

The Productivity of Infinity

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The Productivity of **Infinity**

- ▶ Productivity
 - ▶ Streams, stream specifications
 - ▶ Productivity of stream specifications
 - ▶ Pebbleflow nets (**animation tool demo**)
 - ▶ Productivity prover *ProPro* (**tool demo**)
- ▶ NWO Project **Infinity**
- ▶ Beyond **Infinity** ...



Streams, stream specifications, productivity

- ▶ a **stream** over A is an **infinite sequence** of elements of A .
- ▶ useful for measurements about discrete recurrent phenomena:
 - ▶ heart beats, clock ticks, radiation particles, traffic intensity, ...
 - ▶ heartbeats ('I'm still here'),
communication ping's ('Are you still with me?')
- ▶ streams are frequently denoted in the form: $a_0 : a_1 : a_2 : \dots$
- ▶ many streams can be specified by **programs** (in a finite manner):

$$\text{alt} = 0 : 1 : \text{alt}$$

is a **stream specification** of the stream $0 : 1 : 0 : 1 : 0 : 1 : 0 : 1 : \dots$

- ▶ a stream specification is **productive** if it indeed evaluates to a stream (allows to compute each element of the stream)



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- ▶ a stream specification is **productive** if it indeed evaluates to a stream (allows to compute each element of the stream)



The Thue–Morse Stream

The **Thue–Morse** stream:

0

Example (Stream specification of the Thue–Morse)

$$T \rightarrow 0 : \text{zip}(\text{inv}(T), \text{tail}(T)) \quad \text{stream constant}$$
$$\text{zip}(x : \sigma, \tau) \rightarrow x : \text{zip}(\tau, \sigma)$$
$$\text{tail}(x : \sigma) \rightarrow \sigma \quad \text{stream functions}$$
$$\text{inv}(x : \sigma) \rightarrow i(x) : \text{inv}(\sigma)$$

$$i(0) \rightarrow 1 \quad i(1) \rightarrow 0 \quad \text{data functions}$$

This specification is **productive**.



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Data-Oblivious Productivity

Example (Data-Abstraction of the specification of Thue–Morse)

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The specification of Thue–Morse is **data-obliviously productive**.



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$T \Rightarrow \bullet : \text{zip}(\text{inv}(\bullet : \text{zip}(\text{inv}(T), \text{tail}(T))), \text{tail}(T))$

This specification is **productive**.

The specification of Thue–Morse is **data-obliviously productive**.



Data-Oblivious Productivity

Example (Data-Abstraction of the specification of Thue–Morse)

$T \rightarrow \bullet : \text{zip}(\text{inv}(T), \text{tail}(T))$	<i>stream constant</i>
$\text{zip}(x : \sigma, \tau) \rightarrow x : \text{zip}(\tau, \sigma)$	
$\text{tail}(x : \sigma) \rightarrow \sigma$	<i>stream functions</i>
$\text{inv}(x : \sigma) \rightarrow i(x) : \text{inv}(\sigma)$	
$i(\bullet) \rightarrow \bullet$	<i>data functions</i>

$T \Rightarrow \bullet : \text{zip}(i(\bullet) : \text{zip}(\text{inv}(T), \text{tail}(T)), \text{tail}(T))$

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$\text{zip}(x : \sigma, \tau) \rightarrow x : \text{zip}(\tau, \sigma)$	
$\text{tail}(x : \sigma) \rightarrow \sigma$	<i>stream functions</i>
$\text{inv}(x : \sigma) \rightarrow i(x) : \text{inv}(\sigma)$	
$i(\bullet) \rightarrow \bullet$	<i>data functions</i>

$T \Rightarrow \bullet : \underline{\text{zip}}(\bullet : \text{zip}(\text{inv}(T), \text{tail}(T)), \text{tail}(T))$

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Example (Data-Abstraction of the specification of Thue–Morse)

$T \rightarrow \bullet : \text{zip}(\text{inv}(T), \text{tail}(T))$	<i>stream constant</i>
$\text{zip}(x : \sigma, \tau) \rightarrow x : \text{zip}(\tau, \sigma)$	
$\text{tail}(x : \sigma) \rightarrow \sigma$	<i>stream functions</i>
$\text{inv}(x : \sigma) \rightarrow i(x) : \text{inv}(\sigma)$	
$i(\bullet) \rightarrow \bullet$	<i>data functions</i>

$T \Rightarrow \bullet : \bullet : \text{zip}(\text{tail}(T), \text{zip}(\text{inv}(T), \text{tail}(T)))$

This specification is **productive**.

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Data-Oblivious Productivity

Example (Data-Abstraction of the specification of Thue–Morse)

$T \rightarrow \bullet : \text{zip}(\text{inv}(T), \text{tail}(T))$	<i>stream constant</i>
$\text{zip}(x : \sigma, \tau) \rightarrow x : \text{zip}(\tau, \sigma)$	
$\text{tail}(x : \sigma) \rightarrow \sigma$	<i>stream functions</i>
$\text{inv}(x : \sigma) \rightarrow i(x) : \text{inv}(\sigma)$	
$i(\bullet) \rightarrow \bullet$	<i>data functions</i>

$T \rightsquigarrow \bullet : \dots$

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- Net visualization tool (Java applet) by Ariya Isihara

click&play: <http://infinity.few.vu.nl/productivity>



Productivity Prover *ProPro*

- ▶ Productivity Prover *ProPro* (Haskell-based) by Jörg Endrullis.
Use it at: <http://infinity.few.vu.nl/productivity>

A recent step towards an extension of *ProPro*:

- ▶ Niels Rademaker: 'Productivity of Some Stream Functions'
(CKI-bacheloreindwerkstuk, September 2009)

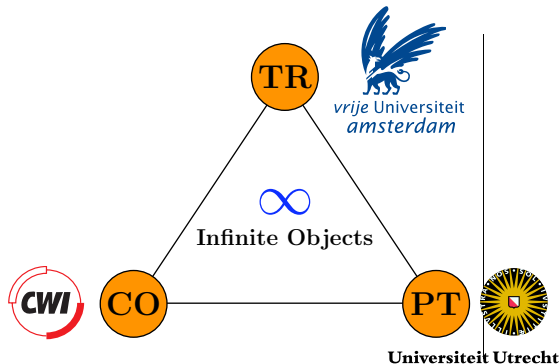


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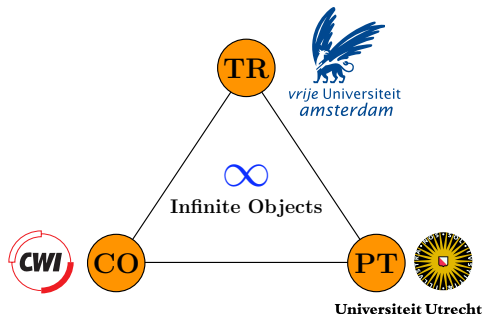
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NWO BRICKS Project *Infinity* (2006–2009)

- ▶ 3 postdocs for 3 years
- ▶ regular meetings every 3 weeks



UU Jeroen Ketema/C.G., Vincent van Oostrom, Albert Visser.

VU Dimitri Hendriks, Jan Willem Klop, Jörg Endrullis,
Roel de Vrijer, Ariya Isihara.

CWI Clemens Kupke/Helle Hansen, Jan Rutten.

Universiteit Utrecht



Beyond Infinity ...



Beyond Infinity Infinity + 2



Beyond Infinity Infinity + 2

NWO Projects (2009–2012):

UU Realising Optimal Sharing (ROS)
(van Oostrom, Swierstra, Dijkstra, G, ...)

VU Lazy Productivity (LaPro)
(Hendriks, Klop, et al)

