An Introduction to Parameterized Complexity

Lecture 1: Fixed-Parameter Tractability

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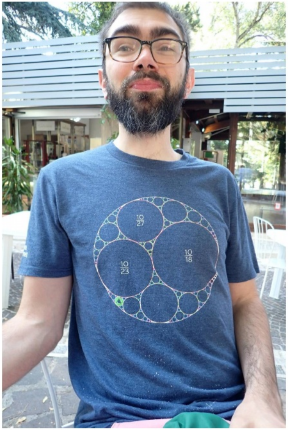
Ph.D. Program, Advanced Period
Gran Sasso Science Institute
L'Aquila, Italy

Monday, June 10, 2024

Course overview

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results		**Algorithmic Meta-Theorems**		
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	GDA	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	GDA	GDA
Algorithmic Techniques		*Formal-Method & Algorithmic Techniques*		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width			**FPT-Intractability Classes & Hierarchies**
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Course developers



Hugo Gilbert
course 2019/20 (Hugo & Clemens)

CG & Alessandro Aloisio
course 2020/21 (Alessandro & C)

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Motivation

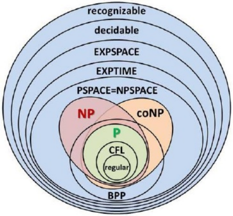
Classical complexity theory

- analyses problems by resource (space or time) needed to solve them on a reasonable machine model
- as a function of the input size $n = |x|$ (Hartmanis/Stearns, 1965)

⇒ variety of complexity classes (P, LOGSPACE, NP, PSPACE, ...)

⇒ tractable problems = polynomial-time computable (in P)

⇒ theory of intractability (reductions, NP completeness)



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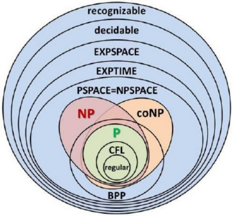

Drawback

- measures problem size $n = |x|$ only in terms of input instances x, and ignores structural information about instances
- sometimes problems are easier to solve for instances if additional structure information is available

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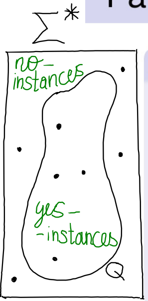
Parameterized complexity

- measures complexity also in terms of a parameter $k = \kappa(x)$
  that may depend on the input x in an arbitrary way

$\Rightarrow$ fixed-parameter tractable problems
relaxes polynomial time solvability to algorithms whose
non-polynomial behavior $f(k) \cdot p(n)$ is restricted by parameter k

$\Rightarrow$ complexity classes (FPT, XP, W[P], W- and A-hierarchies)
$\Rightarrow$ theory of fixed-parameter intractability

Parameterized (versus classical) problems



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- $\kappa : \Sigma^* \to \mathbb{N}$ a function, *the parameterization*.

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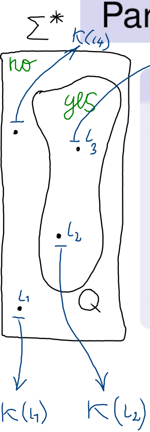

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Assumption

The parameterization κ can be efficiently computed.

Parameterized problems (examples)

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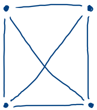
Given: a graph G and an integer k,
Question: Does there exists a clique of size k in G?

Parameter: k.



K_4

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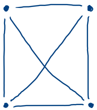

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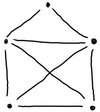

G

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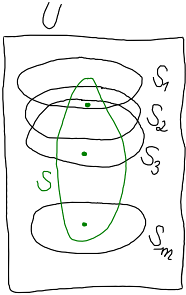

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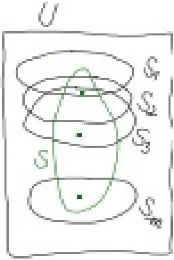

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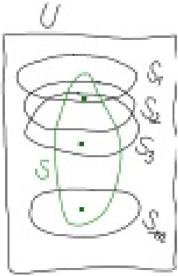

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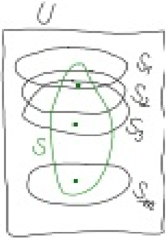

- NP-hard even if $\max |S_i| = 2$,
- is fixed-parameter tractable. for parameter k

The art of parameterization

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There is a hierarchy on parameters.

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  E.g., the number of voters in an election problem.
- Some more structural property of the instance.
  E.g., the diameter of a graph.
- It can be a combination of values, a difference, ...

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- Boolean formulas: number of variables, number of clauses...
- Problems on strings: maximum length of a string, size of the alphabet...

Fixed Parameter Tractability (Class FPT)

Definition

A parameterized problem $\langle Q, \kappa \rangle$ is *fixed-parameter tractable* if:

$$\exists f : \mathbb{N} \to \mathbb{N} \text{ computable } \exists p \in \mathbb{N}[X] \text{ polynomial}$$
$$\exists \mathbb{A} \text{ algorithm, takes inputs in } \Sigma^* \text{ and } \forall x \in \Sigma^*$$
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$\notin$ FPT: $1 \cdot n^{k}$, $n^{\log k}$, $k \cdot 2^{n}$,

$k \cdot 2^{(\log n)^2}$

$= k \cdot n^{\log n}$

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Goal in parameterized algorithmics:

⇒ design FPT algorithms,
⇒ try to make both factors $f(\kappa(x))$ and $p(|x|)$ as small as possible.
⇒ or show (if possible) that finding such factors is impossible

Slices of FPT problems are in P

The *ℓ-th slice* of a parameterized problem $\langle Q, \kappa \rangle$:

$$\langle Q, \kappa \rangle_{\ell} := \{ x \in Q \mid \kappa(x) = \ell \} \qquad \text{(as classical problem)}\,.$$

Proposition

If $\langle Q, \kappa \rangle \in \mathrm{FPT}$, then $\langle Q, \kappa \rangle_{\ell} \in \mathrm{P}$ for all $\ell \in \mathbb{N}$.

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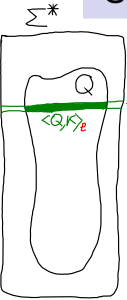

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A problem not in FPT (unless P = NP)

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Application

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Instance: a graph $\mathcal{G}$ and $k \in \mathbb{N}$.

Parameter: k.

Problem: Decide whether $\mathcal{G}$ is k-colorable.

Known: 3-COLORABILITY $\in$ NP-complete (Lovàsz, Stockmeyer, 1973).

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Since 3-COLORABILITY = p-COLORABILITY$_3$,
it follows that p-COLORABILITY $\notin$ FPT (unless P = NP).

Slice-wise polynomial problems (Class XP)

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A parameterized problem $\langle Q, \kappa \rangle$ is *slice-wise polynomial* if:

$$\exists f, g : \mathbb{N} \to \mathbb{N} \text{ computable}$$
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XP/FPT running times: $2^k \cdot n$, $2^{2^{2^k}} \cdot n$, $\left.2^{2^{\cdot^{\cdot^{2}}}}\right\}k \cdot n$

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XP running times: $1 \cdot n^k$, $n^{\log k}$

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Slices of XP problems are in P

The *ℓ-th slice* of a parameterized problem $\langle Q, \kappa \rangle$:

$$\langle Q, \kappa \rangle_{\ell} := \{ x \in Q \mid \kappa(x) = \ell \} \qquad \text{(as classical problem)}\,.$$

Proposition

If $\langle Q, \kappa \rangle \in \mathrm{XP}$, then $\langle Q, \kappa \rangle_{\ell} \in \mathrm{P}$ for all $\ell \in \mathbb{N}$.

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If $\langle Q, \kappa \rangle \in \mathrm{XP}$, then there are a function $f : \mathbb{N} \to \mathbb{N}$ computable, a polynomial p, and an algorithm $\mathbb{A}$ that decides $x \in \Sigma^*$ in running time $\leq f(\kappa(x)) \cdot |x|^{g(\kappa(x))}$ time. This algorithm can be used to decide the ℓ-th slice in time $\leq f(\ell) \cdot |x|^{g(\ell)}$, which for fixed ℓ is a polynomial. $\square$

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Application

p-COLORABILITY
- **Instance:** a graph $\mathcal{G}$ and $k \in \mathbb{N}$.
- **Parameter:** k.
- **Problem:** Decide whether $\mathcal{G}$ is k-colorable.

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Since 3-COLORABILITY = p-COLORABILITY$_3$,
it follows that p-COLORABILITY $\notin$ XP (unless P = NP).

Aims of the course

1. Acquire a basic notions of parameterized complexity.
2. Obtain an introduction to some techniques to derive FPT or XP results.
3. Obtain an introduction to a variety of techniques to prove algorithmic lower bounds and in particular prove parameterized hardness results.

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Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results		**Algorithmic Meta-Theorems**		
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma		1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width		
Algorithmic Techniques		*Formal-Method & Algorithmic Techniques*		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width			**FPT-Intractability Classes & Hierarchies**
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths			motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

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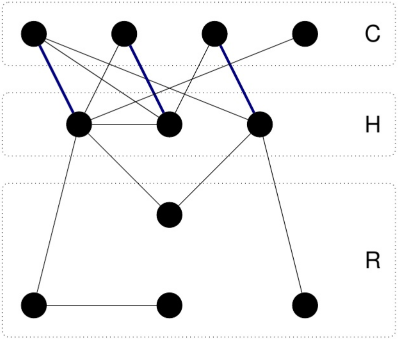
From today's lecture



A **crown decomposition** of a graph G is a partitioning (C, H, R) of $V(G)$, such that:

1. C is nonempty.
2. C is an independent set.
3. H separates C and R.
4. G contains a matching of H into C.

Crown Lemma ($\Leftarrow$ results by König, Hall)

Let G be a graph with no isolated vertices and with at least $3k + 1$ vertices. There is a polynomial-time algorithm that:

- either finds a matching of size $k + 1$ in G;
- or finds a crown decomposition of G.

Tomorrow

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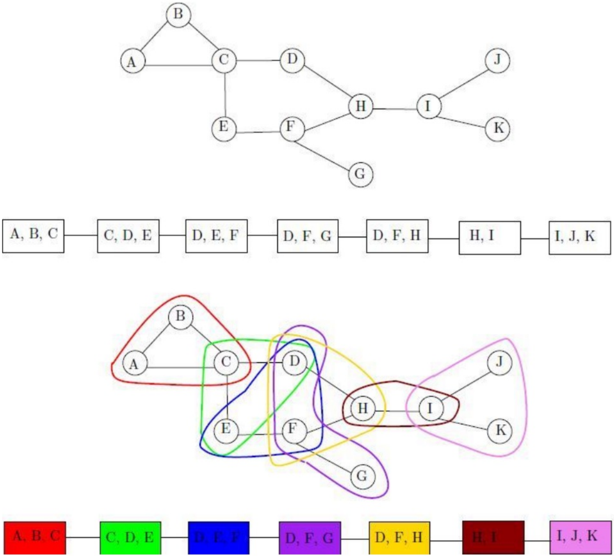
In tomorrow's lecture: a path decomposition of a graph



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In Wednesday's lecture: Monadic second-order logic

$$\psi_3 := \exists C_1 \exists C_2 \exists C_3 \Big(\big(\forall x \bigvee_{i=1}^{3} C_i(x) \big) \wedge \forall x \forall y \big(E(x,y) \to \bigwedge_{i=1}^{3} \neg (C_i(x) \wedge C_i(y)) \big) \Big)$$

$$\mathcal{A}(\mathcal{G}) \models \psi_3 \iff \mathcal{G} \text{ has is } 3\text{-colorable}.$$

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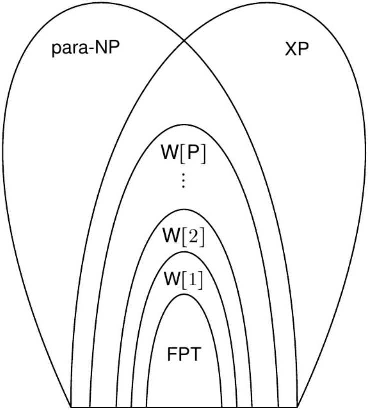
From Friday's lecture: W-Hierarchy

> *'There is no definite single class that can be viewed as "the parameterized* NP*". Rather, there is a whole hierarchy of classes playing this role.* (Flum, Grohe [FG06])



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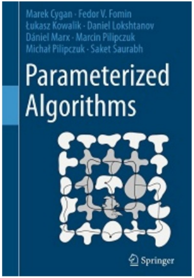
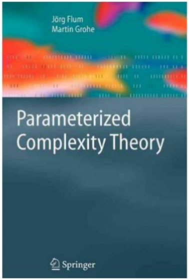
Books





- Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh, *Parameterized Algorithms*, 1st ed., Springer, 2015.
- Jörg Flum and Martin Grohe, *Parameterized Complexity Theory*, Springer, 2006.

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- Sunflower lemma
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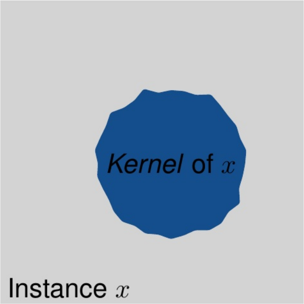
Kernelization methods (informally)

Kernelization is:

- a systematic study of polynomial-time preprocessing algorithms,
- an important tool in the design of parameterized algorithms.



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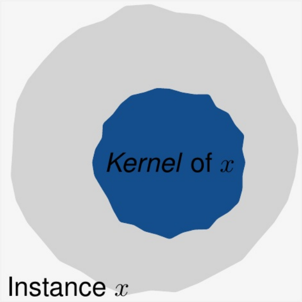

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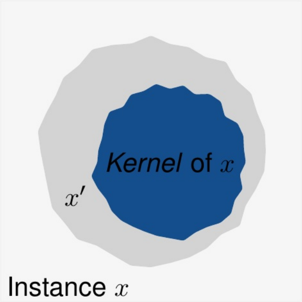

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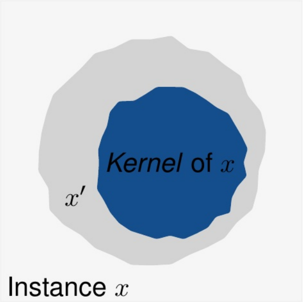

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- Often a collection of efficient preprocessing rules.
- Transform an instance x into a smaller equivalent instance x'.
- Hopefully, $|x'| \leq g(\kappa(x))$.
  $\rightarrow$ use a (non-efficient) exact algorithm.

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$\Rightarrow NP = P$ and we would have solved this famous problem

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That would yield a polynomial alg. for an NP-compl. problem!

Kernelization (formally)

Definition

Let $\langle Q, \kappa \rangle$ be a parameterized problem over Σ.
A *kernelization* of $\langle Q, \kappa \rangle$ is a function $K : \Sigma^* \to \Sigma^*$ such that:

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(K2) K is polynomial-time computable.

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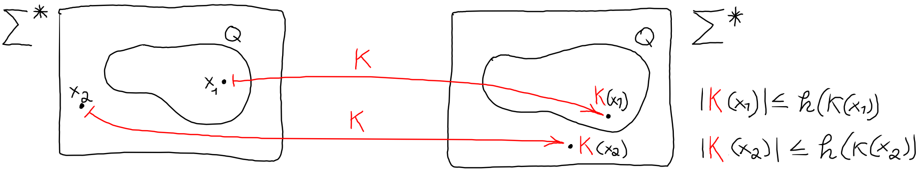

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If $\langle Q, \kappa \rangle \in$ FPT, the $\langle Q, \kappa \rangle$ admits a kernel.

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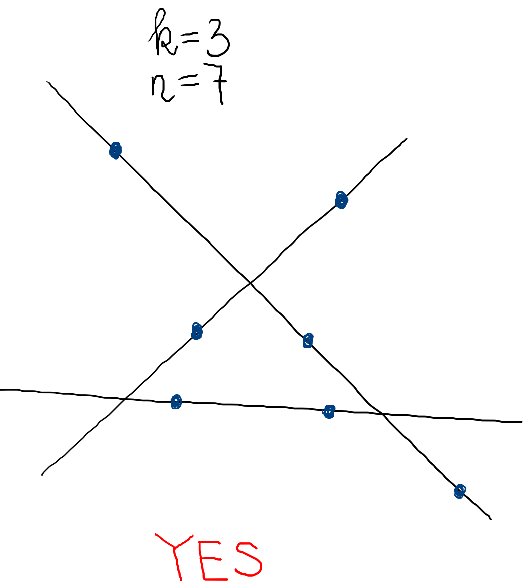

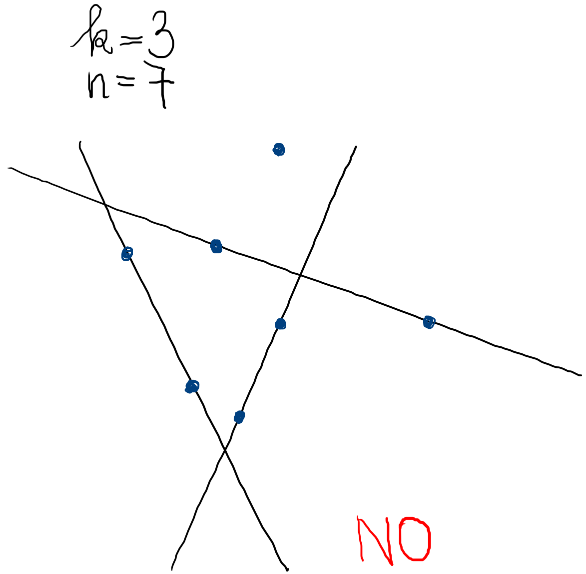

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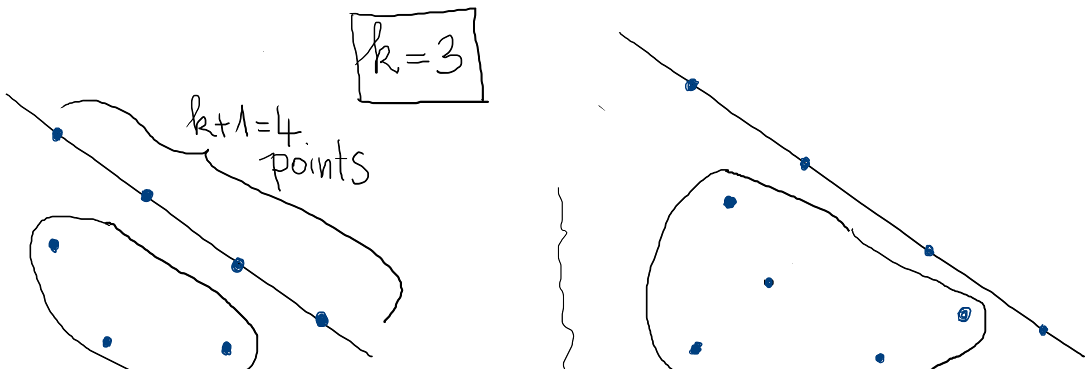

n=7

can be covered by k-1=2 lines | no cover with 2 lines n=9

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If we cannot apply Rule 1, and we have more than k^2 points, **then** say no, and return a trivial no instance.

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Observation: Let (x, κ) be a yes instance of the p-Point-Line-Cover such that Rule 1 cannot be applied. **Then** $n \leq k^2$ holds.

Rule 2:

If we cannot apply Rule 1, and we have more than k^2 points, **then** say no, and return a trivial no instance.

Proposition

p-POINT-LINE-COVER $\in$ FPT: it admits a kernel of size with k^2 points.

Kernelization $\Rightarrow$ FPT

Lemma

If $\langle Q, \kappa \rangle$ admits a kernel and is decidable, then $\langle Q, \kappa \rangle \in$ FPT.

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Definitions

A *kernelization* of $\langle Q, \kappa \rangle$ is a function $K : \Sigma^* \to \Sigma^*$ such that:

(K1) For all $x \in \Sigma^*$: $(x \in Q \iff K(x) \in Q)$.

(K2) K is polynomial-time computable.

(K3) There is a computable function $h : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \Sigma^*$: $|K(x)| \leq h(\kappa(x))$.

A parameterized problem $\langle Q, \kappa \rangle$ is *fixed-parameter tractable* if:

$\exists f : \mathbb{N} \to \mathbb{N}$ computable $\exists p \in \mathbb{N}[X]$ polynomial
$\exists \mathbb{A}$ algorithm, takes inputs in Σ^* and $\forall x \in \Sigma^*$
$\left[\mathbb{A} \text{ decides if } x \in Q \text{ in time } \leq f(\kappa(x)) \cdot p(|x|) \right]$.

FPT := complexity class of all fixed-parameter tractable problems.

Kernelization ⇒ FPT

Lemma

If $\langle Q, \kappa \rangle$ admits a kernel and is decidable, then $\langle Q, \kappa \rangle \in$ FPT.

$\langle Q, \kappa \rangle$ a parameterized problem, $Q \subseteq \Sigma^*$

Definition. $K : \Sigma^* \to \Sigma^*$ a kernelization for $\langle Q, \kappa \rangle$ if:

(K1) $\forall x \in \Sigma^* \, (x \in Q \iff K(x) \in Q)$

(K2) K is poly time computable

(K3) $\exists h : \mathbb{N} \to \mathbb{N} \; \forall x \in \Sigma^* \, (|K(x)| \le h(\kappa(x)))$.

Proposition: If $\langle Q, \kappa \rangle$ is decidable, and has kernelization K, then $\langle Q, \kappa \rangle \in$ FPT

Proof. Since $\langle Q, \kappa \rangle$ is decidable, there is an algorithm A that decides instances $x \in \Sigma^*$ in time $\le f(|x|)$ steps for some computable function $f : \mathbb{N} \to \mathbb{N}$.

Then, assuming a polynomial algorithm A_K for K (time bounded by $p(x)$) we construct an FPT algorithm $A(K)$ for $\langle Q, \kappa \rangle$:

$$\boxed{A(K)} \quad \Sigma^* \ni x \xmapsto[A_K]{K} K(x) \in \Sigma^*$$

A on $K(x)$:

- $K(x) \notin Q$: "no" → "no" $(x \notin Q)$ (K1)
- $K(x) \in Q$: "yes" → "yes" $(x \in Q)$ (K1)

Running time $A(K)$ =

$= \text{time}(A_K) + \text{time}(A(K(x)))$

$= p(|x|) + f(|K(x)|)$

by (K2) by (K3) $\le h(K(x))$

$\le p(|x|) + f(h(\kappa(x)))$

$\le ((f \circ h)(\kappa(x)) + 1) \cdot (1+p)(|x|)$

$= f(\kappa(x)) \cdot \text{poly}(|x|) \in$ FPT.

The (parameterized) Vertex Cover Problem

p-VERTEX-COVER:
Given: A graph G, and an integer k.
Parameter: The integer k.
Question: Does there exists a vertex cover of size at most k?

Definition

Let G be a graph and $S \subseteq V(G)$. The set S is called a vertex cover if for every edge of G at least one of its endpoints is in S.

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Exercise

Find an $O(k^2)$ kernel for p-VERTEX-COVER.

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Theorem (Samuel Buss)

p-VERTEX-COVER $\in$ FPT, *because it admits a kernel with at most $O(k^2)$ vertices and $O(k^2)$ edges.*

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Exercise

Find an $O(k^2)$ kernel for p-VERTEX-COVER.

FPT $\Rightarrow$ Kernelization

Lemma

If $\langle Q, \kappa \rangle \in$ FPT, then $\langle Q, \kappa \rangle$ admits a kernel.

Proof.

Let $\mathbb{A}$ be an algorithm that solves $\langle Q, \kappa \rangle$ in time $f(\kappa(x)) \cdot p(x)$, for all $x \in \Sigma^*$, where $f : \mathbb{N} \to \mathbb{N}$ computable, and $p(n)$ a polynomial.

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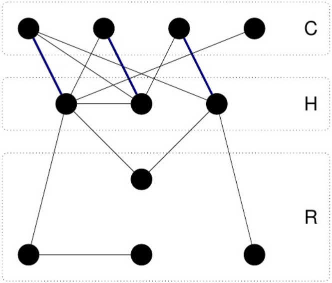
Crown Decomposition and Crown Lemma



A **crown decomposition** of a graph G is a partitioning (C, H, R) of $V(G)$, such that:

1. C is nonempty.
2. C is an independent set.
3. H separates C and R.
4. G contains a matching of H into C.

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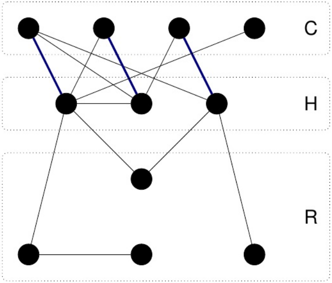

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Crown Lemma ($\Leftarrow$ results by Kőnig, Hall)

Let G be a graph with no isolated vertices and with at least $3k + 1$ vertices. There is a polynomial-time algorithm that:

- either finds a matching of size $k + 1$ in G;
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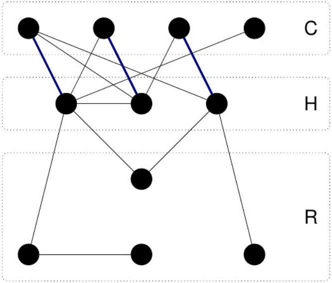

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Exercise

Apply the Crown Lemma to the Vertex Cover Problem.

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- If it returns a matching of size $k + 1$, then conclude that (G, k) is a no-instance
- If it returns a crown decomposition $V(G) = C \cup H \cup R$:
  - Pick the vertices in H in the solution

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  - Reduce $(G - H, k - |H|)$ to $(G - H - C, k - |H|)$
    by using Rule 1 (note that vertices in C are isolated)

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Theorem

p-VERTEX-COVER admits a kernel with at most $3k$ vertices.

The (parameterized) Dual-Coloring Problem

p-COLORABILITY:

Given: A graph $G = \langle V, E\rangle$ on n vertices and an integer k.

Parameter: The integer k.

Question: Is G k-colorable?

Definition

Let $k \in \mathbb{N}$. A graph $G = \langle V, E\rangle$ is k-colorable if there is a function $C : V \to \{1, \ldots, k\}$ such that $C(u) \neq C(v)$ for all edges $\{u, v\} \in E$.

The (parameterized) Dual-Coloring Problem

p-DUAL-COLORABILITY:
Given: A graph $G = \langle V, E \rangle$ on n vertices and an integer k.
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Question: Is G $(n - k)$-colorable?

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Question: Is G $(n - k)$-colorable?

Definition

Let $k \in \mathbb{N}$. A graph $G = \langle V, E \rangle$ is k-colorable if there is a function $C : V \to \{1, \ldots, k\}$ such that $C(u) \neq C(v)$ for all edges $\{u, v\} \in E$.

Exercise

Obtain a kernel with $O(k)$ vertices using crown decomposition.

The Dual-Coloring Problem

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Rule 2: Consider graph $\overline{G}(V, \overline{E})$ obtained from G by saying that $e \in \overline{E}$ iff $e \notin E$.
If $|(V(G))| > 3k$, apply the Crown Lemma to $\overline{G}$.

- If it returns a matching of size $k + 1$, then conclude that (G, k) is a yes-instance

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Theorem

p-DUAL-COLORING *admits a kernel with at most* $3k$ *vertices.*

Sunflower Lemma

Definition

A sunflower with k petals and a core Y is a collection of sets $S_1, \ldots, S_k$ such that $S_i \cap S_j = Y$ for all $i \neq j$. The sets $S_i \setminus Y$ are petals and they must be non-empty.

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A sunflower with 6 petals and a core $Y = \{x_2, x_5\}$.
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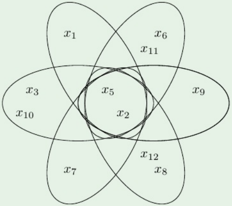
$\ldots$



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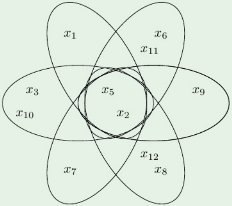

Sunflower Lemma (Erdős, Rado)

Let $\mathcal{A}$ be a family of sets (without duplicates) over a universe U such that each set in $\mathcal{A}$ has cardinality $= d$.
If $|\mathcal{A}| > d!(k-1)^d$, then $\mathcal{A}$ contains a sunflower with k petals which can be computed in time polynomial in $|\mathcal{A}|$, $|U|$, and k.

Application to d-Hitting Set

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Parameterized d-Hitting Set Problem

p-d-HITTING-SET:

Given: A family $\mathcal{A}$ of sets over a universe U, where each set has cardinality $\leq d$ and a positive integer k,

Parameter: The integer k.

Question: Does there exists a subset $H \subseteq U$ of size at most k such that H intersects each set in $\mathcal{A}$?

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Exercise

Apply the sunflower lemma.

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Theorem

p-d-HITTING-SET has a kernel with $\leq d!k^d d$ sets & $\leq d!k^d d^2$ elements.

Application to d-Hitting Set

Observation

If $\mathcal{A}$ contains a sunflower $\mathcal{S} = \{S_1, \ldots, S_{k+1}\}$ of $k+1$ sets, then every hitting set H of $\mathcal{A}$ with $|H| \leq k$ must intersect the core Y of $\mathcal{S}$. Otherwise it is a no-instance, because H cannot intersect each of the $k+1$ petals $S_i \smallsetminus Y$.

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Rule **HS.1**: Let $(U, \mathcal{A}, k)$ be an instance of d-Hitting Set.
Assume that $\mathcal{A}$ contains a sunflower $\mathcal{S} = \{S_1, \ldots, S_{k+1}\}$ of cardinality $k+1$ with core Y.
Then return $(U', \mathcal{A}', k)$, where $\mathcal{A}' := (\mathcal{A} \smallsetminus \mathcal{S}) \cup Y$,
$U' := \bigcup \mathcal{A}' = \bigcup_{X \in \mathcal{A}'} X$.

Proof (kernel of p-d-Hitting-Set with $\le d!k^d d$ sets and $\le d!k^d d^2$ elements).

If for some $d' \in \{1, \ldots, d\}$, the number of sets in $\mathcal{A}$ of size $= d'$ is more than $d'!k^{d'}$, then the sunflower lemma yields a sunflower of size $k+1$.

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If $\varnothing \in \mathcal{A}'$ (a sunflower had an empty core), then it is a no instance. $\square$

Tomorrow

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results		**Algorithmic Meta-Theorems**		
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	GDA	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	GDA	GDA
Algorithmic Techniques		*Formal-Method & Algorithmic Techniques*		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width			**FPT-Intractability Classes & Hierarchies**
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies