Lecture 1: Introduction to Computability

Models of Computation

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Ph.D. Program, Advanced Courses Period
Gran Sasso Science Institute
L'Aquila, Italy

July 7, 2025

Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

Overview

Q's where the A's (may) depend on computation

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A: Yes, if the truth table for ϕ contains (in the row for ϕ) only "T"; no otherwise.

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The decision problem for A in E is solvable (the set A in E is (effectively) calculable) if there exists a decision method for A in E.

(Comput.) What-questions / Computation Problems

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Computing the greatest common divisor
Instance: a pair $\langle a, b \rangle$ of numbers $a, b \in \mathbb{N}$ with $a, b > 0$.
Question: What is $\mathrm{gcd}(a, b)$, the greatest common divisor of a and b?

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Suppose $F : A \to B$ is a mapping, where the elements of A, B are finitely describable objects.

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A mapping F is calculable if there exists a computation method for F.

Representing function

Let $P(a_1, \ldots, a_n)$ be an n-ary number-theoretic predicate.

The representing function f of P:

$$f(a_1, \ldots, a_n) := \begin{cases} 1 & \ldots P(a_1, \ldots, a_n) \text{ is true} \\ 0 & \ldots P(a_1, \ldots, a_n) \text{ is false} \end{cases}$$

Hence:
A decision procedure can be handled as a computation procedure f by taking '0' for 'yes', and '1' for 'no'.

Decision/Computation methods

What is a decision method / computation method?

– A mechanical, algorithmic procedure that:

- can be carried out by a machine (ideal, not limited by resource problems, mechanical breakdown, etc.).
- for computing a function F on an argument a, a is placed on the input device of the machine, which then produces $F(a)$ after finitely many steps.
- for computing a function F, the machine has to be independent of the arguments.

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Due to $3 \mid 12$ and ($\star$) we conclude:

A: Yes. (Infinitely many solutions, e.g. $x = 4$ and $y = -8$.)

Not effectively calculable

Examples (Shoenfield)

- methods that involve chance procedures: tossing a coin
- methods involving magic: asking a fortune teller
- methods that require (unformalised, unmechanised) insight

Effectively calculable?

Example

Hilbert's 10th Problem
Instance: An equation $p(x_1, \ldots, x_n) = 0$, where
p a polynomial with integer coefficients.
Question: Is the equation solvable for $x_1, \ldots, x_n \in \mathbb{Z}$?

Instances based on quadratic polynomials are of the form
$ax^2 + bxy + cy^2 + dx + ey + f = 0$ with $a, b, c, d, e, f \in \mathbb{Z}$.

Effectively calculable? – No!

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Theorem (Matijasevic, 1970)

Hilbert's 10^{th} Problem is unsolvable.

David Hilbert (1862–1943)

Problem (Entscheidungsproblem, 1928)

Is there a method for deciding, given a formula ϕ of the predicate calculus, whether or not ϕ is a tautology?

Timeline: From logic to computability

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1937	Post: machine model; Church's thesis as 'working hypothesis' Turing: convincing analysis of a 'human computer' leading to the 'Turing machine'

Calculable functions?

Questions/Exercises

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2. Suppose $P(a,b)$ is a calculable predicate.
   Why does $(\exists x)P(a,x)$ not have to be calculable?
3. Let $f : \mathbb{N} \to \mathbb{N}$ defined by

$$n \longmapsto \begin{cases} 0 & \ldots n = 0 \;\&\; \text{Goldbach's conjecture is false} \\ 1 & \ldots n = 0 \;\&\; \text{Goldbach's conjecture is true} \\ n+1 & \ldots n > 0 \end{cases}$$

   Is f calculable?

Some Models of Computation

<table>
<tr><th>machine model</th><th>mathematical model</th><th>sort</th></tr>
<tr><td>Turing machine
Post machine
register machine</td><td>Combinatory Logic
λ-calculus
Herbrand–Gödel recursive functions
partial-recursive/μ-recursive functions
Post canonical system (tag system)
Post's Correspondence Problem
Markov algorithms
Lindenmayer systems</td><td>classical</td></tr>
<tr><td></td><td>Fractran</td><td>less well known</td></tr>
<tr><td>cellular automata
neural networks</td><td>term rewrite systems
interaction nets
logic-based models of computation
concurrency and process algebra
ς-calculus
evolutionary programming/genetic algorithms</td><td rowspan="2">modern</td></tr>
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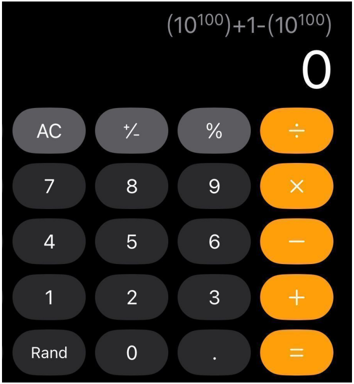

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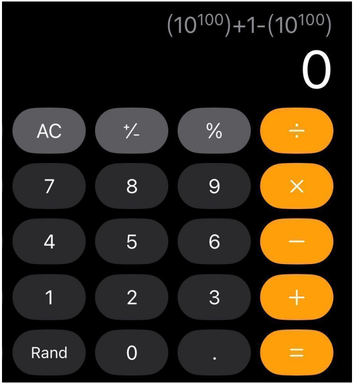

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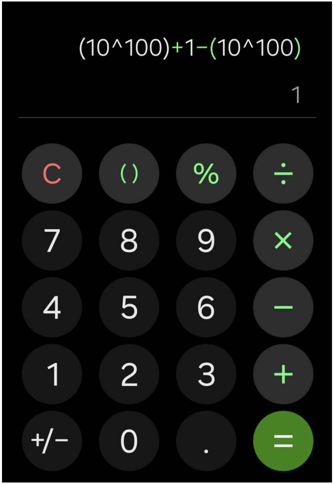
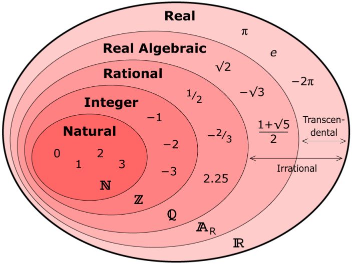


Android

Calculator (2/5): constructive real numbers



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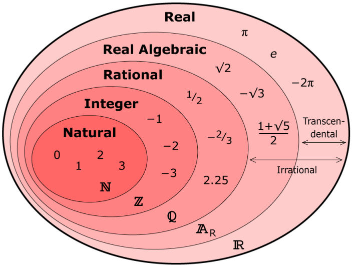

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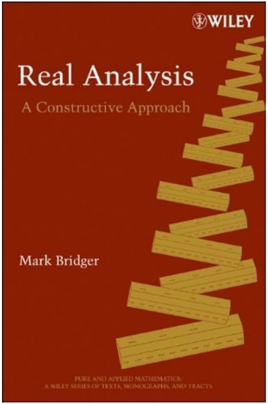

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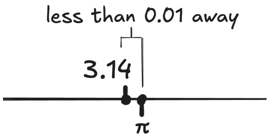

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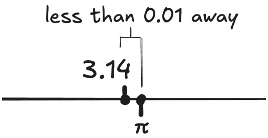

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A real number $x \in \mathbb{R}$ is constructive if:

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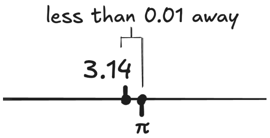

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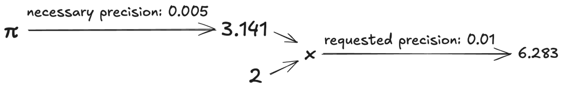

approximating 2π within 0.01

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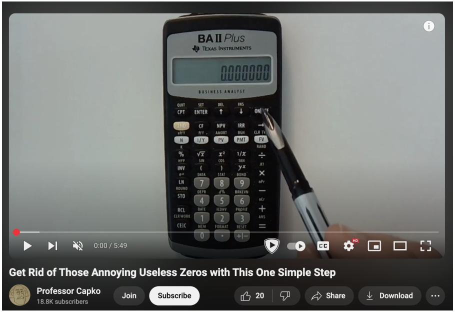

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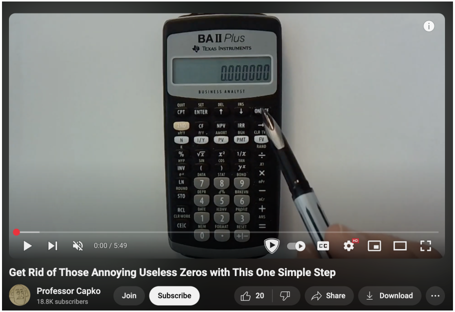

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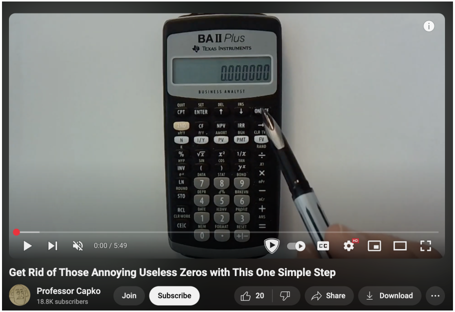

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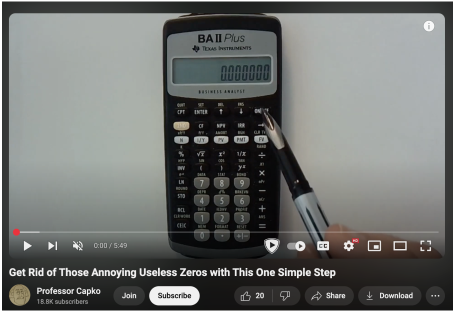

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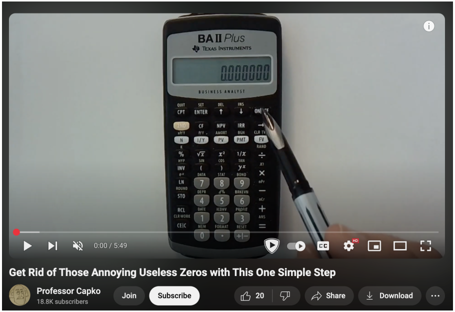

Undecidable problem

Article Talk

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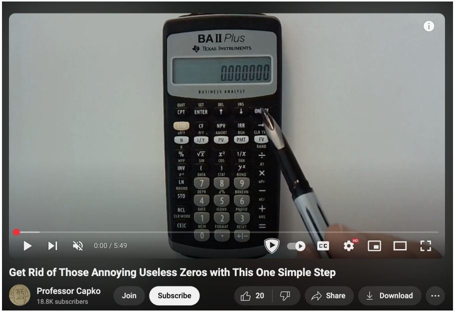

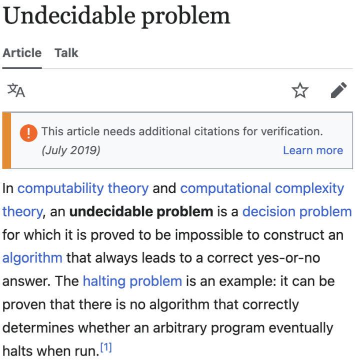

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- Therefore $x - y = 0$ can not always be decided.

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- Credits: tech-blogger Chad Nauseam (link) for post *"A calculator app? Anyone could make that."* (link) [2].

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- study the operational and denotational aspects of MoC's like λ-calculus, CL, string rewriting, term rewriting, interaction nets in a systematic way

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Linguistics

- e.g. formal calculi for discovering the structure of human languages related to subclasses in the Chomsky hierarchy

Recommended reading

1. Post machine: Page 1 + first paragraph on page 2 of:
   - Emil Post: *Finite Combinatory Processes – Formulation 1*, Journal of Symbolic Logic (1936), [3], https://www.wolframscience.com/prizes/tm23/images/Post.pdf.

2. Turing machine motivation: Turing's analysis of a human computer:
   Part I of Section 9, pp. 249–252 of:
   - Alan M. Turing's: *On computable numbers, with an application to the Entscheidungsproblem*', Proceedings of the London Mathematical Society (1936), [4], http://www.wolframscience.com/prizes/tm23/images/Turing.pdf.

Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

References I

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- Chad Nauseam.
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  https://chadnauseam.com/coding/random/calculator-app, 2025.
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  Finite Combinatory Processes – Formulation 1.
  Journal of Symbolic Logic, 1(3):103–105, 1936.
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Proceedings of the London Mathematical Society, 42(2):230–265, 1936.
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