An Introduction to Parameterized Complexity

Lecture 1: Fixed-Parameter Tractability

https://clegra.github.io/paracompl/paracompl.html

Clemens Grabmayer

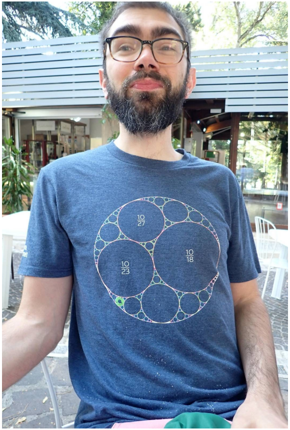
Ph.D. Program, Advanced Period
Gran Sasso Science Institute
L'Aquila, Italy

Monday, July 14, 2025

Course overview

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
Algorithmic Techniques		*Formal-Method & Algorithmic Techniques*		
Introduction & basic FPT results	**Notions of bounded graph width**	**Algorithmic Meta-Theorems**	**FPT-Intractability Classes & Hierarchies**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour

Course developers



Hugo Gilbert

course 2019/20 (Hugo & Clemens)

CG & Alessandro Aloisio

course 2020/21 (Alessandro & C)

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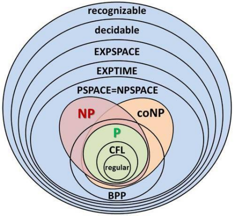
Classical complexity theory

- analyses problems by resource (space or time)
  needed to solve them on a reasonable machine model
- as a function of the input size $n = |x|$ (Hartmanis/Stearns, 1965)

$\Rightarrow$ variety of complexity classes
(P, LOGSPACE, NP, PSPACE, ...)

$\Rightarrow$ tractable problems
= polynomial-time computable (in P)

$\Rightarrow$ theory of intractability
(reductions, NP-completeness)



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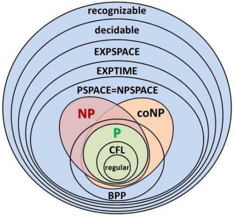

Drawback

- measures problem size $n = |x|$
  only in terms of input instances x,
  and ignores structural information about instances
- sometimes problems are easier to solve
  for instances if additional structure information is available

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Parameterized complexity

- measures complexity also in terms of a parameter $k = \kappa(x)$
  that may depend on the input x in an arbitrary way

$\Rightarrow$ fixed-parameter tractable problems
relaxes polynomial time solvability to algorithms whose
non-polynomial behavior $f(k) \cdot p(n)$ is restricted by parameter k

$\Rightarrow$ complexity classes (FPT, XP, W[P], W- and A-hierarchies)

$\Rightarrow$ theory of fixed-parameter intractability

Parameterized (versus classical) problems

Definition

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- $\kappa : \Sigma^* \to \mathbb{N}$ a function, *the parameterization*.

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Assumption

The parameterization κ can be efficiently computed.

Parameterized problems (examples)

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Given: a graph G and an integer k.
Question: Does there exists a clique of size k in G?

Parameter: k.

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- is fixed-parameter tractable.

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There is a hierarchy on parameters.

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- Some more structural property of the instance.
  E.g., the diameter of a graph.
- It can be a combination of values, a difference, ...

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- Social choice problems: number of voters, candidates, correlation of preferences...
- Boolean formulas: number of variables, number of clauses...
- Problems on strings: maximum length of a string, size of the alphabet...

Fixed Parameter Tractability (Class FPT)

Definition

A parameterized problem $\langle Q, \kappa \rangle$ is *fixed-parameter tractable* if:

$$\exists f : \mathbb{N} \to \mathbb{N} \text{ computable } \exists p \in \mathbb{N}[X] \text{ polynomial}$$
$$\exists \mathbb{A} \text{ algorithm, takes inputs in } \Sigma^* \text{ and } \forall x \in \Sigma^*$$
$$\big[\mathbb{A} \text{ decides if } x \in Q \text{ in time } \leq f(\kappa(x)) \cdot p(|x|)\big].$$

FPT := complexity class of all fixed-parameter tractable problems.

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Goal in parameterized algorithmics:

$\Rightarrow$ design FPT algorithms,
$\Rightarrow$ try to make both factors $f(\kappa(x))$ and $p(|x|)$ as small as possible.
$\Rightarrow$ or show (if possible) that finding such factors is impossible

Slices of FPT problems are in P

The *ℓ-th slice* of a parameterized problem $\langle Q, \kappa \rangle$:

$$\langle Q, \kappa \rangle_{\ell} := \{ x \in Q \mid \kappa(x) = \ell \} \qquad \text{(as classical problem)}\,.$$

Proposition

If $\langle Q, \kappa \rangle \in$ FPT, then $\langle Q, \kappa \rangle_{\ell} \in$ P for all $\ell \in \mathbb{N}$.

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Proof.

If $\langle Q, \kappa \rangle \in$ FPT, then there are a computable function $f : \mathbb{N} \to \mathbb{N}$, a polynomial p, and an algorithm $\mathbb{A}$ that decides $x \in \Sigma^*$ in running time $\leq f(\kappa(x)) \cdot p(|x|)$ time.

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A problem not in FPT (unless P = NP)

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Application

p-COLORABILITY
Instance: a graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Parameter: k.
Problem: Decide whether $\mathcal{G}$ is k-colorable.

Known: 3-COLORABILITY $\in$ NP-complete (Lovàsz, Stockmeyer, 1973).

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Slice-wise polynomial problems (Class XP)

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XP := complexity class of slice-wise polynomial problems.

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Aims of the course

1. Acquire a basic notions of parameterized complexity.
2. Obtain an introduction to some techniques to derive FPT or XP results.
3. Obtain an introduction to a variety of techniques to prove algorithmic lower bounds and in particular prove parameterized hardness results.

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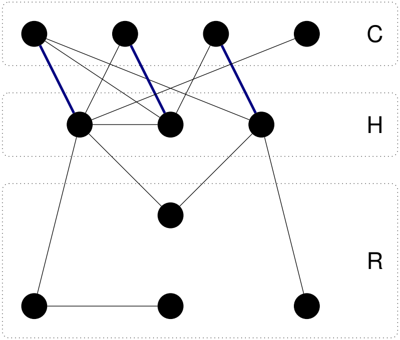

A **crown decomposition** of a graph G is a partitioning (C, H, R) of $V(G)$, such that:

1. C is nonempty.
2. C is an independent set.
3. H separates C and R.
4. G contains a matching of H into C.

Crown Lemma ($\Leftarrow$ results by Kőnig, Hall)

Let G be a graph with no isolated vertices and with at least $3k + 1$ vertices. There is a polynomial-time algorithm that:

- either finds a matching of size $k + 1$ in G;
- or finds a crown decomposition of G.

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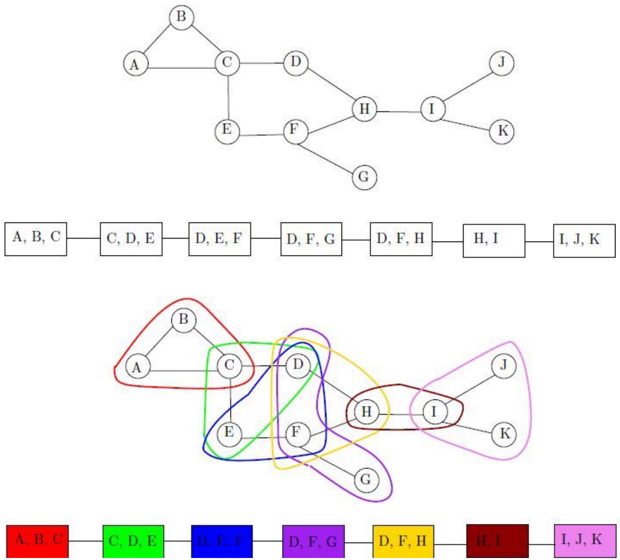
In tomorrow's lecture: a path decomposition of a graph



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In Wednesday's lecture: Monadic second-order logic

$$\psi_3 \;:=\; \exists C_1 \exists C_2 \exists C_3 \Big(\big(\forall x \bigvee_{i=1}^{3} C_i(x)\big) \\ \wedge\; \forall x \forall y \big(E(x,y) \to \bigwedge_{i=1}^{3} \neg(C_i(x) \wedge C_i(y))\big) \Big)$$

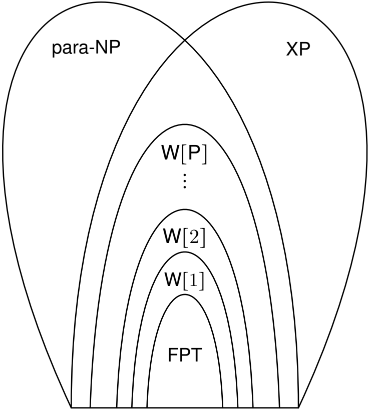
$\mathcal{A}(\mathcal{G}) \models \psi_3 \iff \mathcal{G}$ has is 3-colorable.

Thursday

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
Algorithmic Techniques		*Formal-Method & Algorithmic Techniques*		
Introduction & basic FPT results	**Notions of bounded graph width**	**Algorithmic Meta-Theorems**	**FPT-Intractability Classes & Hierarchies**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour

From Thursday's lecture: W-Hierarchy

'There is no definite single class that can be viewed as "the parameterized NP". Rather, there is a whole hierarchy of classes playing this role. (Flum, Grohe [FG06])



Course overview

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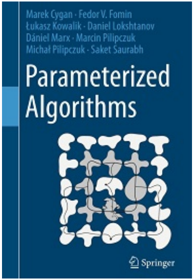
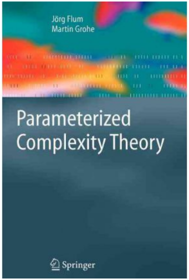
Books





- Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh, *Parameterized Algorithms*, 1st ed., Springer, 2015.
- Jörg Flum and Martin Grohe, *Parameterized Complexity Theory*, Springer, 2006.

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- Idea
- Definition

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- Kernel examples for:
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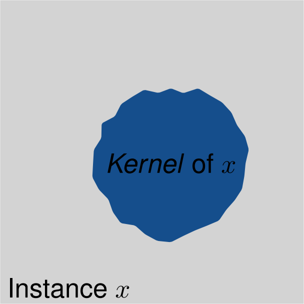
Kernelization methods (informally)

Kernelization is:

- a systematic study of polynomial-time preprocessing algorithms,
- an important tool in the design of parameterized algorithms.



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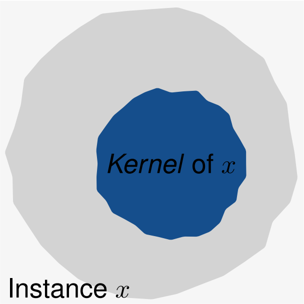

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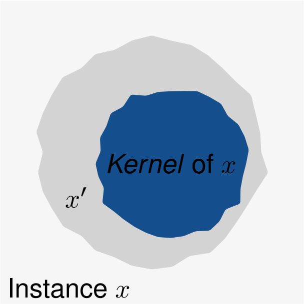

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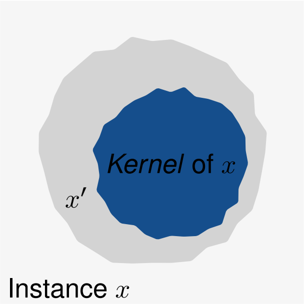

→ Application of rule 1

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- Often a collection of efficient preprocessing rules.
- Transform an instance x into a smaller equivalent instance x'.
- Hopefully, $|x'| \leq g(\kappa(x))$.
  → use a (non-efficient) exact algorithm.

Kernelization (formally)

Definition

Let $\langle Q, \kappa \rangle$ be a parameterized problem over Σ.
A *kernelization* of $\langle Q, \kappa \rangle$ is a function $K : \Sigma^* \to \Sigma^*$ such that:

(K1) For all $x \in \Sigma^*$: $(x \in Q \iff K(x) \in Q)$.

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If $\langle Q, \kappa \rangle \in$ FPT, the $\langle Q, \kappa \rangle$ admits a kernel.

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Proposition

p-POINT-LINE-COVER $\in$ FPT: it admits a kernel of size with k^2 points.

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Given: A graph G, and an integer k.
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Question: Does there exists a vertex cover of size at most k?

Definition

Let G be a graph and $S \subseteq V(G)$. The set S is called a vertex cover if for every edge of G at least one of its endpoints is in S.

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Exercise

Find an $O(k^2)$ kernel for p-VERTEX-COVER.

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Theorem (Samuel Buss)

p-VERTEX-COVER $\in$ FPT, *because it admits a kernel with at most $O(k^2)$ vertices and $O(k^2)$ edges.*

Kernelization ⇒ FPT

Exercise

If $\langle Q, \kappa \rangle$ admits a kernel and is decidable, then $\langle Q, \kappa \rangle \in$ FPT.

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A parameterized problem $\langle Q, \kappa \rangle$ is *fixed-parameter tractable* if:

$$\exists f : \mathbb{N} \to \mathbb{N} \text{ computable } \exists p \in \mathbb{N}[X] \text{ polynomial}$$
$$\exists \mathbb{A} \text{ algorithm, takes inputs in } \Sigma^* \text{ and } \forall x \in \Sigma^*$$
$$\big[\mathbb{A} \text{ decides if } x \in Q \text{ in time} \le f(\kappa(x)) \cdot p(|x|)\big].$$

FPT := complexity class of all fixed-parameter tractable problems.

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$\langle Q, \kappa \rangle$ a parameterized problem, $Q \subseteq \Sigma^*$

Definition. $K : \Sigma^* \longrightarrow \Sigma^*$ a kernelization for $\langle Q, \kappa \rangle$ if:

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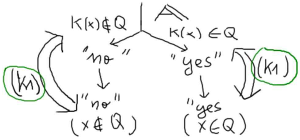
(K2) K is poly time computable

(K3) $\exists h : \mathbb{N} \longrightarrow \mathbb{N} \;\; \forall x \in \Sigma^* (\, |K(x)| \le h(\kappa(x)))$.

Proposition: If $\langle Q, \kappa \rangle$ is decidable, and has kernelization K, then $\langle Q, \kappa \rangle \in$ FPT

Proof. Since $\langle Q, \kappa \rangle$ is decidable, there is an algorithm $\mathbb{A}$ that decides instances $x \in \Sigma^*$ in time $\le f(|x|)$ steps for some computable function $f : \mathbb{N} \to \mathbb{N}$.
Then, assuming a polynomial algorithm $\mathbb{A}_K$ for K (time bounded by $p(x)$)
we construct an FPT algorithm $\mathbb{A}(K)$ for $\langle Q, \kappa \rangle$:

$\boxed{\mathbb{A}(K)} \quad \Sigma^* \ni x \xmapsto[\mathbb{A}_K]{K} K(x) \in \Sigma^*$



Running time $\mathbb{A}(K) =$
$= time(\mathbb{A}_K) + time(\mathbb{A}(K(x)))$
$= p(|x|) + f(|K(x)|)$
by (K2) by (K3) $\le h(\kappa(x))$
$= p(|x|) + f(h(\kappa(x)))$
$= (f \circ h)(\kappa(x)) \cdot (1 + p)(|x|)$
$= \tilde{f}(\kappa(x)) \cdot poly(|x|) \in$ FPT.

FPT $\Rightarrow$ Kernelization

Lemma

If $\langle Q, \kappa \rangle \in$ FPT, then $\langle Q, \kappa \rangle$ admits a kernel.

Proof.

Let $\mathbb{A}$ be an algorithm that solves $\langle Q, \kappa \rangle$ in time $f(\kappa(x)) \cdot p(x)$, for all $x \in \Sigma^*$, where $f : \mathbb{N} \to \mathbb{N}$ computable, and $p(n)$ a polynomial.

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In the last case ($K(x) = x$) we have $p(|x|) \cdot p(|x|) \leq f(\kappa(x)) \cdot p(|x|)$,

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$$K(x) := \begin{cases} x_0 & \ldots\ \mathbb{A} \text{ accepts } x \text{ in } \leq p(|x|) \cdot p(|x|) \text{ steps,} \\ x_1 & \ldots\ \mathbb{A} \text{ rejects } x \text{ in } \leq p(|x|) \cdot p(|x|) \text{ steps,} \\ x & \ldots\ \mathbb{A} \text{ does not terminate in } \leq p(|x|) \cdot p(|x|) \text{ steps.} \end{cases}$$

In the last case ($K(x) = x$) we have $p(|x|) \cdot p(|x|) \leq f(\kappa(x)) \cdot p(|x|)$, and hence $|K(x)| = |x| \leq p(|x|) \leq f(\kappa(x))$.

FPT ⇒ Kernelization

Lemma

If $\langle Q, \kappa \rangle \in$ FPT, then $\langle Q, \kappa \rangle$ admits a kernel.

Proof.

Let $\mathbb{A}$ be an algorithm that solves $\langle Q, \kappa \rangle$ in time $f(\kappa(x)) \cdot p(x)$, for all $x \in \Sigma^*$, where $f : \mathbb{N} \to \mathbb{N}$ computable, and $p(n)$ a polynomial. We can assume $p(n) \geq \max\{n, 1\}$ for all $n \in \mathbb{N}$.
If $Q = \varnothing$ or $Q = \Sigma^*$, then we can defined $K(x) := \epsilon$. Otherwise we have $\varnothing \subsetneq Q \subsetneq \Sigma^*$, and we choose some $x_0 \in Q$, and $x_1 \in \Sigma^* \smallsetminus Q$.
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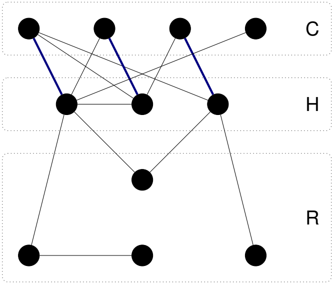
In the last case ($K(x) = x$) we have $p(|x|) \cdot p(|x|) \leq f(\kappa(x)) \cdot p(|x|)$, and hence $|K(x)| = |x| \leq p(|x|) \leq f(\kappa(x))$. Therefore K is a kernel. □

Crown Decomposition and Crown Lemma



A **crown decomposition** of a graph G is a partitioning (C, H, R) of $V(G)$, such that:

1. C is nonempty.
2. C is an independent set.
3. H separates C and R.
4. G contains a matching of H into C.

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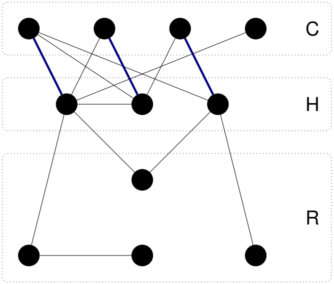

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Crown Lemma ($\Leftarrow$ results by Kőnig, Hall)

Let G be a graph with no isolated vertices and with at least $3k + 1$ vertices. There is a polynomial-time algorithm that:

- either finds a matching of size $k + 1$ in G;
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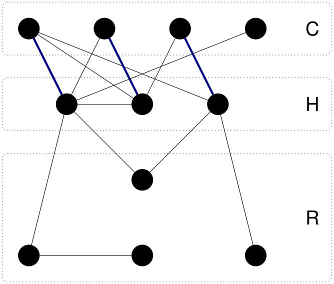

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Exercise

Apply the Crown Lemma to the Vertex Cover Problem.

The (par.) Vertex Cover Problem (smaller kernel)

p-VERTEX-COVER:
Given: A graph G, and an integer k.
Parameter: The integer k.
Question: Does there exists a vertex cover of size at most k?

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Rule 2: If $|(V(G))| \geq 3k + 1$, apply the Crown Lemma.

- If it returns a matching of size $k + 1$,
  then conclude that (G, k) is a no-instance
- If it returns a crown decomposition $V(G) = C \cup H \cup R$:
  - Pick the vertices in H in the solution

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  - Reduce $(G - H, k - |H|)$ to $(G - H - C, k - |H|)$
    by using Rule 1 (note that vertices in C are isolated)

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Theorem

p-VERTEX-COVER admits a kernel with at most $3k$ vertices.

The (parameterized) Dual-Coloring Problem

p-COLORABILITY:
Given: A graph $G = \langle V, E\rangle$ on n vertices and an integer k.
Parameter: The integer k.
Question: Is G k-colorable?

Definition

Let $k \in \mathbb{N}$. A graph $G = \langle V, E\rangle$ is k-colorable if there is a function $C : V \to \{1, \ldots, k\}$ such that $C(u) \neq C(v)$ for all edges $\{u, v\} \in E$.

The (parameterized) Dual-Coloring Problem

p-DUAL-COLORABILITY:
Given: A graph $G = \langle V, E \rangle$ on n vertices and an integer k.
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Question: Is G $(n - k)$-colorable?

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Exercise

Obtain a kernel with $O(k)$ vertices using crown decomposition.

The Dual-Coloring Problem

Rule 1: Let $I \subseteq V(G)$ be the isolated vertices. Remove I from G, and color them with one color. The new instance is $(G - I, k)$

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If $|(V(G))| > 3k$, apply the Crown Lemma to $\overline{G}$.

- If it returns a matching of size $k + 1$,
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Theorem

p-DUAL-COLORING *admits a kernel with at most $3k$ vertices.*

Sunflower Lemma

Definition

A sunflower with k petals and a core Y is a collection of sets $S_1, \ldots, S_k$ such that $S_i \cap S_j = Y$ for all $i \neq j$. The sets $S_i \setminus Y$ are petals and they must be non-empty.

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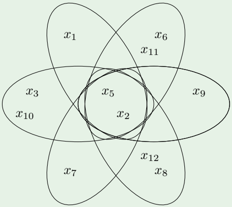
A sunflower with 6 petals and a core $Y = \{x_2, x_5\}$.

$S_1 = \{x_2, x_3, x_5, x_{10}\}$

$S_2 = \{x_1, x_2, x_5\}$

$S_3 = \{x_2, x_5, x_6, x_{11}\}$

$\ldots$



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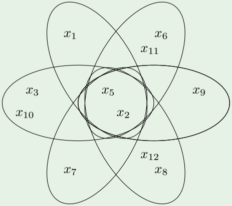

Sunflower Lemma (Erdős, Rado)

Let $\mathcal{A}$ be a family of sets (without duplicates) over a universe U such that each set in $\mathcal{A}$ has cardinality $= d$.
If $|\mathcal{A}| > d!(k-1)^d$, then $\mathcal{A}$ contains a sunflower with k petals which can be computed in time polynomial in $|\mathcal{A}|$, $|U|$, and k.

Application to d-Hitting Set

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Parameterized d-Hitting Set Problem

p-d-HITTING-SET:

Given: A family $\mathcal{A}$ of sets over a universe U, where each set has cardinality $\leq d$ and a positive integer k,

Parameter: The integer k.

Question: Does there exists a subset $H \subseteq U$ of size at most k such that H intersects each set in $\mathcal{A}$?

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Exercise

Apply the sunflower lemma.

Application to d-Hitting Set

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Theorem

*p-d-*HITTING-SET *has a kernel with* $\leq d!k^d d$ *sets &* $\leq d!k^d d^2$ *elements.*

Application to d-Hitting Set

Observation

If $\mathcal{A}$ contains a sunflower $\mathcal{S} = \{S_1, \ldots, S_{k+1}\}$ of $k+1$ sets, then every hitting set H of $\mathcal{A}$ with $|H| \le k$ must intersect the core Y of $\mathcal{S}$. Otherwise it is a no-instance, because H cannot intersect each of the $k+1$ petals $S_i \setminus Y$.

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Rule **HS.1**: Let $(U, \mathcal{A}, k)$ be an instance of d-HITTING SET.
Assume that $\mathcal{A}$ contains a sunflower $\mathcal{S} = \{S_1, \ldots, S_{k+1}\}$ of cardinality $k+1$ with core Y.
Then return $(U', \mathcal{A}', k)$, where $\mathcal{A}' := (\mathcal{A} \smallsetminus \mathcal{S}) \cup Y$,
$U' := \bigcup \mathcal{A}' = \bigcup_{X \in \mathcal{A}'} X$.

Proof (kernel of p-d-HITTING-SET with $\leq d!k^d d$ sets and $\leq d!k^d d^2$ elements).

If for some $d' \in \{1, \ldots, d\}$, the number of sets in $\mathcal{A}$ of size $= d'$ is more than $d'!k^{d'}$, then the sunflower lemma yields a sunflower of size $k+1$.

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If $\mathcal{A}$ contains a sunflower $\mathcal{S} = \{S_1, \ldots, S_{k+1}\}$ of $k+1$ sets, then every hitting set H of $\mathcal{A}$ with $|H| \leq k$ must intersect the core Y of $\mathcal{S}$. Otherwise it is a no-instance, because H cannot intersect each of the $k+1$ petals $S_i \setminus Y$.

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If $\varnothing \in \mathcal{A}'$ (a sunflower had an empty core), then it is a no instance. $\square$

Course overview

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
Algorithmic Techniques		*Formal-Method & Algorithmic Techniques*		
Introduction & basic FPT results	**Notions of bounded graph width**	**Algorithmic Meta-Theorems**	**FPT-Intractability Classes & Hierarchies**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour