

# Lecture 4: Lambda Calculus

## Models of Computation

<https://clegra.github.io/moc/moc.html>

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# Course overview

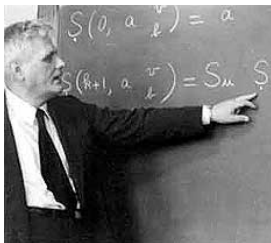
Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
<i>intro</i>	<i>classic models</i>			<i>additional models</i>
<b>Introduction to Computability</b>	<b>Machine Models</b>	<b>Recursive Functions</b>	<b>Lambda Calculus</b>	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = Turing-computable, Church's Thesis	$\lambda$ -terms, $\beta$ -reduction, $\lambda$ -definable functions, partial recursive = $\lambda$ -definable = Turing computable	
	<i>imperative programming</i>	<i>algebraic programming</i>	<i>functional programming</i>	
				14.30 – 16.30
				<b>Three more Models of Computation</b>
				Post's Correspondence Problem, Interaction-Nets, Fractran
				comparing computational power

# Today

## Lambda calculus

- ▶  $\lambda$ -calculus
  - ▶ syntax
  - ▶ reduction rules
- ▶  $\lambda$ -definable functions
- ▶ primitive recursive functions are  $\lambda$ -definable
- ▶  $\mu$ -recursive/partial recursive functions are  $\lambda$ -definable
- ▶  $\lambda$ -definable functions are Turing computable
- ▶ **Hence:  $\lambda$ -definable = partial recursive = Turing-computable**

# Church's Thesis



Alonzo Church (1903 –1995)

## Thesis (Church, 1936)

- ▶ *Every total effectively calculable function is recursive.*
- ▶ *Every effectively calculable partial function is partial-recursive.*

# $\lambda$ -terms

## Definition

- ▶ **variables:**  $x, y, z, x_1, y_1, z_1, \dots \in \Lambda$
- ▶  **$\lambda$ -abstraction:**  $x$  a variable,  $M \in \Lambda \implies (\lambda x.M \in \Lambda)$
- ▶ **application:**  $M, N \in \Lambda \implies (MN) \in \Lambda$

# $\beta$ -reduction

## Definition

- **One-step  $\beta$ -reduction**  $\rightarrow_\beta$  is defined as the application of the rule:

$$(\lambda x.M)N \rightarrow_\beta M[x := N]$$

in  $\lambda$ -terms  $C[(\lambda x.M)N]$  formed by arbitrary  $\lambda$ -term contexts  $C[]$ , where  $(\lambda x.M)N$  is called a **redex**, and furthermore:

$M[x := N]$  := substitution of  $N$  for free occurrences of  $x$  in  $M$   
(using  **$\alpha$ -conversion** to avoid variable capture)

- **Many-step  $\beta$ -reduction**  $\rightarrow_\beta^*$  is defined as the concatenation of zero, one, or more  $\rightarrow_\beta$ -steps.
- A  $\lambda$ -term  $M$  is a **normal form** if it does not contain a redex.

# Church numerals

## Definition

For every  $n \in \mathbb{N}$ , the **Church numeral**  $\ulcorner n \urcorner$  for  $n$  is defined by:

$$\begin{aligned}\ulcorner n \urcorner &:= \lambda f x. f^n x \\ &= \lambda f x. f(\underbrace{f(\dots (f x) \dots)}_n)\end{aligned}$$

## Examples.

$$\ulcorner 0 \urcorner = \lambda f x. x$$

$$\ulcorner 1 \urcorner = \lambda f x. f x$$

$$\ulcorner 2 \urcorner = \lambda f x. f(f x)$$

...



# Turing-computable (total) functions

## Definition

A **total function**  $f : \mathbb{N}^k \rightarrow \mathbb{N}$  is **Turing-computable** if there exists a Turing machine  $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \blacksquare, F \rangle$  and a **calculable** coding function  $\langle \cdot \rangle : \mathbb{N} \rightarrow \Sigma^*$  such that:

- ▶ for all  $n_1, \dots, n_k \in \mathbb{N}$  there exists  $q \in F$  such that:

$$q_0 \langle n_1 \rangle \blacksquare \langle n_2 \rangle \blacksquare \dots \blacksquare \langle n_k \rangle \vdash_M^* q \langle f(n_1, \dots, n_k) \rangle$$



# $\lambda$ -definable functions

## Definition

- Let  $f : \mathbb{N}^n \rightarrow \mathbb{N}$  be total.

A  $\lambda$ -term  $M_f$  **represents**  $f$  if for all  $m_1, \dots, m_n \in \mathbb{N}$ :

$$M_f \ulcorner m_1 \urcorner \dots \ulcorner m_n \urcorner \rightarrow_{\beta}^* \ulcorner f(m_1, \dots, m_n) \urcorner$$

$f$  is  **$\lambda$ -definable** if there exists a  $\lambda$ -term that represents  $f$ .

- Let  $f : \mathbb{N}^n \rightarrow \mathbb{N}$  be a partial function.

A  $\lambda$ -term  $M_f$  **represents**  $f$  if for all  $m_1, \dots, m_n \in \mathbb{N}$ :

$$f(m_1, \dots, m_n) \downarrow \implies M_f \ulcorner m_1 \urcorner \dots \ulcorner m_n \urcorner \rightarrow_{\beta}^* \ulcorner f(m_1, \dots, m_n) \urcorner$$

$$f(m_1, \dots, m_n) \uparrow \implies M_f \ulcorner m_1 \urcorner \dots \ulcorner m_n \urcorner \text{ has no normal form}$$

$f$  is  **$\lambda$ -definable** if there exists a  $\lambda$ -term that represents  $f$ .

# $\lambda$ -definable

## Examples.

- ▶ **successor:**  $M_{\text{succ}} := \lambda n f x. f(n f x)$
- ▶ **addition:**  $M_+ := \lambda m n f x. m f(n f x)$
- ▶ **multiplication:**  $M_\times := \lambda m n f x. m(n f)x$
- ▶ **exponentiation:**  $M_E := \lambda m n f x. m n f x$
- ▶ **unary constant zero function:**  $M_{C_0^1} = \lambda m. \ulcorner 0 \urcorner$
- ▶ **projection function:**  $M_{\pi_i^k} = \lambda n_1 \dots n_k. n_i$

# Pairs in $\lambda$ -calculus

## Definition

For all  $M, N \in \Lambda$  we define the **pair**  $\langle M, N \rangle$  consisting of  $M$  and  $N$ :

$$\langle M, N \rangle := \lambda x. xMN$$

and the **unpairing projections**  $\rho_1$  and  $\rho_2$ :

$$\rho_1 := \lambda p. p(\lambda xy. x)$$

$$\rho_2 := \lambda p. p(\lambda xy. y)$$

## Proposition

For all  $M_1, M_2 \in \Lambda$  and  $i = 1, 2$ :

$$\rho_i \langle M_1, M_2 \rangle \rightarrow_{\beta}^* M_i$$

# True, false, if-then-else, **zero?** in $\lambda$ -calculus

## Definition

$$\mathbf{true} := \lambda xy.x$$
$$\mathbf{false} := \lambda xy.y$$
$$\mathbf{if } P \mathbf{ then } Q \mathbf{ else } R := PQR$$
$$\mathbf{zero?} := \lambda x.x(\lambda y.\mathbf{false})\mathbf{true}$$

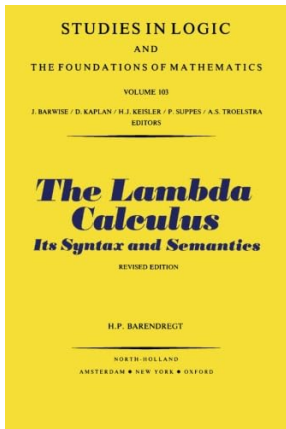
## Proposition

$$\mathbf{if } \mathbf{true} \mathbf{ then } Q \mathbf{ else } R \rightarrow_{\beta}^* Q$$
$$\mathbf{if } \mathbf{false} \mathbf{ then } Q \mathbf{ else } R \rightarrow_{\beta}^* R$$
$$\mathbf{zero?} \ulcorner 0 \urcorner \rightarrow_{\beta}^* \mathbf{true}$$
$$\mathbf{zero?} \ulcorner n + 1 \urcorner \rightarrow_{\beta}^* \mathbf{false}$$

# Typical features of 'computationally complete' MoC's

- ▶ storage (unbounded)
- ▶ control (finite, given)
- ▶ modification
  - ▶ of (immediately accessible) stored data
  - ▶ of control state
- ▶ conditionals
- ▶ loop
- ▶ stopping condition

# The Book



(reference [1])

Hendrik Pieter (Henk) Barendregt

# Exercises

- (1) Describe all possible ways to reduce  $(\lambda xy.x)((\lambda x.xx)(\lambda x.xx))$  to normal form.
- (2) Find two distinct  $\lambda$ -terms representing the successor function on Church-numerals (hint: think of  $n + 1$  and  $1 + n$ ). Prove that your  $\lambda$ -terms are not- $\beta$ -equivalent.
- (3) Try computing the normal form of the  $Y$ -combinator, i.e. of  $AA$  where  $A = \lambda am.m(aam)$ , e.g. by each time selecting the leftmost redex (reducible expression, i.e. subexpression of the shape  $(\lambda x.M)N$ ).

# Primitive recursive functions ( $\mathbb{N}^n \cup \mathbb{N}^0 \rightarrow \mathbb{N}$ )

## Base functions:

- ▶  $\mathcal{O} : \mathbb{N}^0 = \{\emptyset\} \rightarrow \mathbb{N}$ ,  $\emptyset \mapsto 0$  (0-ary constant-0 function)
- ▶  $\text{succ} : \mathbb{N} \rightarrow \mathbb{N}$ ,  $x \mapsto x + 1$  (successor function)
- ▶  $\pi_i^n : \mathbb{N}^n \rightarrow \mathbb{N}$ ,  $\vec{x} = \langle x_1, \dots, x_n \rangle \mapsto x_i$  (projection function)

## Closed under operations:

- ▶ **composition**: if  $f : \mathbb{N}^k \rightarrow \mathbb{N}$ , and  $g_i : \mathbb{N}^n \rightarrow \mathbb{N}$  are prim. rec., then so is  $h = f \circ (g_1 \times \dots \times g_k) : \mathbb{N}^n \rightarrow \mathbb{N}$ :  

$$h(\vec{x}) = f(g_1(\vec{x}), \dots, g_k(\vec{x}))$$
- ▶ **primitive recursion**: if  $f : \mathbb{N}^n \rightarrow \mathbb{N}$ ,  $g : \mathbb{N}^{n+2} \rightarrow \mathbb{N}$  are prim. rec., then so is  $h = \text{pr}(f; g) : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ :

$$h(\vec{x}, 0) = f(\vec{x})$$

$$h(\vec{x}, y + 1) = g(\vec{x}, h(\vec{x}, y), y)$$



# Primitive recursive functions are $\lambda$ -definable

## Proposition

*Every primitive recursive function is  $\lambda$ -definable.*

**Proof** (The case of primitive recursion).

Let  $h := \text{pr}(f; g) : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$  for prim.rec.  $f : \mathbb{N}^n \rightarrow \mathbb{N}$ ,  $g : \mathbb{N}^{n+2} \rightarrow \mathbb{N}$ :

$$h(\vec{x}, 0) = f(\vec{x})$$

$$h(\vec{x}, y + 1) = g(\vec{x}, h(\vec{x}, y), y)$$

Suppose that  $f$  and  $g$  are represented by  $M_f, M_g \in \Lambda$ , respectively.

$$\text{Init} := \langle \ulcorner 0 \urcorner, M_f x_1 \dots x_n \rangle$$

$$\text{Step} := \lambda p. \langle M_{\text{succ}}(\rho_1 p), M_g x_1 \dots x_n (\rho_2 p) (\rho_1 p) \rangle$$

Then the following  $\lambda$ -term  $M_h$  represents  $h$ :

$$M_h := \lambda x_1 \dots x_n x. \rho_2 (x \text{ Step Init})$$



# $\mu$ -recursion, and partial recursive functions

## Definition

A partial function  $f : \mathbb{N}^n \rightarrow \mathbb{N}$  is called **partial recursive** if it can be specified from **base functions** ( $\mathcal{O}$ , **succ**,  $\pi_i^n$ ) by successive applications of **composition**, **primitive recursion**, and **unbounded minimisation**.

A partial recursive function is called **(total) recursive** if it is total.

Let  $f : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$  **total**. Then the partial function defined by:

$$\mu(f) : \mathbb{N}^n \rightarrow \mathbb{N}$$

$$\vec{x} \mapsto \begin{cases} \min(\{y \mid f(\vec{x}, y) = 0\}) & \dots \exists y (f(\vec{x}, y) = 0) \\ \uparrow & \dots \text{else} \end{cases}$$

is called the **unbounded minimisation of  $f$** .

# Reminder: Kleene's normal form theorem

## Theorem

For every *partial recursive* function  $h : \mathbb{N}^n \rightarrow \mathbb{N}$  there exist *primitive recursive* functions  $f : \mathbb{N} \rightarrow \mathbb{N}$  and  $g : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$  such that:

$$h(x_1, \dots, x_n) = (f \circ \mu(g))(x_1, \dots, x_n)$$

# $\mu$ -recursive/partial recursive $\Rightarrow$ $\lambda$ -definable

## Theorem

*Every  $\mu$ -recursive/partial recursive function is  $\lambda$ -definable.*

## Proof.

Let  $h : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$  be partial recursive.

Then by Kleene's normal form theorem there exist  $g : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$  and  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that:

$$h(\vec{x}) = f \circ \mu(g)(\vec{x}) = f(\mu z. [g(\vec{x}, z) = 0])$$

Let  $M_f$  and  $M_g$  be  $\lambda$ -terms representing  $f$  and  $g$ , respectively. Let:

$$W := \lambda y. \text{if } (\text{zero? } M_g x_1 \dots x_n y) \text{ then } (\lambda w. M_f y) \text{ else } (\lambda w. w(M_{\text{succ}} y)w)$$

Then the following  $\lambda$ -term  $M_h$  represents  $h$ :

$$M_h := \lambda x_1 \dots x_n. W \text{ '0' } W$$



# A normalizing reduction strategy

Normal order reduction strategy  $\xrightarrow{n}$  :

only perform  $\rightarrow_{\beta}$ -steps in left-most positions.

## Theorem

The *normal order reduction strategy* in is normalizing in  $\lambda$ -calculus, that is:

$$M \rightarrow_{\beta}^* N \wedge N \text{ is a normal form} \implies M \xrightarrow{n}^* N$$

# $\lambda$ -definable $\Rightarrow$ Turing-computable

## Theorem

*Every  $\lambda$ -definable function is Turing computable.*

## Idea of the Proof.

Let  $f : \mathbb{N}^n \rightarrow \mathbb{N}$  be a partial function that is  $\lambda$ -definable. Then there exists a  $\lambda$ -term  $M_f$  that represents  $f$ .

To compute  $f$ , one can build a Turing machine  $M$  that, for given  $m_1, \dots, m_n \in \mathbb{N}$ :

- ▶ simulates a normal order rewrite sequence on  $M_f \ulcorner m_1 \urcorner \dots \ulcorner m_n \urcorner$  to obtain the normal form  $\ulcorner f(m_1, \dots, m_n) \urcorner$  □

# Summary

## Lambda calculus

- ▶  $\lambda$ -calculus
  - ▶ syntax
  - ▶ reduction rules
- ▶  $\lambda$ -definable functions
- ▶ primitive recursive functions are  $\lambda$ -definable
- ▶  $\mu$ -recursive/partial recursive functions are  $\lambda$ -definable
- ▶  $\lambda$ -definable functions are Turing computable
- ▶ Hence:  $\lambda$ -definable = partial recursive = Turing-computable

# Suggested reading

- ▶ **Interaction-Based Models of Computation:**

Chapter 7, **The Lambda Calculus** of the book:

- ▶ Maribel Fernández [2]: *Models of Computation (An Introduction to Computability Theory)*, Springer-Verlag London, 2009.

- ▶ **Post's Correspondence Problem**

- ▶ see paper link webpage

- ▶ **Fractran**

- ▶ see paper and video link webpage



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computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = Turing-computable, Church's Thesis	$\lambda$ -terms, $\beta$ -reduction, $\lambda$ -definable functions, partial recursive = $\lambda$ -definable = Turing computable	
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# References



[Henk Pieter Barendregt.](#)

*The Lambda Calculus (Its Syntax and Semantics)*, volume 103 of *Studies in Logic and the Foundations of Mathematics*.  
Elsevier, 1984.



[Maribel Fernández.](#)

*Models of Computation (An Introduction to Computability Theory)*.  
Springer, Dordrecht Heidelberg London New York, 2009.