Lecture 3: Algorithmic Meta-Theorems

(A Short Introduction to Parameterized Complexity)

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Ph.D. Program, Advanced Courses Period

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Course overview

Monday, June 16 10.30 – 12.30	Tuesday, June 17 10.30 – 12.30	Wednesday, June 18	Thursday, June 19 10.30 – 12.30	Friday, June 20
Algorithmic Techniques			*Formal-Method* & *Algorithmic Techniques*	
Introduction & basic FPT results	**Notions of bounded graph width**		**Algorithmic Meta-Theorems**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width		1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	(Fair Division)
				14.30 – 16.30
				FPT-Intractability Classes & Hierarchies
			(Fair Division)	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Overview

- Courcelle's theorem
  - FPT-results by model-checking MSO-formulas
    - for graphs / structures with bounded tree-width
    - for maximization problems over graphs of bounded tree-width
    - for graphs of bounded clique-width
  - applications to concrete problems

Overview

- logic preliminaries
  - first-order logic
    - expressing graph problems by f-o formulas
  - monadic second-order logic (MSO)
    - expressing graph problems by MSO formulas
  - complexity of evaluation and model checking problems
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- graph minors
- meta-theorems for first-order model-checking: an example

Meta-theorems: idea, benefits and limitations

idea:

- express a problem P by a logical formula φ_P (of 'short' size)
- use model checking of φ_P
  on logical structures of bounded width k (tree-, clique-width, ...)
  - is time bounded depending on k, size of φ_P, size of the structure
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- a quick and easy way to show
  that [some problems] are fixed-parameter tractable,
- without working out the tedious details
  of a dynamic programming algorithm.

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limitations:

- algorithms obtained by meta-theorems
  cannot be expected to be optimal.
- a careful analysis of a specific problem at hand
  will usually yield more efficient fpt-algorithms

Logical preliminaries

First-order logic (formula example)

$$\begin{aligned}\varphi_3 \;:=\; \exists x_1 \exists x_2 \exists x_3 \big(\, & \neg(x_1 = x_2) \wedge \neg E(x_1, x_2) \\ & \wedge \neg(x_1 = x_3) \wedge \neg E(x_1, x_3) \\ & \wedge \neg(x_2 = x_3) \wedge \neg E(x_2, x_3) \,\big)\end{aligned}$$

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$\mathcal{A}(\mathcal{G}) \models \varphi_3 \quad \Longleftrightarrow \quad \mathcal{G}$ has a 3-element independent set.

$S \subseteq V$ is independent set in $\mathcal{G} = \langle V, E \rangle$ $:\Longleftrightarrow\ \forall e = \{u, v\} \in E\ (\,\neg(u \in S \wedge v \in S)\,)$
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$\mathcal{A}(\mathcal{G}) \vDash \varphi_3 \iff \mathcal{G}$ has a 3-element independent set.

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$\mathcal{A}(\mathcal{G}) \vDash \varphi_k \iff \mathcal{G}$ has a k-element independent set.

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- *sentences*: formulas without *free* variables.

First-order logic: semantics (structures)

Definition

Let $\tau = \{R_1, \ldots, R_n\}$ be a vocabulary.
A *τ-structure* is a tuple $\mathcal{A} = \langle A; R_1^{\mathcal{A}}, \ldots R_n^{\mathcal{A}} \rangle$ consisting of:

- the *universe* A,
- *interpretations* ${R_i}^{\mathcal{A}} \subseteq A^{ar(R_i)} = \overbrace{A \times \ldots \times A}^{ar(R_i)}$ for each of the relation symbols R_i in τ, where $i \in \{1, \ldots, n\}$.

Examples

Let $\tau_{\mathsf{G}} = \{E/_2\}$ vocabulary with binary edge relation.
The *standard structure* for a graph $\mathcal{G} = \langle V, E \rangle$:

$$\mathcal{A}_{\tau_{\mathsf{G}}}(\mathcal{G}) := \langle V; E^{\mathsf{symm}} \rangle \,.$$

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Let $\tau_{\mathsf{HG}} = \{\mathit{VERT}/_1, \mathit{EDGE}/_1, \mathit{INC}/_2\}$ vocabulary (for hypergraphs).
The *hypergraph structure* for a graph $\mathcal{G} = \langle V, E \rangle$:

$$\mathcal{A}_{\tau_{\mathsf{HG}}}(\mathcal{G}) := \langle V \cup E;\ V,\ E,\ \{\langle v, e \rangle \mid v \in V,\ e \in E,\ v \in e\} \rangle\,.$$

Interpretation of first-order formulas in structures

Let $\mathcal{A} = \langle A; \{R^{\mathcal{A}}\}_{R\in\tau} \rangle$ be a τ-structure. For a τ-formula $\varphi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$ its *interpretation* $\varphi(\mathcal{A}) \subseteq A^k$ in $\mathcal{A}$ is defined by:

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$$\varphi(\mathcal{A}) \coloneqq \Big\{\langle a_1,\ldots,a_k\rangle \in A^k \;\Big|\; \begin{array}{l}\text{there exists } a_{k+1} \in A \\ \text{such that } \langle a_{i_1},\ldots,a_{i_\ell}\rangle \in \varphi_0(\mathcal{A})\end{array}\Big\}$$

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- $\mathcal{A} \vDash \varphi(a_1,\ldots,a_k)$ will mean: $\langle a_1,\ldots,a_k\rangle \in \varphi(\mathcal{A})$.
- For a sentence φ, $\mathcal{A} \vDash \varphi$ will mean $\varphi(\mathcal{A}) \neq \varnothing$ (then $\varphi(\mathcal{A}) = \{\langle\rangle\}$).

Expressing graph properties by first-order formulas

Exercise

For given formulas $\varphi(x)$ and for all $k \in \mathbb{N}$, $k \geq 1$ define formulas $\exists^{\geq k} x\, \varphi(x)$, $\exists^{<k} x\, \varphi(x)$, $\exists^{=k} x\, \varphi(x)$, such that in a given τ-structure $\mathcal{A} = \langle A; \{R^{\mathcal{A}}\}_{R\in\tau}\rangle$:

$$\begin{aligned}
\mathcal{A} \models \exists^{\geq k} x\, \varphi(x) &\iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| \geq k \\
\mathcal{A} \models \exists^{<k} x\, \varphi(x) &\iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| < k \\
\mathcal{A} \models \exists^{=k} x\, \varphi(x) &\iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| = k
\end{aligned}$$

Expressing graph properties by first-order formulas

Exercise

For given formulas $\varphi(x)$ and for all $k \in \mathbb{N}$, $k \geq 1$ define formulas $\exists^{\geq k} x\, \varphi(x)$, $\exists^{<k} x\, \varphi(x)$, $\exists^{=k} x\, \varphi(x)$, such that in a given τ-structure $\mathcal{A} = \langle A; \{R^{\mathcal{A}}\}_{R\in\tau}\rangle$:

$$\mathcal{A} \models \exists^{\geq k} x\, \varphi(x) \iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| \geq k$$
$$\mathcal{A} \models \exists^{<k} x\, \varphi(x) \iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| < k$$
$$\mathcal{A} \models \exists^{=k} x\, \varphi(x) \iff |\{a \in A \mid \mathcal{A} \models \varphi(a)\}| = k$$

Expressing graph properties by first-order formulas

Exercise

Express by a first-order formula with the vocabulary $\tau_{\mathsf{G}} = \{E/_2\}$ for graphs that:

(i) a graph $\mathcal{G}$ contains a clique with precisely k elements,

(ii) a graph $\mathcal{G}$ has a dominating set with less or equal to k elements,

(iii) a graph $\mathcal{G}$ has a dominating set with precisely k elements,

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Recall:

$$\varphi_k \;\coloneqq\; \exists x_1 \ldots \exists x_k \Big(\bigwedge_{1 \le i < j \le k} (\neg(x_i = y_i) \wedge \neg E(x_i, x_j)) \Big)$$

$$\mathcal{A}_{\tau_{\mathsf{G}}}(\mathcal{G}) \models \varphi_k \quad \Longleftrightarrow \quad \mathcal{G} \text{ has a } k\text{-element independent set}\,.$$

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Exercise

Express by a first-order formula with the vocabulary with vocabulary $\tau_{\text{HG}} = \{\mathit{VERT}/_1, \mathit{EDGE}/_1, \mathit{INC}/_2\}$ for hypergraphs that:

(i) a graph $\mathcal{G}$ contains a clique with precisely k elements,

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Expressing graph properties by first-order formulas

Exercise

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Evaluation and model checking (first-order logic)

Let Φ be a class of first-order formulas.
The *evaluation problem* for Φ:

EVAL(Φ)

Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.
Problem: Compute $\varphi(\mathcal{A})$.

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Width of formula φ: maximal number of free variables in a subformula of φ.

Theorem

EVAL(FO) *and* MC(FO) *can be solved in time* $O(|\varphi| \cdot |A|^w \cdot w)$, *where* w *is the width of the input formula* φ.

Monadic second-order logic (formula example)

$$\psi_3 \;:=\; \exists C_1 \exists C_2 \exists C_3 \Big(\big(\forall x \big(\bigvee_{i=1}^{3} C_i(x)\big)\big) \wedge \forall x \big(\bigwedge_{1 \leq i < j \leq 3} \neg\big(C_i(x) \wedge C_j(x) \big)\big)$$

$$\wedge\; \forall x \forall y \big(E(x,y) \rightarrow \bigwedge_{i=1}^{3} \neg\big(C_i(x) \wedge C_i(y)\big)\big) \Big)$$

$\mathcal{A}(\mathcal{G}) \models \psi_3 \iff \mathcal{G}$ has is 3-colorable.

Monadic second-order logic

- language based on:
  - a *vocabulary* $\tau = \{R_1, \ldots, R_n\}$ of *predicate symbols* R_i together with arity *ar*$(R_i) \in \mathbb{N}$
  - the binary equality predication $=$
  - first-order variable symbols: $x, y, z, w, x_1, y_1, z_1, w_1, x_2, \ldots$
  - monadic second-order variable symbols (symbolizing variables for unary predicate symbols): $X, Y, Z, W, X_1, Y_1, Z_1, W_1, X_1, \ldots,$
  - propositional connectives: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
  - existential quantifier $\exists$, universal quantifier $\forall$

Interpretation of MSO-formulas in first-order structures

Let $\mathcal{A} = \langle A; \{R^{\mathcal{A}}\}_{R\in\tau}\rangle$ be a τ-structure.
For a MSO(τ)-formula $\varphi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k,\boldsymbol{X}_1,\ldots,\boldsymbol{X}_\ell)$ its *interpretation* $\varphi(\mathcal{A}) \subseteq A^k \times \mathcal{P}(A)^\ell$ in $\mathcal{A}$ is defined by:

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$$\varphi(\mathcal{A}) := \left\{ \langle a_1, \ldots, a_k, P_1, \ldots, P_\ell \rangle \in A^k \times \mathcal{P}(A)^\ell \mid a_j \in P_i \right\}$$

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- If $\varphi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k,\boldsymbol{X}_1,\ldots,\boldsymbol{X}_\ell) \equiv \exists \boldsymbol{X}_{k+1}\varphi_0(\boldsymbol{x}_{i_1},\ldots,\boldsymbol{x}_{i_{k'}},\boldsymbol{X}_{j_1},\ldots,\boldsymbol{X}_{j_{\ell'}})$ with $i_1,\ldots,i_{k'} \in [k]$, and $j_1,\ldots,j_{\ell'} \in [\ell+1]$ then:
$$\varphi(\mathcal{A}) := \{\langle a_1,\ldots,a_k,P_1,\ldots,P_\ell\rangle \in A^k \times \mathcal{P}(A)^\ell \mid \text{there exists } P_{\ell+1} \in \mathcal{P}(A) \text{ such that } \langle a_{i_1},\ldots,a_{i_{k'}},P_{j_1},\ldots,P_{j_{\ell'}}\rangle \in \varphi_0(\mathcal{A})\}$$

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- $\mathcal{A} \models \varphi(a_1,\ldots,a_k,P_1,\ldots,P_\ell)$
  will mean: $\langle a_1,\ldots,a_k,P_1,\ldots,P_\ell\rangle \in \varphi(\mathcal{A})$.

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  $$\varphi(\mathcal{A}) \coloneqq \Big\{\langle a_1,\ldots,a_k,P_1,\ldots,P_\ell\rangle \in A^k \times \mathcal{P}(A)^\ell \,\Big|\, \text{there exists } P_{\ell+1} \in \mathcal{P}(A) \text{ such that } \langle a_{i_1},\ldots,a_{i_{k'}},P_{j_1},\ldots,P_{j_{\ell'}}\rangle \in \varphi_0(\mathcal{A})\Big\}$$

- $\mathcal{A} \vDash \varphi(a_1,\ldots,a_k,P_1,\ldots,P_\ell)$
  will mean: $\langle a_1,\ldots,a_k,P_1,\ldots,P_\ell\rangle \in \varphi(\mathcal{A})$.
- For a sentence φ, $\mathcal{A} \vDash \varphi$ will mean $\varphi(\mathcal{A}) \neq \varnothing$ (then $\varphi(\mathcal{A}) = \{\langle\rangle\}$).

Monadic second-order logic (formula example)

$$\psi_3 \; := \; \exists C_1 \exists C_2 \exists C_3 \Big(\big(\forall x \big(\bigvee_{i=1}^{3} C_i(x) \big) \big) \wedge \forall x \big(\bigwedge_{1 \leq i < j \leq 3} \neg \big(C_i(x) \wedge C_j(x) \big) \big) \\ \wedge \forall x \forall y \big(E(x,y) \rightarrow \bigwedge_{i=1}^{3} \neg (C_i(x) \wedge C_i(y)) \big) \Big)$$

$\mathcal{A}(\mathcal{G}) \vDash \psi_3 \quad \Longleftrightarrow \quad \mathcal{G}$ has is 3-colorable .

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$$\begin{aligned}
\equiv\; \exists C_1 \exists C_2 \exists C_3 \Big(& \forall x \big(C_1(x) \vee C_2(x) \vee C_3(x) \big) \\
& \wedge \forall x \Big(\neg\big(C_1(x) \wedge C_2(x)\big) \wedge \neg\big(C_1(x) \wedge C_3(x)\big) \\
& \qquad \wedge \neg\big(C_2(x) \wedge C_3(x)\big) \Big) \\
& \wedge \forall x \forall y \Big(E(x,y) \to \neg\big(C_1(x) \wedge C_1(y)\big) \\
& \qquad \wedge \neg\big(C_2(x) \wedge C_2(y)\big) \\
& \qquad \wedge \neg\big(C_3(x) \wedge C_3(y)\big) \Big) \Big)
\end{aligned}$$

$\mathcal{A}(\mathcal{G}) \vDash \psi_3 \iff \mathcal{G}$ has is 3-colorable.

Expressing graph properties by MSO formulas (1)

Exercise

Express by a monadic second-order formula $\varphi(X)$ with one free unary predicate variable X over the vocabulary $\tau_{\mathsf{G}} = \{E/2\}$ for graphs that for all graphs $\mathcal{G} = \langle V, E\rangle$:

$$\mathcal{A}_{\tau_{\mathsf{G}}}(\mathcal{G}) \vDash \varphi(S) \iff S \subseteq V \text{ is an independent set in } \mathcal{G}$$

Recall:

$$\begin{aligned} S \subseteq V \text{ is independent set in } \mathcal{G} :\iff & \forall e = \{u, v\} \in E\,(\,\neg(u \in S \wedge v \in S)\,) \\ \iff & \forall u, v \in S(\,u \neq v \Rightarrow \{u, v\} \notin E\,) \end{aligned}$$

Exercise

Express the independent set property by a MSO(τ_{HG}) formula ψ with vocabulary $\tau_{\mathsf{HG}} = \{\,\mathit{VERT}/1, \mathit{EDGE}/1, \mathit{INC}/2\}$ for hypergraphs:

$$\mathcal{A}_{\tau_{\mathsf{HG}}}(\mathcal{G}) \vDash \psi(S) \iff S \subseteq V \text{ is an independent set in } \mathcal{G}$$

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Expressing graph properties by MSO formulas (2)

Exercise

Express by a monadic second-order formula *feedback*(X) with one free unary predicate variable X over $\tau_{\text{HG}} = \{\textit{VERT}/_1, \textit{EDGE}/_1, \textit{INC}/_2\}$, the vocabulary for graphs, that for all hypergraphs $\mathcal{G} = \langle V, E \rangle$:

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \vDash \textit{feedback}(S) \iff S \subseteq V \text{ is a feedback vertex set}$$

(A set $S \subseteq V$ is a feedback vertex set in $\mathcal{G}$ if S contains a vertex of every cycle of $\mathcal{G}$.)

Steps:

- Construct a formula *cycle-family*(X) that expresses the property of a set being the union of cycles.
- Using *cycle-family*(X), construct *feedback*(X).

MSO for graphs and hypergraphs

- MSO(τ_{G}) : MSO with vocabulary $\tau_{\mathrm{G}} = \{E/_2\}$

MSO for graphs and hypergraphs

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MSO for graphs and hypergraphs

- $\mathrm{MSO}(\tau_{\mathrm{G}})$: MSO with vocabulary $\tau_{\mathrm{G}} = \{E/_2\}$
- $\mathrm{MSO}(\tau_{\mathrm{HG}})$: MSO with vocab. $\tau_{\mathrm{HG}} = \{\mathit{VERT}/_1, \mathit{EDGE}/_1, \mathit{INC}/_2\}$
- MSO_1 :
  - vocabulary: $\{\mathit{INC}/_2\}$
  - quantifications: $\exists_{\text{(vert)}} x$ / $\forall_{\text{(vert)}} x$, $\exists_{\text{(edge)}} x$ / $\forall_{\text{(edge)}} x$, $\exists_{\text{(vert)}} X$ / $\forall_{\text{(vert)}} X$

MSO for graphs and hypergraphs

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- MSO(τ_{HG}) : MSO with vocab. $\tau_{\mathsf{HG}} = \{ VERT/_1, EDGE/_1, INC/_2 \}$
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  - quantifications: $\exists_{\text{(vert)}} x$ / $\forall_{\text{(vert)}} x$, $\exists_{\text{(edge)}} x$ / $\forall_{\text{(edge)}} x$,
    $\exists_{\text{(vert)}} X$ / $\forall_{\text{(vert)}} X$
- MSO_2 :
  - vocabulary: $\{ INC/_2 \}$
  - quantifications: $\exists_{\text{(vert)}} x$ / $\forall_{\text{(vert)}} x$, $\exists_{\text{(edge)}} x$ / $\forall_{\text{(edge)}} x$,
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- MSO(τ_G) : MSO with vocabulary $\tau_G = \{E/_2\}$
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  - vocabulary: $\{INC/_2\}$
  - quantifications: $\exists_{(vert)} x$ / $\forall_{(vert)} x$, $\exists_{(edge)} x$ / $\forall_{(edge)} x$,
    $\exists_{(vert)} X$ / $\forall_{(vert)} X$
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Correspondences

MSO(τ_G) corresponds to MSO$_1$

where 'corresponds to' means: 'is equally expressive as'.

MSO for graphs and hypergraphs

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    $\exists_{(\mathrm{vert})} X$ / $\forall_{(\mathrm{vert})} X$, $\exists_{(\mathrm{edge})} X$ / $\forall_{(\mathrm{edge})} X$

Correspondences

$\mathrm{MSO}(\tau_{\mathrm{G}})$	corresponds to	MSO_1
$\mathrm{MSO}(\tau_{\mathrm{HG}})$	corresponds to	MSO_2

where 'corresponds to' means: 'is equally expressive as'.

MSO for graphs and hypergraphs

- MSO(τ_{G}): MSO with vocabulary $\tau_{\mathsf{G}} = \{E/_2\}$
- MSO(τ_{HG}): MSO with vocab. $\tau_{\mathsf{HG}} = \{VERT/_1, EDGE/_1, INC/_2\}$
- MSO_1 :
  - vocabulary: $\{INC/_2\}$
  - quantifications: $\exists_{(\mathsf{vert})}x$ / $\forall_{(\mathsf{vert})}x$, $\exists_{(\mathsf{edge})}x$ / $\forall_{(\mathsf{edge})}x$, $\exists_{(\mathsf{vert})}X$ / $\forall_{(\mathsf{vert})}X$
- MSO_2 :
  - vocabulary: $\{INC/_2\}$
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Correspondences

MSO(τ_{G})	corresponds to	MSO_1
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where 'corresponds to' means: 'is equally expressive as'.

Note:

MSO(τ_{HG}) / MSO_2 are more expressive than MSO(τ_{G}) / MSO_1.

MSO for graphs and hypergraphs

- $\mathrm{MSO}(\tau_{\mathrm{G}})$: MSO with vocabulary $\tau_{\mathrm{G}} = \{E/_2\}$
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    $\exists_{\text{(vert)}} X$ / $\forall_{\text{(vert)}} X$
- MSO_2 :
  - vocabulary: $\{INC/_2\}$
  - quantifications: $\exists_{\text{(vert)}} x$ / $\forall_{\text{(vert)}} x$, $\exists_{\text{(edge)}} x$ / $\forall_{\text{(edge)}} x$,
    $\exists_{\text{(vert)}} X$ / $\forall_{\text{(vert)}} X$, $\exists_{\text{(edge)}} X$ / $\forall_{\text{(edge)}} X$

Correspondences

$\mathrm{MSO}(\tau_{\mathrm{G}})$	corresponds to	MSO_1
$\mathrm{MSO}(\tau_{\mathrm{HG}})$	corresponds to	MSO_2

where 'corresponds to' means: 'is equally expressive as'.

Note:

We use MSO for either logic, restrict to $\mathrm{MSO}(\tau_{\mathrm{G}})$ / MSO_1 if needed.

MSO for graphs and hypergraphs

- MSO(τ_{G}) : MSO with vocabulary $\tau_{\mathrm{G}} = \{E/_2\}$
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- MSO$_2$:
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    $\exists_{\text{(vert)}} X$ / $\forall_{\text{(vert)}} X$, $\exists_{\text{(edge)}} X$ / $\forall_{\text{(edge)}} X$

Correspondences

MSO(τ_{G}) corresponds to MSO$_1$

MSO(τ_{HG}) corresponds to MSO$_2$

where 'corresponds to' means: 'is equally expressive as'.

Note:

MSO(τ_{HG}) / MSO$_2$ are more expressive than MSO(τ_{G}) / MSO$_1$.

Expressing graph properties by MSO formulas (5)

Exercise

Express by a MSO(τ_{HG}) formula *conn*(X) with one free unary predicate variable X over $\tau_{\mathrm{HG}} = \{\mathit{VERT}/1, \mathit{EDGE}/1, \mathit{INC}/2\}$, the vocabulary for graphs, that for all hypergraphs $\mathcal{G} = \langle V, E \rangle$:

$$\mathcal{A}_{\tau_{\mathrm{HG}}}(\mathcal{G}) \models \mathit{hamiltonian} \iff \text{there is a Hamiltonian path in } \mathcal{G}.$$

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Note:

- This property is not expressible by a (single) MSO(τ_{G}) formula.

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Note:

- This property is not expressible by a (single) MSO(τ_{G}) formula.
- Other properties that are not MSO(τ_{G}) expressible:
  - balanced bipartite graphs
  - existence of a perfect matching
  - simple graphs (graphs with no parallel edges)
  - existence of spanning trees with maximum degree 3

Expressing graph properties by MSO formulas (5)

Exercise

$\mathcal{A}_{\tau_{\mathsf{HG}}}(\mathcal{G}) \models$ *hamiltonian* $\iff$ there is a Hamiltonian path in $\mathcal{G}$.

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Evaluation and model checking (MSO)

The *model checking problem* for MSO-formulas on labeled, ordered unranked trees:

MC(MSO, TREE_{lo})

Instance: A labeled, ordered, unranked Σ-tree $\mathcal{T}$, and a MSO(τ^u_Σ)-formula φ

Problem: Decide whether $\mathcal{T} \vDash \varphi$.

where for given alphabet Σ, $\tau^u_\Sigma := \{E/_2, N/_2\} \cup \{P_a/_1 \mid a \in \Sigma\}$.

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Theorem

MC(MSO, TREE$_{lo}$) $\in$ FPT*.*
More precisely, there is a computable function $f : \mathbb{N} \to \mathbb{N}$ *such that* MC(MSO, TREE$_{lo}$) *can be decided in time* $\leq O(f(|\varphi|) + \|\mathcal{T}\|)$*.*

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Note that here: $f(k) \ge 2^{\cdot^{\cdot^{\cdot^{2}}}}\}k$ (a non-elementary function).

Courcelle's Theorem

Courcelle's Theorem for graphs

p^*-*tw*-MC(MSO)

Instance: A graph $\mathcal{G}$ and an MSO(τ_{HG})-sentence φ.
Parameter: $tw(\mathcal{G}) + |\varphi|$ (where $tw(\mathcal{G})$ the tree-width of $\mathcal{G}$)
Problem: Decide whether $\mathcal{A}(\mathcal{G}) \models \varphi$.

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Theorem (special case of Courcelle's Theorem)

p^*-*tw*-MC(MSO) $\in$ FPT. *More precisely, the problem is decidable, for some computable and non-decreasing function* $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *by an algorithm in time:*

$$f(k_1, k_2) \cdot n\,, \quad \textit{where } k_1 := \mathit{tw}(\mathcal{A}),\ k_2 := |\varphi|,\ n := |V(\mathcal{G})|$$

Courcelle's Theorem: applications (1)

p^*-*tw*-COLORABILITY $\in$ FPT
Instance: A graph $\mathcal{G}$ and $\ell \in \mathbb{N}$.
Parameter: *tw*$(\mathcal{C})$
Problem: Decide whether is $\mathcal{G}$ ℓ-colorable.

Example

- p^*-*tw*-3-COLORABILITY $\in$ FPT.

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Courcelle's Theorem: applications (2)

p^*-*tw*-HAMILTONICITY

Instance: A graph $\mathcal{G}$

Parameter: $tw(\mathcal{C})$

Problem: Decide whether $\mathcal{G}$ is a hamiltonian (that is, contains a cyclic path that visits every vertex precisely once).

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Example

p^*-*tw*-HAMILTONICITY $\in$ FPT.

Tree decompositions, and tree-width for graphs

Definition (recalling tree-width for graphs)

A *tree decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a pair $\langle \mathcal{T}, \{B_t\}_{t\in T} \rangle$ where $\mathcal{T} = \langle T, F \rangle$ a (undirected, unrooted) tree, and $B_t \subseteq V$ for all $t \in T$ such that:

(T1) $A = \bigcup_{t\in T} B_t$ (every vertex of $\mathcal{G}$ is in some bag).

(T2) $(\forall \{u,v\} \in E)\,(\exists t \in T)\big[\, \{u,v\} \subseteq B_t \,\big]$
(the vertices of every edge of $\mathcal{G}$ are realized in some bag).

(T3) $(\forall v \in V)\,\big[$ subgraph of $\mathcal{T}$ defd. by $\{t \in T \mid v \in B_t\}$ is connected $\big]$
(the tree vertices (in $\mathcal{T}$) whose bags contain some vertex of $\mathcal{G}$ induce a subgraph of $\mathcal{T}$ that is connected).

The *width* of a tree dec. $\langle \mathcal{T}, \{B_t\}_{t\in T} \rangle$ is $\max\{|B_t| - 1 \mid t \in T\}$.

The *tree-width tw*$(\mathcal{A})$ of a τ-structure $\mathcal{A}$ is defined by:

tw$(\mathcal{A})$:= minimal width of a tree decomposition of $\mathcal{A}$.

Tree decompositions, and tree-width for structures

Definition (extension of tree-width to structures)

A *tree decomposition* of a τ-structure $\mathcal{A} = \langle A; \{R^{\mathcal{A}}\}_{R\in\tau}\rangle$ is a pair $\langle \mathcal{T}, \{B_t\}_{t\in T}\rangle$ where $\mathcal{T} = \langle T, F\rangle$ a (undirected, unrooted) tree, and $B_t \subseteq V$ for all $t \in T$ such that:

(T1) $A = \bigcup_{t\in T} B_t$ (every element of the universe of $\mathcal{A}$ is in some bag).

(T2) $(\forall R \in \tau)\,(\forall \langle a_1, \ldots, a_r\rangle \in R^{\mathcal{A}})(\exists t \in T)\big[\,\{a_1, \ldots, a_r\} \subseteq B_t\,\big]$
(the vertices of every 'hyperedge' in $R^{\mathcal{A}}$ are realized in some bag).

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p^*-*tw*-MC(MSO)

Instance: A structure $\mathcal{A}$ and an MSO-sentence φ.
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Problem: Decide whether $\mathcal{A} \models \varphi$.

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p^*-*tw*-MC(MSO) $\in$ FPT. *More precisely, the problem is decidable by an algorithm in time:*

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f *computable and non-decreasing*

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$$\leq (f(k_1, k_2) + c) \cdot \|\mathcal{A}\|$$

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f(k_1, k_2) \cdot |A| + O(\|\mathcal{A}\|) &\leq f(k_1, k_2) \cdot |A| + c \cdot \|\mathcal{A}\| && \text{with some } c > 0 \\
&\leq (f(k_1, k_2) + c) \cdot \|\mathcal{A}\| \\
&\leq g(k) \cdot (\|\mathcal{A}\| + |\varphi|) && \text{for } g(x) := f(x, x) + c \\
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\end{aligned}$$

Vertex Cover (first attempt)

Let $\mathcal{G} = \langle V, E \rangle$ a graph. For all $S \subseteq V$:

S is a vertex cover of $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \in E \, (\, u \in S \lor v \in S))$

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p^*-*tw*-VERTEX-COVER

Instance: A graph $\mathcal{G} = \langle V, E\rangle$, and $\ell \in \mathbb{N}$.

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Problem: Does $\mathcal{G}$ have a vertex cover of size at most ℓ?

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Courcelle's Theorem: Refinement 1

p^*-*tw*-MC$^{\leq}$(MSO)

Instance: A structure $\mathcal{A}$, an $\varphi(X)$, and $m \in \mathbb{N}$.

Parameter: *tw*$(\mathcal{A}) + |\varphi(X)|$.

Problem: Decide whether $\mathcal{A} \models \exists X(\, \mathit{card}^{\leq m}(X) \wedge \varphi(X)\,)$.

Refinement 1 of Courcelle's Theorem

p^*-*tw*-MC$^{\leq}$(MSO) $\in$ FPT. More precisely, the problem is decidable by an algorithm in time:

$$f(k_1, k_2) \cdot |A| + O(\|\mathcal{A}\|), \quad \text{where } k_1 := \mathit{tw}(\mathcal{A}), \text{ and } k_2 := |\varphi|,$$

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Example

p^*-*tw*-VERTEX-COVER $\in$ FPT.

Vertex Cover

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p^*-*tw*-VERTEX-COVER
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Example

p^*-*tw*-VERTEX-COVER $\in$ FPT.

Courcelle's Theorem: Refinement 2

p^*-*tw*-MC$^=$(MSO)
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Parameter: *tw*$(\mathcal{A}) + |\varphi(X)|$.
Problem: Decide whether $\mathcal{A} \vDash \exists X(\, \mathit{card}^{=m}(X) \wedge \varphi(X)\,)$.

Refinement 2 of Courcelle's Theorem

p^*-*tw*-MC$^=$(MSO) $\in$ FPT. More precisely, the problem is decidable by an algorithm in time:

$$f(k_1, k_2) \cdot |A|^2 + O(\|\mathcal{A}\|)\,,$$

where $k_1 :=$ *tw*$(\mathcal{A})$, and $k_2 := |\varphi|$, f computable and non-decreasing

Courcelle's Theorem Ref. 3: Optimization version

p^*-*tw*-opt-MC(MSO)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, an MSO-sentence $\varphi(X_1, \ldots, X_p)$.

Parameter: $tw(\mathcal{G}) + |\varphi(X_1, \ldots, X_p)|$.

Compute: $\begin{matrix}\max\\ \min\end{matrix} \left\{ \alpha(|X_1|, \ldots, |X_p|) \;\middle|\; \begin{matrix} X_1, \ldots, X_p \subseteq V \cup E \\ \mathcal{A}(\mathcal{G}) \models \varphi(X_1, \ldots, X_p). \end{matrix} \right\}$.

where α is an affine function $\alpha(x_1, \ldots, x_p) = a_0 + \sum_{i=1}^{p} a_i \cdot x_i$.

Optimization version of Courcelle's Theorem

p^*-*tw*-opt-MC(MSO) $\in$ FPT, and it is decidable by an algorithm in time:

$f(tw(\mathcal{G}), |\varphi|) \cdot |V|$, where f computable and non-decreasing.

Maximum 2-edge colorable subgraphs

p^*-*tw*-max-2-edge-colorable-subgraph

Instance: A graph $\mathcal{G} = \langle V, E \rangle$.

Parameter: *tw*($\mathcal{G}$).

Compute: Maximum number of edges in a 2-edge colored subgraph of G.

Example [AA & Vahan Mkrtchyan]

p^*-*tw*-max-2-edge-colorable-subgraph $\in$ FPT.

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Example [AA & Vahan Mkrtchyan]

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Courcelle's Theorem: applications (3)

p^*-*tw*-INDEPENDENT-SET

Instance: A graph $\mathcal{G}$, a number $\ell \in \mathbb{N}$.
Parameter: *tw*$(\mathcal{G})$
Problem: Decide whether $\mathcal{G}$ has an independent set of ℓ elements.

Example

p^*-*tw*-INDEPENDENT-SET $\in$ FPT.

Courcelle's Theorem: applications (3)

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Instance: A graph $\mathcal{G}$, a number $\ell \in \mathbb{N}$.

Parameter: $tw(\mathcal{G})$

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Example

p^*-*tw*-INDEPENDENT-SET $\in$ FPT.

Courcelle's Theorem: applications (4)

p^*-*tw*-FEEDBACK-VERTEX-SET
Instance: A graph $\mathcal{G}$ and $\ell \in \mathbb{N}$.
Parameter: $tw(\mathcal{C})$
Problem: Decide whether $\mathcal{G}$ has a feedback vertex set of ℓ elements.

Example

p^*-*tw*-FEEDBACK-VERTEX-SET $\in$ FPT.

Courcelle's Theorem: applications (4)

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Instance: A graph $\mathcal{G}$ and $\ell \in \mathbb{N}$.
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Problem: Decide whether $\mathcal{G}$ has a feedback vertex set of ℓ elements.

Example

p^*-*tw*-FEEDBACK-VERTEX-SET $\in$ FPT.

Courcelle's Theorem: applications (5)

p^*-*tw*-CROSSING-NUMBER
Instance: A graph $\mathcal{G}$, and $k \in \mathbb{N}$
Parameter: $tw(\mathcal{G}) + k$
Problem: Decide whether the crossing number of $\mathcal{G}$ is k.

Example

p^*-*tw*-CROSSING-NUMBER $\in$ FPT.

The *crossing number* is the least number of edge crossings required to draw the graph in the plane such that at each point at most two edges cross.

Courcelle's Theorem: applications (5)

Definition

Let $\mathcal{G}_1 = \langle V_1, E_1 \rangle$ and $\mathcal{G}_2 = \langle V_2, E_2 \rangle$ be graphs.
$\mathcal{G}_1$ is a *subdivision* of $\mathcal{G}_2$ if:

- $\mathcal{G}_1$ arises by splitting the edges of $\mathcal{G}_2$ into paths with intermediate vertices.

$\mathcal{H}$ is a *topological subgraph* of $\mathcal{G}$ if $\mathcal{G}$ has a subgraph that is a subdivision of $\mathcal{H}$.

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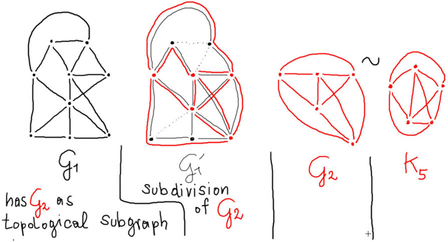
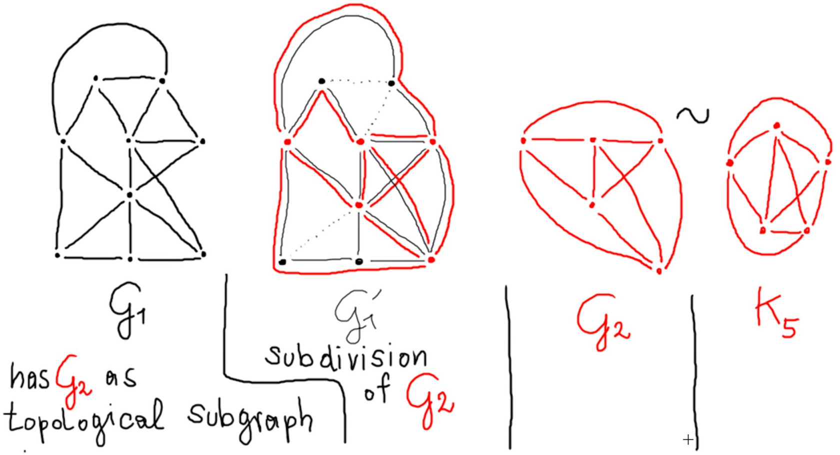


Courcelle's Theorem: applications (5)



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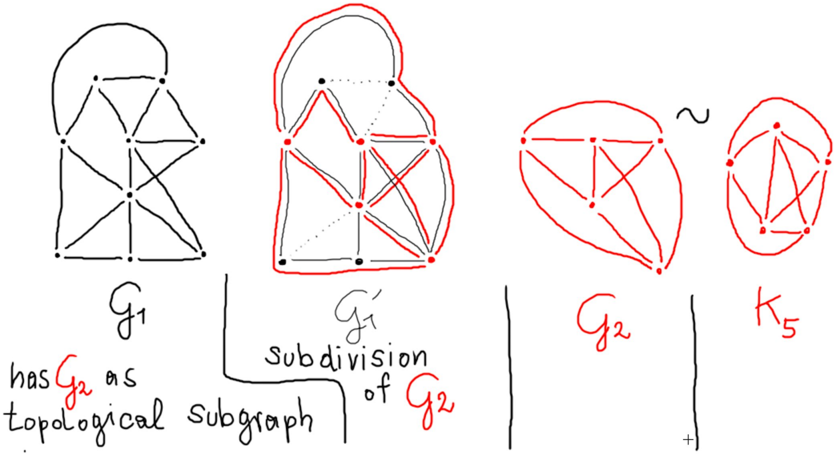

Theorem (Kuratowski)

A graph is planar if and only if it contains neither $\mathcal{K}_5$ nor $\mathcal{K}_{3,3}$ as topological subgraph.

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Lemma

There is a MSO(τ_{HG}) formula *top-sub*$_{\mathcal{H}}$ such that for every graph $\mathcal{G}$:

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Using MSO(τ_{HG}) formula *path*(x, y, Z) that Z is a path from x to y.

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A graph is planar if and only if it contains neither $\mathcal{K}_5$ nor $\mathcal{K}_{3,3}$ as topological subgraph.

Lemma

There is a MSO(τ_{HG}) formula *top-sub*$_{\mathcal{H}}$ such that for every graph $\mathcal{G}$:

$$\mathcal{A}_{\tau_{\mathsf{HG}}}(\mathcal{G}) \models \textit{top-sub}_{\mathcal{H}} \iff \mathcal{H} \text{ is a topological subgraph of } \mathcal{G}.$$

Using MSO(τ_{HG}) formula *path*(x, y, Z) that Z is a path from x to y.

Lemma

There is a MSO(τ_{HG}) formula *cross*$_k$ such that for every graph $\mathcal{G}$:

$$\mathcal{A}_{\tau_{\mathsf{HG}}}(\mathcal{G}) \models \textit{cross}_k \iff \text{the crossing number of } \mathcal{G} \text{ is at most } k.$$

Proof: By induction, where $\textit{cross}_0 := \neg\textit{top-sub}_{\mathcal{K}_5} \wedge \neg\textit{top-sub}_{\mathcal{K}_{3,3}}$.

Computably boundedness between notions of width

(from Sasák, [Sásak, 2010])

$$g(wd_1) \geq wd_2 \;:\Leftrightarrow\; wd_1 \xrightarrow{g} wd_2$$

- FPT-results
  transfer upwards
  (and conversely to $\xrightarrow{g}$)
- ($\notin$ FPT)-results
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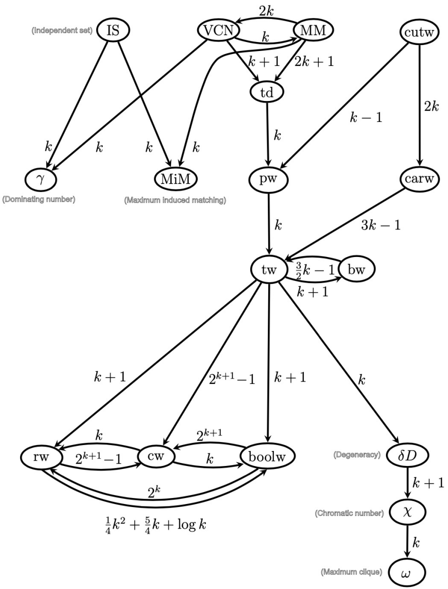

Comparing parameterizations

Definition (computably bounded below)

Let $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ parameterizations.

- $\kappa_1 \succeq \kappa_2 \;:\Longleftrightarrow\; \exists g : \mathbb{N} \to \mathbb{N}$ computable $\forall x \in \Sigma^* \big[\, g(\kappa_1(x)) \geq \kappa_2(x) \,\big]$.
- $\kappa_1 \approx \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \kappa_2 \succeq \kappa_1$.
- $\kappa_1 \succ \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \neg(\kappa_2 \succeq \kappa_1)$.

Proposition

For all parameterized problems $\langle Q, \kappa_1\rangle$ and $\langle Q, \kappa_2\rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ with $\kappa_1 \succeq \kappa_2$:

$$\langle Q, \kappa_1\rangle \in \text{FPT} \quad \Longleftarrow \quad \langle Q, \kappa_2\rangle \in \text{FPT}$$
$$\langle Q, \kappa_1\rangle \notin \text{FPT} \quad \Longrightarrow \quad \langle Q, \kappa_2\rangle \notin \text{FPT}$$

Courcelle's Theorem for clique-width

Recall that MSO(τ_{G}) ~ MSO_1 (quantification over sets of vertices, but not sets of edges).

p^*-*clw*-MC(MSO(τ_{G})/MSO_1)
Instance: A graph $\mathcal{G}$ and an MSO(τ_{G})-sentence φ.
Parameter: *clw*($\mathcal{G}$) + $|\varphi|$.
Problem: Decide whether $\mathcal{A}(\mathcal{G}) \vDash \varphi$.

Theorem ([Courcelle et al., 2000])

p^*-*clw*-MC(MSO(τ_{G})/MSO_1) $\in$ FPT.

Also, there is a maximization version of this theorem.

Courcelle's Theorem for clique-width (example)

Let $\mathcal{G} = \langle V, E \rangle$ a graph. For all $S \subseteq V$:

S is a vertex cover of $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \in E \,(\, u \in S \vee v \in S))$

p^*-*clw*-VERTEX-COVER

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and $\ell \in \mathbb{N}$.

Instance: *clw*$(\mathcal{G})$.

Problem: Does $\mathcal{G}$ have a vertex cover of size at most ℓ?

Example

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Application to maximum 2-edge colorable subgraphs?

p^*-*clw*-max-2-edge-colorable-subgraph
Instance: A graph $\mathcal{G} = \langle V, E \rangle$.
Parameter: *clw*$(\mathcal{G})$.
Compute: Maximum number of edges
in a 2-edge colored subgraph of G.

Open problem [AA, Vahan Mkrtchyan]

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Open problem [AA, Vahan Mkrtchyan]

p^*-*clw*-max-2-edge-colorable-subgraph $\in$ FPT ?

We saw that there is a MSO_2 formula $\varphi(X)$ such that:

$$\mathcal{A}_{\tau_{\text{HG}}}(\mathcal{G}) \vDash \varphi(S) \iff S \subseteq E \text{ is an 2-colorable set of edges in } \mathcal{G}$$

But there seems not to be such an MSO_1 formula.

Courcelle's Theorem for clique-width (non-example)

p^*-*clw*-HAMILTONICITY

Instance: A graph $\mathcal{G}$

Parameter: *clw*$(\mathcal{C})$

Problem: Decide whether $\mathcal{G}$ is a hamiltonian (that is, contains a cyclic path that visits every vertex precisely once).

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Recall

There is no MSO_1 formula that expresses Hamiltonicity.

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Hence we cannot apply Courcelle's Theorem for clique-width.

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Recall

There is no MSO_1 formula that expresses Hamiltonicity.

Hence we cannot apply Courcelle's Theorem for clique-width. Indeed:

Theorems

(T1) p^*-*clw*-HAMILTONICITY $\notin$ FPT,
since it is not decidable in time $\notin n^{o(\mathit{clw}(\mathcal{C}))}$ (Fomin et al, 2014).

(T2) p^*-*clw*-HAMILTONICITY $\in O(n^{o(\mathit{clw}(\mathcal{C}))})$
(Bergougnoux, Kanté, Kwon, 2020).

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(from Sasák, [Sásak, 2010])

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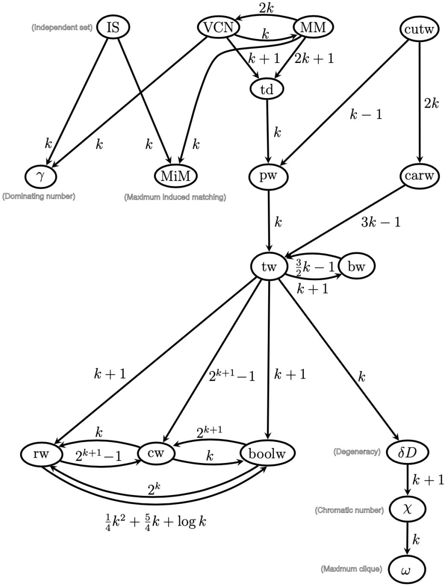

Graph Minors

Graph minors

Definition

A graph $\mathcal{G}_0$ is a *minor* of a graph $\mathcal{G}$ if $\mathcal{G}_0$ is obtained by:

- deleting some edges,
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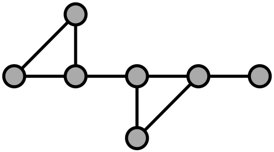
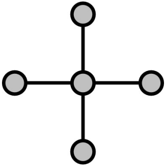


$\mathcal{G}$



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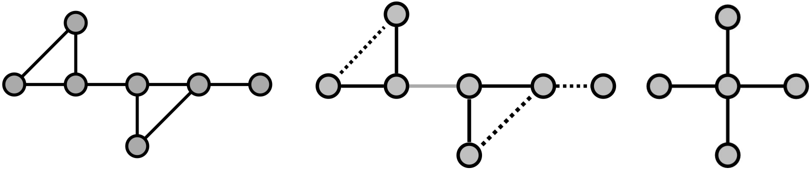

$\mathcal{G}$ $\mathcal{G}_0$

Excluded minors

Definition (minor closed)

A class $\boldsymbol{\mathcal{G}}$ is *minor closed* if for every $\mathcal{G} \in \boldsymbol{\mathcal{G}}$ all minors of $\mathcal{G}$ are in $\boldsymbol{\mathcal{G}}$.

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We say that a class $\boldsymbol{\mathcal{G}}$ is characterized by excluded minors in $\boldsymbol{\mathcal{H}}$ if:

$$\boldsymbol{\mathcal{G}} := \text{Excl}(\boldsymbol{\mathcal{H}}) := \{\mathcal{G} \mid \mathcal{G} \text{ does not have a minor in } \boldsymbol{\mathcal{H}}\}$$

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Theorem (Graph Minor Theorem (Robertson–Seymour, 1983–2004))

Every class of graphs that is minor closed can be characterized by finitely many excluded minors. That is, for every class $\boldsymbol{\mathcal{G}}$ of minor closed graphs there are graphs $\mathcal{H}_1, \ldots, \mathcal{H}_m$ such that:

$$\boldsymbol{\mathcal{G}} = \mathrm{Excl}(\{\mathcal{H}_1, \ldots, \mathcal{H}_m\}) .$$

Deciding minor closed classes

p-MINOR

Instance: Graphs $\mathcal{G}$ and $\mathcal{H}$.
Parameter: $\|\mathcal{G}\|$
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Corollary

Every minor-closed class of graphs is decidable in cubic time.

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Every minor-closed class of graphs is decidable in cubic time.

Corollary

Let $\langle Q, \kappa \rangle$ be a parameterized problem on graphs such that for every $k \in \mathbb{N}$, either $\{\mathcal{G} \in Q \mid \kappa(\mathcal{G}) = k\}$ or $\{\mathcal{G} \notin Q \mid \kappa(\mathcal{G}) = k\}$ is minor closed.

Then every slice $\langle Q, \kappa \rangle_k$ is decidable in cubic time. In this case we can say that $\langle Q, \kappa \rangle$ is nonuniformly fixed-parameter tractable.

Non-uniformly fixed-parameter tractable

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *fixed-parameter tractable* if:

$\exists f : \mathbb{N} \to \mathbb{N}$ computable $\exists p \in \mathbb{N}[X]$ polynomial

$\exists \mathbb{A}$ algorithm, takes inputs in Σ^*

$\forall x \in \Sigma^* \big[$ $\mathbb{A}$ decides whether $x \in Q$ holds
in time $\leq f(\kappa(x)) \cdot p(|x|) \big]$

Non-uniformly fixed-parameter tractable

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Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *non-uniformly fixed-parameter tractable* (in nu-FPT) if:

$$\exists f : \mathbb{N} \to \mathbb{N} \text{ computable } \exists p \in \mathbb{N}[X] \text{ polynomial}$$
$$\exists \{\mathbb{A}_k\}_{k \in \mathbb{N}} \text{ algorithms, takes inputs in } \Sigma^*$$
$$\forall x \in \Sigma^* \big[\, \mathbb{A}_{\kappa(x)} \text{ decides whether } x \in Q \text{ holds in time} \le f(\kappa(x)) \cdot p(|x|) \,\big]$$

Using minor-closed classes for FPT results

Corollary

Let $\langle Q, \kappa \rangle$ be a parameterized problem on graphs such that for every $k \in \mathbb{N}$, either $\{\mathcal{G} \in Q \mid \kappa(\mathcal{G}) = k\}$ or $\{\mathcal{G} \notin Q \mid \kappa(\mathcal{G}) = k\}$ is minor closed. Then $\langle Q, \kappa \rangle$ is non-uniformly fixed-parameter tractable (in nu-FPT).

Applications:

- p-VERTEX-COVER $\in$ nu-FPT (p-VERTEX-COVER is minor closed).

Using minor-closed classes for FPT results

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Applications:

- p-VERTEX-COVER $\in$ nu-FPT (p-VERTEX-COVER is minor closed).
- p-FEEDBACK-VERTEX-SET $\in$ nu-FPT (problem is minor closed).

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- p-VERTEX-COVER $\in$ nu-FPT (p-VERTEX-COVER is minor closed).
- p-FEEDBACK-VERTEX-SET $\in$ nu-FPT (problem is minor closed).
- p-DISJOINT-CYCLES
  - **Instance:** A graph $\mathcal{G}$, and $k \in \mathbb{N}$.
  - **Parameter:** k.
  - **Problem:** Decide whether $\mathcal{G}$ has k disjoint cycles.

  p-DISJOINT-CYCLES $\in$ nu-FPT, since the class of graphs that do not have k disjoint cycles is minor closed.

First-Order Meta-Theorem (example)

Seese's theorem

A class $\boldsymbol{\mathcal{G}}$ of graphs has *bounded degree* if there is $d \in \mathbb{N}$ such that $\Delta(\mathcal{G}) \le d$ for all $\mathcal{G} \in \boldsymbol{\mathcal{G}}$ (where $\Delta(\mathcal{G})$ = max. degree of vertex in $\mathcal{G}$).

p-MC(FO, $\boldsymbol{\mathcal{G}}$)

Instance: A graph $\mathcal{G} \in \boldsymbol{\mathcal{G}}$, and a f-o formula φ over τ_{HG}

Parameter: $|\varphi|$.

Problem: Decide whether $\mathcal{A}(\mathcal{G}) \models \varphi$.

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Theorem ([Seese, 1995])

p-MC(FO, $\boldsymbol{\mathcal{G}}$) $\in$ FPT *for every class* $\boldsymbol{\mathcal{G}}$ *of bounded degree. This model checking problem can be solved in time* $f(|\varphi|) \cdot |\mathcal{G}|$, *(linear in* $|\mathcal{G}|$*).*

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Theorem (for comparison, we saw it earlier)

EVAL(FO) *and* MC(FO) *can be solved in time* $O(|\varphi| \cdot |A|^w \cdot w)$*, where* w *is the width of the input formula* φ*.*

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Example

On the class of graphs of bounded degree:

- p-CLIQUE $\in$ FPT ('is there a clique of size k')?

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On the class of graphs of bounded degree:

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- p-VERTEX-COVER $\in$ FPT ('vertex cover of size at most k?')

First-order metatheorems: reference

A good reference for other meta-theorems for first-order logic is:

[Kreutzer, 2009]: Stephan Kreutzer: *Algorithmic Meta-Theorems*.

Summary

- Logic preliminaries
  - first-order logic
    - expressing graph problems by f-o formulas
  - monadic second-order logic (MSO)
    - expressing graph problems by MSO formulas
  - complexity of evaluation and model checking problems
- Courcelle's theorem
  - FPT-results by model-checking MSO-formulas
    - for graphs with bounded tree-width
    - for structures with bounded tree-width
    - for graphs of bounded clique-width
  - applications to concrete problems
- graph minors
- meta-theorems for first-order model-checking: an example

Course overview

Monday, June 16 10.30 – 12.30	Tuesday, June 17 10.30 – 12.30	Wednesday, June 18	Thursday, June 19 10.30 – 12.30	Friday, June 20
Algorithmic Techniques			*Formal-Method* & *Algorithmic Techniques*	
Introduction & basic FPT results	**Notions of bounded graph width**		**Algorithmic Meta-Theorems**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width		1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	(Fair Division)
				14.30 – 16.30
				FPT-Intractability Classes & Hierarchies
			(Fair Division)	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Example suggestions

Examples

1. Find a first-order logic formula over τ_{G} that expresses that a graph has a cycle of length precisely k.
2. Find an MSO_1 or $\mathsf{MSO}(\tau_{\mathsf{G}})$ formula that expresses that a graph has a dominating set of $\leq k$ elements.
3. Find an MSO_2 or $\mathsf{MSO}(\tau_{\mathsf{HG}})$ formula *feedback*(S) that expresses that $S \subseteq V$ is a feedback vertex set.
4. $(^{\star})$ Find an MSO_1 or $\mathsf{MSO}(\tau_{\mathsf{G}})$ formula that expresses that a graph is connected.
5. $(^{\star})$ Find an MSO_2 or $\mathsf{MSO}(\tau_{\mathsf{HG}})$ formula *path*(x, y, Z) that expresses that Z is a set of edges that forms a path from x to y.

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