Lecture 1: Introduction to Computability

Models of Computation

https://clegra.github.io/moc/moc.html

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L'Aquila, Italy

July 7, 2025

Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
intro		*classic models*		*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

Overview

- What is computation?
  - questions where the answer may depend on computation
  - algorithm examples
  - unsolvable problems

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- references

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A: Yes, if the truth table for ϕ contains (in the row for ϕ) only "T"; no otherwise.

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A decision method for A in E is a method by which, given an element $a \in E$, we can decide in a finite number of steps whether or not $a \in A$.

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The decision problem for A in E is solvable (the set A in E is (effectively) calculable) if there exists a decision method for A in E.

(Comput.) What-questions / Computation Problems

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Computing the greatest common divisor
Instance: a pair $\langle a, b \rangle$ of numbers $a, b \in \mathbb{N}$ with $a, b > 0$.
Question: What is $\gcd(a, b)$, the greatest common divisor of a and b?

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Suppose $F : A \to B$ is a mapping, where the elements of A, B are finitely describable objects.

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A mapping F is calculable if there exists a computation method for F.

Representing function

Let $P(a_1, \ldots, a_n)$ be an n-ary number-theoretic predicate.

The representing function f of P:

$$f(a_1, \ldots, a_n) := \begin{cases} 1 & \ldots P(a_1, \ldots, a_n) \text{ is true} \\ 0 & \ldots P(a_1, \ldots, a_n) \text{ is false} \end{cases}$$

Hence:
A decision procedure can be handled as a computation procedure f by taking '0' for 'yes', and '1' for 'no'.

Decision/Computation methods

What is a decision method / computation method?

– A mechanical, algorithmic procedure that:

- can be carried out by a machine (ideal, not limited by resource problems, mechanical breakdown, etc.).
- for computing a function F on an argument a, a is placed on the input device of the machine, which then produces $F(a)$ after finitely many steps.
- for computing a function F, the machine has to be independent of the arguments.

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Due to $3 \mid 12$ and ($\star$) we conclude:

A: Yes. (Infinitely many solutions, e.g. $x = 4$ and $y = -8$.)

Not effectively calculable

Examples (Shoenfield)

- methods that involve chance procedures: tossing a coin
- methods involving magic: asking a fortune teller
- methods that require (unformalised, unmechanised) insight

Effectively calculable?

Example

Hilbert's 10th Problem

Instance: An equation $p(x_1, \ldots, x_n) = 0$, where
p a polynomial with integer coefficients.

Question: Is the equation solvable for $x_1, \ldots, x_n \in \mathbb{Z}$?

Instances based on quadratic polynomials are of the form
$ax^2 + bxy + cy^2 + dx + ey + f = 0$ with $a, b, c, d, e, f \in \mathbb{Z}$.

Effectively calculable? – No!

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Theorem (Matijasevic, 1970)

Hilbert's 10th Problem is unsolvable.

David Hilbert (1862–1943)

Problem (Entscheidungsproblem, 1928)

Is there a method for deciding, given a formula ϕ of the predicate calculus, whether or not ϕ is a tautology?

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1937	Post: machine model; Church's thesis as 'working hypothesis' Turing: convincing analysis of a 'human computer' leading to the 'Turing machine'

Calculable functions?

Questions/Exercises

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2. Suppose $P(a,b)$ is a calculable predicate.
   Why does $(\exists x)P(a,x)$ not have to be calculable?
3. Let $f : \mathbb{N} \to \mathbb{N}$ defined by

$$n \longmapsto \begin{cases} 0 & \ldots n = 0 \;\&\; \text{Goldbach's conjecture is false} \\ 1 & \ldots n = 0 \;\&\; \text{Goldbach's conjecture is true} \\ n+1 & \ldots n > 0 \end{cases}$$

   Is f calculable?

Some Models of Computation

machine model	mathematical model	sort
Turing machine Post machine register machine	Combinatory Logic λ-calculus Herbrand–Gödel recursive functions partial-recursive/μ-recursive functions Post canonical system (tag system) Post's Correspondence Problem Markov algorithms Lindenmayer systems	*classical*
	Fractran	*less well known*
cellular automata neural networks	term rewrite systems interaction nets logic-based models of computation concurrency and process algebra ς-calculus evolutionary programming/genetic algorithms	*modern*
abstract state machines		*modern*
hypercomputation		*speculative*
quantum computing bio-computing reversible computing		*physics-/biology-inspired*

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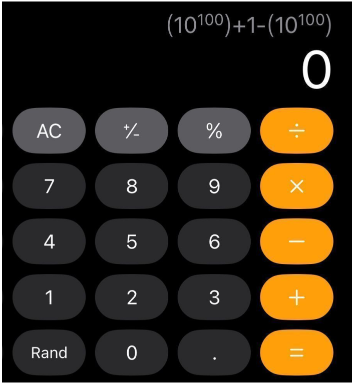

iOS

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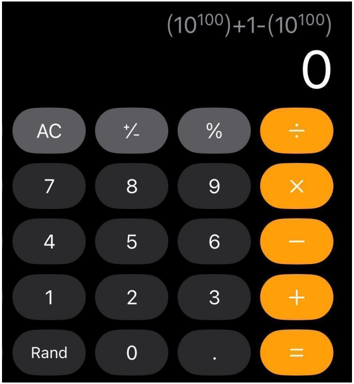

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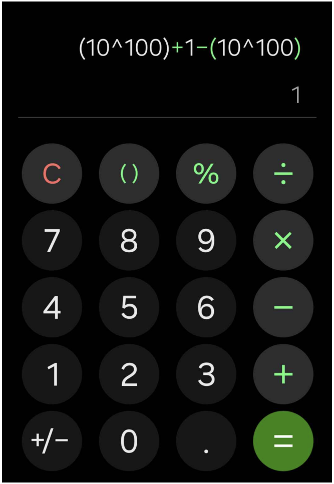
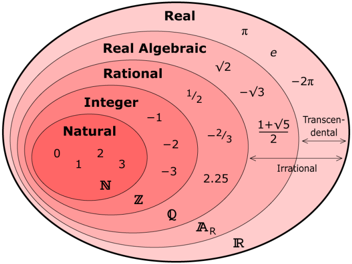


Android

Calculator (2/5): constructive real numbers



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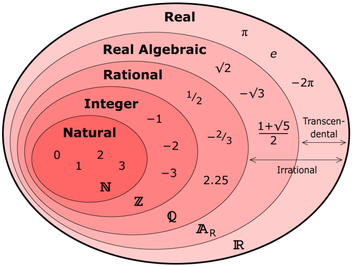

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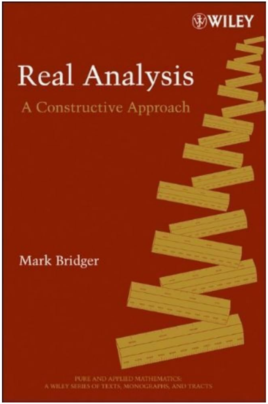

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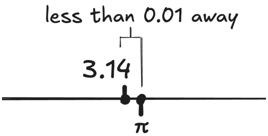

approximating π within 0.01

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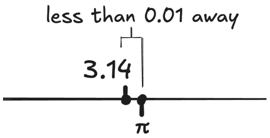

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- there exists a program P_x that for every bound $0 < \delta \in \mathbb{Q}$ returns a rational approximation $P_x(\delta) \in \mathbb{Q}$ of x with $|x - P_x(\delta)| < \delta$.

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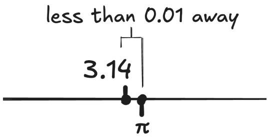

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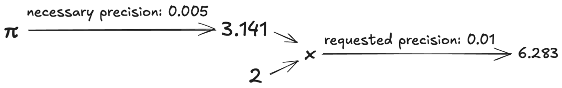

approximating 2π within 0.01

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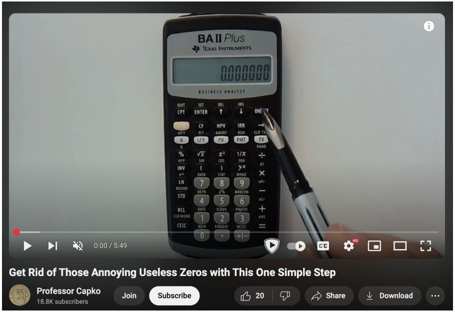

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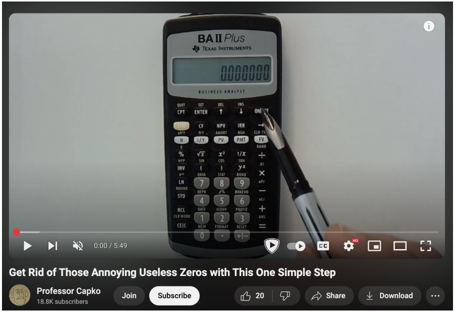

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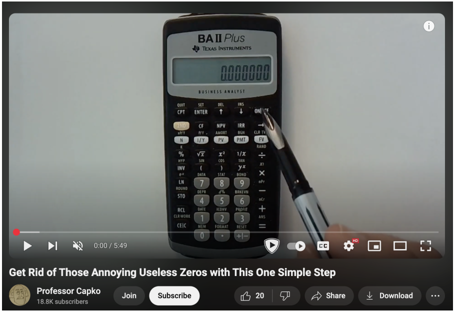

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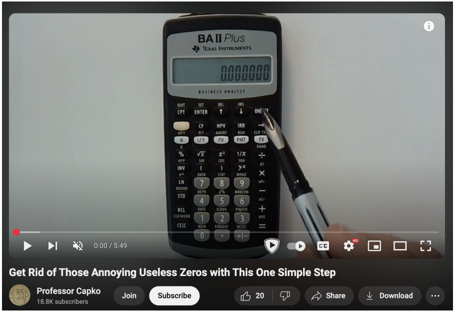

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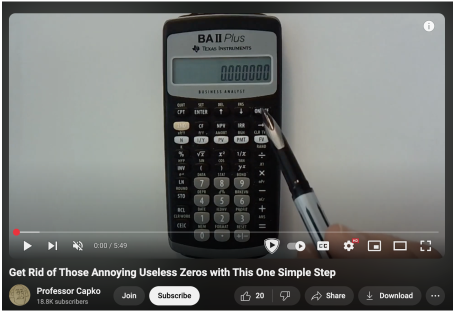

Undecidable problem

Article Talk

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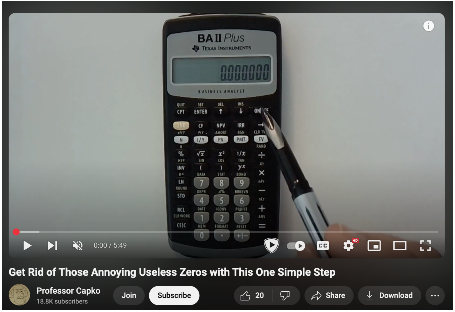

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- Credits: tech-blogger Chad Nauseam (link) for post *"A calculator app? Anyone could make that."* (link) [2].

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Rewriting

- study the operational and denotational aspects of MoC's like λ-calculus, CL, string rewriting, term rewriting, interaction nets in a systematic way

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Linguistics

- e.g. formal calculi for discovering the structure of human languages related to subclasses in the Chomsky hierarchy

Recommended reading

1. Post machine: Page 1 + first paragraph on page 2 of:
   - Emil Post: *Finite Combinatory Processes – Formulation 1*, Journal of Symbolic Logic (1936), [3], https://www.wolframscience.com/prizes/tm23/images/Post.pdf.

2. Turing machine motivation: Turing's analysis of a human computer:
   Part I of Section 9, pp. 249–252 of:
   - Alan M. Turing's: *On computable numbers, with an application to the Entscheidungsproblem*', Proceedings of the London Mathematical Society (1936), [4], http://www.wolframscience.com/prizes/tm23/images/Turing.pdf.

Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

References I

- Maribel Fernández.
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  Springer, Dordrecht Heidelberg London New York, 2009.

- Chad Nauseam.
  A calculator app? Anyone could make that.".
  https://chadnauseam.com/coding/random/calculator-app, 2025.
  Accessed: 29 June 2025.

- Emil Leon Post.
  Finite Combinatory Processes – Formulation 1.
  Journal of Symbolic Logic, 1(3):103–105, 1936.
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Alan M. Turing.
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