

# Lecture 3: Algorithmic Meta-Theorems

## (A Short Introduction to Parameterized Complexity)

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# Course overview

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
<i>Algorithmic Techniques</i>		<i>Formal-Method &amp; Algorithmic Techniques</i>		
<b>Introduction &amp; basic FPT results</b> motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	<b>Notions of bounded graph width</b> path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	<b>Algorithmic Meta-Theorems</b> 1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	<b>FPT-Intractability Classes &amp; Hierarchies</b> motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour

# Overview

- ▶ logic preliminaries
  - ▶ first-order logic
    - ▶ expressing graph problems by f-o formulas
  - ▶ monadic second-order logic (MSO)
    - ▶ expressing graph problems by MSO formulas
  - ▶ complexity of evaluation and model checking problems
- ▶ Courcelle's theorem
  - ▶ FPT-results by model-checking MSO-formulas
    - ▶ for graphs / structures with bounded tree-width
    - ▶ for maximization problems over graphs of bounded tree-width
    - ▶ for graphs of bounded clique-width
  - ▶ applications to concrete problems
- ▶ graph minors
- ▶ meta-theorems for first-order model-checking: an example

# Meta-theorems: idea, benefits and limitations

## idea:

- ▶ express a problem  $P$  by a logical formula  $\varphi_P$  (of 'short' size)
- ▶ use **model checking** of  $\varphi_P$   
on logical structures of **bounded width  $k$**  (tree-, clique-width, ...)
  - ▶ is time bounded depending on  $k$ , size of  $\varphi_P$ , size of the structure
  - ▶ this often facilitates FPT-results

## benefits:

- ▶ a quick and easy way to show  
that [some problems] are fixed-parameter tractable,
- ▶ without working out the tedious details  
of a dynamic programming algorithm.

## limitations:

- ▶ algorithms obtained by meta-theorems  
cannot be expected to be optimal.
- ▶ a careful analysis of a specific problem at hand  
will usually yield more efficient fpt-algorithms

# Logical preliminaries

# First-order logic (formula example)

$$\varphi_3 := \exists x_1 \exists x_2 \exists x_3 \left( \neg(x_1 = x_2) \wedge \neg E(x_1, x_2) \right. \\ \left. \wedge \neg(x_1 = x_3) \wedge \neg E(x_1, x_3) \right. \\ \left. \wedge \neg(x_2 = x_3) \wedge \neg E(x_2, x_3) \right)$$

$$\mathcal{A}(\mathcal{G}) \models \varphi_3 \iff \mathcal{G} \text{ has a 3-element independent set.}$$

$$\varphi_k := \exists x_1 \dots \exists x_k \left( \bigwedge_{1 \leq i < j \leq k} (\neg(x_i = x_j) \wedge \neg E(x_i, x_j)) \right)$$

$$\mathcal{A}(\mathcal{G}) \models \varphi_k \iff \mathcal{G} \text{ has a } k\text{-element independent set.}$$

$$S \subseteq V \text{ is independent set in } \mathcal{G} = \langle V, E \rangle : \iff \forall e = \{u, v\} \in E \ ( \neg(u \in S \wedge v \in S) ) \\ \iff \forall u, v \in S \ ( u \neq v \Rightarrow \{u, v\} \notin E )$$

# First-order logic: syntax (language)

- ▶ language based on:
  - ▶ a *vocabulary*  $\tau = \{R_1, \dots, R_n\}$  of *predicate symbols*  $R_i$  together with arity  $\text{ar}(R_i) \in \mathbb{N}$
  - ▶ the binary equality predication  $=$
  - ▶ (first-order) variable symbols:  $x, y, z, w, x_1, y_1, z_1, w_1, x_2, \dots$
  - ▶ propositional connectives:  $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
  - ▶ existential quantifier  $\exists$ , universal quantifier  $\forall$
- ▶ *atomic formulas (atoms)*: a formula  $x = y$  or  $R(x_1 \dots x_n)$  for  $R \in \tau$
- ▶ *quantifier-free formula*: atoms, literals (= negated atoms), formulas built up from atoms by using propositional connectives
- ▶ *quantifications* over (first-order variables):
  - ▶ existential quantifications  $\exists x$  and universal quantifications  $\forall x$
- ▶ *formulas*:
 
$$\varphi ::= x = y \mid R(x_1, \dots, x_{\text{ar}(R)}) \quad (\text{where } R \in \tau)$$

$$\mid \neg \varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \rightarrow \varphi_2 \mid \varphi_1 \leftrightarrow \varphi_2$$

$$\mid \exists x \varphi \mid \forall x \varphi$$
- ▶ *sentences*: formulas without *free* variables.

# First-order logic: semantics (structures)

## Definition

Let  $\tau = \{R_1, \dots, R_n\}$  be a vocabulary.

A  $\tau$ -*structure* is a tuple  $\mathcal{A} = \langle A; R_1^{\mathcal{A}}, \dots, R_n^{\mathcal{A}} \rangle$  consisting of:

- ▶ the *universe*  $A$ ,
- ▶ *interpretations*  $R_i^{\mathcal{A}} \subseteq A^{ar(R_i)} = \overbrace{A \times \dots \times A}^{ar(R_i)}$  for each of the relation symbols  $R_i$  in  $\tau$ , where  $i \in \{1, \dots, n\}$ .

## Examples

Let  $\tau_G = \{E/2\}$  vocabulary with binary edge relation.

The *standard structure* for a graph  $\mathcal{G} = \langle V, E \rangle$ :

$$\mathcal{A}_{\tau_G}(\mathcal{G}) := \langle V; E^{\text{symm}} \rangle.$$

## Example

Let  $\tau_{\text{HG}} = \{VERT/1, EDGE/1, INC/2\}$  vocabulary (for hypergraphs).

The *hypergraph structure* for a graph  $\mathcal{G} = \langle V, E \rangle$ :

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) := \langle V \cup E; V, E, \{\langle v, e \rangle \mid v \in V, e \in E, v \in e\} \rangle.$$



# Interpretation of first-order formulas in structures

Let  $\mathcal{A} = \langle A; \{R^{\mathcal{A}}\}_{R \in \tau} \rangle$  be a  $\tau$ -structure. For a  $\tau$ -formula  $\varphi(x_1, \dots, x_k)$  its *interpretation*  $\varphi(\mathcal{A}) \subseteq A^k$  in  $\mathcal{A}$  is defined by:

- If  $\varphi(x_1, \dots, x_k) \equiv R(x_{i_1}, \dots, x_{i_r})$  with  $i_1, \dots, i_r \in [k]$ , then:

$$\varphi(\mathcal{A}) := \{ \langle a_1, \dots, a_k \rangle \in A^k \mid \langle a_{i_1}, \dots, a_{i_r} \rangle \in R^{\mathcal{A}} \}$$

- If  $\varphi(x_1, \dots, x_k) \equiv \varphi_1(x_{i_1}, \dots, x_{i_l}) \wedge \varphi_2(x_{j_1}, \dots, x_{j_m})$  with  $i_1, \dots, i_l, j_1, \dots, j_m \in [k]$ , then:

$$\begin{aligned} \varphi(\mathcal{A}) := & \{ \langle a_1, \dots, a_k \rangle \in A^k \mid \langle a_{i_1}, \dots, a_{i_l} \rangle \in \varphi_1(\mathcal{A}) \} \\ & \cap \{ \langle a_1, \dots, a_k \rangle \in A^k \mid \langle a_{j_1}, \dots, a_{j_m} \rangle \in \varphi_2(\mathcal{A}) \} \end{aligned}$$

- If  $\varphi(x_1, \dots, x_k) \equiv \exists x_{k+1} \varphi_0(x_{i_1}, \dots, x_{i_\ell})$  with  $i_1, \dots, i_\ell \in [k+1]$ , then:

$$\begin{aligned} \varphi(\mathcal{A}) := & \{ \langle a_1, \dots, a_k \rangle \in A^k \mid \text{there exists } a_{k+1} \in A \\ & \text{such that } \langle a_{i_1}, \dots, a_{i_\ell} \rangle \in \varphi_0(\mathcal{A}) \} \end{aligned}$$

- $\mathcal{A} \models \varphi(a_1, \dots, a_k)$  will mean:  $\langle a_1, \dots, a_k \rangle \in \varphi(\mathcal{A})$ .
- For a sentence  $\varphi$ ,  $\mathcal{A} \models \varphi$  will mean  $\varphi(\mathcal{A}) \neq \emptyset$  (then  $\varphi(\mathcal{A}) = \{ \langle \rangle \}$ ).

# Expressing graph properties by first-order formulas

## Exercise

For given formulas  $\varphi(x)$  and for all  $k \in \mathbb{N}$ ,  $k \geq 1$  define formulas  $\exists^{\geq k} x \varphi(x)$ ,  $\exists^{< k} x \varphi(x)$ ,  $\exists^{=k} x \varphi(x)$ , such that in a given  $\tau$ -structure  $\mathcal{A} = \langle A; \{R^{\mathcal{A}}\}_{R \in \tau} \rangle$ :

$$\mathcal{A} \models \exists^{\geq k} x \varphi(x) \iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| \geq k$$

$$\mathcal{A} \models \exists^{< k} x \varphi(x) \iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| < k$$

$$\mathcal{A} \models \exists^{=k} x \varphi(x) \iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| = k$$

# Expressing graph properties by first-order formulas

## Exercise

Express by a first-order formula with the vocabulary  $\tau_G = \{E/2\}$  for graphs that:

- (i) a graph  $\mathcal{G}$  contains a **clique** with precisely  $k$  elements,
- (ii) a graph  $\mathcal{G}$  has a **dominating set** with less or equal to  $k$  elements,
- (iii) a graph  $\mathcal{G}$  has a **dominating set** with precisely  $k$  elements,

Recall:

$$\varphi_k := \exists x_1 \dots \exists x_k \left( \bigwedge_{1 \leq i < j \leq k} (\neg(x_i = x_j) \wedge \neg E(x_i, x_j)) \right)$$

$$\mathcal{A}_{\tau_G}(\mathcal{G}) \models \varphi_k \iff \mathcal{G} \text{ has a } k\text{-element independent set.}$$

# Expressing graph properties by first-order formulas

## Exercise

Express by a first-order formula with the vocabulary with vocabulary  $\tau_{\text{HG}} = \{ \text{VERT}/_1, \text{EDGE}/_1, \text{INC}/_2 \}$  for hypergraphs that:

- (i) a graph  $\mathcal{G}$  contains a **clique** with precisely  $k$  elements,
- (ii) a graph  $\mathcal{G}$  has a **dominating set** with less or equal to  $k$  elements,
- (iii) a graph  $\mathcal{G}$  has a **dominating set** with precisely  $k$  elements.

# Evaluation and model checking (first-order logic)

Let  $\Phi$  be a class of **first-order** formulas.

The **evaluation problem** for  $\Phi$ :

**EVAL**( $\Phi$ )

**Instance:** A structure  $\mathcal{A}$  and a formula  $\varphi \in \Phi$ .

**Problem:** Compute  $\varphi(\mathcal{A})$ .

The **model checking problem** for  $\Phi$ :

**MC**( $\Phi$ )

**Instance:** A structure  $\mathcal{A}$  and a formula  $\varphi \in \Phi$ .

**Problem:** Decide whether  $\mathcal{A} \models \varphi$  (that is,  $\varphi(\mathcal{A}) \neq \emptyset$ ).

**Width** of formula  $\varphi$ : maximal number of free variables in a subformula of  $\varphi$ .

## Theorem

**EVAL(FO)** and **MC(FO)** can be solved in time  $O(|\varphi| \cdot |\mathcal{A}|^w \cdot w)$ , where  $w$  is the width of the input formula  $\varphi$ .

# Monadic second-order logic (formula example)

$$\begin{aligned}
 \psi_3 := & \exists C_1 \exists C_2 \exists C_3 \left( \left( \forall x \left( \bigvee_{i=1}^3 C_i(x) \right) \right) \wedge \forall x \left( \bigwedge_{1 \leq i < j \leq 3} \neg (C_i(x) \wedge C_j(x)) \right) \right) \\
 & \wedge \forall x \forall y \left( E(x, y) \rightarrow \bigwedge_{i=1}^3 \neg (C_i(x) \wedge C_i(y)) \right)
 \end{aligned}$$

$$\begin{aligned}
 \equiv & \exists C_1 \exists C_2 \exists C_3 \left( \forall x (C_1(x) \vee C_2(x) \vee C_3(x)) \right. \\
 & \wedge \forall x \left( \neg (C_1(x) \wedge C_2(x)) \wedge \neg (C_1(x) \wedge C_3(x)) \right. \\
 & \qquad \qquad \qquad \left. \left. \wedge \neg (C_2(x) \wedge C_3(x)) \right) \right) \\
 & \wedge \forall x \forall y \left( E(x, y) \rightarrow \neg (C_1(x) \wedge C_1(y)) \right. \\
 & \qquad \qquad \qquad \wedge \neg (C_2(x) \wedge C_2(y)) \\
 & \qquad \qquad \qquad \left. \left. \wedge \neg (C_3(x) \wedge C_3(y)) \right) \right)
 \end{aligned}$$

$$\mathcal{A}(\mathcal{G}) \models \psi_3 \iff \mathcal{G} \text{ has is 3-colorable.}$$

# Monadic second-order logic

► language based on:

- a *vocabulary*  $\tau = \{R_1, \dots, R_n\}$  of *predicate symbols*  $R_i$  together with arity  $ar(R_i) \in \mathbb{N}$
- the binary equality predication  $=$
- first-order variable symbols:  $x, y, z, w, x_1, y_1, z_1, w_1, x_2, \dots$
- *monadic second-order variable symbols* (symbolizing variables for unary predicate symbols):  $X, Y, Z, W, X_1, Y_1, Z_1, W_1, X_2, \dots$
- propositional connectives:  $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
- existential quantifier  $\exists$ , universal quantifier  $\forall$

► *atomic formulas (atoms)*:  $x = y \mid R(x_1 \dots x_n) \mid X(x)$  (for  $R \in \tau$ )

► *quantifications* :

- first-order existential quantifications  $\exists x$  and universal quant.  $\forall x$
- second-order existential quantific.  $\exists X$  and universal quantif.  $\forall X$

► *formulas*:

$$\begin{aligned} \varphi ::= & x = y \mid R(x_1, \dots, x_{ar(R)}) \mid X(x) \\ & \mid \neg \varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \rightarrow \varphi_2 \mid \varphi_1 \leftrightarrow \varphi_2 \\ & \mid \exists x \varphi \mid \forall x \varphi \mid \exists X \varphi \mid \forall X \varphi \end{aligned}$$

# Interpretation of MSO-formulas in first-order structures

Let  $\mathcal{A} = \langle A; \{R^{\mathcal{A}}\}_{R \in \tau} \rangle$  be a  $\tau$ -structure.

For a  $\text{MSO}(\tau)$ -formula  $\varphi(x_1, \dots, x_k, X_1, \dots, X_\ell)$  its *interpretation*  $\varphi(\mathcal{A}) \subseteq A^k \times \mathcal{P}(A)^\ell$  in  $\mathcal{A}$  is defined by:

- ▶ similar clauses as before, plus:
- ▶ If  $\varphi(x_1, \dots, x_k, X_1, \dots, X_\ell) \equiv X_i(x_j)$  with  $i \in [k]$  and  $j \in [\ell]$ , then:

$$\varphi(\mathcal{A}) := \{ \langle a_1, \dots, a_k, P_1, \dots, P_\ell \rangle \in A^k \times \mathcal{P}(A)^\ell \mid a_j \in P_i \}$$

- ▶ If  $\varphi(x_1, \dots, x_k, X_1, \dots, X_\ell) \equiv \exists X_{k+1} \varphi_0(x_{i_1}, \dots, x_{i_{k'}}, X_{j_1}, \dots, X_{j_{\ell'}})$  with  $i_1, \dots, i_{k'} \in [k]$ , and  $j_1, \dots, j_{\ell'} \in [\ell + 1]$  then:

$$\varphi(\mathcal{A}) := \{ \langle a_1, \dots, a_k, P_1, \dots, P_\ell \rangle \in A^k \times \mathcal{P}(A)^\ell \mid \text{there exists } P_{\ell+1} \in \mathcal{P}(A) \text{ such that } \langle a_{i_1}, \dots, a_{i_{k'}}, P_{j_1}, \dots, P_{j_{\ell'}} \rangle \in \varphi_0(\mathcal{A}) \}$$

- ▶  $\mathcal{A} \models \varphi(a_1, \dots, a_k, P_1, \dots, P_\ell)$   
will mean:  $\langle a_1, \dots, a_k, P_1, \dots, P_\ell \rangle \in \varphi(\mathcal{A})$ .

- ▶ For a sentence  $\varphi$ ,  $\mathcal{A} \models \varphi$  will mean  $\varphi(\mathcal{A}) \neq \emptyset$  (then  $\varphi(\mathcal{A}) = \{ \langle \rangle \}$ ).



# Monadic second-order logic (formula example)

$$\psi_3 := \exists C_1 \exists C_2 \exists C_3 \left( \left( \forall x \left( \bigvee_{i=1}^3 C_i(x) \right) \right) \wedge \forall x \left( \bigwedge_{1 \leq i < j \leq 3} \neg (C_i(x) \wedge C_j(x)) \right) \right) \\ \wedge \forall x \forall y \left( E(x, y) \rightarrow \bigwedge_{i=1}^3 \neg (C_i(x) \wedge C_i(y)) \right)$$

$$\equiv \exists C_1 \exists C_2 \exists C_3 \left( \forall x (C_1(x) \vee C_2(x) \vee C_3(x)) \right. \\ \wedge \forall x (\neg (C_1(x) \wedge C_2(x)) \wedge \neg (C_1(x) \wedge C_3(x)) \\ \wedge \neg (C_2(x) \wedge C_3(x))) \\ \wedge \forall x \forall y (E(x, y) \rightarrow \neg (C_1(x) \wedge C_1(y)) \\ \wedge \neg (C_2(x) \wedge C_2(y)) \\ \wedge \neg (C_3(x) \wedge C_3(y))) \left. \right)$$

$$\mathcal{A}(\mathcal{G}) \models \psi_3 \iff \mathcal{G} \text{ has is } 3\text{-colorable.}$$

# Expressing graph properties by MSO formulas (1)

## Exercise

Express by a monadic second-order formula  $\varphi(X)$  with one free unary predicate variable  $X$  over the vocabulary  $\tau_G = \{E/2\}$  for graphs that for all graphs  $\mathcal{G} = \langle V, E \rangle$ :

$$\mathcal{A}_{\tau_G}(\mathcal{G}) \models \varphi(S) \iff S \subseteq V \text{ is an independent set in } \mathcal{G}$$

Recall:

$$\begin{aligned} S \subseteq V \text{ is independent set in } \mathcal{G} &\iff \forall e = \{u, v\} \in E \ ( \neg(u \in S \wedge v \in S) ) \\ &\iff \forall u, v \in S \ ( u \neq v \Rightarrow \{u, v\} \notin E ) \end{aligned}$$

## Exercise

Express the independent set property by a  $\text{MSO}(\tau_{\text{HG}})$  formula  $\psi$  with vocabulary  $\tau_{\text{HG}} = \{ \text{VERT}/1, \text{EDGE}/1, \text{INC}/2 \}$  for hypergraphs:

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \models \psi(S) \iff S \subseteq V \text{ is an independent set in } \mathcal{G}$$

# Expressing graph properties by MSO formulas (2)

## Exercise

Express by a monadic second-order formula  $\text{feedback}(X)$  with one free unary predicate variable  $X$  over  $\tau_{\text{HG}} = \{\text{VERT}/_1, \text{EDGE}/_1, \text{INC}/_2\}$ , the vocabulary for graphs, that for all hypergraphs  $\mathcal{G} = \langle V, E \rangle$ :

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \models \text{feedback}(S) \iff S \subseteq V \text{ is a feedback vertex set}$$

(A set  $S \subseteq V$  is a feedback vertex set in  $\mathcal{G}$  if  $S$  contains a vertex of every cycle of  $\mathcal{G}$ .)

Steps:

- ▶ Construct a formula  $\text{cycle-family}(X)$  that expresses the property of a set being the union of cycles.
- ▶ Using  $\text{cycle-family}(X)$ , construct  $\text{feedback}(X)$ .

# MSO for graphs and hypergraphs

- ▶  $\text{MSO}(\tau_G)$ : MSO with vocabulary  $\tau_G = \{E/2\}$
- ▶  $\text{MSO}(\tau_{\text{HG}})$ : MSO with vocab.  $\tau_{\text{HG}} = \{\text{VERT}/1, \text{EDGE}/1, \text{INC}/2\}$
- ▶  $\text{MSO}_1$ :
  - ▶ vocabulary:  $\{\text{INC}/2\}$
  - ▶ quantifications:  $\exists_{(\text{vert})}x / \forall_{(\text{vert})}x, \exists_{(\text{edge})}x / \forall_{(\text{edge})}x,$   
 $\exists_{(\text{vert})}X / \forall_{(\text{vert})}X$
- ▶  $\text{MSO}_2$ :
  - ▶ vocabulary:  $\{\text{INC}/2\}$
  - ▶ quantifications:  $\exists_{(\text{vert})}x / \forall_{(\text{vert})}x, \exists_{(\text{edge})}x / \forall_{(\text{edge})}x,$   
 $\exists_{(\text{vert})}X / \forall_{(\text{vert})}X, \exists_{(\text{edge})}X / \forall_{(\text{edge})}X$

## Correspondences

$\text{MSO}(\tau_G)$  corresponds to  $\text{MSO}_1$   
 $\text{MSO}(\tau_{\text{HG}})$  corresponds to  $\text{MSO}_2$

where 'corresponds to' means: 'is equally expressive as'.

### Note:

We use MSO for either logic, restrict to  $\text{MSO}(\tau_G)$  /  $\text{MSO}_1$  if needed.

# Expressing graph properties by MSO formulas (5)

## Exercise

Express by a  $\text{MSO}(\tau_{\text{HG}})$  formula  $\text{conn}(X)$  with one free unary predicate variable  $X$  over  $\tau_{\text{HG}} = \{\text{VERT}/_1, \text{EDGE}/_1, \text{INC}/_2\}$ , the vocabulary for graphs, that for all hypergraphs  $\mathcal{G} = \langle V, E \rangle$ :

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \models \text{hamiltonian} \iff \text{there is a Hamiltonian path in } \mathcal{G}.$$

## Note:

- ▶ This property is not expressible by a (single)  $\text{MSO}(\tau_{\text{G}})$  formula.
- ▶ Other properties that are **not**  $\text{MSO}(\tau_{\text{G}})$  expressible:
  - ▶ balanced bipartite graphs
  - ▶ existence of a perfect matching
  - ▶ simple graphs (graphs with no parallel edges)
  - ▶ existence of spanning trees with maximum degree 3

# Expressing graph properties by MSO formulas (5)

## Exercise

$\mathcal{A}_{\text{THG}}(\mathcal{G}) \models \textit{hamiltonian} \iff$  there is a Hamiltonian path in  $\mathcal{G}$ .

# Evaluation and model checking (MSO)

The *model checking problem* for MSO-formulas on labeled, ordered unranked trees:

MC(MSO, TREE<sub>lo</sub>)

**Instance:** A labeled, ordered, unranked  $\Sigma$ -tree  $\mathcal{T}$ ,  
and a MSO( $\tau_\Sigma^u$ )-formula  $\varphi$

**Problem:** Decide whether  $\mathcal{T} \models \varphi$ .

where for given alphabet  $\Sigma$ ,  $\tau_\Sigma^u := \{E/2, N/2\} \cup \{P_a/1 \mid a \in \Sigma\}$ .

## Theorem

MC(MSO, TREE<sub>lo</sub>)  $\in$  FPT.

More precisely, there is a computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that MC(MSO, TREE<sub>lo</sub>) can be decided in time  $\leq O(f(|\varphi|) + \|\mathcal{T}\|)$ .

**Note** that here:  $f(k) \geq 2^{\cdot^{\cdot^{\cdot^2}}}$  (a non-elementary function).

# Courcelle's Theorem



# Courcelle's Theorem for graphs

$p^*$ -*tw*-MC(MSO)

**Instance:** A graph  $\mathcal{G}$  and an  $\text{MSO}(\tau_{\text{HG}})$ -sentence  $\varphi$ .

**Parameter:**  $\text{tw}(\mathcal{G}) + |\varphi|$  (where  $\text{tw}(\mathcal{G})$  the tree-width of  $\mathcal{G}$ )

**Problem:** Decide whether  $\mathcal{A}(\mathcal{G}) \models \varphi$ .

## Theorem (special case of Courcelle's Theorem)

$p^*$ -*tw*-MC(MSO)  $\in \text{FPT}$ . More precisely, the problem is decidable, for some computable and non-decreasing function  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  by an algorithm in time:

$$f(k_1, k_2) \cdot n, \quad \text{where } k_1 := \text{tw}(\mathcal{A}), k_2 := |\varphi|, n := |V(\mathcal{G})|$$

# Courcelle's Theorem: applications (1)

$p^*$ - $tw$ -COLORABILITY  $\in$  **FPT**

**Instance:** A graph  $\mathcal{G}$  and  $\ell \in \mathbb{N}$ .

**Parameter:**  $tw(\mathcal{C})$

**Problem:** Decide whether is  $\mathcal{G}$   $\ell$ -colorable.

## Example

- ▶  $p^*$ - $tw$ -3-COLORABILITY  $\in$  **FPT**.
- ▶  $p^*$ - $tw$ -COLORABILITY  $\in$  **FPT**.

# Courcelle's Theorem: applications (2)

$p^*$ - $tw$ -HAMILTONICITY

**Instance:** A graph  $\mathcal{G}$

**Parameter:**  $tw(\mathcal{C})$

**Problem:** Decide whether  $\mathcal{G}$  is a hamiltonian (that is, contains a cyclic path that visits every vertex precisely once).

Example

$p^*$ - $tw$ -HAMILTONICITY  $\in$  FPT.

# Tree decompositions, and tree-width for graphs

## Definition (recalling tree-width for graphs)

A **tree decomposition** of a graph  $\mathcal{G} = \langle V, E \rangle$  is a pair  $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$  where  $\mathcal{T} = \langle T, F \rangle$  a (undirected, unrooted) tree, and  $B_t \subseteq V$  for all  $t \in T$  such that:

- (T1)  $A = \bigcup_{t \in T} B_t$  (every vertex of  $\mathcal{G}$  is in some bag).
- (T2)  $(\forall \{u, v\} \in E) (\exists t \in T) [\{u, v\} \subseteq B_t]$   
(the vertices of **every edge** of  $\mathcal{G}$  are realized in some bag).
- (T3)  $(\forall v \in V) [\text{subgraph of } \mathcal{T} \text{ defd. by } \{t \in T \mid v \in B_t\} \text{ is connected}]$   
(the tree vertices (in  $\mathcal{T}$ ) whose bags contain some vertex of  $\mathcal{G}$  induce a subgraph of  $\mathcal{T}$  that is connected).

The **width** of a tree dec.  $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$  is  $\max \{|B_t| - 1 \mid t \in T\}$ .

The **tree-width**  $tw(\mathcal{A})$  of a  $\tau$ -structure  $\mathcal{A}$  is defined by:

$tw(\mathcal{A}) :=$  minimal width of a tree decomposition of  $\mathcal{A}$ .

# Tree decompositions, and tree-width for structures

## Definition (extension of tree-width to structures)

A **tree decomposition** of a  $\tau$ -structure  $\mathcal{A} = \langle A; \{R^A\}_{R \in \tau} \rangle$  is a pair  $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$  where  $\mathcal{T} = \langle T, F \rangle$  a (undirected, unrooted) tree, and  $B_t \subseteq V$  for all  $t \in T$  such that:

- (T1)  $A = \bigcup_{t \in T} B_t$  (every element of the universe of  $\mathcal{A}$  is in some bag).
- (T2)  $(\forall R \in \tau) (\forall \langle a_1, \dots, a_r \rangle \in R^A) (\exists t \in T) [\{a_1, \dots, a_r\} \subseteq B_t]$   
(the vertices of every 'hyperedge' in  $R^A$  are realized in some bag).
- (T3)  $(\forall v \in V) [\text{subgraph of } \mathcal{T} \text{ defd. by } \{t \in T \mid v \in B_t\} \text{ is connected}]$   
(the tree vertices (in  $\mathcal{T}$ ) whose bags contain some vertex of  $\mathcal{G}$  induce a subgraph of  $\mathcal{T}$  that is connected).

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The **tree-width**  $tw(\mathcal{A})$  of a  $\tau$ -structure  $\mathcal{A}$  is defined by:

$tw(\mathcal{A}) :=$  minimal width of a tree decomposition of  $\mathcal{A}$ .

# Courcelle's Theorem

$p^*$ -tw-MC(MSO)

**Instance:** A structure  $\mathcal{A}$  and an MSO-sentence  $\varphi$ .

**Parameter:**  $tw(\mathcal{A}) + |\varphi|$ .

**Problem:** Decide whether  $\mathcal{A} \models \varphi$ .

Theorem ([Courcelle, 1990])

$p^*$ -tw-MC(MSO)  $\in$  FPT. More precisely, the problem is decidable by an algorithm in time:

$f(k_1, k_2) \cdot |A| + O(\|\mathcal{A}\|)$ , where  $k_1 := tw(\mathcal{A})$ , and  $k_2 := |\varphi|$ ,  
 $f$  computable and non-decreasing

$$\begin{aligned}
 f(k_1, k_2) \cdot |A| + O(\|\mathcal{A}\|) &\leq f(k_1, k_2) \cdot |A| + c \cdot \|\mathcal{A}\| && \text{with some } c > 0 \\
 &\leq (f(k_1, k_2) + c) \cdot \|\mathcal{A}\| \\
 &\leq g(k) \cdot (\|\mathcal{A}\| + |\varphi|) && \text{for } g(x) := f(x, x) + c \\
 & && k := k_1 + k_2 \\
 &\leq g(k) \cdot n && \text{where } n := \|\mathcal{A}\| + |\varphi|
 \end{aligned}$$

# Vertex Cover (first attempt)

Let  $\mathcal{G} = \langle V, E \rangle$  a graph. For all  $S \subseteq V$ :

$S$  is a **vertex cover** of  $\mathcal{G} : \iff \forall e = \{u, v\} \in E (u \in S \vee v \in S)$

$p^*$ -tw-VERTEX-COVER

**Instance:** A graph  $\mathcal{G} = \langle V, E \rangle$ , and  $\ell \in \mathbb{N}$ .

**Instance:**  $tw(\mathcal{G})$ .

**Problem:** Does  $\mathcal{G}$  have a vertex cover of size at most  $\ell$ ?

# Courcelle's Theorem: Refinement 1

$p^*\text{-tw-MC}^{\leq}(\text{MSO})$

**Instance:** A structure  $\mathcal{A}$ , an  $\varphi(X)$ , and  $m \in \mathbb{N}$ .

**Parameter:**  $\text{tw}(\mathcal{A}) + |\varphi(X)|$ .

**Problem:** Decide whether  $\mathcal{A} \models \exists X ( \text{card}^{\leq m}(X) \wedge \varphi(X) )$ .

## Refinement 1 of Courcelle's Theorem

$p^*\text{-tw-MC}^{\leq}(\text{MSO}) \in \text{FPT}$ . More precisely, the problem is decidable by an algorithm in time:

$$f(k_1, k_2) \cdot |A| + O(\|\mathcal{A}\|), \quad \text{where } k_1 := \text{tw}(\mathcal{A}), \text{ and } k_2 := |\varphi|, \\ f \text{ computable and non-decreasing}$$



# Vertex Cover

Let  $\mathcal{G} = \langle V, E \rangle$  a graph. For all  $S \subseteq V$ :

$S$  is a **vertex cover** of  $\mathcal{G} : \iff \forall e = \{u, v\} \in E (u \in S \vee v \in S)$

$p^*$ -tw-VERTEX-COVER

**Instance:** A graph  $\mathcal{G} = \langle V, E \rangle$ , and  $\ell \in \mathbb{N}$ .

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Example

$p^*$ -tw-VERTEX-COVER  $\in$  FPT.

# Vertex Cover

Let  $\mathcal{G} = \langle V, E \rangle$  a graph. For all  $S \subseteq V$ :

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**Instance:** A graph  $\mathcal{G} = \langle V, E \rangle$ , and  $\ell \in \mathbb{N}$ .

**Instance:**  $tw(\mathcal{G})$ .

**Problem:** Does  $\mathcal{G}$  have a vertex cover of size at most  $\ell$ ?

Example

$p^*$ -tw-VERTEX-COVER  $\in$  FPT.

# Courcelle's Theorem: Refinement 2

$p^*\text{-tw-MC}^=(\text{MSO})$

**Instance:** A structure  $\mathcal{A}$ , an MSO-sentence  $\varphi(X)$ , and  $m \in \mathbb{N}$ .

**Parameter:**  $\text{tw}(\mathcal{A}) + |\varphi(X)|$ .

**Problem:** Decide whether  $\mathcal{A} \models \exists X ( \text{card}^m(X) \wedge \varphi(X) )$ .

## Refinement 2 of Courcelle's Theorem

$p^*\text{-tw-MC}^=(\text{MSO}) \in \text{FPT}$ . More precisely, the problem is decidable by an algorithm in time:

$f(k_1, k_2) \cdot |A|^{k_1} + O(\|\mathcal{A}\|)$ , where  $k_1 := \text{tw}(\mathcal{A})$ , and  $k_2 := |\varphi|$ ,  
 $f$  computable and non-decreasing

# Courcelle's Theorem Ref. 3: Optimization version

$p^*$ -**tw-opt-MC**(**MSO**)

**Instance:** A graph  $\mathcal{G} = \langle V, E \rangle$ , an **MSO**-sentence  $\varphi(X_1, \dots, X_p)$ .

**Parameter:**  $\text{tw}(\mathcal{G}) + |\varphi(X_1, \dots, X_p)|$ .

**Compute:**  $\max_{\min} \left\{ \alpha(|X_1|, \dots, |X_p|) \mid \begin{array}{l} X_1, \dots, X_p \subseteq V \cup E \\ \mathcal{A}(\mathcal{G}) \models \varphi(X_1, \dots, X_p) \end{array} \right\}$ .

where  $\alpha$  is an affine function  $\alpha(x_1, \dots, x_p) = a_0 + \sum_{i=1}^p a_i \cdot x_i$ .

## Optimization version of Courcelle's Theorem

$p^*$ -**tw-opt-MC**(**MSO**)  $\in$  **FPT**, and it is decidable by an algorithm in time:

$f(\text{tw}(\mathcal{G}), |\varphi|) \cdot |V|$ , where  $f$  computable and non-decreasing.

# Maximum 2-edge colorable subgraphs

$p^*$ - $tw$ -max-2-edge-colorable-subgraph

**Instance:** A graph  $\mathcal{G} = \langle V, E \rangle$ .

**Parameter:**  $tw(\mathcal{G})$ .

**Compute:** Maximum number of edges  
in a 2-edge colored subgraph of  $G$ .

Example [AA & Vahan Mkrtchyan]

$p^*$ - $tw$ -max-2-edge-colorable-subgraph  $\in$  FPT.

# Maximum 2-edge colorable subgraphs

$p^*$ -tw-max-2-edge-colorable-subgraph

**Instance:** A graph  $\mathcal{G} = \langle V, E \rangle$ .

**Parameter:**  $tw(\mathcal{G})$ .

**Compute:** Maximum number of edges  
in a 2-edge colored subgraph of  $\mathcal{G}$ .

Example [AA & Vahan Mkrtchyan]

$p^*$ -tw-max-2-edge-colorable-subgraph  $\in$  FPT.

# Courcelle's Theorem: applications (3)

$p^*$ - $tw$ -INDEPENDENT-SET

**Instance:** A graph  $\mathcal{G}$ , a number  $\ell \in \mathbb{N}$ .

**Parameter:**  $tw(\mathcal{G})$

**Problem:** Decide whether  $\mathcal{G}$  has an independent set of  $\ell$  elements.

## Example

$p^*$ - $tw$ -INDEPENDENT-SET  $\in$  FPT.

# Courcelle's Theorem: applications (4)

$p^*$ - $tw$ -FEEDBACK-VERTEX-SET

**Instance:** A graph  $\mathcal{G}$  and  $\ell \in \mathbb{N}$ .

**Parameter:**  $tw(\mathcal{C})$

**Problem:** Decide whether  $\mathcal{G}$  has a feedback vertex set of  $\ell$  elements.

## Example

$p^*$ - $tw$ -FEEDBACK-VERTEX-SET  $\in$  FPT.



# Courcelle's Theorem: applications (5)

$p^*$ - $tw$ -CROSSING-NUMBER

**Instance:** A graph  $\mathcal{G}$ , and  $k \in \mathbb{N}$

**Parameter:**  $tw(\mathcal{G}) + k$

**Problem:** Decide whether the crossing number of  $\mathcal{G}$  is  $k$ .

## Example

$p^*$ - $tw$ -CROSSING-NUMBER  $\in$  FPT.

The *crossing number* is the least number of edge crossings required to draw the graph in the plane such that at each point at most two edges cross.

# Courcelle's Theorem: applications (5)

## Definition

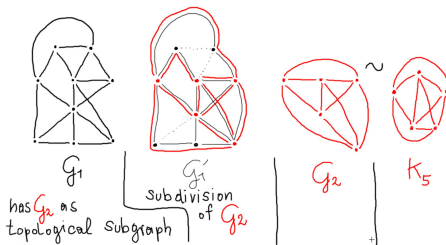
Let  $\mathcal{G}_1 = \langle V_1, E_1 \rangle$  and  $\mathcal{G}_2 = \langle V_2, E_2 \rangle$  be graphs.

$\mathcal{G}_1$  is a *subdivision* of  $\mathcal{G}_2$  if:

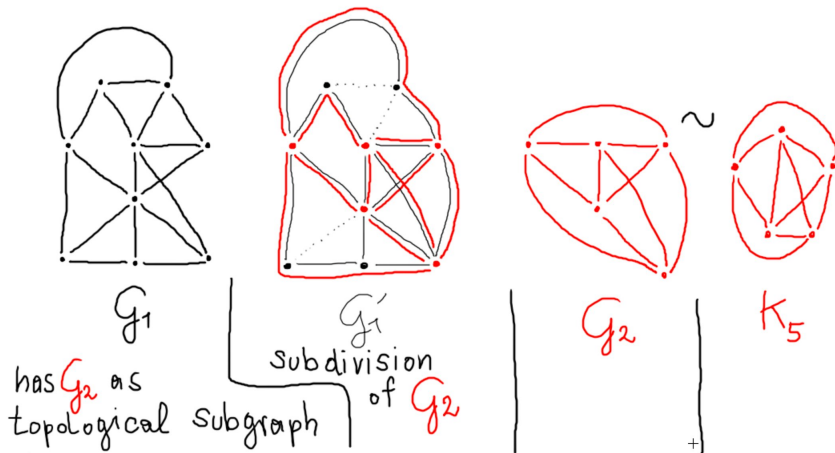
- ▶  $\mathcal{G}_1$  arises by splitting the edges of  $\mathcal{G}_2$  into paths with intermediate vertices.

$\mathcal{H}$  is a *topological subgraph* of  $\mathcal{G}$

if  $\mathcal{G}$  has a subgraph that is a subdivision of  $\mathcal{H}$ .



# Courcelle's Theorem: applications (5)



## Theorem (Kuratowski)

*A graph is planar if and only if it contains neither  $K_5$  nor  $K_{3,3}$  as topological subgraph.*

# Courcelle's Theorem: applications (5)

## Theorem (Kuratowski)

*A graph is planar if and only if it contains neither  $\mathcal{K}_5$  nor  $\mathcal{K}_{3,3}$  as topological subgraph.*

## Lemma

There is a **MSO**( $\tau_{\text{HG}}$ ) formula  $\text{top-sub}_{\mathcal{H}}$  such that for every graph  $\mathcal{G}$ :

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \models \text{top-sub}_{\mathcal{H}} \iff \mathcal{H} \text{ is a topological subgraph of } \mathcal{G}.$$

Using **MSO**( $\tau_{\text{HG}}$ ) formula  $\text{path}(x, y, Z)$  that  $Z$  is a path from  $x$  to  $y$ .

## Lemma

There is a **MSO**( $\tau_{\text{HG}}$ ) formula  $\text{cross}_k$  such that for every graph  $\mathcal{G}$ :

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \models \text{cross}_k \iff \text{the crossing number of } \mathcal{G} \text{ is at most } k.$$

**Proof:** By induction, where  $\text{cross}_0 := \neg \text{top-sub}_{\mathcal{K}_5} \wedge \neg \text{top-sub}_{\mathcal{K}_{3,3}}$ .

# Computably boundedness between notions of width

(from Sasák, [Sásak, 2010])

$$g(wd_1) \geq wd_2 : \Leftrightarrow wd_1 \xrightarrow{g} wd_2$$

► FPT-results

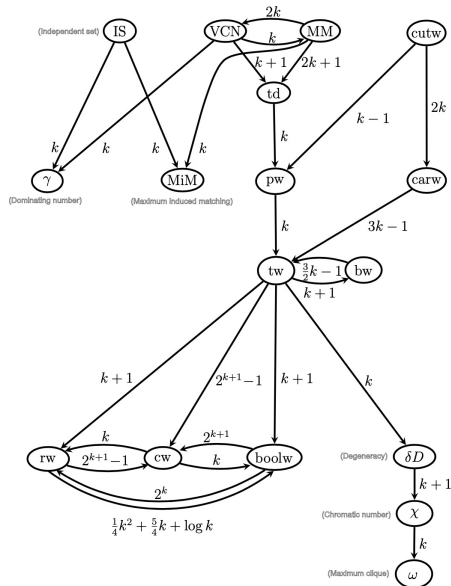
transfer upwards

(and conversely to  $\xrightarrow{g}$ )

► ( $\notin$  FPT)-results

transfer downwards

(and along  $\xrightarrow{g}$ )



# Comparing parameterizations

## Definition (computably bounded below)

Let  $\kappa_1, \kappa_2 : \Sigma^* \rightarrow \mathbb{N}$  parameterizations.

- ▶  $\kappa_1 \geq \kappa_2 : \iff \exists g : \mathbb{N} \rightarrow \mathbb{N} \text{ computable } \forall x \in \Sigma^* [g(\kappa_1(x)) \geq \kappa_2(x)]$ .
- ▶  $\kappa_1 \approx \kappa_2 : \iff \kappa_1 \geq \kappa_2 \wedge \kappa_2 \geq \kappa_1$ .
- ▶  $\kappa_1 > \kappa_2 : \iff \kappa_1 \geq \kappa_2 \wedge \neg(\kappa_2 \geq \kappa_1)$ .

## Proposition

For all parameterized problems  $\langle Q, \kappa_1 \rangle$  and  $\langle Q, \kappa_2 \rangle$  with parameterizations  $\kappa_1, \kappa_2 : \Sigma^* \rightarrow \mathbb{N}$  with  $\kappa_1 \geq \kappa_2$ :

$$\langle Q, \kappa_1 \rangle \in \text{FPT} \iff \langle Q, \kappa_2 \rangle \in \text{FPT}$$

$$\langle Q, \kappa_1 \rangle \notin \text{FPT} \implies \langle Q, \kappa_2 \rangle \notin \text{FPT}$$

# Courcelle's Theorem for clique-width

Recall that  $\text{MSO}(\tau_G) \sim \text{MSO}_1$  (quantification over sets of vertices, but not sets of edges).

$p^*$ - $\text{clw}$ -MC( $\text{MSO}(\tau_G)/\text{MSO}_1$ )

**Instance:** A graph  $\mathcal{G}$  and an  $\text{MSO}(\tau_G)$ -sentence  $\varphi$ .

**Parameter:**  $\text{clw}(\mathcal{G}) + |\varphi|$ .

**Problem:** Decide whether  $\mathcal{A}(\mathcal{G}) \models \varphi$ .

Theorem ([Courcelle et al., 2000])

$p^*$ - $\text{clw}$ -MC( $\text{MSO}(\tau_G)/\text{MSO}_1$ )  $\in \text{FPT}$ .

Also, there is a **maximization version** of this theorem.

# Courcelle's Theorem for clique-width (example)

Let  $\mathcal{G} = \langle V, E \rangle$  a graph. For all  $S \subseteq V$ :

$S$  is a **vertex cover** of  $\mathcal{G} : \iff \forall e = \{u, v\} \in E (u \in S \vee v \in S)$

$p^*$ -**clw**-VERTEX-COVER

**Instance:** A graph  $\mathcal{G} = \langle V, E \rangle$ , and  $\ell \in \mathbb{N}$ .

**Instance:** **clw**( $\mathcal{G}$ ).

**Problem:** Does  $\mathcal{G}$  have a vertex cover of size at most  $\ell$ ?

## Example

$p^*$ -**clw**-VERTEX-COVER  $\in$  **FPT**.



# Application to maximum 2-edge colorable subgraphs?

$p^*$ -**clw**-max-2-edge-colorable-subgraph

**Instance:** A graph  $\mathcal{G} = \langle V, E \rangle$ .

**Parameter:** **clw**( $\mathcal{G}$ ).

**Compute:** Maximum number of edges  
in a 2-edge colored subgraph of  $\mathcal{G}$ .

Open problem [AA, Vahan Mkrtchyan]

$p^*$ -**clw**-max-2-edge-colorable-subgraph  $\in$  **FPT** ?

We saw that there is a **MSO**<sub>2</sub> formula  $\varphi(X)$  such that:

$$\mathcal{A}_{\text{THG}}(\mathcal{G}) \models \varphi(S) \iff S \subseteq E \text{ is an 2-colorable set of edges in } \mathcal{G}$$

But there seems not to be such an **MSO**<sub>1</sub> formula.

# Courcelle's Theorem for clique-width (non-example)

$p^*$ -**clw**-HAMILTONICITY

**Instance:** A graph  $\mathcal{G}$

**Parameter:** **clw**( $\mathcal{C}$ )

**Problem:** Decide whether  $\mathcal{G}$  is a hamiltonian (that is, contains a cyclic path that visits every vertex precisely once).

## Recall

There is no **MSO**<sub>1</sub> formula that expresses Hamiltonicity.

Hence we cannot apply Courcelle's Theorem for clique-width. Indeed:

## Theorems

- (T1)  $p^*$ -**clw**-HAMILTONICITY  $\notin$  FPT,  
since it is not decidable in time  $\notin n^{o(\text{clw}(\mathcal{C}))}$  (Fomin et al, 2014).
- (T2)  $p^*$ -**clw**-HAMILTONICITY  $\in O(n^{o(\text{clw}(\mathcal{C}))})$   
(Bergougnoux, Kanté, Kwon, 2020).

# Computably boundedness between notions of width

(from Sasák, [Sásak, 2010])

$$g(wd_1) \geq wd_2 : \Leftrightarrow wd_1 \xrightarrow{g} wd_2$$

► FPT-results

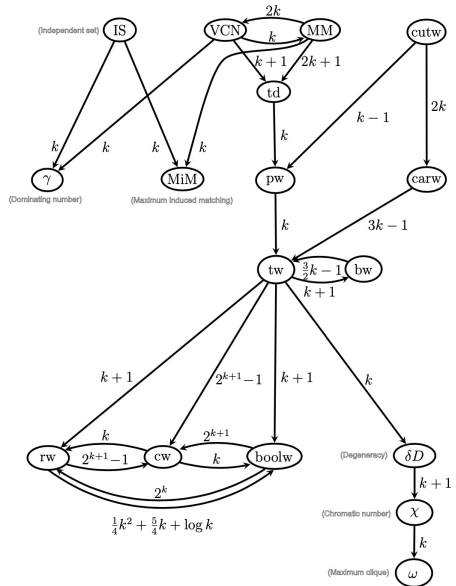
transfer upwards

(and conversely to  $\xrightarrow{g}$ )

► ( $\notin$  FPT)-results

transfer downwards

(and along  $\xrightarrow{g}$ )



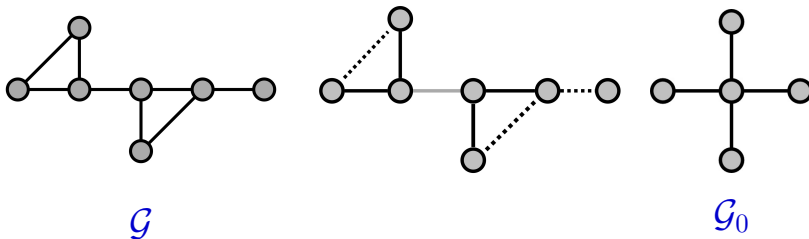
# Graph Minors

# Graph minors

## Definition

A graph  $\mathcal{G}_0$  is a *minor* of a graph  $\mathcal{G}$  if  $\mathcal{G}_0$  is obtained by:

- ▶ deleting some edges,
- ▶ deleting arising isolated vertices,
- ▶ contracting edges in  $\mathcal{G}$ .



# Excluded minors

## Definition (minor closed)

A class  $\mathcal{G}$  is *minor closed* if for every  $\mathcal{G} \in \mathcal{G}$  all minors of  $\mathcal{G}$  are in  $\mathcal{G}$ .

We say that a class  $\mathcal{G}$  is characterized by excluded minors in  $\mathcal{H}$  if:

$$\mathcal{G} := \text{Excl}(\mathcal{H}) := \{\mathcal{G} \mid \mathcal{G} \text{ does not have a minor in } \mathcal{H}\}$$

## Theorem (Graph Minor Theorem (Robertson–Seymour, 1983–2004))

*Every class of graphs that is minor closed can be characterized by finitely many excluded minors. That is, for every class  $\mathcal{G}$  of minor closed graphs there are graphs  $\mathcal{H}_1, \dots, \mathcal{H}_m$  such that:*

$$\mathcal{G} = \text{Excl}(\{\mathcal{H}_1, \dots, \mathcal{H}_m\}).$$

# Deciding minor closed classes

$p$ -MINOR

**Instance:** Graphs  $\mathcal{G}$  and  $\mathcal{H}$ .

**Parameter:**  $\|\mathcal{G}\|$

**Problem:** Decide whether  $\mathcal{G}$  is a minor of  $\mathcal{H}$ .

## Theorem

$p$ -MINOR  $\in$  **FPT**, decidable in time  $f(k) \cdot n^3$  where  $k = \|\mathcal{G}\|$ , and  $n$  is the number of vertices of  $\mathcal{H}$ .

## Corollary

Every minor-closed class of graphs is decidable in cubic time.

## Corollary

Let  $\langle Q, \kappa \rangle$  be a parameterized problem on graphs such that for every  $k \in \mathbb{N}$ , either  $\{\mathcal{G} \in Q \mid \kappa(\mathcal{G}) = k\}$  or  $\{\mathcal{G} \notin Q \mid \kappa(\mathcal{G}) = k\}$  is minor closed.

Then every slice  $\langle Q, \kappa \rangle_k$  is decidable in cubic time. In this case we can say that  $\langle Q, \kappa \rangle$  is **nonuniformly fixed-parameter tractable**.

# Non-uniformly fixed-parameter tractable

A parameterized problem  $\langle Q, \Sigma, \kappa \rangle$  is *fixed-parameter tractable* if:

$\exists f : \mathbb{N} \rightarrow \mathbb{N}$  computable  $\exists p \in \mathbb{N}[X]$  polynomial

$\exists \mathbb{A}$  algorithm, takes inputs in  $\Sigma^*$

$\forall x \in \Sigma^* [ \mathbb{A} \text{ decides whether } x \in Q \text{ holds}$   
in time  $\leq f(\kappa(x)) \cdot p(|x|) ]$

## Definition

A parameterized problem  $\langle Q, \Sigma, \kappa \rangle$  is *non-uniformly fixed-parameter tractable* (in *nu-FPT*) if:

$\exists f : \mathbb{N} \rightarrow \mathbb{N}$  computable  $\exists p \in \mathbb{N}[X]$  polynomial

$\exists \{ \mathbb{A}_k \}_{k \in \mathbb{N}}$  algorithms, takes inputs in  $\Sigma^*$

$\forall x \in \Sigma^* [ \mathbb{A}_{\kappa(x)} \text{ decides whether } x \in Q \text{ holds}$   
in time  $\leq f(\kappa(x)) \cdot p(|x|) ]$



# Using minor-closed classes for FPT results

## Corollary

Let  $\langle Q, \kappa \rangle$  be a parameterized problem on graphs such that for every  $k \in \mathbb{N}$ , either  $\{\mathcal{G} \in Q \mid \kappa(\mathcal{G}) = k\}$  or  $\{\mathcal{G} \notin Q \mid \kappa(\mathcal{G}) = k\}$  is minor closed. Then  $\langle Q, \kappa \rangle$  is *non-uniformly fixed-parameter tractable* (in *nu-FPT*).

Applications:

- ▶  $p$ -VERTEX-COVER  $\in$  *nu-FPT* ( $p$ -VERTEX-COVER is minor closed).
- ▶  $p$ -FEEDBACK-VERTEX-SET  $\in$  *nu-FPT* (problem is minor closed).



$p$ -DISJOINT-CYCLES

**Instance:** A graph  $\mathcal{G}$ , and  $k \in \mathbb{N}$ .

**Parameter:**  $k$ .

**Problem:** Decide whether  $\mathcal{G}$  has  $k$  disjoint cycles.

$p$ -DISJOINT-CYCLES  $\in$  *nu-FPT*, since the class of graphs that do not have  $k$  disjoint cycles is minor closed.

# First-Order Meta-Theorem (example)

# Seese's theorem

A class  $\mathcal{G}$  of graphs has *bounded degree* if there is  $d \in \mathbb{N}$  such that  $\Delta(\mathcal{G}) \leq d$  for all  $\mathcal{G} \in \mathcal{G}$  (where  $\Delta(\mathcal{G}) = \max.$  degree of vertex in  $\mathcal{G}$ ).

$p\text{-MC}(\text{FO}, \mathcal{G})$

**Instance:** A graph  $\mathcal{G} \in \mathcal{G}$ , and a f-o formula  $\varphi$  over  $\tau_{\text{HG}}$

**Parameter:**  $|\varphi|$ .

**Problem:** Decide whether  $\mathcal{A}(\mathcal{G}) \models \varphi$ .

**Theorem ([Seese, 1995])**

$p\text{-MC}(\text{FO}, \mathcal{G}) \in \text{FPT}$  for every class  $\mathcal{G}$  of bounded degree. This model checking problem can be solved in time  $f(|\varphi|) \cdot |\mathcal{G}|$ , (linear in  $|\mathcal{G}|$ ).

**Theorem (for comparison, we saw it earlier)**

$\text{EVAL}(\text{FO})$  and  $\text{MC}(\text{FO})$  can be solved in time  $O(|\varphi| \cdot |A|^w \cdot w)$ , where  $w$  is the width of the input formula  $\varphi$ .

# First-order metatheorems: reference

A good reference for other meta-theorems for first-order logic is:

[[Kreutzer, 2009](#)]: Stephan Kreutzer: *Algorithmic Meta-Theorems*.

# Summary

- ▶ Logic preliminaries
  - ▶ first-order logic
    - ▶ expressing graph problems by f-o formulas
  - ▶ monadic second-order logic (MSO)
    - ▶ expressing graph problems by MSO formulas
  - ▶ complexity of evaluation and model checking problems
- ▶ Courcelle's theorem
  - ▶ FPT-results by model-checking MSO-formulas
    - ▶ for graphs with bounded tree-width
    - ▶ for structures with bounded tree-width
    - ▶ for graphs of bounded clique-width
  - ▶ applications to concrete problems
- ▶ graph minors
- ▶ meta-theorems for first-order model-checking: an example

# Course overview

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
<i>Algorithmic Techniques</i>		<i>Formal-Method &amp; Algorithmic Techniques</i>		
<b>Introduction &amp; basic FPT results</b> motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	<b>Notions of bounded graph width</b> path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	<b>Algorithmic Meta-Theorems</b> 1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	<b>FPT-Intractability Classes &amp; Hierarchies</b> motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour

# Example suggestions

## Examples

1. Find a **first-order logic formula** over  $\tau_G$  that expresses that a graph has a **cycle of length precisely  $k$** .
2. Find an  **$MSO_1$**  or  **$MSO(\tau_G)$**  formula that expresses that a graph has a **dominating set of  $\leq k$**  elements.
3. Find an  **$MSO_2$**  or  **$MSO(\tau_{HG})$**  formula ***feedback***( $S$ ) that expresses that  $S \subseteq V$  is a feedback vertex set.
4. (\*) Find an  **$MSO_1$**  or  **$MSO(\tau_G)$**  formula that expresses that a graph is **connected**.
5. (\*) Find an  **$MSO_2$**  or  **$MSO(\tau_{HG})$**  formula ***path***( $x, y, Z$ ) that expresses that  $Z$  is a set of edges that forms a path from  $x$  to  $y$ .

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