

The Graph Structure of Process Interpretations of Regular Expressions

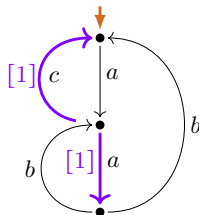
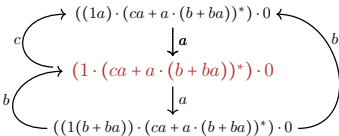
Clemens Grabmayer

<https://clegra.github.io>

Department of Computer Science



L'Aquila, Italy



Computer Science Seminar

University of Novi Sad

March 2026

Overview

- ▶ regular expressions (with unary/binary star, under-star-1-free $(*/\perp)$)
- ▶ Milner's process interpretation P /semantics $\llbracket \cdot \rrbracket_P$
 - ▶ P -/ $\llbracket \cdot \rrbracket_P$ -expressible graphs (\leadsto expressibility question)
 - ▶ axioms for $\llbracket \cdot \rrbracket_P$ -identity (\leadsto completeness question)
- ▶ loop existence and elimination (LEE)
 - ▶ defined by loop elimination rewrite system, its completion
 - ▶ describes interpretations of $(*/\perp)$ reg. expr.s (extraction possible)
 - ▶ LEE-witnesses: labelings of process graphs with LEE
 - ▶ LEE is preserved under bisimulation collapse (stepwise collapse)
- ▶ 1-LEE = sharing via 1-transitions facilitates LEE
- ▶ LEE/1-LEE characterize image of P^\bullet (restricted/unrestricted)
 - ▶ where P^\bullet a compact (sharing-increased) refinement of P
- ▶ outlook on work-to-do

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 - ▶ describes interpretations of $(*/\perp)$ reg. expr.s (extraction possible)
 - ▶ LEE-witnesses: labelings of process graphs with LEE
 - ▶ LEE is preserved under bisimulation collapse (stepwise collapse)
- ▶ 1-LEE = sharing via 1-transitions facilitates LEE
 - ▶ describes interpretations of all reg. expr.s (extraction possible)
 - ▶ not preserved under bisimulation collapse (approximation possible)
- ▶ LEE/1-LEE characterize image of P^\bullet (restricted/unrestricted)
 - ▶ where P^\bullet a compact (sharing-increased) refinement of P
 - ▶ via refined extraction using LEE/1-LEE
- ▶ outlook on work-to-do

Regular Expressions

Definition (*~ Copi-Elgot-Wright, 1958*)

Regular expressions over alphabet A with unary Kleene star:

$e, e_1, e_2 ::= 0 \mid a \mid e_1 + e_2 \mid e_1 \cdot e_2 \mid e^*$ (for $a \in A$).

- ▶ symbol **0** instead of \emptyset , symbol **1** instead of $\{\epsilon\}$

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Regular Expressions

Definition (*~ Kleene, 1951, ~ Copi-Elgot-Wright, 1958*)

Regular expressions over alphabet A with unary / binary Kleene star:

$$e, e_1, e_2 ::= 0 \mid a \mid e_1 + e_2 \mid e_1 \cdot e_2 \mid e^* \quad (\text{for } a \in A).$$

$$e, e_1, e_2 ::= 0 \mid 1 \mid a \mid e_1 + e_2 \mid e_1 \cdot e_2 \mid e_1 \oplus e_2 \quad (\text{for } a \in A).$$

- ▶ symbol **0** instead of \emptyset , symbol **1** instead of $\{\epsilon\}$
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Regular Expressions (1-free)

Definition (~ Kleene, 1951, ~ Copi-Elgot-Wright, 1958)

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Definition (for process interpretation)

1-free regular expressions over alphabet A with binary Kleene star:

$$f, f_1, f_2 ::= 0 \mid a \mid f_1 + f_2 \mid f_1 \cdot f_2 \mid f_1^{\oplus} f_2 \quad (\text{for } a \in A).$$

Regular Expressions (1-free)

Definition (~ Kleene, 1951, ~ Copi-Elgot-Wright, 1958)

Regular expressions over alphabet A with unary / binary Kleene star:

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Definition (for process interpretation)

1-free regular expressions over alphabet A with unary / binary Kleene star:

$$f, f_1, f_2 ::= 0 \mid a \mid f_1 + f_2 \mid f_1 \cdot f_2 \mid (f_1^*) \cdot f_2 \quad (\text{for } a \in A),$$

$$f, f_1, f_2 ::= 0 \mid a \mid f_1 + f_2 \mid f_1 \cdot f_2 \mid f_1^{\otimes} f_2 \quad (\text{for } a \in A).$$

Regular Expressions (under-star-/1-free)

Definition (\sim Kleene, 1951, \sim Copi-Elgot-Wright, 1958)

Regular expressions over alphabet A with unary / binary Kleene star:

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- ▶ symbol **0** instead of \emptyset , symbol **1** instead of $\{\epsilon\}$
- ▶ with unary star * : **1** is definable as **0** *
- ▶ with binary star $^{\oplus}$: **1** is **not** definable (in its absence)

Definition (for process interpretation)

The set $RExp^{(+)}(A)$ of **1-free regular expressions** over A is defined by:

$$f, f_1, f_2 ::= 0 \mid a \mid f_1 + f_2 \mid f_1 \cdot f_2 \mid f_1^* \cdot f_2 \quad (\text{for } a \in A),$$

the set $RExp^{(*/+)}(A)$ of **under-star-1-free regular expressions** over A by:

$$uf, uf_1, uf_2 ::= 0 \mid 1 \mid a \mid uf_1 + uf_2 \mid uf_1 \cdot uf_2 \mid f^* \quad (\text{for } a \in A).$$

Process interpretation P of regular expressions *(Milner, 1984)*

$0 \xrightarrow{P}$ deadlock δ , no termination

$1 \xrightarrow{P}$ empty-step process ϵ , then terminate

$a \xrightarrow{P}$ atomic action a , then terminate

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$e_1 + e_2 \xrightarrow{P}$ (*choice*) execute $P(e_1)$ or $P(e_2)$

$e_1 \cdot e_2 \xrightarrow{P}$ (*sequentialization*) execute $P(e_1)$, then $P(e_2)$

$e^* \xrightarrow{P}$ (*iteration*) repeat (terminate or execute $P(e)$)

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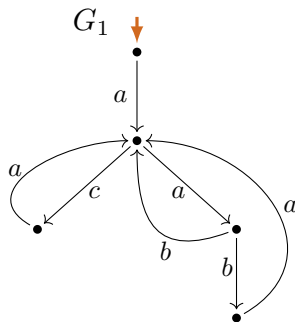
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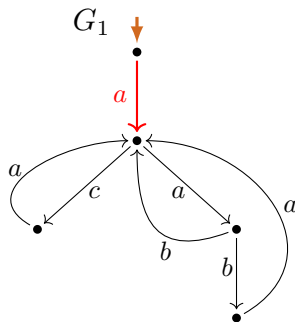
$\llbracket e \rrbracket_P := [P(e)]_{\Leftrightarrow}$ (*bisimilarity* equivalence class of process $P(e)$)

P -expressibility and $[[\cdot]]_P$ -expressibility (example, informally)



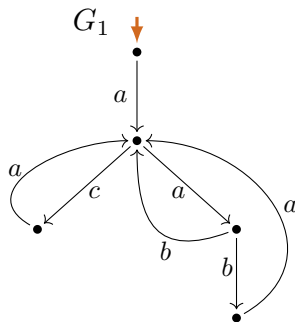
$$P\left(\overbrace{(a \cdot (c \cdot a + a \cdot (b + b \cdot a)))^*}^f \cdot 0 \right)$$

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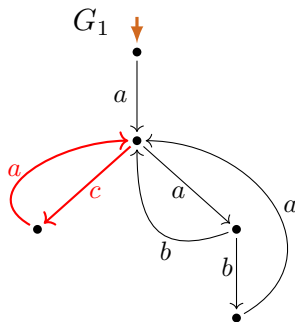
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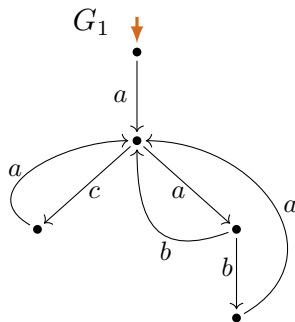
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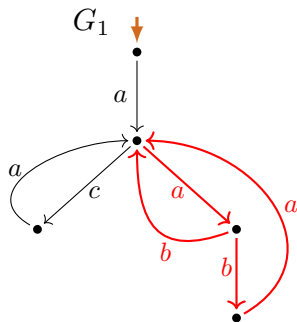
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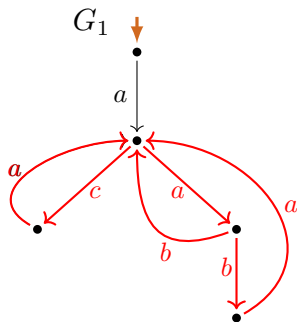
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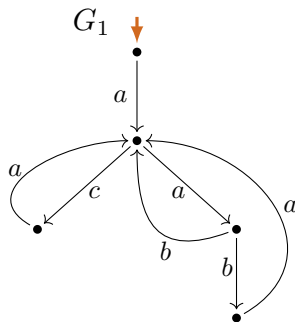
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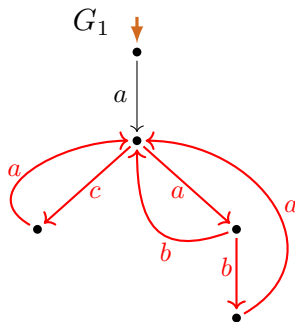
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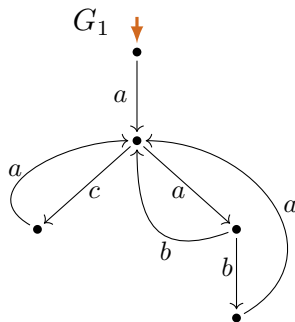
$$\begin{array}{c}
 \overbrace{P\left((a \cdot (c \cdot a + a \cdot (b + b \cdot a)))^* \cdot 0 \right)}^f \\
 P\left(a \cdot (c \cdot a + a \cdot (b + b \cdot a))^{\otimes} 0 \right)
 \end{array}$$

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$$\begin{array}{c}
 \overbrace{\hspace{10em}}^f \\
 P\left((a \cdot (c \cdot a + a \cdot (b + b \cdot a))^*) \cdot 0 \right) \\
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P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (example, informally)

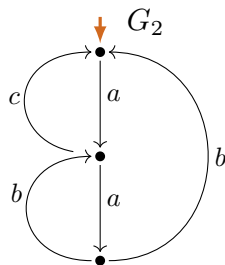
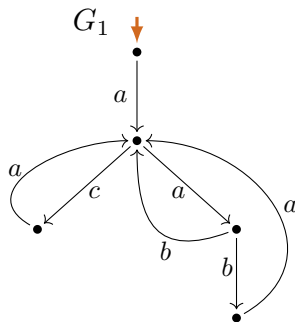


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$$G_1 \in \llbracket f \rrbracket_P$$

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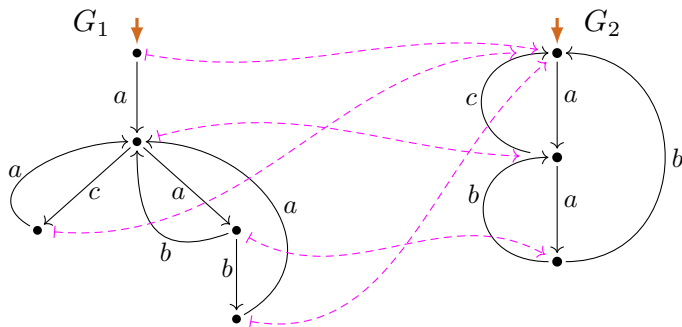


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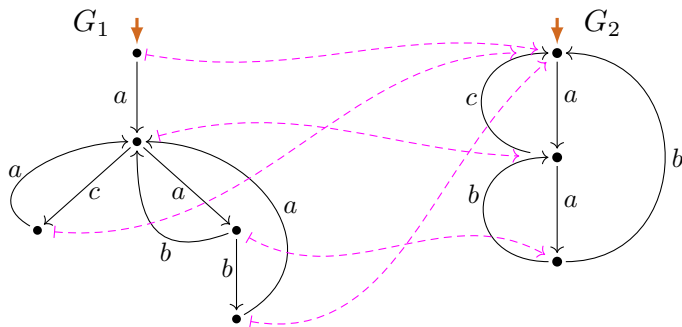


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P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (example, informally)



$$P\left(\overbrace{(a \cdot (c \cdot a + a \cdot (b + b \cdot a)))^*}^f \cdot 0 \right)$$

$$P\left(a \cdot (c \cdot a + a \cdot (b + b \cdot a))^{\otimes} 0 \right)$$

$$G_2 \in \llbracket f \rrbracket_P$$

$$G_1 \in \llbracket f \rrbracket_P$$

Process interpretation P (formally)

Definition (Transition system specification \mathcal{T})

$$\frac{}{a \xrightarrow{a} 1} \quad \frac{e_i \xrightarrow{a} e'_i}{e_1 + e_2 \xrightarrow{a} e'_i} \quad (i \in \{1, 2\})$$

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 \\
 \frac{e \xrightarrow{a} e'}{e^* \xrightarrow{a} e' \cdot e^*}
 \end{array}$$

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Definition (Transition system specification \mathcal{T})

$$\begin{array}{c}
 \frac{}{1 \Downarrow} \qquad \frac{e_i \Downarrow}{(e_1 + e_2) \Downarrow} \ (i \in \{1, 2\}) \qquad \frac{e_1 \Downarrow \quad e_2 \Downarrow}{(e_1 \cdot e_2) \Downarrow} \qquad \frac{}{(e^*) \Downarrow} \\
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 \overline{1 \Downarrow} \qquad \overline{e_i \Downarrow} \quad (i \in \{1, 2\}) \quad \overline{e_1 \Downarrow \quad e_2 \Downarrow} \quad \overline{(e^*) \Downarrow} \\
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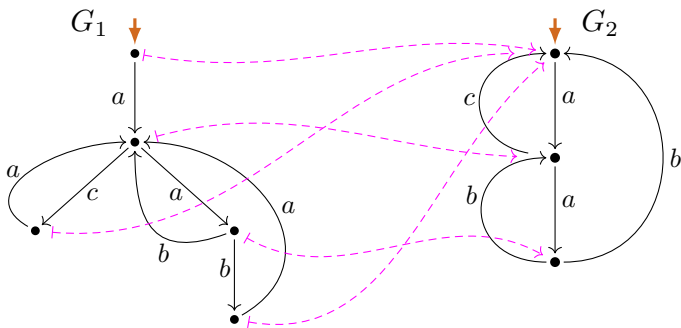
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 \\
 \frac{e_1 \xrightarrow{a} e'_1}{e_1 \cdot e_2 \xrightarrow{a} e'_1 \cdot e_2} \qquad \frac{e_1 \Downarrow \quad e_2 \xrightarrow{a} e'_2}{e_1 \cdot e_2 \xrightarrow{a} e'_2} \qquad \frac{e \xrightarrow{a} e'}{e^* \xrightarrow{a} e' \cdot e^*}
 \end{array}$$

Definition

The **process (graph) interpretation** $P(e)$ of a regular expression e :

$P(e) :=$ **labeled transition graph** generated by e by derivations in \mathcal{T} .

P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (example, **informally**)

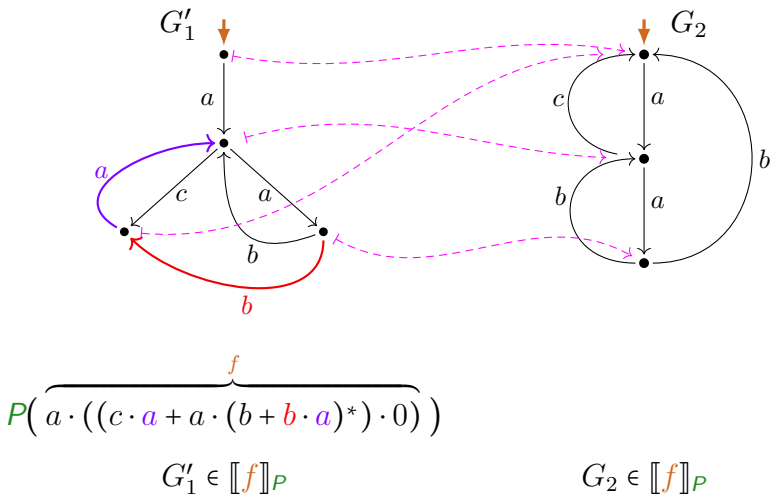


$$P\left(\overbrace{a \cdot ((c \cdot a + a \cdot (b + b \cdot a)^*) \cdot 0))}^f\right)$$

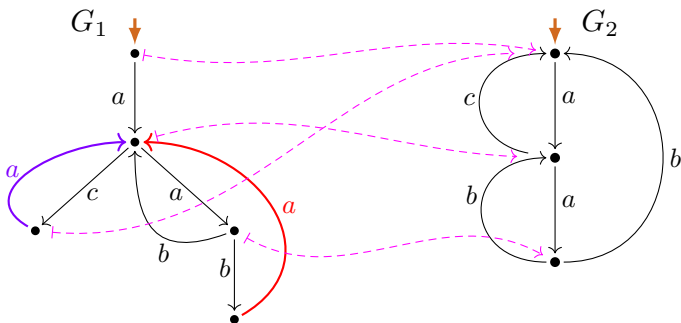
$$G_1 \in \llbracket f \rrbracket_P$$

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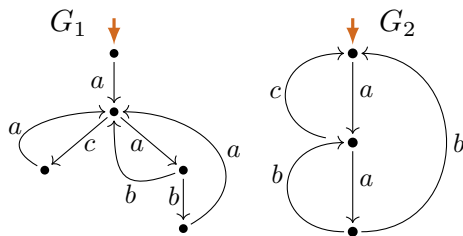


$$P\left(\overbrace{a \cdot ((c \cdot a + a \cdot (b + b \cdot (a + a)))^*)}^f \cdot 0\right)$$

$$G_1 \in \llbracket f \rrbracket_P$$

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P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (examples)

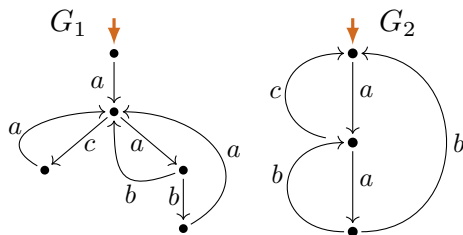


P -expressible

$\llbracket \cdot \rrbracket_P$ -expressible

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P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (examples)



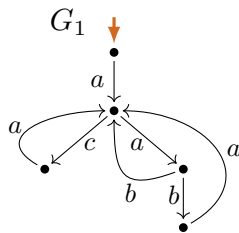
P -expressible

?

$\llbracket \cdot \rrbracket_P$ -expressible

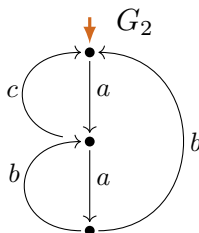
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P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (examples)



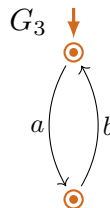
P -expressible

$\llbracket \cdot \rrbracket_P$ -expressible



?

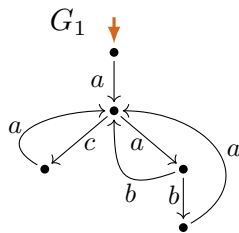
$\llbracket \cdot \rrbracket_P$ -expressible



not P -expressible

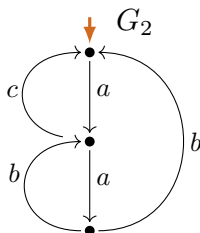
not $\llbracket \cdot \rrbracket_P$ -expressible

P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (examples)



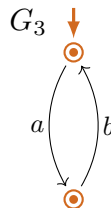
P -expressible

$\llbracket \cdot \rrbracket_P$ -expressible



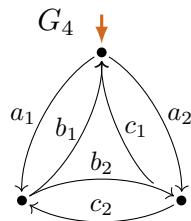
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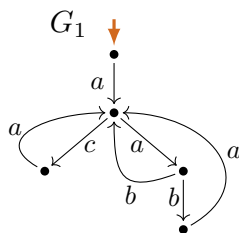


not P -expressible

not $\llbracket \cdot \rrbracket_P$ -expressible

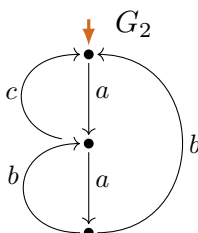


P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (examples)



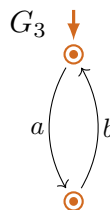
P -expressible

$\llbracket \cdot \rrbracket_P$ -expressible



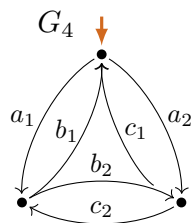
?

$\llbracket \cdot \rrbracket_P$ -expressible



not P -expressible

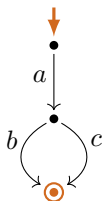
not $\llbracket \cdot \rrbracket_P$ -expressible



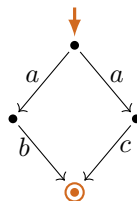
Q2: How can P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility be characterized?

Process semantics equality $=_{\llbracket \cdot \rrbracket_P}$

- Fewer identities hold for $=_{\llbracket \cdot \rrbracket_P}$ than for $=_{\llbracket \cdot \rrbracket_L}$:



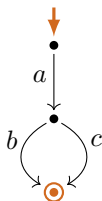
$$P(a \cdot (b + c))$$



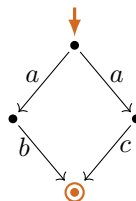
$$P(a \cdot b + a \cdot c)$$

Process semantics equality $=_{\llbracket \cdot \rrbracket_P}$

- **Fewer** identities hold for $=_{\llbracket \cdot \rrbracket_P}$ than for $=_{\llbracket \cdot \rrbracket_L}$:



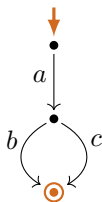
$$P(a \cdot (b + c))$$



$$P(a \cdot b + a \cdot c)$$

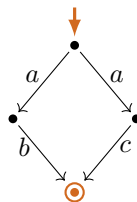
Process semantics equality $=_{\llbracket \cdot \rrbracket_P}$

- Fewer identities hold for $=_{\llbracket \cdot \rrbracket_P}$ than for $=_{\llbracket \cdot \rrbracket_L}$:



$$a \cdot (b + c)$$

$\neq_{\llbracket \cdot \rrbracket_P}$

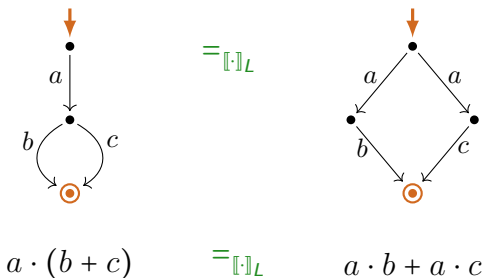


$$a \cdot b + a \cdot c$$

$\neq_{\llbracket \cdot \rrbracket_P}$

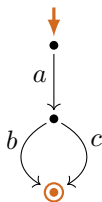
Process semantics equality $=_{\llbracket \cdot \rrbracket_P}$

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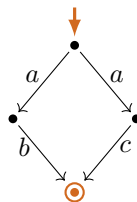
Process semantics equality $=_{[\cdot]_P}$

► Fewer identities hold for $=_{[\cdot]_P}$ than for $=_{[\cdot]_L}$: $=_{[\cdot]_P} \not\subseteq =_{[\cdot]_L}$.



$a \cdot (b + c)$

$\neq_{[\cdot]_P}$



$a \cdot b + a \cdot c$

$\neq_{[\cdot]_P}$

Milner's proof system **Mil**

Axioms:

$$(A1) \quad e + (f + g) = (e + f) + g$$

$$(A2) \quad e + 0 = e$$

$$(A3) \quad e + f = f + e$$

$$(A4) \quad e + e = e$$

$$(A5) \quad e \cdot (f \cdot g) = (e \cdot f) \cdot g$$

$$(A6) \quad (e + f) \cdot g = e \cdot g + f \cdot g$$

$$(A7) \quad e = 1 \cdot e$$

$$(A8) \quad e = e \cdot 1$$

$$(A9) \quad 0 = 0 \cdot e$$

$$(A10) \quad e^* = 1 + e \cdot e^*$$

$$(A11) \quad e^* = (1 + e)^*$$

$$\text{But: } e \cdot (f + g) \neq e \cdot f + e \cdot g$$

$$\text{But: } e \cdot 0 \neq 0$$

Inference rules: rules of equational logic *plus*

$$\frac{e = f \cdot e + g}{e = f^* \cdot g} \text{RSP}^* \text{ (if } f \text{ does not terminate immediately)}$$

Milner's Question (Q1)

Is **Mil** complete with respect to $=_{[\cdot]_P}$? (Does $e =_{[\cdot]_P} f \implies e =_{\text{Mil}} f$ hold?)

Milner's questions

(Q1) Complete axiomatization:

Is the proof system Mil complete for $\llbracket \cdot \rrbracket_P$?

(Q2) $\llbracket \cdot \rrbracket_P$ -Expressibility:

What structural property characterizes process graphs that are $\llbracket \cdot \rrbracket_P$ -expressible?

Milner's questions

(Q1) Complete axiomatization:

Is the proof system *Mil complete* for $=_{\llbracket \cdot \rrbracket_P}$?

(Q2) $\llbracket \cdot \rrbracket_P$ -Expressibility:

What *structural property* characterizes
process graphs that are $\llbracket \cdot \rrbracket_P$ -expressible ?

- is decidable (Baeten/Corradini/G, 2007)

Milner's questions

(Q1) Complete axiomatization:

*Is the proof system **Mil complete** for $=_{\llbracket \cdot \rrbracket_P}$?*

(Q2) $\llbracket \cdot \rrbracket_P$ -Expressibility:

*What **structural property** characterizes
process graphs that are $\llbracket \cdot \rrbracket_P$ -expressible ?*

- ▶ is decidable (Baeten/Corradini/G, 2007)
- ▶ partial new answer (G/Fokkink, 2020):
 - ▶ bisimulation collapse has **loop existence & elimination property (LEE)**
if expressible by **under-star-1-free** regular expression

Milner's questions

(Q1) Complete axiomatization:

Is the proof system *Mil complete* for $\llbracket \cdot \rrbracket_P$?

- ▶ series of partial completeness results for:
 - ▶ exitless iterations (Fokkink, 1998)
 - ▶ with a stronger fixed-point rule (G, 2006)
 - ▶ under-star 1-free, and without 0 (Corradini/de Nicola/Labella, 2004)
 - ▶ with 0 but under-star-1-free (G/Fokkink, 2020)

(Q2) $\llbracket \cdot \rrbracket_P$ -Expressibility:

What *structural property* characterizes process graphs that are $\llbracket \cdot \rrbracket_P$ -expressible?

- ▶ is decidable (Baeten/Corradini/G, 2007)
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Milner's questions

(Q1) Complete axiomatization:

Is the proof system *Mil complete* for $\llbracket \cdot \rrbracket_P$?

- ▶ Yes! (G, 2022, proof summary, employing LEE and crystallization)
- ▶ series of partial completeness results for:
 - ▶ exitless iterations (Fokkink, 1998)
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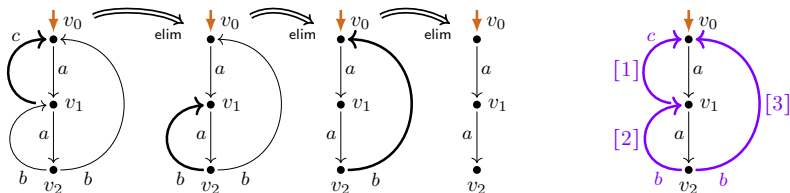
Question (Q2) specialized

(Q2)₀ P -Expressibility and P - $(*/1)$ -Expressibility:

What *structural property* characterizes:

- ▶ process graphs that are P -expressible ?
(... in the *image of P* ?)
- ▶ process graphs that are P -expressible by $(*/1)$ regular expressions?
(... in the *image of $(*/1)$ expressions under P* ?)

Loop Existence and Elimination (LEE)



Loop graphs (interpretations of innermost iterations without 1)

Definition

A process graph is a **loop graph** if:

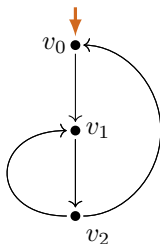
- (L1) There is an infinite path from the **start vertex**.
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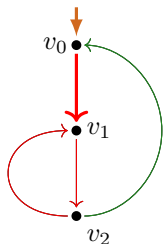


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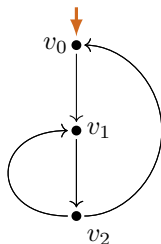
(L1), ~~(L2)~~

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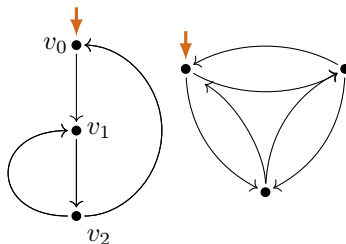
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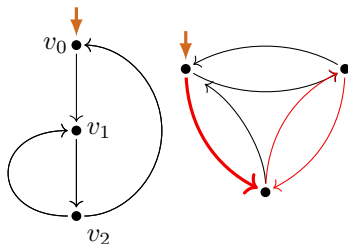
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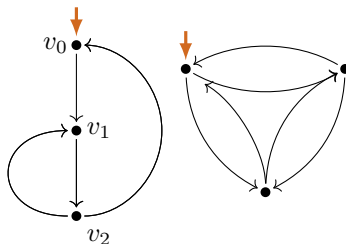
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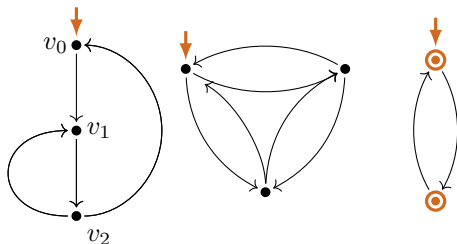
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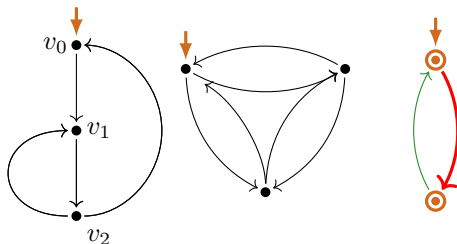
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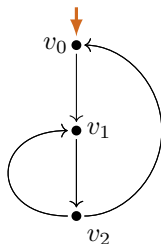
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Loop graphs (interpretations of innermost iterations without 1)

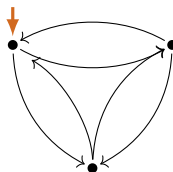
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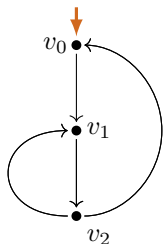


Loop graphs (interpretations of innermost iterations without 1)

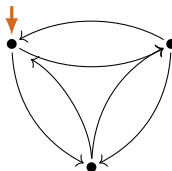
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(L1), (L2), ~~(L3)~~

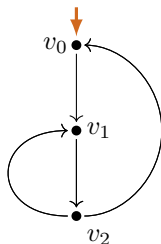


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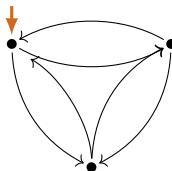
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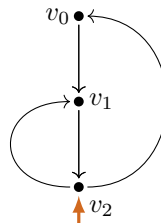
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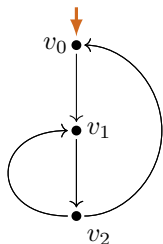


Loop graphs (interpretations of innermost iterations without 1)

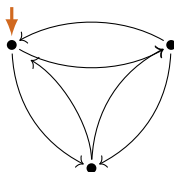
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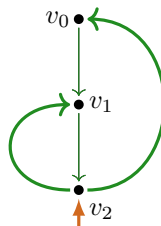
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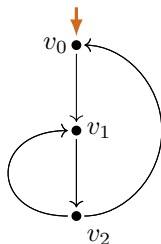


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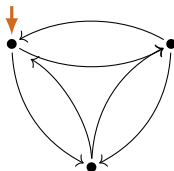
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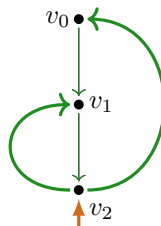
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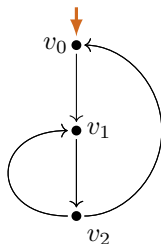
loop chart

Loop graphs (interpretations of innermost iterations without 1)

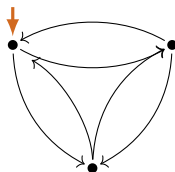
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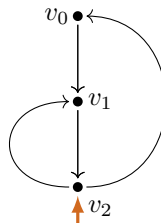
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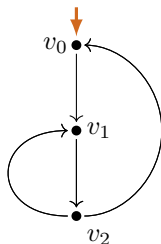
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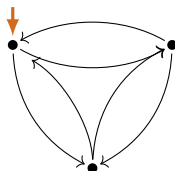
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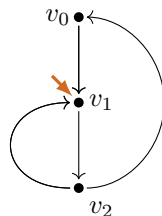
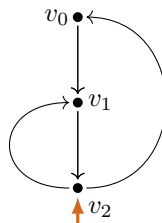
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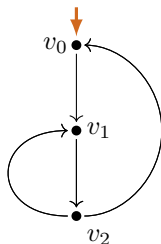


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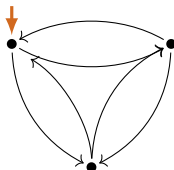
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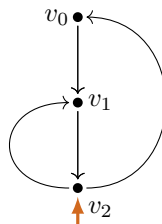
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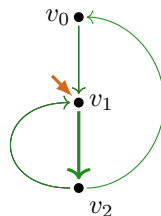
(L1), ~~(L2)~~



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loop chart

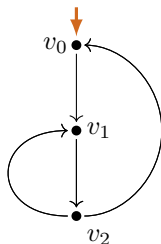


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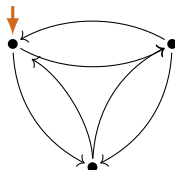
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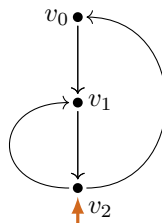
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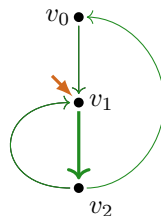
(L1), ~~(L2)~~



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loop chart



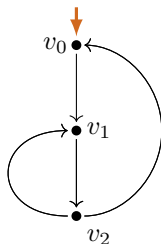
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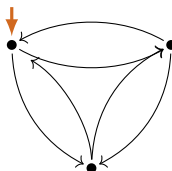
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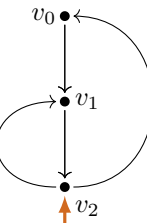
- (L1) There is an infinite path from the **start vertex**.
- (L2) Every infinite path from the **start vertex** returns to it.
- (L3) Termination is **only** possible at the **start vertex**.



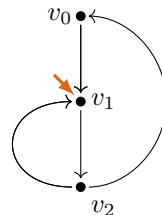
(L1), ~~(L2)~~



(L1), (L2), ~~(L3)~~



loop chart



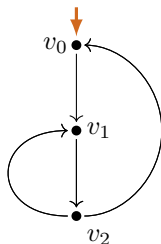
loop chart

Loop graphs (interpretations of innermost iterations without 1)

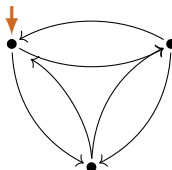
Definition

A process graph is a **loop graph** if:

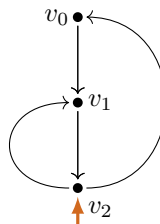
- (L1) There is an infinite path from the **start vertex**.
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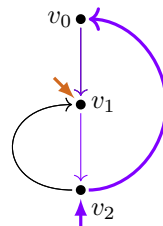
(L1), ~~(L2)~~



(L1), (L2), ~~(L3)~~

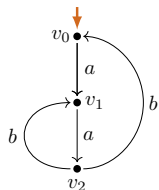


loop chart

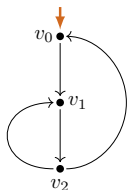


loop subchart

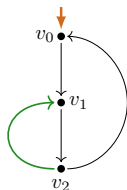
Loop existence and elimination (example 1)



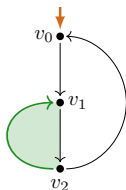
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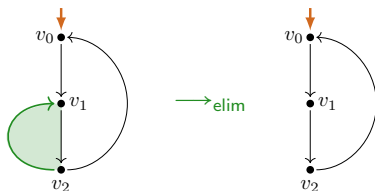
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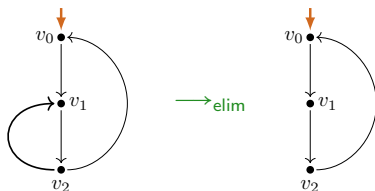
Loop existence and elimination (example 1)



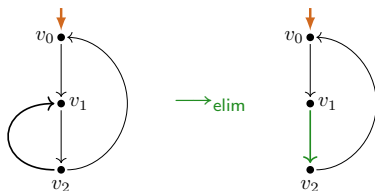
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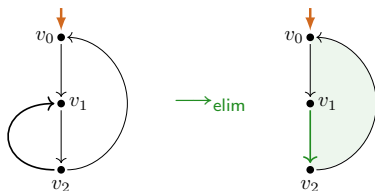
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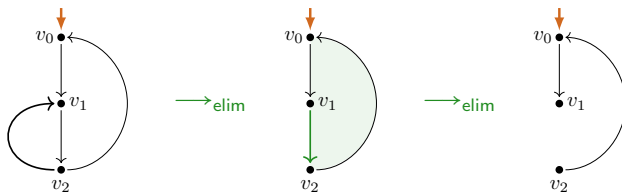
Loop existence and elimination (example 1)



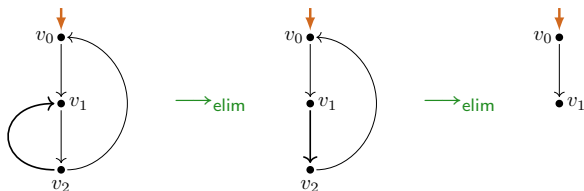
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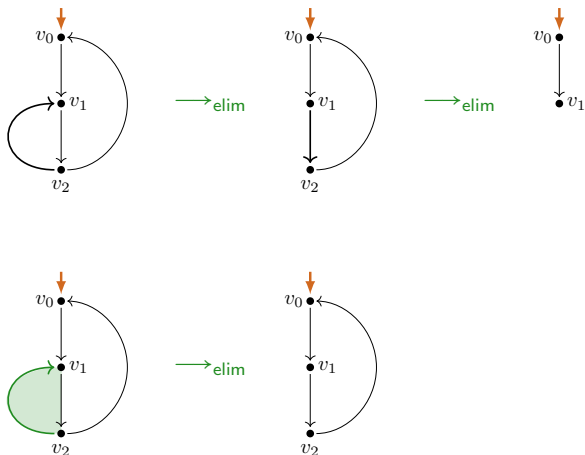
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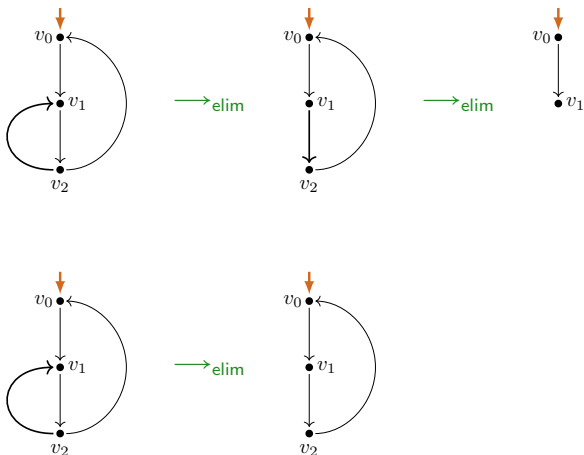
Loop existence and elimination (example 1)



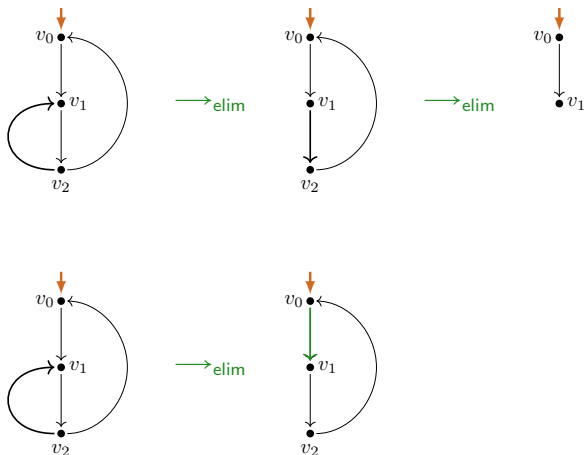
Loop existence and elimination (example 1)



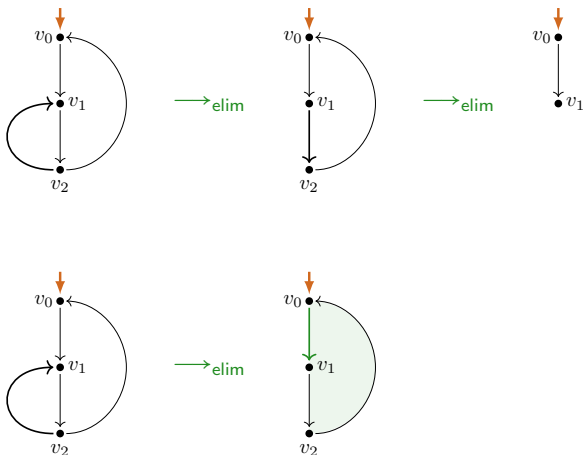
Loop existence and elimination (example 1)



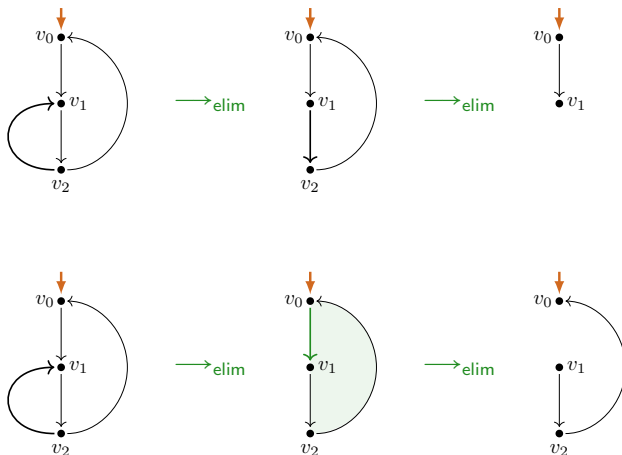
Loop existence and elimination (example 1)



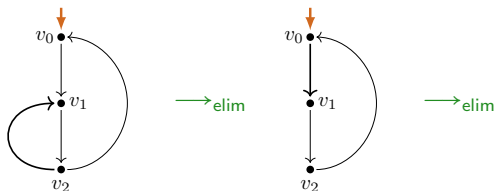
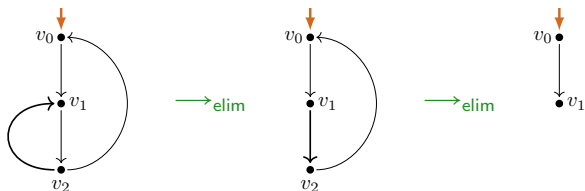
Loop existence and elimination (example 1)



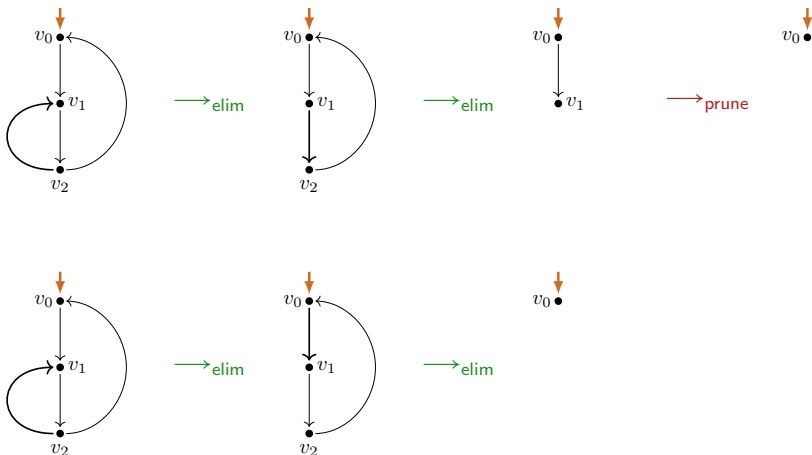
Loop existence and elimination (example 1)



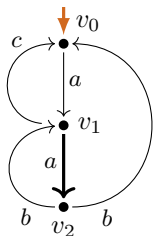
Loop existence and elimination (example 1)



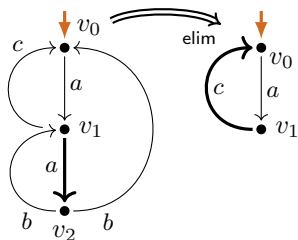
Loop existence and elimination (example 1)



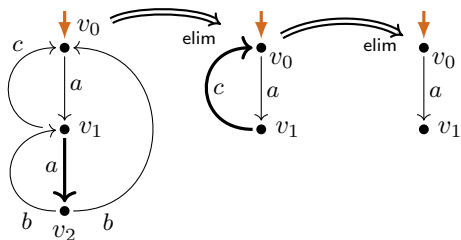
Loop existence and elimination (example 2)



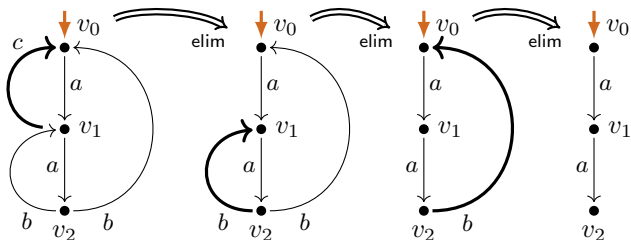
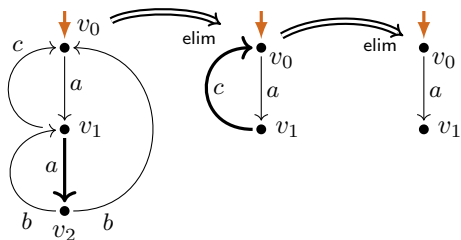
Loop existence and elimination (example 2)



Loop existence and elimination (example 2)



Loop existence and elimination (example 2)



LEE

Definition

A chart \mathcal{C} satisfies **LEE** (*loop existence and elimination*) if:

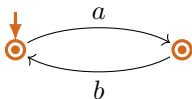
$$\exists \mathcal{C}_0 \left(\mathcal{C} \xrightarrow[\text{elim}]{*} \mathcal{C}_0 \not\rightarrow_{\text{elim}} \wedge \mathcal{C}_0 \text{ permits no infinite path} \right).$$

LEE

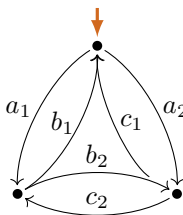
Definition

A chart \mathcal{C} satisfies **LEE** (*loop existence and elimination*) if:

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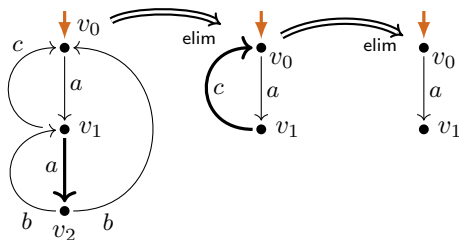


LEE

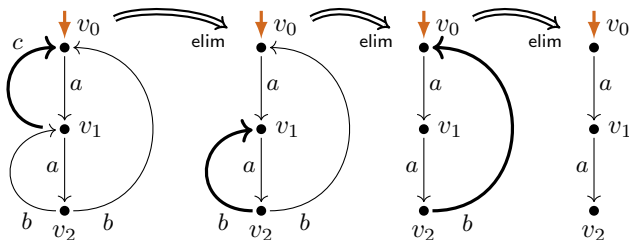
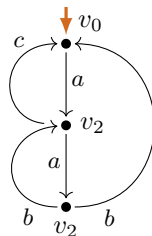


LEE

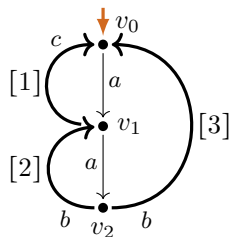
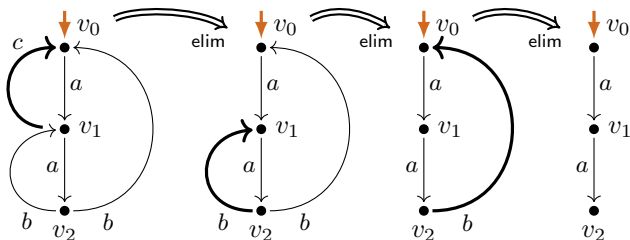
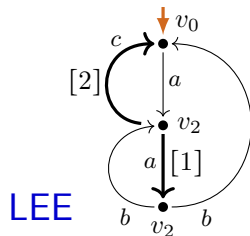
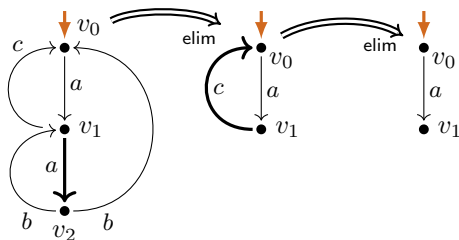
LEE



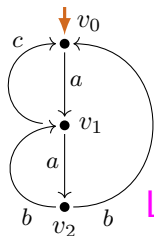
LEE



LEE

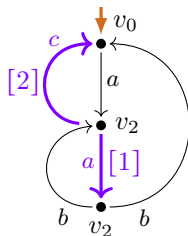


LEE

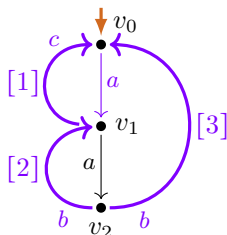


LEE-graph

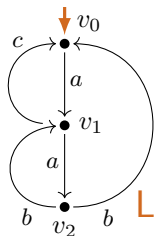
LEE-witness



LEE-witness

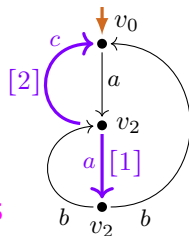


Layered LEE

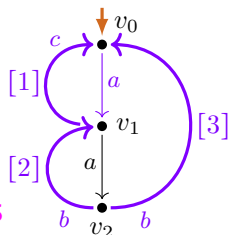


LLEE-graph

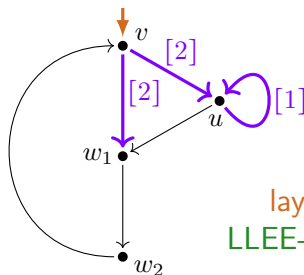
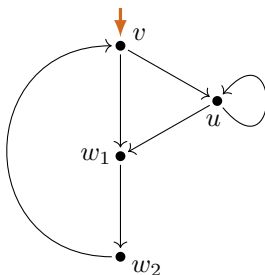
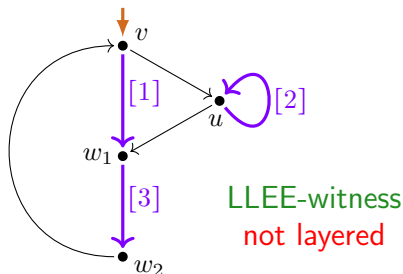
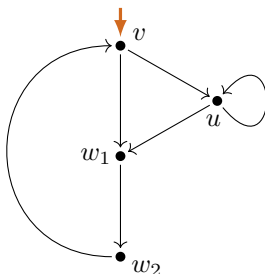
layered
LLEE-witness



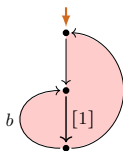
layered
LLEE-witness



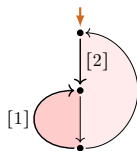
Layered LEE-witness (LLEE-witness)



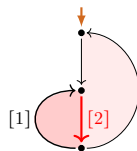
7 LEE-witnesses



layered

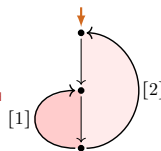


layered

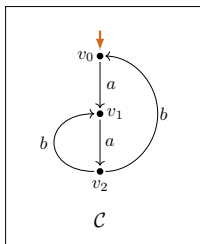


not layered

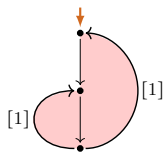
make layered



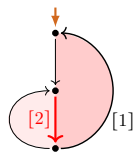
layered



\mathcal{C}

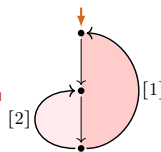


layered



not layered

make layered



layered

Loop elimination: properties

$\xrightarrow{\text{elim}}$: eliminate a transition-induced loop by:

- ▶ removing the loop-entry transition(s)
- ▶ garbage collection

$\xrightarrow{\text{prune}}$: remove a transition to a deadlocking state

Lemma

(i) $\xrightarrow{\text{elim}} \cup \xrightarrow{\text{prune}}$ *is terminating*.

Loop elimination: properties

$\xrightarrow{\text{elim}}$: eliminate a transition-induced loop by:

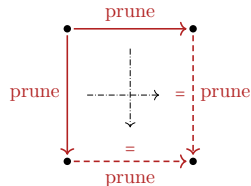
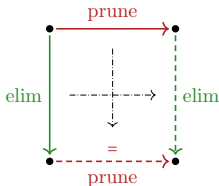
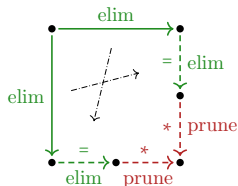
- ▶ removing the loop-entry transition(s)
- ▶ garbage collection

$\xrightarrow{\text{prune}}$: remove a transition to a deadlocking state

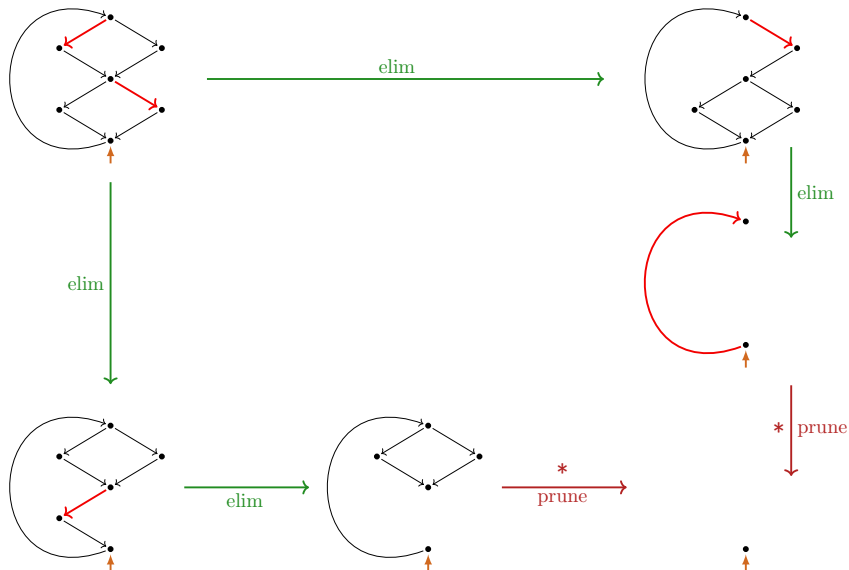
Lemma

(i) $\xrightarrow{\text{elim}} \cup \xrightarrow{\text{prune}}$ is terminating.

(ii) $\xrightarrow{\text{elim}} \cup \xrightarrow{\text{prune}}$ is decreasing [Van Oostrom, de Bruijn]



'Critical pair': bi-loop elimination



Loop elimination, and properties

$\xrightarrow{\text{elim}}$: eliminate a transition-induced loop by:

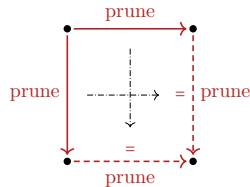
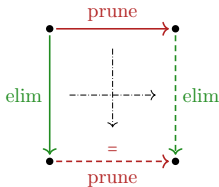
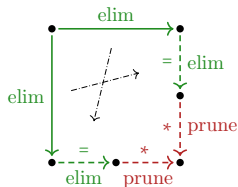
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Loop elimination, and properties

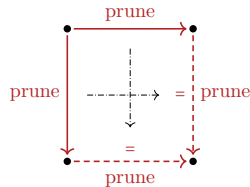
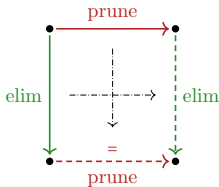
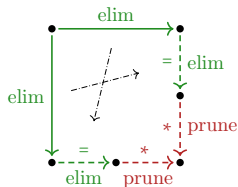
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- (iii) $\xrightarrow{\text{elim}} \cup \xrightarrow{\text{prune}}$ is confluent.



Structure property LEE

$$\text{LEE}(G) : \Longleftrightarrow \exists G_0 \left(G \xrightarrow{*}_{\text{elim}} G_0 \not\rightarrow_{\text{elim}} \wedge G_0 \text{ has no infinite trace} \right).$$

Lemma (by using termination and confluence)

For every process graph G the following are equivalent:

- (i) $\text{LEE}(G)$.
- (ii) *There is an $\xrightarrow{\text{elim}}$ normal form without an infinite trace.*

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Theorem (efficient decidability)

The problem of deciding $\text{LEE}(G)$ for process graphs G is in PTIME.

Interpretation/extraction correspondences with LEE

(\Leftarrow G/Fokkink 2020, G 2021)

(Int)_P^(*/ \pm): *P*-(*/ \pm)-expressible graphs have the *structural property* LEE.

Process *interpretations* $P(e)$ of (*/ \pm) regular expressions e are finite process graphs that *satisfy* LEE.

(Extr)_P: LEE *implies* $\llbracket \cdot \rrbracket_P$ -*expressibility*

From every finite process graph G with LEE
a regular expression e can be *extracted* such that $G \rightleftarrows P(e)$.

Interpretation/extraction correspondences with LEE

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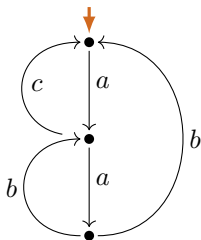
From every finite process graph G with LEE
a regular expression e can be *extracted* such that $G \Leftrightarrow P(e)$.

(Coll): LEE *is preserved under collapse*

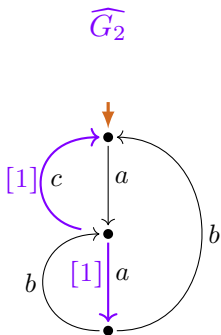
The class of finite process graphs with LEE
is *closed under bisimulation collapse*.

Expression extraction using **LLEE** (G/Fokkink 2020, G 2021/22)

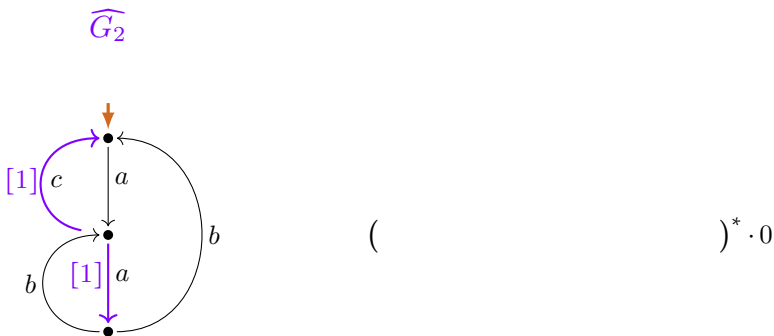
G_2



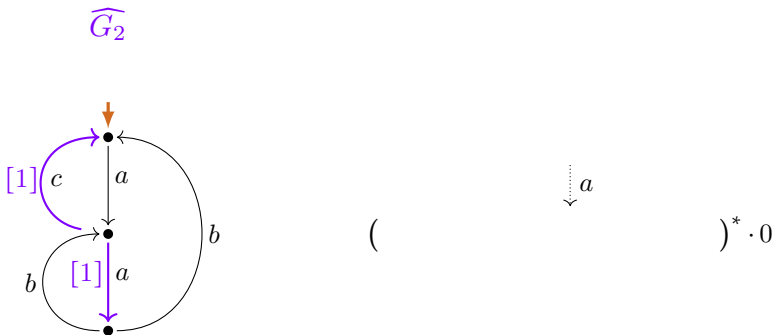
Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)



Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

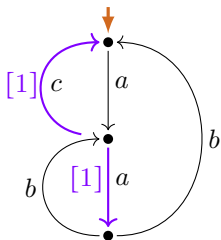


Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)



Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

\widehat{G}_2

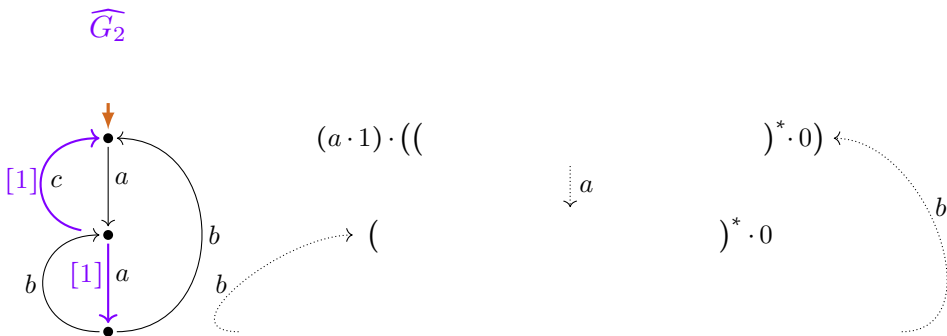


$$(a \cdot 1) \cdot ((\quad)^* \cdot 0)$$

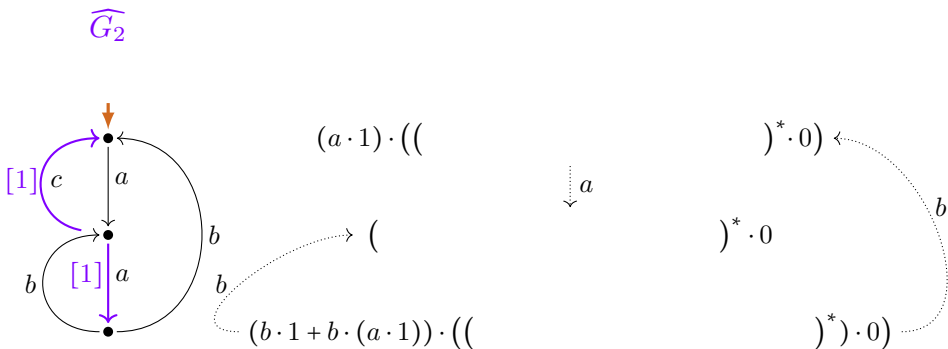
$$(\quad)^* \cdot 0$$

$\downarrow a$

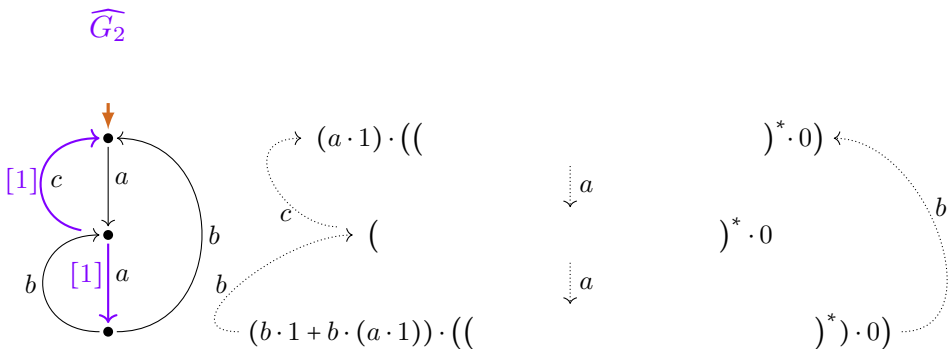
Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)



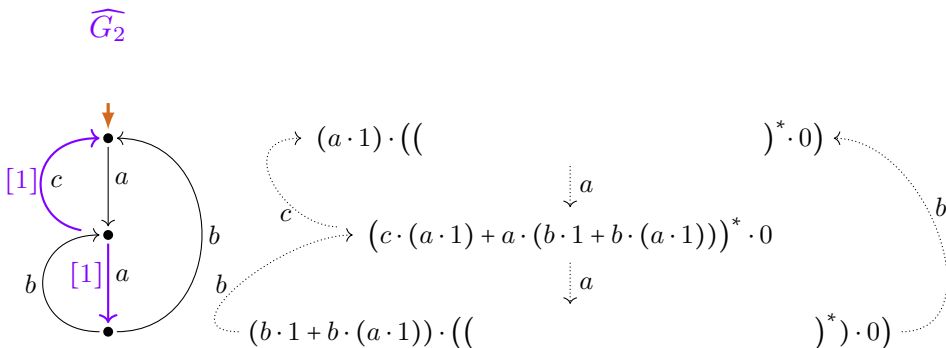
Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)



Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

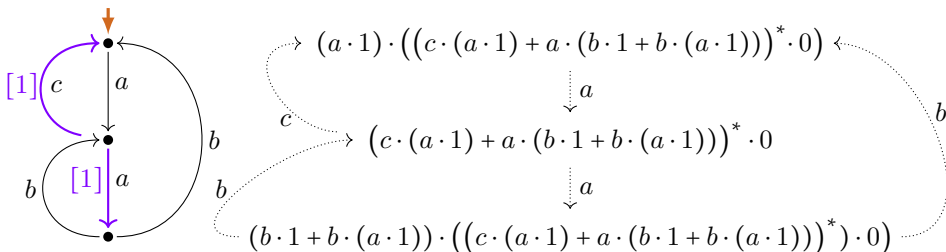


Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

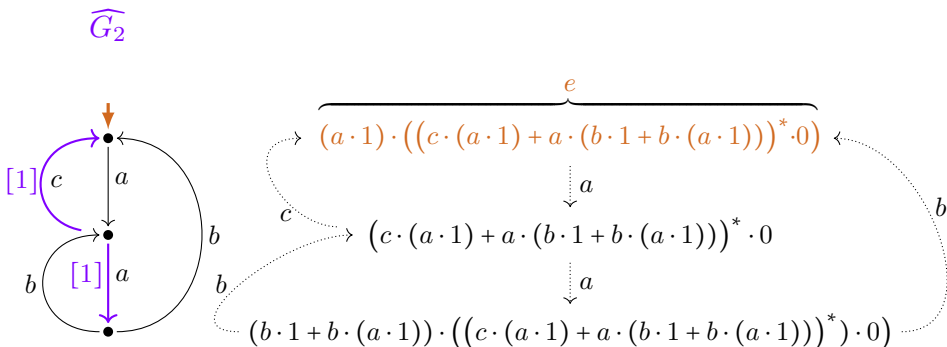


Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

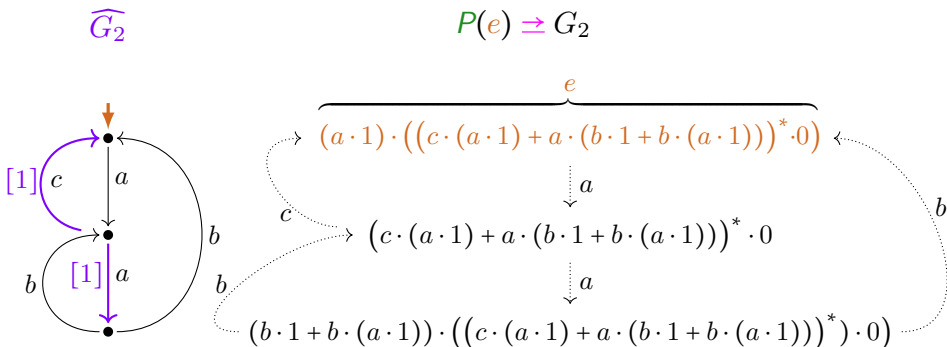
\widehat{G}_2



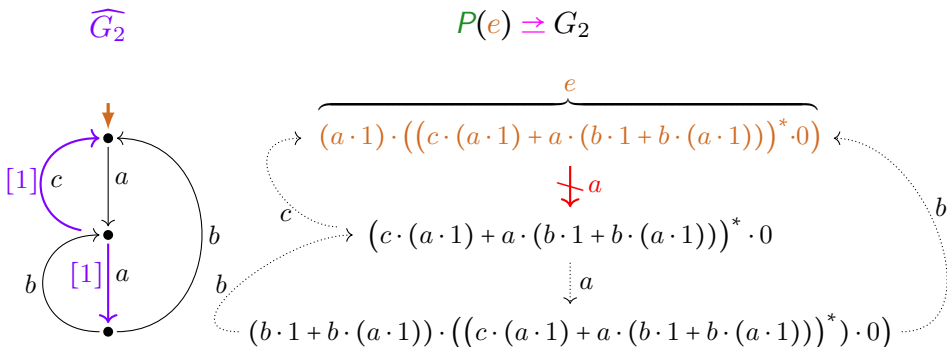
Expression extraction using **LLEE** (G/Fokkink 2020, G 2021/22)



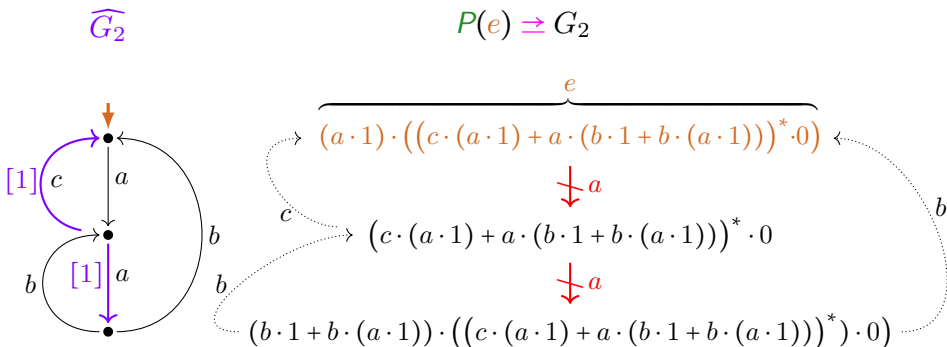
Expression extraction using **LLEE** (G/Fokkink 2020, G 2021/22)



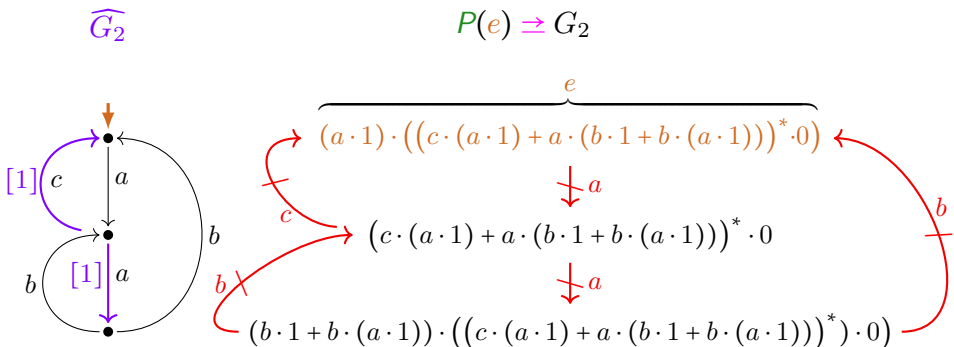
Expression extraction using **LLEE** (G/Fokkink 2020, G 2021/22)



Expression extraction using **LLEE** (G/Fokkink 2020, G 2021/22)

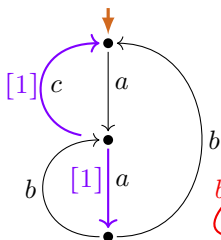


Expression extraction using **LLEE** (G/Fokkink 2020, G 2021/22)

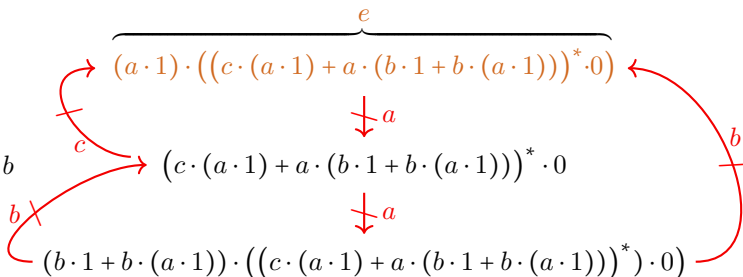


Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

\widehat{G}_2



$P(e) \Rightarrow G_2 \not\Rightarrow P(e)$



Interpretation of extracted expression

G'_2

$P(e) = G'_2$

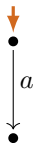


$$\overbrace{(a \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)}^e$$

Interpretation of extracted expression

G'_2

$P(e) = G'_2$



$$\begin{array}{c}
 \overbrace{(a \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)}^e \\
 \downarrow a \\
 (1 \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)
 \end{array}$$

Interpretation of extracted expression

G'_2

$$P(e) = G'_2$$



$$\overbrace{(a \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)}^e$$

$\downarrow a$

$$(1 \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)$$

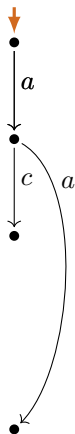
$\downarrow c$

$$((1 \cdot (a \cdot 1)) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1))))^* \cdot 0$$

Interpretation of extracted expression

G'_2

$P(e) = G'_2$



$$\overbrace{(a \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)}^e$$

$\downarrow a$

$$(1 \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)$$

$\downarrow c$

$$((1 \cdot (a \cdot 1)) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0$$

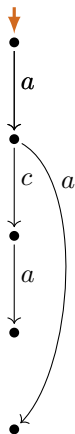
$\swarrow a$

$$((1 \cdot (b \cdot 1 + b \cdot (a \cdot 1))) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0$$

Interpretation of extracted expression

G'_2

$P(e) = G'_2$



$$\overbrace{(a \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)}^e$$

$\downarrow a$

$$(1 \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)$$

$\downarrow c$

$$((1 \cdot (a \cdot 1)) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)$$

$\downarrow a$

$$((1 \cdot 1) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)$$

a

$$((1 \cdot (b \cdot 1 + b \cdot (a \cdot 1))) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)$$

Interpretation of extracted expression

G'_2

$P(e) = G'_2$

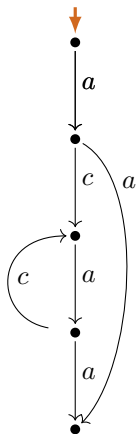


$$\begin{array}{c}
 \overbrace{(a \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)}^e \\
 \downarrow a \\
 (1 \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0) \\
 \downarrow c \\
 ((1 \cdot (a \cdot 1)) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0) \\
 \downarrow a \\
 ((1 \cdot 1) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0) \\
 \downarrow a \\
 ((1 \cdot (b \cdot 1 + b \cdot (a \cdot 1))) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)
 \end{array}$$

A curved arrow labeled 'a' connects the second expression to the final expression.

Interpretation of extracted expression

G'_2



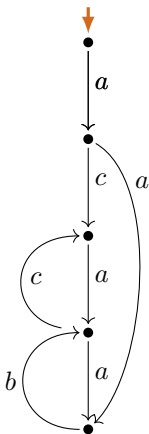
$P(e) = G'_2$

$$\begin{array}{c}
 \overbrace{(a \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)}^e \\
 \downarrow a \\
 (1 \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0) \\
 \downarrow c \\
 ((1 \cdot (a \cdot 1)) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0) \\
 \downarrow a \\
 ((1 \cdot 1) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0) \\
 \downarrow a \\
 ((1 \cdot (b \cdot 1 + b \cdot (a \cdot 1))) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)
 \end{array}$$

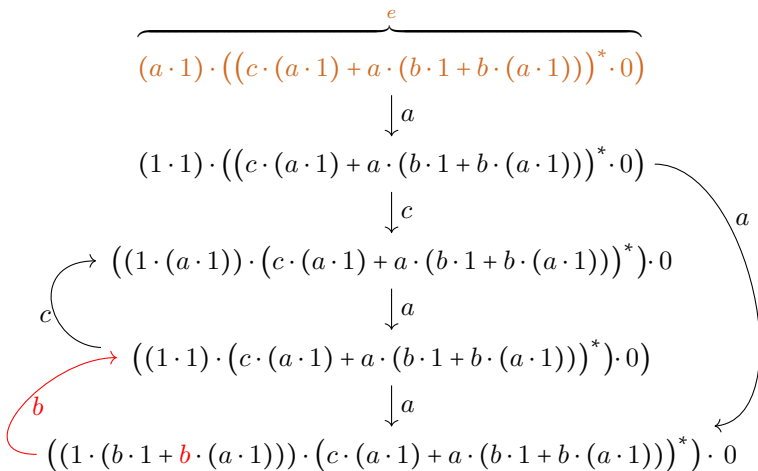
Red curved arrows indicate transitions from the third and fourth expressions to the fifth expression, corresponding to the self-loop and curved transition in the graph.

Interpretation of extracted expression

G'_2



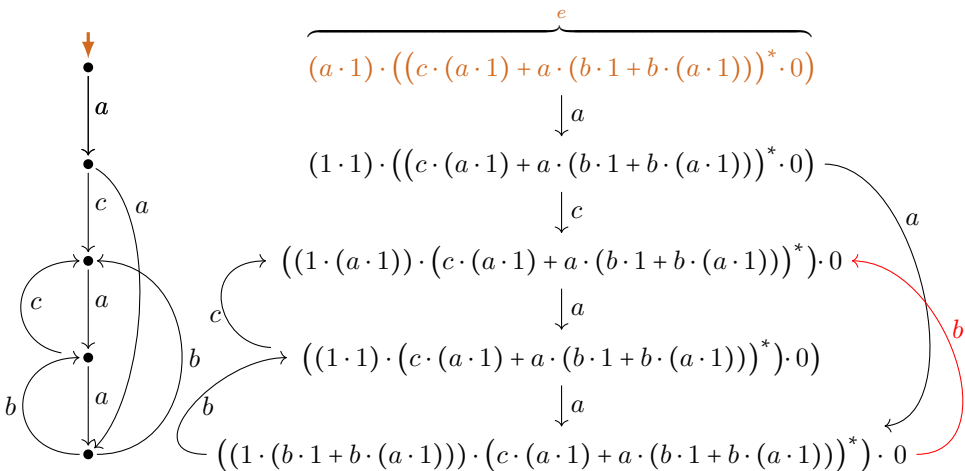
$P(e) = G'_2$



Interpretation of extracted expression

G'_2

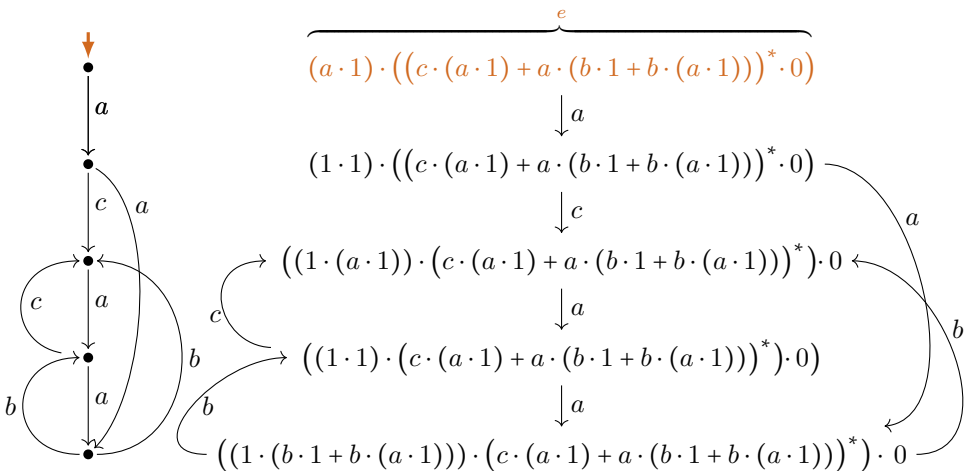
$P(e) = G'_2$



Interpretation of extracted expression

G'_2

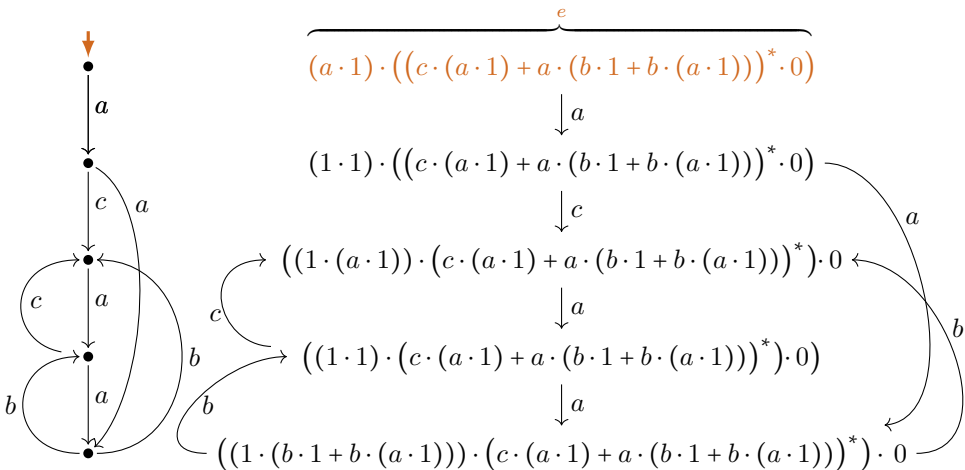
$$P(e) = G'_2 \xrightarrow{\text{pink}} G_2$$



Interpretation of extracted expression

G'_2

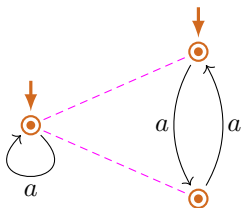
$$P(e) = G'_2 \xrightarrow{\text{pink}} G_2 \not\equiv G'_2$$



LEE under bisimulation

Observation

- LEE is **not** invariant under bisimulation.



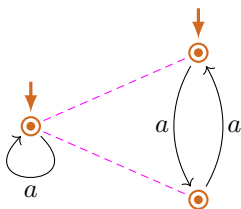
LEE

¬LEE

LEE under bisimulation

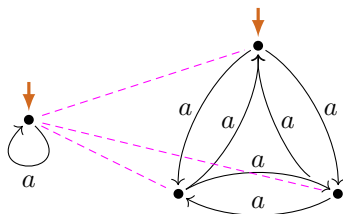
Observation

- LEE is **not** invariant under bisimulation.



LEE

¬LEE



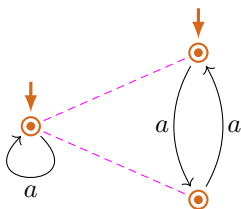
LEE

¬LEE

LEE under bisimulation

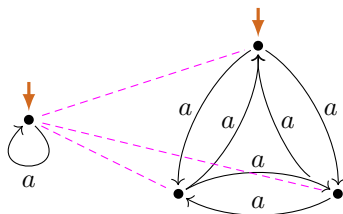
Observation

- ▶ LEE is **not** invariant under bisimulation.
- ▶ LEE is **not** preserved by converse functional bisimulation.



LEE

¬LEE



LEE

¬LEE

LEE under functional bisimulation

Lemma

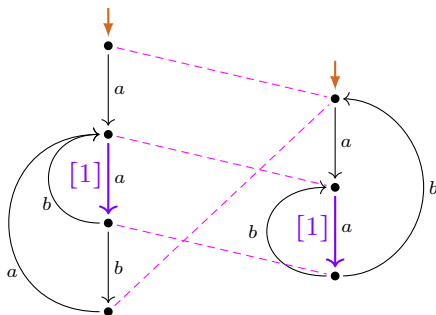
(i) LEE is preserved by *functional bisimulations*:

$$\text{LEE}(G_1) \wedge G_1 \rightsquigarrow G_2 \implies \text{LEE}(G_2).$$

Proof (Idea).

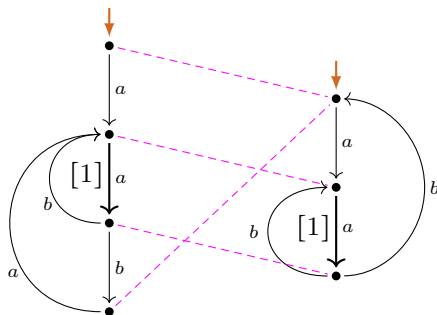
Use loop elimination in G_1 to carry out loop elimination in G_2 .

Collapsing LEE-witnesses

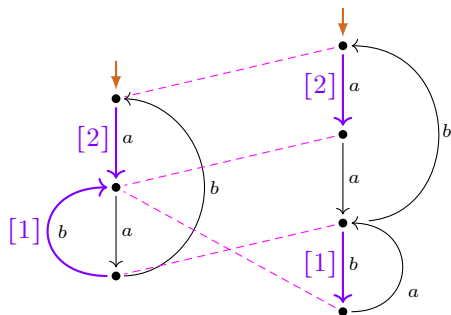


$$P(a(a(b + ba))^* \cdot 0)$$

Collapsing LEE-witnesses



$$P(a(a(b + ba))^* \cdot 0)$$



$$P((aa(ba))^* \cdot b)^* \cdot 0)$$

LEE under functional bisimulation / bisimulation collapse

Lemma

(i) LEE is preserved by *functional bisimulations*:

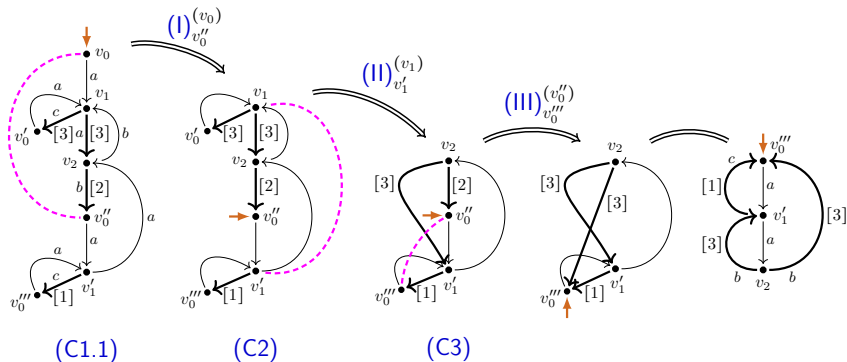
$$\text{LEE}(G_1) \wedge G_1 \rightrightarrows G_2 \implies \text{LEE}(G_2).$$

(ii) LEE is preserved from a process graph to its *bisimulation collapse*:

$$\text{LEE}(G) \wedge G \text{ has bisimulation collapse } C \implies \text{LEE}(C).$$

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20)

(no 1-transitions!)

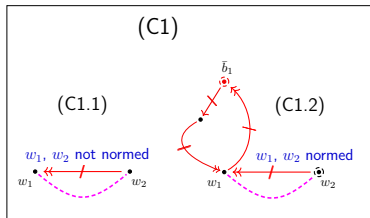


Lemma

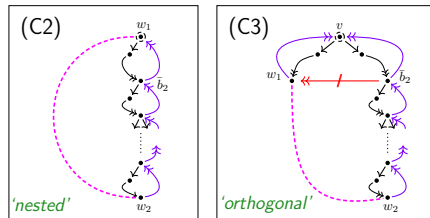
The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

Reduced bisimilarity redundancies in LLEE-graphs (no 1-trans.!) (G/Fokkink, LICS'20)

w_1, w_2 in different scc's



w_1, w_2 in the same scc

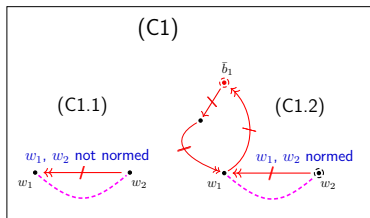


Lemma

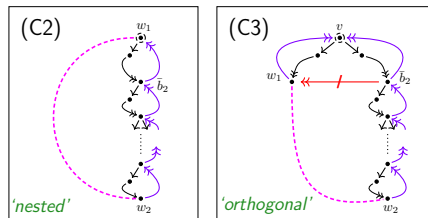
Every *not collapsed* LLEE-graph contains bisimilar vertices $w_1 \neq w_2$ of kind (C1), (C2), or (C3) (a *reduced bisimilarity redundancy* $\langle w_1, w_2 \rangle$):

Reduced bisimilarity redundancies in LLEE-graphs (no 1-trans.!) (G/Fokkink, LICS'20)

w_1, w_2 in different scc's



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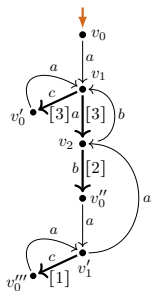
Lemma

Every **not collapsed** LLEE-graph contains bisimilar vertices $w_1 \neq w_2$ of kind (C1), (C2), or (C3) (a **reduced bisimilarity redundancy** $\langle w_1, w_2 \rangle$):

Lemma

Every **reduced bisimilarity redundancy** in a LLEE-graph can be eliminated LLEE-preservingly.

LL_{EE}-preserving collapse of LL_{EE}-charts (G/Fokkink, LICS'20)
(no 1-transitions!)



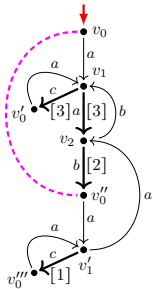
(C1.1)

Lemma

The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20)

(no 1-transitions!)



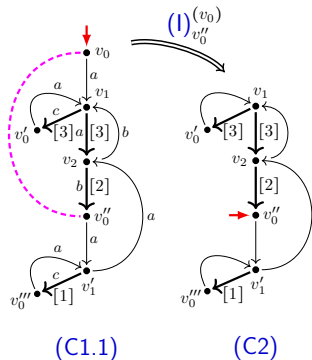
(C1.1)

Lemma

The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20)

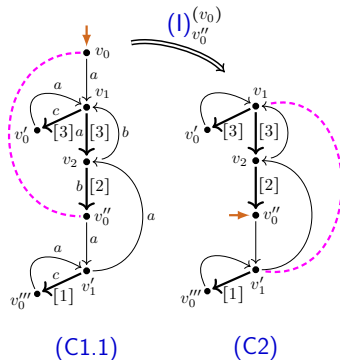
(no 1-transitions!)



Lemma

The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20) (no 1-transitions!)

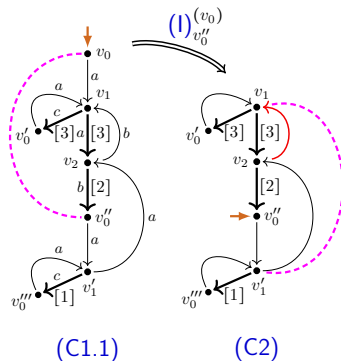


Lemma

The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20)

(no 1-transitions!)

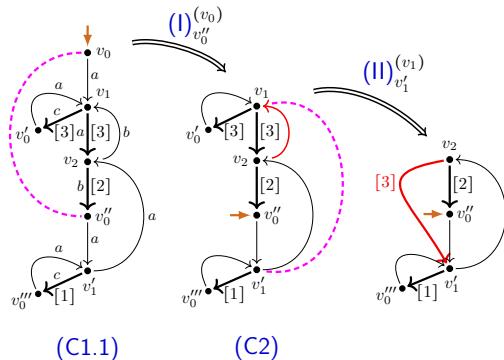


Lemma

The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20)

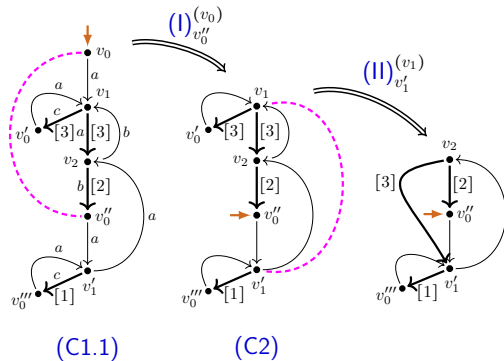
(no 1-transitions!)



Lemma

The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

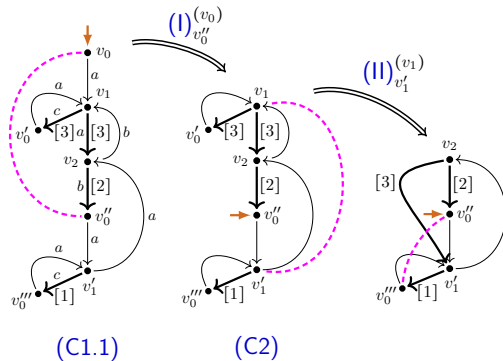
LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20) (no 1-transitions!)



Lemma

The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20) (no 1-transitions!)

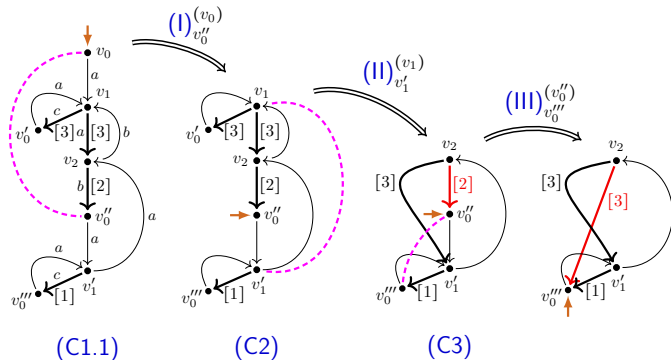


Lemma

The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20)

(no 1-transitions!)

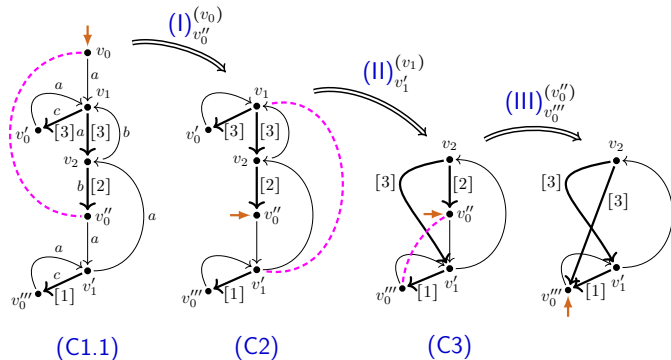


Lemma

The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20)

(no 1-transitions!)



Lemma

The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

Properties of LEE-charts

Theorem (\Leftarrow G/Fokkink, 2020)

A process graph G

is $\llbracket \cdot \rrbracket_P$ -expressible by an under-star-1-free regular expression

(i.e. P -expressible modulo bisimilarity by an $(\pm \backslash *)$ reg. expr.)

if and only if

the bisimulation collapse of G satisfies LEE.

Properties of LEE-charts

Theorem (\Leftarrow G/Fokkink, 2020)

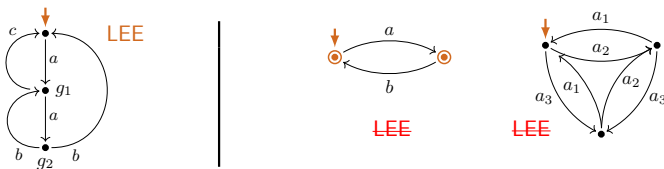
A process graph G

is $[[\cdot]]_P$ -expressible by an under-star-1-free regular expression
(i.e. P -expressible modulo bisimilarity by an $(\pm\backslash*)$ reg. expr.)

if and only if

the bisimulation collapse of G satisfies LEE.

Hence $[[\cdot]]_P$ -expressible | **not** $[[\cdot]]_P$ -expressible by 1-free regular expressions:



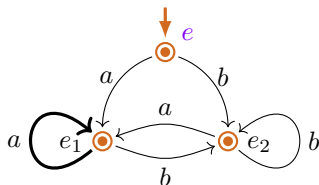
1-LEE

$\hat{=}$ sharing via 1-transitions facilitates LEE

Failure of LEE in general (example)

$(a^* \cdot b^*)^*$ not $(*/\perp)$!

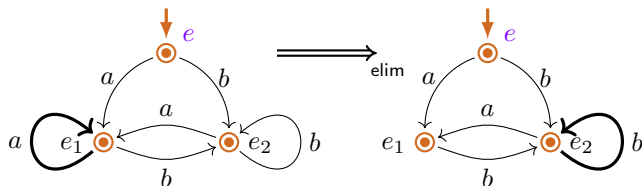
$P((a^* \cdot b^*)^*)$



Failure of LEE in general (example)

$(a^* \cdot b^*)^*$ not $(*/\perp)$!

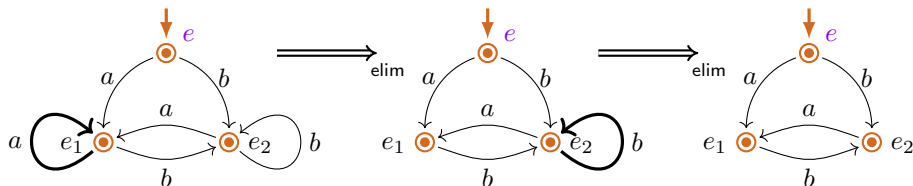
$P((a^* \cdot b^*)^*)$



Failure of LEE in general (example)

$(a^* \cdot b^*)^*$ not $(*/\perp)$!

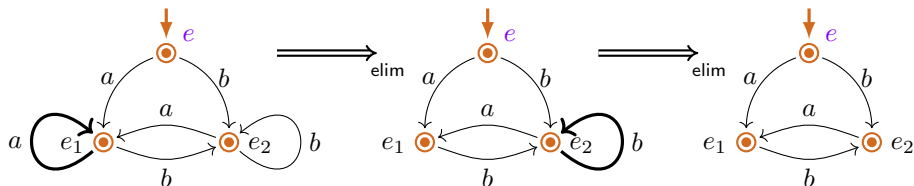
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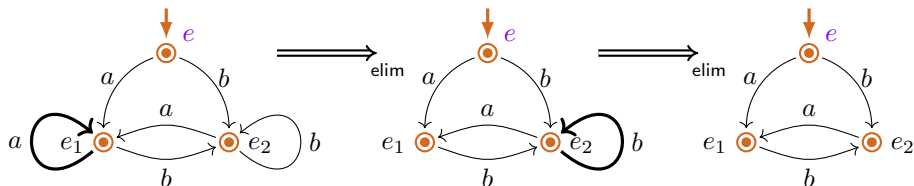


no loop subchart,
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LEE

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1-Graphs and induced graphs

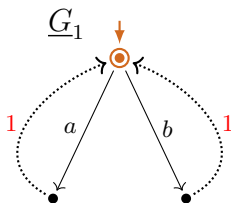
Definition

$$\xrightarrow{1} \cdot \dots \cdot \xrightarrow{1} \cdot \xrightarrow{a} \quad \hat{=} \quad \xrightarrow{(a)}$$

induced a -transitions, for $a \in A$

$$\xrightarrow{1} \cdot \dots \cdot \xrightarrow{1} \cdot \Downarrow \quad \hat{=} \quad \Downarrow^{(1)}$$

induced termination.



1-Graphs and induced graphs

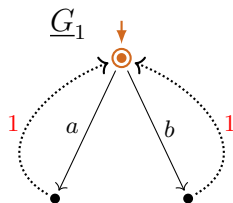
Definition

$$v_1 \xrightarrow{1} \cdot \dots \cdot \xrightarrow{1} \cdot \xrightarrow{a} v_2 \quad \hat{=} \quad v_1 \xrightarrow{(a)} v_2$$

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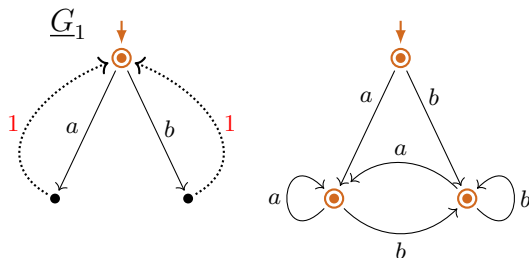
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1-Graphs and induced graphs

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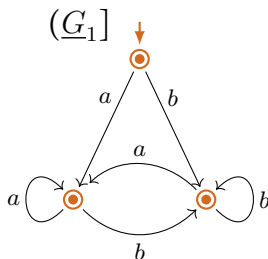
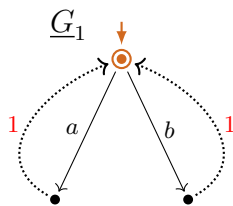
$$\begin{aligned}
 v_1 \xrightarrow{1} \cdot \dots \cdot \xrightarrow{1} \cdot \xrightarrow{a} v_2 &\equiv v_1 \xrightarrow{(a)} v_2 && \text{induced } a\text{-transitions, for } a \in A \\
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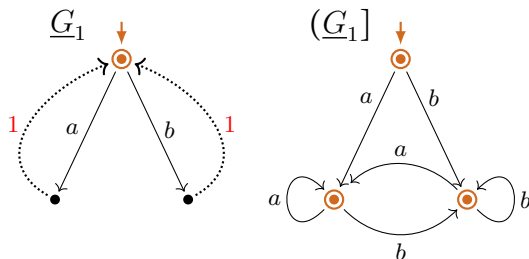
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Definition

The induced (process) graph of a 1-graph $\underline{G} = \langle V, A, 1, v_s, \rightarrow, \Downarrow \rangle$ is:

$$(\underline{G}) = \langle V, A, v_s, \xrightarrow{[\cdot]}, \Downarrow^{(1)} \rangle.$$



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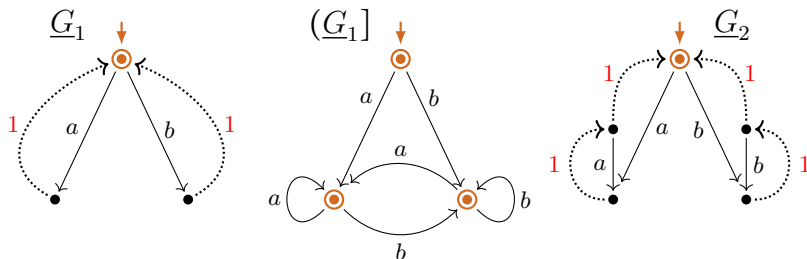
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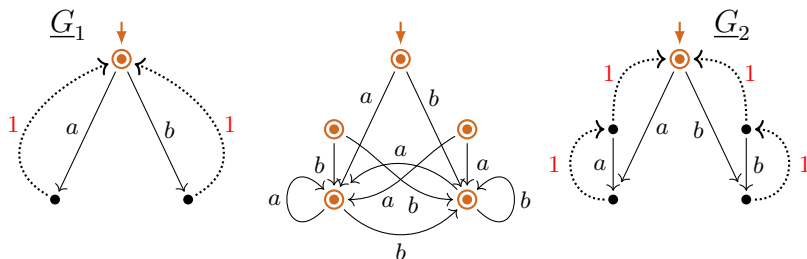
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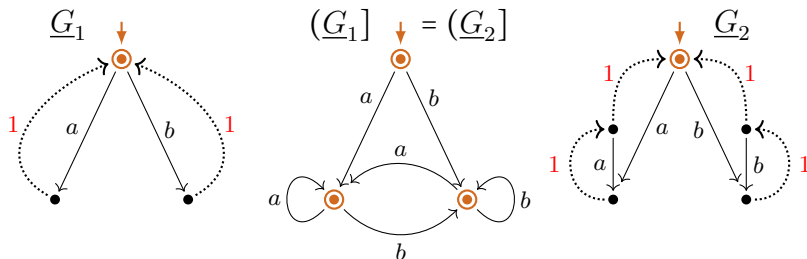
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1-LEE

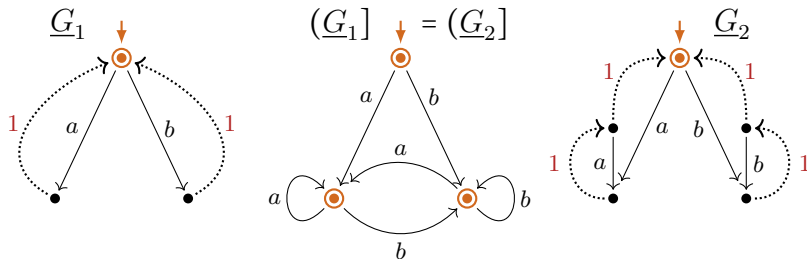
Definition

$1\text{-LEE}(G)$ holds for a graph G ,
if $G = (\underline{G}]$ for some weakly-guarded 1-graph \underline{G} with $\text{LEE}(\underline{G})$.

1-LEE

Definition

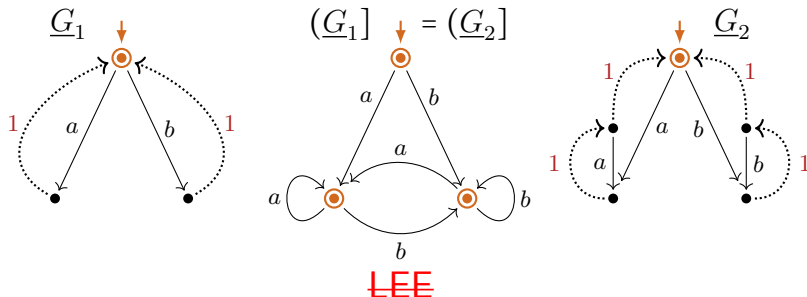
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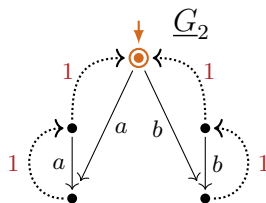
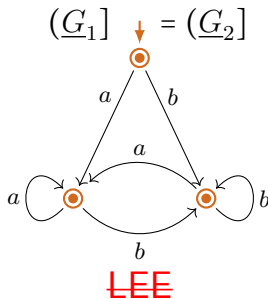
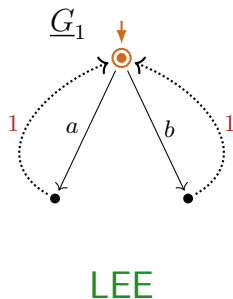
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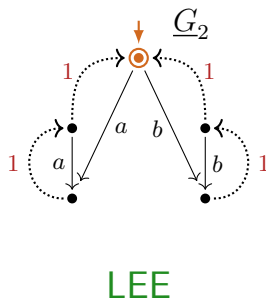
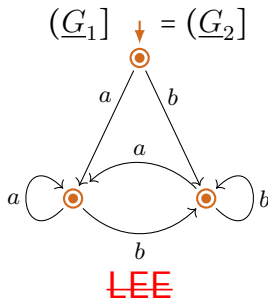
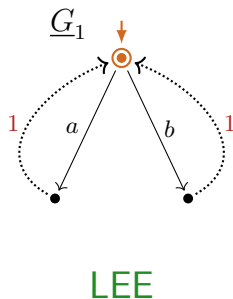
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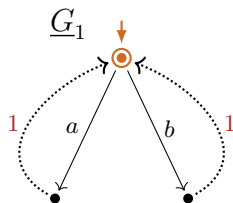
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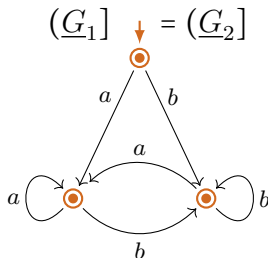
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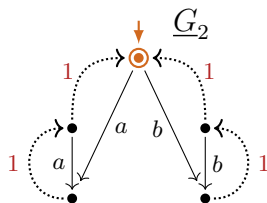


LEE



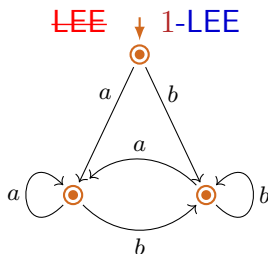
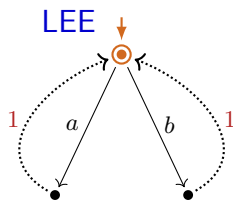
~~LEE~~

1-LEE



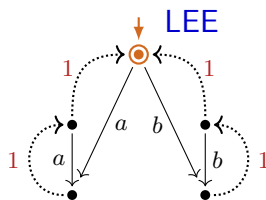
LEE

1-LEE holds for process interpretations



$P((a^* \cdot b^*)^*)$

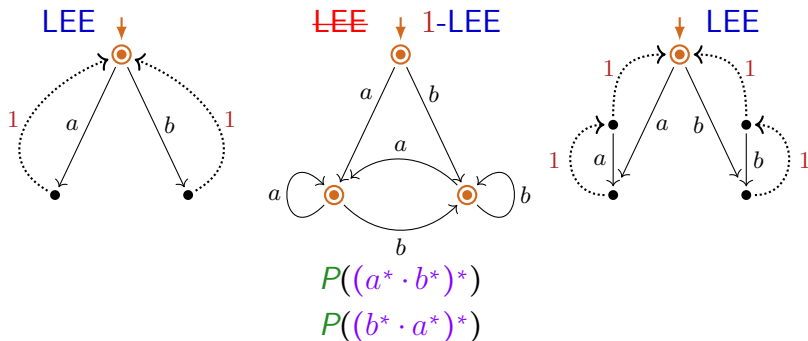
$P((b^* \cdot a^*)^*)$



1-LEE holds for process interpretations

Lemma

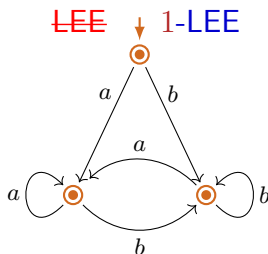
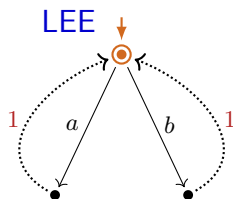
There is a **1-graph interpretation** \underline{P} of reg. expression e as 1-graphs $\underline{P}(e)$ such that for all $e \in RExp$: (i): $LEE(\underline{P}(e))$, (ii): $(\underline{P}(e)) = \underline{P}(e)$.



1-LEE holds for process interpretations

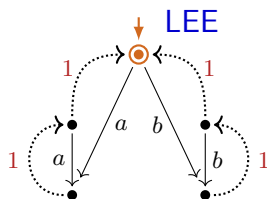
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$\underline{P}((a^* \cdot b^*)^*)$

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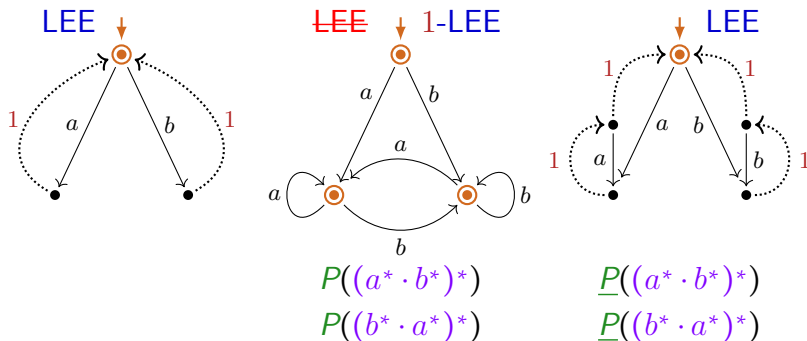
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Theorem

1-LEE($\underline{P}(e)$) holds for all regular expressions e .



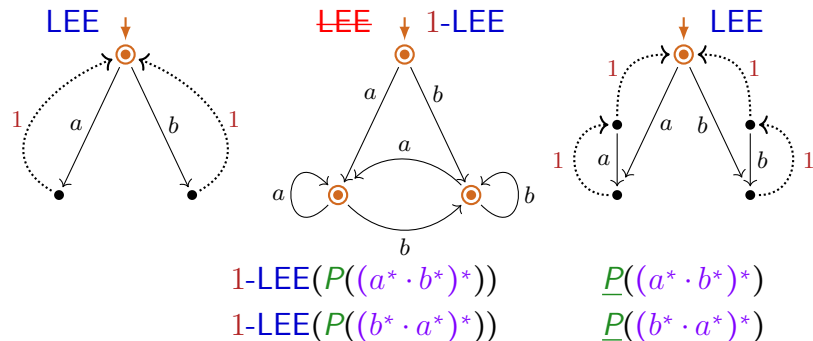
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Interpretation/extraction correspondences with 1-LEE

(\Leftarrow G 2021/22/23)

(Int)_P: *P-expressible* graphs have the *structural property* 1-LEE

Process **interpretations** $P(e)$ of regular expressions e are finite process graphs that satisfy 1-LEE.

(Extr)_P: 1-LEE implies $\llbracket \cdot \rrbracket_P$ -*expressibility*

From every finite 1-process-graph \underline{G} with 1-LEE a regular expression e can be **extracted** such that $\underline{G} \Leftrightarrow P(e)$.

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(Coll): 1-LEE is *not preserved under collapse*

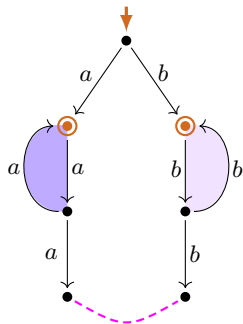
The class of finite process graphs with 1-LEE is **not closed under bisimulation collapse**.

1-LEE/ LEE characterize

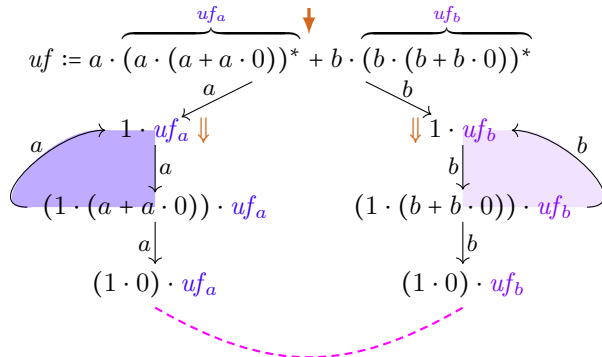
the un-/restricted image of compact version P^\bullet of P

Image of P is **not closed** under bisimulation collapse
not even for $(*/\perp)$ regular expressions (example)

$P(uf)$



$P(uf)$



Compact process interpretation P^\bullet

Definition (Transition system specification \mathcal{T})

$$\begin{array}{c}
 \overline{1 \Downarrow} \qquad \frac{e_i \Downarrow}{(e_1 + e_2) \Downarrow} \ (i \in \{1, 2\}) \qquad \frac{e_1 \Downarrow \quad e_2 \Downarrow}{(e_1 \cdot e_2) \Downarrow} \qquad \overline{(e^*) \Downarrow} \\
 \\
 \overline{a \xrightarrow{a} 1} \qquad \frac{e_i \xrightarrow{a} e'_i}{e_1 + e_2 \xrightarrow{a} e'_i} \ (i \in \{1, 2\}) \\
 \\
 \frac{e_1 \xrightarrow{a} e'_1}{e_1 \cdot e_2 \xrightarrow{a} e'_1 \cdot e_2} \qquad \frac{e_1 \Downarrow \quad e_2 \xrightarrow{a} e'_2}{e_1 \cdot e_2 \xrightarrow{a} e'_2} \qquad \frac{e \xrightarrow{a} e'}{e^* \xrightarrow{a} e' \cdot e^*}
 \end{array}$$

Compact process interpretation P^\bullet

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Compact process interpretation P^\bullet

Definition (Transition system specification \mathcal{T}^\bullet , changed rules w.r.t. \mathcal{T})

$$\frac{e_1 \xrightarrow{a} e'_1}{e_1 \cdot e_2 \xrightarrow{a} e'_1 \cdot e_2} \text{ (if } e'_1 \text{ is normed)}$$

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Compact process interpretation P^\bullet

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The compact process (graph) interpretation $P^\bullet(e)$ of a reg. expr's e :

$P^\bullet(e) :=$ labeled transition graph generated by e by derivations in \mathcal{T}^\bullet .

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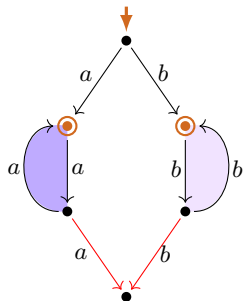
$P^\bullet(e) :=$ labeled transition graph generated by e by derivations in \mathcal{T}^\bullet .

Lemma (P^\bullet increases sharing; P^\bullet, P have same bisimulation semantics)

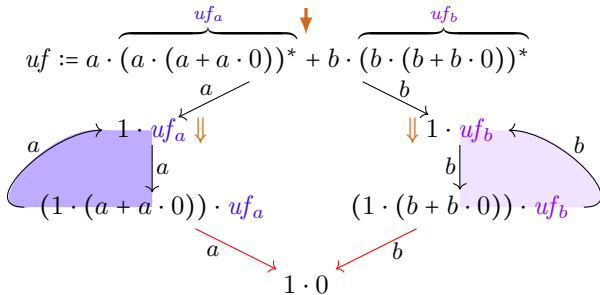
- (i) $P(e) \Rightarrow P^\bullet(e)$ for all regular expressions e .
- (ii) (G is $\llbracket \cdot \rrbracket_{P^\bullet}$ -expressible $\iff G$ is $\llbracket \cdot \rrbracket_P$ -expressible) for all graphs G .

Image of P restricted to $(*/1)$ regular expressions ... contains all of its bisimulation collapses (example)

$P^\bullet(uf)$



$P^\bullet(uf)$



Interpretation correspondence of P^\bullet with LEE

(Int) $_{P^\bullet}^{(*/+)}:$ By *under-star-1-free* expressions P^\bullet -expressible graphs satisfy LEE:

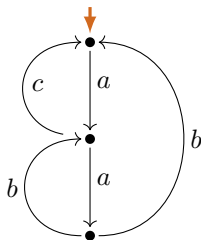
Compact process interpretations $P^\bullet(uf)$
 of *under-star-1-free* regular expressions uf
 are finite process graphs that satisfy LEE.

(Extr) $_{P^\bullet}^{(*/+)}:$ LEE implies $\llbracket \cdot \rrbracket_{P^\bullet}$ -expressibility by *under-star-1-free* reg. expr's:

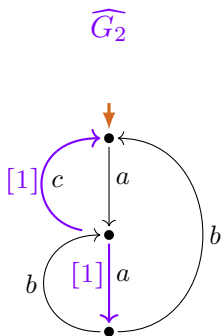
From every finite process graph G with LEE
 an *under-star-1-free* regular expression uf can be extracted
 such that $G \rightrightarrows P^\bullet(uf)$.

Refined extraction expression (example)

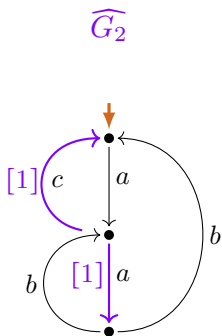
G_2



Refined extraction expression (example)

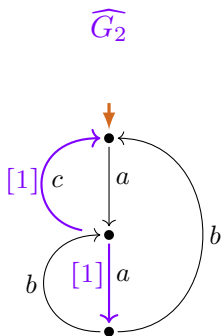


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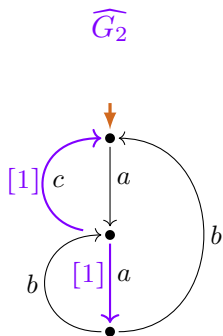
$$(1 \cdot (\quad)^*) \cdot 0$$

Refined extraction expression (example)



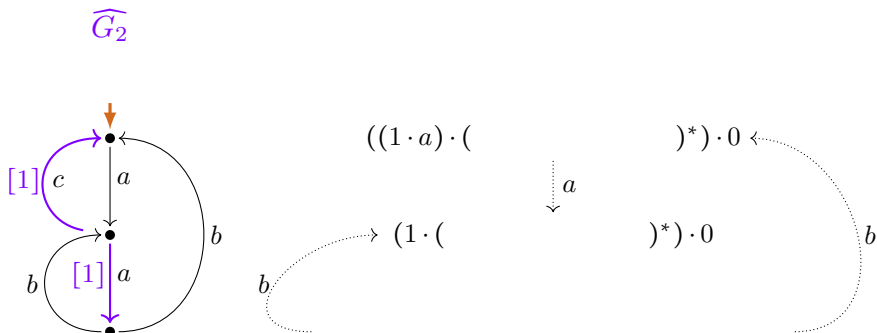
$$(1 \cdot (\begin{array}{c} \vdots \\ a \\ \vee \end{array})^*) \cdot 0$$

Refined extraction expression (example)

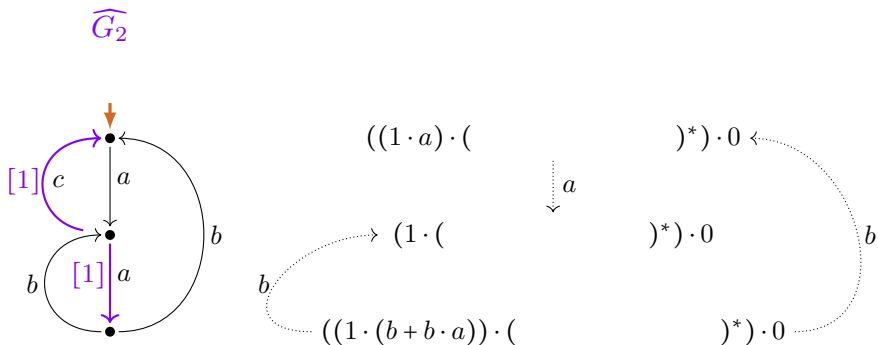


$$\begin{array}{c}
 ((1 \cdot a) \cdot (\quad)^*) \cdot 0 \\
 \downarrow a \\
 (1 \cdot (\quad)^*) \cdot 0
 \end{array}$$

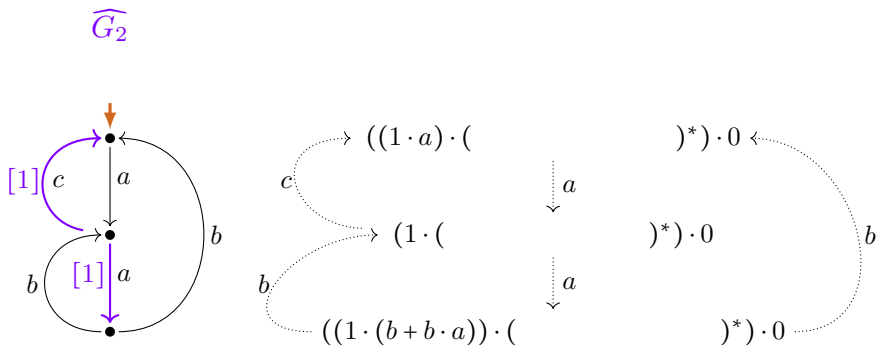
Refined extraction expression (example)



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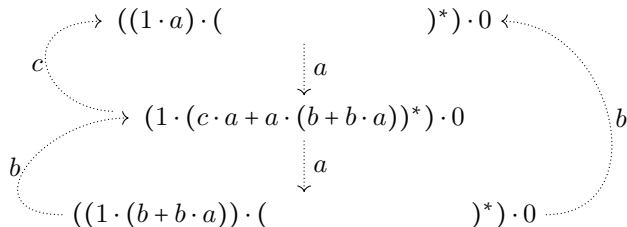
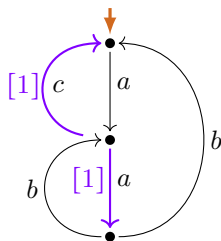


Refined extraction expression (example)



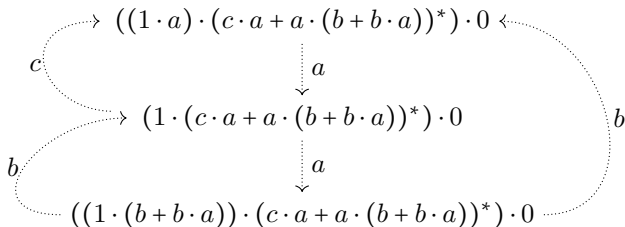
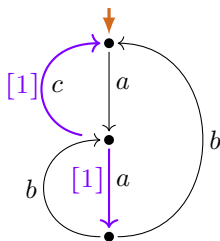
Refined extraction expression (example)

\widehat{G}_2



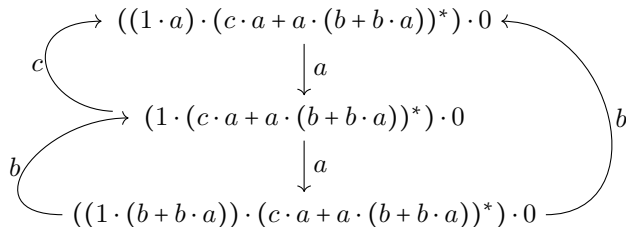
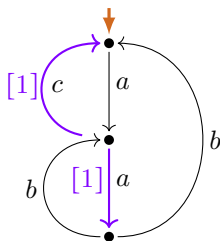
Refined extraction expression (example)

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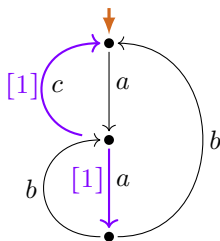
Refined extraction expression (example)

\widehat{G}_2

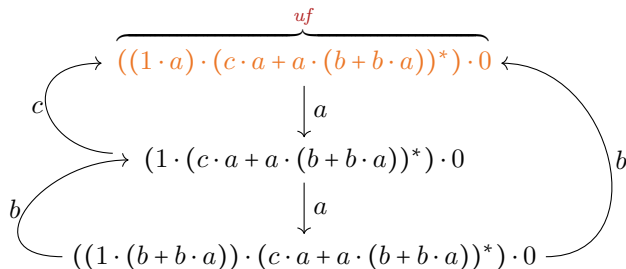


Refined extraction expression (example)

\widehat{G}_2



$$P^\bullet(uf) = P(uf) \simeq G_2$$



Interpretation/extraction correspondences of P^\bullet with LEE

(Int) $_{P^\bullet}^{(*/\pm)}$: By *under-star-1-free* expressions P^\bullet -expressible graphs satisfy LEE:

Compact process interpretations $P^\bullet(uf)$
 of *under-star-1-free* regular expressions uf
 are finite process graphs that satisfy LEE.

(Extr) $_{P^\bullet}^{(*/\pm)}$: LEE implies $\llbracket \cdot \rrbracket_{P^\bullet}$ -expressibility by *under-star-1-free* reg. expr's:

From every finite process graph G with LEE
 an *under-star-1-free* regular expression uf can be extracted
 such that $G \rightrightarrows P^\bullet(uf)$.

From every finite collapsed process graph G with LEE
 an *under-star-1-free* regular expression uf can be extracted
 such that $G \simeq P^\bullet(uf)$.

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(ImColl) $_{P^\bullet}^{(*/\pm)}$: The image of P^\bullet ,
 restricted to *under-star-1-free* regular expressions,
 is closed under bisimulation collapse.

Interpretation/extraction correspondences of P^\bullet with 1-LEE

(Int) $_{P^\bullet}$: P^\bullet -expressible graphs satisfy 1-LEE:

Compact process interpretations $P^\bullet(e)$ of regular expressions e are finite process graphs that satisfy 1-LEE.

(Extr) $_{P^\bullet}$: LEE implies $\llbracket \cdot \rrbracket_{P^\bullet}$ -expressibility:

From every finite process graph G with 1-LEE
an regular expression e can be extracted
such that $G \rightrightarrows P^\bullet(e)$.

From every finite collapsed process graph G with 1-LEE
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Interpretation/extraction correspondences of P^\bullet with 1-LEE

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From every finite collapsed process graph G with 1-LEE
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(ImColl) $_{P^\bullet}$: The image of P^\bullet is not closed under bisimulation collapse.

$$\text{LEE} \stackrel{\wedge}{=} \text{image of } P^\bullet \big|_{\text{RExp}^{(*/\perp)}}$$

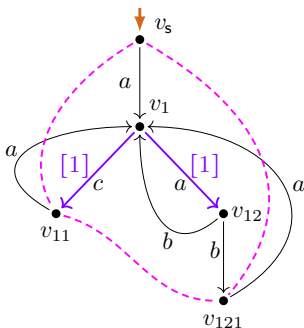
Theorem

For every process graph G TFAE:

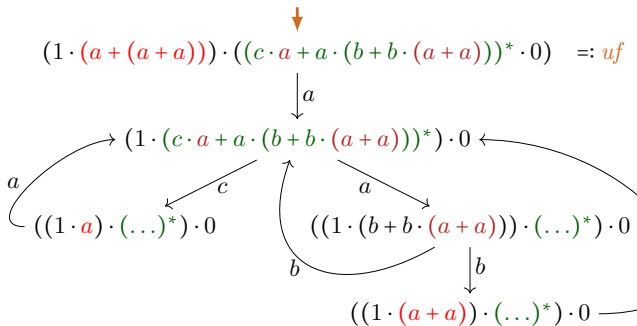
- (i) $\text{LEE}(G)$.
- (ii) G is P^\bullet -expressible by an $(*/\perp)$ regular expression
(i.e. $G \simeq P^\bullet(e)$ for some $e \in \text{RExp}^{(*/\perp)}$).
- (iii) G is isomorphic to a graph in the image of P^\bullet on $(*/\perp)$ reg. expr's
(i.e. $G \simeq G'$ for some $G' \in \text{im}(P^\bullet \big|_{\text{RExp}^{(*/\perp)})}$).

Adapted (refined) extraction from LLEE-graph

$$G_1 / \widehat{G}_1$$



$$P^\bullet(uf) = P(uf) \simeq G_1$$



1-LEE $\stackrel{\wedge}{=}$ image of P^\bullet

Theorem

For every process graph G TFAE:

- (i) 1-LEE(G)
(i.e. $G = (\underline{G}]$ for some 1-transition-process-graph \underline{G} with LEE(\underline{G})).
- (ii) G is P^\bullet -expressible by a regular expression
(i.e. $G \simeq P^\bullet(e)$ for some $e \in RExp$).
- (iii) G is isomorphic to a graph in the image of P^\bullet
(i.e. $G \simeq G'$ for some $G' \in im(P^\bullet)$).

Summary

- ▶ Characterizations of the image of P^\bullet (refinement of P):
 - ▶ LEE $\hat{=}$ image of $P^\bullet|_{\text{RExp}(*/\dagger)} \not\equiv$ image of $P|_{\text{RExp}(*/\dagger)}$
 - ▶ 1-LEE $\hat{=}$ image of $P^\bullet \not\equiv$ image of P

Summary

- ▶ process interpretation P /semantics $\llbracket \cdot \rrbracket_P$ of regular expressions
 - ▶ expressibility and **completeness** questions
- ▶ **loop existence and elimination (LEE)**
 - ▶ loop elimination **rewrite system** can be **completed**
 - ▶ **interpretation/extraction** correspondences with $(*/\pm)$ reg. expr.s
 - ▶ **LEE-witnesses**: **labelings** of graphs with **LEE**
 - ▶ **stepwise LEE**-preserving bisimulation collapse
- ▶ **1-LEE** = sharing via **1-transitions** facilitates **LEE**
 - ▶ **interpretation/extraction** correspondences with **all** regular expressions
 - ▶ **not** preserved under bisim. collapse (**approximation** possible)
- ▶ **Characterizations** of the **image of P^\bullet** (**refinement** of P):
 - ▶ $\text{LEE} \triangleq \text{image of } P^\bullet|_{\text{RExp}(*/\pm)} \not\equiv \text{image of } P|_{\text{RExp}(*/\pm)}$
 - ▶ $\text{1-LEE} \triangleq \text{image of } P^\bullet \not\equiv \text{image of } P$
- ▶ outlook on work-to-do

My next aims

Completeness problem, solution:

A1: graph structure of regular expression processes ([LEE](#)/[1-LEE](#))

A2: motivation of crystallization

A4: details of crystallization procedure,
and completeness of Milner's proof system

Expressibility problem

A3: [LEE](#) is decidable in polynomial time.

Q: Is [1-LEE](#) decidable in polynomial time?

P: Is expressibility by a regular expression, for a finite process graph,
decidable in polynomial time/fixed-parameter tractable time?

Resources

- ▶ Slides/abstract on clegra.github.io
 - ▶ slides: [.../1f/IFIP-1_6-2024.pdf](#)
 - ▶ abstract: [.../1f/abstract-IFIP-1_6-2024.pdf](#)
- ▶ CG: Closing the Image of the Process Interpretation of 1-Free Regular Expressions Under Bisimulation Collapse
 - ▶ TERMGRAPH 2024, [extended abstract](#).
- ▶ CG: The Image of the Process Interpretation of Regular Expressions is Not Closed under Bisimulation Collapse,
 - ▶ [arXiv:2303.08553](#), 2021/2023.
- ▶ CG: Milner's Proof System for Regular Expressions Modulo Bisimilarity is Complete,
 - ▶ LICS 2022, [arXiv:2209.12188](#), [poster](#).
- ▶ CG, Wan Fokkink: A Complete Proof System for 1-Free Regular Expressions Modulo Bisimilarity,
 - ▶ LICS 2020, [arXiv:2004.12740](#), [video on youtube](#).
- ▶ CG: Modeling Terms by Graphs with Structure Constraints,
 - ▶ TERMGRAPH 2018, [EPTCS 288](#), [arXiv:1902.02010](#).

Language semantics $\llbracket \cdot \rrbracket_L$ of reg. expr's *(Copi-Elgot-Wright, 1958)*

$0 \xrightarrow{L} \text{empty language } \emptyset$

$1 \xrightarrow{L} \{\epsilon\} \quad (\epsilon \text{ the empty word})$

$a \xrightarrow{L} \{a\}$

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$e_1 + e_2 \xrightarrow{L} \text{union of } L(e_1) \text{ and } L(e_2)$

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$e^* \xrightarrow{L} \text{set of words formed by concatenating words in } L(e),$
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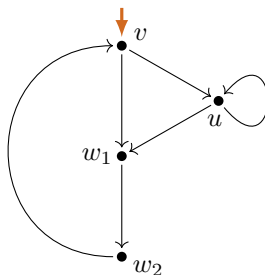
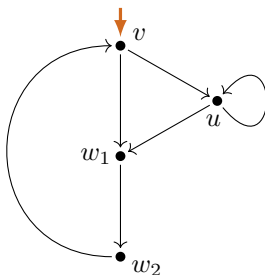
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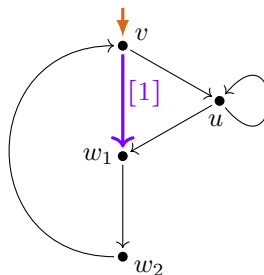
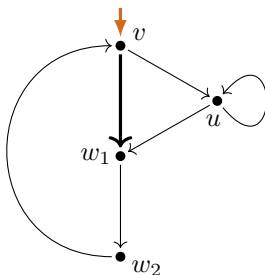
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$\llbracket e \rrbracket_L := L(e) \quad (\text{language defined by } e)$

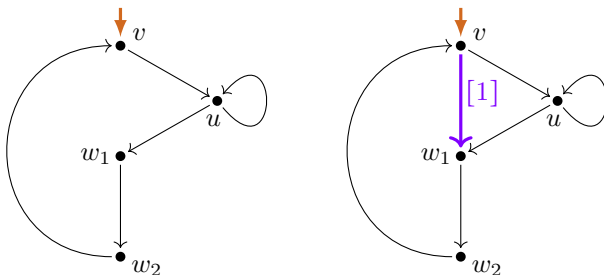
Layered LEE-witness (LLEE-witness)



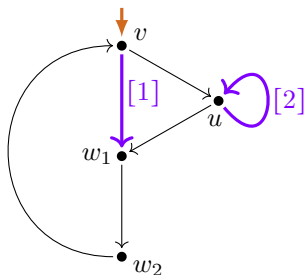
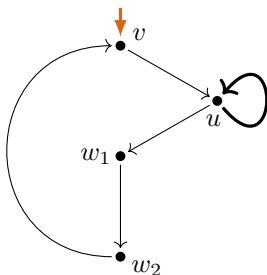
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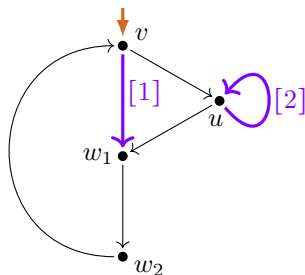
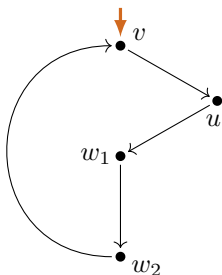
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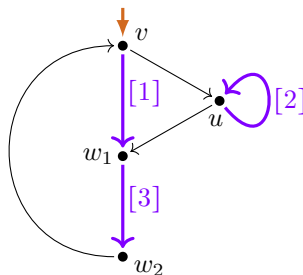
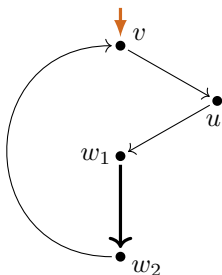
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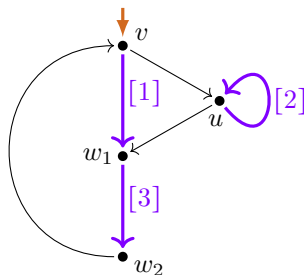
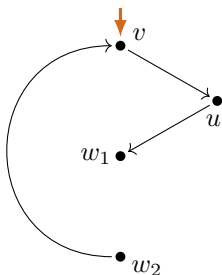
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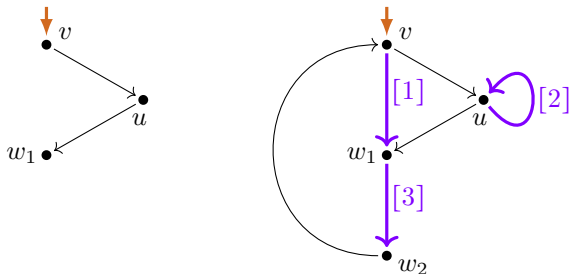
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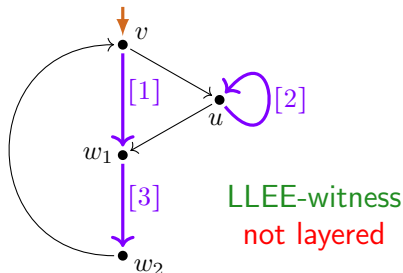
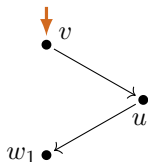
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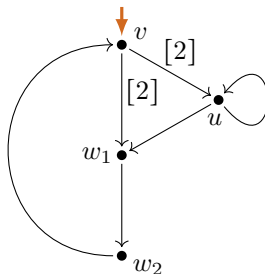
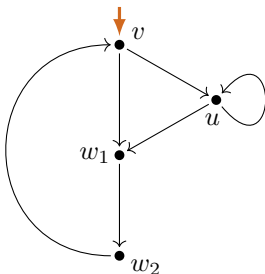
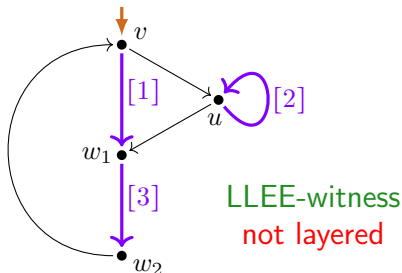
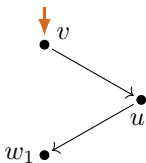
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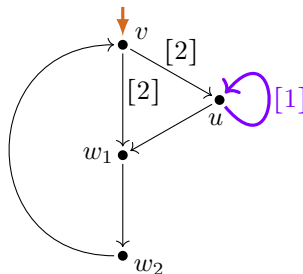
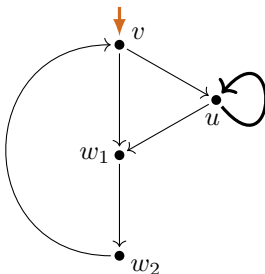
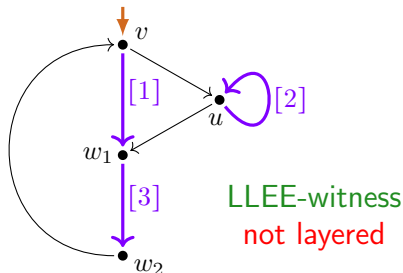
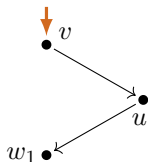
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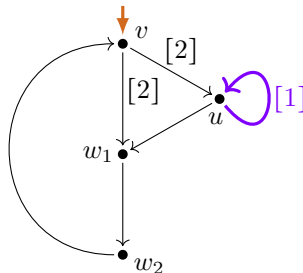
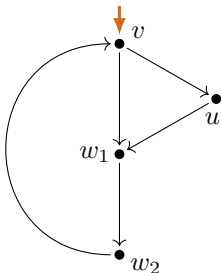
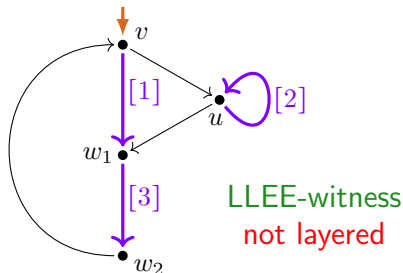
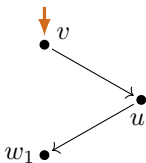
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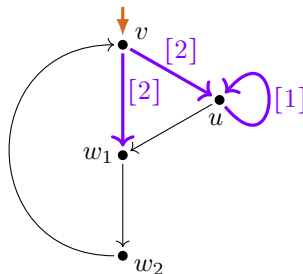
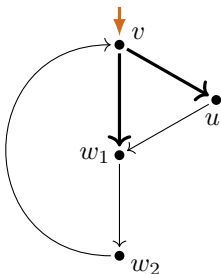
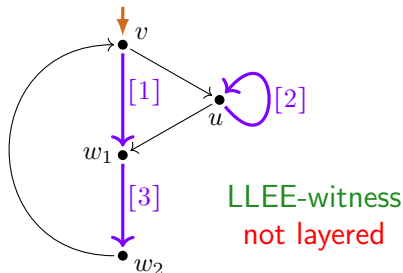
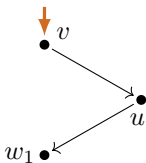
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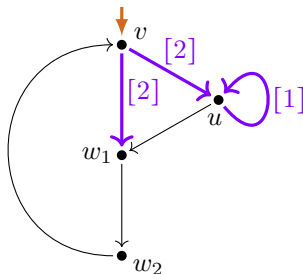
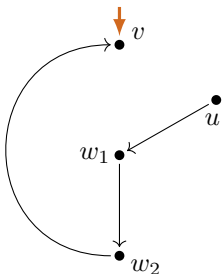
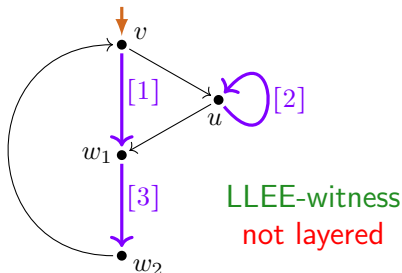
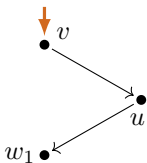
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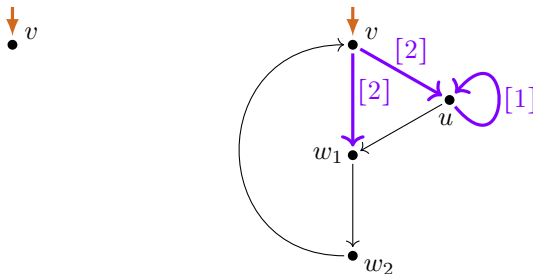
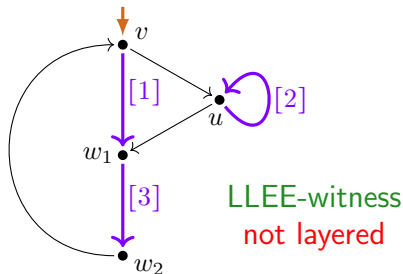
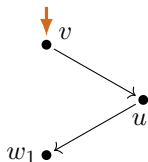
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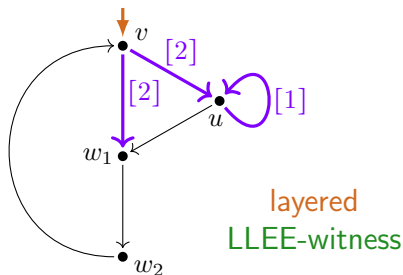
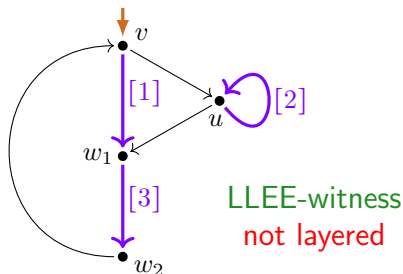
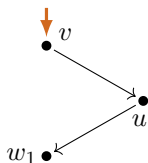
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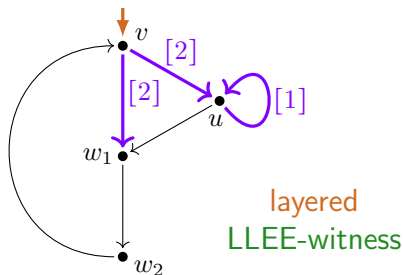
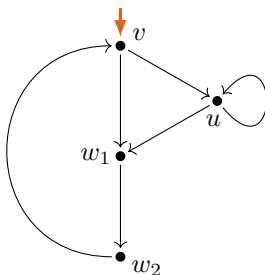
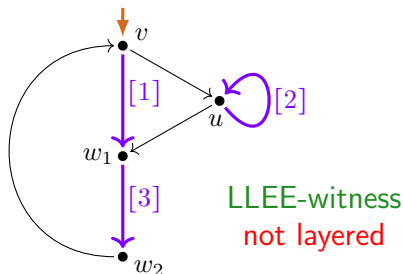
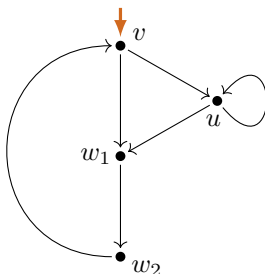
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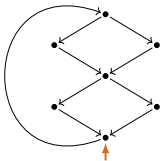
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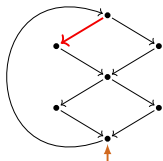
Layered LEE-witness (LLEE-witness)



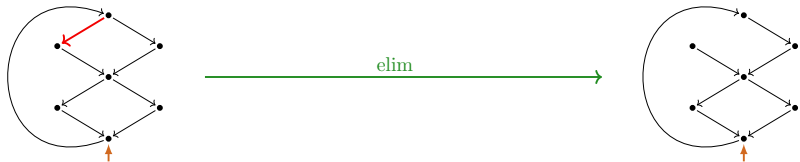
'Critical pair': bi-loop elimination



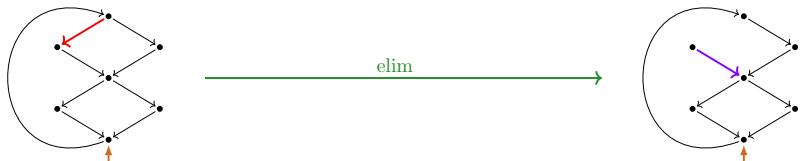
'Critical pair': bi-loop elimination



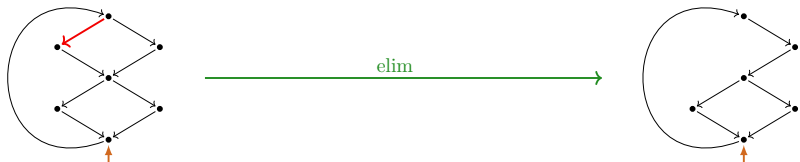
'Critical pair': bi-loop elimination



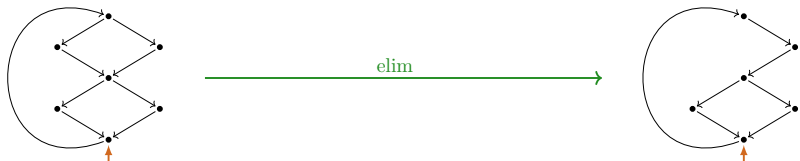
'Critical pair': bi-loop elimination



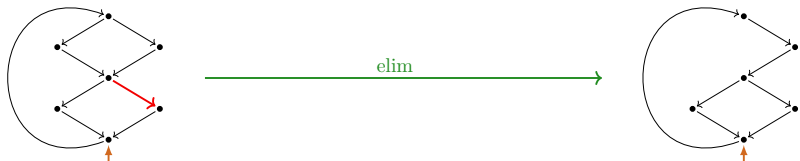
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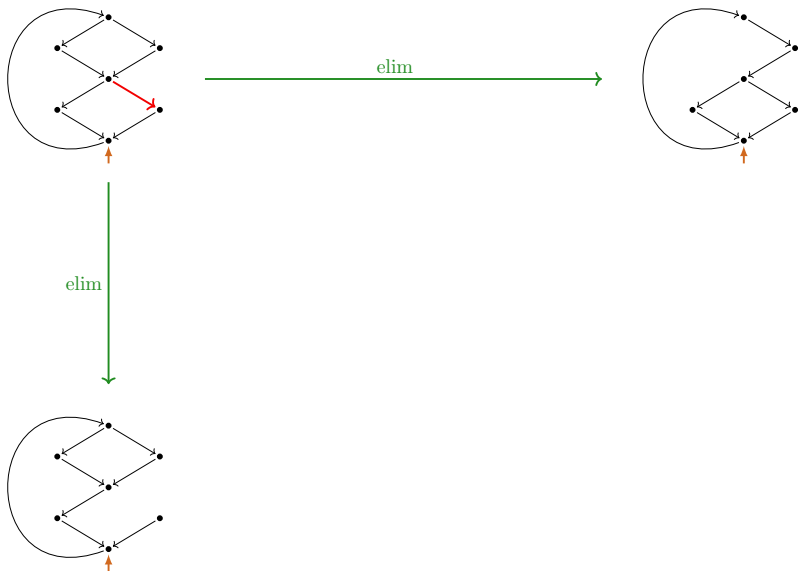
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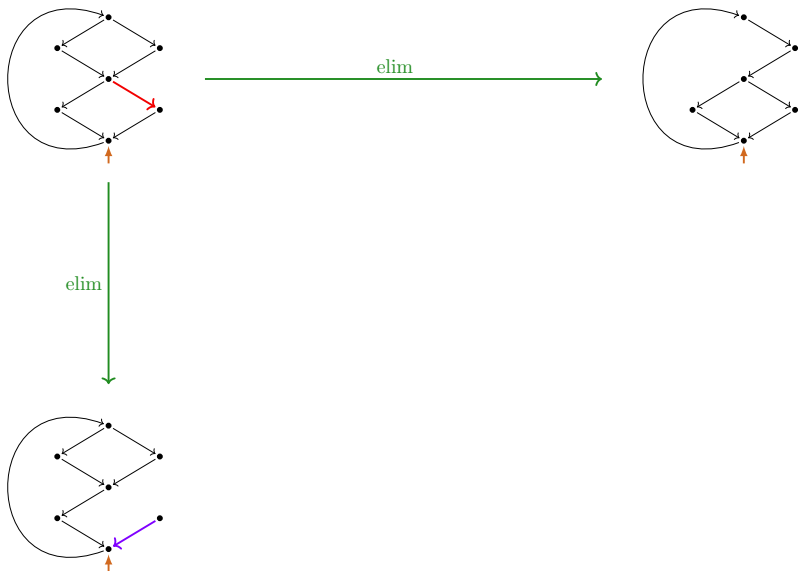
'Critical pair': bi-loop elimination



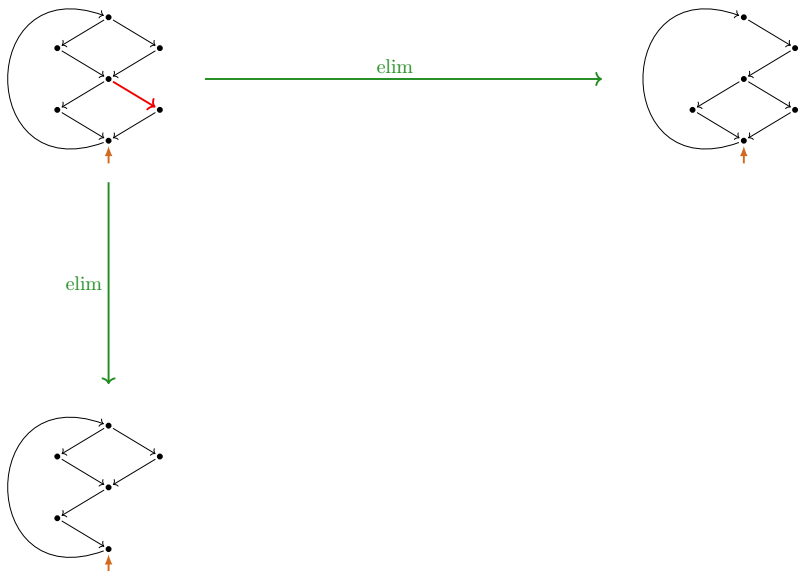
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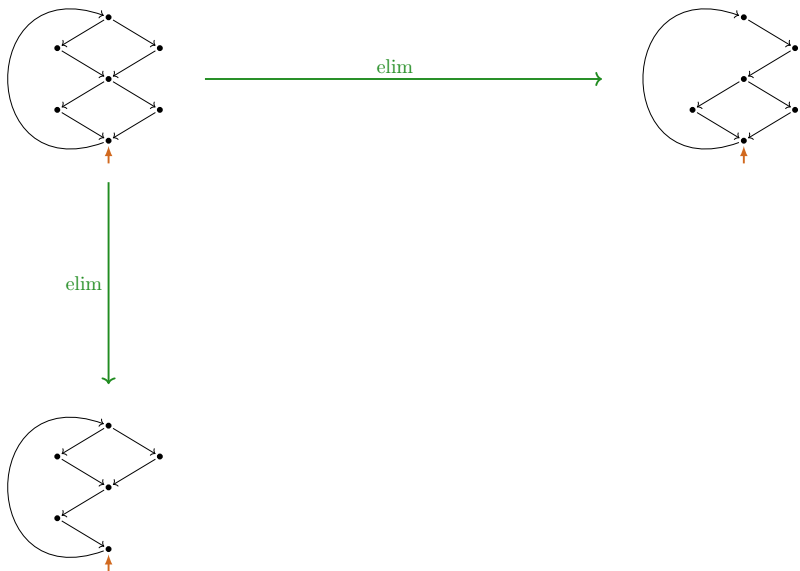
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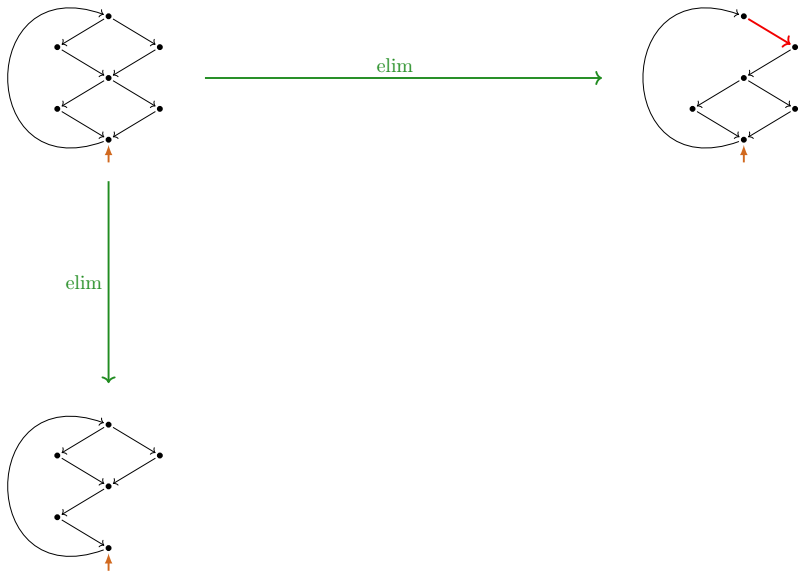
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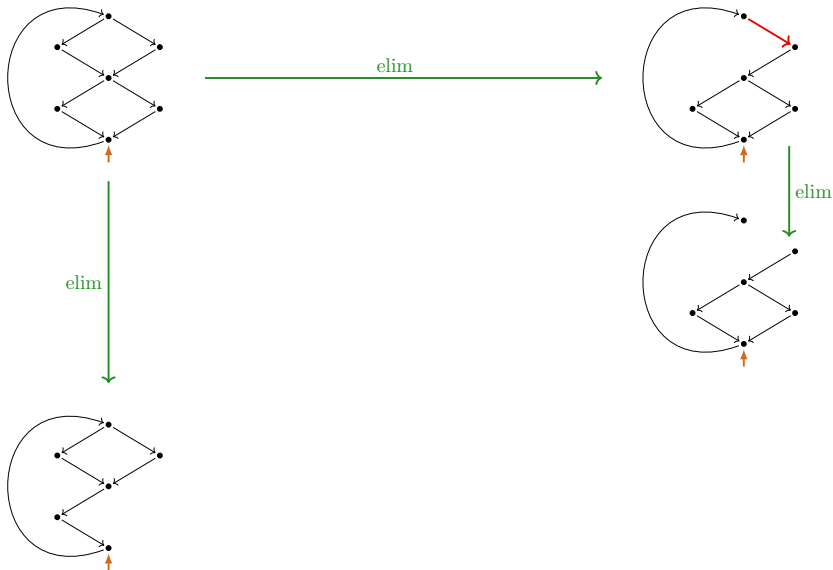
'Critical pair': bi-loop elimination



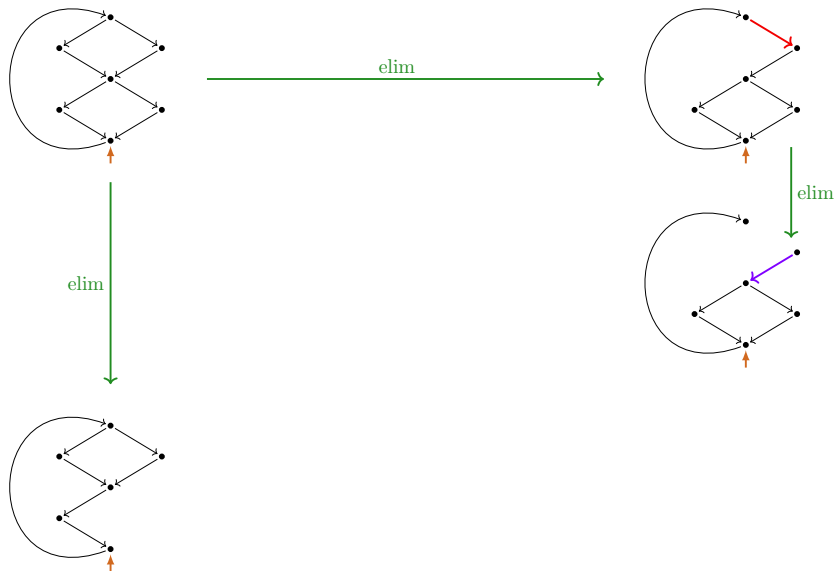
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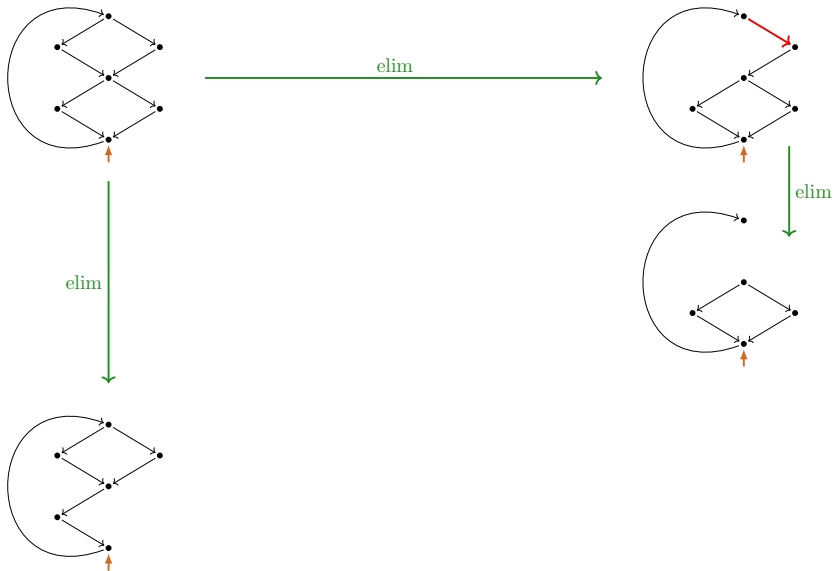
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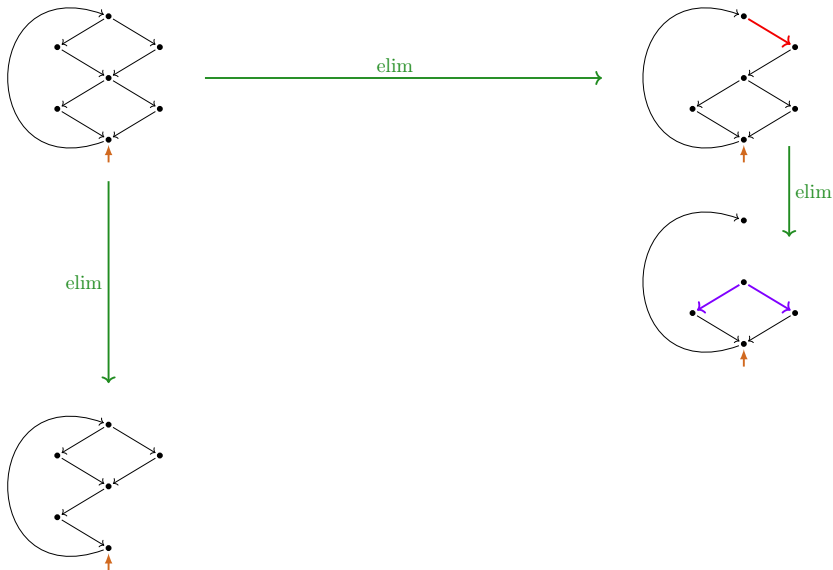
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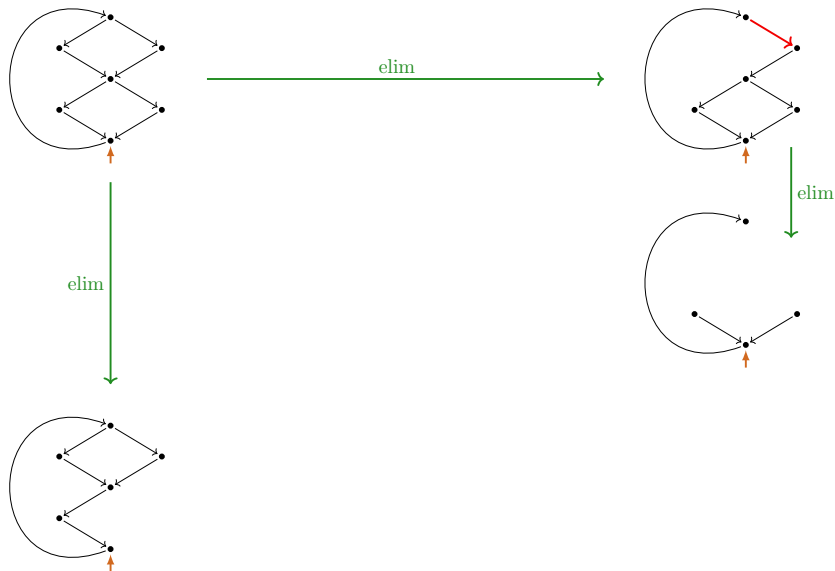
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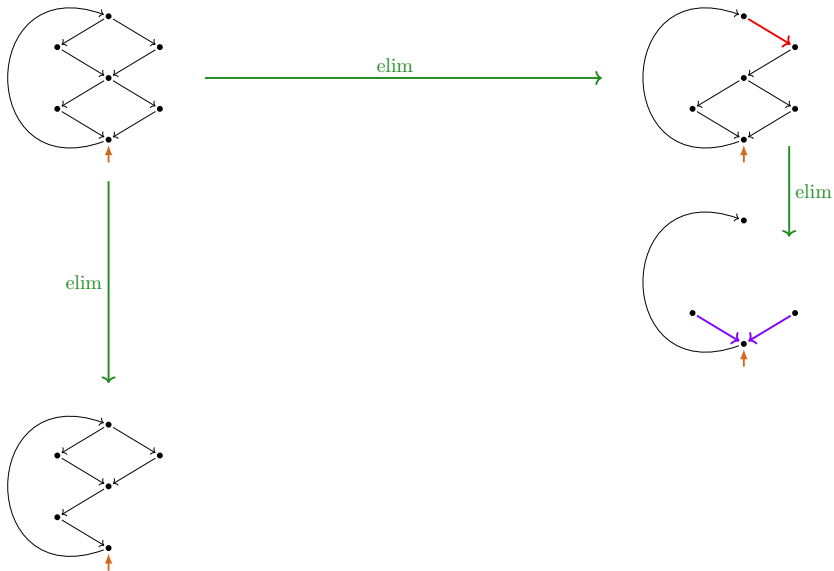
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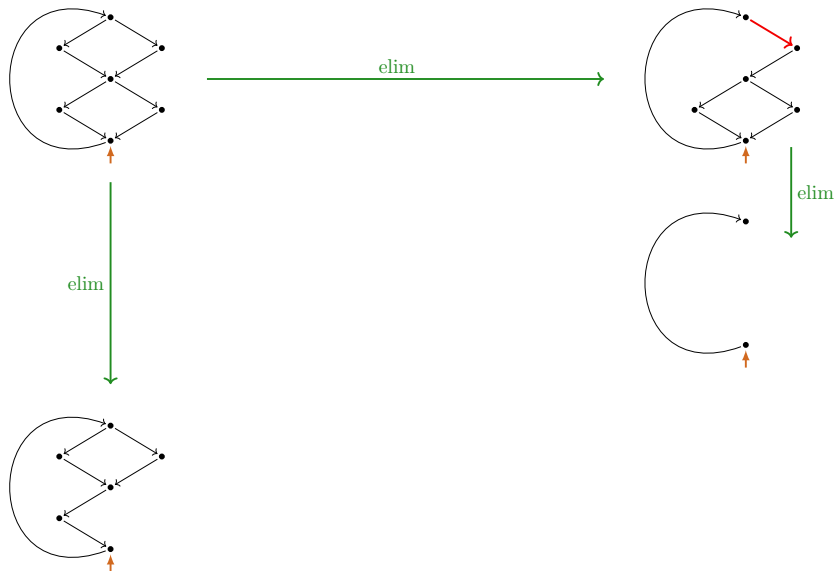
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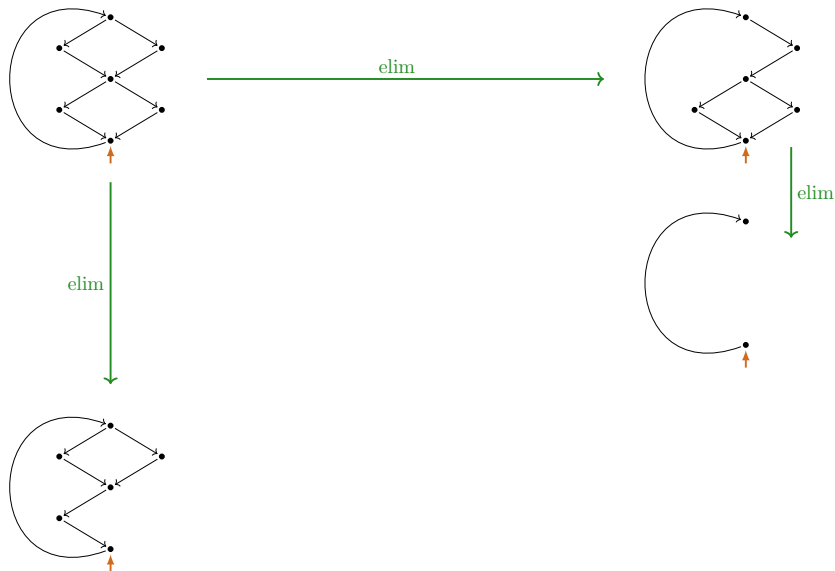
'Critical pair': bi-loop elimination



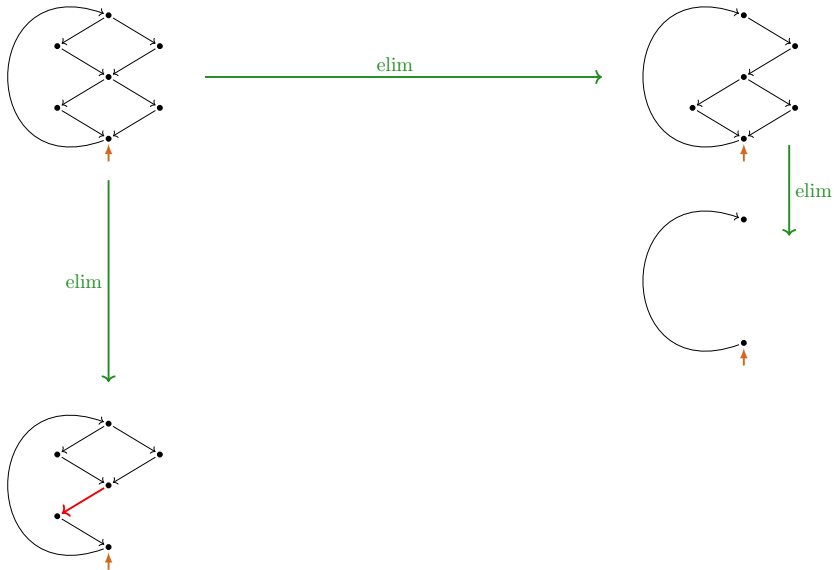
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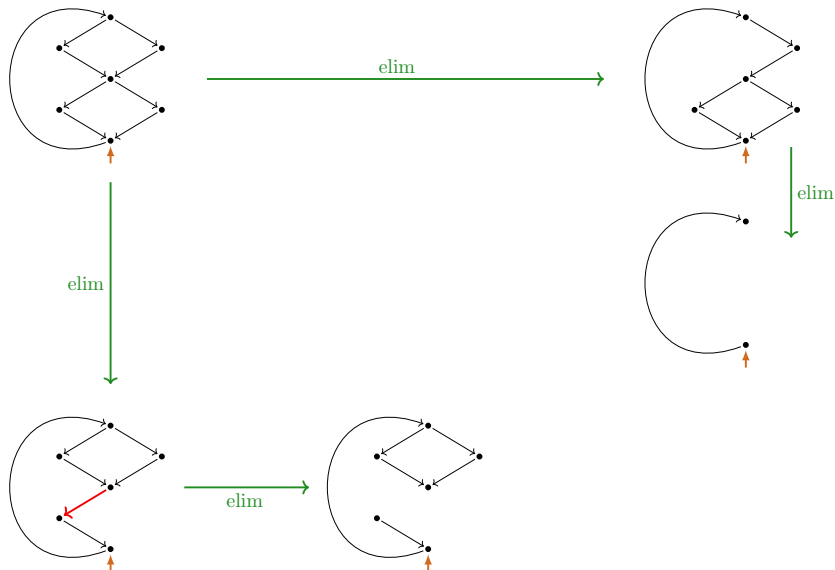
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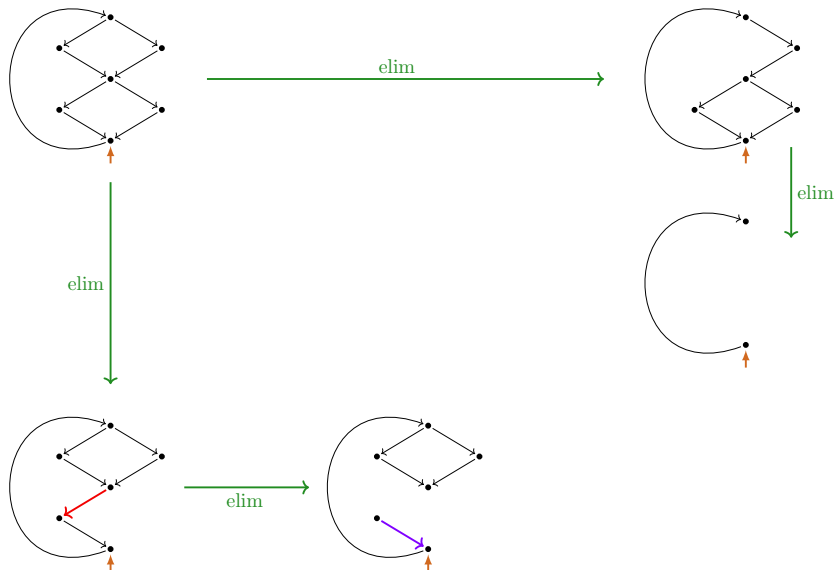
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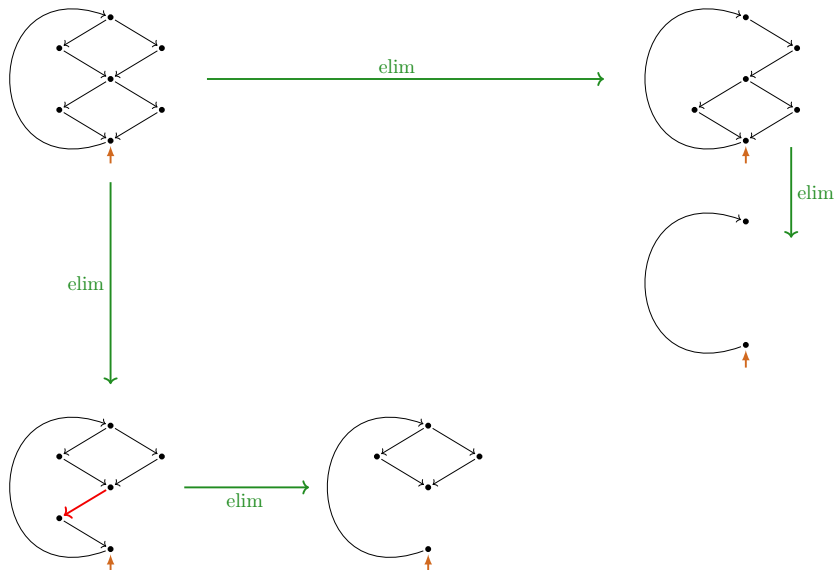
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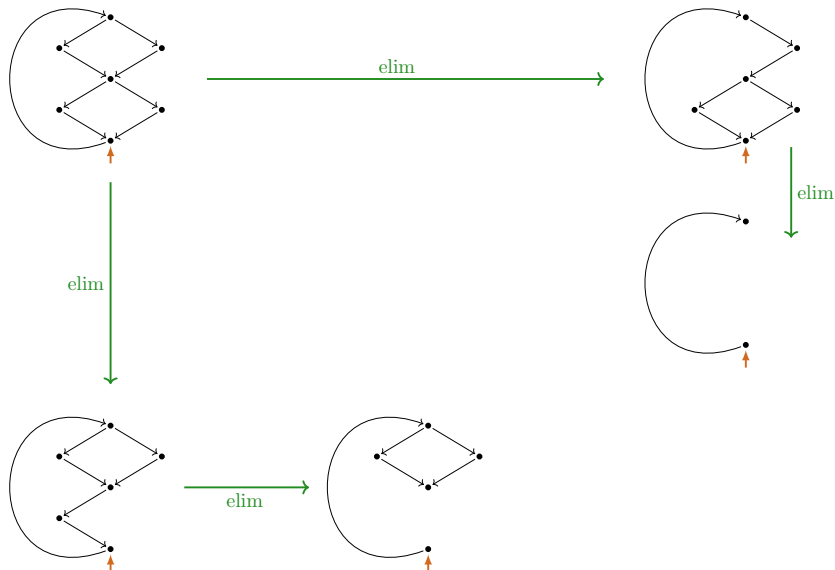
'Critical pair': bi-loop elimination



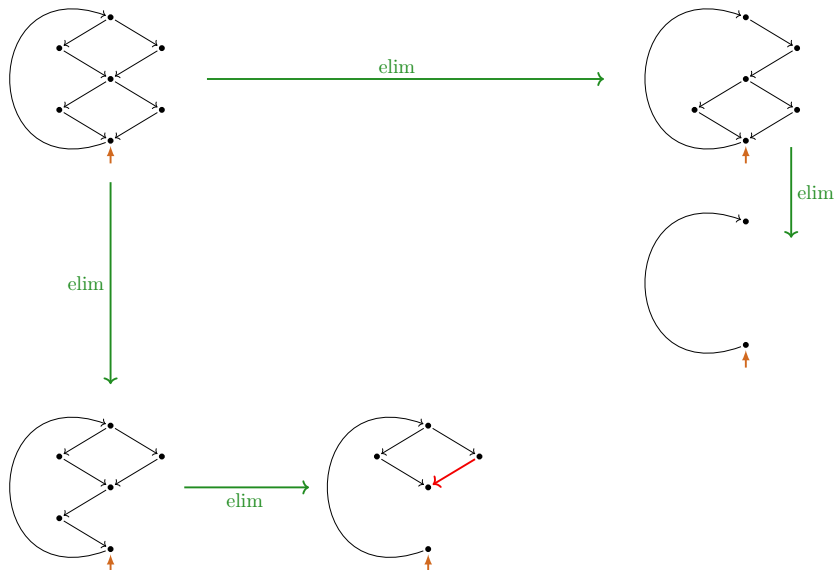
'Critical pair': bi-loop elimination



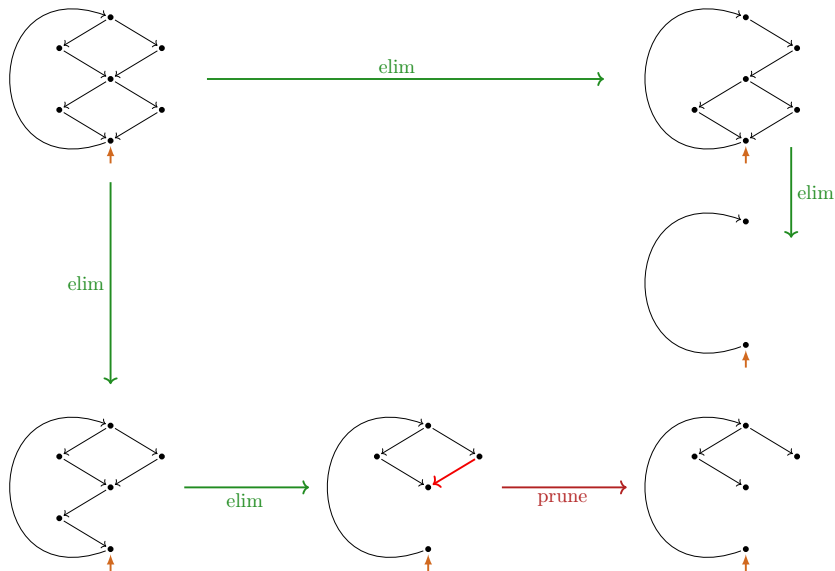
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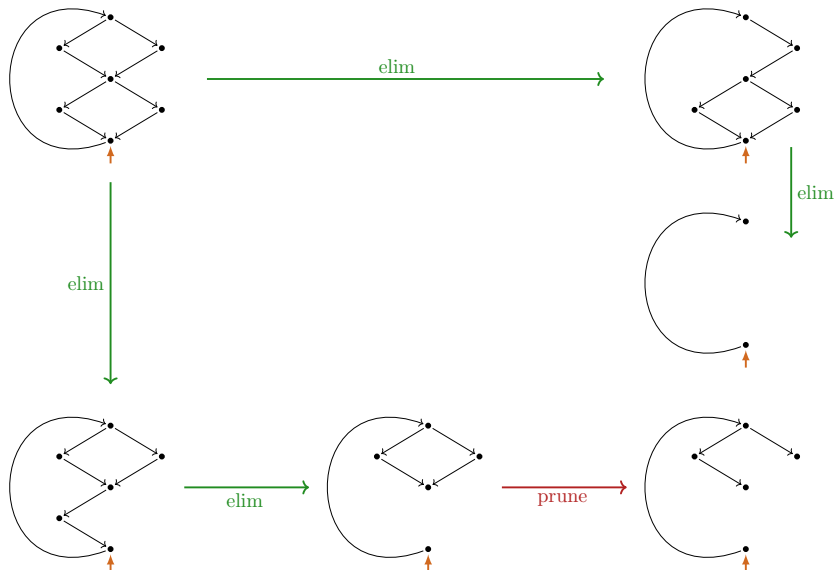
'Critical pair': bi-loop elimination



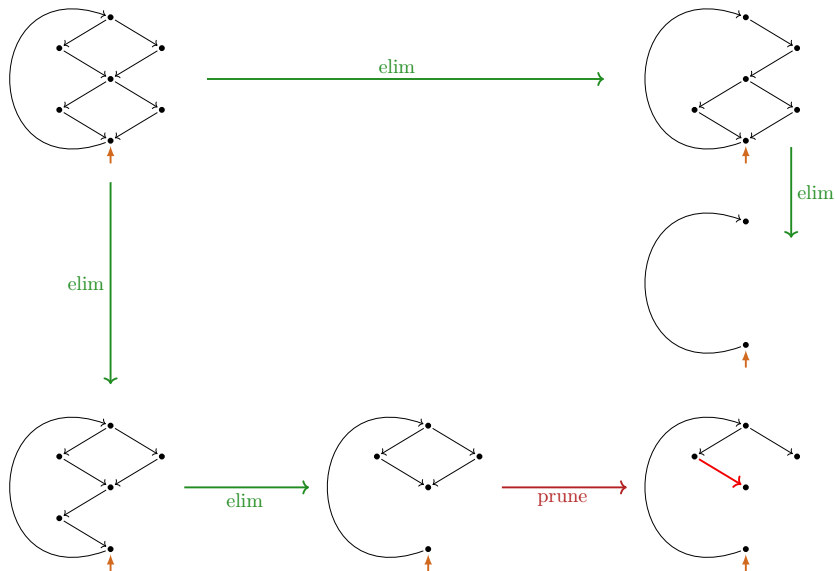
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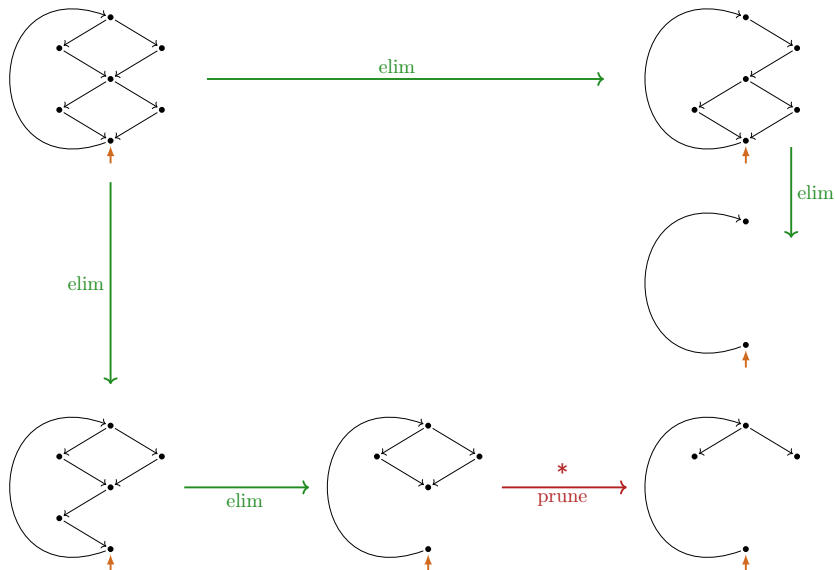
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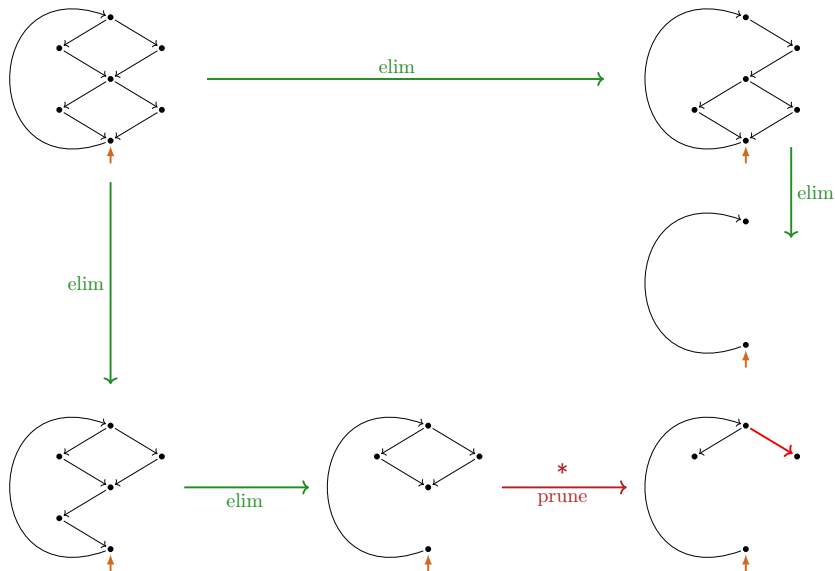
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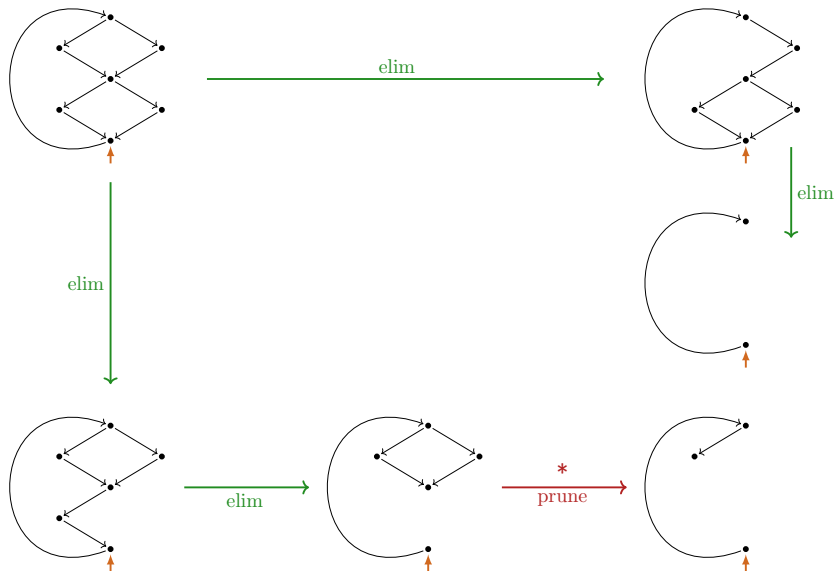
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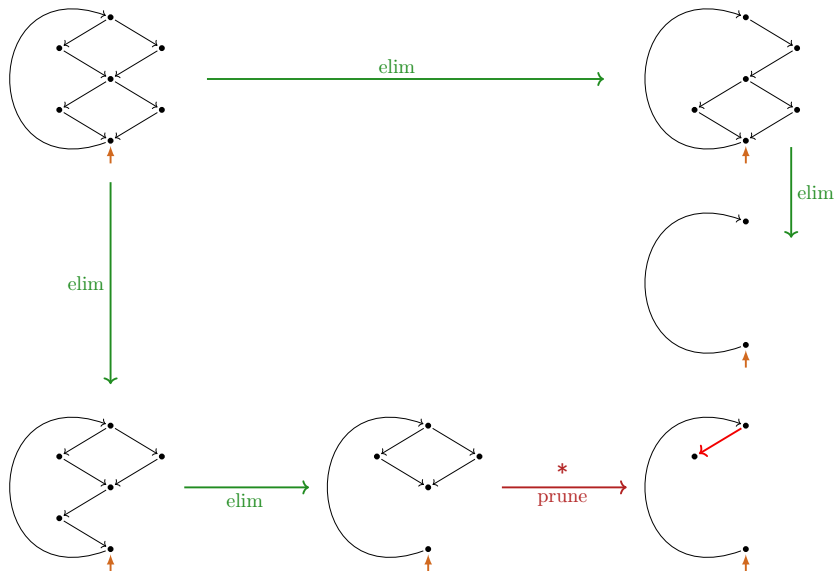
'Critical pair': bi-loop elimination



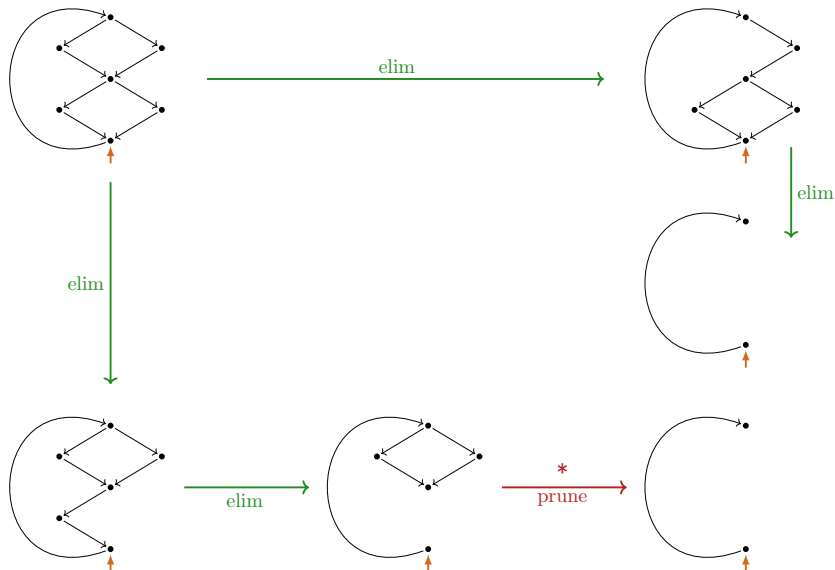
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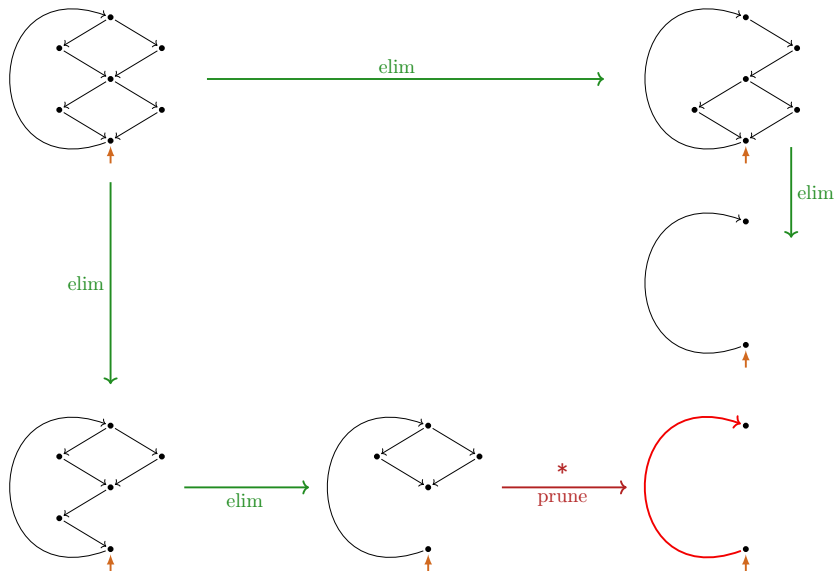
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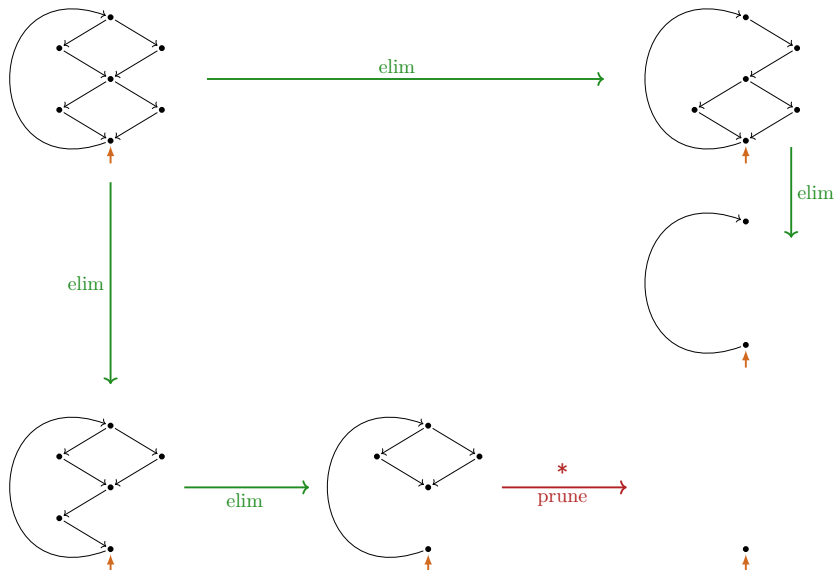
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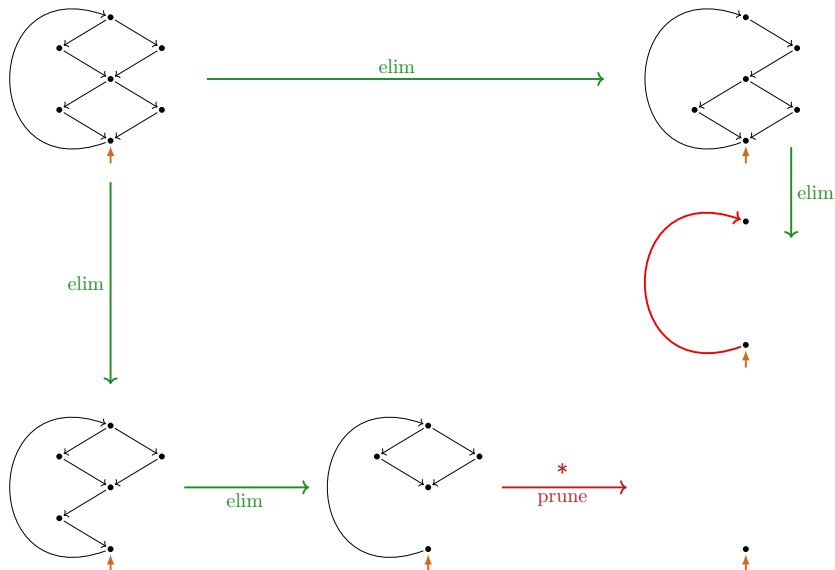
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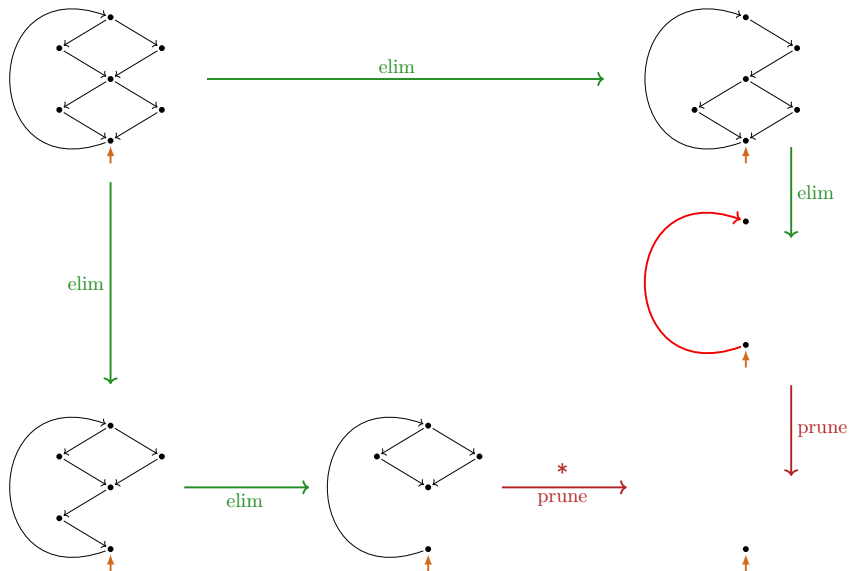
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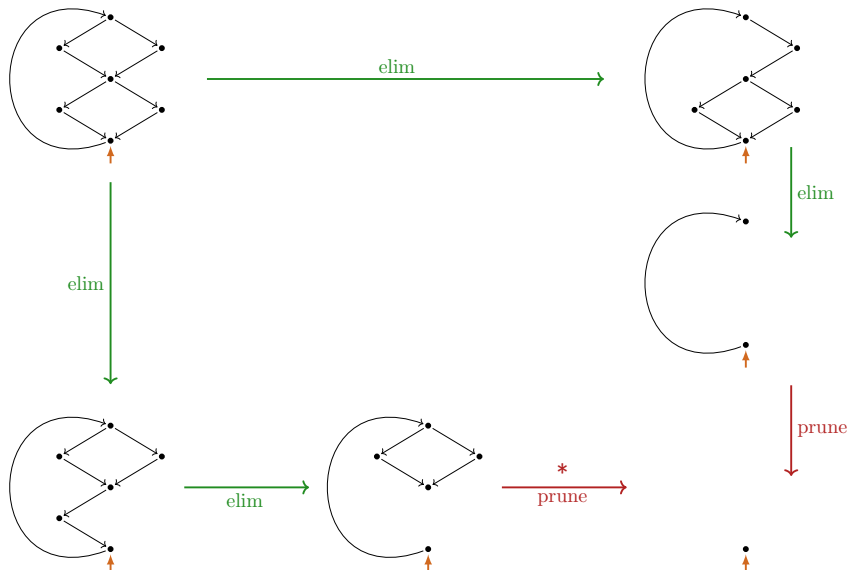
'Critical pair': bi-loop elimination



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