Lecture 1: Introduction to Computability

Models of Computation

https://clegra.github.io/moc/moc.html

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L'Aquila, Italy

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Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
intro		*classic models*		*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

Today

- What is computation?
  - questions where the answer may depend on computation
  - algorithm examples
  - unsolvable problems

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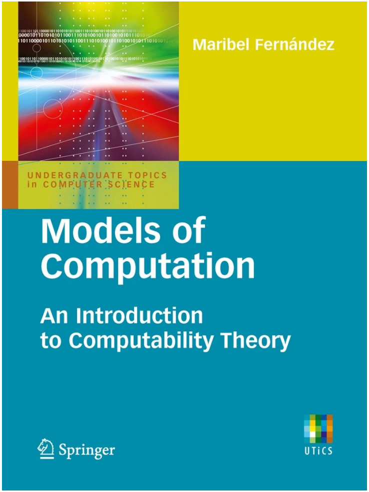
- recommended reading
- references

Book


Maribel Fernández
UNDERGRADUATE TOPICS in COMPUTER SCIENCE
Models of Computation
An Introduction to Computability Theory
Springer
UTiCS

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A: Yes, if the truth table for ϕ contains (in the row for ϕ) only "T"; no otherwise.

(Comput.) Yes-or-no-questions / Decision problems

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A decision method for A in E is a method by which, given an element $a \in E$, we can decide in a finite number of steps whether or not $a \in A$.

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The decision problem for A in E is solvable (the set A in E is (effectively) calculable) if there exists a decision method for A in E.

(Comput.) What-questions / Computation Problems

Example

Computing the greatest common divisor
Instance: a pair $\langle a, b \rangle$ of numbers $a, b \in \mathbb{N}$ with $a, b > 0$.
Question: What is $\mathrm{gcd}(a, b)$, the greatest common divisor of a and b?

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Suppose $F : A \to B$ is a mapping, where the elements of A, B are finitely describable objects.

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A mapping F is calculable if there exists a computation method for F.

Representing function

Let $P(a_1, \ldots, a_n)$ be an n-ary number-theoretic predicate.

The representing function f of P:

$$f(a_1, \ldots, a_n) := \begin{cases} 1 & \ldots P(a_1, \ldots, a_n) \text{ is true} \\ 0 & \ldots P(a_1, \ldots, a_n) \text{ is false} \end{cases}$$

Hence:
A decision procedure can be handled as a computation procedure f by taking ‘0’ for ‘yes’, and ‘1’ for ‘no’.

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– Similar for a decision methods.

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Due to $3 \mid 12$ and ($\star$) we conclude:

A: Yes. (Infinitely many solutions, e.g. $x = 4$ and $y = -8$.)

Not effectively calculable

Examples (Shoenfield)

- methods that involve chance procedures: tossing a coin

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- methods that involve chance procedures: tossing a coin
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- methods that require (unformalised, unmechanised) insight

Effectively calculable?

Example

Hilbert's 10th Problem

Instance: An equation $p(x_1, \ldots, x_n) = 0$, where
p a polynomial with integer coefficients.

Question: Is the equation solvable for $x_1, \ldots, x_n \in \mathbb{Z}$?

Instances based on quadratic polynomials are of the form
$ax^2 + bxy + cy^2 + dx + ey + f = 0$ with $a, b, c, d, e, f \in \mathbb{Z}$.

Effectively calculable? – No!

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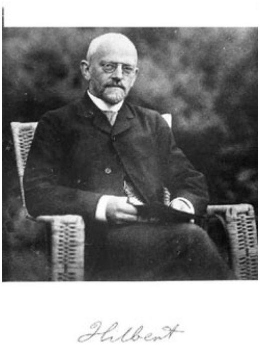
Theorem (Matijasevic, 1970)

Hilbert's 10th Problem is unsolvable.

David Hilbert (1862–1943)



Problem (Entscheidungsproblem, 1928)

Is there a method for deciding, given a formula ϕ of the predicate calculus, whether or not ϕ is a tautology?

Timeline: From logic to computability

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1937	Post: machine model; Church's thesis as 'working hypothesis' Turing: convincing analysis of a 'human computer' leading to the 'Turing machine'

Calculable functions?

Questions/Exercises

1. Suppose $P(a,b)$ is a calculable predicate.
   Why does $(\exists x)P(a,x)$ not have to be calculable?

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3. Can computation problems for mappings $F : \mathbb{N}^n \to \mathbb{N}^m$ always be represented by decision problems?

Some Models of Computation

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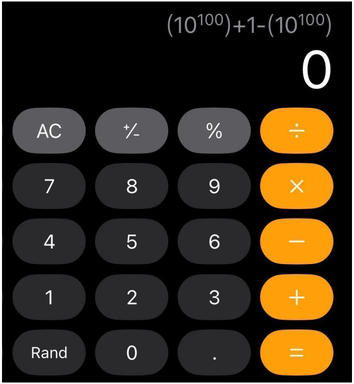

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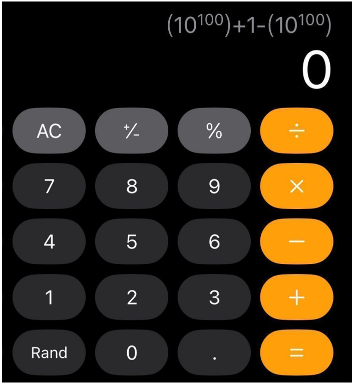

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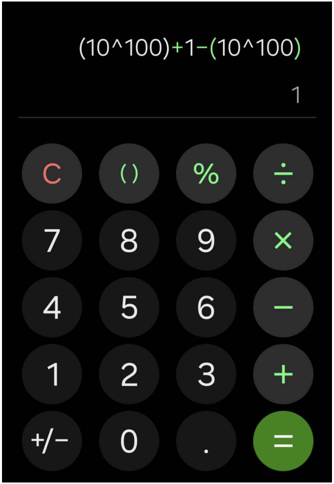
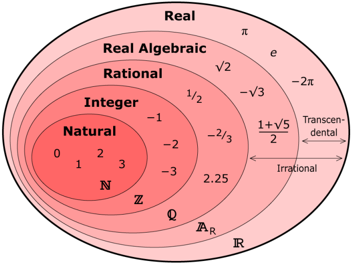


Android

Calculator (2/5): constructive real numbers



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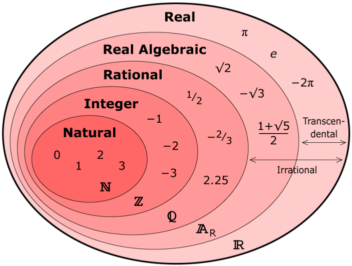

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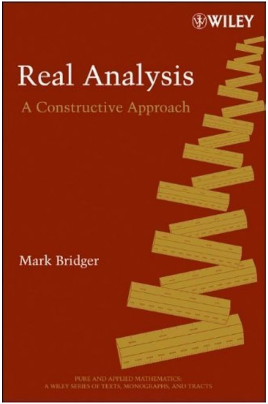

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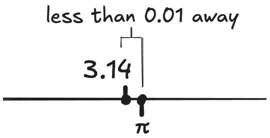

approximating π within 0.01

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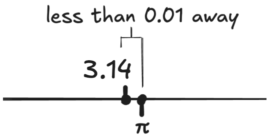

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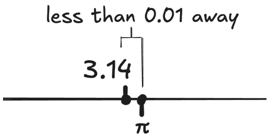

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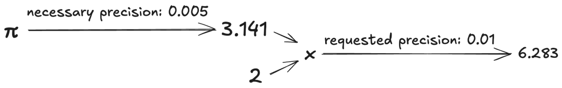

approximating 2π within 0.01

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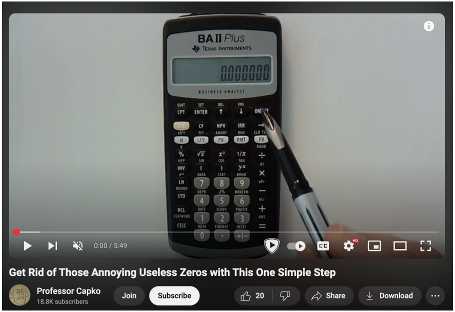

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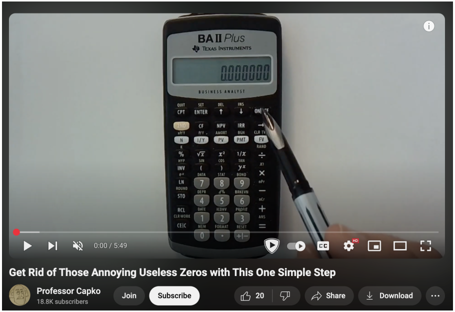

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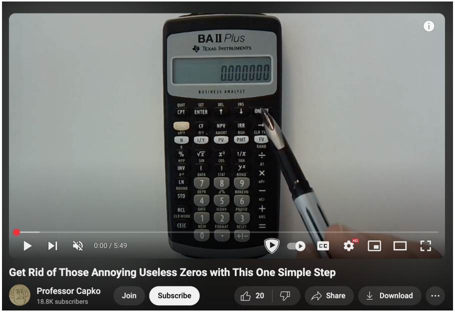

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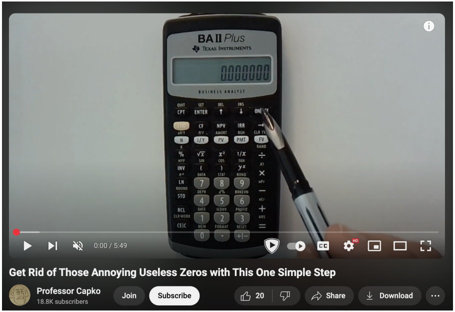

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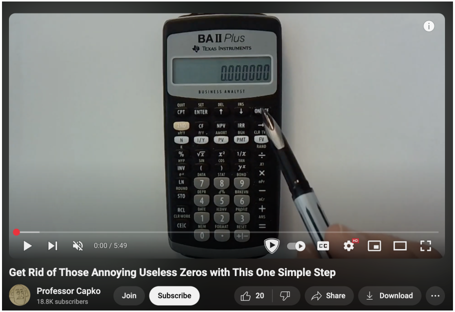

Undecidable problem

Article Talk

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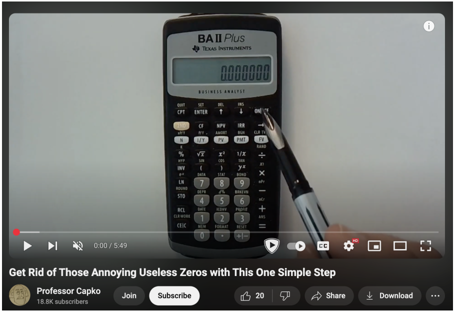

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- Credits: tech-blogger Chad Nauseam (link) for post *"A calculator app? Anyone could make that."* (link) [2].

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Rewriting

- study in a systematic way the operational and denotational aspects of MoC's like λ-calculus, CL, string rewriting, term rewriting, interaction nets

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Linguistics

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Recommended reading

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2. Turing machine motivation: Turing's analysis of a human computer: Part I of Section 9, pp. 249–252 of:
   - Alan M. Turing's: *On computable numbers, with an application to the Entscheidungsproblem*', Proceedings of the London Mathematical Society (1936), [4], http://www.wolframscience.com/prizes/tm23/images/Turing.pdf.

Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
intro		*classic models*		*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

References I

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