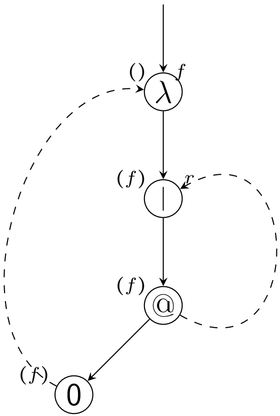

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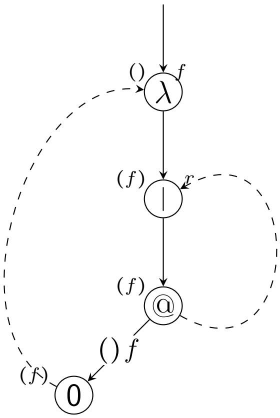

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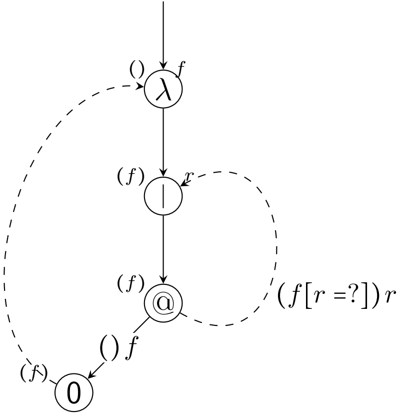

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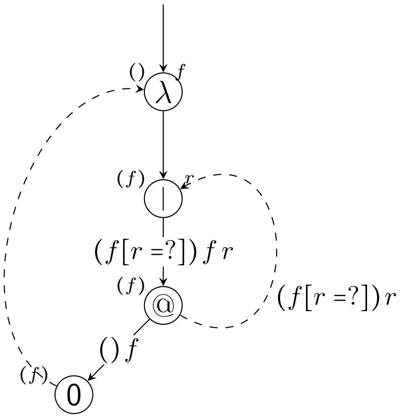

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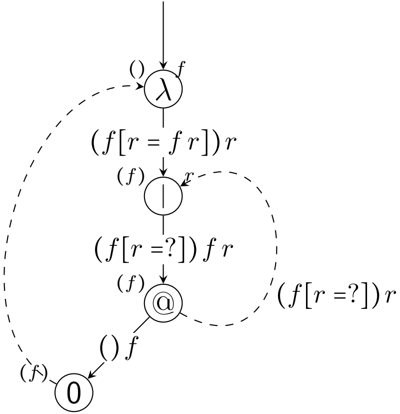

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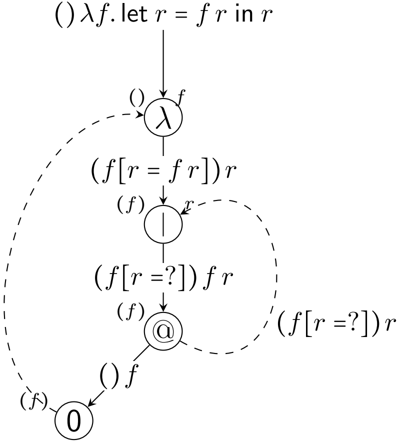

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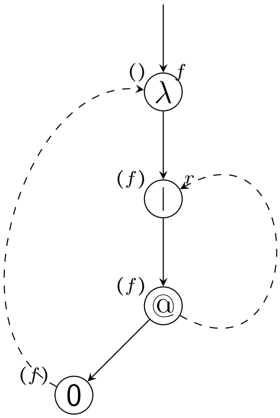

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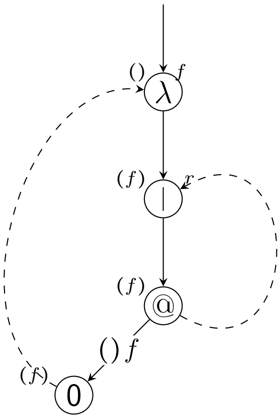

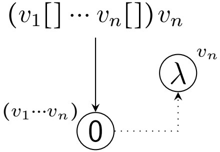

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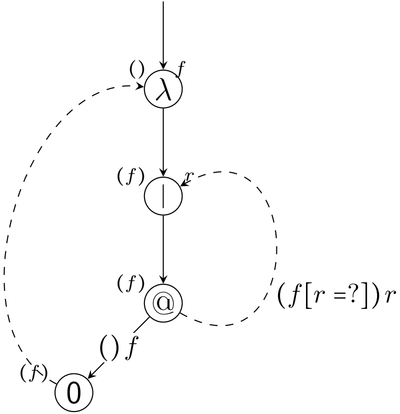

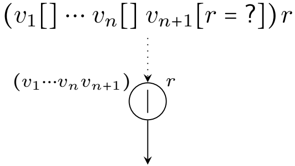

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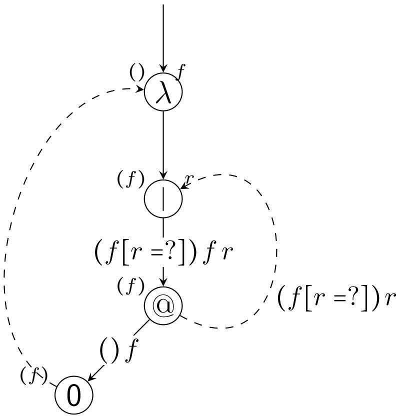

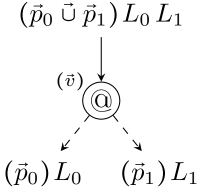

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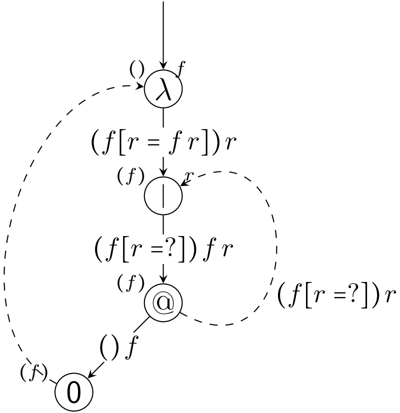

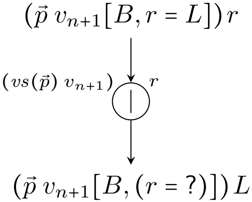

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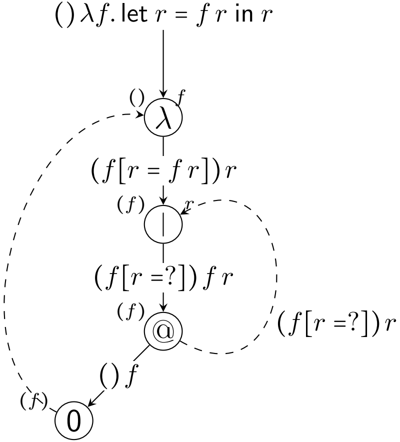
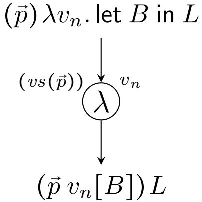




Maximal sharing: complexity

$$\begin{tikzcd}
L \arrow[r, mapsto, bend left, "{[\![\cdot]\!]_{\mathcal{T}}}"] & G \arrow[d, Rightarrow] \\
L_0 & G_0 \arrow[l, mapsto, bend left, "\mathrm{rb}"]
\end{tikzcd}$$

1. interpretation

   of $\boldsymbol{\lambda}_{\mathsf{letrec}}$-term L with $|L| = n$

   as λ-term-graph $G = [\![L]\!]_{\mathcal{T}}$

   - in time $O(n^2)$, size $|G| \in O(n^2)$.

2. bisimulation collapse $|\!\downarrow$

   of f-o term graph G into G_0

   - in time $O(|G| \log |G|) = O(n^2 \log n)$

3. readback rb

   of f-o term graph G_0

   yielding $\boldsymbol{\lambda}_{\mathsf{letrec}}$-term $L_0 = \mathrm{rb}(G_0)$.

   - in time $O(|G| \log |G|) = O(n^2 \log n)$

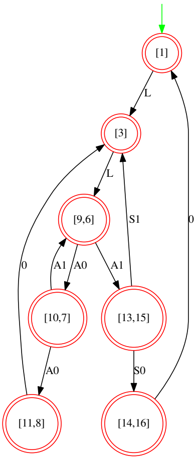
Theorem

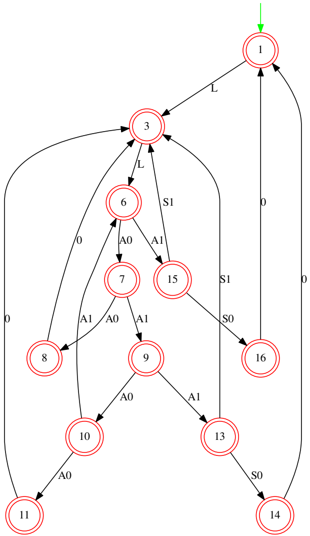
Computing a maximally compact form $L_0 = (\mathrm{rb} \circ |\!\downarrow \circ [\![\cdot]\!]_{\mathcal{T}})(L)$ of L for a $\boldsymbol{\lambda}_{\mathsf{letrec}}$-term L requires time $O(n^2 \log n)$, where $|L| = n$.

Demo: console output

```
jan:~/papers/maxsharing-ICFP/talks/ICFP-2014> maxsharing running.l
λ-letrec-term:
λx. λf. let r = f (f r x) x in r

derivation:
                ---------- 0                 ------- 0
                (x f[r]) f    (x f[r]) r     (x) x
                ------------------------- @  ---------- S
                (x f[r]) f r                 (x f[r]) x
------------ 0  ----------------------------------------- @    ------- 0
(x f[r]) f      (x f[r]) f r x                                 (x) x
----------------------------------------------------------- @  ---------- S
(x f[r]) f (f r x)                                             (x f[r]) x
--------------------------------------------------------------------------- @
(x f[r]) f (f r x) x                                                           (x f[r]) r
----------------------------------------------------------------------------------------- let
(x f) let r = f (f r x) x in r
--------------------------------------------------------------------------------------------- λ
(x) λf. let r = f (f r x) x in r
----------------------------------------------------------------------------------------------- λ
() λx. λf. let r = f (f r x) x in r

writing DFA to file: running-dfa.pdf

readback of DFA:
λx. λy. let F = y (y F x) x in F

writing minimised DFA to file: running-mindfa.pdf

readback of minimised DFA:
λx. λy. let F = y F x in F
jan:~/papers/maxsharing-ICFP/talks/ICFP-2014> 
```

Demo: generated λ-NFAs



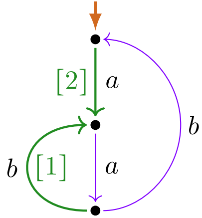


Resources (maximal sharing)

- tool maxsharing on `hackage.haskell.org`
- papers and reports
  - Maximal Sharing in the Lambda Calculus with Letrec
    - ICFP 2014 paper
    - accompanying report arXiv:1401.1460
  - Term Graph Representations for Cyclic Lambda Terms
    - TERMGRAPH 2013 proceedings
    - extended report arXiv:1308.1034
  - Vincent van Oostrom, CG: Nested Term Graphs
    - TERMGRAPH 2014 post-proceedings in EPTCS 183
- thesis Jan Rochel
  - Unfolding Semantics of the Untyped λ-Calculus with letrec
    - Ph.D. Thesis, Utrecht University, 2016

Process interpretation of regular expressions

(based on joint work with Wan Fokkink)



Regular expressions *(S.C. Kleene, 1951)*

Definition

The set $\mathsf{Reg}(A)$ of regular expressions over alphabet A is defined by the grammar:

$$e, f \;::=\; 0 \mid 1 \mid a \mid (e + f) \mid (e \cdot f) \mid (e^{\star}) \qquad \text{(for } a \in A\text{).}$$

Regular expressions *(S.C. Kleene, 1951)*

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Note, here:

- symbol 0 instead of $\varnothing$
- symbol 1 used (often dropped, definable as $0^{\star}$)
- no complementation operation $\overline{e}$
  - which is not expressible under language interpretation

Language semantics $[\![\cdot]\!]_L$ of reg. expr's *(Copi–Elgot–Wright, 1958)*

$\mathbf{0} \xmapsto{L}$ empty language $\varnothing$

$\mathbf{1} \xmapsto{L} \{\epsilon\}$ (ϵ the empty word)

$a \xmapsto{L} \{a\}$

Language semantics $[\![\cdot]\!]_L$ of reg. expr's *(Copi–Elgot–Wright, 1958)*

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$e + f \xmapsto{L}$ union of $L(e)$ and $L(f)$

$e \cdot f \xmapsto{L}$ element-wise concatenation of $L(e)$ and $L(f)$

$e^* \xmapsto{L}$ set of words formed by concatenating words in $L(e)$, and adding the empty word ϵ

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and adding the empty word ϵ

$[\![e]\!]_L := L(e)$ (language defined by e)

Process semantics of regular expressions $[\![\cdot]\!]_P$ (Milner, 1984)

$\mathbf{0} \overset{P}{\longmapsto}$ deadlock $\boldsymbol{\delta}$, no termination

$\mathbf{1} \overset{P}{\longmapsto}$ empty-step process $\boldsymbol{\epsilon}$, then terminate

$a \overset{P}{\longmapsto}$ atomic action a, then terminate

Process semantics of regular expressions $[\![\cdot]\!]_P$ *(Milner, 1984)*

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$e + f \stackrel{P}{\longmapsto}$ (*choice*) execute $[\![e]\!]_P$ or $[\![f]\!]_P$

$e \cdot f \stackrel{P}{\longmapsto}$ (*sequentialization*) execute $[\![e]\!]_P$, then $[\![f]\!]_P$

$e^* \stackrel{P}{\longmapsto}$ (*iteration*) repeat (terminate or execute $[\![e]\!]_P$)

Process semantics of regular expressions $[\![\cdot]\!]_P$ *(Milner, 1984)*

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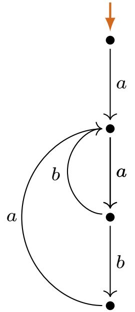
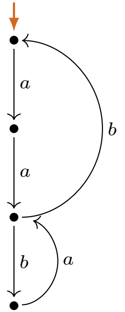
$e^* \overset{P}{\longmapsto}$ (*iteration*) repeat (terminate or execute $[\![e]\!]_P$)

$[\![e]\!]_P \ := \ [P(e)]_{\underline{\leftrightarrow}}$ (bisimilarity equivalence class of process $P(e)$)

Process interpretation of regular expressions

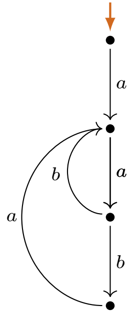


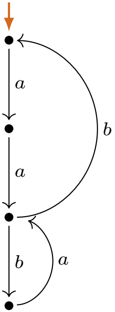
$P(a(a(b+ba))^*0)$



$P((aa(ba)^*b)^*0)$

Process interpretation of regular expressions

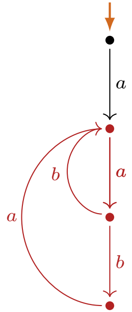




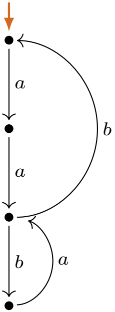
$P(a \cdot (a \cdot (b + b \cdot a))^* \cdot 0)$

$P((a \cdot a \cdot (b \cdot a)^* \cdot b)^* \cdot 0)$

Process interpretation of regular expressions

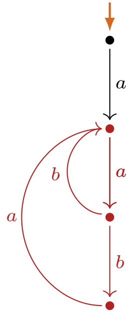


$P(a \cdot (a \cdot (b + b \cdot a))^{*} \cdot 0)$



$P((a \cdot a \cdot (b \cdot a)^{*} \cdot b)^{*} \cdot 0)$

Process interpretation of regular expressions



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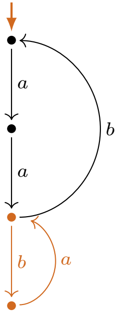

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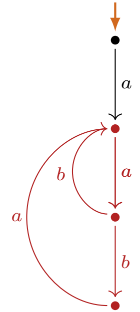

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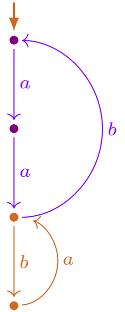

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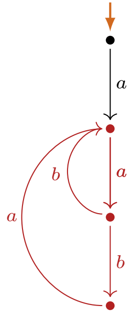
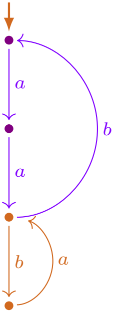
Process interpretation of regular expressions

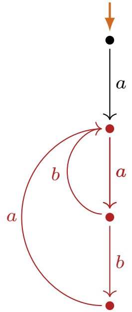


$$P(a \cdot (a \cdot (b + b \cdot a))^* \cdot 0)$$

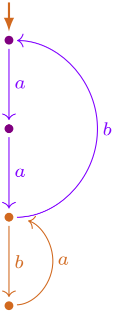


$$P((a \cdot a \cdot (b \cdot a)^* \cdot b)^* \cdot 0)$$

Process interpretation of regular expressions

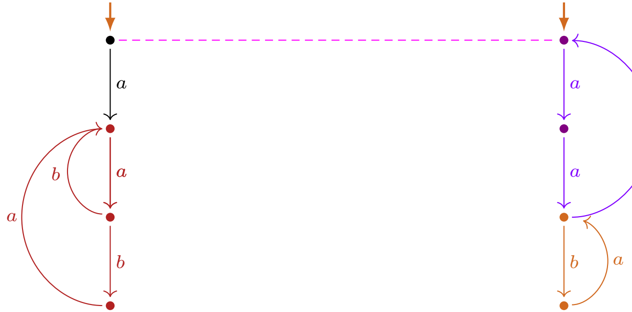


$$P(a(a(b+ba))^*0)$$



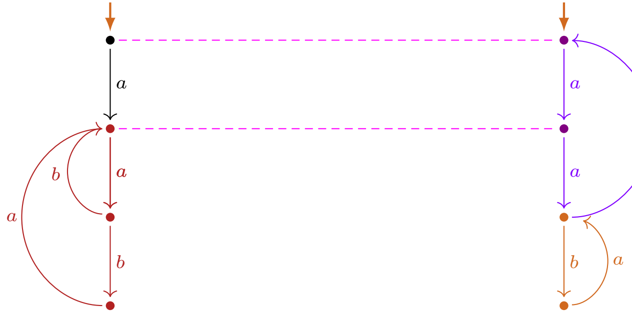
$$P((aa(ba)^*b)^*0)$$

Process interpretation of regular expressions



$P(a(a(b+ba))^{*}0)$ $\quad$ $P((aa(ba)^{*}b)^{*}0)$

Process interpretation of regular expressions



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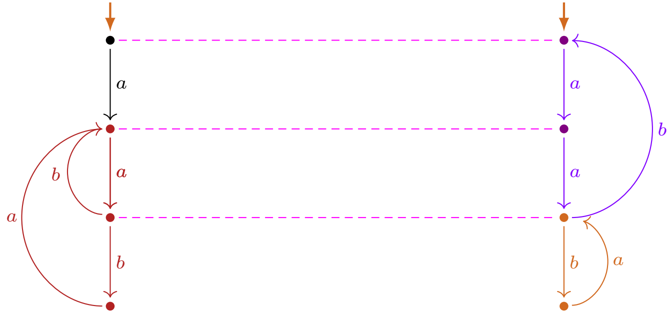
Process interpretation of regular expressions



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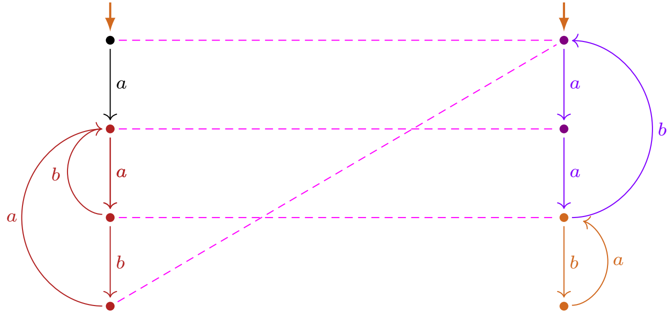

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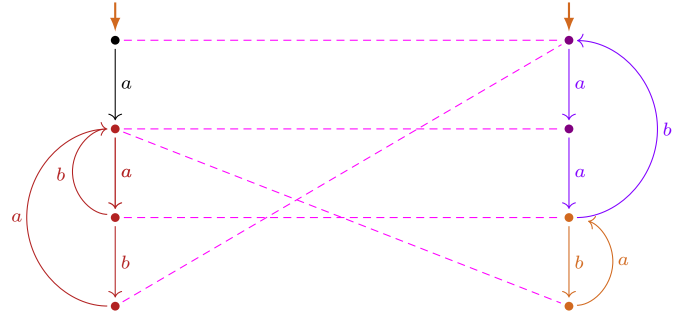

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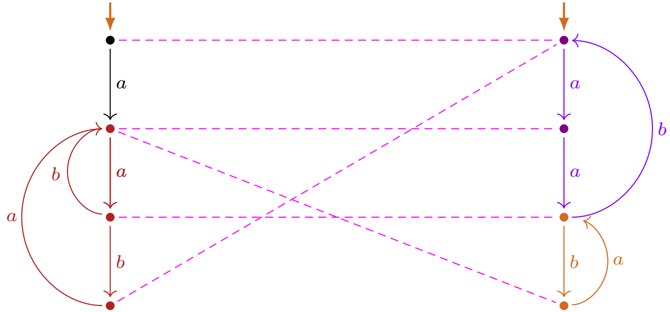

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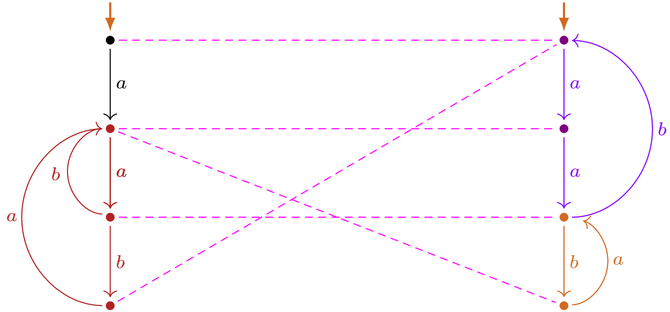

$$a(a(b+ba))^{*}0 \qquad \underline{\leftrightarrow}_{P} \qquad (aa(ba)^{*}b)^{*}0$$

Process graphs and NFAs

Definition

A process graph over actions in A is a tuple $G = \langle V, v_{\mathsf{s}}, T, E \rangle$ where:

- V is a set of *vertices*,
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- $T \subseteq V \times A \times V$ the set of *transitions*,
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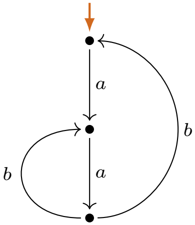
Antimirov (1996): NFA-definition of $P(\cdot)$ via partial derivatives.

Expressible process graphs (under bisimulation $\underline{\leftrightarrow}$)



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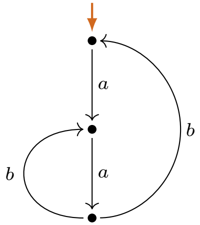

$? \in im(P(\cdot))\ ?$

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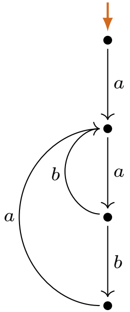
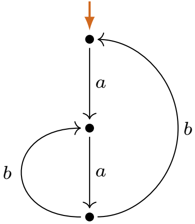


$\in im(P(\cdot))$

$P(\cdot)$-expressible



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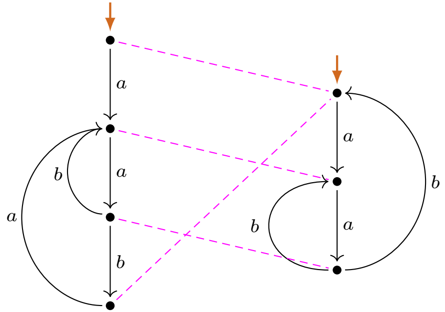

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$P(\cdot)$-expressible

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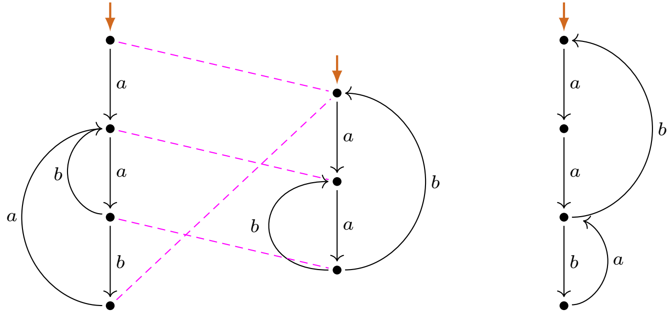

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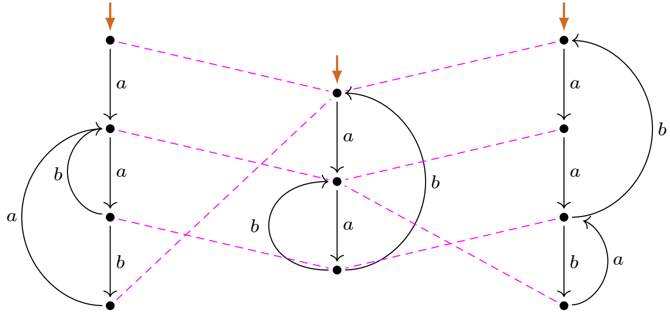

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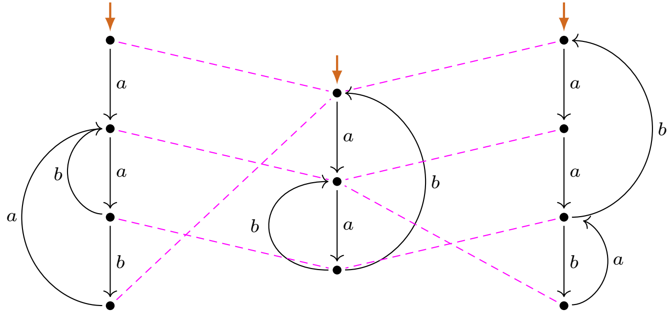

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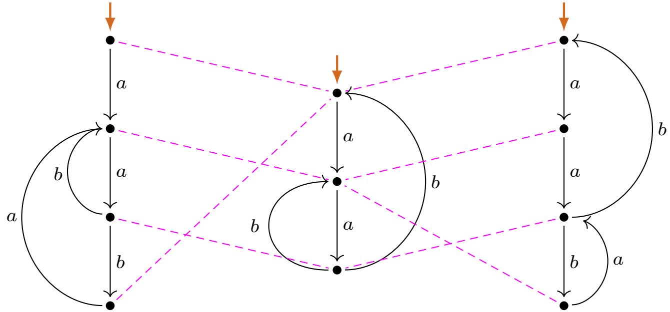

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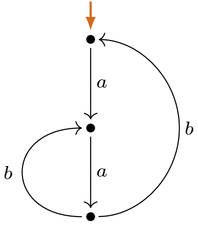
$[\![\cdot]\!]_P$-expressible | $[\![\cdot]\!]_P$-expressible | $[\![\cdot]\!]_P$-expressible

Properties of P and $[\![\cdot]\!]_P$

- Not every finite-state process is $P(\cdot)$-expressible.



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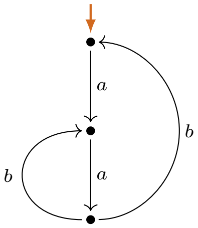

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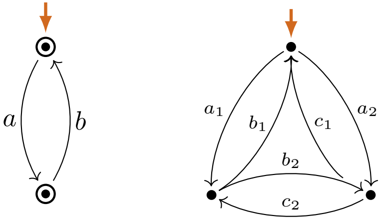

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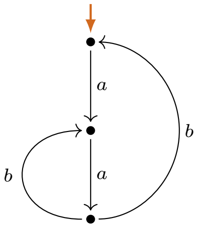

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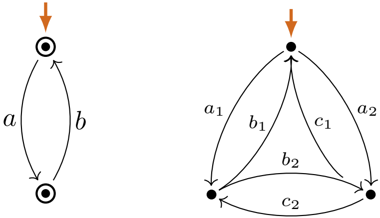

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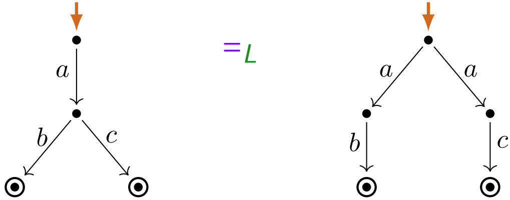

$a \cdot (b + c) \quad =_L \quad a \cdot b + a \cdot c$

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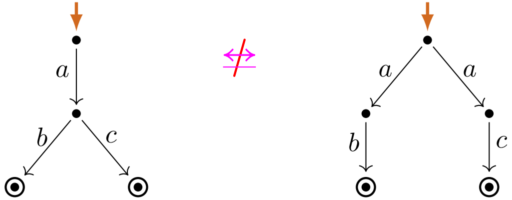

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Complete axiomatization of $=_L$ *(Aanderaa/Salomaa, 1965/66)*

Axioms :

(B1)	$e + (f + g) = (e + f) + g$	(B7)	$e \cdot 1 = e$
(B2)	$(e \cdot f) \cdot g = e \cdot (f \cdot g)$	(B8)	$e \cdot 0 = 0$
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Inference rules : equational logic *plus*

$$\frac{e = f \cdot e + g}{e = f^* \cdot g}\ \mathsf{FIX} \quad (\text{if } \underbrace{\{\epsilon\} \notin L(f)}_{\textit{non-empty-word property}})$$

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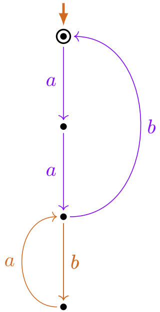
- Mil⁻ + one of two stronger rules (than RSP*) is complete *(G, 2006)*

Well-behaved form, looping palm trees



$P((aa(ba)^* b)^*)$

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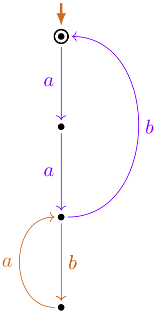

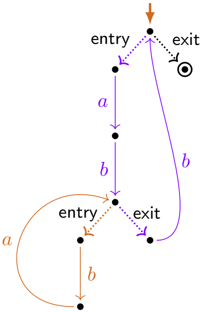

well-behaved form

(Corradini, Baeten)

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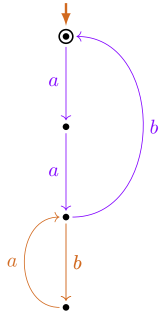

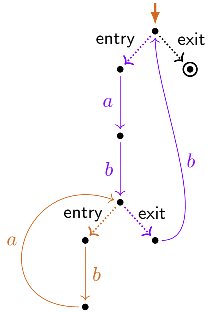

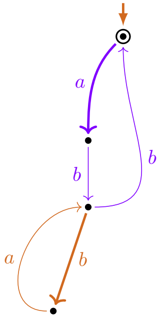

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Loop chart

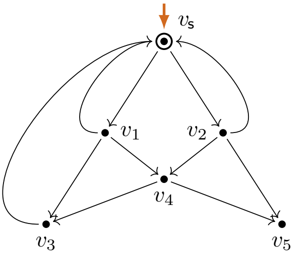
Definition

A process graph is a loop chart if:

L-1.

L-2.

L-3.



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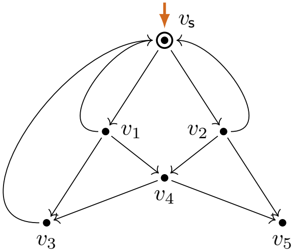

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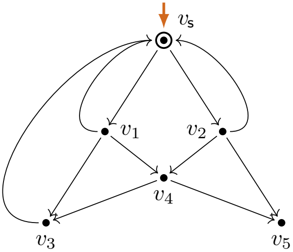

Loop chart

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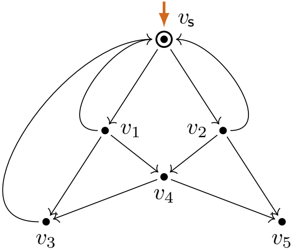
L-3. Termination is only possible at the start vertex.



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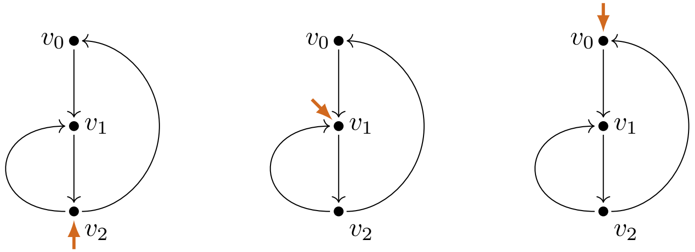

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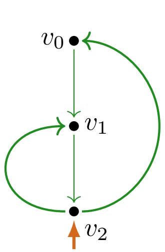

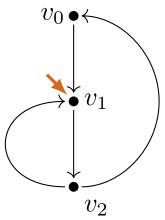

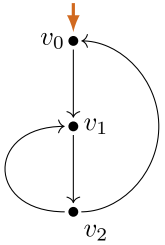

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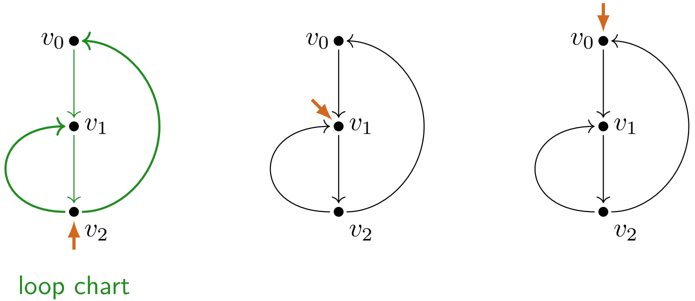

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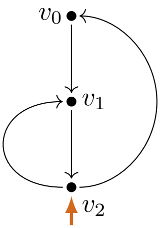

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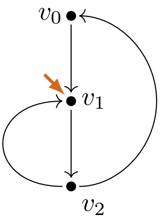

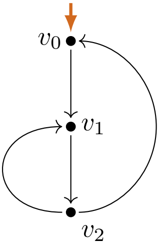

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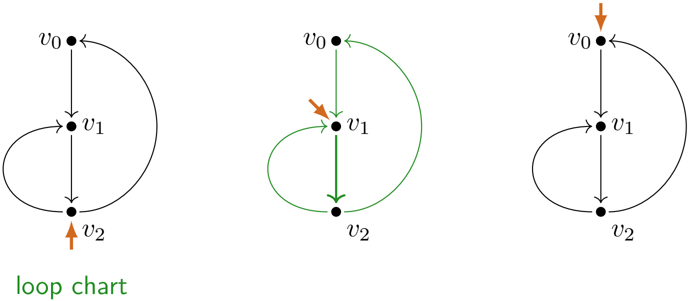

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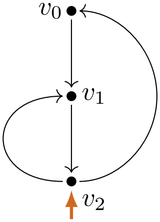

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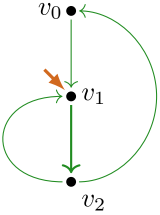

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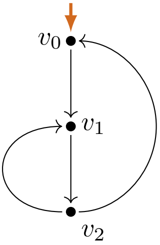

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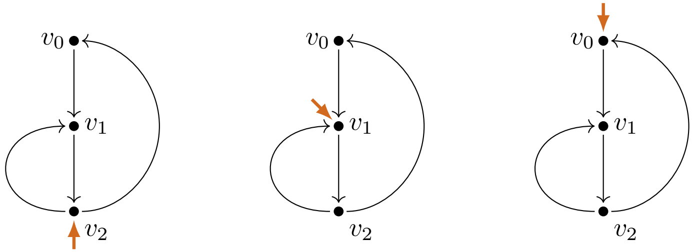

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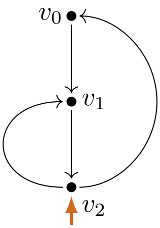

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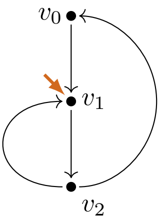

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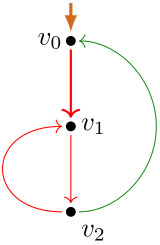

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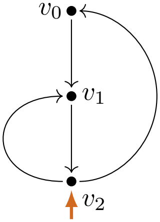

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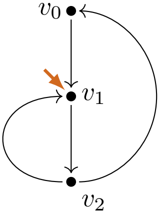

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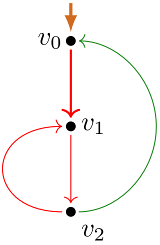

no loop chart

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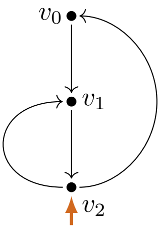

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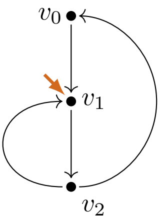

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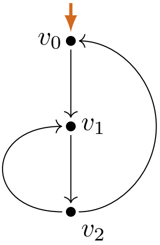
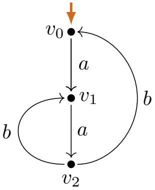


no loop chart

Loop elimination



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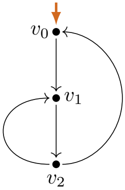

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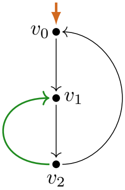

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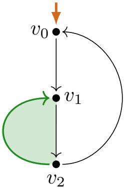

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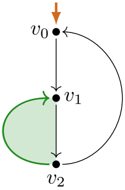

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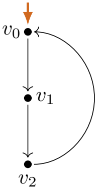

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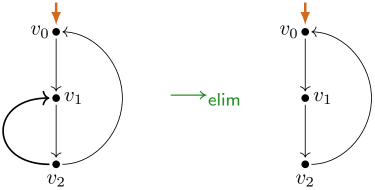

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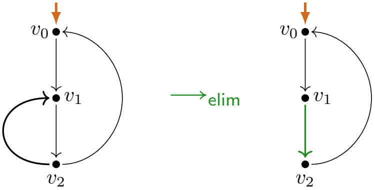

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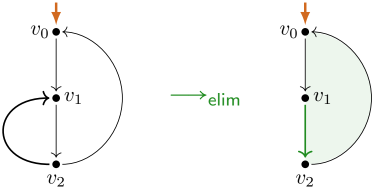

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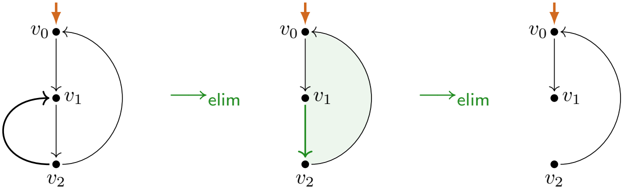

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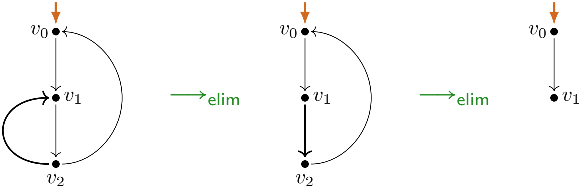

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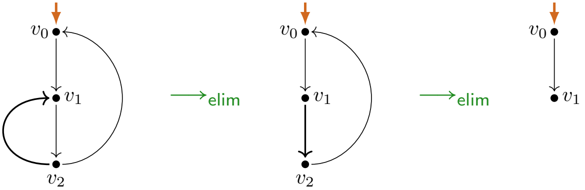

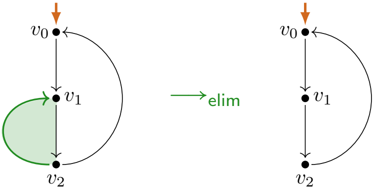

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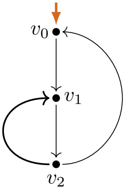

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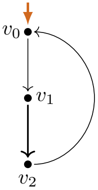

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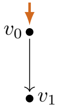

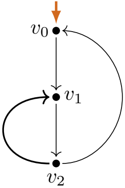

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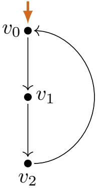

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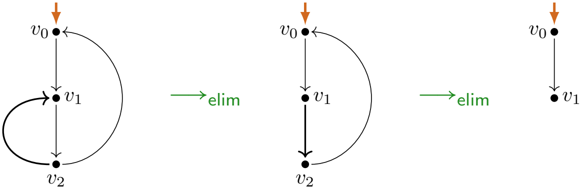

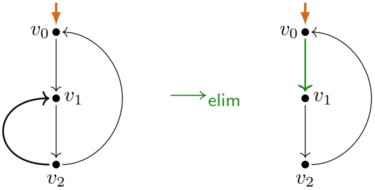

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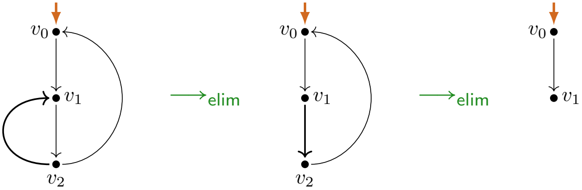

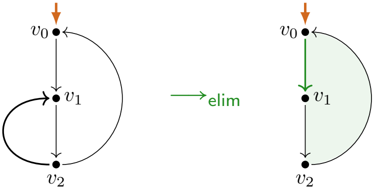

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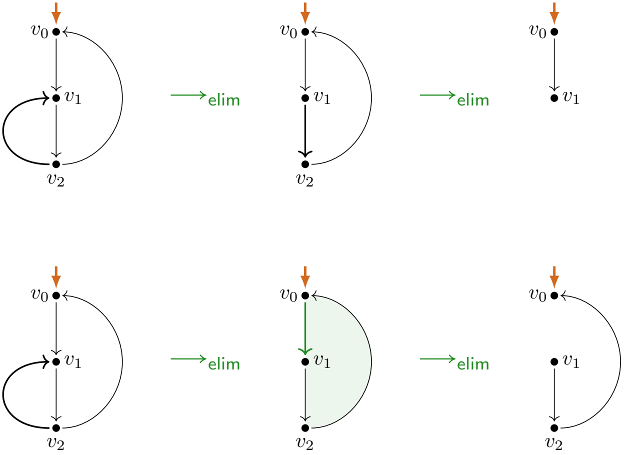

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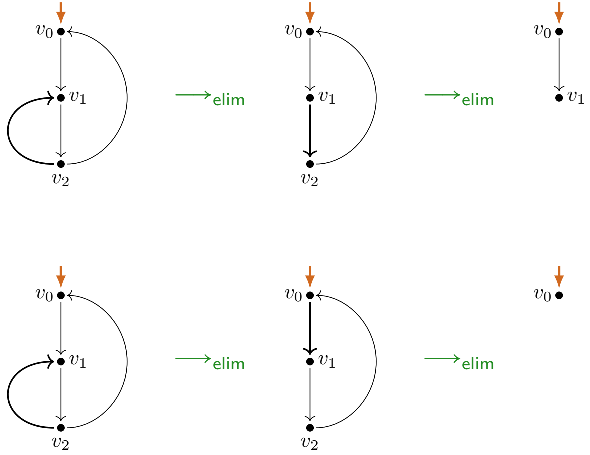

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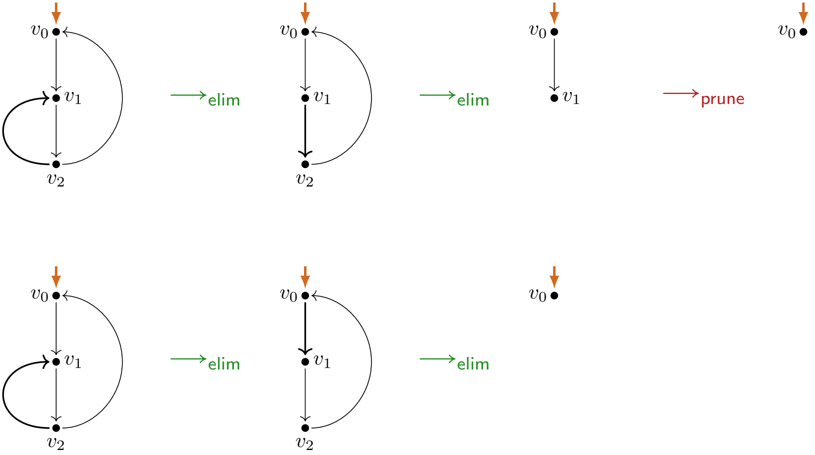

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$\longrightarrow_{\text{elim}}$: eliminate a transition-induced loop by:

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$\longrightarrow_{\text{prune}}$: remove a transition to a deadlocking state

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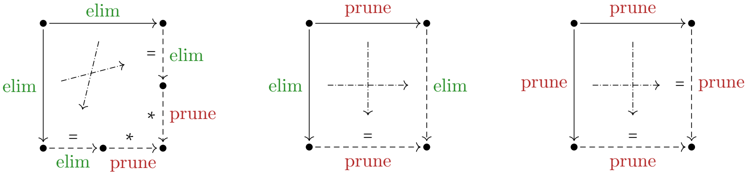

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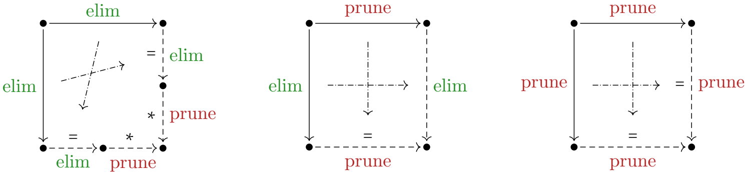

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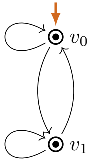

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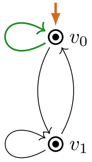

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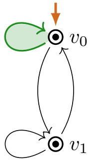

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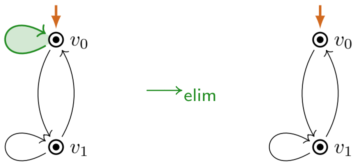

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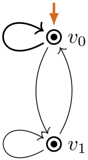

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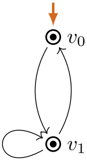

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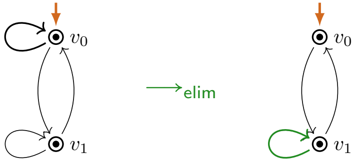

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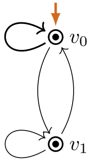

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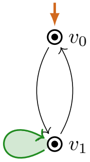

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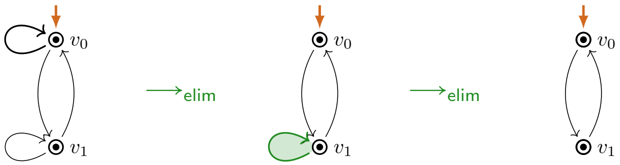

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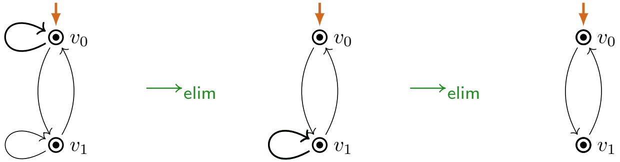

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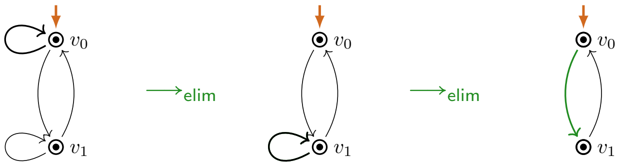

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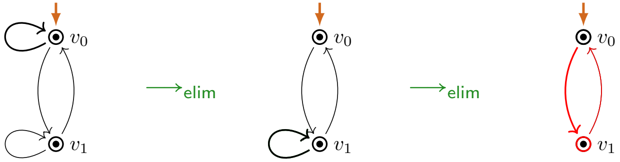

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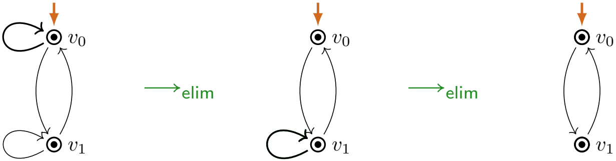

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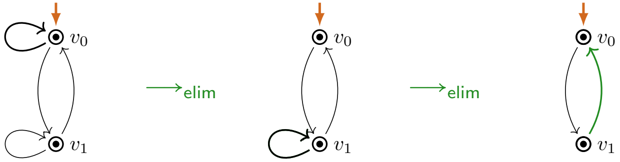

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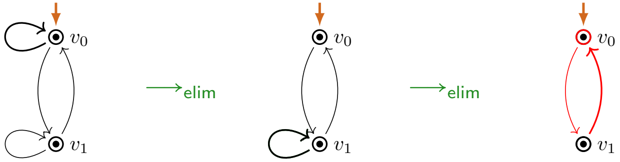
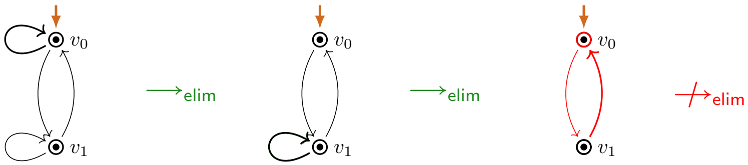
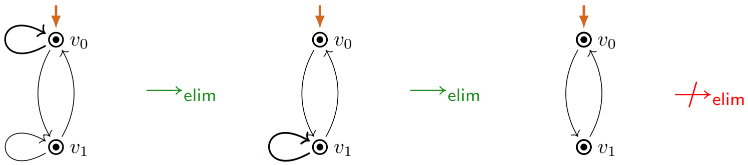


Loop elimination



Loop elimination



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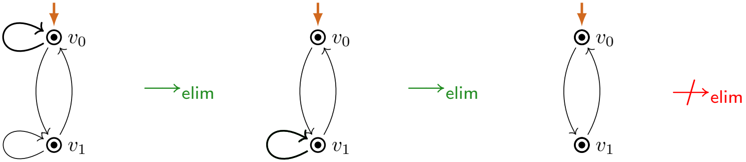
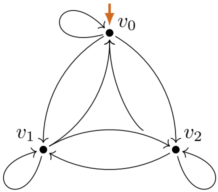
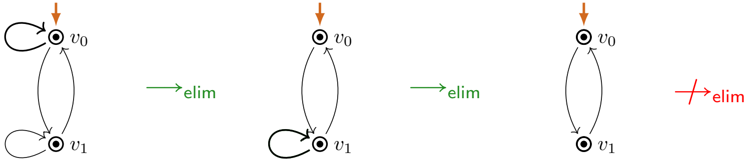
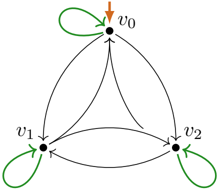


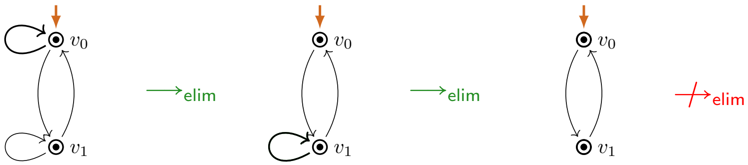


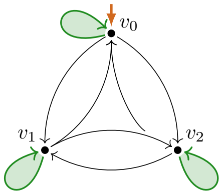
Loop elimination



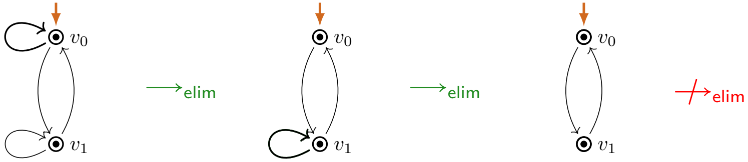


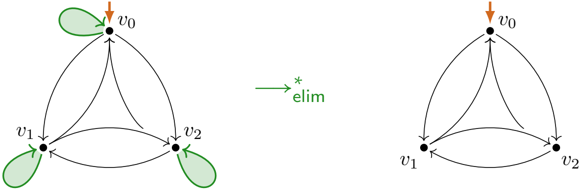
Loop elimination



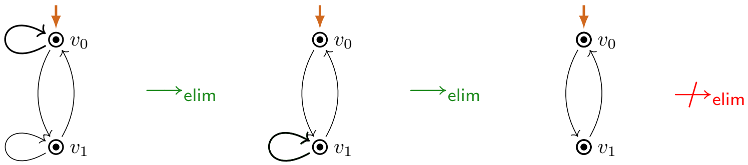


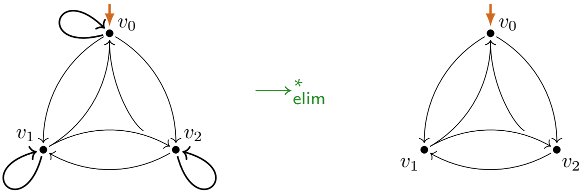
Loop elimination



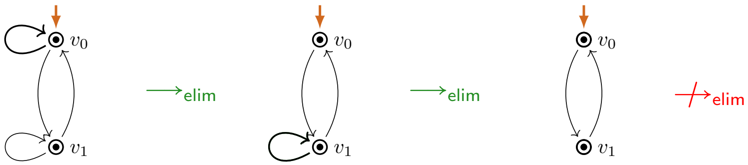


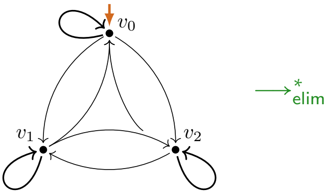
Loop elimination



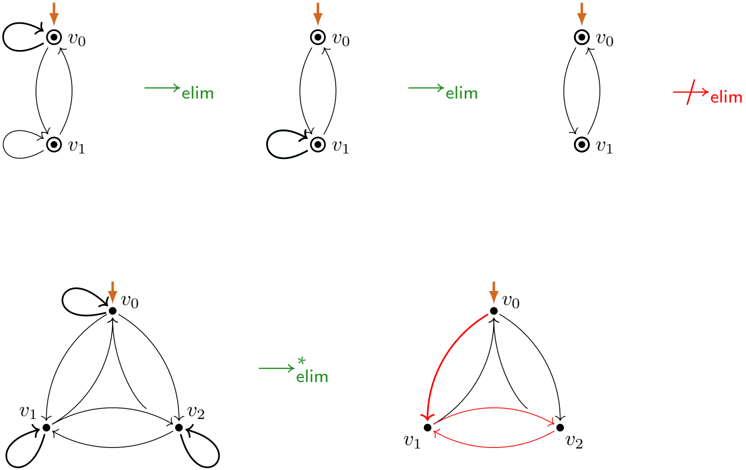


Loop elimination

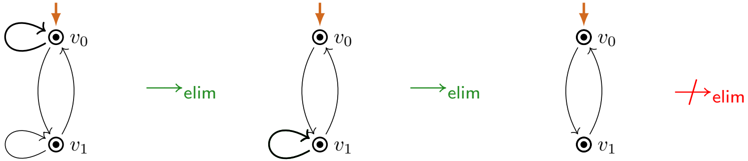


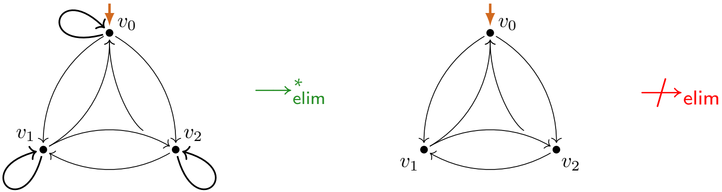


Loop elimination



Loop elimination





Structure property LEE

Definition

A process graph G satisfies LEE (*loop existence and elimination*) if:

$$\exists G_0 \left(G \longrightarrow^{*}_{\text{elim}} G_0 \not\longrightarrow_{\text{elim}} \right.$$
$$\left. \wedge \; G_0 \text{ has no infinite trace} \right) .$$

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Lemma (by using termination and confluence)

For every process graph G the following are equivalent:

(i) LEE(G).

(ii) *There is an $\longrightarrow_{\text{elim}}$ normal form without an infinite trace.*

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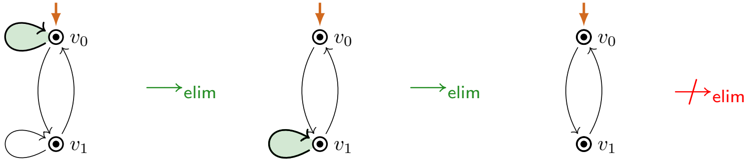
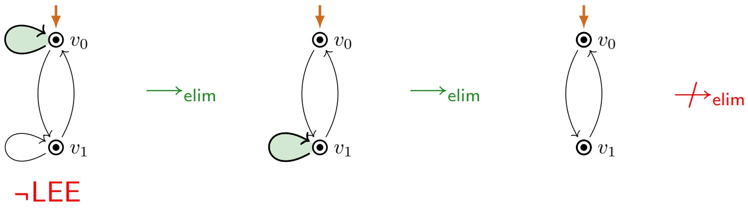
Theorem (efficient decidability)

The problem of deciding LEE(G) for process graphs G is in P.

LEE fails

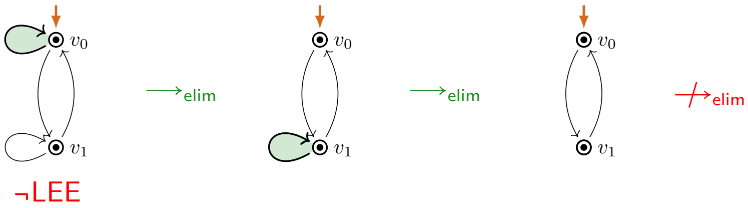

v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$

LEE fails

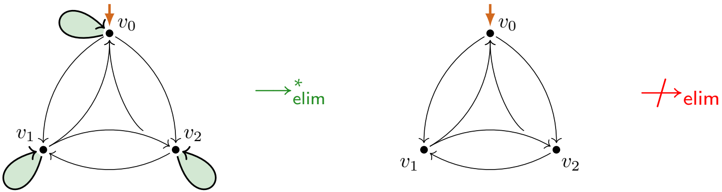


v_0
v_1
$\neg$LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$

LEE fails

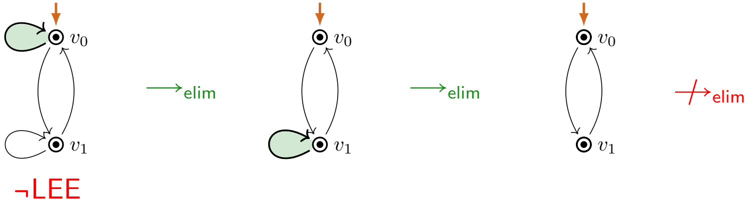


v_0
v_1
$\neg$LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$



v_0
v_1
v_2
$\longrightarrow^{*}_{\text{elim}}$
v_0
v_1
v_2
$\not\longrightarrow_{\text{elim}}$

LEE fails



v_0
v_1
$\neg$LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
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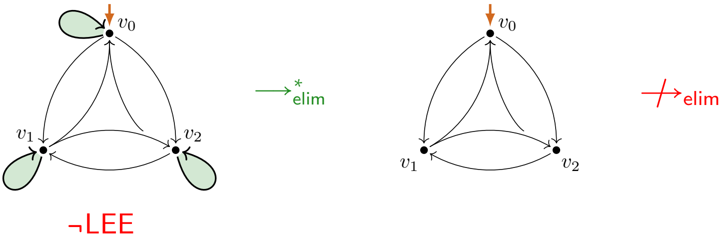
v_0
v_1
$\not\longrightarrow_{\text{elim}}$



v_0
v_1
v_2
$\neg$LEE
$\longrightarrow^{*}_{\text{elim}}$
v_0
v_1
v_2
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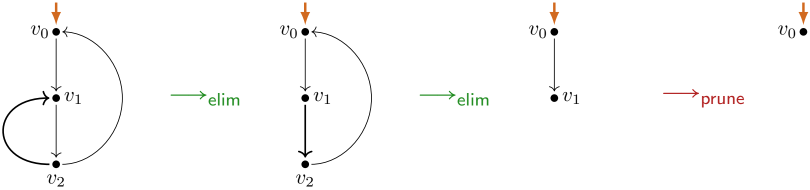
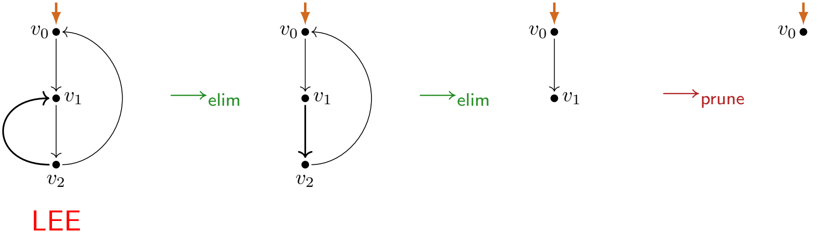
LEE holds

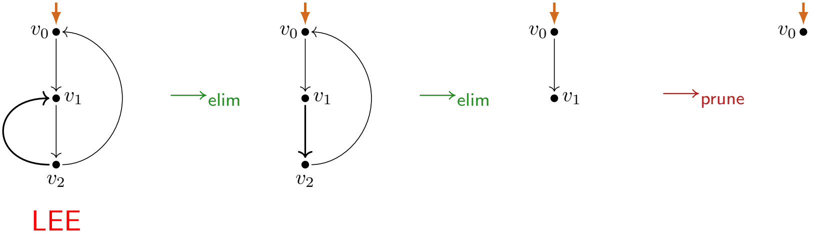


LEE holds



v_0
v_1
v_2
LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{prune}}$
v_0

LEE holds / Recording loop elimination



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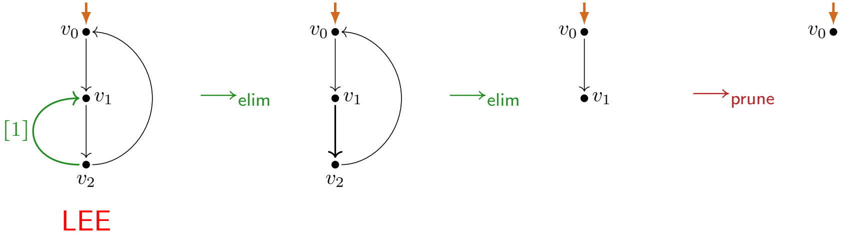
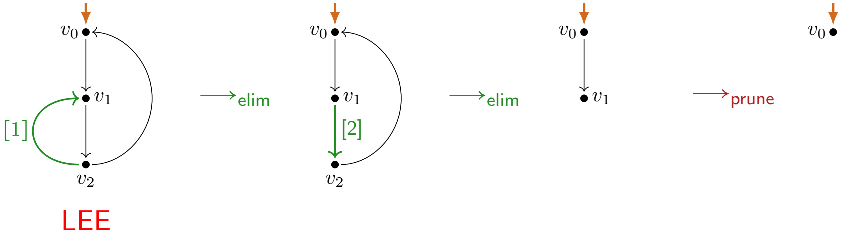


LEE holds / Recording loop elimination



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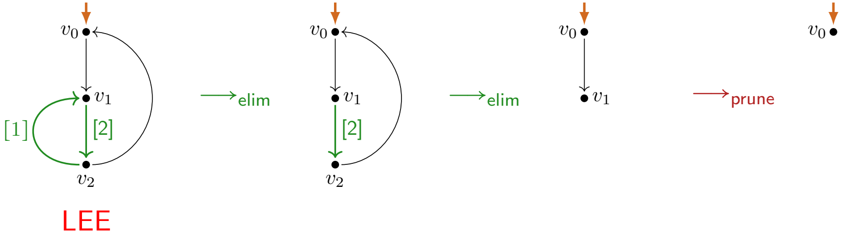

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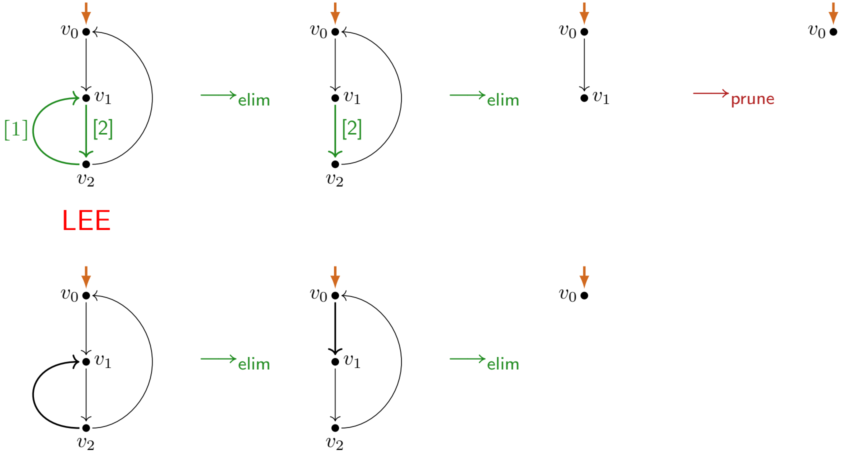

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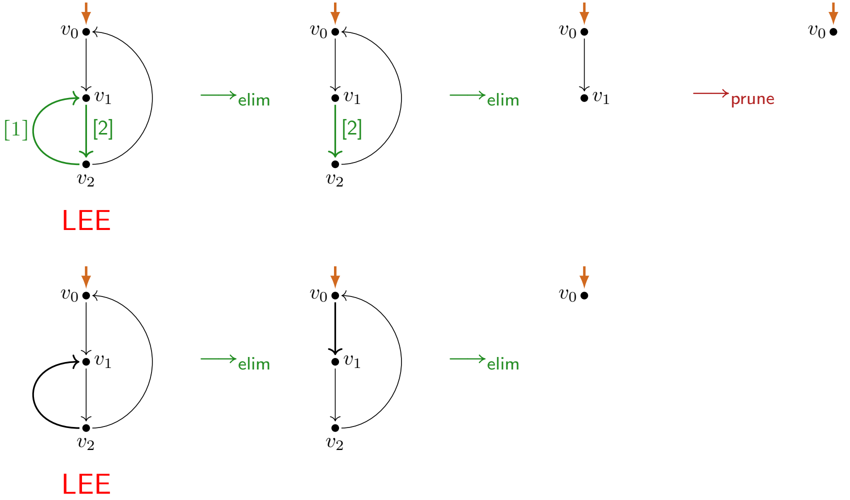

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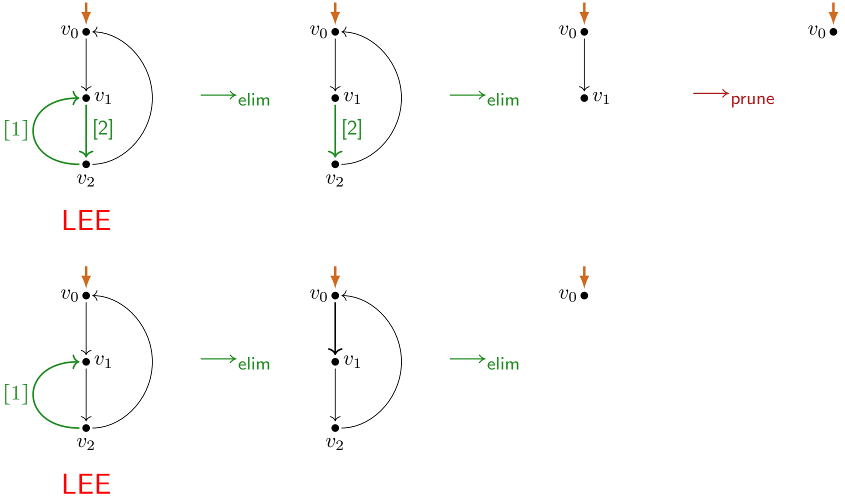

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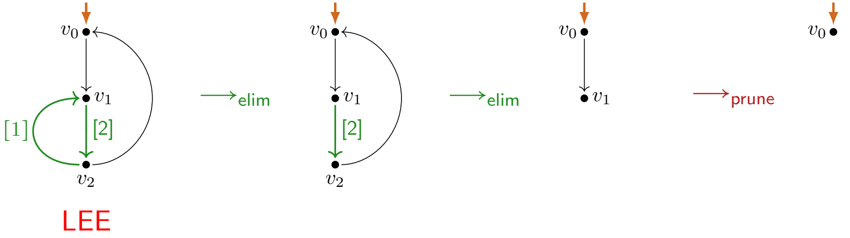

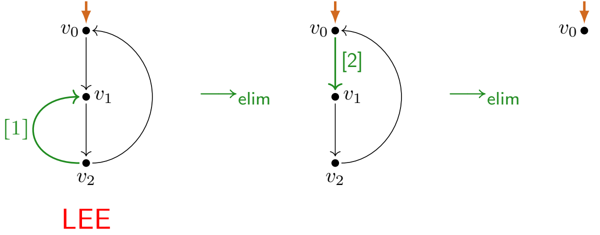

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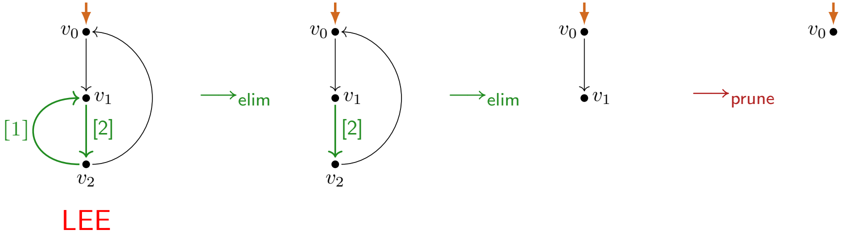
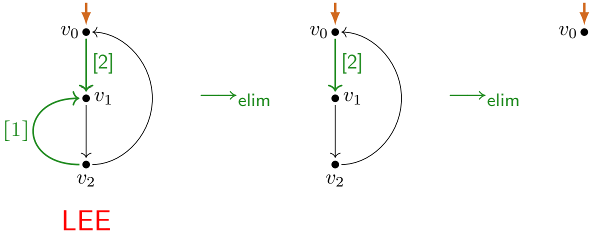
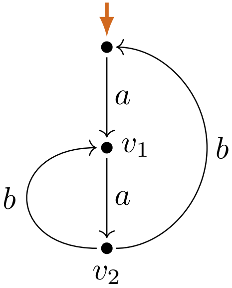


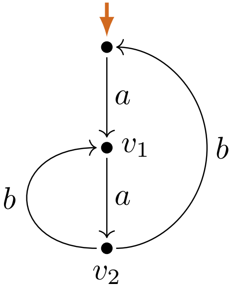


LEE-witness



LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:



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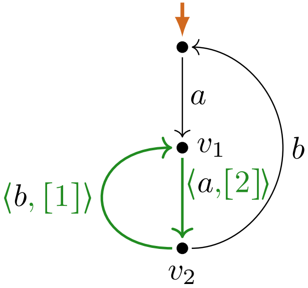
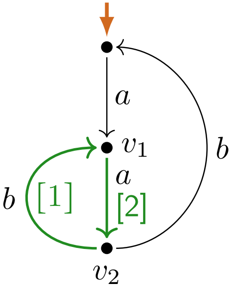


LEE-witness



loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

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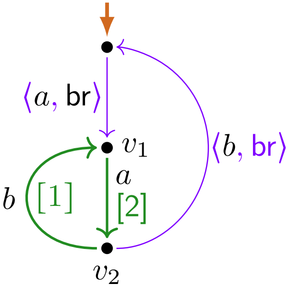
LEE-witness



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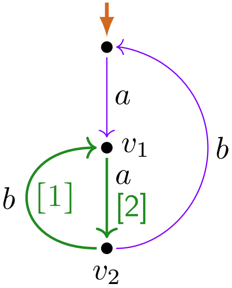

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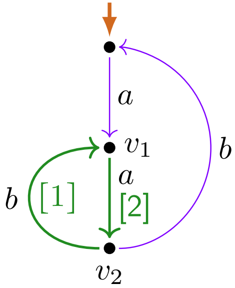
LEE-witness



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Definition

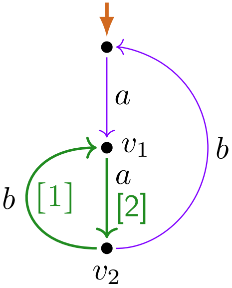
A loop–branch labeling is a LEE-witness, if:

L1.

L2.

L3.

LEE-witness



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L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\text{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\text{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

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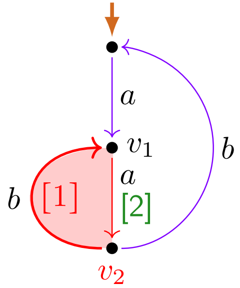

$\mathcal{L}(v_2, \to_{[1]}, \to_{\mathsf{br},[>1]})$

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L2.

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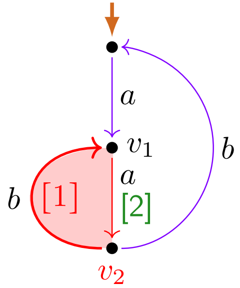

$\mathcal{L}(v_2, \to_{[1]}, \to_{\mathsf{br},[>1]})$
is loop subchart

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L2.

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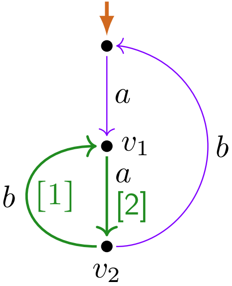
LEE-witness



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L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
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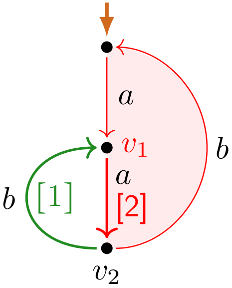
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness



$\mathcal{L}(v_1, \rightarrow_{[2]}, \rightarrow_{\mathsf{br},[>2]})$

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
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Definition

A loop–branch labeling is a LEE-witness, if:

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L2.

L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
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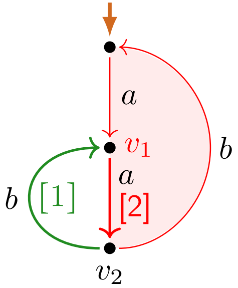
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness



$\mathcal{L}(v_1, \to_{[2]}, \to_{\text{br},[>2]})$
is loop subchart

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
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Definition

A loop–branch labeling is a LEE-witness, if:

L1. $\forall n \in \mathbb{N} \forall v \in V \left(v \to_{[n]} \Rightarrow \mathcal{L}(v, \to_{[n]}, \to_{\text{br},[>n]}) \text{ is a loop subchart} \right)$.

L2.

L3.

$\mathcal{L}(v, \to_{[n]}, \to_{\text{br},[>n]})$:= subchart induced
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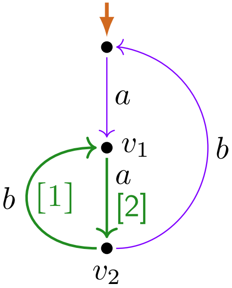
or entry steps $\to_{[m]}$ with $m > n$,
until v is reached again

LEE-witness



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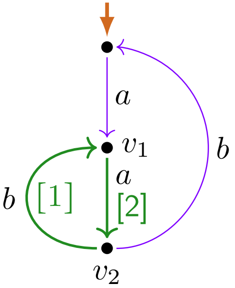

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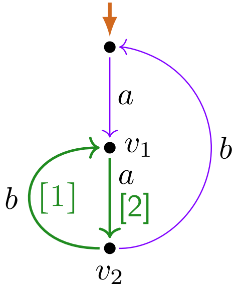

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L3. Overlapping/touching loop subcharts gen. <u>from different vertices</u> have different entry-step levels.

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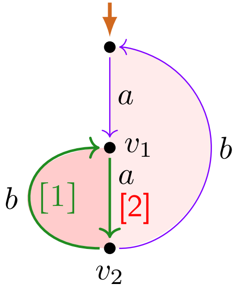

$\mathcal{L}(v_2, \to_{[1]}, \to_{\text{br},[>1]})$

$\mathcal{L}(v_1, \to_{[2]}, \to_{\text{br},[>2]})$

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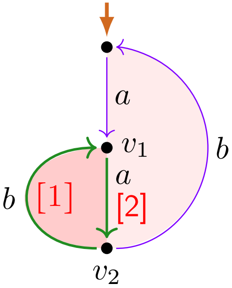

$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$

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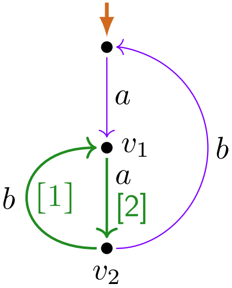

LEE-witness

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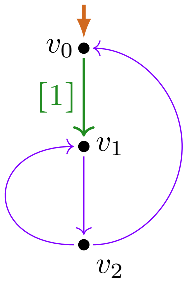
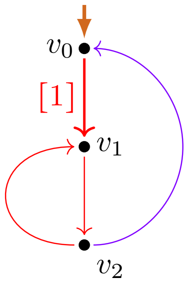
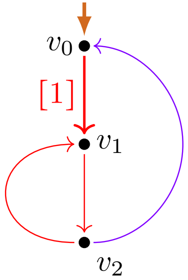


LEE-witness ?



LEE-witness ?



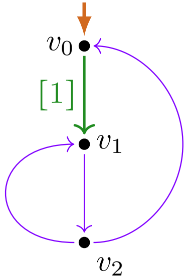
no!

(L1.) violated:

$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$

not a loop chart

LEE-witness ?



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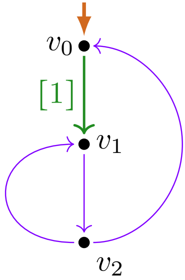
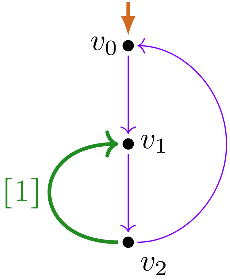
not a loop chart

LEE-witness ?





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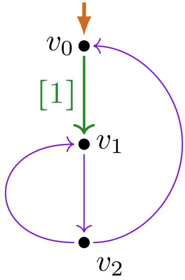

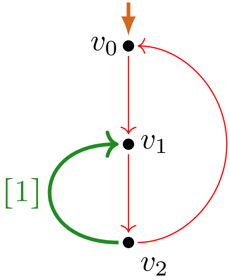

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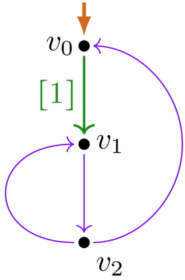

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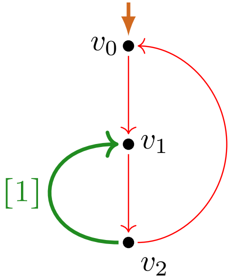


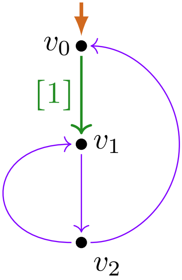
no!

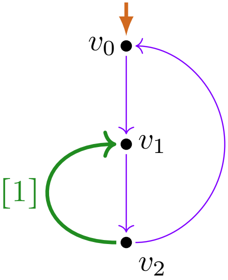
(L2.) violated:

infinite $\rightarrow_{\mathsf{br}}$ path

from start vertex

LEE-witness ?





no!

(L1.) violated:

$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$
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no!

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infinite $\rightarrow_{\mathsf{br}}$ path
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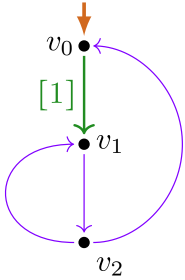
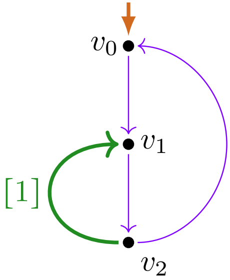
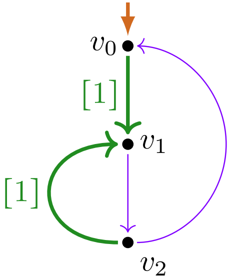
LEE-witness ?







no!

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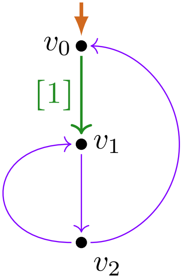

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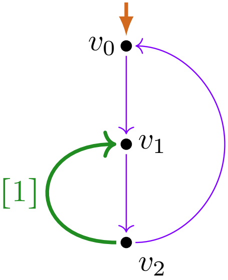

no!

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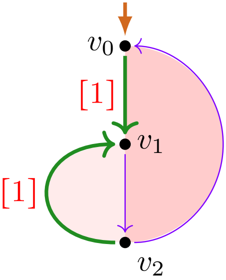

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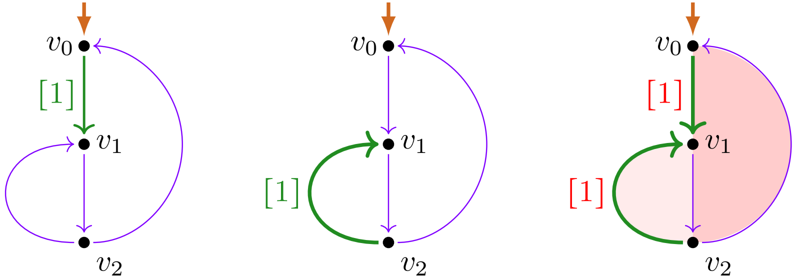
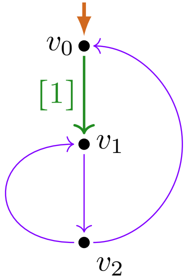


no!	no!	no!
(L1.) violated:	(L2.) violated:	(L3.) violated:
$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$ not a loop chart	infinite $\rightarrow_{\mathsf{br}}$ path from start vertex	overlapping loop charts have same level

LEE-witness ?



no!

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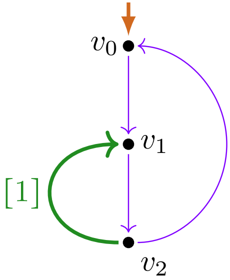

no!

(L2.) violated:

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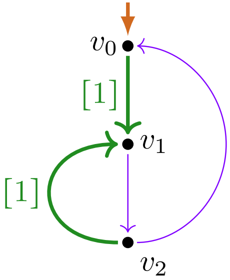

no!

(L3.) violated:

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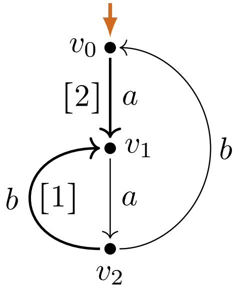
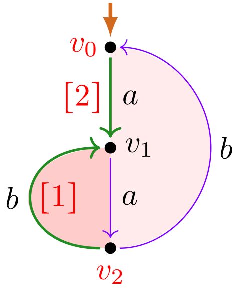


LEE-witness



$\mathcal{L}(v_2, \to_{[1]}, \to_{\text{br},[>1]})$

$\mathcal{L}(v_0, \to_{[2]}, \to_{\text{br},[>2]})$

LEE-witness

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L2. No infinite $\to_{\text{br}}$ path from start vertex.

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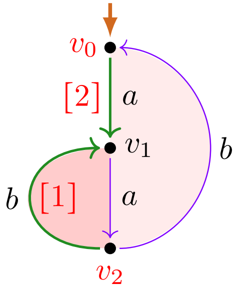
Layered LEE-witness



$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$

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Definition

A loop–branch labeling is a layered LEE-witness, if:

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l-L2. No infinite $\rightarrow_{\mathsf{br}}$ path from start vertex.

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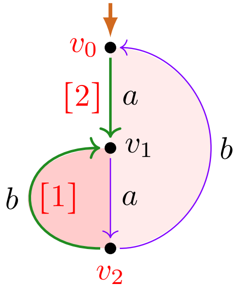
Layered LEE-witness



$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$

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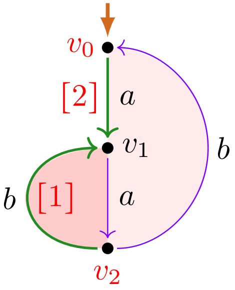
Layered LEE-witness



$\mathcal{L}(v_2, \to_{[1]}, \to_{\mathsf{br}})$
$\mathcal{L}(v_0, \to_{[2]}, \to_{\mathsf{br}})$

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l-L1. $\forall n \in \mathbb{N} \forall v \in V \left(v \to_{[n]} \Rightarrow \mathcal{L}(v, \to_{[n]}, \to_{\mathsf{br}}) \text{ is a loop subchart} \right)$.

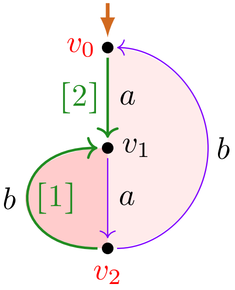
l-L2. No infinite $\to_{\mathsf{br}}$ path from start vertex.

l-L3. $\mathcal{L}(w_i, \to_{[n_i]}, \to_{\mathsf{br}})$ for $i \in \{1,2\}$ loop charts $\wedge\, w_1 \neq w_2 \,\wedge\, w_1 \in \mathcal{L}(w_2, \ldots, \ldots) \implies n_1 < n_2$.

$\mathcal{L}(v, \to_{[n]}, \to_{\mathsf{br}})$:= subchart induced
by entry steps $\to_{[n]}$ from v
followed by branch steps $\to_{\mathsf{br}}$

until v is reached again

Layered LEE-witness



$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\mathsf{br}})$
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loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

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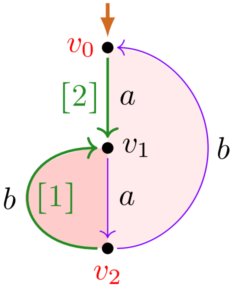
Layered LEE-witness



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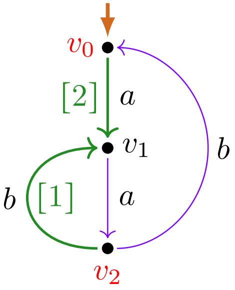

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LEE versus LEE-witness

Theorem

For every process graph G :

$$\mathsf{LEE}(G) \iff G \textit{ has a } \mathsf{LEE}\textit{-witness.}$$

LEE versus LEE-witness

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Proof.

$\Rightarrow$: record loop elimination

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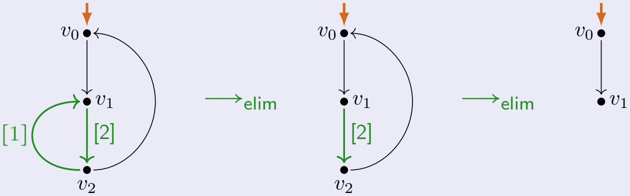
Proof.

$\Rightarrow$: record loop elimination

$\Leftarrow$: carry out loop-elimination as indicated in the LEE-witness, in *inside–out* direction, e.g.:



LEE and (layered) LEE-witness

Lemma

Every layered LEE-witness is a LEE-witness.

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Lemma

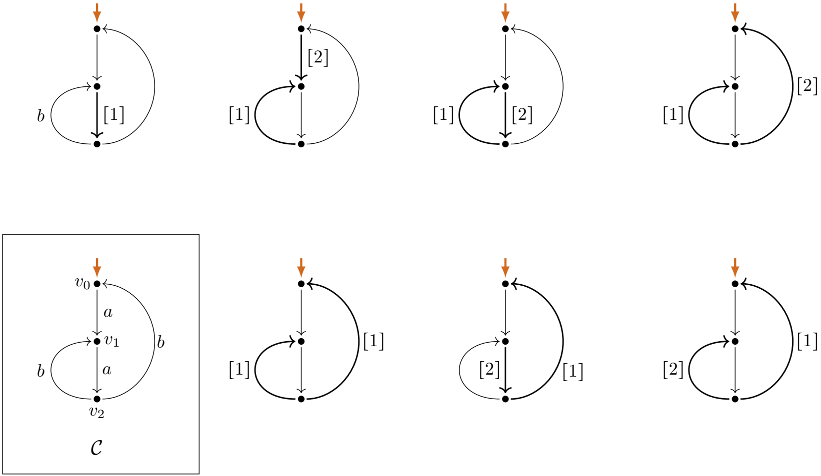
For every process graph G the following are equivalent:

(i) LEE(G).

(ii) G *has a LEE-witness.*

(iii) G *has a layered LEE-witness.*

7 LEE-witnesses



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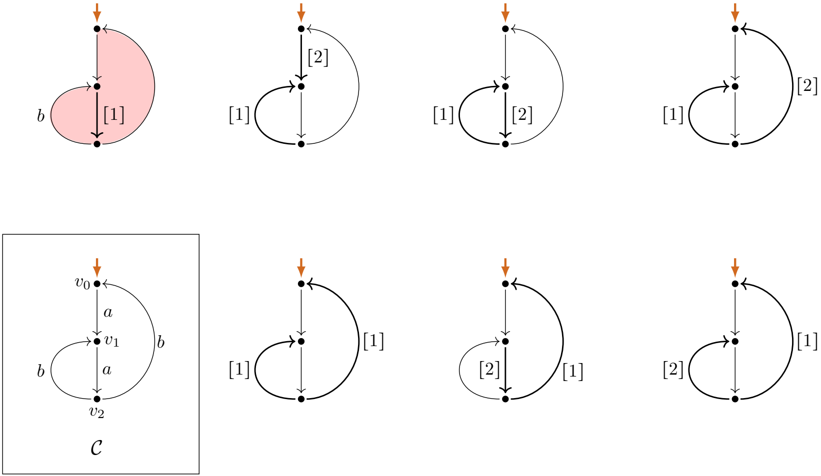

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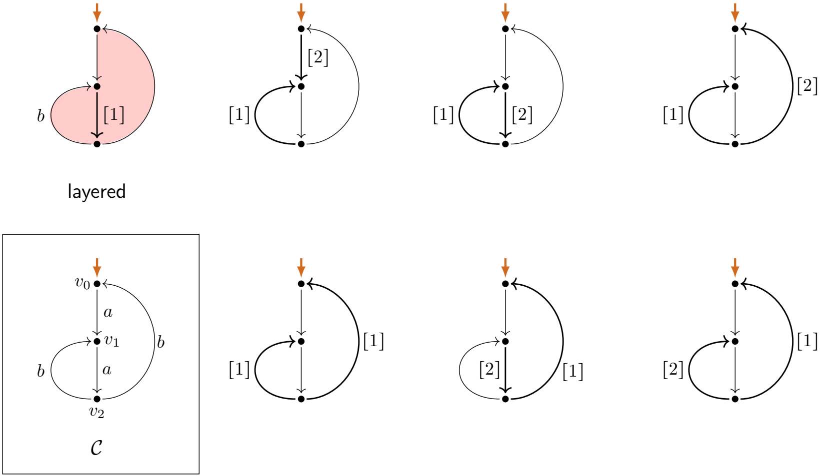

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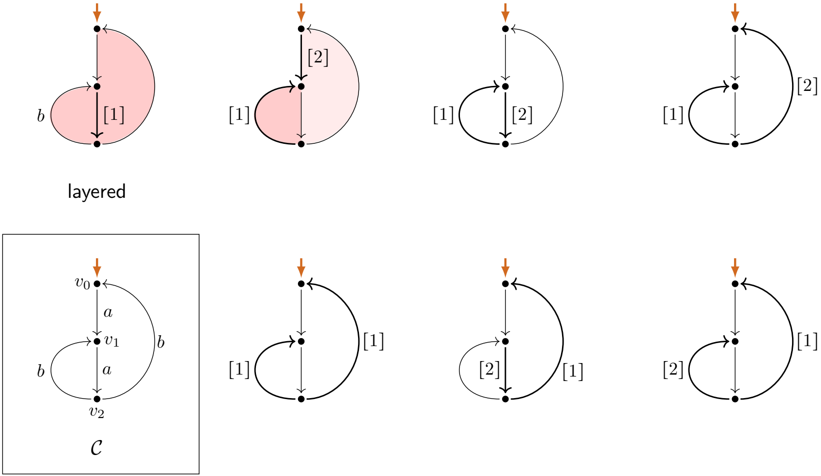

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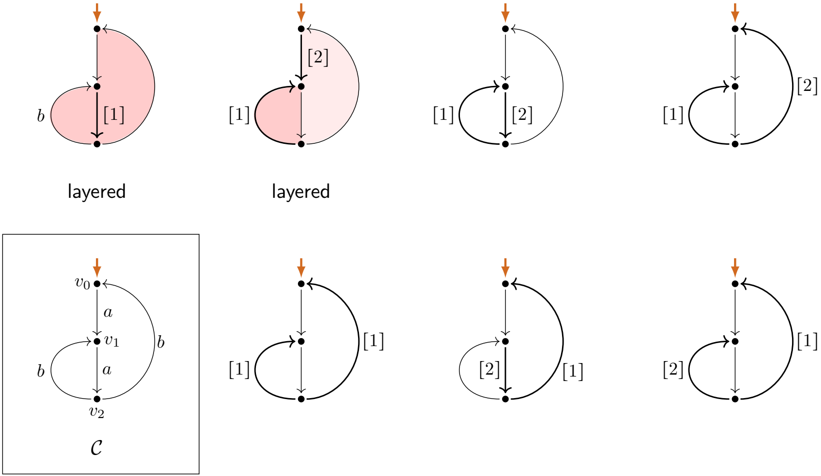

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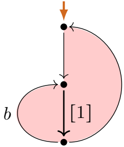

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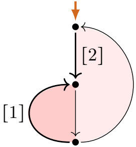

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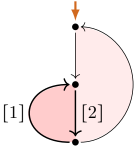

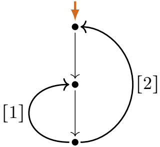

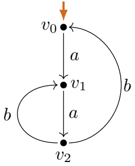

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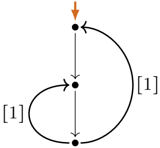

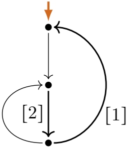

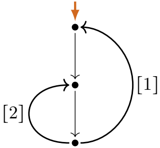

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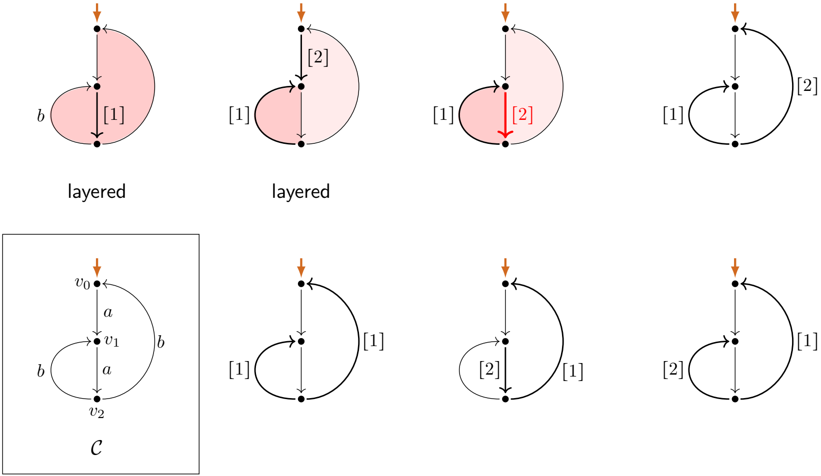

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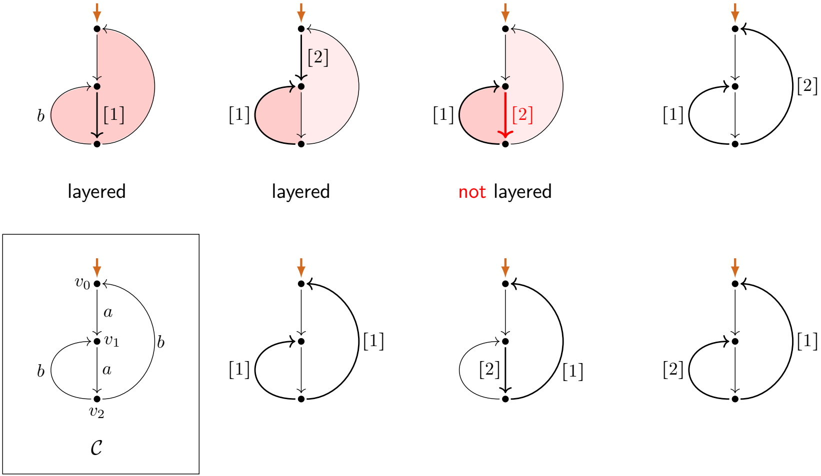

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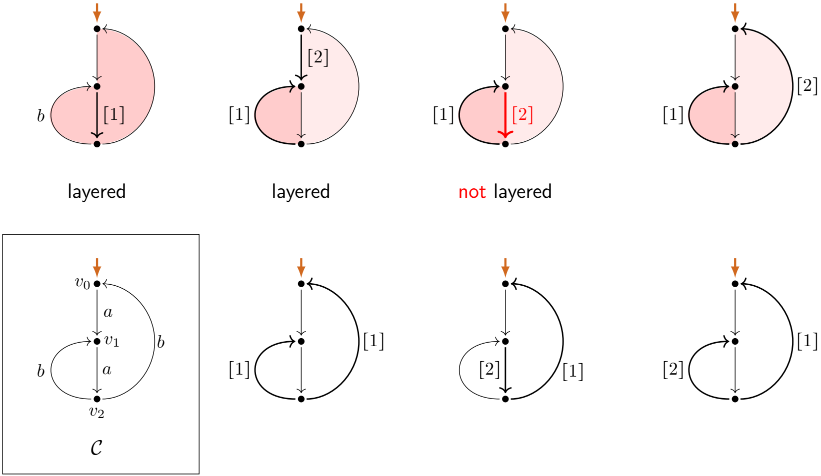

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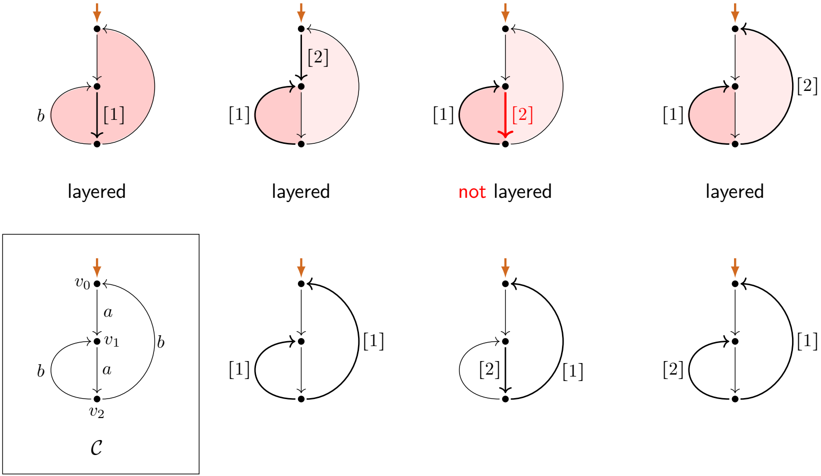

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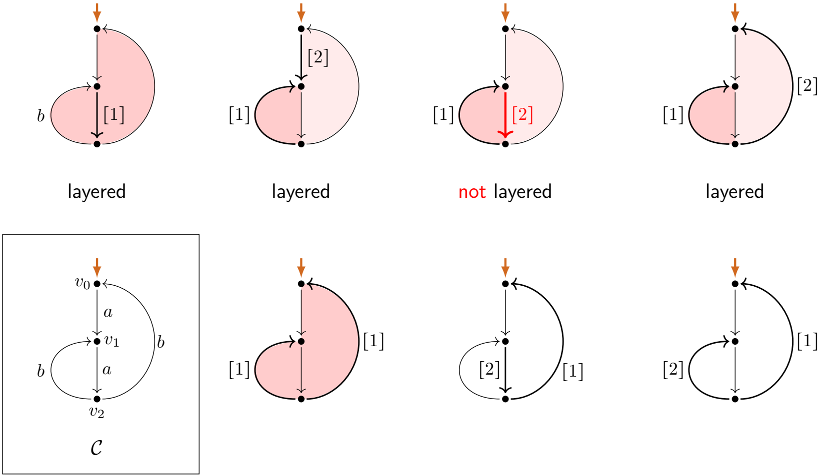

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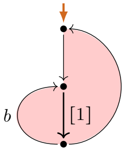

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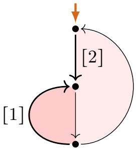

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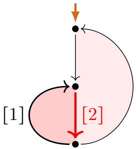

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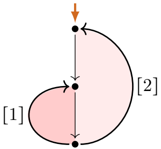

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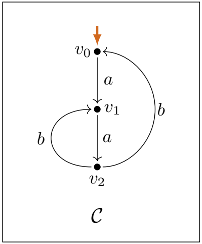

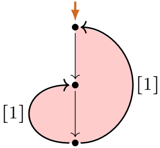

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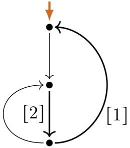

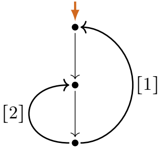

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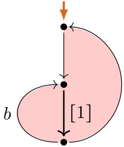

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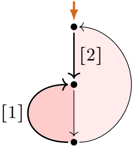

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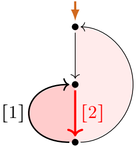

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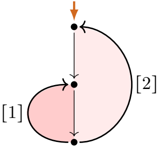

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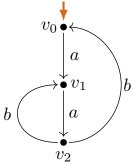

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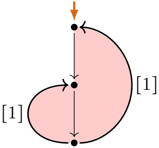

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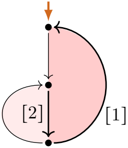

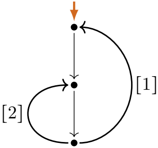

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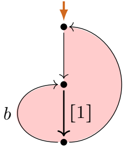

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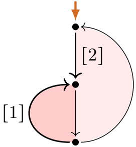

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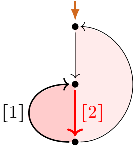

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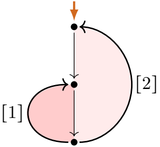

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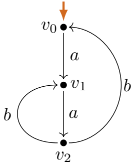

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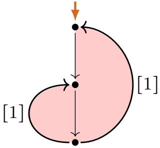

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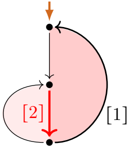

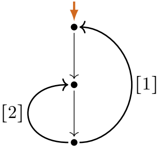

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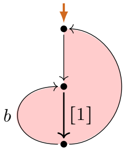

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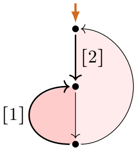

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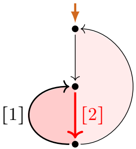

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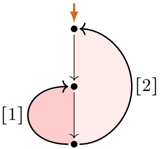

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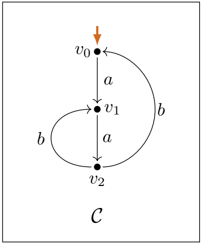

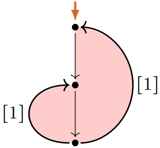

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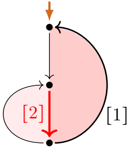

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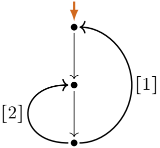

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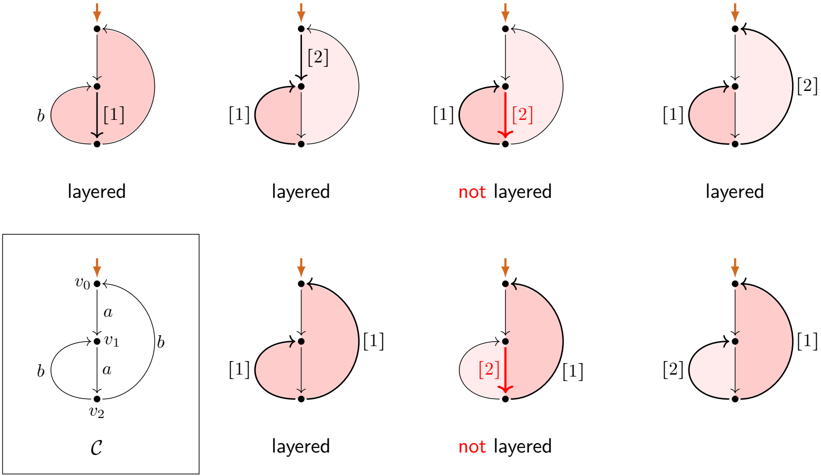

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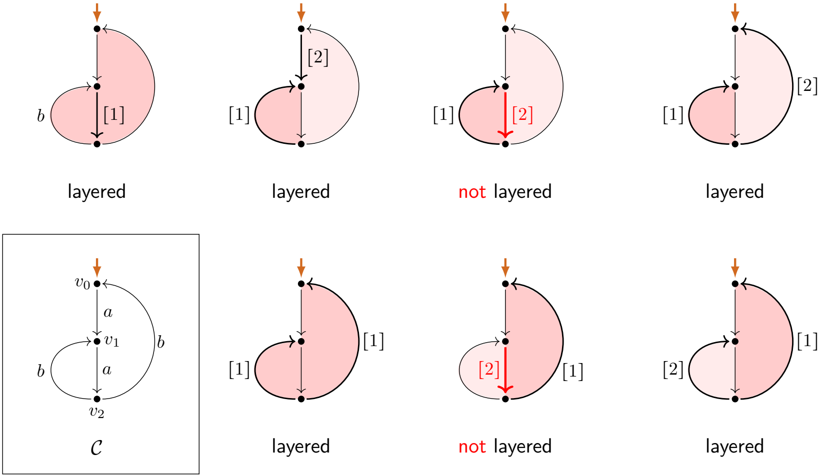

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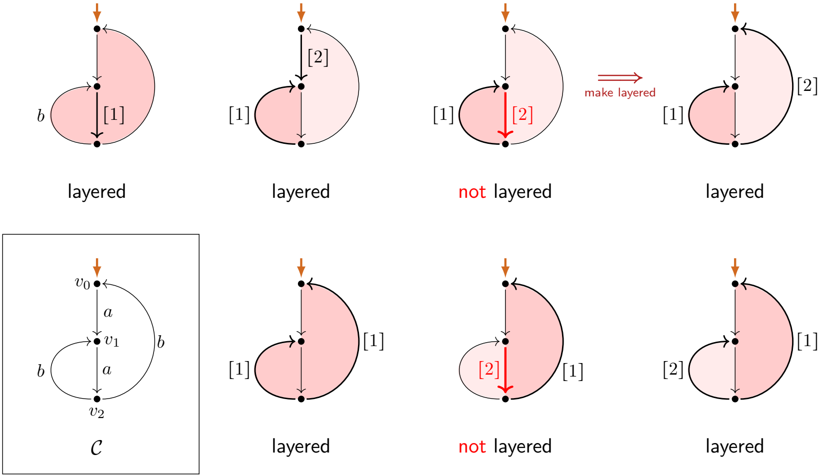

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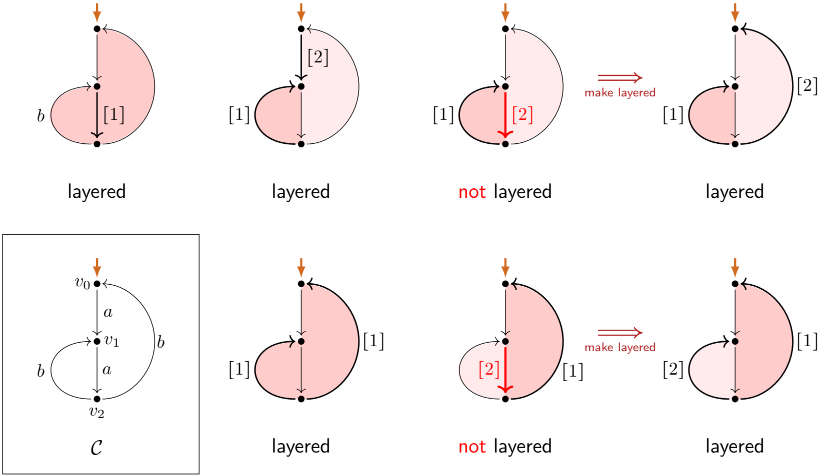

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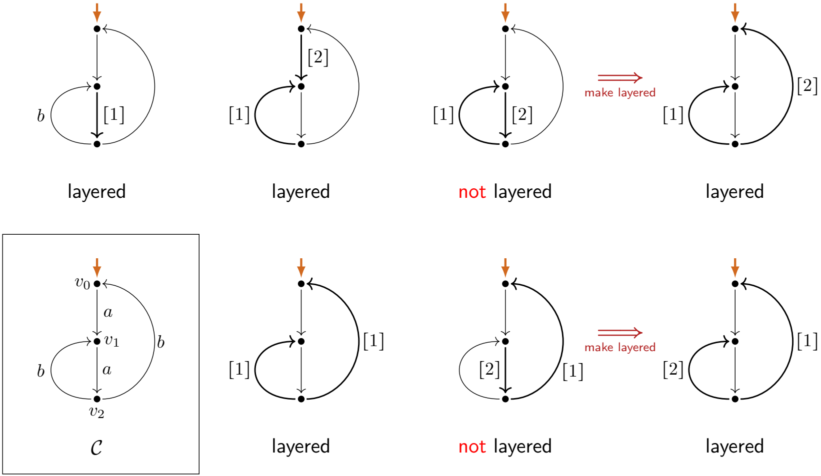


LEE under bisimulation?

LEE under bisimulation

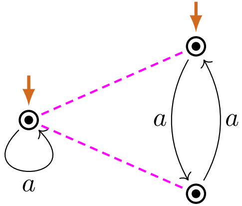
Observation

- LEE is not invariant under bisimulation.

LEE under bisimulation

Observation

- LEE is not invariant under bisimulation.

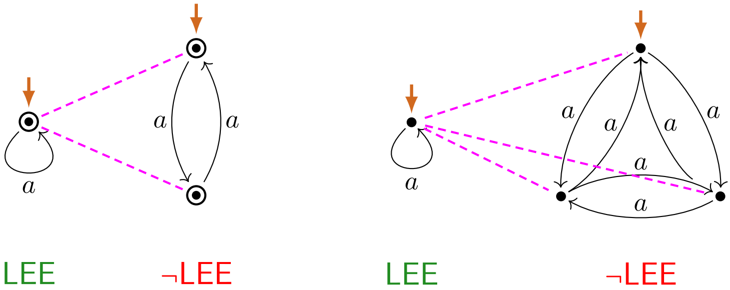


LEE ¬LEE

LEE under bisimulation

Observation

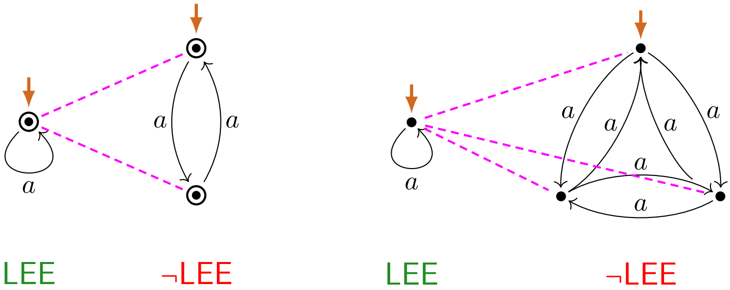
- LEE is not invariant under bisimulation.



LEE under bisimulation

Observation

- LEE is not invariant under bisimulation.
- LEE is not preserved by converse functional bisimulation.



LEE under functional bisimulation

Lemma

(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \mathrel{\underline{\rightarrow}} G_2 \implies \mathsf{LEE}(G_2)\ .$$

LEE under functional bisimulation

Lemma

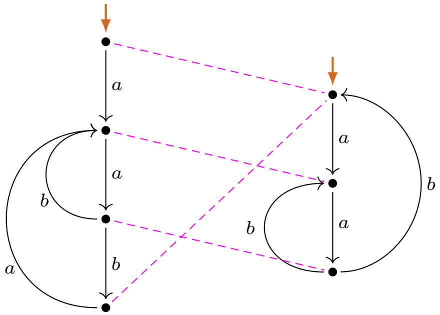
(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \mathrel{\underline{\rightarrow}} G_2 \implies \mathsf{LEE}(G_2)\ .$$

Proof (Idea).

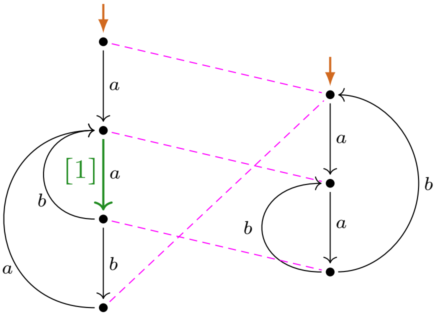
Use loop elimination in G_1 to carry out loop elimination in G_2.

Collapsing LEE-witnesses



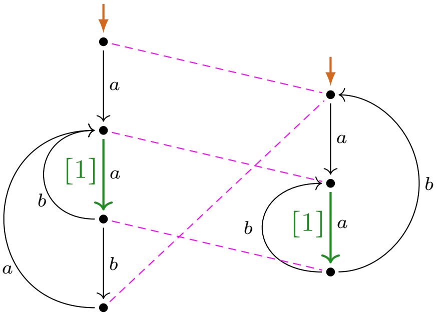
$P(a(a(b+ba))^*0)$

Collapsing LEE-witnesses



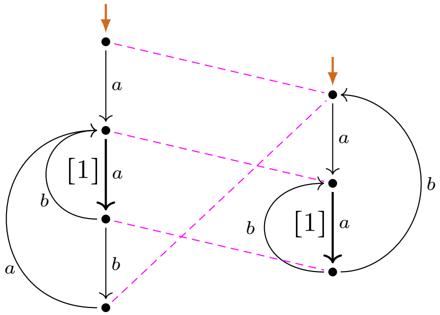
$P(a(a(b+ba))^*0)$

Collapsing LEE-witnesses



$P(a(a(b+ba))^*0)$

Collapsing LEE-witnesses



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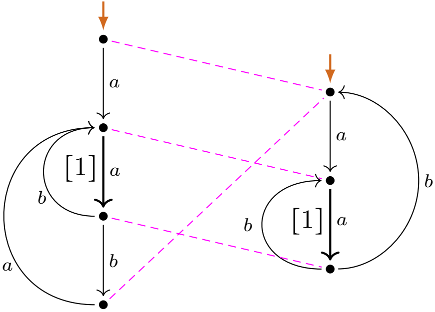
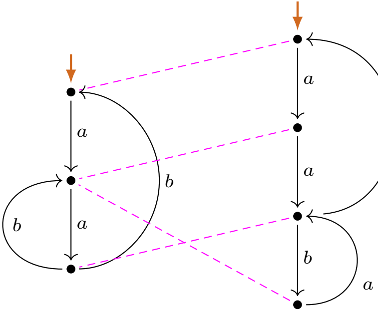
Collapsing LEE-witnesses

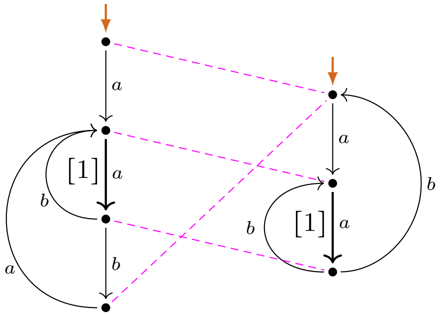


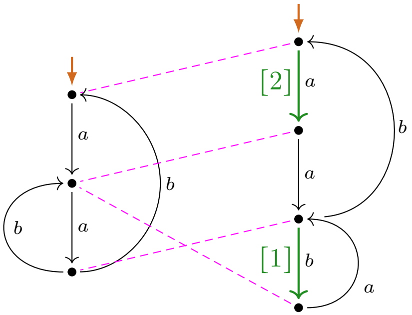


$P(a(a(b+ba))^*0)$

$P((aa(ba)^*b)^*0)$

Collapsing LEE-witnesses

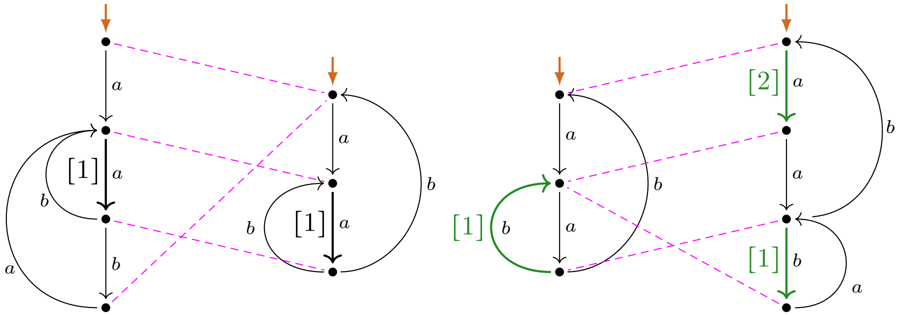




$P(a(a(b+ba))^{*}0)$

$P((aa(ba)^{*}b)^{*}0)$

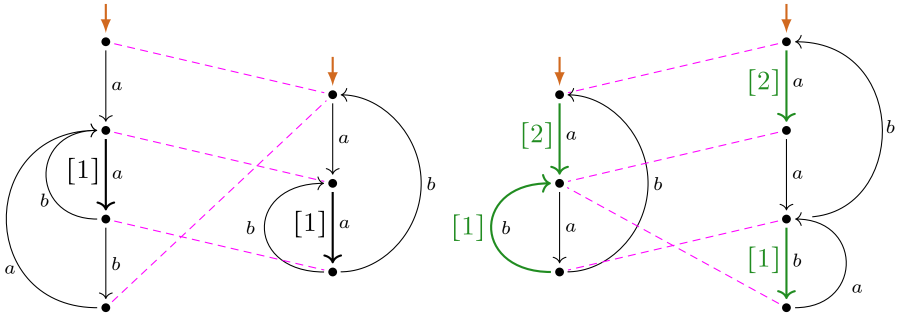
Collapsing LEE-witnesses



$P(a(a(b+ba))^*0)$

$P((aa(ba)^*b)^*0)$

Collapsing LEE-witnesses



$P(a(a(b + ba))^*0)$

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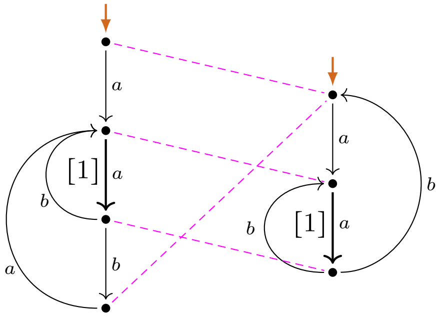
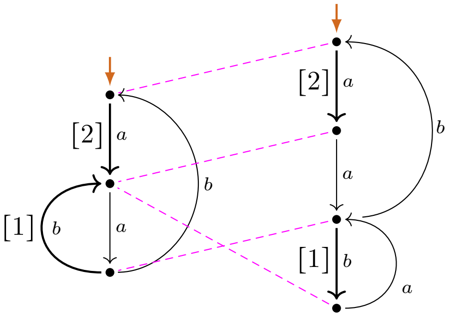
Collapsing LEE-witnesses



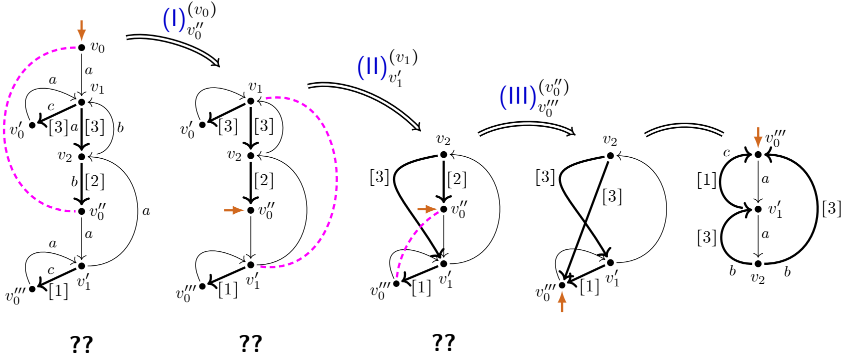
$P(a(a(b+ba))^{*}0)$



$P((aa(ba)^{*}b)^{*}0)$

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20)

(no 1-transitions!)



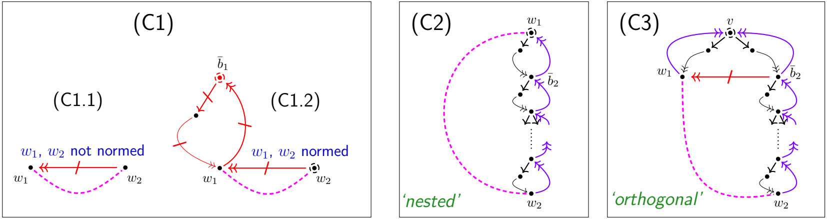
Lemma

The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

Reduced bisimilarity redundancies in LLEE-charts (no 1-trans.!)

(G/Fokkink, LICS'20)

w_1, w_2 in different scc's — w_1, w_2 in the same scc



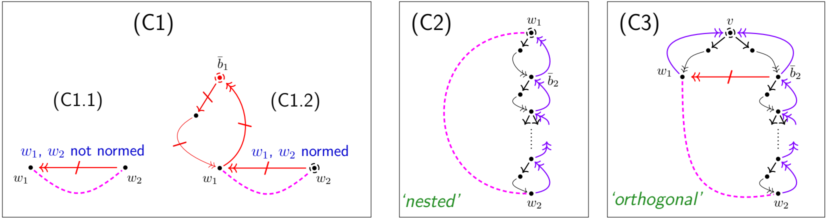
Lemma

Every not collapsed LLEE-chart contains bisimilar vertices $w_1 \neq w_2$ of kind **??**, **??**, *or* **??** *(a reduced bisimilarity redundancy $\langle w_1, w_2 \rangle$):*

Reduced bisimilarity redundancies in LLEE-charts (no 1-trans.!)

(G/Fokkink, LICS'20)

w_1, w_2 in different scc's $\qquad$ w_1, w_2 in the same scc



Lemma

Every not collapsed LLEE-chart contains bisimilar vertices $w_1 \neq w_2$ of kind **??**, **??**, *or* **??** *(a reduced bisimilarity redundancy $\langle w_1, w_2 \rangle$):*

Lemma

Every reduced bisimilarity redundancy in a LLEE-chart can be eliminated LLEE-preservingly.

LEE under functional bisimulation

Lemma

(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \mathrel{\underline{\to}} G_2 \implies \mathsf{LEE}(G_2) \; .$$

Idea of Proof for (i)

Use loop elimination in G_1 to carry out loop elimination in G_2.

LEE under functional bisimulation / bisimulation collapse

Lemma

(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \mathrel{\underline{\rightarrow}} G_2 \implies \mathsf{LEE}(G_2) .$$

(ii) LEE *is preserved from a process graph to its bisimulation collapse:*

$$\mathsf{LEE}(G) \wedge C \textit{ is bisimulation collapse of } G \implies \mathsf{LEE}(C) .$$

Idea of Proof for (i)

Use loop elimination in G_1 to carry out loop elimination in G_2.

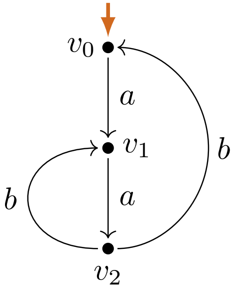
Readback

Lemma

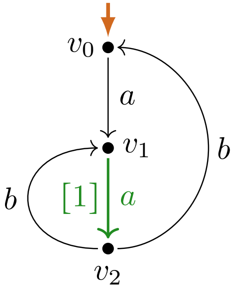
Process graphs with LEE are $P(\cdot)$-expressible:

$$\mathsf{LEE}(G) \implies \exists e \in \mathsf{Reg}(A) \left(G \leftrightarrows P(e) \right) .$$

Readback from layered LEE-witness (example)

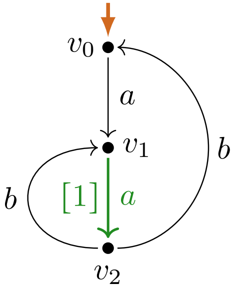


Readback from layered LEE-witness (example)



layered
LEE-witness

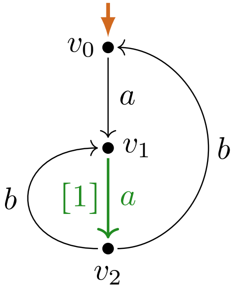
Readback from layered LEE-witness (example)



layered LEE-witness

$$
\begin{aligned}
s(v_0) &= 0^* \cdot a \cdot s(v_1) \\
&=_{\mathsf{Mil}^-} a \cdot s(v_1) \\
&=_{\mathsf{Mil}^-} a \cdot \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
s(v_1) &= \big(a \cdot s(v_2, v_1)\big)^* \cdot 0 \\
&=_{\mathsf{Mil}^-} \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
s(v_2, v_1) &= 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big) \\
&=_{\mathsf{Mil}^-} 0^* \cdot \big(b \cdot 1 + b \cdot a\big) \\
&=_{\mathsf{Mil}^-} b + b \cdot a \\
s(v_1, v_1) &= 1 \\
s(v_0, v_1) &= 0^* \cdot a \cdot s(v_1, v_1) \\
&= 0^* \cdot a \cdot 1 \\
&=_{\mathsf{Mil}^-} a
\end{aligned}
$$

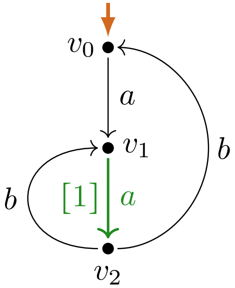
Readback from layered LEE-witness (example)



layered
LEE-witness

$$s(v_0) = \quad 0^* \cdot a \cdot s(v_1)$$

Readback from layered LEE-witness (example)

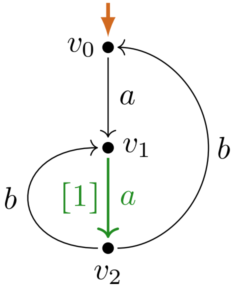


layered
LEE-witness

$$s(v_0) = \quad 0^* \cdot a \cdot s(v_1)$$

$$s(v_1) = \quad \big(a \cdot s(v_2, v_1)\big)^* \cdot 0$$

Readback from layered LEE-witness (example)



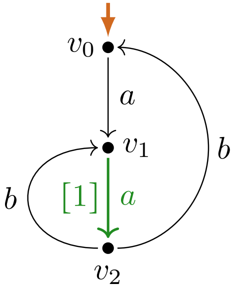
layered
LEE-witness

$$s(v_0) = \quad 0^* \cdot a \cdot s(v_1)$$

$$s(v_1) = \quad \big(a \cdot s(v_2, v_1)\big)^* \cdot 0$$

$$s(v_2, v_1) = \quad 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big)$$

Readback from layered LEE-witness (example)



layered
LEE-witness

$$s(v_0) = 0^* \cdot a \cdot s(v_1)$$

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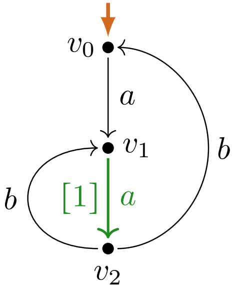
$$s(v_2, v_1) = 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big)$$

$$s(v_1, v_1) = 1$$

Readback from layered LEE-witness (example)



layered
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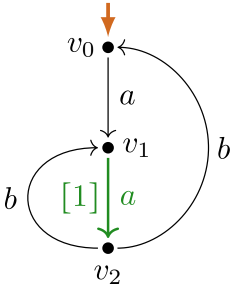
Readback from layered LEE-witness (example)



layered
LEE-witness

$$s(v_0) = 0^* \cdot a \cdot s(v_1)$$

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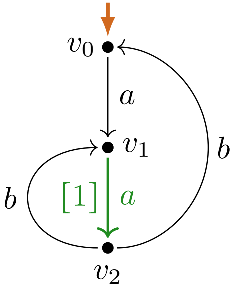
$$s(v_1, v_1) = 1$$

$$s(v_0, v_1) = 0^* \cdot a \cdot s(v_1, v_1)$$

$$= 0^* \cdot a \cdot 1$$

Readback from layered LEE-witness (example)



layered
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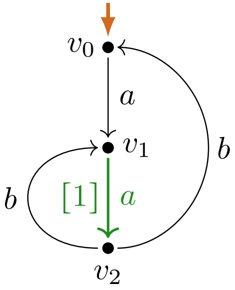
$$s(v_2, v_1) = 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big)$$

$$\begin{aligned} s(v_1, v_1) &= 1 \\ s(v_0, v_1) &= 0^* \cdot a \cdot s(v_1, v_1) \\ &= 0^* \cdot a \cdot 1 \\ &=_{\mathsf{Mil}^-} a \end{aligned}$$

Readback from layered LEE-witness (example)



layered
LEE-witness

$$
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s(v_1) &= \quad \big(a \cdot s(v_2, v_1)\big)^* \cdot 0 \\
s(v_2, v_1) &= \quad 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big) \\
&=_{\mathsf{Mil}^-} 0^* \cdot \big(b \cdot 1 + b \cdot a\big) \\
s(v_1, v_1) &= \quad 1 \\
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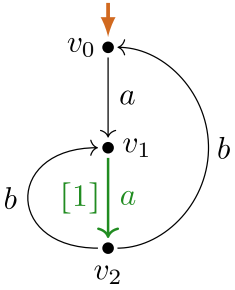
\end{aligned}
$$

Readback from layered LEE-witness (example)



layered
LEE-witness

$$s(v_0) = 0^* \cdot a \cdot s(v_1)$$

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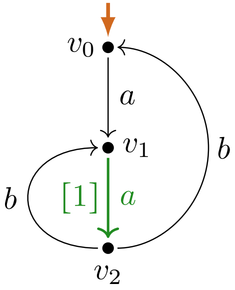
\end{aligned}$$

Readback from layered LEE-witness (example)



layered
LEE-witness

$$s(v_0) = 0^* \cdot a \cdot s(v_1)$$

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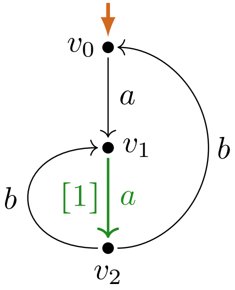
Readback from layered LEE-witness (example)



layered
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$$

$$
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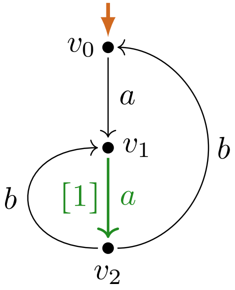
Readback from layered LEE-witness (example)



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&=_{\mathsf{Mil}^-} \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
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\end{aligned}
$$

1-return-less regular expressions

Lemma

Process graphs with LEE are $P(\cdot)$-expressible:

$$\mathsf{LEE}(G) \implies \exists e \in \mathrm{Reg}(A)\,\big(\, G \,\underline{\leftrightarrow}\, P(e) \,\big)\;.$$

1-return-less regular expressions

Lemma

Process graphs with LEE are $[\![\cdot]\!]_{P}^{\cancel{\mathbf{1r}}\backslash\star}$-expressible:

$$\mathsf{LEE}(G) \implies \exists\, e \in \mathit{Reg}^{\cancel{\mathbf{1r}}\backslash\star}(A)\,\big(\, G \underline{\leftrightarrow} P(e) \,\big)\,.$$

1-return-less regular expressions

Lemma

Process graphs with LEE are $[\![\cdot]\!]_{P}^{\cancel{1r}\backslash\star}$-expressible:

$$\mathsf{LEE}(G) \implies \exists e \in \mathit{Reg}^{\cancel{1r}\backslash\star}(A)\,\big(\, G \mathrel{\underline{\leftrightarrow}} P(e) \,\big)\;.$$

Definition (Corradini, De Nicola, Labella (here intuitive version))

A regular expression e is 1-return-less(-under-$\star$) ($e \in \mathit{Reg}^{\cancel{1r}\backslash\star}(A)$) if:

1-return-less regular expressions

Lemma

Process graphs with LEE are $[\![\cdot]\!]_{P}^{\cancel{1r}\backslash\star}$-expressible:

$$\mathsf{LEE}(G) \implies \exists e \in \mathrm{Reg}^{\cancel{1r}\backslash\star}(A)\,\big(\,G \;\underline{\leftrightarrow}\; P(e)\,\big)\;.$$

Definition (Corradini, De Nicola, Labella (here intuitive version))

A regular expression e is 1-return-less(-under-$\star$) ($e \in \mathrm{Reg}^{\cancel{1r}\backslash\star}(A)$) if:

- for no iteration subexpression f^* of e does $P(f)$ proceed to a process p such that:
  - p has the option to immediately terminate, and
  - p has the option to do a proper step, and terminate later.

Non-/Examples of 1-return-less regular expressions

- $(a \cdot (1 + b))^*$

1-return-less regular expressions

Lemma

Process graphs with LEE are $[\![\cdot]\!]_{P}^{\cancel{1r}\backslash\star}$-expressible:

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Definition (Corradini, De Nicola, Labella (here intuitive version))

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Non-/Examples of 1-return-less regular expressions

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1-return-less regular expressions

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Characterization of expressibility$^{\cancel{\mathbf{1r}}\backslash\star}$

Theorem

For every process graph G with bisimulation collapse C the following are equivalent:

(i) G *is* $[\![\cdot]\!]_P^{\cancel{\mathbf{1r}}\backslash\star}$*-expressible.*

(ii) $\mathsf{LEE}(C)$.

(iii) C *has a* LEE*-witness.*

(iv) C *has a layered* LEE*-witness.*

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Milners characterization question:

Q1. Which structural property of finite process graphs characterizes $P(\cdot)$-expressibility?

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Milners characterization question restricted:

$\mathrm{Q1}'$. Which structural property of finite process graphs characterizes $[\![\cdot]\!]_P^{\cancel{\mathbf{1r}}\backslash\star}$-expressibility?

Characterization of expressibility$^{\cancel{\mathbf{1r}}\backslash\star}$

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Milners characterization question restricted, and adapted:

Q1′′. Which structural property of collapsed finite process graphs characterizes $[\![\cdot]\!]_P^{\cancel{\mathbf{1r}}\backslash\star}$-expressibility?

Characterization of expressibility$^{\cancel{\mathbf{1r}}\backslash\star}$

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Answering Milners characterization question restricted, and adapted:

Q1′′. Which structural property of collapsed finite process graphs characterizes $[\![\cdot]\!]_{P}^{\cancel{\mathbf{1r}}\backslash\star}$-expressibility?

- The loop-existence and elimination property LEE.

Characterization of expressibility$^{\cancel{\mathbf{1r}}\backslash\star}$

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For every process graph G with bisimulation collapse C the following are equivalent:

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Answering Milners characterization question restricted, and adapted:

Q1″. Which structural property of collapsed finite process graphs characterizes $[\![\cdot]\!]_P^{\cancel{\mathbf{1r}}\backslash\star}$-expressibility?

- The loop-existence and elimination property LEE.

Also yields: efficient decision method of $[\![\cdot]\!]_P^{\cancel{\mathbf{1r}}\backslash\star}$-expressibility?

Structure constrained finite process graphs

graphs with LEE / a (layered) LEE-witness

Benefits of the class of process graphs with LEE:

- is closed under $\Rightarrow$
- forth-/back-correspondence with 1-return-less regular expressions

Structure constrained finite process graphs

graphs with LEE / a (layered) LEE-witness

$\subsetneqq$ graphs whose collapse satisfies LEE

$=$ graphs that are $[\![\cdot]\!]_{P}^{\cancel{1r}\backslash\star}$-expressible

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$\llbracket\cdot\rrbracket_{P}^{\cancel{1r}\backslash\star}$-expressible graphs

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Structure constrained finite process graphs

$$
\begin{array}{ll}
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Benefits of the class of process graphs with LEE:

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Structure constrained finite process graphs

loop–exit palm trees	$\subsetneqq$	$[\![\cdot]\!]_P^{\cancel{1r}\backslash\star}$-expressible graphs
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Structure constrained finite process graphs

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Application to Milner's questions yields partial results:

Q1: characterization/efficient decision of $[\![\cdot]\!]_{P}^{\cancel{1r}\backslash\star}$-expressibility

Q2: alternative compl. proof of Mil on 1-return-less expressions (C/DN/L)

Comparison results: structure-constrained graphs

λ-calculus with letrec under $=_{\boldsymbol{\lambda}^{\infty}}$

Not available: graph interpretation that is studied under $\underline{\leftrightarrow}$

Regular expressions under $\underline{\leftrightarrow}_P$

Given: graph interpretation $P(\cdot)$, studied under bisimulation $\underline{\leftrightarrow}$

- not closed under $\underline{\to}$, and $\underline{\leftrightarrow}$, incomplete under $\underline{\leftrightarrow}$

Comparison results: structure-constrained graphs

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Not available: graph interpretation that is studied under $\underline{\leftrightarrow}$

Defined: int's $[\![\cdot]\!]_{\mathcal{H}}/[\![\cdot]\!]_{\mathcal{T}}$ as higher-order/first-order λ-term graphs

- closed under $\underline{\to}$ (hence under collapse)
- back-/forth correspondence with λ-calculus with letrec
  - efficient translation and readback
  - translation is inverse of readback

Regular expressions under $\underline{\leftrightarrow}_P$

Given: graph interpretation $P(\cdot)$, studied under bisimulation $\underline{\leftrightarrow}$

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Regular expressions under $\leftrightarrow_P$

Given: graph interpretation $P(\cdot)$, studied under bisimulation $\leftrightarrow$

- not closed under $\Rightarrow$, and $\leftrightarrow$, incomplete under $\leftrightarrow$

Defined: class of process graphs with LEE / (layered) LEE-witness

- closed under $\Rightarrow$ (hence under collapse)
- back-/forth correspondence with 1-return-less expr's
- contains the collapse of a process graph G
  $\iff$ G is $[\![\cdot]\!]_P^{\cancel{1r}\backslash\star}$-expressible modulo $\leftrightarrow$