

Lecture 2: Graph width notions, dynamical programming

An Introduction to Parameterized Complexity

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Ph.D. Program, Advanced Period
Gran Sasso Science Institute
L'Aquila, Italy

Tuesday, June 11, 2024

Course overview

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results motivation for FPT kernelization, Crown Lemma, Sunflower Lemma		Algorithmic Meta-Theorems 1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width		
	GDA		GDA	GDA
<i>Algorithmic Techniques</i>		<i>Formal-Method & Algorithmic Techniques</i>		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths			FPT-Intractability Classes & Hierarchies motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies
		GDA	GDA	

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- ▶ dynamical programming on trees, example:
 - ▶ WEIGHTED-INDEPENDENT-SET (and VERTEX-COVER)
- ▶ path-width
 - ▶ example: fpt-algorithm for bounded path-width
- ▶ tree-width
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 - ▶ clique-width
 - ▶ using f -width to define:
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 - ▶ rank-width

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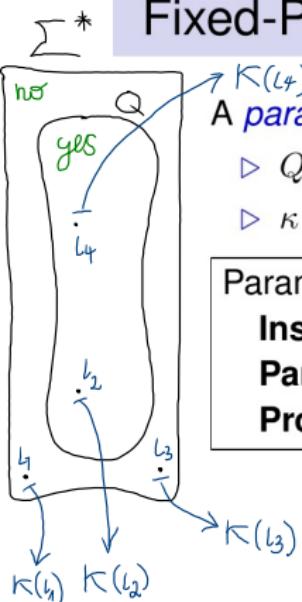
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Fixed-Parameter tractable

A *parameterized problem* is a triple $\langle Q, \Sigma, \kappa \rangle$ (short: $\langle Q, \kappa \rangle$) where:

- ▷ $Q \subseteq \Sigma^*$ is the set of *(classical) problem instances*,
- ▷ $\kappa : \Sigma^* \rightarrow \mathbb{N}$ is a (general) function, *the parameterization*.

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Parameterized problem $\langle Q, \Sigma, \kappa \rangle$

Instance: $x \in \Sigma^*$.

Parameter: $\kappa(x)$.

Problem: Is $x \in Q$?

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Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *fixed-parameter tractable* (is in FPT) if:

$\exists f : \mathbb{N} \rightarrow \mathbb{N}$ computable $\exists p \in \mathbb{N}[X]$ polynomial

$\exists \mathbb{A}$ algorithm, takes inputs in Σ^*

$\forall x \in \Sigma^* \left[\mathbb{A} \text{ decides whether } x \in Q \text{ holds in time } \leq f(\kappa(x)) \cdot p(|x|) \right]$

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†) Assumptions for a robust fpt-theory

$\kappa(x)$ is *polynomially computable*, or itself *fpt-computable*: for all $x \in \Sigma^*$ in time $\leq g(\kappa(x)) \cdot q(|x|)$ for g computable, $q \in \mathbb{N}[X]$.

Yesterday:

p-VERTEX-COVER \in FPT

kernel sizes $O(k^2), O(k)$



brute-force-algorithm
 $O(n^{k+2})$

FPT-alg's : $O(k^{2k+4} \cdot n^2)$ $O(k^{k+2} \cdot n^2)$

Comparing parameterizations

Definition (computably bounded below)

Let $\kappa_1, \kappa_2 : \Sigma^* \rightarrow \mathbb{N}$ parameterizations.

- ▶ $\kappa_1 \geq \kappa_2 : \iff \exists g : \mathbb{N} \rightarrow \mathbb{N} \text{ computable } \forall x \in \Sigma^* [g(\kappa_1(x)) \geq \kappa_2(x)]$.
- ▶ $\kappa_1 \approx \kappa_2 : \iff \kappa_1 \geq \kappa_2 \wedge \kappa_2 \geq \kappa_1$.
- ▶ $\kappa_1 > \kappa_2 : \iff \kappa_1 \geq \kappa_2 \wedge \neg(\kappa_2 \geq \kappa_1)$.

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$$\begin{aligned} & \text{A decides } x \in Q \text{ in } \leq f(\kappa_2(x)).\top(|x|) \text{ steps} \\ \Rightarrow & \text{A decides } x \in Q \text{ in } \leq (f \circ g)(\kappa_1(x)).\top(|x|) \text{ steps} \end{aligned}$$

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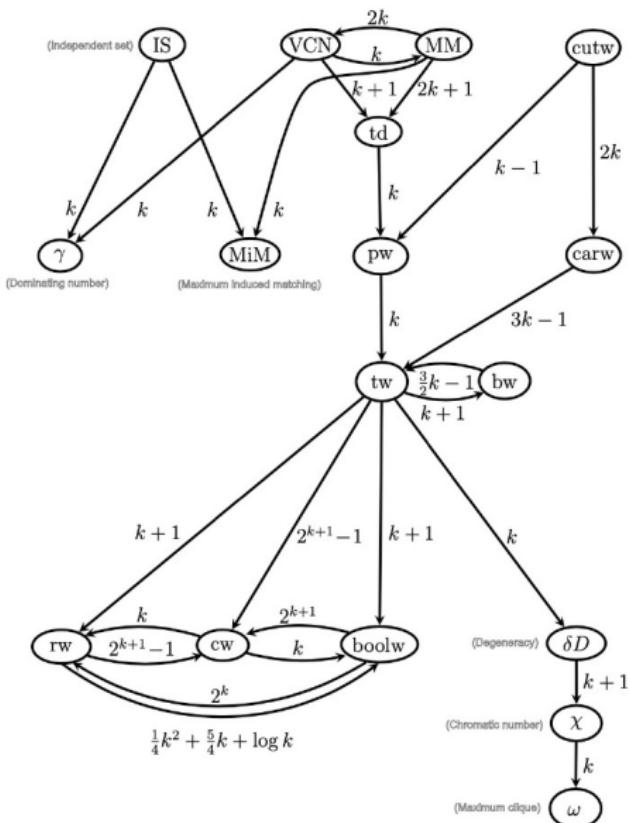
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Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \geq wd_2 : \Leftrightarrow wd_1 \xrightarrow{g} wd_2$$

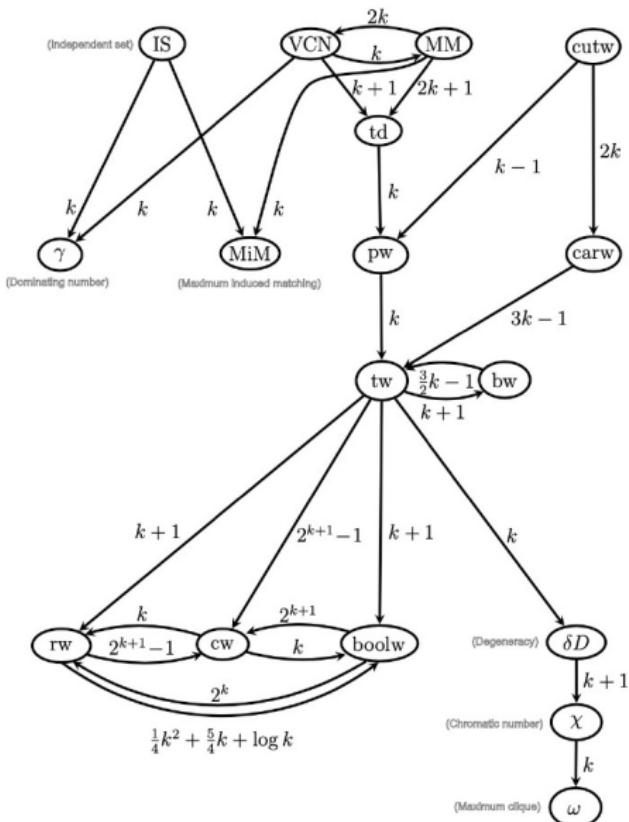


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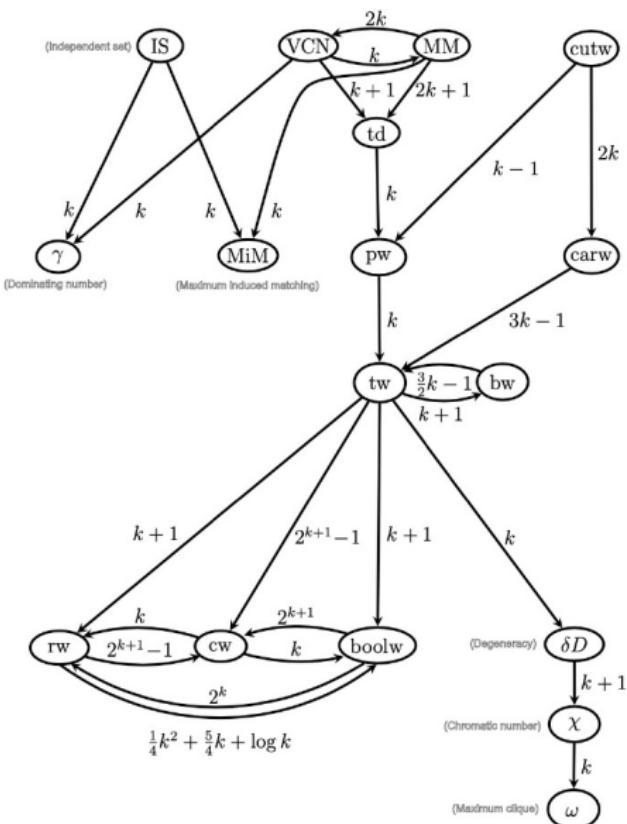


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- ▶ (\notin FPT)-results transfer downwards
(and along \xrightarrow{g})



You Always Walk Alone (with your children)

Attività motoria con i figli:

'la possibilità di uscire con i figli minori è consentita a un solo genitore per camminare purché questo avvenga in prossimità della propria abitazione'

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PHYSICAL-DISTANCE-WALKING

Instance: Graph $\mathcal{G} = \langle V, E \rangle$ with V people who want to go for a walk in the next hour in a radius of 200m of their home, and edges in E between them if they live closer than 400m of each other. A number $\ell \in \mathbb{N}$.

Problem:

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corresponds to: INDEPENDENT-SET

Weighted Independent Set, and Vertex Cover

Let $\mathcal{G} = \langle V, E \rangle$ a graph. For all $S \subseteq V$:

S is independent set in \mathcal{G} : $\iff \forall e = \{u, v\} \in E (\neg(u \in S \wedge v \in S))$
 $\iff \forall e = \{u, v\} \in E (u \notin S \vee v \notin S)$

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Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $w : V \rightarrow \mathbb{R}_0^+$.

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$S \subseteq V$ is minimal vertex cover $\iff V \setminus S$ is maximal independent set
Hence: solution of WEIGHTED-INDEPENDENT-SET
 \implies solution of VERTEX-COVER.

Weighted Ind. Set / Vertex Cover, width-parameterized

p^* -WEIGHTED-INDEPENDENT-SET

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Parameter: path-width / tree-width k .

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Dynamical programming on trees (example)

WEIGHTED-INDEPENDENT-SET

Instance: A tree $\mathcal{T} = \langle T, F \rangle$, and a weight function $w : T \rightarrow \mathbb{R}_0^+$.

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Dynamical programming on trees (example)

↳ undirected, connected graphs without cycles

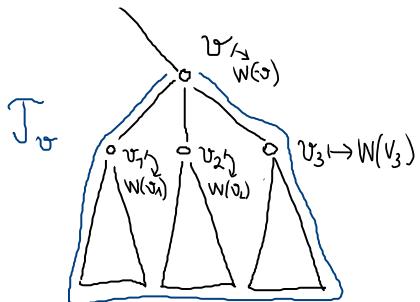
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rooted
directed tree

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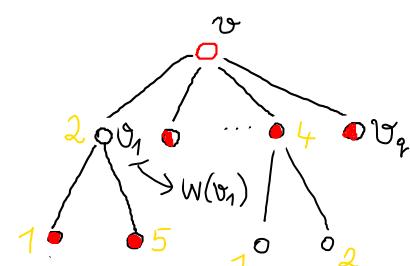
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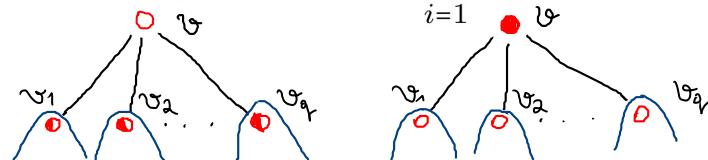
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$$B[v] = \sum_{i=1}^q A[v_i], \quad A[v] = \max\left\{B[v], w(v) + \sum_{i=1}^q B[v_i]\right\}.$$

Solution: value of $A[r]$, can be computed bottom-up in linear time.

Dynamical programming on trees (example)

WEIGHTED-INDEPENDENT-SET

Instance: A tree $\mathcal{T} = \langle T, F \rangle$, and a weight function $w : T \rightarrow \mathbb{R}_0^+$.

Problem: What is the max. weight of an independent set of \mathcal{T} ?

Theorem

On *trees* with n nodes,

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VERTEX-COVER

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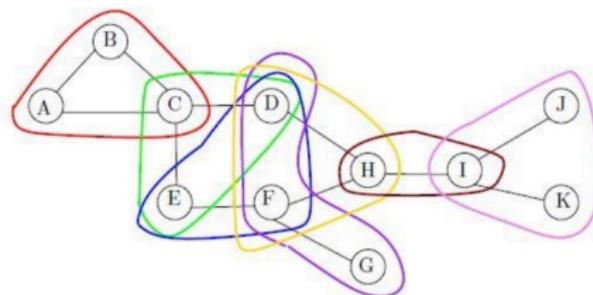
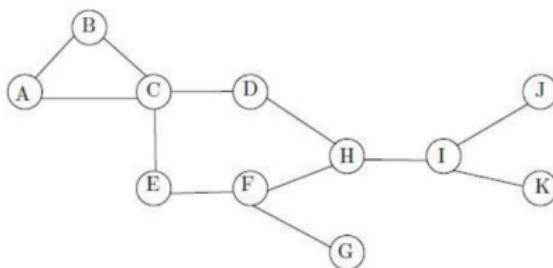
Problem: Does \mathcal{T} have a vertex cover of size at most ℓ ?

Corollary

On **trees** with n nodes,

VERTEX-COVER $\in \text{DTIME}(O(n))$.

Path-decomposition (example)



Path decompositions, and path-width

Definition (Robertson–Seymour, 1983)

A *path decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a sequence $\{B_1, B_2, \dots, B_r\}$ of bags $B_i \subseteq V$ such that:

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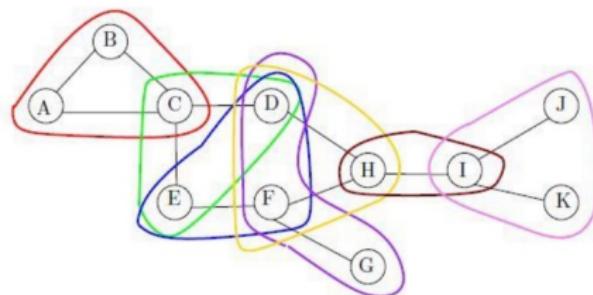
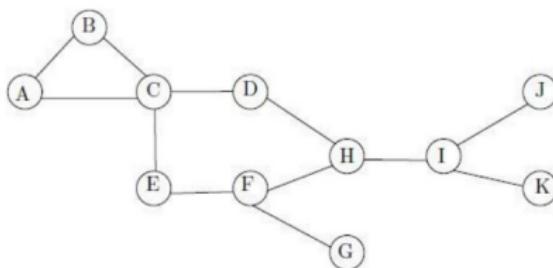
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The *path-width* $pw(\mathcal{G})$ of a graph $\mathcal{G} = \langle V, E \rangle$ is defined by:

$pw(\mathcal{G}) :=$ minimal width of a path decomposition of \mathcal{G} .

Path-decomposition (example)



Path decomposition defines separations

Lemma

Let $\langle B_1, B_2, \dots, B_r \rangle$ be a path decomposition of a graph $\mathcal{G} = \langle V, E \rangle$.

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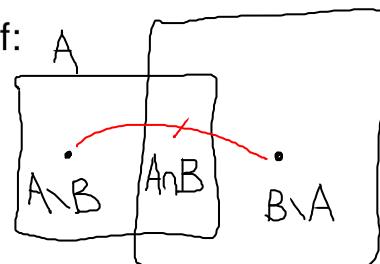
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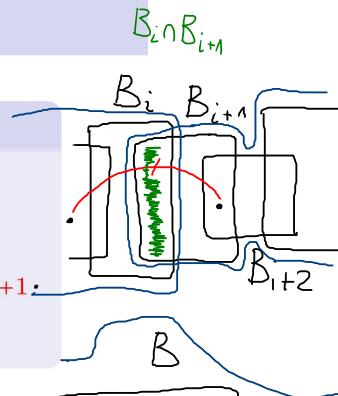


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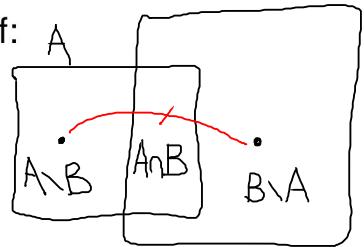
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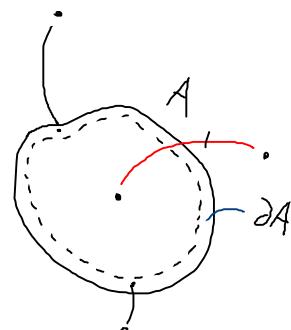
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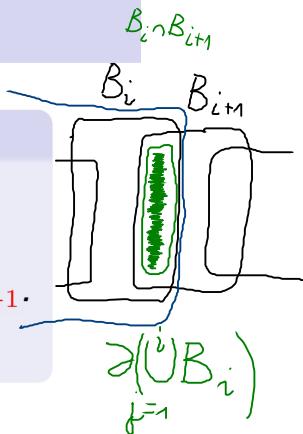
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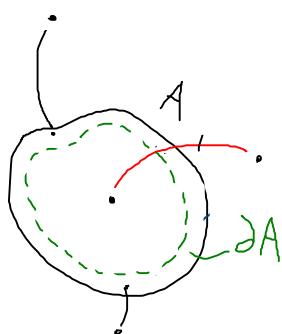


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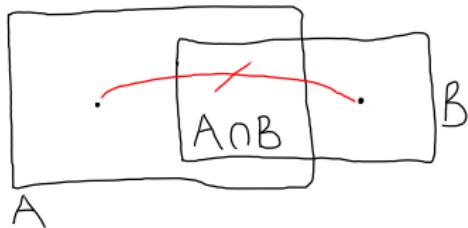
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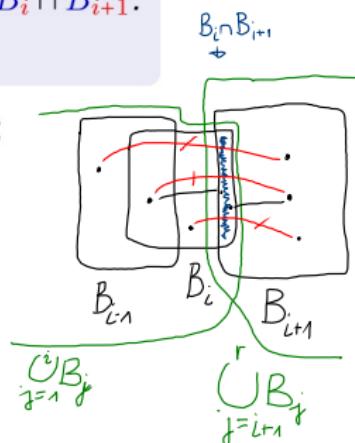
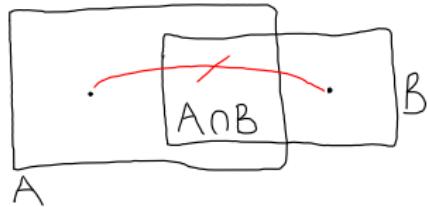
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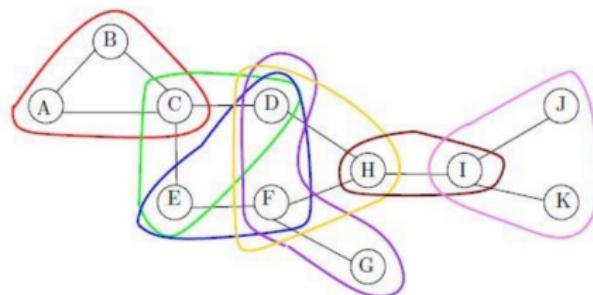
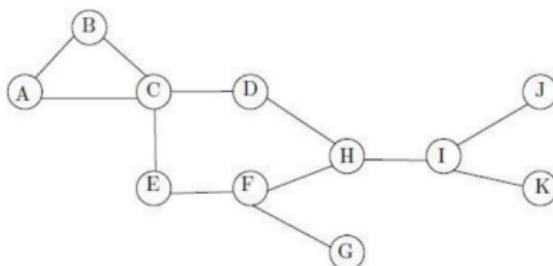
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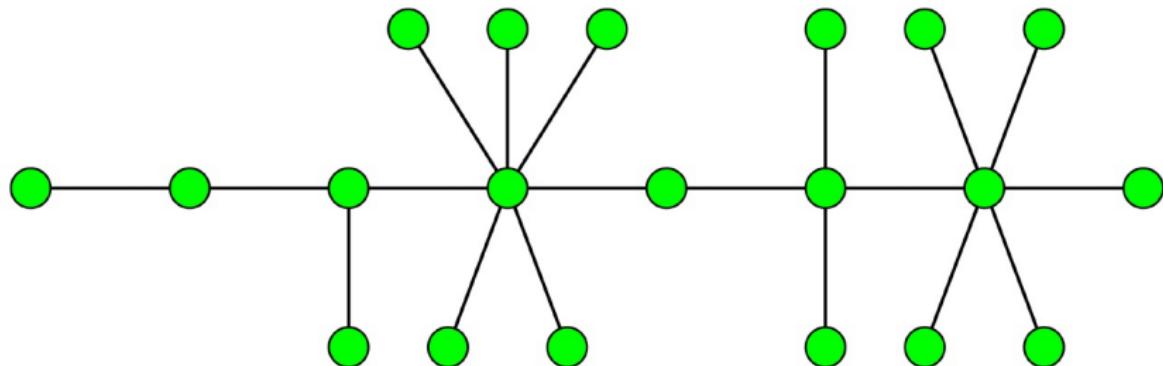
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Path-decomposition (example)



Caterpillar

Path-width?



Nice path decomposition

Definition

A *path decomposition* $\langle B_1, B_2, \dots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if:

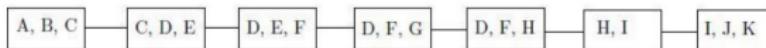
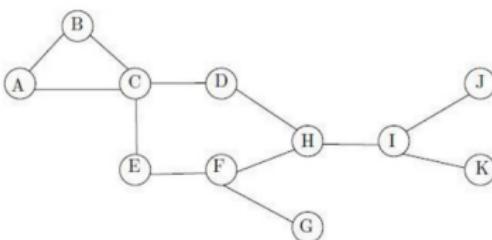
- ▶ $B_1 = B_r = \emptyset$
- ▶ Every index $i > 1$ is either of:
 - ▶ **introduce index**: there is $v \in V$ such that $B_{i+1} = B_i \cup \{v\}$ and $v \notin B_i$,
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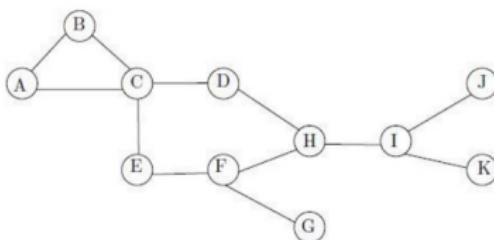


Nice path decomposition

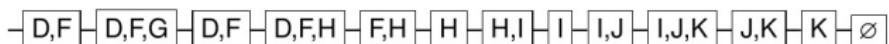
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Lemma

From every *path decomposition* $\langle B_1, B_2, \dots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ of width k a *nice path decomposition* $\langle B'_1, B'_2, \dots, B'_{r'} \rangle$ of width k can be constructed in time $O(k^2 \cdot \max\{r, n\})$ where $n := |V|$.



Weighted Independent Set

Let $\mathcal{G} = \langle V, E \rangle$ a graph.

$S \subseteq V$ is independent set in \mathcal{G} : $\iff \forall e = \{u, v\} (\neg(u \in S \wedge v \in S))$.

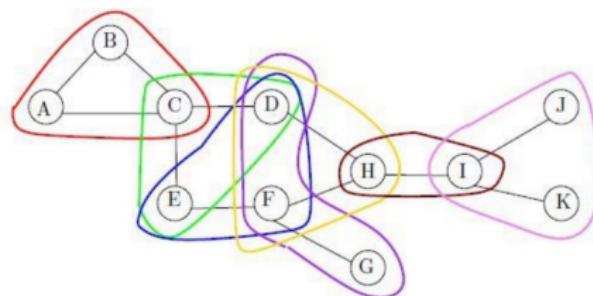
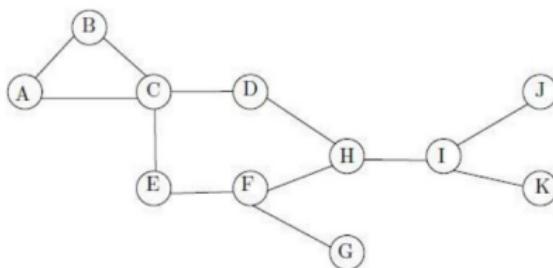
WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $w : V \rightarrow \mathbb{R}_0^+$.

Parameter: path-width k .

Problem: What is the max. weight of an independent set of \mathcal{G} ?

Path-decomposition (example)

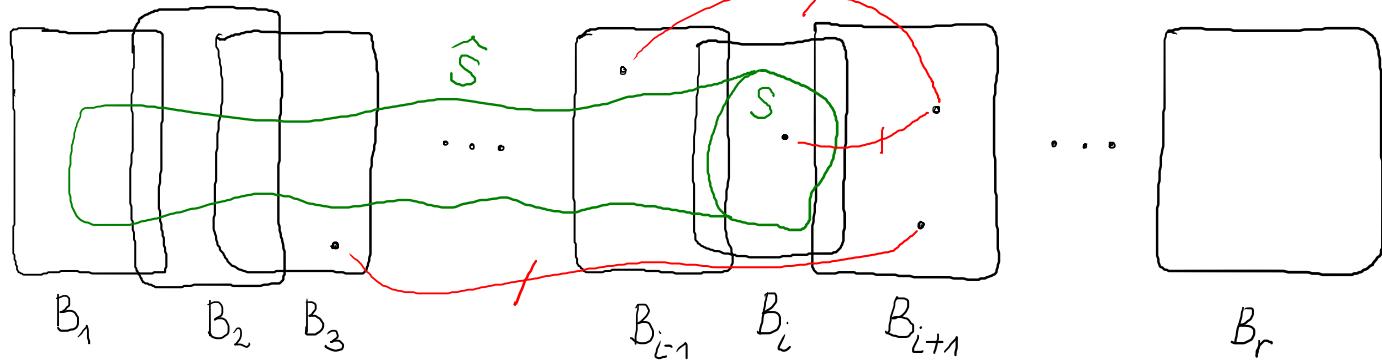


Dyn. programming using path-width (Weigh. Ind. Set)

Let $\langle B_1, \dots, B_r \rangle$ be a nice path decomposition of $\mathcal{G} = \langle V, E \rangle$.
 Then for every $i \in \{1, \dots, r\}$, and every $S \subseteq B_i$, we define:

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path decomposition $\langle B_1, B_2, \dots, B_r \rangle$



G

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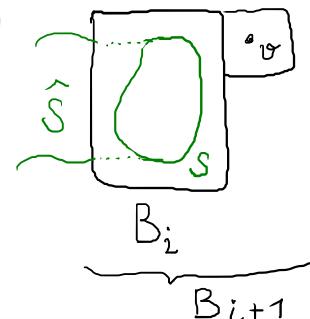
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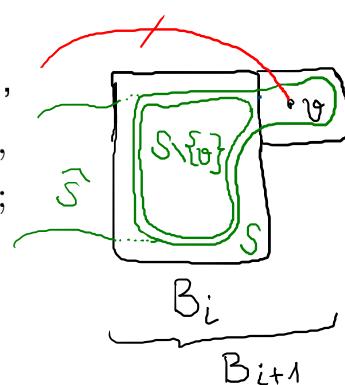
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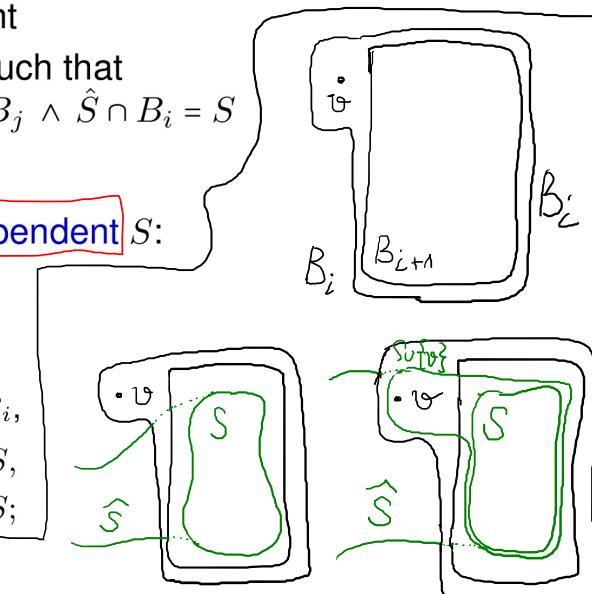
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\Rightarrow the time for computing all values at r :

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Dynamical programming with path width (example)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and path-width $pw(\mathcal{G}) = k$,
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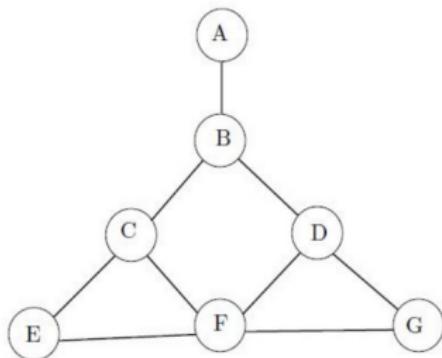
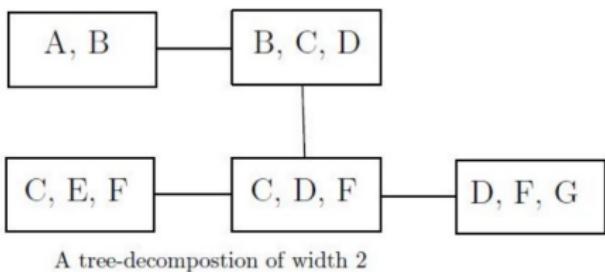
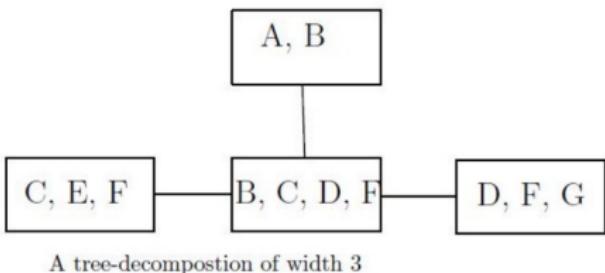
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Tree decomposition (example)

The original graph G 

Tree decompositions, and tree-width

Definition (Bertelé–Brioschi, 1972, Halin, 1976, Robertson–Seymour, 1984)

A *tree decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a pair $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ where $\mathcal{T} = \langle T, F \rangle$ a (undirected, unrooted) tree, and $B_t \subseteq V$ such that:

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The *width* of a tree decomposition $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ is

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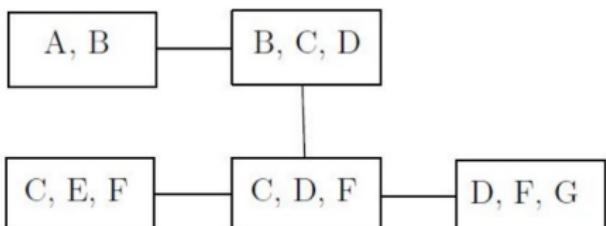
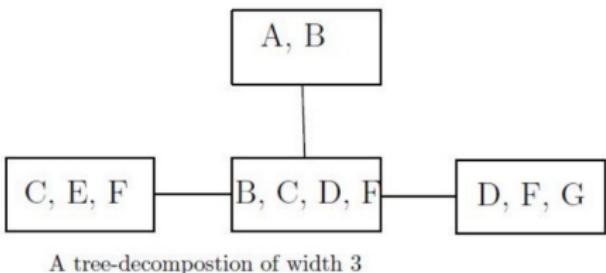
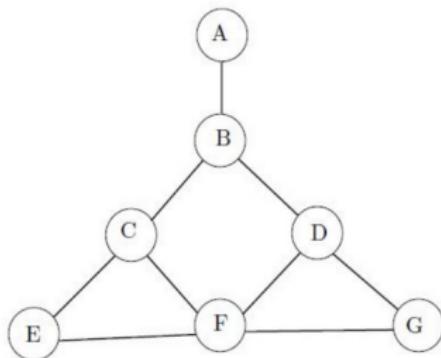
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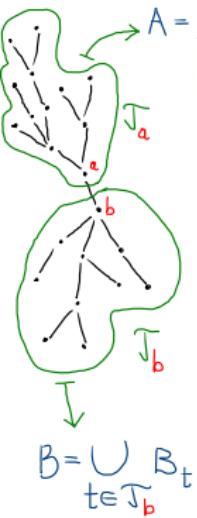
The *tree-width tw(\mathcal{G})* of a graph $\mathcal{G} = \langle V, E \rangle$ is defined by:

$\text{tw}(\mathcal{G}) := \text{minimal width of a tree decomposition of } \mathcal{G}.$

Tree decomposition (example)



Tree decomposition defines separations



Lemma

Let $\langle T, \{B_t\}_{t \in T} \rangle$ be a tree decomposition of a graph $G = \langle V, E \rangle$.

Let $e = \langle a, b \rangle$ be an edge of T . The $T \setminus e$ is the union of a tree T_a containing a , and a tree T_b containing b .

Then for $A := \bigcup_{t \in V(T_a)} B_t$ and $B := \bigcup_{t \in V(T_b)} B_t$ it holds:

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Recall, for a graph $\mathcal{G} = \langle V, E \rangle$:

- ▶ A pair $\langle A, B \rangle$ of subsets $A, B \subseteq V$ is a *separation* of \mathcal{G} if:
 - ▶ $V = A \cup B$
 - ▶ there is no edge between $A \setminus B$ and $B \setminus A$.

$A \cap B$ is called the *separator* of a separation $\langle A, B \rangle$, and $|A \cap B|$ is called its *order*.

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- ▶ The *border (vertices)* $\partial(A)$ of a set $A \subseteq V$ of vertices consists of all vertices that have a neighbor in $V \setminus A$.

Computing tree-width

TREE-WIDTH

Instance: A graph \mathcal{G} and $k \in \mathbb{N}$.

Problem: Decide whether $tw(\mathcal{G}) = k$.

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Theorem

p -TREE-WIDTH is fixed-parameter tractable,
in time $2^{p(k)} \cdot n$ where $n := |V|$.

Nice tree decomposition

Definition

A *tree decomposition* $(\mathcal{T}, \{B_t\}_{t \in T})$ of graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if it is based on the choice of a leaf as *root* r and the parent–children relation away from r such that:

- ▶ $B_r = \emptyset$, and $B_\ell = \emptyset$ for every leaf $\ell \in T$.
- ▶ Every non-leaf node $t \in T$ is of one of three types:
 - ▶ **introduce node**: t has exactly one child t' such that $B_t = B_{t'} \cup \{v\}$; we say v is **introduced** at t .
 - ▶ **forget node**: t has exactly one child t' such that $B_t = B_{t'} \setminus \{w\}$ for some $w \in B_{t'}$; we say w is **forgotten** at t .
 - ▶ **join node**: a node t with two children t_1, t_2 such that $B_t = B_{t_1} = B_{t_2}$.

Nice tree decomposition

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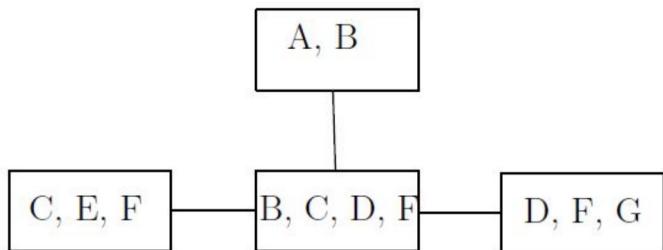
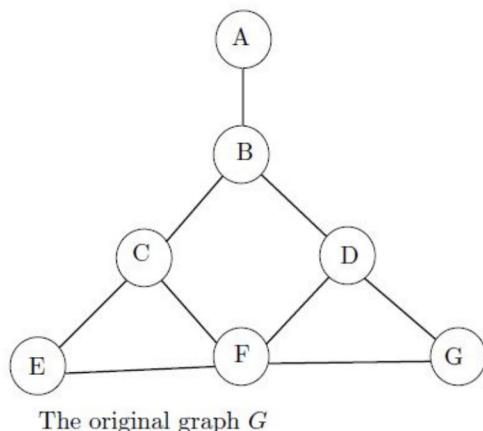
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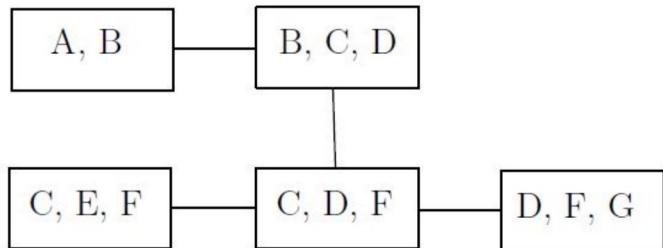
Lemma

From every *tree decomposition* $(\mathcal{T}, \{B_t\}_{t \in T})$ of a graph $\mathcal{G} = \langle V, E \rangle$ of width k a *nice tree decomposition* $(\mathcal{T}', \{B'_t\}_{t \in T'})$ of width k and with $r := |V(\mathcal{T})| \in O(kn)$ vertices can be constructed in time $O(k^2 \cdot \max\{r, n\})$ where $n := |V|$.

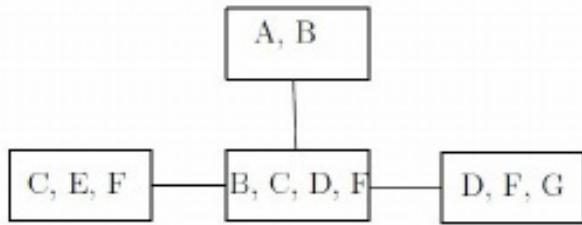
Tree decomposition (example)



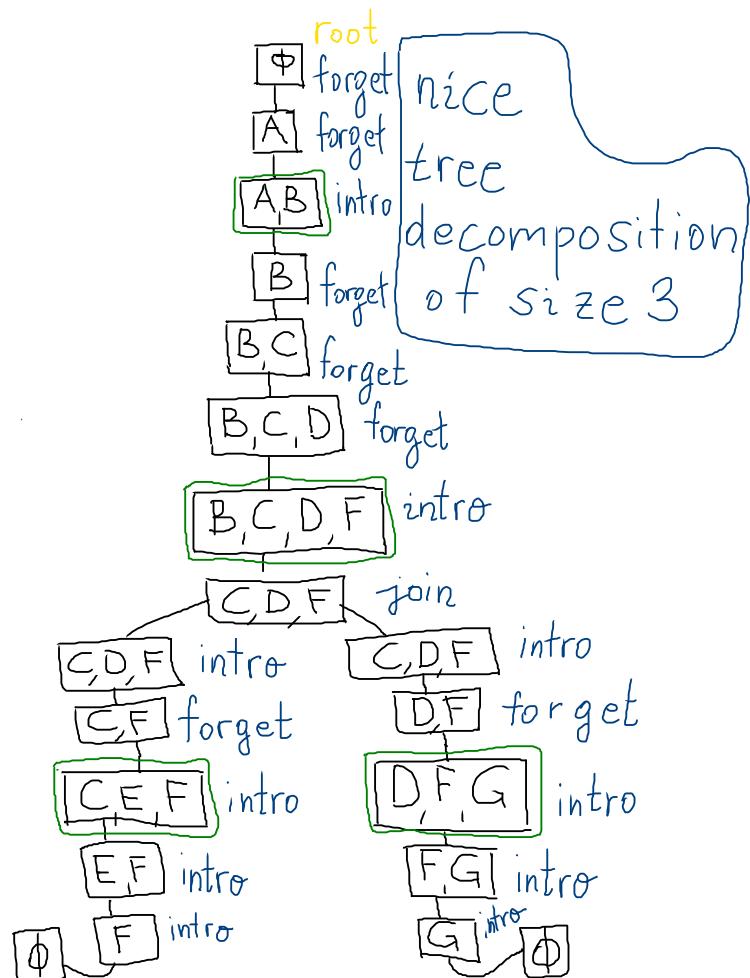
A tree-decomposition of width 3

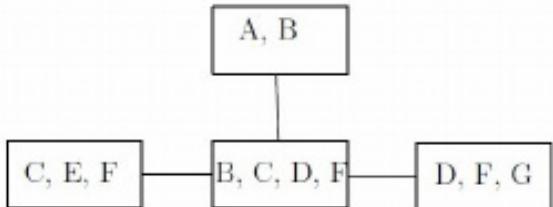


A tree-decomposition of width 2

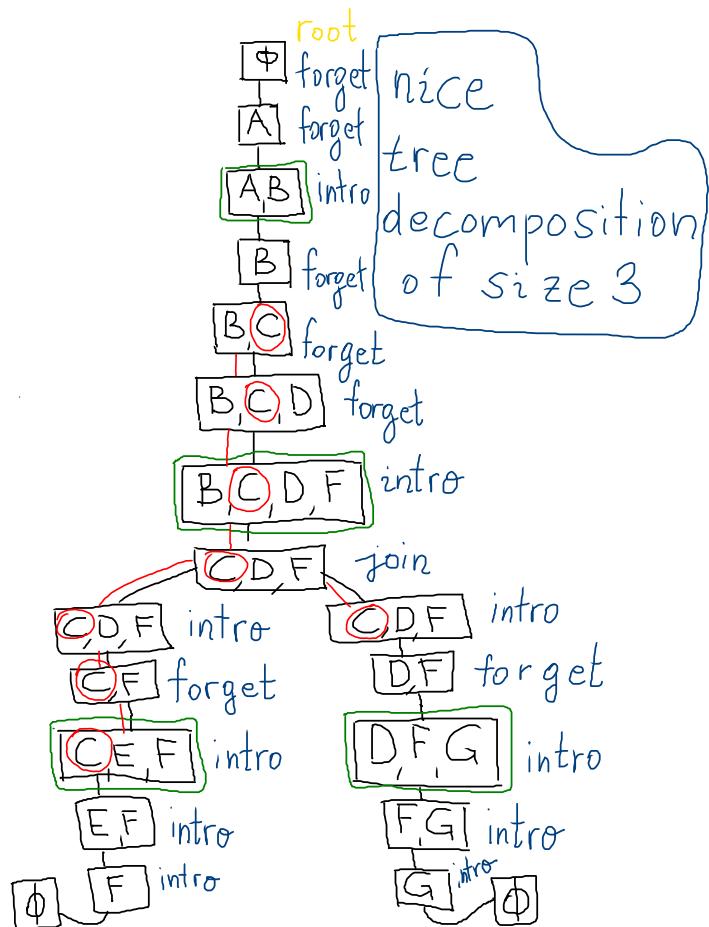
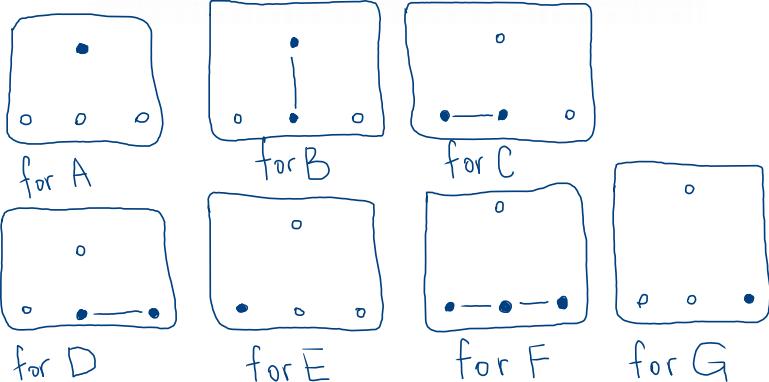


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A tree-decomposition of width 3



Tree decompositions, and tree-width

Definition (Bertelé–Brioschi, 1972, Halin, 1976, Robertson–Seymour, 1984)

A *tree decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a pair $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ where $\mathcal{T} = \langle T, F \rangle$ a (undirected, unrooted) tree, and $B_t \subseteq V$ such that:

(T1) $V = \bigcup_{t \in T} B_t$ (every vertex of \mathcal{G} is in some bag).

(T2) $(\forall \{u, v\} \in E) (\exists t \in T) [\{u, v\} \subseteq B_t]$

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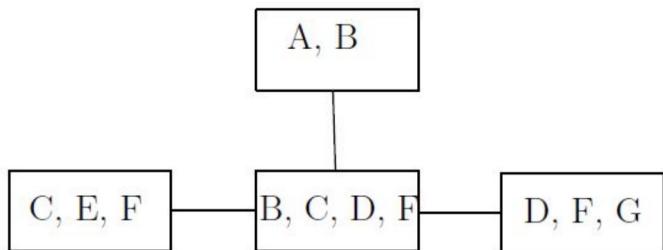
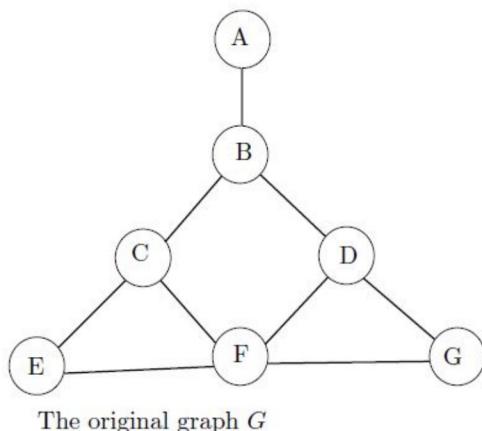
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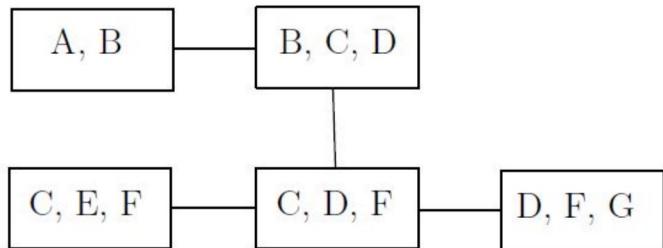
The *tree-width tw(\mathcal{G})* of a graph $\mathcal{G} = \langle V, E \rangle$ is defined by:

$\text{tw}(\mathcal{G}) :=$ minimal width of a tree decomposition of \mathcal{G} .

Tree decomposition (example)



A tree-decomposition of width 3



A tree-decomposition of width 2

Weighted Independent Set

Let $\mathcal{G} = \langle V, E \rangle$ a graph.

$S \subseteq V$ is independent set in \mathcal{G} : $\iff \forall e = \{u, v\} (\neg(u \in S \wedge v \in S))$.

WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $w : V \rightarrow \mathbb{R}_0^+$.

Parameter: tree-width k .

Problem: What is the max. weight of an independent set of \mathcal{G} ?

Dynamical programming using tree-width (example)

For every node t of a nice tree decomposition, and every $S \subseteq B_t$, we define:

$$c[t, S] := \begin{cases} -\infty & \text{if } S \text{ is not independent,} \\ \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent } \wedge S \subseteq \hat{S} \subseteq V_t \wedge \hat{S} \cap B_t = S \\ & \text{if } S \text{ is independent.} \end{cases}$$

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- ▶ leaf node t : $c[t, \emptyset] = 0$

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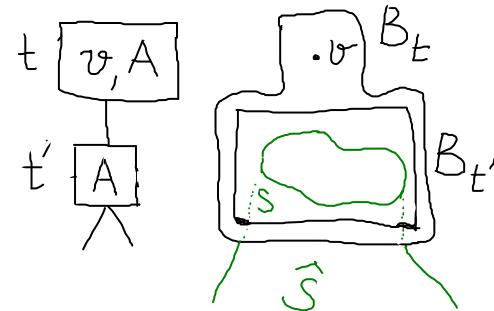
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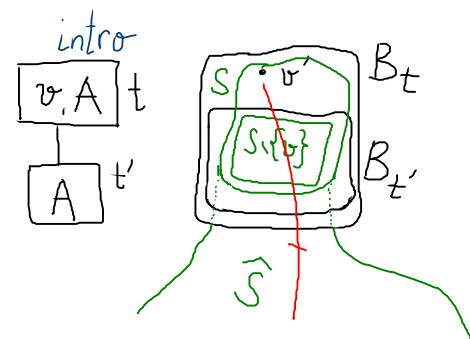
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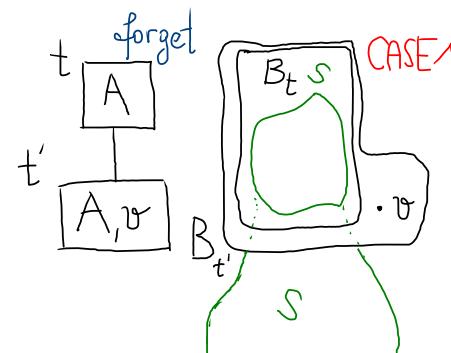
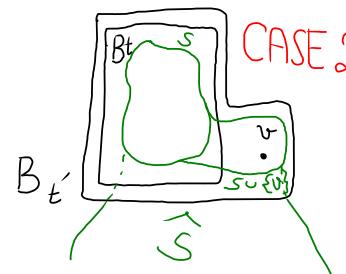
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CASE 1

CASE 2



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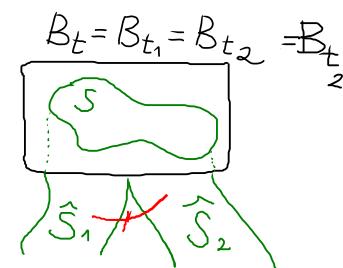
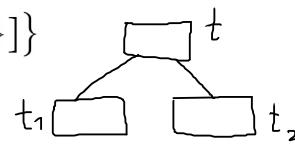
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$$c[t, S] := \begin{cases} -\infty & \text{if } S \text{ is not independent,} \\ \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent } \wedge S \subseteq \hat{S} \subseteq V_t \wedge \hat{S} \cap B_t = S \\ & \text{if } S \text{ is independent.} \end{cases}$$

Recursive equations for computing $c[t, S]$ for independent S :

- ▶ leaf node t : $c[t, \emptyset] = 0$
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$$c[t, S] = \begin{cases} c[t', S] & \text{if } v \notin S \\ \dots & \end{cases}$$

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- ▶ join node t with children t_1 and t_2 :

$$c[t, S] = c[t_1, S] + c[t_2, S] - w(S)$$

Dyn. programming using tree-width (Weigh. Ind. Set)

Let $\langle \mathcal{T} = \langle T, F, \textcolor{blue}{r} \rangle, \{B_t\}_{t \in T} \rangle$ be a nice tree decomposition of $\mathcal{G} = \langle V, E \rangle$ of width k . For every $t \in T$, and every independent $S \subseteq B_t$:

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$$= \max_{S \subseteq B_{\textcolor{blue}{r}}} c[\textcolor{blue}{r}, S] = c[\textcolor{blue}{r}, \emptyset]$$

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Then for all $t \in T$:

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\Rightarrow the time for computing all values at the root $\textcolor{blue}{r}$:

$$(2^{\textcolor{red}{k}+1} \cdot O(k^2)) \cdot |T| + O(\textcolor{red}{k}^{O(1)} \cdot n) \in \textcolor{brown}{O}(2^k \cdot k^{O(1)} \cdot n), \text{ since } |T| \in O(\textcolor{red}{k} \cdot n).$$

Dynamical programming with tree width (example)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and tree-width $tw(\mathcal{G}) = k$,
 p^* -WEIGHTED-INDEPENDENT-SET $\in \text{DTIME}(2^k \cdot k^{O(1)} \cdot n)$.

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Dyn. programming with tree-width: general strategy

We consider problem P for graphs $\mathcal{G} = \langle V, E \rangle$ of size n and nice tree decompositions $\langle \mathcal{T} = \langle T, F, r \rangle, \{B_t\}_{t \in T} \rangle$ of tree width k .

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- ▶ Formulate a family of properties that can be restricted to subtrees of \mathcal{T} such that
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- ▶ Obtain an estimate of the time needed to compute the properties in a node t depending on n and k .
- ▶ Sum up the time needed to compute the solution(s) at root r of \mathcal{T} .
- ▶ Add time needed to obtain the solution of P from properties at r .

Dynamical programming: similar results (I)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and $\text{tw}(\mathcal{G}) = k$,

- ▶ p^* -VERTEX-COVER, INDEPENDENT-SET $\in \text{DTIME}(2^k \cdot k^{O(1)} \cdot n)$,
- ▶ p^* -DOMINATING-SET $\in \text{DTIME}(4^k \cdot k^{O(1)} \cdot n)$,
- ▶ p^* -ODD CYCLE TRAVERSAL $\in \text{DTIME}(3^k \cdot k^{O(1)} \cdot n)$,
- ▶ p^* -MAXCUT $\in \text{DTIME}(2^k \cdot k^{O(1)} \cdot n)$,
- ▶ p^* - q -COLORABILITY $\in \text{DTIME}(q^k \cdot k^{O(1)} \cdot n)$.

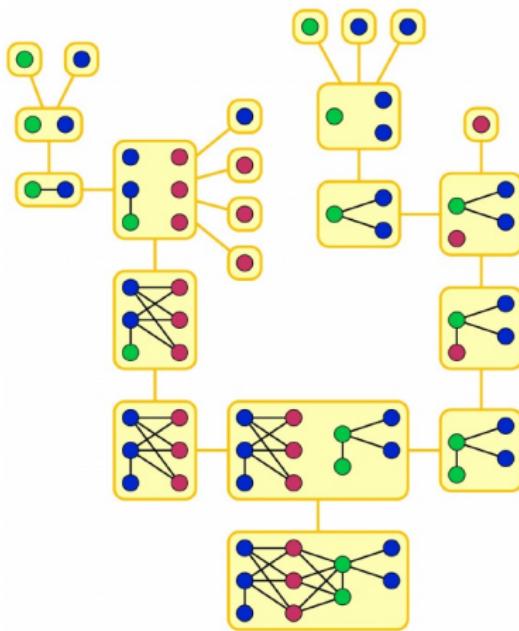
Dynamical programming: similar results (II)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and $\text{tw}(\mathcal{G}) = k$, the following problems are in $\text{DTIME}(k^{O(k)} \cdot n)$:

- ▶ p^* -STEINER-TREE,
- ▶ p^* -FEEDBACK-VERTEX-SET,
- ▶ p^* -HAMILTONIAN-PATH and p^* -LONGEST-PATH,
- ▶ p^* -HAMILTONIAN-CYCLE and p^* -LONGEST-CYCLE,
- ▶ p^* -CHROMATIC-NUMBER,
- ▶ p^* -CYCLE-PACKING,
- ▶ p^* -CONNECTED-VERTEX-COVER,
- ▶ p^* -CONNECTED-FEEDBACK-VERTEX-SET.

Clique width (example)



Clique-Width

For $k \in \mathbb{N}$, the k -expressions are defined by:

$$\varphi, \varphi_1, \varphi_2 ::= i \mid \text{edge}_{i-j}(\varphi) \mid \text{recolor}_{i \rightarrow j}(\varphi) \mid (\varphi_1 \oplus \varphi_2)$$

for $i, j \in [k]$ with $i \neq j$.

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- ▷ $\mathcal{G}(i)$ is the graph with a single vertex of color i .
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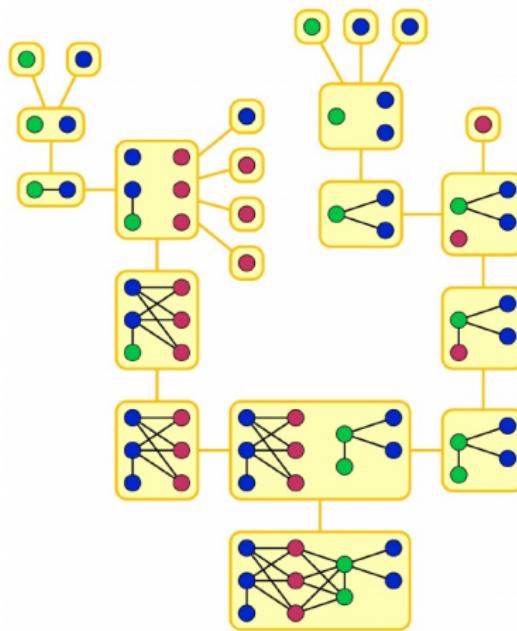
Definition (Courcelle, Engelfriet, Rozenberg, 1993, [2])

The *clique-width* $\text{clw}(\mathcal{G})$ of $\mathcal{G} = \langle V, E \rangle$ is defined by:

$\text{clw}(\mathcal{G}) :=$ the least $k \in \mathbb{N}$ such that, for some k -expression φ ,
 $\mathcal{G} = \mathcal{G}(\varphi)$ (when removing colors)

Clique width (example)

Building a graph \mathcal{G} of clique-width $c/w(\mathcal{G}) = 3$:



Clique-Width (examples, properties, computability)

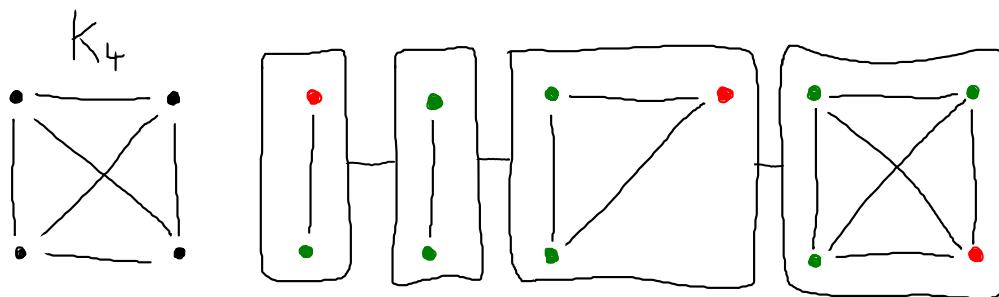
Example

- ▶ The class of cliques has clique-width 2.

Clique-Width (examples, properties, computability)

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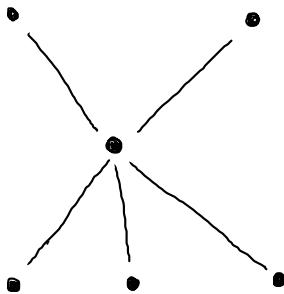
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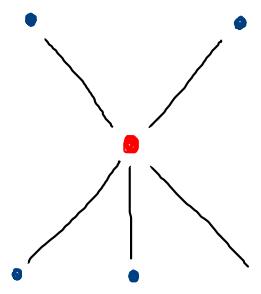
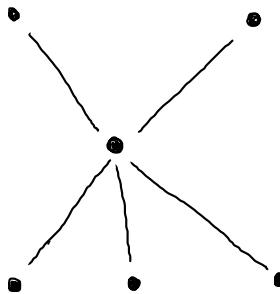
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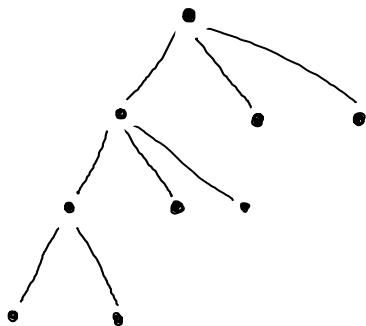
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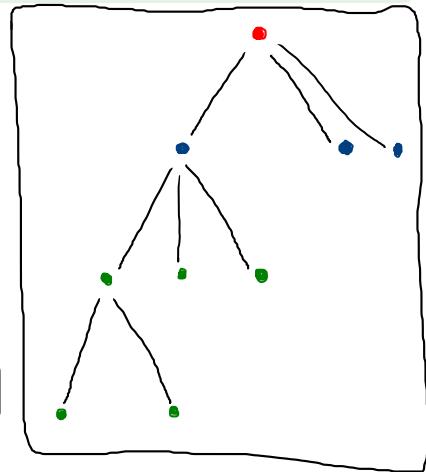
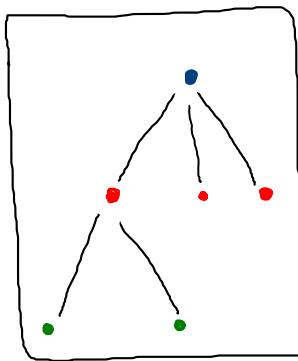
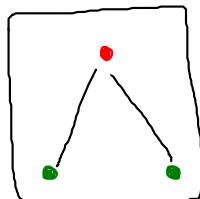
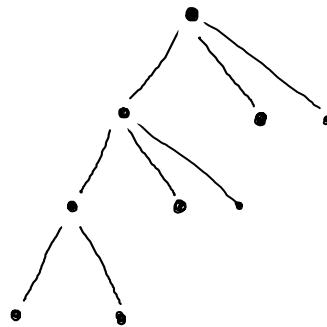
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- ▶ Every graph property expressible in **MSO (monadic second-order logic)** can be decided in linear time w.r.t. the graph's clique-width.

f -Width (of sets)

By a *cut function* or a *connectivity function* we mean a function $f : 2^U \rightarrow \mathbb{R}_0^+$ such that:

f is *symmetric*: $\iff \forall X \subseteq U [f(X) = f(U \setminus X)]$;

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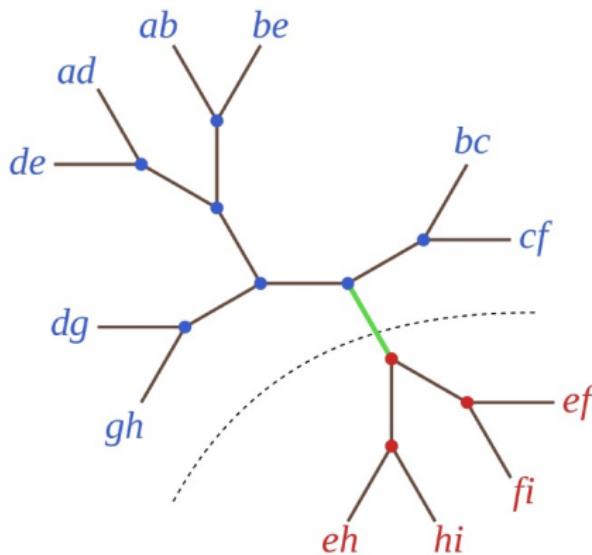
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A *branch decomposition* of U is a pair $\langle \mathcal{T}, \eta \rangle$ where:

- ▷ $\mathcal{T} = \langle T, F \rangle$ a tree.
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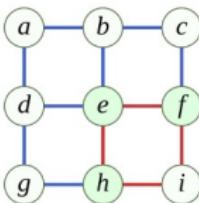
$w_f(U) := \underline{\text{minimum width}}$ of branch decomp's of U w.r.t. f .

Branch-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph. The *border (vertices)* of a set $X \subseteq E$ of edges is defined by:

$$\partial(X) := \left\{ v \in V \mid \exists e_1 \in X \exists e_2 \in E \setminus X \left[v \text{ is incident to } e_1 \text{ and } e_2 \right] \right\}$$



Branch-Width

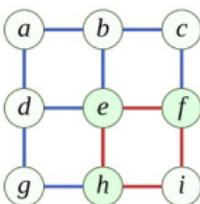
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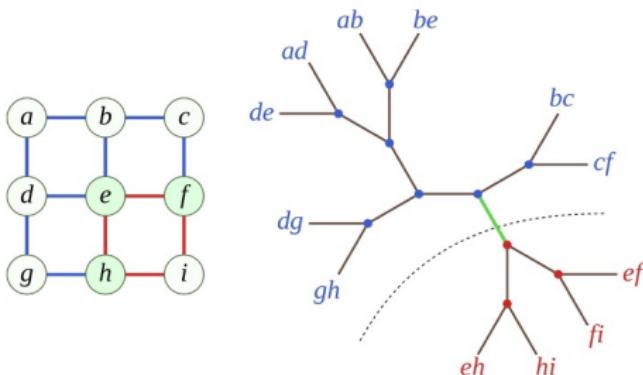
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Proposition

$bw(\mathcal{G}) \approx tw(\mathcal{G})$, for every graph; more precisely:

$$bw(\mathcal{G}) \leq tw(\mathcal{G}) + 1 \leq \frac{3}{2} \cdot bw(\mathcal{G}).$$

Rank-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.

For $X \subseteq V$ we define the $GF(2)$ -matrix:

$B_{\mathcal{G}}(X) := (b_{x,y})_{x \in X, y \in V \setminus X}$, where, for all $x \in X, y \in V \setminus X$:

$$b_{x,y} = 1 \iff \{x, y\} \in E.$$

($B_{\mathcal{G}}(X)$ is the adjacency matrix of the bipartite graph induced by \mathcal{G} between X and $V \setminus X$.)

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The *rank-width* $rw(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is:

$$rw(\mathcal{G}) := w_{\rho_{\mathcal{G}}}(E) \text{ for } \rho_{\mathcal{G}} : 2^V \rightarrow \mathbb{N}_0, X \mapsto \text{rank of } \mathcal{B}_{\mathcal{G}}(X)$$

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The *rank-width* $\text{rw}(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is:

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Properties

- ▶ $\text{rw}(\mathcal{G}) \leq \text{tw}(\mathcal{G})$.
- ▶ tree-width cannot be bounded functionally by rank-width:
 $\text{rw}(K_n) = 1$, but $\text{tw}(K_n) = n - 1$.

Carving-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.

For $X \subseteq V$ the *edge-cut* of X is:

$$\text{cut}_{\mathcal{G}}(X) := \{e = \{u, v\} \in E \mid u \in X, v \in V \setminus X\} .$$

The *carving-width* $\text{carw}(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is:

$$\text{carw}(\mathcal{G}) := w_{\text{cut}}(E) \quad \text{for } \text{cut}: 2^V \rightarrow \mathbb{N}_0, X \mapsto |\text{cut}_{\mathcal{G}}(X)| .$$

Carving-Width and Cut-Width

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Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph with $n = |V|$.

For a permutation $\pi : \{1, \dots, n\} \rightarrow V$ on V we define:

$$\text{width}(\pi) := \max_{1 \leq i \leq n} \text{cut}_{\mathcal{G}}(\{\pi(j) \mid 1 \leq j \leq i\}).$$

The *cut-width* $\text{cutw}(\mathcal{G})$ of \mathcal{G} is:

$$\text{cutw}(\mathcal{G}) := \min_{\pi \text{ perm. of } V} \text{width}(\pi).$$

Coverage in Multi-Interface Networks



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$CMI(p)$ (for $p \in \mathbb{N}$)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, $W : V \rightarrow 2^{\{1, \dots, a\}}$ available-interface allocation, $c : \{1, \dots, a\} \rightarrow \mathbb{R}^+$ interface cost function.

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Solution: An allocation $W_A : V \rightarrow 2^{\{1, \dots, a\}}$ of active interfaces covering \mathcal{G} such that $W_A(v) \subseteq W(v)$, and $|W_A(v)| \leq p$ for all $v \in V$, if possible; otherwise, a negative answer.

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Problem: Obtain, if possible, a minimal solution with respect to the total cost of the interfaces that are activated, that is, $c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i)$.

Coverage in Multi-Interface Networks

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$CMI(2)$ ∈ NP-complete, also for graphs with max. node degree ≥ 4 .

Coverage in Multi-Interface Networks (parameterized)

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 $c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i)$.

Corollary

$(p^*)'$ -CMI(p) $\in \text{FPT}$.

Comparing parameterizations

Definition (computably bounded)

Let $\kappa_1, \kappa_2 : \Sigma^* \rightarrow \mathbb{N}$ parameterizations.

- ▶ $\kappa_1 \geq \kappa_2 : \iff \exists g : \mathbb{N} \rightarrow \mathbb{N} \text{ computable } \forall x \in \Sigma^* [g(\kappa_1(x)) \geq \kappa_2(x)]$.
- ▶ $\kappa_1 \approx \kappa_2 : \iff \kappa_1 \geq \kappa_2 \wedge \kappa_2 \geq \kappa_1$.
- ▶ $\kappa_1 > \kappa_2 : \iff \kappa_1 \geq \kappa_2 \wedge \neg(\kappa_2 \geq \kappa_1)$.

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Proposition

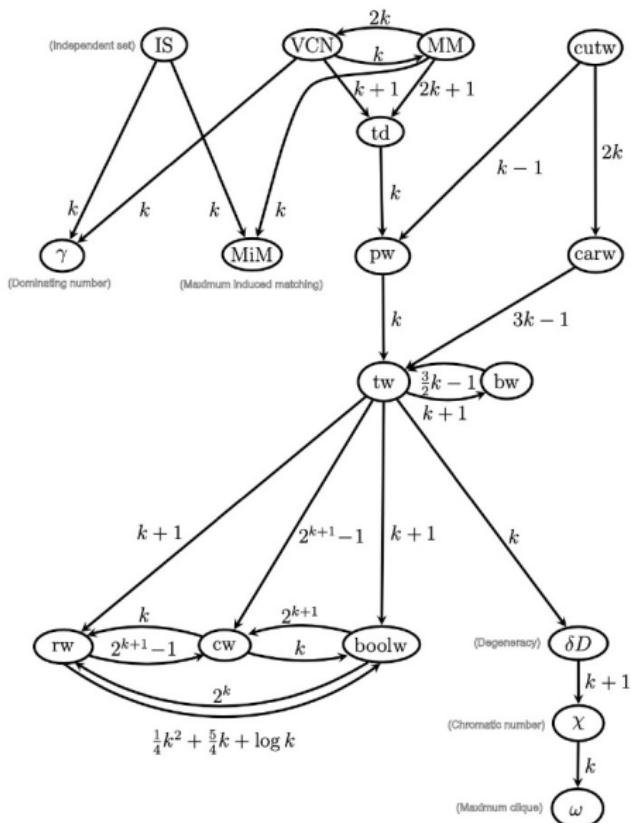
For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \rightarrow \mathbb{N}$ with $\kappa_1 \geq \kappa_2$:

$$\begin{aligned}\langle Q, \kappa_1 \rangle \in \text{FPT} &\iff \langle Q, \kappa_2 \rangle \in \text{FPT} \\ \langle Q, \kappa_1 \rangle \notin \text{FPT} &\implies \langle Q, \kappa_2 \rangle \notin \text{FPT}\end{aligned}$$

Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \geq wd_2 : \Leftarrow wd_1 \xrightarrow{g(k)} wd_2$$

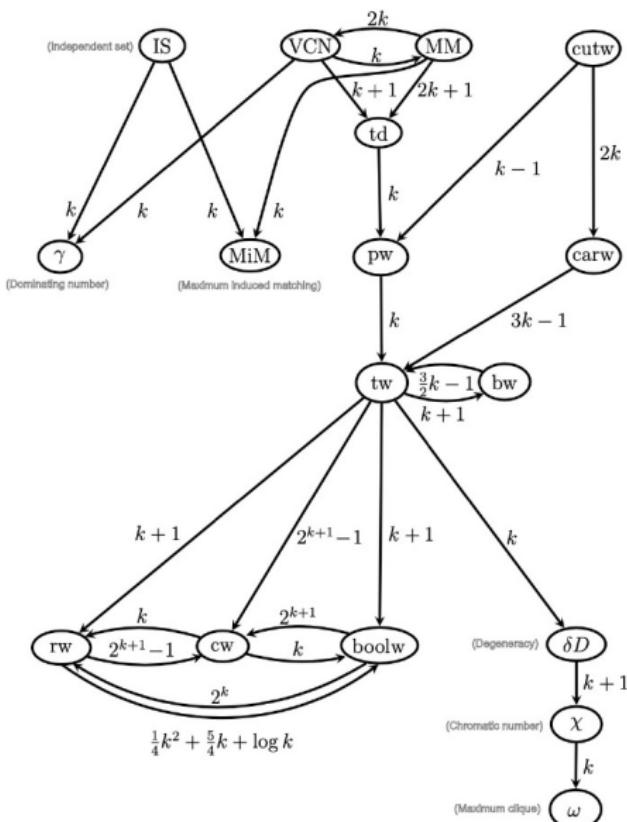


Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \geq wd_2 : \Leftarrow wd_1 \xrightarrow{g(k)} wd_2$$

- ▶ FPT-results
transfer upwards
(and conversely to $\overset{g}{\rightarrow}$)

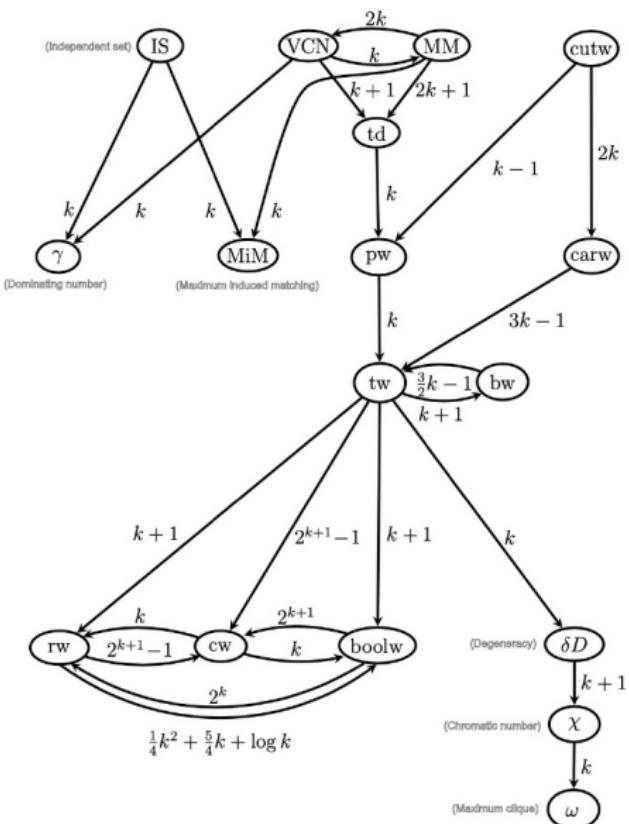


Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \geq wd_2 : \Leftarrow wd_1 \xrightarrow{g(k)} wd_2$$

- ▶ FPT-results
transfer upwards
(and conversely to \xrightarrow{g})
- ▶ (\notin FPT)-results
transfer downwards
(and along \xrightarrow{g})



Summary

- ▶ comparing parameterizations
- ▶ dynamical programming on trees, example:
 - ▶ WEIGHTED-INDEPENDENT-SET (and VERTEX-COVER)
- ▶ path-width
 - ▶ example: fpt-algorithm for bounded path-width
- ▶ tree-width
 - ▶ example: fpt-algorithm for bounded path-width
- ▶ fpt-results for other problems, obtained similarly
- ▶ other notions of width
 - ▶ clique-width
 - ▶ using f -width to define:
 - ▶ carving-width (and cut-width)
 - ▶ branch-width
 - ▶ rank-width
- ▶ example problem: coverage in multi-interface networks
- ▶ comparing width-notions

Tomorrow

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results		Algorithmic Meta-Theorems		
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	GDA	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	GDA	GDA
<i>Algorithmic Techniques</i>		<i>Formal-Method & Algorithmic Techniques</i>		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width			FPT-Intractability Classes & Hierarchies
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Tomorrow

- ▶ recalling notions from logic:
 - ▶ propositional, and first-order logic
 - ▶ monadic second-order logic (**MSO**)
- ▶ **Courcelle's Theorem:** obtaining FPT-results by
 - ▶ model-checking of **MSO**-properties
on graphs and structures of **bounded tree-/clique-width**

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