

Lecture 4: Fixed-Parameter Intractability

(A Short Introduction to Parameterized Complexity)

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Course overview

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
<i>Algorithmic Techniques</i>		<i>Formal-Method & Algorithmic Techniques</i>		
Introduction & basic FPT results motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	Notions of bounded graph width path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	Algorithmic Meta-Theorems 1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	FPT-Intractability Classes & Hierarchies motivation for FP-intractability results, FPT-reductions, class XP (slice-wise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour

Overview

- ▶ Motivation for fixed-parameter intractability
- ▶ Fixed parameter reductions
- ▶ The classes [para-NP](#) and [XP](#)
- ▶ The class [W\[P\]](#)
- ▶ Logic preliminaries (continued)
- ▶ [W-hierarchy](#)
 - ▶ definitions
 - ▶ with Boolean circuits
 - ▶ as parameterized weighted Fagin definability problems
- ▶ [A-hierarchy](#)
 - ▶ definition as parameterized model-checking problems
- ▶ picture overview of these classes

Two classical problems

QUERIES

Instance: a relational database D , a conjunctive query α .

Compute: answer to query α from database D .

- ▶ QUERIES \in NP-complete.

LTL-MODEL-CHECKING

Instance: a Kripke structure (state space) \mathcal{K} , an LTL formula φ

Parameter: size $|\varphi|$ of formula φ

Question: Does $\mathcal{K} \models \varphi$ hold?

- ▶ LTL-MODEL-CHECKING \in PSPACE-complete.

Comparing their parameterizations

QUERIES

Instance: a relational database D , a conjunctive query α .

Parameter: size $k = |\alpha|$ of query α

Compute: answer to query α from database D .

- ▶ $\text{QUERIES} \in \text{NP-complete}$.
- ▶ $\text{QUERIES} \in O(n^k)$ for $n = \|D\|$, which does **not** give an **FPT** result.

LTL-MODEL-CHECKING

Instance: a Kripke structure (state space) \mathcal{K} , an LTL formula φ

Parameter: size $k = |\varphi|$ of formula φ

Question: Does $\mathcal{K} \models \varphi$ hold?

- ▶ $\text{LTL-MODEL-CHECKING} \in \text{PSPACE-complete}$,
- ▶ $\text{LTL-MODEL-CHECKING} \in O(k \cdot 2^{2k} \cdot n) \in \text{FPT}$ for $n = \|\mathcal{K}\|$.

Fixed-parameter intractability

‘The purpose [...] is to give evidence that certain problems are not fixed-parameter tractable (just as the main purpose of the theory of NP-completeness is to give evidence that certain problems are not polynomial time computable.)’

In classical theory, the notion of NP-completeness is central to a nice, simple, and far-reaching theory for intractable problems.

Unfortunately, the world of parameterized intractability is more complex: There is a big variety of seemingly different classes of intractable parameterized problems.’

(Flum, Grohe [2])

Fixed-Parameter tractable

Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *fixed-parameter tractable* (is in **FPT**) if:

$\exists f : \mathbb{N} \rightarrow \mathbb{N}$ computable $\exists p \in \mathbb{N}[X]$ polynomial

$\exists \mathbb{A}$ algorithm, takes inputs in Σ^*

$\forall x \in \Sigma^* [\mathbb{A} \text{ decides whether } x \in Q \text{ holds}$
 $\text{in time } \leq f(\kappa(x)) \cdot p(|x|)]$

Slices of parameterized problems

The ℓ -th slice, for $\ell \in \mathbb{N}$, of a parameterized problem $\langle Q, \kappa \rangle$ is:

$$\langle Q, \kappa \rangle_{\ell} := \{x \in Q \mid \kappa(x) = \ell\}.$$

Proposition (slices of FPT problems are in PTIME)

Let $\langle Q, \kappa \rangle$ be a parameterized problem, and $\ell \in \mathbb{N}$.

If $\langle Q, \kappa \rangle \in \text{FPT}$, then $\langle Q, \kappa \rangle_{\ell} \in \text{PTIME}$.

Proof

Let ℓ be fixed. Then for all $x \in \Sigma^*$:

Decide $x \in Q, \kappa(x) = \ell$ in time $\leq f(\kappa(x)) \cdot p(|x|) = f(\ell) \cdot p(|x|) \in \text{PTIME}$.

A problem not in FPT

The ℓ -th slice, for $\ell \in \mathbb{N}$, of a parameterized problem $\langle Q, \kappa \rangle$ is:

$$\langle Q, \kappa \rangle_{\ell} := \{x \in Q \mid \kappa(x) = \ell\}.$$

Slices of FPT problems are in PTIME

If $\langle Q, \kappa \rangle \in \text{FPT}$, then $\langle Q, \kappa \rangle_{\ell} \in \text{PTIME}$.

p -COLORABILITY

Instance: A graph \mathcal{G} , and $\ell \in \mathbb{N}$.

Parameter: ℓ .

Problem: Decide whether \mathcal{G} is ℓ -colorable.

Consequence: p -COLORABILITY \notin FPT (unless $P = NP$).

It is well-known: 3-COLORABILITY \in NP-complete. Now since 3-COLORABILITY is the third slice of p -COLORABILITY, the proposition entails p -COLORABILITY \notin FPT unless $P = NP$.

Polynomial reductions / hardness / completeness

Definition

Let $\langle Q_1, \Sigma_1 \rangle, \langle Q_2, \Sigma_2 \rangle$ be classical problems.

An **polynomial-time reduction** from $\langle Q_1, \Sigma_1 \rangle$ to $\langle Q_2, \Sigma_2 \rangle$ is a mapping $R: \Sigma_1^* \rightarrow \Sigma_2^*$:

R1. $(x \in Q_1 \iff R(x) \in Q_2)$ for all $x \in \Sigma_1^*$.

R2. R is **computable by a polynomial-time algorithm**: there is a polynomial $p(X)$ such that R is computable in time $p(|x|)$.

$\langle Q_1, \Sigma_1 \rangle \leq_{\text{pol}} \langle Q_2, \Sigma_2 \rangle :=$
there is a polynomial-time reduction from $\langle Q_1, \Sigma_1 \rangle$ to $\langle Q_2, \Sigma_2 \rangle$.

Proposition

If $\langle Q_1, \Sigma_1 \rangle \leq_{\text{pol}} \langle Q_2, \Sigma_2 \rangle$, then: $\langle Q_1, \Sigma_1 \rangle \in \mathbf{P} \iff \langle Q_2, \Sigma_2 \rangle \in \mathbf{P}$.
 $\langle Q_1, \Sigma_1 \rangle \notin \mathbf{P} \implies \langle Q_2, \Sigma_2 \rangle \notin \mathbf{P}$.

Let \mathbf{C} be class of classical problems.

- ▶ $\langle Q, \Sigma \rangle$ is **C-hard**: if, for all $\langle Q', \Sigma' \rangle \in \mathbf{C}$, $\langle Q', \Sigma' \rangle \leq_{\text{pol}} \langle Q, \Sigma \rangle$.
- ▶ $\langle Q, \Sigma \rangle$ is **C-complete**: if $\langle Q, \Sigma \rangle$ is C-hard, and $\langle Q, \Sigma \rangle \in \mathbf{C}$.

Fixed-parameter tractable reductions

Definition

Let $\langle Q_1, \Sigma_1, \kappa \rangle, \langle Q_2, \Sigma_2, \kappa_2 \rangle$ be parameterized problems.

An **fpt-reduction** from $\langle Q_1, \kappa_1 \rangle$ to $\langle Q_2, \kappa_2 \rangle$ is a mapping

$R: \Sigma_1^* \rightarrow (\Sigma_2)^*$:

R1. $(x \in Q_1 \iff R(x) \in Q_2)$ for all $x \in \Sigma_1^*$.

R2. R is **computable by a fpt-algorithm** (with respect to κ): there are f computable and $p(X)$ polynomial such that R is computable in time $f(\kappa_1(x)) \cdot p(|x|)$.

R3. $\kappa_2(R(x)) \leq g(\kappa_1(x))$ for all $x \in \Sigma_1^*$, for some computable function $g: \mathbb{N} \rightarrow \mathbb{N}$.

$\langle Q_1, \kappa_1 \rangle \leq_{\text{fpt}} \langle Q_2, \kappa_2 \rangle :=$ there is an fpt-red. from $\langle Q_1, \kappa_1 \rangle$ to $\langle Q_2, \kappa_2 \rangle$.

Proposition

If $\langle Q_1, \kappa_1 \rangle \leq_{\text{fpt}} \langle Q_2, \kappa_2 \rangle$, then: $\langle Q_1, \kappa_1 \rangle \in \text{FPT} \iff \langle Q_2, \kappa_2 \rangle \in \text{FPT}$.
 $\langle Q_1, \kappa_1 \rangle \notin \text{FPT} \implies \langle Q_2, \kappa_2 \rangle \notin \text{FPT}$.

Comparing parameterizations (revisited)

Definition (computably bounded below)

Let $\kappa_1, \kappa_2 : \Sigma^* \rightarrow \mathbb{N}$ parameterizations.

- ▶ $\kappa_1 \succeq \kappa_2 : \iff \exists g : \mathbb{N} \rightarrow \mathbb{N} \text{ computable } \forall x \in \Sigma^* [g(\kappa_1(x)) \geq \kappa_2(x)]$.
- ▶ $\kappa_1 \approx \kappa_2 : \iff \kappa_1 \succeq \kappa_2 \wedge \kappa_2 \succeq \kappa_1$.
- ▶ $\kappa_1 \succ \kappa_2 : \iff \kappa_1 \succeq \kappa_2 \wedge \neg(\kappa_2 \succeq \kappa_1)$.

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with $\kappa_1 \succeq \kappa_2$:

$$\begin{aligned} \langle Q, \kappa_1 \rangle \in \text{FPT} &\iff \langle Q, \kappa_2 \rangle \in \text{FPT}, \\ \langle Q, \kappa_1 \rangle \notin \text{FPT} &\implies \langle Q, \kappa_2 \rangle \notin \text{FPT}. \end{aligned}$$

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with $Q \subseteq \Sigma^*$:

$$\kappa_1 \succeq \kappa_2 \iff \langle Q, \kappa_1 \rangle \leq_{\text{fpt}} \langle Q, \kappa_2 \rangle \text{ via } R : \Sigma^* \rightarrow \Sigma^*, x \mapsto x.$$

Fixed-parameter tractable reductions

Examples

- ▶ p -CLIQUE \equiv_{fpt} p -INDEPENDENT-SET.
- ▶ p -DOMINATING-SET \equiv_{fpt} p -HITTING-SET.

Non-Example

- ▶ For graphs $\mathcal{G} = \langle V, E \rangle$, and sets $X \subseteq V$:
 X is independent set of $\mathcal{G} \iff V \setminus X$ is a vertex cover of \mathcal{G}
 yields a **polynomial reduction** between p -INDEPENDENT-SET and p -VERTEX-COVER, but **does not yield an fpt-reduction**.

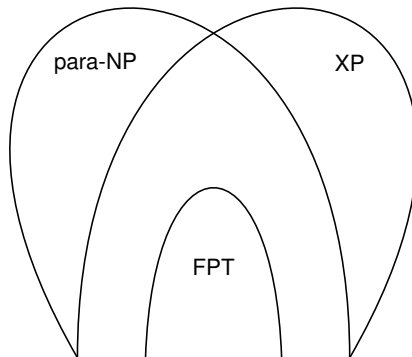
Fpt-reduction closure / hardness / reducibility

Let \mathbf{C} be a class of parameterized problems.

We define for all parameterized problems $\langle Q, \kappa \rangle$:

- ▶ $[\langle Q, \kappa \rangle]^{\text{fpt}} := \{ \langle Q', \kappa' \rangle \mid \langle Q', \kappa' \rangle \leq_{\text{fpt}} \langle Q, \kappa \rangle \}.$
- ▶ $[\mathbf{C}]^{\text{fpt}} := \bigcup_{\langle Q, \kappa \rangle \in \mathbf{C}} [\langle Q, \kappa \rangle]^{\text{fpt}}$
is the *closure of \mathbf{C} under fpt-reductions*.
- ▶ $\langle Q, \kappa \rangle$ is *\mathbf{C} -hard under fpt-reductions*
if every problem in \mathbf{C} is fpt-reducible to $\langle Q, \kappa \rangle$
that is: $\mathbf{C} \subseteq [\langle Q, \kappa \rangle]^{\text{fpt}}$, and hence $[\mathbf{C}]^{\text{fpt}} \subseteq [\langle Q, \kappa \rangle]^{\text{fpt}}.$
- ▶ $\langle Q, \kappa \rangle$ is *\mathbf{C} -complete under fpt-reductions*
if $\langle Q, \kappa \rangle \in \mathbf{C}$ and $\langle Q, \kappa \rangle$ is \mathbf{C} -hard under fpt-reductions,
and then: $[\mathbf{C}]^{\text{fpt}} = [\langle Q, \kappa \rangle]^{\text{fpt}}.$

para-NP and XP



para-NP

Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is in **para-NP** if there is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$, and a polynomial $p \in \mathbb{N}[X]$ such that there is a **non-deterministic** algorithm Δ such that:

- ▶ Δ decides, for all $x \in \Sigma^*$, whether $x \in Q$ in $\leq f(\kappa(x)) \cdot p(|x|)$ steps.
- ▶ para-NP is closed under fpt-reductions.
- ▶ $\text{NP} \subseteq \text{para-NP}$.

Example

- ▶ $p\text{-CLIQUE}$, $p\text{-INDEPENDENT-SET}$, $p\text{-DOMINATING-SET}$, $p\text{-HITTING-SET}$, $p\text{-COLORABILITY} \in \text{para-NP}$.
- ▶ $\text{FPT} = \text{para-NP}$ if and only if $\text{PTIME} = \text{NP}$.
- ▶ A non-trivial problem $\langle Q, \kappa \rangle$ is **para-NP-complete** for fpt-reductions if and only if the **union of finitely many slices of $\langle Q, \kappa \rangle$ is NP-complete**. Hence a non-trivial problem with at least one NP-complete slice is **para-NP-complete**.
 - ▶ $p\text{-COLORABILITY} \in \text{para-NP-complete}$.

XP (slicewise polynomial problems)

Recall: slices of FPT-problems are in PTIME. This suggests a class:

XP_{nu} , *non-uniform XP*: the class of parameterized problems $\langle Q, \kappa \rangle$, whose slices $\langle Q, \kappa \rangle_k$ are all in PTIME.

- ▶ **But:** XP_{nu} contains undecidable problems:
 - ▶ Let $Q \subseteq \{1\}^*$ be an undecidable set. Let $\kappa : \{1\}^* \rightarrow \mathbb{N}$, $x \mapsto \kappa(x) := \max \{1, |x|\}$. Then $\langle Q, \kappa \rangle \in \text{XP}_{\text{nu}}$.

Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is in **XP** if there is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that there is an algorithm \mathbb{A} such that:

- ▶ \mathbb{A} decides $x \in Q$, for all $x \in \Sigma^*$, in $\leq f(\kappa(x)) \cdot |x|^{f(\kappa(x))}$ steps;

equivalently, if in addition to computable $f : \mathbb{N} \rightarrow \mathbb{N}$ there are polynomials $p_k \in \mathbb{N}[X]$ for all $k \in \mathbb{N}$ such that:

- ▶ \mathbb{A} decides $x \in Q$, for all $x \in \Sigma^*$, in $\leq f(\kappa(x)) \cdot p_{\kappa(x)}(|x|)$ steps.

XP (slicewise polynomial problems)

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Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is in **XP** if there is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that there is an algorithm \mathbb{A} such that:

- ▶ \mathbb{A} decides $x \in Q$, for all $x \in \Sigma^*$, in $\leq f(\kappa(x)) + |x|^{f(\kappa(x))}$ steps;

equivalently, if in addition to computable $f : \mathbb{N} \rightarrow \mathbb{N}$ there are polynomials $p_k \in \mathbb{N}[X]$ for all $k \in \mathbb{N}$ such that:

- ▶ \mathbb{A} decides $x \in Q$, for all $x \in \Sigma^*$, in $\leq f(\kappa(x)) \cdot p_{\kappa(x)}(|x|)$ steps.

XP (slicewise polynomial problems)

Example

- ▶ p -CLIQUE, p -INDEPENDENT-SET, p -DOMINATING-SET, p -HITTING-SET \in XP.
- ▶ p -COLORABILITY \notin XP, because 3-COLORABILITY \in NP-complete.

Proposition

If $\text{PTIME} \neq \text{NP}$, then $\text{para-NP} \not\subseteq \text{XP}$.

Proof.

If $\text{para-NP} \subseteq \text{XP}$, then p -COLORABILITY \in XP. But then it follows that 3-COLORABILITY \in PTIME, and as 3-COLORABILITY is NP-complete, that $\text{PTIME} = \text{NP}$. □

Proposition

$\text{FPT} \subsetneq \text{XP}$.

Model checking

The *model checking problem* for a class Φ of first-order formulas:

$\text{MC}(\Phi)$

Instance: A structure \mathcal{A} and a formula $\varphi \in \Phi$.

Problem: Decide whether $\mathcal{A} \models \varphi$ (that is, $\varphi(\mathcal{A}) \neq \emptyset$).

Theorem

$\text{MC}(\text{FO})$ can be solved in time $O(|\varphi| \cdot |\mathcal{A}|^w \cdot w)$, where w is the width of the input formula φ (max. no. of free variables in a subformula of φ).

The *parameterized model checking problem* for a class Φ of formulas:

$p\text{-MC}(\Phi)$.

Instance: A structure \mathcal{A} and a formula $\varphi \in \Phi$.

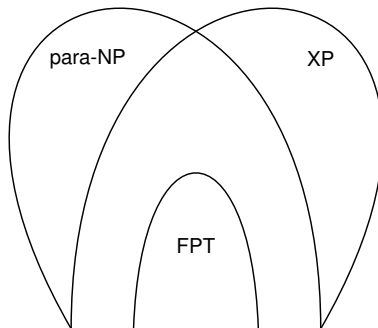
Parameter: $|\varphi|$.

Problem: Decide whether $\mathcal{A} \models \varphi$.

Theorem

$p\text{-MC}(\Phi) \in \text{XP}$.

FPT versus para-NP and XP



Proposition

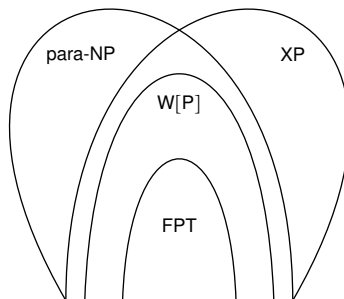
- ▶ $\text{FPT} \subseteq \text{para-NP}$, and:
 $\text{FPT} = \text{para-NP}$ if and only if $\text{PTIME} = \text{NP}$.
- ▶ $\text{para-NP} \not\subseteq \text{XP}$ if $\text{PTIME} \neq \text{NP}$.
- ▶ $\text{FPT} \subsetneq \text{XP}$.

W[P]

‘There is no definite single class that can be viewed as “the parameterized NP”. Rather, there is a whole hierarchy of classes playing this role.

*The class **W[P]** can be placed on top of this hierarchy. It is one of the most important parameterized complexity classes.’*

(Flum, Grohe [2])



W[P] and limited non-determinism

$\langle Q, \Sigma \rangle \in \text{NP}[f]$ means:

$\iff \exists p(X)$ polynomial $\exists \mathbb{M}$ non-deterministic Turingmachine

$(\forall x \in \Sigma^* ((x \in Q \iff \mathbb{M} \text{ accepts } x)$

\wedge on input x , \mathbb{M} halts in $\leq p(|x|)$ steps, of which

at most $\leq f(|x|)$ are non-deterministic)

$\text{NP}[\mathcal{F}] := \bigcup_{f \in \mathcal{F}} \text{NP}[f]$ for class of functions \mathcal{F} .

Fact

$$\text{NP}[\log n] = \text{P}, \quad \text{NP}[n^{O(1)}] = \text{NP}.$$

W[P]

Definition

- ▶ Let Σ be an alphabet, and $\kappa : \Sigma^* \rightarrow \mathbb{N}$ a parameterization.
 A nondeterministic Turing machine M with input alphabet Σ is κ -restricted if there are computable functions $f, h : \mathbb{N} \rightarrow \mathbb{N}$ and a polynomial $p \in \mathbb{N}_0[x]$ such that on every run with input $x \in \Sigma^*$ the machine M performs
 - ▷ at most $f(\kappa(x)) \cdot p(|x|)$ steps,
 - ▷ at most $h(\kappa(x)) \cdot \log |x|$ of them being nondeterministic,
- ▶ $W[P]$ contains all problems $\langle Q, \kappa \rangle$ that can be decided by a κ -restricted nondeterministic Turing machine.

W[P] (properties)

Theorems

- T1. $\text{FPT} \subseteq \text{W}[P] \subseteq \text{XP} \cap \text{para-NP}$
- T2. $\text{W}[P]$ is closed under fpt-reductions.
- T3. p -CLIQUE, p -INDEPENDENT-SET, p -DOMINATING-SET, and p -HITTING-SET are in $\text{W}[P]$.

The W-hierarchy – Boolean circuits

A (*Boolean*) *circuit* is a DAG in which nodes are labeled:

- ▶ nodes of in-degree > 1 as *and-node* or as *or-node*,
- ▶ nodes of in-degree $= 1$ as *negation nodes*,
- ▶ nodes of in-degree $= 0$ as *Boolean constants* 0 or 1, or *input node* (we assume input nodes to be numbered $1, \dots, n$),
- ▶ one node of out-degree 0 is labeled as *output node*.

A circuit \mathcal{C} with n input nodes defines a function $\mathcal{C}(\cdot) : \{0, 1\}^n \rightarrow \{0, 1\}$ (a *Boolean function*) in the natural way.

- ▶ If $\mathcal{C}(x) = 1$, for $x \in \{0, 1\}^n$, we say that x *satisfies* \mathcal{C} .
- ▶ The *weight* of a tuple $x = \langle x_1, \dots, x_n \rangle \in \{0, 1\}^*$ is $\sum_{i=1}^n x_i$.

Definition

We say that \mathcal{C} is *k-satisfiable* if \mathcal{C} is satisfied by a tuple of weight k .

W[P] complete problems

p -WSAT(CIRC)

Instance: A circuit \mathcal{C} and $k \in \mathbb{N}$

Parameter: k .

Problem: Decide whether \mathcal{C} is k -satisfiable.

Theorem

p -WSAT(CIRC) is W[P]-complete under fpt-reductions.

Definition

The **depth** of the circuit is the max. length of a path from an input node to the output node. **Small nodes** have indegree at most 2 while **large nodes** have indegree > 2 . The **weft** of a circuit is the max. number of **large nodes** on a path from an input node to the output node. We denote by $\text{CIRC}_{t,d}$ the class of circuits with weft $\leq t$ and depth $\leq d$.

Application

p -DOMINATING-SET \in W[P], since it reduces to p -WSAT(CIRC_{2,3}).

Limited non-determinism (classically)

$\langle Q, \Sigma \rangle \in \text{NP}[f]$ means:

$\iff \exists p(X)$ polynomial $\exists \mathbb{M}$ non-deterministic Turingmachine

$(\forall x \in \Sigma^* ((x \in Q \iff \mathbb{M} \text{ accepts } x)$

\wedge on input x , \mathbb{M} halts in $\leq p(|x|)$ steps, of which

at most $\leq f(|x|)$ are non-deterministic)

$\text{NP}[\mathcal{F}] := \bigcup_{f \in \mathcal{F}} \text{NP}[f]$ for class of functions \mathcal{F} .

Fact

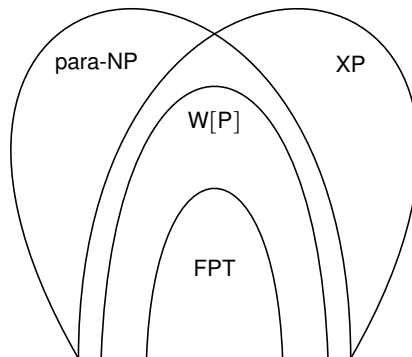
$$\text{NP}[\log n] = \text{P}, \quad \text{NP}[n^{O(1)}] = \text{NP}.$$

Theorem (Cai, Chen, 1997)

The following are equivalent:

- (i) $\text{FPT} = \text{W[P]}$.
- (ii) *There is a computable, nondecreasing, unbounded function $\iota : \mathbb{N} \rightarrow \mathbb{N}$ such that $\text{P} = \text{NP}[\iota(n) \cdot \log n]$.*

FPT and $W[P]$ versus para-NP and XP



Proposition

$FPT \subseteq W[P] \subseteq XP \cap \text{para-NP}$.

Why is the theory of W[P]/W/A-hardness important?

- ▶ Prevents from **wasting hours** tackling a problem which is **fundamentally difficult**;
- ▶ Finding results on a problem is always a **ping-pong game** between trying to design a hardness/FPT result;
- ▶ There is a **hierarchy on parameters** and it is worth knowing which is the smallest one such that the problem remains FPT;
- ▶ There is a **hierarchy on complexity classes** and it is worth noting to which extent a problem is hard.

Logic preliminaries (continued)

- ▶ *atomic formulas/atoms*: a formula $x = y$ or $Rx_1 \dots x_n$
- ▶ *literal*: an atom or a negated atom
- ▶ *quantifier-free formula*: a formula without quantifiers
- ▶ formula in *negation-normal form*:
negations only occur in front of atoms
- ▶ formula in *prenex normal form*: formula of the form
 $Q_1x_1 \dots Q_kx_k \psi$, where ψ is quantifier-free
and $Q_1, \dots, Q_k \in \{\exists, \forall\}$
- ▶ Σ_0 and Π_0 : the class of quantifier-free formulas
- ▶ Σ_{t+1} : class of all formulas $\exists x_1 \dots \exists x_k \varphi$ where $\varphi \in \Pi_t$
- ▶ Π_{t+1} : class of all formulas $\forall x_1 \dots \forall x_k \varphi$ where $\varphi \in \Sigma_t$

Weighted Fagin definability

Let $\varphi(X)$ be a f-o formula with a free relation variable X with arity s .
 Let τ be a vocabulary for φ , plus a relation symbol R of arity s .

A *solution for φ in a τ -structure \mathcal{A}* is a relation $S \subseteq A^s$ such that $\mathcal{A} \models \varphi(\overline{S})$.

The *weighted Fagin definability problem* for $\varphi(X)$ is:

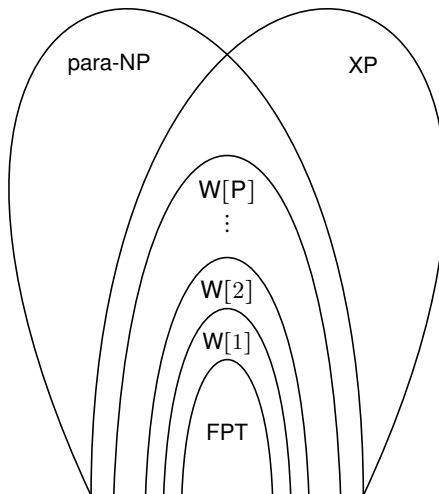
WD_φ

Instance: A structure \mathcal{A} and $k \in \mathbb{N}$.

Problem: Decide whether there is a solution $S \subseteq A^s$ for φ
 of cardinality $|S| = k$.

WD_Φ : the class of all problems WD_φ with $\varphi \in \Phi$, where Φ is a class of first-order formulas with free relation variable X .

W-Hierarchy



W-Hierarchy

$p\text{-WD}_\varphi$ (φ a fo-formula with free relation variable X of arity s)

Instance: A structure \mathcal{A} and $k \in \mathbb{N}$.

Parameter: k .

Problem: Is there a relation $S \subseteq A^s$ of cardinality $|S| = k$ with $\mathcal{A} \models \varphi(S)$.

$p\text{-WD-}\Phi$: the class of all problems $p\text{-WD-}\varphi$ with $\varphi \in \Phi$, Φ is a class of first-order formulas.

Definition (Downey–Fellows, 1995)

$W[t] := [p\text{-WD-}\Pi_t]^\text{fpt}$, for $t \geq 1$, form the *W-hierarchy*.

Examples

- ▶ $p\text{-CLIQUE} \in W[1]$.
- ▶ $p\text{-DOMINATING-SET} \in W[2]$.
- ▶ $p\text{-HITTING-SET} \in W[2]$.

W-Hierarchy

$p\text{-WD}_\varphi$ (φ a fo-formula with free relation variable X of arity s)

Instance: A structure \mathcal{A} and $k \in \mathbb{N}$.

Parameter: k .

Problem: Is there a relation $S \subseteq A^s$ of cardinality $|S| = k$ with $\mathcal{A} \models \varphi(S)$.

$p\text{-WD-}\Phi$: the class of all problems $p\text{-WD-}\varphi$ with $\varphi \in \Phi$, Φ is a class of first-order formulas.

Definition (Downey–Fellows, 1995)

$W[t] := [p\text{-WD-}\Pi_t]^\text{fpt}$, for $t \geq 1$, form the *W-hierarchy*.

Examples

- ▶ $p\text{-CLIQUE} \in W[1]$.
- ▶ $p\text{-DOMINATING-SET} \in W[2]$.
- ▶ $p\text{-HITTING-SET} \in W[2]$.

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W-hierarchy

Definition

(**W-hierarchy**) For $t \geq 1$, a parameterized problem $\langle Q, \kappa \rangle$ **belongs to the class $W[t]$** if there is a parameterized reduction from $\langle Q, \kappa \rangle$ to $p\text{-WSAT}(\text{CIRC}_{t,d})$ (with parameter t) for some $d \geq 1$.

$$\text{FPT} \subseteq W[1] \subseteq W[2] \dots$$

- ▶ $p\text{-CLIQUE}$, $p\text{-INDEPENDENT-SET}$ are $W[1]$ -Complete.
- ▶ $p\text{-DOMINATING-SET}$, $p\text{-HITTING-SET}$ are $W[2]$ -Complete.

Hypothesis: $W[1] \neq \text{FPT}$

Proposition

This definition of the W-hierarchy is equivalent to the one here before. That is, it holds, for all $t \geq 1$:

$$W[t] = \left[\{p\text{-WSAT}(\text{CIRC}_{t,d}) \mid d \geq 1\} \right]^{\text{fpt}}.$$

W-Hierarchy (properties)

Immediate from definition follows: $[p\text{-WD-FO}]^{\text{fpt}} = \bigcup_{i=1}^{\infty} W[i]$.

Theorems

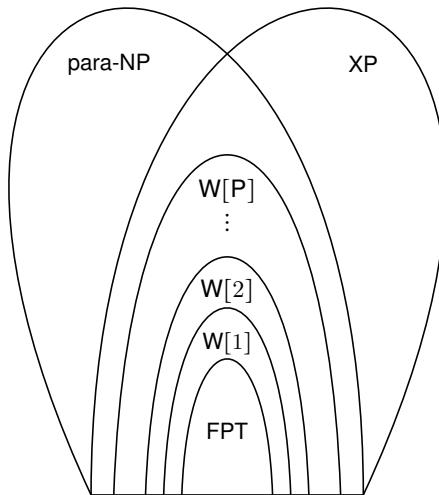
T1. $p\text{-WD-FO} \subseteq W[P]$, and hence $W[t] \subseteq W[P]$ for all $t \geq 1$.

T2. $p\text{-WD-}\Sigma_1 \subseteq \text{FPT}$.

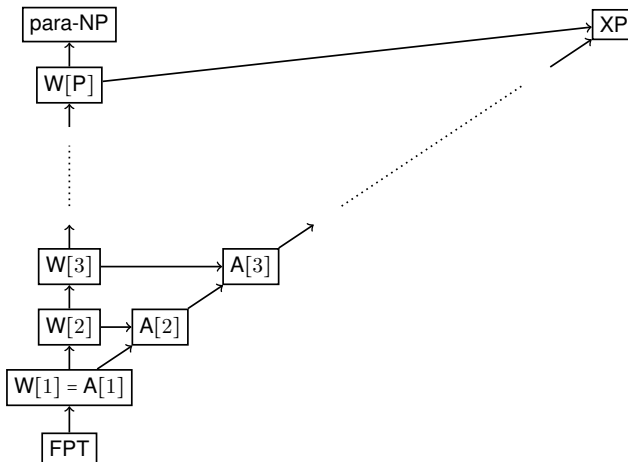
T3. $p\text{-WD-}\Sigma_{t+1} \subseteq p\text{-WD-}\Pi_t$, for all $t \geq 1$.

T4. $W[t] = [p\text{-WD-}\Sigma_{t+1}]^{\text{fpt}}$ for all $t \geq 1$.

W-Hierarchy versus para-NP and XP



A-Hierarchy



A-Hierarchy (definition and examples 1,2)

The parameterized model checking problem for a class Φ of formulas:

$p\text{-MC}(\Phi)$

Instance: A structure \mathcal{A} and a formula $\varphi \in \Phi$.

Parameter: $|\varphi|$.

Problem: Decide whether $\varphi(\mathcal{A}) \neq \emptyset$.

Definition (Flum, Grohe, 2001)

$A[t] := [p\text{-MC}(\Sigma_t)]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

- ▶ $p\text{-CLIQUE} \in A[1]$.
- ▶ $p\text{-DOMINATING-SET} \in A[2]$.

A-Hierarchy (definition and examples 3,4)

The parameterized model checking problem for a class Φ of formulas:

$p\text{-MC}(\Phi)$

Instance: A structure \mathcal{A} and a formula $\varphi \in \Phi$.

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Problem: Decide whether $\varphi(\mathcal{A}) \neq \emptyset$.

Definition (Flum, Grohe, 2001)

$A[t] := [p\text{-MC}(\Sigma_t)]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

- ▶ $p\text{-HITTING-SET} \in A[2]$.
- ▶ $p\text{-SUBGRAPH-ISOMORPHISM} \in A[1]$.

A-Hierarchy (example 5)

The parameterized model checking problem for a class Φ of formulas:

$p\text{-MC}(\Phi)$

Instance: A structure \mathcal{A} and a formula $\varphi \in \Phi$.

Parameter: $|\varphi|$.

Problem: Decide whether $\varphi(\mathcal{A}) \neq \emptyset$.

Definition (Flum, Grohe, 2001)

$A[t] := [p\text{-MC}(\Sigma_t)]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

► $p\text{-SUBGRAPH-ISOMORPHISM} \in A[1]$.

$p\text{-SUBGRAPH-ISOMORPHISM}$

Instance: Graphs \mathcal{G} and \mathcal{H} .

Parameter: The number of vertices of \mathcal{H} .

Problem: Does \mathcal{G} have a subgraph isomorphic to \mathcal{H} .

A-Hierarchy (example 6)

The parameterized model checking problem for a class Φ of formulas:

$p\text{-MC}(\Phi)$

Instance: A structure \mathcal{A} and a formula $\varphi \in \Phi$.

Parameter: $|\varphi|$.

Problem: Decide whether $\varphi(\mathcal{A}) \neq \emptyset$.

Definition (Flum, Grohe, 2001)

$A[t] := [p\text{-MC}(\Sigma_t)]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

► $p\text{-VERTEX-DELETION} \in A[2]$.

$p\text{-VERTEX-DELETION}$

Instance: Graphs \mathcal{G} and \mathcal{H} , and $k \in \mathbb{N}$.

Parameter: $k + \ell$, where ℓ the number of vertices of \mathcal{H} .

Problem: Is it possible to delete at most k vertices from \mathcal{G} such that the resulting graph has no subgraph isomorphic to \mathcal{H} ?

A-Hierarchy (example 7)

The parameterized model checking problem for a class Φ of formulas:

$p\text{-MC}(\Phi)$

Instance: A structure \mathcal{A} and a formula $\varphi \in \Phi$.

Parameter: $|\varphi|$.

Problem: Decide whether $\varphi(\mathcal{A}) \neq \emptyset$.

Definition (Flum, Grohe, 2001)

$A[t] := [p\text{-MC}(\Sigma_t)]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

► $p\text{-CLIQUE-DOMINATING-SET} \in A[2]$.

$p\text{-CLIQUE-DOMINATING-SET}$

Instance: Graphs \mathcal{G} , and $k, \ell \in \mathbb{N}$.

Parameter: $k + \ell$, where ℓ the number of vertices of \mathcal{H} .

Problem: Decide whether \mathcal{G} contains a set of k vertices from \mathcal{G} that dominates every clique of ℓ elements.

A-Hierarchy (properties)

Theorems

T1. $A[1] \subseteq W[P]$.

T2. $W[t] \subseteq A[t]$, for all $t \in \mathbb{N}$.

- **Unlikely:** $A[t] \subseteq W[t]$, for $t > 1$.

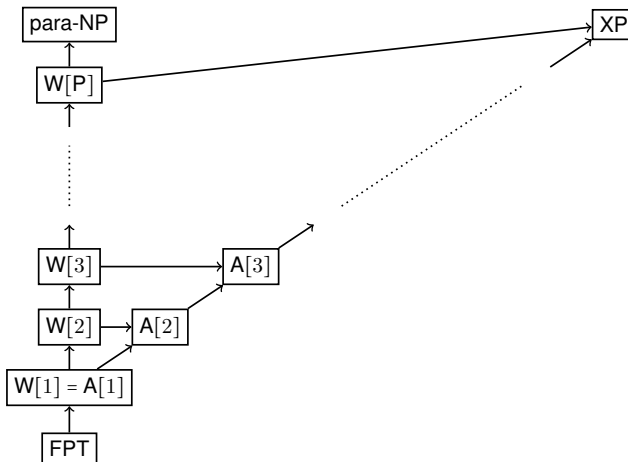
Reason:

- the A-hierarchy are parameterizations of problems that are complete for the levels of the polynomial hierarchy
- the W-hierarchy is a refinement of NP in parameterized complexity

- **Unlikely:** $[p\text{-MC(FO)}]^{\text{fpt}} = \bigcup_{i=1}^{\infty} A[i]$,

contrasting with: $[p\text{-WD-FO}]^{\text{fpt}} = \bigcup_{i=1}^{\infty} W[i]$.

W-Hierarchy and A-Hierarchy versus para-NP and XP



Revisiting the two problems at start today

QUERIES

Instance: a relational database D , a conjunctive query α .

Parameter: size $k = |\alpha|$ of query α

Compute: answer to query α from database D .

- ▶ QUERIES \in NP-complete.
- ▶ QUERIES $\in O(n^k)$ for $n = \|D\|$, which does **not** give an **FPT** result.
- ▶ QUERIES \in **W[1]** (= strong evidence for it **likely not to be** in **FPT**).

LTL-MODEL-CHECKING

Instance: a Kripke structure (state space) \mathcal{K} , an LTL formula φ

Parameter: size $k = |\varphi|$ of formula φ

Question: Does $\mathcal{K} \models \varphi$ hold?

- ▶ LTL-MODEL-CHECKING \in PSPACE-complete,
- ▶ LTL-MODEL-CHECKING $\in O(k \cdot 2^{2k} \cdot n) \in$ **FPT** for $n = \|\mathcal{K}\|$.

Summary

- ▶ Motivation for fixed-parameter intractability
- ▶ Fixed parameter reductions
- ▶ The classes **para-NP** and **XP**
- ▶ The class **W[P]**
- ▶ Logic preliminaries (continued)
- ▶ **W-hierarchy**
 - ▶ definitions
 - ▶ with Boolean circuits
 - ▶ as parameterized weighted Fagin definability problems
- ▶ **A-hierarchy**
 - ▶ definition as parameterized model-checking problems
- ▶ picture overview of these classes

Course overview

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
<i>Algorithmic Techniques</i>		<i>Formal-Method & Algorithmic Techniques</i>		
Introduction & basic FPT results motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	Notions of bounded graph width path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	Algorithmic Meta-Theorems 1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	FPT-Intractability Classes & Hierarchies motivation for FP-intractability results, FPT-reductions, class XP (slice-wise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour

Example suggestions

Examples

1. **FPT** results transfer backwards over fpt-reductions:
 If $\langle Q_1, \kappa_1 \rangle \leq_{\text{fpt}} \langle Q_2, \kappa_2 \rangle$, then $Q_2 \in \text{FPT}$ implies $Q_1 \in \text{FPT}$.
2. Find the idea for:
 $p\text{-DOMINATING-SET} \equiv_{\text{fpt}} p\text{-HITTING-SET}$.
- 3.

References



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