Lecture 3: Recursive Functions

Models of Computation

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Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

Overview

Today

Recursive functions

- primitive recursive functions
- Gödel–Herbrand(–Kleene) general recursive functions
- partial recursive functions
  - defined with μ-recursion (unbounded minimisation)
- Partial recursive functions = Turing computable functions

Timeline: From logic to computability

1900	Hilbert's 23 Problems in mathematics
1910/12/13	Russell/Whitehead: Principia Mathematica
1928	Hilbert/Ackermann: formulate completeness/decision problems for the predicate calculus (the latter called 'Entscheidungsproblem')
1929	Presburger: completeness/decidability of theory of addition on $\mathbb{Z}$
1930	Gödel: completeness theorem of predicate calculus
1931	Gödel: incompleteness theorems for first-order arithmetic
1932	Church: λ-calculus
1933/34	Herbrand/Gödel: general recursive functions
1936	Church/Kleene: λ-definable $\sim$ general recursive Church Thesis: 'effectively calculable' be defined as either Church shows: the 'Entscheidungsproblem' is unsolvable Post: machine model; Church's thesis as 'working hypothesis'
1937	Turing: convincing analysis of a 'human computer' leading to the 'Turing machine'

Turing-computable (total) functions

Definition

A total function $f : \mathbb{N}^k \to \mathbb{N}$ is Turing-computable if there exists a Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \text{b\!\!/}, F \rangle$ and a calculable coding function $\langle \cdot \rangle : \mathbb{N} \to \Sigma^*$ such that:

- for all $n_1, \ldots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:

$$q_0 \langle n_1 \rangle \text{b\!\!/} \langle n_2 \rangle \text{b\!\!/} \ldots \text{b\!\!/} \langle n_k \rangle \vdash_M^* q \langle f(n_1, \ldots, n_k) \rangle$$

Recursive Functions

Primitive recursive functions defined by recursive equations:

like e.g. functions $+, \cdot, (\cdot)^{\cdot} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, and $(\cdot)! : \mathbb{N} \to \mathbb{N}$:

$$
\begin{aligned}
n + 0 &= n & n \cdot 0 &= 0 \\
n + (m+1) &= (n+m) + 1 & n \cdot (m+1) &= n \cdot m + n \\
n^0 &= 1 & 0! &= 1 \\
n^{m+1} &= n^m \cdot n & (n+1)! &= (n+1) \cdot n!
\end{aligned}
$$

primitive recursive functions: defined by such equations (termination of the evaluation process guaranteed)

general recursive functions: defined by more general systems of equations

μ-recursive (partial recursive) functions: extend the primitive recursive functions by a μ-operator that allows to construct partial functions

Rósza Péter

Rósza Péter (1905–1977)

Primitive recursive functions ($\mathbb{N}^k \to \mathbb{N}$)

Base functions:

- $\mathcal{O} : \mathbb{N}^0 = \{\varnothing\} \to \mathbb{N}\,,\ \varnothing \mapsto 0$ (0-ary constant-0 function)
- succ $: \mathbb{N} \to \mathbb{N}\,,\ x \mapsto x + 1$ (successor function)
- $\pi_i^n : \mathbb{N}^n \to \mathbb{N}\,,\ \vec{x} = \langle x_1, \dots, x_n \rangle \mapsto x_i$ (projection function)

Closed under operations:

- composition: if $f : \mathbb{N}^k \to \mathbb{N}$, and $g_i : \mathbb{N}^n \to \mathbb{N}$ are prim. rec., then so is $h = f \circ (g_1 \times \ldots \times g_k) : \mathbb{N}^n \to \mathbb{N}$:
$$h(\vec{x}) = f(g_1(\vec{x}), \dots, g_k(\vec{x}))$$
- primitive recursion: if $f : \mathbb{N}^n \to \mathbb{N}$, $g : \mathbb{N}^{n+2} \to \mathbb{N}$ are prim. rec., then so is $h = \mathrm{pr}(f; g) : \mathbb{N}^{n+1} \to \mathbb{N}$:
$$h(\vec{x}, 0) = f(\vec{x})$$
$$h(\vec{x}, y + 1) = g(\vec{x}, h(\vec{x}, y), y)$$

Primitive recursive functions ($\mathbb{N}^n \to \mathbb{N}^l$)

Base functions:

- $\mathcal{O} : \mathbb{N}^0 = \{\varnothing\} \to \mathbb{N}$, $\varnothing \mapsto 0$ (0-ary constant-0 function)
- succ $: \mathbb{N} \to \mathbb{N}$, $x \mapsto x + 1$ (successor function)
- $\pi^n_i : \mathbb{N}^n \to \mathbb{N}$, $\vec{x} = \langle x_1, \ldots, x_n \rangle \mapsto x_i$ (projection function)
- for $n > 1$: $\mathrm{id}^n : \mathbb{N}^n \to \mathbb{N}^n$, $\vec{x} = \langle x_1, \ldots, x_n \rangle \mapsto \vec{x}$ (n-ary identity f.)

Closed under operations:

- composition: if $f : \mathbb{N}^{km} \to \mathbb{N}^l$, and $g_i : \mathbb{N}^n \to \mathbb{N}^m$ are prim. rec., then so is $h = f \circ (g_1 \times \ldots \times g_k) : \mathbb{N}^n \to \mathbb{N}^l$:

$$h(\vec{x}) = f(g_1(\vec{x}), \ldots, g_k(\vec{x}))$$

- primitive recursion: if $f : \mathbb{N}^n \to \mathbb{N}^l$, $g : \mathbb{N}^{n+l+1} \to \mathbb{N}^l$ are prim. rec., then so is $h = \mathrm{pr}(f; g) : \mathbb{N}^{n+1} \to \mathbb{N}^l$:

$$h(\vec{x}, 0) = f(\vec{x})$$
$$h(\vec{x}, y + 1) = g(\vec{x}, h(\vec{x}, y), y)$$

Primitive recursive functions (exercises)

Exercise

Show that the following functions are primitive recursive:

- addition
- constant functions
- multiplication
- (positive) sign-function
- the representing functions $\chi_{=}$ and $\chi_{<}$ for the predicates $=$ and $<$.

Try-yourself-Examples

Show that the following functions are primitive recursive:

- exponentiation
- factorial

Admissible operations for primitive recursive functions

Proposition

1. *definition by case distinction:*

$$f(\vec{x}) := \begin{cases} f_1(\vec{x}) & \ldots P_1(\vec{x}) \\ f_2(\vec{x}) & \ldots \wedge P_2(\vec{x}) \neg P_1(\vec{x}) \\ \ldots \\ f_k(\vec{x}) & \ldots \wedge P_k(\vec{x}) \wedge \neg P_{k-1}(\vec{x}) \wedge \ldots \neg P_1(\vec{x}) \\ f_{k+1}(\vec{x}) & \ldots \wedge \neg P_k(\vec{x}) \wedge \ldots \neg P_1(\vec{x}) \end{cases}$$

2. *definition by bounded recursion:*

$$\mu z_{\leq y}.\,[P(x_1,\ldots,x_n,z)] :=$$

$$\begin{cases} z & \ldots \neg P(x_1,\ldots,x_n,i) \text{ for } 0 \leq i < z \leq y, \\ & \qquad \text{and } P(x_1,\ldots,x_n,z) \\ y+1 & \ldots \neg \exists z.\ \wedge 0 \leq z \leq y P(x_1,\ldots,x_n,z) \end{cases}$$

Properties of primitive recursive functions

Proposition

1. *Every primitive recursive function is total.*
2. *Every primitive recursive function is Turing-computable.*

Proof.

For (2):

- the base functions are Turing-computable
- the Turing-computible functions are closed under the schemes composition and primitive recursion

□

Turing-computable (total) functions

Definition

A total function $f : \mathbb{N}^k \to \mathbb{N}$ is Turing-computable if there exists a Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \not{b}, F \rangle$ and a calculable coding function $\langle \cdot \rangle : \mathbb{N} \to \Sigma^*$ such that:

- for all $n_1, \ldots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:

$$q_0 \langle n_1 \rangle \not{b} \langle n_2 \rangle \not{b} \ldots \not{b} \langle n_k \rangle \vdash_M^* q \langle f(n_1, \ldots, n_k) \rangle$$

Features of computationally complete MoC's present?

- storage (unbounded)
- control (finite, given)
- modification
  - of (immediately accessible) stored data
  - of control state
- conditionals
- loop (unbounded)
- stopping condition

Features of computationally complete MoC's present?

- storage (unbounded) ✓
- control (finite, given) ✓
- modification ✓
  - of (immediately accessible) stored data
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- conditionals ✓
- loop ✓ (unbounded)
- stopping condition ✓

Features of computationally complete MoC's present?

- storage (unbounded) ✓
- control (finite, given) ✓
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  - of (immediately accessible) stored data
  - of control state
- conditionals ✓
- loop ✓ (unbounded) ✗
- stopping condition ✓

Not primitive recursive (I)

Proposition

There exist calculable/Turing-computable functions that are not primitive recursive.

Proof.

By diagonalisation. □

Not primitive recursive (II): Ackermann function

Wilhelm Ackermann (1896–1962)

Not primitive recursive (II): Ackermann function

Ackermann function $\mathsf{A} : \mathbb{N}^2 \to \mathbb{N}$ (simplified version by Rósza Péter):

$$\mathsf{A}(0, x) = \mathsf{Succ}(x)$$

$$\mathsf{A}(x+1, 0) = \mathsf{A}(x, \mathsf{Succ}(0))$$

$$\mathsf{A}(x+1, y+1) = \mathsf{A}(x, \mathsf{A}(x+1, y))$$

A is not primitive recursive, it grows too fast:

$$\mathsf{A}(0, n) = n + 1$$

$$\mathsf{A}(1, n) = n + 2$$

$$\mathsf{A}(2, n) = 2n + 3$$

$$\mathsf{A}(3, n) = 2^{n+3} - 2$$

$$\mathsf{A}(4, n) = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{16}}}}}}}_{n} - 3$$

$$\ldots$$

Not primitive recursive (II): Ackermann function

Ackermann function $\mathsf{A} : \mathbb{N}^2 \to \mathbb{N}$ (simplified version by Rósza Péter):

$$\mathsf{A}(0, x) = \mathsf{Succ}(x)$$
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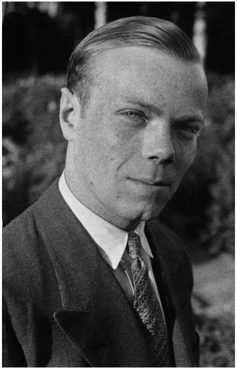
$$\mathsf{A}(x+1, y+1) = \mathsf{A}(x, \mathsf{A}(x+1, y))$$

A grows faster than every primitive recursive function:

Theorem

For every primitive recursive $f : \mathbb{N} \to \mathbb{N}$ there exists some $i \in \mathbb{N}$ such that $f(i) < \mathsf{A}(i, i)$.

Jacques Herbrand



Jacques Herbrand (1908–1931)

Kurt Gödel

Kurt Gödel (1906–1978)

Gödel–Herbrand general recursive function

Defined by systems of recursion equations like that for the Ackermann function:

$$\begin{aligned}
\mathsf{A}(0, x) &= \mathsf{Succ}(x) \\
\mathsf{A}(\mathsf{Succ}(x), 0) &= \mathsf{A}(x, \mathsf{Succ}(0)) \\
\mathsf{A}(\mathsf{Succ}(x), \mathsf{Succ}(y)) &= \mathsf{A}(x, \mathsf{A}(\mathsf{Succ}(x), y))
\end{aligned}$$

Gödel–Herbrand general recursive function

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$$\mathsf{A}(0, x) = \mathsf{Succ}(x)$$
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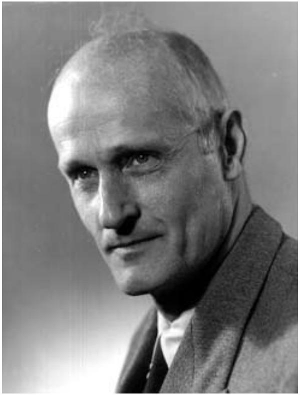
$$\mathsf{A}(\mathsf{Succ}(x), \mathsf{Succ}(y)) = \mathsf{A}(x, \mathsf{A}(\mathsf{Succ}(x), y))$$

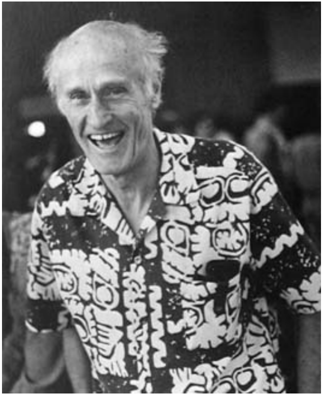
Numerals: $\langle 0 \rangle := 0$, and $\langle n \rangle := \underbrace{\mathsf{Succ}(\ldots \mathsf{Succ}(0))}_{n}$ for $n > 1$.

Definition

A function $f : \mathbb{N}^k \to \mathbb{N}$ is called general recursive if it can be defined by (such a) system $\mathcal{S}$ of recursion equations via a function symbol F if for all $n_1, \ldots, n_k \in \mathbb{N}$, the expression $F(\langle n_1 \rangle, \ldots, \langle n_k \rangle)$ evaluates according to $\mathcal{S}$ to a unique numeral $\langle n \rangle$, and such that furthermore: $n = f(n_1, \ldots, n_k)$.

Stephen Cole Kleene





Stephen Cole Kleene (1906–1994)

Unbounded minimisation (μ-recursion)

Let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ total. Then the partial function defined by:

$$\mu(f) : \mathbb{N}^k \rightharpoonup \mathbb{N}$$
$$\vec{x} \mapsto \begin{cases} \text{min}\{y \mid f(\vec{x}, y) = 0\} & \ldots\ \exists y\,(f(\vec{x}, y) = 0) \\ \uparrow & \ldots\ \text{else} \end{cases}$$

is called the unbounded minimisation of f.

Unbounded minimisation (μ-recursion)

Let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ total. Then the partial function defined by:

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is called the unbounded minimisation of f.

Let $f : \mathbb{N}^{k+1} \rightharpoonup \mathbb{N}$ partial. Then the partial function $\mu(f)$:

$$\begin{aligned} \mu(f) : \mathbb{N}^k &\rightharpoonup \mathbb{N} \\ \vec{x} &\mapsto \begin{cases} z & \ldots f(\vec{x}, z) = 0 \;\wedge\; \forall y\,\big(0 \le y < z \to (f(\vec{x}, y){\downarrow} \neq 0)\big) \\ \uparrow & \ldots \neg\exists y\,\big(f(\vec{x}, y) = 0 \;\wedge\; \forall z\,(0 \le z < y \to (f(\vec{x}, z){\downarrow})\big) \end{cases} \end{aligned}$$

is called the unbounded minimisation of f.

Partial, and total, recursive functions

Definition

A partial function $f : \mathbb{N}^n \rightharpoonup \mathbb{N}^l$ is called partial recursive if it can be specified from base functions ($\mathcal{O}$, succ, π_i^n, and id^n) by successive applications of composition, primitive recursion, and unbounded minimisation.

A partial recursive function is called (total) recursive if it is total.

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Proposition

Every partial recursive function is Turing-computable.

Turing-computable functions

Definition

1. A total function $f : \mathbb{N}^k \to \mathbb{N}$ is Turing-computable if there exists a Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \not{b}, F \rangle$ and a calculable coding function $\langle\cdot\rangle : \mathbb{N} \to \Sigma^*$ such that:
   - for all $n_1, \ldots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:
   $$q_0 \langle n_1 \rangle \not{b} \langle n_2 \rangle \not{b} \ldots \not{b} \langle n_k \rangle \vdash_M^* q \langle f(n_1, \ldots, n_k) \rangle$$

Turing-computable functions

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   - for all $n_1, \ldots, n_k \in \mathbb{N}$:
   $$M \text{ accepts } \langle n_1 \rangle \not{b} \langle n_2 \rangle \not{b} \ldots \not{b} \langle n_k \rangle \iff f(n_1, \ldots, n_k)\!\downarrow$$
   - for all $n_1, \ldots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:
   $$f(n_1, \ldots, n_k)\!\downarrow \implies q_0 \langle n_1 \rangle \not{b} \langle n_2 \rangle \not{b} \ldots \not{b} \langle n_k \rangle \vdash_M^* q \langle f(n_1, \ldots, n_k) \rangle$$

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Partial recursive vs. Turing-computable functions

Lemma

Every Turing-computable function is partial recursive.

Partial recursive vs. Turing-computable functions

Lemma

Every Turing-computable function is partial recursive.

Proof by arithmetization of Turing machines, showing:

Theorem (Kleene's normal form theorem)

For every Turing-computable, partial function (and hence for every partial recursive function) $h : \mathbb{N}^k \to \mathbb{N}$ there exist primitive recursive functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N}^{k+1} \to \mathbb{N}$ such that:

$$h(x_1, \ldots, x_n) = (f \circ \mu(g))(x_1, \ldots, x_n)$$

Partial recursive vs. Turing-computable functions

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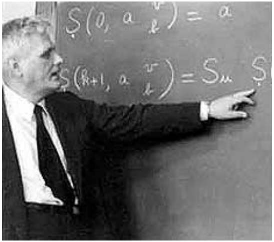
$$h(x_1, \ldots, x_n) = (f \circ \mu(g))(x_1, \ldots, x_n)$$

Theorem

The Turing-computable (partial) functions coincide with the partial recursive functions.

Church’s Thesis

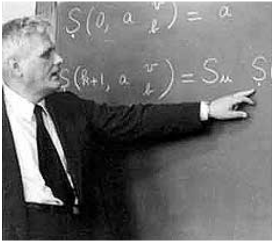


Alonzo Church (1903 –1995)

Thesis (Church, 1936)

Every effectively calculable function is general recursive.

λ-calculus



Alonzo Church (1903 –1992)

Theorem (Kleene/Church, 1935)

Every λ-definable function is general recursive, and vice versa.

Typical features of ‘computationally complete’ MoC’s

- storage (unbounded)

Typical features of ‘computationally complete’ MoC’s

- storage (unbounded)
- control (finite, given)

Typical features of ‘computationally complete’ MoC’s

- storage (unbounded)
- control (finite, given)
- modification

Typical features of ‘computationally complete’ MoC’s

- storage (unbounded)
- control (finite, given)
- modification
  - of (immediately accessible) stored data

Typical features of ‘computationally complete’ MoC’s

- storage (unbounded)
- control (finite, given)
- modification
  - of (immediately accessible) stored data
  - of control state

Typical features of ‘computationally complete’ MoC’s

- storage (unbounded)
- control (finite, given)
- modification
  - of (immediately accessible) stored data
  - of control state
- conditionals

Typical features of ‘computationally complete’ MoC’s

- storage (unbounded)
- control (finite, given)
- modification
  - of (immediately accessible) stored data
  - of control state
- conditionals
- loop

Typical features of ‘computationally complete’ MoC’s

- storage (unbounded)
- control (finite, given)
- modification
  - of (immediately accessible) stored data
  - of control state
- conditionals
- loop
- stopping condition

Summary

-
-
-
-
-
-
- A-hierarchy
-

Recommended reading

1. Recursive and primitive-recursive functions:
   Chapter 3, The Lambda Calculus of the book:
   - Maribel Fernández [1]: *Models of Computation (An Introduction to Computability Theory)*, Springer-Verlag London, 2009.

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Example suggestions

Examples

1.
2.
3.

References

Maribel Fernández.
Models of Computation (An Introduction to Computability Theory).
Springer, Dordrecht Heidelberg London New York, 2009.