Lecture 3: Recursive Functions

Models of Computation

https://clegra.github.io/moc/moc.html

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Course overview

🟩	🟩	🟪	🟩	🟩
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	**Three more Models of Computation**
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	Post's Correspondence Problem, Interaction-Nets, Fractran
	imperative programming	*algebraic programming*	*functional programming*	

Calculable function?

Questions/Exercises

2. Let $f : \mathbb{N} \to \mathbb{N}$ defined by

$$n \longmapsto \begin{cases} 0 & \ldots\ n = 0\ \&\ \text{Goldbach's conjecture is false} \\ 1 & \ldots\ n = 0\ \&\ \text{Goldbach's conjecture is true} \\ n+1 & \ldots\ n > 0 \end{cases}$$

Is f calculable?

Calculable function?

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Is f calculable?

Answer: Yes, because it is one of two calculable functions.
(We just do no know which one.)

Summary

Recursive functions

- primitive recursive functions

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- Gödel–Herbrand(–Kleene) general recursive functions

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  - effectively calculable functions $\triangleq$ partial-recursive functions

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Recursive functions

- primitive recursive functions
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  - defined with μ-recursion (unbounded minimisation)
- Partial recursive functions = Turing computable functions
- Church's thesis
  - effectively calculable functions $\triangleq$ partial-recursive functions
  - some debate

Timeline: From logic to computability

1900	Hilbert's 23 Problems in mathematics
1910/12/13	Russell/Whitehead: Principia Mathematica
1928	Hilbert/Ackermann: formulate completeness/decision problems for the predicate calculus (the latter called 'Entscheidungsproblem')
1929	Presburger: completeness/decidability of theory of addition on $\mathbb{Z}$
1930	Gödel: completeness theorem of predicate calculus
1931	Gödel: incompleteness theorems for first-order arithmetic
1932	Church: λ-calculus
1933/34	Herbrand/Gödel: general recursive functions
1936	Church/Kleene: λ-definable $\sim$ general recursive Church Thesis: 'effectively calculable' be defined as either Church shows: the 'Entscheidungsproblem' is unsolvable Post: machine model; Church's thesis as 'working hypothesis'
1937	Turing: convincing analysis of a 'human computer' leading to the 'Turing machine'

Turing-computable (total) functions

Definition

A total function $f : \mathbb{N}^k \to \mathbb{N}$ is Turing-computable if there exists a Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \not{b}, F \rangle$ and a calculable coding function $\langle \cdot \rangle : \mathbb{N} \to \Sigma^*$ such that:

- for all $n_1, \ldots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:

$$q_0 \langle n_1 \rangle \not{b} \langle n_2 \rangle \not{b} \ldots \not{b} \langle n_k \rangle \vdash_M^* q \langle f(n_1, \ldots, n_k) \rangle$$

Busy Beaver is not computable

Definition (Tibor Radó, 1962)

$BB(n) :=$ the largest number of steps any n-state Turing machine with tape alphabet $\{0, 1\}$ and blank symbol 0 can run before eventually halting, when started on an empty tape.

Proposition

The busy beaver function $BB : \mathbb{N} \to \mathbb{N}$, $n \mapsto BB(n)$ is not (Turing-)computable.
More generally: one can solve the Halting Problem, given oracle access to any function $b : \mathbb{N} \to \mathbb{N}$ such that $b(n) \geq BB(n)$ for all $n \in \mathbb{N}$.

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Proposition

Let $f : \mathbb{N} \to \mathbb{N}$ *be a Turing-computable function.*
Then there exists an $n_f \in \mathbb{N}$ *such that* *BB*$(n) > f(n)$ *for all* $n \geq n_f$.

Recursive Functions

Functions defined by recursive equations:
like e.g. functions $+, \cdot, (\cdot)^{\cdot} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, and $(\cdot)! : \mathbb{N} \to \mathbb{N}$:

$$\begin{aligned} n + 0 &= n \\ n + (m+1) &= (n+m) + 1 \end{aligned}$$

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General recursive functions: defined by more general systems of equations

μ-Recursive (partial recursive) functions:
extend the primitive recursive functions by a μ-operator
that allows to obtain partial functions

Rósza Péter

Rósza Péter (1905–1977)

Primitive recursive functions ($\mathbb{N}^k \to \mathbb{N}$)

Base functions:

- $\mathcal{O} : \mathbb{N}^0 = \{\varnothing\} \to \mathbb{N}\,,\ \varnothing \mapsto 0$ (0-ary constant-0 function)
- $\mathsf{succ} : \mathbb{N} \to \mathbb{N}\,,\ x \mapsto x + 1$ (successor function)
- $\pi^n_i : \mathbb{N}^n \to \mathbb{N}\,,\ \vec{x} = \langle x_1, \ldots, x_n \rangle \mapsto x_i$ (projection function)

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Closed under operations:

- composition: if $f : \mathbb{N}^k \to \mathbb{N}$, and $g_i : \mathbb{N}^n \to \mathbb{N}$ are prim. rec., then so is $h = f \circ (g_1 \times \dots \times g_k) : \mathbb{N}^n \to \mathbb{N}$:
$$h(\vec{x}) = f(g_1(\vec{x}), \dots, g_k(\vec{x}))$$

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- primitive recursion: if $f : \mathbb{N}^n \to \mathbb{N}$, $g : \mathbb{N}^{n+2} \to \mathbb{N}$ are prim. rec., then so is $h = \mathrm{pr}(f; g) : \mathbb{N}^{n+1} \to \mathbb{N}$:
$$h(\vec{x}, 0) = f(\vec{x})$$
$$h(\vec{x}, y+1) = g(\vec{x}, h(\vec{x}, y), y)$$

Primitive recursive functions ($\mathbb{N}^n \to \mathbb{N}^l$)

Base functions:

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Closed under operations:

- composition: if $f : \mathbb{N}^{km} \to \mathbb{N}^l$, and $g_i : \mathbb{N}^n \to \mathbb{N}^m$ are prim. rec., then so is $h = f \circ (g_1 \times \ldots \times g_k) : \mathbb{N}^n \to \mathbb{N}^l$:
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- primitive recursion: if $f : \mathbb{N}^n \to \mathbb{N}^l$, $g : \mathbb{N}^{n+l+1} \to \mathbb{N}^l$ are prim. rec., then so is $h = \mathsf{pr}(f; g) : \mathbb{N}^{n+1} \to \mathbb{N}^l$:
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- $\pi_i^n : \mathbb{N}^n \to \mathbb{N}\,,\ \vec{x} = \langle x_1, \ldots, x_n \rangle \mapsto x_i$ (projection function)
- for $n > 1$: $\mathrm{id}^n : \mathbb{N}^n \to \mathbb{N}^n\,,\ \vec{x} = \langle x_1, \ldots, x_n \rangle \mapsto \vec{x}$ (n-ary identity f.)

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Primitive recursive functions (exercises)

Exercise

Show that the following functions are primitive recursive:

- addition
- constant functions
- multiplication
- (positive) sign-function
- the representing functions $\chi_=$ and $\chi_<$ for the predicates $=$ and $<$.

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Try-yourself-Examples

Show that the following functions are primitive recursive:

- exponentiation
- factorial

Admissible operations for primitive recursive functions

Proposition

1. *definition by case distinction:*

$$f(\vec{x}) := \begin{cases} f_1(\vec{x}) & \ldots P_1(\vec{x}) \\ f_2(\vec{x}) & \ldots P_2(\vec{x}) \wedge \neg P_1(\vec{x}) \\ \ldots & \\ f_k(\vec{x}) & \ldots P_k(\vec{x}) \wedge \neg P_{k-1}(\vec{x}) \wedge \ldots \wedge \neg P_1(\vec{x}) \\ f_{k+1}(\vec{x}) & \ldots \neg P_k(\vec{x}) \wedge \ldots \wedge \neg P_1(\vec{x}) \end{cases}$$

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2. *definition by bounded recursion:*

$$\mu z_{\leq y}.\,[P(x_1, \ldots, x_n, z)] :=$$

$$\begin{cases} z & \ldots\ \neg P(x_1, \ldots, x_n, i) \text{ for } 0 \leq i < z \leq y, \\ & \quad\ \text{and } P(x_1, \ldots, x_n, z) \\ y + 1 & \ldots\ \neg \exists z. \big(0 \leq z \leq y \wedge P(x_1, \ldots, x_n, z) \big) \end{cases}$$

Properties of primitive recursive functions

Proposition

1. *Every primitive recursive function is total.*
2. *Every primitive recursive function is Turing-computable.*

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Proof.

For (2):

- the base functions are Turing-computable
- the Turing-computible functions are closed under the schemes composition and primitive recursion

□

Turing-computable (total) functions

Definition

A total function $f : \mathbb{N}^k \to \mathbb{N}$ is Turing-computable if there exists a Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \not{b}, F \rangle$ and a calculable coding function $\langle \cdot \rangle : \mathbb{N} \to \Sigma^*$ such that:

- for all $n_1, \ldots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:

$$q_0 \langle n_1 \rangle \not{b} \langle n_2 \rangle \not{b} \ldots \not{b} \langle n_k \rangle \vdash_M^* q \langle f(n_1, \ldots, n_k) \rangle$$

Features of computationally complete MoC's present?

- storage (unbounded)
- control (finite, given)
- modification
  - of (immediately accessible) stored data
  - of control state
- conditionals
- loop (unbounded)
- stopping condition

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- stopping condition ✓

Not primitive recursive (I)

Proposition

There exist calculable/Turing-computable functions that are not primitive recursive.

Proof.

By diagonalisation. □

Not primitive recursive (II): Ackermann function

Wilhelm Ackermann (1896–1962)

Not primitive recursive (II): Ackermann function

Ackermann function $\mathsf{A} : \mathbb{N}^2 \to \mathbb{N}$ (simplified version by Rósza Péter):

$$\mathsf{A}(0, x) = \mathsf{Succ}(x)$$

$$\mathsf{A}(x+1, 0) = \mathsf{A}(x, \mathsf{Succ}(0))$$

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\end{aligned}$$

A is not primitive recursive, it grows too fast:

$$\begin{aligned}
\mathsf{A}(0, n) &= n + 1 \\
\mathsf{A}(1, n) &= n + 2 \\
\mathsf{A}(2, n) &= 2n + 3 \\
\mathsf{A}(3, n) &= 2^{n+3} - 2 \\
\mathsf{A}(4, n) &= \underbrace{2^{2^{\cdot^{\cdot^{2^{16}}}}}}_{n} - 3
\end{aligned}$$

$$\dots$$

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A grows faster than every primitive recursive function:

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Theorem

For every primitive recursive $f : \mathbb{N} \to \mathbb{N}$ there exists some $i \in \mathbb{N}$ such that $f(i) < \mathsf{A}(i,i)$.

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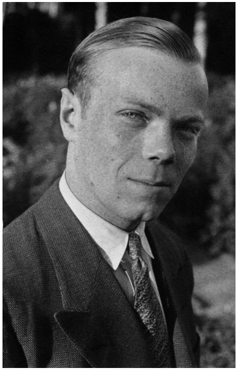
Theorem (Green, 1964)

For all $n \in \mathbb{N}$:

$$\mathit{BB}(2n) \geq \mathsf{A}(n, n)\,.$$

Jacques Herbrand



Jacques Herbrand (1908–1931)

Kurt Gödel

Kurt Gödel (1906–1978)

Gödel–Herbrand general recursive function

Defined by systems of recursion equations like that for the Ackermann function:

$$\begin{aligned} \mathsf{A}(0, y) &= \mathsf{Succ}(y) \\ \mathsf{A}(\mathsf{Succ}(x), 0) &= \mathsf{A}(x, \mathsf{Succ}(0)) \\ \mathsf{A}(\mathsf{Succ}(x), \mathsf{Succ}(y)) &= \mathsf{A}(x, \mathsf{A}(\mathsf{Succ}(x), y)) \end{aligned}$$

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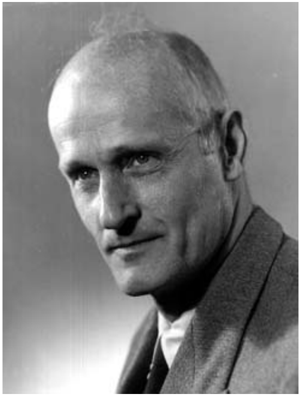
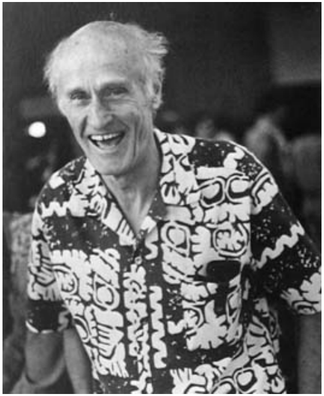
Numerals: $\langle 0 \rangle := 0$, and $\langle n \rangle := \underbrace{\mathsf{Succ}(\ldots\mathsf{Succ}(0))}_{n}$ for $n > 1$.

Definition

A function $f : \mathbb{N}^k \to \mathbb{N}$ is called general recursive if it can be defined by (such a) system $\mathcal{S}$ of recursion equations via a function symbol F if for all $n_1, \ldots, n_k \in \mathbb{N}$, the expression $F(\langle n_1 \rangle, \ldots, \langle n_k \rangle)$ evaluates according to $\mathcal{S}$ to a unique numeral $\langle n \rangle$, and such that furthermore: $n = f(n_1, \ldots, n_k)$.

Stephen Cole Kleene

Stephen Cole Kleene (1906–1994)

Unbounded minimisation (μ-recursion)

Let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ total. Then the partial function defined by:

$$\mu(f) : \mathbb{N}^k \rightharpoonup \mathbb{N}$$
$$\vec{x} \mapsto \begin{cases} \text{min}\{y \in \mathbb{N} \mid f(\vec{x}, y) = 0\} & \dots\ \exists y\,(f(\vec{x}, y) = 0) \\ \uparrow & \dots\ \text{else} \end{cases}$$

is called the unbounded minimisation of f.

Unbounded minimisation (μ-recursion)

Let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ total. Then the partial function defined by:

$$\mu(f) : \mathbb{N}^k \rightharpoonup \mathbb{N}$$

$$\vec{x} \mapsto \begin{cases} \min\{y \in \mathbb{N} \mid f(\vec{x}, y) = 0\} & \ldots \exists y \, (f(\vec{x}, y) = 0) \\ \uparrow & \ldots \text{else} \end{cases}$$

is called the unbounded minimisation of f.

Let $f : \mathbb{N}^{k+1} \rightharpoonup \mathbb{N}$ partial. Then the partial function $\mu(f)$:

$$\mu(f) : \mathbb{N}^k \rightharpoonup \mathbb{N}$$

$$\vec{x} \mapsto \begin{cases} z & \ldots f(\vec{x}, z) = 0 \;\wedge\; \forall y \left(0 \le y < z \to (f(\vec{x}, y)\downarrow \neq 0)\right) \\ \uparrow & \ldots \neg\exists y \left(f(\vec{x}, y) = 0 \;\wedge\; \forall z \, (0 \le z < y \to (f(\vec{x}, z)\downarrow)\right) \end{cases}$$

is called the unbounded minimisation of f.

Partial, and total, recursive functions

Definition

A partial function $f : \mathbb{N}^n \rightharpoonup \mathbb{N}^l$ is called partial recursive if it can be specified from base functions ($\mathcal{O}$, succ, π^n_i, and id^n) by successive applications of composition, primitive recursion, and unbounded minimisation.

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Proposition

Every partial recursive function is Turing-computable.

Primitive recursive

- storage (unbounded) ✓
- control (finite, given) ✓
- modification ✓
  - of (immediately accessible) stored data
  - of control state
- conditionals ✓
- loop ✓ (unbounded) ✗
- stopping condition ✓

Partial recursive = prim. rec. + unbounded minimization

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Turing-computable functions

Definition

1. A total function $f : \mathbb{N}^k \to \mathbb{N}$ is Turing-computable if there exists a Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \not{b}, F\rangle$ and a calculable coding function $\langle\cdot\rangle : \mathbb{N} \to \Sigma^*$ such that:
   - for all $n_1, \ldots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:

$$q_0\langle n_1\rangle\not{b}\langle n_2\rangle\not{b}\ldots\not{b}\langle n_k\rangle \vdash_M^* q\langle f(n_1,\ldots,n_k)\rangle$$

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   - for all $n_1, \ldots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:
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Every Turing-computable function is partial recursive.

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Proof by arithmetization of Turing machines, showing:

Theorem (Kleene's normal form theorem)

For every Turing-computable, partial function (and hence for every partial recursive function) $h : \mathbb{N}^k \to \mathbb{N}$ there exist primitive recursive functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N}^{k+1} \to \mathbb{N}$ such that:

$$h(x_1, \ldots, x_n) = (f \circ \mu(g))(x_1, \ldots, x_n)$$

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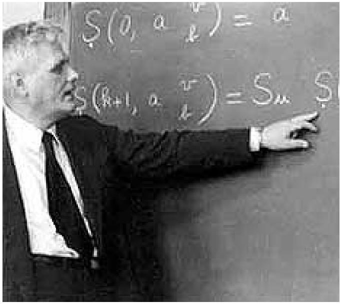
Theorem

The Turing-computable (partial) functions coincide with the partial recursive functions.

Alonzo Church



Alonzo Church (1903–1995)

Effectively calculable functions

Alonzo Church (1936):

> *"We now define the notion [. . .] of an effectively calculable function of positive integers by identifying it with the notion of a recursive function of positive integers (or a λ-definable function of positive integers). This definition is thought to be justified by the considerations which follow, so far as positive justification can ever be obtained for the selection of formal definition to correspond to an intuitive notion."*

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Definition (Church)

For every total function $f : \mathbb{N} \to \mathbb{N}$, and partial function $g : \mathbb{N} \rightharpoonup \mathbb{N}$,

f is effectively calculable $\;:\Longleftrightarrow\;$ f is recursive

g is effectively calculable $\;:\Longleftrightarrow\;$ g is partial-recursive

Church's Thesis

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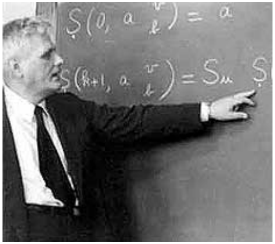
λ-calculus


$S(0, a \overset{v}{k}) = a$
$S(k+1, a \overset{v}{k}) = S_u$

Alonzo Church (1903 –1992)

Theorem (Kleene/Church, 1935)

Every λ-definable function is general recursive, and vice versa.

Recommended reading

1. Recursive and primitive-recursive functions:
   Chapter 3, The Lambda Calculus of the book:
   - Maribel Fernández [2]: *Models of Computation (An Introduction to Computability Theory)*, Springer-Verlag London, 2009.

Post's 'working hypothesis'

E.L. Post in his 1936 article (Post machines):

> *"The writer expects the present formulation to turn out to be logically equivalent to recursiveness in the sense of the Gödel–Church development. Its purpose, however, is not only to present a system of a certain logical potency but also, in its restricted field, of psychological fidelity. In the latter sense wider and wider formulations are contemplated. On the other hand, our aim will be to show that all such are logically reducible to formulation 1 [Post machines]. We offer this conclusion at the present moment as a working hypothesis. And to our mind such is Church's identification of effective calculability with recursiveness."*

Church on Post's 'working hypothesis'

Alonzo Church in his review (1937) of Post's 1936 article:

> *"The author proposes a definition of "finite 1-process" which is similar in formulation, and in fact equivalent, to computation by a Turing machine (see the preceding review). He does not, however, regard his formulation as certainly to be identified with effectiveness in the ordinary sense, but takes this identification as a "working hypothesis" in need of continual verification. To this the reviewer would object that effectiveness in the ordinary sense has not been given an exact definition, and hence the working hypothesis in question has not an exact meaning. To define effectiveness as computability by an arbitrary machine, subject to restrictions of finiteness, would seem to be an adequate representation of the ordinary notion, and if this is done the need for a working hypothesis disappears."*

Church on Turing's paper

A. Church in his review (1937) of Turing's 1936 article:

> *"The author proposes as a criterion that an infinite sequence of digits 0 and 1 be "computable" that it shall be possible to devise a computing machine, occupying a finite space and with working parts of finite size, which will write down the sequence to any desired number of terms if allowed to run for a sufficiently long time. As a matter of convenience, certain further restrictions are imposed on the character of the machine, but these are of such a nature as obviously to cause no loss of generality—in particular, a human calculator, provided with pencil and paper and explicit instructions, can be regarded as a kind of Turing machine. It is thus immediately clear that computability, so defined, can be identified with (especially, is no less general than) the notion of effectiveness as it appears in certain mathematical problems [. . .].*

Busy Beaver and axiomatic-theory consistency

Proposition

Let T be a computable and arithmetically sound axiomatic theory. Then there exists a constant n_T such that for all $n \geq n_T$, no statement of the form "BB(n) = k" can be proved in T.

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There is an explicit 748-state Turing machine that halts iff the set theory ZF is inconsistent. Hence, assuming that ZF is consistent, ZF cannot prove the value of BB(748).

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There is an explizit 27-state Turing machine that halts iff Goldbach's Conjecture is false.

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Theorem (Matiyasevich, O'Rear, Aaronson)

There is an explicit 744-state Turing machine that halts iff the Riemann Hypothesis is false.

Summary

Recursive functions

- primitive recursive functions
- Gödel–Herbrand(–Kleene) general recursive functions
- partial recursive functions
  - defined with μ-recursion (unbounded minimisation)
- Partial recursive functions = Turing computable functions
- Church's thesis
  - effectively calculable functions $\triangleq$ partial-recursive functions
  - some debate

Course overview

🟩	🟩	🟩	🟪	🟩
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	**Three more Models of Computation**
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing’s analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct’s, partial recursive = = Turing-computable, Church’s Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	Post’s Correspondence Problem, Interaction-Nets, Fractran
	imperative programming	*algebraic programming*	*functional programming*	

References I

Alonzo Church.
An Unsolvable Problem of Elementary Number Theory.
American Journal of Mathematics, 58(2):345–363, April 1936.

Maribel Fernández.
Models of Computation (An Introduction to Computability Theory).
Springer, Dordrecht Heidelberg London New York, 2009.

Emil Leon Post.
Finite Combinatory Processes – Formulation 1.
Journal of Symbolic Logic, 1(3):103–105, 1936.
`https://www.wolframscience.com/prizes/tm23/images/Post.pdf`.

References II

Alan M. Turing.
On Computable Numbers, with an Application to the Entscheidungsproblem.
Proceedings of the London Mathematical Society, 42(2):230–265, 1936.
`http://www.wolframscience.com/prizes/tm23/images/Turing.pdf`.