

# An Introduction to Parameterized Complexity

## Lecture 1: Fixed-Parameter Tractability

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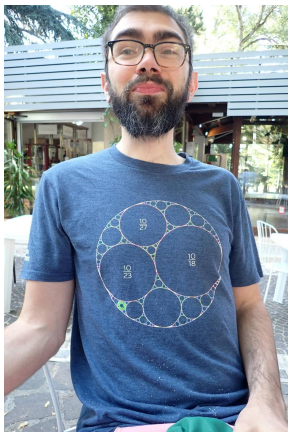
L'Aquila, Italy

Monday, June 10, 2024

# Course overview

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
<b>Introduction &amp; basic FPT results</b> motivation for FPT kernelization, Crown Lemma, Sunflower Lemma		<b>Algorithmic Meta-Theorems</b> 1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width		
	GDA		GDA	GDA
<i>Algorithmic Techniques</i>		<i>Formal-Method &amp; Algorithmic Techniques</i>		
	14.30 – 16.30			14.30 – 16.30
	<b>Notions of bounded graph width</b>			<b>FPT-Intractability Classes &amp; Hierarchies</b>
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

# Course developers



Hugo Gilbert  
course 2019/20 (Hugo & Clemens)



CG & Alessandro Aloisio  
course 2020/21 (Alessandro & C)

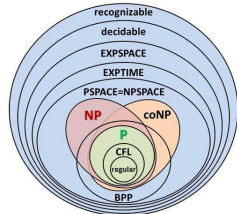
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# Motivation

## Classical complexity theory

- ▶ analyses problems by **resource** (space or time)  
needed to solve them on a **reasonable machine model**
- ▶ as a function of the **input size**  $n = |x|$  (Hartmanis/Stearns, 1965)
- ⇒ variety of **complexity classes**  
(P, LOGSPACE, NP, PSPACE, ...)
- ⇒ **tractable problems**  
= polynomial-time computable (in P)
- ⇒ **theory of intractability**  
(reductions, NP completeness)



## Drawback

- ▶ measures problem size  $n = |x|$   
only in terms of input instances  $x$ ,  
and **ignores structural information** about instances
- ▶ sometimes problems are **easier to solve**  
for instances if additional structure information is available

# Motivation

## Classical complexity theory

- ▶ analyses problems by **resource** (space or time)  
needed to solve them on a **reasonable machine model**
- ▶ as a function of the **input size**  $n = |x|$  (Hartmanis/Stearns, 1965)
- ⇒ variety of **complexity classes** (P, LOGSPACE, NP, PSPACE, ...)
- ⇒ **tractable problems** = polynomial-time computable (in P)
- ⇒ **theory of intractability** (reductions, NP completeness)

## Parameterized complexity

- ▶ measures complexity also in terms of a parameter  $k = \kappa(x)$   
that may depend on the input  $x$  in an arbitrary way
- ⇒ **fixed-parameter tractable problems**  
relaxes polynomial time solvability to algorithms whose  
non-polynomial behavior  $f(k) \cdot p(n)$  is restricted by parameter  $k$
- ⇒ complexity classes (FPT, XP, W[P], W- and A-hierarchies)
- ⇒ **theory of fixed-parameter intractability**

# Parameterized (versus classical) problems

## Definition

A **classical (decision) problem** is a pair  $\langle \Sigma, Q \rangle$  where:

- ▷  $Q \subseteq \Sigma^*$  the set of **problem yes-instances** over a finite alphabet  $\Sigma$

A **parameterized (decision) problem** is a triple  $\langle \Sigma, Q, \kappa \rangle$  where:

- ▷  $Q \subseteq \Sigma^*$  the set of **problem yes-instances** over a finite alphabet  $\Sigma$ ,
- ▷  $\kappa : \Sigma^* \rightarrow \mathbb{N}$  a function, **the parameterization**.

We regularly shorten  $\langle \Sigma, Q, \kappa \rangle$  to a pair  $\langle Q, \kappa \rangle$ .

## Assumption

The parameterization  $\kappa$  can be **efficiently** computed.

# Parameterized problems (examples)

## A Parameterized Clique Problem

p-CLIQUE:

**Given:** a graph  $G$  and an integer  $k$ ,

**Question:** Does there exists a clique of size  $k$  in  $G$ ?

**Parameter:**  $k$ .

## A Parameterized Hitting Set Problem

p-HITTING SET

**Given:** a universe  $U = \{x_1, \dots, x_n\}$ , a collection of sets  $S = (S_1, \dots, S_m)$  where  $S_i \subseteq U$  and an integer  $k$ ,

**Question:** Does there exists a set  $S \subseteq U$  such that  $|S| \leq k$  and  $S \cap S_i \neq \emptyset, \forall i \in \{1, \dots, m\}$ .

**Parameter:**  $\max |S_i|$ .

- ▶ NP-hard even if  $\max |S_i| = 2$ ,
- ▶ is **fixed-parameter tractable**.



# The art of parameterization

What is a **good parameter**?

- ▶ We should have reasons to believe that the parameter is “**small**” for some applications.
- ▶ It is better if the parameter is **intuitive**.
- ▶ It is better if the parameter is **efficiently computable**.

*There is a hierarchy on parameters.*

# The art of parameterization

There are many **different types** of parameters!

- ▶ The **size of the solution** we are looking for.
- ▶ The **size of some parts** of the instance.  
E.g., the number of voters in an election problem.
- ▶ Some more **structural property** of the instance.  
E.g., the diameter of a graph.
- ▶ It can be a **combination** of values, a **difference**, ...

# The art of parameterization

- ▶ **Graph problems**: maximum degree, treewidth, diameter...
- ▶ **Social choice problems**: number of voters, candidates, correlation of preferences...
- ▶ **Boolean formulas**: number of variables, number of clauses...
- ▶ **Problems on strings**: maximum length of a string, size of the alphabet...

# Fixed Parameter Tractability (Class FPT)

## Definition

A parameterized problem  $\langle Q, \kappa \rangle$  is *fixed-parameter tractable* if:

$$\begin{aligned} &\exists f : \mathbb{N} \rightarrow \mathbb{N} \text{ computable } \exists p \in \mathbb{N}[X] \text{ polynomial} \\ &\quad \exists \text{ algorithm, takes inputs in } \Sigma^* \text{ and } \forall x \in \Sigma^* \\ &\quad \quad [\text{algorithm decides if } x \in Q \text{ in time } \leq f(\kappa(x)) \cdot p(|x|)]. \end{aligned}$$

**FPT** := complexity class of all fixed-parameter tractable problems.

## Assumption for a robust fpt-theory:

$\kappa$  is *polynomially computable*, or itself *fpt-computable*.

## Goal in parameterized algorithmics:

- $\Rightarrow$  design FPT algorithms,
- $\Rightarrow$  try to make both factors  $f(\kappa(x))$  and  $p(|x|)$  as small as possible.
- $\Rightarrow$  or show (if possible) that finding such factors is impossible

# Slices of FPT problems are in P

The  $\ell$ -th slice of a parameterized problem  $\langle Q, \kappa \rangle$ :

$$\langle Q, \kappa \rangle_{\ell} := \{x \in Q \mid \kappa(x) = \ell\} \quad (\text{as classical problem}).$$

## Proposition

If  $\langle Q, \kappa \rangle \in \text{FPT}$ , then  $\langle Q, \kappa \rangle_{\ell} \in \text{P}$  for all  $\ell \in \mathbb{N}$ .

## Proof.

If  $\langle Q, \kappa \rangle \in \text{FPT}$ , then there are a computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , a polynomial  $p$ , and an algorithm  $\mathbb{A}$  that decides  $x \in \Sigma^*$  in running time  $\leq f(\kappa(x)) \cdot p(|x|)$  time. This algorithm can also be used to decide the  $\ell$ -th slice in time  $\leq f(\ell) \cdot p(|x|)$ , which for fixed  $\ell$  is a polynomial.  $\square$

# A problem not in FPT (unless $P = NP$ )

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## Application

$p$ -COLORABILITY

**Instance:** a graph  $\mathcal{G}$  and  $k \in \mathbb{N}$ .

**Parameter:**  $k$ .

**Problem:** Decide whether  $\mathcal{G}$  is  $k$ -colorable.

Known: 3-COLORABILITY  $\in$  NP-complete (Lovász, Stockmeyer, 1973).

Since 3-COLORABILITY =  $p$ -COLORABILITY<sub>3</sub>,

it follows that  $p$ -COLORABILITY  $\notin$  FPT (unless  $P = NP$ ).

# Slice-wise polynomial problems (Class XP)

## Definition

A parameterized problem  $\langle Q, \kappa \rangle$  is *slice-wise polynomial* if:

$\exists f, g : \mathbb{N} \rightarrow \mathbb{N}$  computable

$\exists \mathbb{A}$  algorithm, takes inputs in  $\Sigma^*$  and  $\forall x \in \Sigma^*$

$\left[ \mathbb{A} \text{ decides if } x \in Q \text{ in time } \leq f(\kappa(x)) \cdot |x|^{g(\kappa(x))} \right].$

**XP** := complexity class of slice-wise polynomial problems.

# Slices of XP problems are in P

The  $\ell$ -th slice of a parameterized problem  $\langle Q, \kappa \rangle$ :

$$\langle Q, \kappa \rangle_{\ell} := \{x \in Q \mid \kappa(x) = \ell\} \quad (\text{as classical problem}).$$

## Proposition

If  $\langle Q, \kappa \rangle \in \text{XP}$ , then  $\langle Q, \kappa \rangle_{\ell} \in \text{P}$  for all  $\ell \in \mathbb{N}$ .

## Proof.

If  $\langle Q, \kappa \rangle \in \text{XP}$ , then there are a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  computable, a polynomial  $p$ , and an algorithm  $\mathbb{A}$  that decides  $x \in \Sigma^*$  in running time  $\leq f(\kappa(x)) \cdot |x|^{g(\kappa(x))}$  time. This algorithm can be used to decide the  $\ell$ -th slice in time  $\leq f(\ell) \cdot |x|^{g(\ell)}$ , which for fixed  $\ell$  is a polynomial.  $\square$



# A problem not in XP (unless $P = NP$ )

The  $\ell$ -th slice of a parameterized problem  $\langle Q, \kappa \rangle$ :

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# Aims of the course

- 1 Acquire a **basic notions** of parameterized complexity.
- 2 Obtain an introduction to some techniques to derive **FPT or XP results**.
- 3 Obtain an introduction to a variety of techniques to prove **algorithmic lower bounds** and in particular prove **parameterized hardness results**.

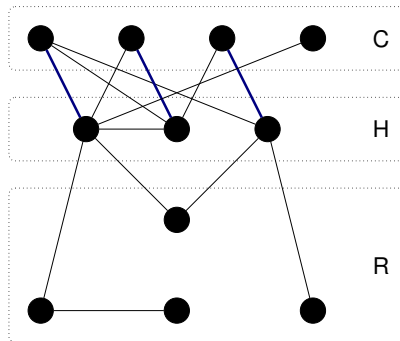
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# Today

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# From today's lecture



A **crown decomposition** of a graph  $G$  is a partitioning  $(C, H, R)$  of  $V(G)$ , such that:

- 1  $C$  is nonempty.
- 2  $C$  is an independent set.
- 3  $H$  separates  $C$  and  $R$ .
- 4  $G$  contains a matching of  $H$  into  $C$ .

## Crown Lemma ( $\Leftarrow$ results by König, Hall)

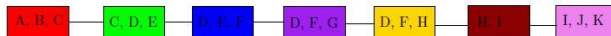
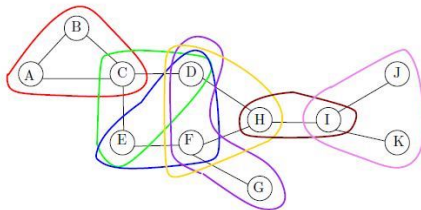
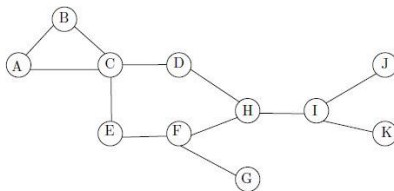
Let  $G$  be a graph with no isolated vertices and with at least  $3k + 1$  vertices. There is a polynomial-time algorithm that:

- ▶ either finds a matching of size  $k + 1$  in  $G$ ;
- ▶ or finds a crown decomposition of  $G$ .

# Tomorrow

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# In tomorrow's lecture: a path decomposition of a graph



# Wednesday

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# In Wednesday's lecture: Monadic second-order logic

$$\psi_3 := \exists C_1 \exists C_2 \exists C_3 \left( \left( \forall x \bigvee_{i=1}^3 C_i(x) \right) \right. \\ \left. \wedge \forall x \forall y \left( E(x, y) \rightarrow \bigwedge_{i=1}^3 \neg (C_i(x) \wedge C_i(y)) \right) \right)$$

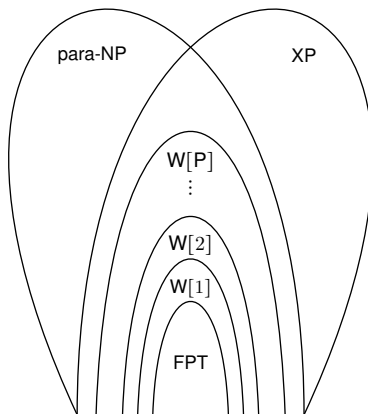
$$\mathcal{A}(\mathcal{G}) \models \psi_3 \iff \mathcal{G} \text{ has is } 3\text{-colorable.}$$

# Friday

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# From Friday's lecture: W-Hierarchy

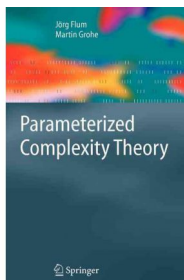
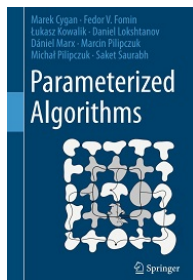
*‘There is no definite single class that can be viewed as “the parameterized NP”. Rather, there is a whole hierarchy of classes playing this role.* (Flum, Grohe [FG06])



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# Books



Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh, *Parameterized Algorithms*, 1st ed., Springer, 2015.



Jörg Flum and Martin Grohe, *Parameterized Complexity Theory*, Springer, 2006.

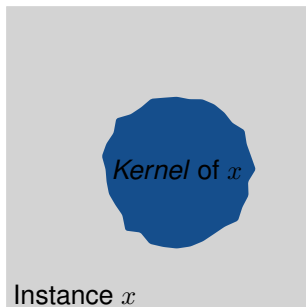
# Kernelization

- ▶ Idea
- ▶ Definition
- ▶ Kernel examples for:
  - ▶ point line cover problem
  - ▶ vertex cover problem
- ▶ Kernelization  $\Leftrightarrow$  FPT
- ▶ Crown lemma and crown decomposition
  - ▶ smaller kernel for vertex cover problem
  - ▶ kernel for dual colorability problem
- ▶ Sunflower lemma
  - ▶ kernel for hitting set problem

# Kernelization methods (informally)

Kernelization is:

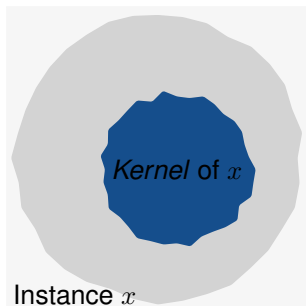
- ▶ a systematic study of polynomial-time preprocessing algorithms,
- ▶ an important tool in the design of parameterized algorithms.



# Kernelization methods (informally)

Kernelization is:

- ▶ a systematic study of **polynomial-time preprocessing algorithms**,
- ▶ an important tool in the design of parameterized algorithms.



→ Application of rule 1

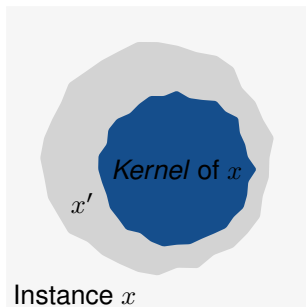
- ▶ Often a collection of efficient **preprocessing rules**.



# Kernelization methods (informally)

Kernelization is:

- ▶ a systematic study of **polynomial-time preprocessing algorithms**,
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→ Application of rule 1

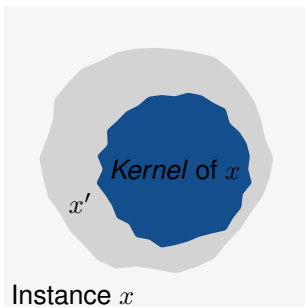
→ Application of rule 2

- ▶ Often a collection of efficient **preprocessing rules**.
- ▶ Transform an instance  $x$  into a smaller equivalent instance  $x'$ .

# Kernelization methods (informally)

Kernelization is:

- ▶ a systematic study of polynomial-time preprocessing algorithms,
- ▶ an important tool in the design of parameterized algorithms.



→ Application of rule 1

→ Application of rule 2

- ▶ Often a collection of efficient preprocessing rules.
- ▶ Transform an instance  $x$  into a smaller equivalent instance  $x'$ .
- ▶ Hopefully,  $|x'| \leq g(\kappa(x))$ .  
→ use a (non-efficient) exact algorithm.

# Kernelization (formally)

## Definition

Let  $\langle Q, \kappa \rangle$  be a parameterized problem over  $\Sigma$ .

A *kernelization* of  $\langle Q, \kappa \rangle$  is a function  $K: \Sigma^* \rightarrow \Sigma^*$  such that:

- (K1) For all  $x \in \Sigma^* : (x \in Q \iff K(x) \in Q)$ .
- (K2)  $K$  is polynomial-time computable.
- (K3) There is a computable function  $h: \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $x \in \Sigma^* : |K(x)| \leq h(\kappa(x))$ .

We say that such a kernelization  $K$  is *polynomial* (resp. *linear*) (and that  $Q$  *has a polynomial* (resp. *linear*) kernel) if the function  $h$  is *polynomial* (resp. *linear*).

## Lemma

*If  $\langle Q, \kappa \rangle$  admits a kernel and is decidable, then  $\langle Q, \kappa \rangle \in \text{FPT}$ .*

## Lemma

*If  $\langle Q, \kappa \rangle \in \text{FPT}$ , the  $\langle Q, \kappa \rangle$  admits a kernel.*

# The (parameterized) Point Line Cover Problem

## p-POINT-LINE-COVER:

**Given:**  $n$  points in the plane and an integer  $k$ .

**Parameter:** The integer  $k$ .

**Question:** Do there exist  $k$  lines that cover all points?

### Rule 1:

If we have a line that hits  $k + 1$  or more points, **then**:

- i) include it in the solution;
- ii) remove the points hit by the line;
- iii) set  $k := k - 1$ .

**Observation:** Let  $(x, \kappa)$  be a **yes instance** of the p-Point-Line-Cover such that Rule 1 cannot be applied. **Then**  $n \leq k^2$  holds.

### Rule 2:

If we cannot apply Rule 1, and we have more than  $k^2$  points, **then** say **no**, and return a **trivial no instance**.

## Proposition

p-POINT-LINE-COVER  $\in$  **FPT**: it admits a kernel of size with  $k^2$  points.

# The (parameterized) Vertex Cover Problem

## p-VERTEX-COVER:

**Given:** A graph  $G$ .

**Parameter:** The integer  $k$ .

**Question:** Does there exists a vertex cover of size at most  $k$ ?

## Definition

Let  $G$  be a graph and  $S \subseteq V(G)$ . The set  $S$  is called a **vertex cover** if for every edge of  $G$  at least one of its endpoints is in  $S$ .

## Exercise

Find an  $O(k^2)$  kernel for p-VERTEX-COVER.

# The (parameterized) Vertex Cover Problem *(Buss kernel)*

**Rule 1:** If  $G$  contains an isolated vertex  $v$ , delete  $v$  from  $G$ .  
The new instance is  $(G \setminus v, k)$

**Rule 2:** If there is a vertex  $v$  of degree at least  $k + 1$ , then delete  $v$  (and its incident edges) from  $G$  and decrement the parameter  $k$  by 1.  
The new instance is  $(G \setminus v; k - 1)$

## Observations

- ▶ After exhaustive application of Rule 1 and Rule 2 all vertices have degree between 1 and  $k$ .
- ▶ If  $G$  has maximum degree  $d$ ,  $k$  vertices can cover  $\leq k \cdot d$  edges.
- ▶ If  $G$  has a vertex cover of  $\leq k$  vertices after exhaustive application of Rules 1 & 2, then  $G$  has  $\leq k^2$  edges (and  $\leq k^2 + k$  vertices).

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## Observations

- ▶ If  $G$  has a vertex cover of  $\leq k$  vertices after exhaustive application of Rules 1 & 2, then  $G$  has  $\leq k^2$  edges (and  $\leq k^2 + k$  vertices).

**Rule 3:** Let  $(G, k)$  be an instance to which Rules 1 & 2 are not applicable. If  $G$  has  $> k^2 + k$  vertices, or  $> k^2$  edges, then  $(G, k)$  is a **no-instance** that can be replaced by a **trivial no-instance**.

## Theorem (Samuel Buss)

**$p$ -VERTEX-COVER**  $\in$  **FPT**, because it admits a kernel with at most  $O(k^2)$  vertices and  $O(k^2)$  edges.

# Kernelization $\Rightarrow$ FPT

## Exercise

If  $\langle Q, \kappa \rangle$  admits a kernel and is decidable, then  $\langle Q, \kappa \rangle \in \text{FPT}$ .

## Definitions

A *kernelization* of  $\langle Q, \kappa \rangle$  is a function  $K: \Sigma^* \rightarrow \Sigma^*$  such that:

- (K1) For all  $x \in \Sigma^*$ :  $(x \in Q \iff K(x) \in Q)$ .
- (K2)  $K$  is polynomial-time computable.
- (K3) There is a computable function  $h: \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $x \in \Sigma^*$ :  $|K(x)| \leq h(\kappa(x))$ .

A parameterized problem  $\langle Q, \kappa \rangle$  is *fixed-parameter tractable* if:

$$\begin{aligned} &\exists f: \mathbb{N} \rightarrow \mathbb{N} \text{ computable } \exists p \in \mathbb{N}[X] \text{ polynomial} \\ &\quad \exists \text{ algorithm, takes inputs in } \Sigma^* \text{ and } \forall x \in \Sigma^* \\ &\quad \left[ \text{algorithm decides if } x \in Q \text{ in time } \leq f(\kappa(x)) \cdot p(|x|) \right]. \end{aligned}$$

**FPT** := complexity class of all fixed-parameter tractable problems.



# Kernelization $\Rightarrow$ FPT

## Lemma

If  $\langle Q, \kappa \rangle$  admits a kernel and is decidable, then  $\langle Q, \kappa \rangle \in \text{FPT}$ .

$\langle Q, \kappa \rangle$  a parameterized problem,  $Q \subseteq \Sigma^*$

Definition:  $K: \Sigma^* \rightarrow \Sigma^*$  a kernelization for  $\langle Q, \kappa \rangle$  if:

(K1)  $\forall x \in \Sigma^* (x \in Q \Leftrightarrow K(x) \in Q)$

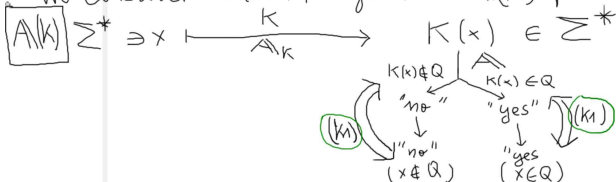
(K2)  $K$  is poly time computable

(K3)  $\exists h: \mathbb{N} \rightarrow \mathbb{N} \forall x \in \Sigma^* (|K(x)| \leq h(\kappa(x)))$ .

Proposition: If  $\langle Q, \kappa \rangle$  is decidable, and has kernelization  $K$ , then  $\langle Q, \kappa \rangle \in \text{FPT}$

Proof. Since  $\langle Q, \kappa \rangle$  is decidable, there is an algorithm  $A$  that decides instances  $x \in \Sigma^*$  in time  $\leq f(|x|)$  steps for some computable function  $f: \mathbb{N} \rightarrow \mathbb{N}$ .

Then, assuming a polynomial algorithm  $A_K$  for  $K$  (time bounded by  $P(x)$ ) we construct an FPT algorithm  $A(K)$  for  $\langle Q, \kappa \rangle$ :



$$\begin{aligned}
 \text{Running time } A(K) &= \text{time}(A_K) + \text{time}(A(K(x))) \\
 &= p(|x|) + f(\kappa(K(x))) \\
 &\stackrel{\text{by (K2)}}{=} p(|x|) + f(h(\kappa(x))) \\
 &= (f \circ h)(\kappa(x)) \cdot (1+p)(|x|) \\
 &= \tilde{f}(\kappa(x)) \cdot \text{poly}(|x|) \in \text{FPT}.
 \end{aligned}$$

# FPT $\Rightarrow$ Kernelization

## Lemma

If  $\langle Q, \kappa \rangle \in \text{FPT}$ , then  $\langle Q, \kappa \rangle$  admits a *kernel*.

## Proof.

Let  $\mathbb{A}$  be an algorithm that solves  $\langle Q, \kappa \rangle$  in time  $f(\kappa(x)) \cdot p(x)$ , for all  $x \in \Sigma^*$ , where  $f : \mathbb{N} \rightarrow \mathbb{N}$  computable, and  $p(n)$  a polynomial. We can assume  $p(n) \geq \max\{n, 1\}$  for all  $n \in \mathbb{N}$ .

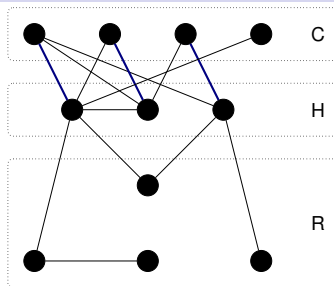
If  $Q = \emptyset$  or  $Q = \Sigma^*$ , then we can defined  $K(x) := \epsilon$ . Otherwise we have  $\emptyset \subsetneq Q \subsetneq \Sigma^*$ , and we choose some  $x_0 \in Q$ , and  $x_1 \in \Sigma^* \setminus Q$ .

We define the *polynomial-time computable function*  $K : \Sigma^* \rightarrow \Sigma^*$  by:

$$K(x) := \begin{cases} x_0 & \dots \mathbb{A} \text{ accepts } x \text{ in } \leq p(|x|) \cdot p(|x|) \text{ steps,} \\ x_1 & \dots \mathbb{A} \text{ rejects } x \text{ in } \leq p(|x|) \cdot p(|x|) \text{ steps,} \\ x & \dots \mathbb{A} \text{ does not terminate in } \leq p(|x|) \cdot p(|x|) \text{ steps.} \end{cases}$$

In the last case ( $K(x) = x$ ) we have  $p(|x|) \cdot p(|x|) \leq f(\kappa(x)) \cdot p(|x|)$ , and hence  $|K(x)| = |x| \leq p(|x|) \leq f(\kappa(x))$ . Therefore  $K$  is a *kernel*.  $\square$

# Crown Decomposition and Crown Lemma



A **crown decomposition** of a graph  $G$  is a partitioning  $(C, H, R)$  of  $V(G)$ , such that:

- 1  $C$  is nonempty.
- 2  $C$  is an independent set.
- 3  $H$  separates  $C$  and  $R$ .
- 4  $G$  contains a matching of  $H$  into  $C$ .

## Crown Lemma ( $\Leftarrow$ results by Kőnig, Hall)

Let  $G$  be a graph with no isolated vertices and with at least  $3k + 1$  vertices. There is a polynomial-time algorithm that:

- ▶ either finds a matching of size  $k + 1$  in  $G$ ;
- ▶ or finds a crown decomposition of  $G$ .

## Exercise

Apply the Crown Lemma to the Vertex Cover Problem.

# The (par.) Vertex Cover Problem (smaller kernel)

## p-VERTEX-COVER:

**Given:** A graph  $G$ .

**Parameter:** The integer  $k$ .

**Question:** Does there exist a vertex cover of size at most  $k$ ?

**Rule 1:** If  $G$  contains an isolated vertex  $v$ , delete  $v$  from  $G$ .  
The new instance is  $(G - v, k)$

**Rule 2:** If  $|V(G)| \geq 3k + 1$ , apply the Crown Lemma.

- ▶ If it returns a matching of size  $k + 1$ ,  
then conclude that  $(G, k)$  is a **no-instance**
- ▶ If it returns a **crown decomposition**  $V(G) = C \cup H \cup R$ :
  - ▶ Pick the vertices in  $H$  in the solution
  - ▶ Reduce  $(G, k)$  to  $(G - H, k - |H|)$
  - ▶ Reduce  $(G - H, k - |H|)$  to  $(G - H - C, k - |H|)$   
by using Rule 1 (note that vertices in  $C$  are isolated)

## Theorem

**p-VERTEX-COVER** admits a kernel with at most  $3k$  vertices.

# The (parameterized) Dual-Coloring Problem

## p-COLORABILITY:

**Given:** A graph  $G = \langle V, E \rangle$  on  $n$  vertices and an integer  $k$ .

**Parameter:** The integer  $k$ .

**Question:** Is  $G$   $k$ -colorable?

## Definition

Let  $k \in \mathbb{N}$ . A graph  $G = \langle V, E \rangle$  is  $k$ -colorable if there is a function  $C : V \rightarrow \{1, \dots, k\}$  such that  $C(u) \neq C(v)$  for all edges  $\{u, v\} \in E$ .

## Exercise

Obtain a kernel with  $O(k)$  vertices using crown decomposition.

# The Dual-Coloring Problem

**Rule 1:** Let  $I \subseteq V(G)$  be the isolated vertices. Remove  $I$  from  $G$ , and color them with one color. The new instance is  $(G - I, k)$

**Rule 2:** Consider graph  $\overline{G}(V, \overline{E})$  obtained from  $G$  by saying that  $e \in \overline{E}$  iff  $e \notin E$ .

If  $|V(G)| > 3k$ , apply the Crown Lemma to  $\overline{G}$ .

- ▶ If it returns a matching of size  $k + 1$ , then conclude that  $(G, k)$  is a **yes-instance**
- ▶ If it returns **crown decomposition**  $V(G) = V(\overline{G}) = C \cup H \cup R$ :
  - ▶ The vertices in  $H$  can be saved.
  - ▶ Reduce  $(G, k)$  to  $(G - H - C, k - |H|)$  if  $|H| < k$ , and otherwise to a **yes-instance**
  - ▶ Note that the vertices in  $C$  belong to a clique in  $G(V, E)$ , that is we need  $|C|$  colors, and that we need different colors for  $R$ .

## Theorem

**$p$ -DUAL-COLORING** admits a kernel with at most  $3k$  vertices.

# Sunflower Lemma

## Definition

A **sunflower** with  $k$  **petals** and a **core**  $Y$  is a collection of sets  $S_1, \dots, S_k$  such that  $S_i \cap S_j = Y$  for all  $i \neq j$ . The sets  $S_i \setminus Y$  are petals and they must be non-empty.

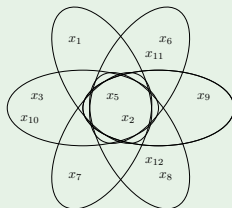
A sunflower with 6 petals and a core  $Y = \{x_2, x_5\}$ .

$$S_1 = \{x_2, x_3, x_5, x_{10}\}$$

$$S_2 = \{x_1, x_2, x_5\}$$

$$S_3 = \{x_2, x_5, x_6, x_{11}\}$$

...



## Sunflower Lemma (Erdős, Rado)

Let  $\mathcal{A}$  be a family of sets (without duplicates) over a universe  $U$  such that each set in  $\mathcal{A}$  has cardinality  $= d$ .

If  $|\mathcal{A}| > d!(k-1)^d$ , then  $\mathcal{A}$  contains a sunflower with  $k$  petals which can be computed in time polynomial in  $|\mathcal{A}|$ ,  $|U|$ , and  $k$ .

# Application to $d$ -Hitting Set

## Sunflower Lemma (Erdős, Rado)

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## Parameterized $d$ -Hitting Set Problem

### p-d-HITTING-SET:

**Given:** A family  $\mathcal{A}$  of sets over a universe  $U$ , where each set has cardinality  $\leq d$  and a positive integer  $k$ ,

**Parameter:** The integer  $k$ .

**Question:** Does there exist a subset  $H \subseteq U$  of size at most  $k$  such that  $H$  intersects each set in  $\mathcal{A}$ ?

## Exercise

Apply the sunflower lemma.



# Application to $d$ -Hitting Set

## Observation

If  $\mathcal{A}$  contains a sunflower  $\mathcal{S} = \{S_1, \dots, S_{k+1}\}$  of  $k+1$  sets, then every hitting set  $H$  of  $\mathcal{A}$  with  $|H| \leq k$  must intersect the core  $Y$  of  $\mathcal{S}$ . Otherwise it is a **no-instance**, because  $H$  cannot intersect each of the  $k+1$  petals  $S_i \setminus Y$ .

**Rule HS.1:** Let  $(U, \mathcal{A}, k)$  be an instance of  $d$ -HITTING SET.

Assume that  $\mathcal{A}$  contains a sunflower  $\mathcal{S} = \{S_1, \dots, S_{k+1}\}$  of cardinality  $k+1$  with core  $Y$ .

Then return  $(U', \mathcal{A}', k)$ , where  $\mathcal{A}' := (\mathcal{A} \setminus \mathcal{S}) \cup Y$ ,  
 $U' := \bigcup \mathcal{A}' = \bigcup_{X \in \mathcal{A}'} X$ .

**Proof** (kernel of  $p$ - $d$ -HITTING-SET with  $\leq d!k^d d$  sets and  $\leq d!k^d d^2$  elements).

If for some  $d' \in \{1, \dots, d\}$ , the number of sets in  $\mathcal{A}$  of size  $= d'$  is more than  $d'!k^{d'}$ , then the sunflower lemma yields a sunflower of size  $k+1$ .

Rule **HS.1** applies. By applying this rule exhaustively, we obtain a new family of sets  $\mathcal{A}'$  with  $\leq d'!k^{d'}$  sets of size  $= d'$  for every  $d' \in \{1, \dots, d\}$ . Hence  $|\mathcal{A}'| \leq d!k^d d$  and  $|U'| = d!k^d d^2$ .

If  $\emptyset \in \mathcal{A}'$  (a sunflower had an empty core), then it is a **no instance**.  $\square$

# Tomorrow

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
<b>Introduction &amp; basic FPT results</b> motivation for FPT kernelization, Crown Lemma, Sunflower Lemma		<b>Algorithmic Meta-Theorems</b> 1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width		
	GDA		GDA	GDA
Algorithmic Techniques		Formal-Method & Algorithmic Techniques		
	14.30 – 16.30			14.30 – 16.30
	<b>Notions of bounded graph width</b>			<b>FPT-Intractability Classes &amp; Hierarchies</b>
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies