

From Abstract Rewriting Systems to Abstract Proof Systems

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Abstract. Some personal recollections on the introduction of ‘abstract proof systems’ as a framework for formulating syntax-independent, general results about rule derivability and admissibility. With a particular eye on the inspiration I owe to Roel de Vrijer: the analogy with abstract rewriting systems.

*For Roel de Vrijer with my best wishes
on the occasion of his 60th birthday*

1 Introduction

As a first aim¹ in my Ph.D. project as *AIO* at the VU-Amsterdam, Jan Willem Klop suggested to me to investigate the proof-theoretic relationships between three kinds of proof systems for recursive type equivalence: a Hilbert-style proof system AC by Amadio and Cardelli (1993), a coinductively motivated proof system BH by Brandt and Henglein (1998), and a proof system AK by Ariola and Klop (1995) for consistency-checking with respect to recursive type equivalence. While originally formulated as a sequent-style calculus, the system of Brandt and Henglein has a straightforward reformulation in natural-deduction style. The system of Ariola and Klop can be formulated as a tableau system.

The easiest kind of relationship turned out to hold between the Brandt–Henglein and Ariola–Klop systems: proofs in BH are, basically, mirror-images of ‘consistency unfoldings’ (successful finite consistency-checks, comparable to tableaux proofs) in AK.² The proof-theoretic relationships of both systems with the Amadio–Cardelli system, however, are considerably less straightforward. Although also in this case I found proof-transformations in both directions fairly soon, these transformations were quite complicated, and at least in part had the flavour of *ad hoc* solutions. The situation was theoretically unsatisfying insofar as a frequently encountered situation, at the time of constructing these proof-transformations, was the following. While having shown that a particular

¹ It turned out that work at this ‘first aim’ eventually developed into my thesis [4].

² Jan Willem Klop had observed this mirroring property in examples before, and pointed me to this phenomenon.

rule R of a proof system \mathcal{S}_1 can always be eliminated from a derivation in an extension \mathcal{S}'_2 of another system \mathcal{S}_2 by an effective procedure, I was not sure at all whether applying such an elimination-procedure was in fact essential. Perhaps instances of R could always be modeled by derivations in \mathcal{S}'_2 , and I only had failed to find some very easy demonstration of this fact. In other words, by having only shown ‘admissibility’ of R in \mathcal{S}'_2 , I had not excluded the possibility that R was actually ‘derivable’ in \mathcal{S}'_2 .

For this reason I got interested in the relationship between, on the one hand, the notions of rule derivability and rule admissibility, and on the other hand, the precise manner of how rules can be eliminated from derivations: either by ‘mimicking’ derivations or by elimination procedures. The notions of derivability and admissibility of inference rules date back at least to Kleene [7] (1952) and Lorenzen [8] (1955), and have since then been used for the analysis of concrete proof calculi, and, this predominantly concerns rule admissibility, of what kinds of inferences a particular semantically given logic admits³. Notwithstanding the familiarity of rule derivability and admissibility, the impression I got from the literature was that these notions are usually defined only for concrete proof systems, and not in an abstract way. Furthermore, I did not find definitions that applied to natural-deduction style systems directly (definitions which I could immediately have applied for the Brandt–Henglein system BH), nor a presentation that explains the general practical relevance of rule derivability and admissibility for *interpretational proof theory*, and that is, for ‘syntactical translations of one formal theory into the other’ [11].

However, I came across the definitions of rule derivability and admissibility in the book [6, p.70] Hindley and Seldin, and found it appealing because it is based on an abstract notion of proof system (‘formal system’). While this definition only applies to Hilbert-style proof systems, and while only basic properties are stated for rule derivability and admissibility in [6], two lemmas there (Lemma 6.14 and Lemma 6.15) stimulated me to think about definitions for natural-deduction style systems, and about how the mentioned lemmas could be extended.

In autumn 2002 I picked up again on my earlier triggered interest in rule derivability and admissibility, trying to investigate, in an abstract framework close to the one chosen by Hindley and Seldin, the relationship between derivability and admissibility of rules and the ways how rules can be eliminated from derivations. Having gathered a number of, elementary, results about derivability and admissibility in natural-deduction style proof systems, I gave two talks on this topic in the regular TCS-seminar of our theory group at the VU on Friday afternoons (January 24, and February 7, 2003). While finding my results interesting, I recall that Roel de Vrijer reacted strongly against the concrete formulation of rules as extensional ‘rule descriptions’ that I employed.

³ An impressive body of work concerning this question is the study of admissible rules in intuitionistic propositional logic (IPC) by, to mention only a few names, Friedman, Rasiowa, Harrop, Rybakov, Visser, de Jongh, Ghilardi, and Iemhoff.

Overview. In Section 2 Roel’s⁴ remarks about two ‘naive’ abstract notions of inference rule are explained, which had highlighted to me significant problems of those notions. Section 3 is concerned with the concept of ‘abstract rewriting system’ in the theory of rewriting, which Roel suggested me to look at. In Section 4 the definition of ‘abstract pure Hilbert system’ (APHS) is given and explained, which I formulated as a direct consequence of Roel’s remarks. In Section 5 derivability and admissibility of rules in APHSs are defined, and some results about these notions are stated. In Section 6 the idea of the notion of rules in ‘abstract natural-deduction systems’ is outlined. Finally in Section 7 a short summary is given, and an idea for further research is mentioned.

2 What is an inference rule, formally?

For this section heading I borrowed from the title ‘*What is an inference rule?*’ of the article [2] by Fagin, Halpern, and Vardi. There, the investigation concentrates on the question of what kinds of inferences semantically defined logics give rise to, and how these inferences can be classified even with little specific knowledge about the underlying semantics. Here, however, the focus is on the ways how rules are formulated operationally, and how abstract rule definitions can be obtained.

Inference rules in logical calculi are defined in a variety of ways. Here are just a few examples:

$$\begin{array}{c}
 \frac{A \rightarrow B \quad A}{B} \text{MP} \qquad \frac{[A]^u \quad \mathcal{D}_1}{\frac{B}{A \rightarrow B}} \rightarrow\text{I}, u \qquad \frac{A[x := t], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta} \text{L}\exists \\
 \\
 \frac{p_1 \xrightarrow{a} p_2}{p_1 + q \xrightarrow{a} p_2} \text{L+} \qquad \frac{\tau_1 = \tau[\alpha := \tau_1] \quad \tau_2 = \tau[\alpha := \tau_2]}{\tau_1 = \tau_2} \text{UFP}
 \end{array}$$

The rules in the first row are taken from proof calculi for, in any case, classical predicate logic: *modus ponens* in a Hilbert-system, the \rightarrow -introduction rule in a natural-deduction system, and the left- \exists -introduction rule in a sequent-style Gentzen system. The rule L+ in the second row is taken from a transition system specification (TSS) of a process algebra containing alternative composition $+$, and UFP is the unique-fixed-point rule in the Amadio–Cardelli system AC for recursive type equivalence.

As in the case of the examples above, rules in logical formalisms are usually defined as schemes using a meta-language of the formula language of the theory to be formalised such that the instances can be obtained by meta-level substitution. In the easiest case of a Hilbert-style system (that is ‘pure’, see Section 4) a rule R is usually represented by a scheme expression of the form

⁴ I hope to stay true to the spirit of this *liber amicorum* by switching to using only Roel de Vrijer’s first name from now on.

$A_1, \dots, A_n/A$ that consists of premise expressions A_1, \dots, A_n , and a conclusion expression A , where A_1, \dots, A_n, A are expressions in an extension of the formula language of the theory \mathcal{T} to be formalised with, for example, meta-variables for formulas and terms in \mathcal{T} , and notation for object-language substitution in \mathcal{T} . The instances of R are then defined to be the inferences of the form $(\bar{\sigma}(A_1))^*, \dots, (\bar{\sigma}(A_n))^*/(\bar{\sigma}(\sigma))^*$ where σ is a substitution that assigns formulas and terms of \mathcal{T} to meta-variables of according sort, $\bar{\sigma}$ is a homomorphic extension of σ to formula expressions including meta-variables, and where $(\cdot)^*$ is an operation that evaluates formula expressions containing substitution notation over the formula language of \mathcal{T} .

A precise formal introduction of an abstract notion of rule that takes the syntactic view on rule definition seriously (along considerations just sketched) seems to call for adopting concepts like the LF logical framework [5]. However, this was not the path on which I proceeded in work for my thesis. Then I decided to leave the syntactic view on rule definition out of consideration, and concentrate instead on abstract notions of inference rule that do not require syntactic restrictions to be imposed on the formula language.

An abstract definition of rule is given by Hindley and Seldin in [6] as ‘rule descriptions’ (called ‘rules’ in [6]): partial functions that map sequences of premise formulas to a single conclusion formula.

Definition 1. Let Fo be a set, and $n \in \mathbb{N}$. A *rule description* for an n -premise rule in a pure Hilbert system with set Fo of formulas is a partial function $\Phi : (Fo)^n \rightarrow Fo$. A rule description Φ over Fo describes the n -premise rule R_Φ on Fo whose *instances* are defined, for all $A_1, \dots, A_n, A \in Fo$, by:

$$\frac{A_1 \dots A_n}{A} \text{ is an instance of } R_\Phi \iff \Phi(A_1, \dots, A_n) = A.$$

In the first of my earlier mentioned talks in 2003, I used rule descriptions according to this definition, and even generalisations for describing rules in natural-deduction systems, to formulate results on rule derivability and admissibility. During my presentation Roel immediately noticed, and acutely remarked, the obvious drawback of rule descriptions according to Definition 1: they are not able to describe rules that allow more than one conclusion to be inferred from a given sequence of premises. Rules such as for example:

$$\frac{A}{A \vee B} \vee I_R \qquad \frac{\forall x A}{A[x := t]} \vee E$$

the \vee -introduction rule, and the \forall -elimination rule in natural-deduction systems for classical or intuitionistic logic.

Now it is an easy remedy to let rule descriptions for n -premise rules, in the case of pure Hilbert systems, be defined as functions $\Phi : (Fo)^n \rightarrow \mathcal{P}(Fo)$, and stipulate that such a rule description Φ defines the rule R_Φ with the property that $A_1, \dots, A_n/A$ is an instance of R_Φ if and only if $A \in \Phi(A_1, \dots, A_n)$. While this is the change I carried out for my second talk, it is perhaps more natural to drop the functional aspect in the formalisation of rule descriptions altogether, and view

rules as relations between premise formulas and a conclusion formula. Troelstra and Schwichtenberg have chosen such a formulation in [11] for the definition below of rules in ‘LR-systems’: systems with ‘local rules’ that correspond to pure Hilbert systems (for the latter see the explanation at the start of Section 4).

Definition 2. Let Fo be a set, and $n \in \mathbb{N}$. An n -premise *LR-system rule* on (formulas of) Fo is a set of sequences $\langle A_1, \dots, A_n, A \rangle$ in Fo of length $n + 1$. Elements $\langle A_1, \dots, A_n, A \rangle$ of such a rule are called *instances* and usually written as $A_1, \dots, A_n/A$.

While having carried out a change in the definition of rules with the effect of Definition 2, I vividly remember that Roel had a further objection⁵ to the changed definition: It is conceivable that a concrete syntactic definition of a rule allows two or more instances with the same sequence of premises and the same conclusion. There may be reasons to preserve the existence of such extensionally, but not intensionally, equivalent instances of rules.

A possible example is the following formulation of an \wedge -elimination rule:

$$\frac{A_1 \wedge A_2}{A_i} \wedge E \ (i \in \{1, 2\})$$

for a proof system for propositional or predicate logic. This rule has two intensionally different instances:

$$\frac{(x = 0) \wedge (x = 0)}{x = 0} \wedge E \qquad \frac{(x = 0) \wedge (x = 0)}{x = 0} \wedge E$$

whereas an LR-rule according to Definition 2 could only contain a single instance of the form $\langle x = 0, x = 0, x = 0 \rangle$.

However, it has to be remarked that the rule $\wedge E$ is usually split into the two parts $\wedge E_L$ and $\wedge E_R$ that as their instances have the instances of $\wedge E$ with i chosen as 1 or 2, respectively. Furthermore, it seems to me that logicians usually take care to avoid rules that exhibit a behaviour similar as the \wedge -elimination rule above. At least I am presently unaware of a comparable example from ‘real-life’ proof-theory. Being interested to hear from others, I formulate the following question.

Question 3. Do there exist generally accepted proof calculi that contain rules with instances that are extensionally, but not intensionally, equivalent?

I think that researchers in proof theory working on ‘deep inference’ calculi, in which rules are allowed to manipulate the syntactic structure of expressions not only at their outer symbols but also inside of contexts (therefore these proof systems possess a TRS-like feature), could provide an answer quickly.

Roel’s second objection seemed to call for an entirely different framework for abstract rule definitions. I took his suggestion to heart to compare the situation with the definition of ‘abstract rewriting systems’, which had proved to be very useful in the theory of rewriting.

⁵ As far as I remember, this must have been in a discussion days after my second talk.

3 Abstract rewriting systems

Rewriting theory knows many kinds of rewriting systems such as λ -calculus, term rewriting systems (TRSs), combinatory reduction systems (CRSs), higher-order rewriting systems (HRSs), graph rewriting systems, and more. In addition to the development of theory that is specific to each kind of rewriting system, it turned out to be very useful to develop also an abstract rewriting theory, in which the concrete form or structure of objects in a particular rewriting system is not taken into account. It turns out that even in a fully abstract framework interesting statements can be formulated, for example, about how properties of the reflexive-transitive closure \rightarrow^* of a rewrite relation \rightarrow follow from properties of \rightarrow , or from properties of the set of rewrite steps that induces \rightarrow .

Among the first concepts in abstract rewriting theory were the results for ‘replacements systems’ by Staples (1975), and for ‘abstract reduction systems’ by Klop (1980). In their most simple formulation, the abstract reductions systems [10] are defined as follows.

Definition 4. An *abstract reduction system* is a structure $\langle A, \rightarrow \rangle$ consisting of a set A with a binary *reduction relation* \rightarrow on A .

In a more general definition of ‘abstract rewriting system’ (see also [10]), this notion denotes a system $\langle A, \{\rightarrow_\alpha\}_{\alpha \in I} \rangle$ consisting of a set and a family of binary reduction relations on A . However, the results in [10, Ch. 1] bear witness to the fact that even the simple version of abstract reduction system is a very fruitful concept that leads to many interesting results, which due to their abstract character can be applied to rewrite systems in general.

But abstract rewriting systems in the simple form as in Definition 4 have an evident drawback: they do not allow to distinguish between two different rewrite steps from an object a to an object b .

Example 5. Consider the TRS \mathcal{T} with as single rule $f(x) \rightarrow x$ and with the induced rewrite relation $\rightarrow_{\mathcal{T}}$ on the set Ter of terms over the signature $\Sigma = \{f\}$. In this TRS, the term $f(f(a))$ contains two redexes, which give rise to the steps:

$$f(f(a)) \rightarrow f(a) \qquad f(f(a)) \rightarrow f(a) ,$$

respectively. But in the extensional description of \mathcal{T} as an abstract reduction system $\langle Ter, \rightarrow_{\mathcal{T}} \rangle$, these steps cannot be distinguished, since both are witnessed by the same fact: $f(f(a)) \rightarrow_{\mathcal{T}} f(a)$. This phenomenon is called a *syntactic accident* (J.J. Lévy).

Preservation of the identity of rewrite steps under projecting to an abstract formalism is essential for proving abstract results on e.g. tracing steps under rewrite sequences, or on residuals of steps under other steps. This calls for an abstract framework in which ‘steps are first-class citizens’ (van Oostrom), and where rewrite relations are only secondary notions. Such a framework is that of ‘abstract rewriting systems’, which date back to Newman [9], and in the formulation by van Oostrom and Roel de Vrijer [12, 10] are defined as follows.

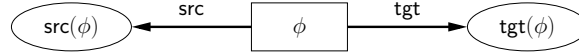


Figure 1: Visualisation as a hypergraph hyperedge of a step $\phi \in \Phi$ in an abstract rewriting system $\mathcal{A} = \langle A, \Phi, \text{src}, \text{tgt} \rangle$ with source formula $\text{src}(\phi) \in A$ and target formula $\text{tgt}(\phi) \in A$.

Definition 6. An *abstract rewriting system (ARS)* is a quadruple of the form $\mathcal{A} = \langle A, \Phi, \text{src}, \text{tgt} \rangle$ where:

- A and Φ are sets, the set of *objects*, and the set of *steps*, of \mathcal{A} , respectively;
- $\text{src}, \text{tgt} : \Phi \rightarrow A$ are functions, the *source* and the *target function* of \mathcal{A} .

(See Figure 1 for a visualisation a step in an ARS as a hyperedge for a hypergraph.)

Newman [9] attempted⁶ to show the confluence property of the λI -calculus by using abstract rewriting theory in systems comparable to abstract rewriting systems. An example of an important result that is based on abstract rewriting systems is the theory of ‘abstract residual systems’ [10, Ch. 8], which allows to give uniform proofs for results in the theory of residuals in orthogonal first-order and higher-order term rewriting systems (TRSs and HRSs).

I think that it is likely that Roel’s work as a co-author of [12] and his familiarity with abstract rewriting systems have informed his remarks to me about the problems with abstract definitions of rules explained in the previous section.

4 Abstract pure Hilbert systems

In this section the definition of ‘abstract pure Hilbert system’ will be given, which I developed starting in [3] and then in my thesis [4] as a reaction to and a direct consequence of Roel’s remarks as explained in Section 2. Before turning to that definition, the concept of ‘pure Hilbert system’ needs to be explained.

In a strict interpretation of this concept, a ‘Hilbert system’ is a proof system for classical, intuitionistic, or minimal predicate (or propositional) logic that consists of axiom schemes, and of the single rule *modus ponens* (MP):

$$\frac{A \rightarrow B \quad A}{B} \text{ MP}$$

In a more liberal interpretation, a Hilbert system may contain arbitrary rules ρ whose instances are of the form:

$$\frac{A_1 \quad \dots \quad A_n}{A} \rho \tag{1}$$

⁶ It is pointed out in [12, cf. Rem. 6.14, (ii)] that Newman’s attempt actually fails because for showing confluence of the λI -calculus he uses statements that depend on abstract rewriting properties which do not hold for the λI -calculus.

A Hilbert system \mathcal{H} (in the liberal sense) is called ‘pure’ if rule applications within derivations in \mathcal{H} do not depend on the presence or absence of assumption occurrences in the immediate subderivations. More precisely, the following must hold: Suppose that ρ is a rule in a Hilbert system \mathcal{H} . And suppose that, for some formulas A, A_1, \dots, A_n of \mathcal{H} , (1) is an instance of ρ . Furthermore, let $\mathcal{D}_1, \dots, \mathcal{D}_n$ be derivations in \mathcal{H} (which may contain unproven assumptions) with conclusions A_1, \dots, A_n , respectively. Then the proof tree \mathcal{D} of the form:

$$\frac{\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \dots & A_n \end{array} \quad \rho}{A}$$

is a derivation in \mathcal{H} , irrespectively of whether or not assumptions occur in $\mathcal{D}_1, \dots, \mathcal{D}_n$, and how these do actually look like.

Typical examples of pure Hilbert systems are Hilbert systems in the strict sense for propositional or predicate logic. Examples of non-pure Hilbert systems are the well-known proof systems for modal logics such as **K**, **T**, **S4**, and **S5**, which contain the *necessitation rule* (NR) with instances of the form:

$$\frac{A}{\Box A} \text{ NR} \tag{2}$$

The applicability of NR in derivations of these proof systems is restricted to situations in which the immediate subderivation of the premise does not contain any unproven assumptions (and hence this premise must be a theorem). Therefore instances of NR of the form (2) are subject to a side-condition on the absence of assumptions in the immediate subderivation of the premise, and hence Hilbert systems including this rule are impure, that is, not pure.

In the stipulation for the property ‘pure’ of Hilbert systems I followed an analogous stipulation that Avron uses in [1] for ‘Hilbert-type systems for consequence’. In [1] a nice characterisation of Hilbert systems and natural-deduction systems among all sequent-style Gentzen systems is given. In Avron’s terminology, pure Hilbert systems as explained above correspond⁷ to pure, single-conclusioned Hilbert-type systems for consequence, which formally are sequent-style systems.

Pure Hilbert systems are among the easiest kinds of proof systems. Therefore it was natural for me to start with formulating an abstract version of these systems, before later going on to define abstract versions of systems of natural deduction. The concept of ‘abstract pure Hilbert system (APHS)’, which will be defined below, is analogous to abstract rewriting systems in the sense that steps between objects (here, inference steps between formulas) are central, not the relations between objects (but these relations are induced by the steps). However, somewhat different from the definition of ARSs, in which the notion of a (rewrite) rule does not figure at all, in the concept of an APHS the distinction between different inference rules is retained. The foremost reason for this was my wish to let derivations in APHSs closely resemble proof trees in an usual sense with rule names displayed next to the derivations’ inferences.

⁷ This correspondence is made explicit and proved in Appendix E of [3].

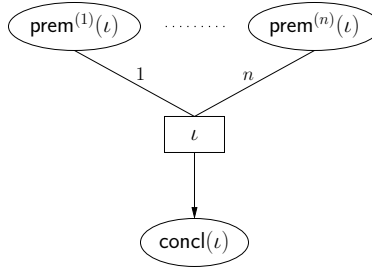


Figure 2: Visualisation as hypergraph hyperedge of an instance ι of an APHS-rule with arity $\text{arity}(\iota) = n$.

As a consequence it is a prerequisite for the definition of APHSs (see Definition 8 below) to introduce first the concept of ‘unnamed APHS-rule’. Here, the following notation will be used: For all sets X , by $\text{Seqs}_f(X)$ the set of all finite-length sequences of elements of X is meant; the empty sequence is denoted by $\langle \rangle$, and n -element sequences in $\text{Seqs}_f(X)$ are written as $\langle x_1, \dots, x_n \rangle$, where $x_1, \dots, x_n \in X$.

Definition 7. Let Fo be a set. An *unnamed APHS-rule* (an unnamed rule for an ‘abstract pure Hilbert system’) on (the formulas of) Fo is a triple $R = \langle \text{Insts}, \text{prem}, \text{concl} \rangle$ where:

- Insts is a set whose elements are called the *instances* of R ,
- $\text{prem} : \text{Insts} \rightarrow \text{Seqs}_f(Fo)$ and $\text{concl} : \text{Insts} \rightarrow Fo$ are the *premise function* and the *conclusion function* of R , respectively.

For all sets Fo , by $\mathcal{R}(Fo)$ the class of unnamed APHS-rules on Fo is denoted.

Note that unnamed APHS-rules are allowed to have applications with different arities. In addition to the functions prem and concl associated with a rule, for an unnamed APHS-rule $R = \langle \text{Insts}, \text{prem}, \text{concl} \rangle$

$$\begin{array}{l} \text{the function} \quad \text{arity} : \text{Insts} \rightarrow \mathbb{N} \quad \quad \quad \text{and} \\ \text{the partial functions} \quad \text{prem}^{(i)} : \text{Insts} \rightarrow Fo \quad (\text{for all } i \in \mathbb{N}) \end{array}$$

are defined as follows: arity assigns to an instance ι of R the number of its premises, that is, the length of the sequence $\text{prem}(\iota)$. For all $i \in \mathbb{N}, i \geq 1$, $\text{prem}^{(i)}(\iota)$ assigns to an instance ι of R its i -th premise, that is, the i -th formula A_i in $\text{prem}(\iota) = \langle A_1, \dots, A_i, \dots \rangle$ if it exists; otherwise, $\text{prem}^{(i)}(\iota)$ is undefined.

Using these definitions, a visualisation of an instance $\iota \in \text{Insts}$ with $\text{arity}(\iota) = n$ of an unnamed APHS-rule $R = \langle \text{Insts}, \text{prem}, \text{concl} \rangle$ as a hypergraph hyperedge is given in Figure 2. In a proof tree such an instance can be written as:

$$\frac{\text{prem}^{(1)}(\iota) \quad \dots \quad \text{prem}^{(n)}(\iota)}{\text{concl}(\iota)}$$

While rules *for* APHSs do not carry names, rules *in* abstract pure Hilbert systems as defined below *do* carry names.

Definition 8. An *abstract pure Hilbert system* (an *APHS*) \mathcal{H} is a quadruple $\langle Fo, Na, nAx, nRu \rangle$ where:

- Fo, Na, nAx and nRu are sets whose elements are called the *formulas* of \mathcal{H} , the *names* (for axioms and rules) *in* \mathcal{H} , the *named axioms* of \mathcal{H} , and the *named rules* of \mathcal{H} , respectively; we demand $Fo \neq \emptyset$, i.e. that the formula set be nonempty;
- $nAx \subseteq Fo \times Na$, i.e. the named axioms of \mathcal{H} are tuples with formulas of \mathcal{H} as first, and names in \mathcal{H} as second, components;
- $nRu \subseteq \mathcal{R}(Fo) \times Na$, i.e. the named rules of \mathcal{H} are tuples that have unnamed APHS-rules on Fo as their first, and names as their second components;
- for the named axioms and the named rules of \mathcal{H} the following holds:
 - (i) names of named axioms in nAx are different from names of named rules,
 - (ii) different named rules in nRu carry different names (but for an unnamed APHS-rule R and different names $na_1, na_2 \in Na$, it is possible that $\langle R, na_1 \rangle, \langle R, na_2 \rangle \in nRu$).

This definition of APHSs, which is essentially the formulation in my thesis [4], is actually a slight reformulation of the notion of ‘abstract Hilbert system with names for axioms and rules’ (n-AHS) in [3] after discussions with Roel in my last year as Ph.D. student, when I was writing down my thesis. In [3] an n-AHS is defined as a quintuple $\langle Fo, Ax, Ru, Na, \text{name} \rangle$ consisting of a set Fo of formulas, a set $Ax \subseteq Fo$ of axioms, a set $Ru \subseteq \mathcal{R}(Fo)$ of unnamed APHS-rules, a set Na of names, and a name function $\text{name} : Ru \rightarrow Na$. Roel remarked that this use of a name function on a set of unnamed APHS-rules prevents the possibility that the same unnamed APHS-rule could occur twice, under different names, in an APHS. He convinced me that the formalisation should not a priori exclude systems in which an initially unnamed rule occurs under more than one name.

Let $\mathcal{H} = \langle Fo, Na, nAx, nRu \rangle$ be an APHS. In proof trees, an instance $\iota \in Insts_R$ of an unnamed rule R such that $\langle R, na \rangle \in nRu$, is written as:

$$\frac{\text{prem}^{(1)}(\iota) \quad \dots \quad \text{prem}^{(n)}(\iota)}{\text{concl}(\iota)} \quad na$$

A *derivation in* \mathcal{H} is defined as a proof tree in the sense of [11]: a tree in which the nodes are labelled by formulas and in which the edges make part of rule instances and are not drawn, but are replaced by horizontal lines that represent rule instances and carry the name of the rule that is applied. Axioms and assumptions appear as top nodes; lower nodes are formed by applications of rules; the bottommost formula is the conclusion. If \mathcal{D} is a derivation in \mathcal{H} , then by $\text{assm}(\mathcal{D})$ the set of assumptions of \mathcal{D} , and by $\text{concl}(\mathcal{D})$ the conclusion of \mathcal{D} will be denoted. Let R be an APHS-rule on the formula set of \mathcal{H} . A derivation \mathcal{D} in \mathcal{H} is called a *mimicking derivation* for an instance ι of R if $\text{assm}(\mathcal{D}) \subseteq \text{set}(\text{prem}(\iota))$ and $\text{concl}(\mathcal{D}) = \text{concl}(\iota)$.

The (usual, standard) consequence relation $\vdash_{\mathcal{H}}$ in an APHS \mathcal{H} with formula set Fo is a binary relation between sets of formulas and formulas of \mathcal{H} . For all formulas $A \in Fo$, and sets $\Sigma \subseteq Fo$ of formulas, in \mathcal{H} , $\Sigma \vdash_{\mathcal{H}} A$ holds if there is a derivation \mathcal{D} in \mathcal{H} such that the assumptions of \mathcal{D} are contained in Σ , and A is the conclusion of \mathcal{D} . By $\vdash_{\mathcal{H}} A$ the statement $\emptyset \vdash_{\mathcal{H}} A$ is abbreviated. A formula A is a *theorem* of an \mathcal{H} if $\vdash_{\mathcal{H}} A$ holds, that is, if there is a derivation in \mathcal{H} without assumptions that has A as its conclusion.

Let \mathcal{H} be an APHS, R an unnamed APHS-rule over the set of formulas of \mathcal{H} , and $\hat{R} = \langle R, na \rangle$ a named version of R . Then the result of adding \hat{R} to the named rules of \mathcal{H} is denoted by $\mathcal{H} + \hat{R}$ given that it is an APHS.

5 Rule derivability and admissibility in APHSs

In this section the definitions of derivability and admissibility of arbitrary APHS-rules in an APHS are introduced, based on the stipulations at the end of the previous section. Furthermore, basic results about these notions are stated, together with results that I have obtained in [3] and [4].

In the definitions below of rule derivability and admissibility in APHSs the following notation will be used: for a set X and a sequence $\sigma \in Seq_f(X)$, $set(\sigma)$ denotes the set of elements of X that occur in σ .

Let \mathcal{H} be an abstract Hilbert system, and let $R = \langle Insts, prem, concl \rangle$ an unnamed APHS-rule over the set of formulas of \mathcal{H} . Then R is called *derivable in \mathcal{H}* if:

$$\text{for all } \iota \in Insts: set(prems(\iota)) \vdash_{\mathcal{H}} concl(\iota). \quad (3)$$

R is called *correct for \mathcal{H}* if:

$$\text{for all } \iota \in Insts: [(\text{for all } A \in set(prems(\iota)): \vdash_{\mathcal{H}} A) \implies \vdash_{\mathcal{H}} concl(\iota)]. \quad (4)$$

And R is stipulated to be *admissible in \mathcal{H}* if every (or equivalent: one) extension $\mathcal{H} + \hat{R}$ of \mathcal{H} by adding a named version \hat{R} of R has the same theorems as \mathcal{H} . A named rule $\hat{R} = \langle R, na \rangle$ is derivable/correct/admissible in \mathcal{H} if the underlying unnamed rule R is derivable/correct/admissible in \mathcal{H} , respectively.

Each of these definitions can be reformulated in terms of ‘mimicking derivations’. For example, an unnamed APHS-rule R is derivable in an APHS \mathcal{H} if and only if for every instance of R there exists a mimicking derivation in \mathcal{H} .

The following proposition gathers some of the most basic properties of rule derivability, correctness, and admissibility, and of their interrelations. Items (i)–(iii) of this statement are a reformulation for APHSs of a lemma by Hindley and Seldin (Lemma 6.14 on p. 70 in [6]). Item (iv) is taken from [3] (see Theorem 3.5 on p. 18–19 there).

Proposition 9. *Let \mathcal{H} be an APHS, and let R be an unnamed APHS-rule on the set of formulas of APHS. Then the following statements hold:*

- (i) *R is admissible in \mathcal{H} if and only if R is correct for \mathcal{H} .*

- (ii) *If R is derivable in \mathcal{H} , then R is admissible in \mathcal{H} . But the converse implication does not hold in general.*
- (iii) *If R is derivable in \mathcal{H} , then R is derivable in every extension of \mathcal{H} by adding new formulas, new axioms, and/or new rules.*
- (iv) *R is derivable in \mathcal{H} if and only if R is admissible in every extension of \mathcal{H} by adding new formulas, new axioms, and/or new rules.*

Further results that I have obtained in [3] include the following:

1. Let \mathcal{H} be an APHS, R an unnamed APHS-rule, and $\mathcal{H}+\hat{R}$ an extension of \mathcal{H} by adding a named version $\hat{R} = \langle R, na \rangle$ of R . If R is admissible in \mathcal{H} , then every derivation \mathcal{D} in $\mathcal{H}+\hat{R}$ without assumptions can be replaced by a mimicking derivation \mathcal{D}' in \mathcal{H} (without assumptions). If R is derivable in \mathcal{H} , then every derivation \mathcal{D} in $\mathcal{H}+\hat{R}$ can be replaced by a mimicking derivation \mathcal{D}' in \mathcal{H} , which moreover can be found by stepwise replacements of \hat{R} -instances in \mathcal{D} by mimicking derivations in \mathcal{H} .
2. Let $\mathcal{H}_1, \mathcal{H}_2$ be APHSs that possess the same set of formulas. The following statements are equivalent with the statement that \mathcal{H}_1 and \mathcal{H}_2 have the same admissible rules: (i) the rules of \mathcal{H}_1 are admissible in \mathcal{H}_2 , and vice versa, and (ii) \mathcal{H}_1 and \mathcal{H}_2 have the same theorems. Statements equivalent with the statement that \mathcal{H}_1 and \mathcal{H}_2 have the same derivable rules are: (i)' the rules of \mathcal{H}_1 are derivable in \mathcal{H}_2 , and vice versa, and (ii)' \mathcal{H}_1 and \mathcal{H}_2 induce the same consequence relation.
3. Let \mathcal{H} be an APHS, R an unnamed APHS-rule, and $\mathcal{H}+\hat{R}$ an extension of \mathcal{H} by adding a named version $\hat{R} = \langle R, na \rangle$ of R . If R is derivable in \mathcal{H} , then \hat{R} -elimination for derivations in $\mathcal{H}+\hat{R}$ can be performed effectively: For a given derivation \mathcal{D} in $\mathcal{H}+\hat{R}$, pick an arbitrary instance of \hat{R} in the derivation and replace it, in the derivation, by a mimicking derivation; carry out such *mimicking steps* repeatedly until no further applications of \hat{R} are present, and a derivation \mathcal{D}' in \mathcal{H} has been reached. This nondeterministic procedure is strongly normalising.

Apart from the notions of rule derivability and admissibility defined above that are based on the standard notion of consequence relation, in [3] two variant notions are studied that refer to variant consequence relations. For these variant notions similar results are obtained; their interconnections with each other and with the standard notions are studied with the result of ‘interrelation prisms’.

6 Abstract natural-deduction systems

In this section the notion of rule in an ‘abstract natural-deduction system’ (ANDS) is only hinted at. For the details I refer to Chapter 4 and Appendix B of my thesis [4].

Rules in natural-deduction style proof systems typically have a more complex form than rules in Hilbert systems: An instance ι of a rule in a natural-deduction system is usually not exclusively determined by a sequence of premises and a

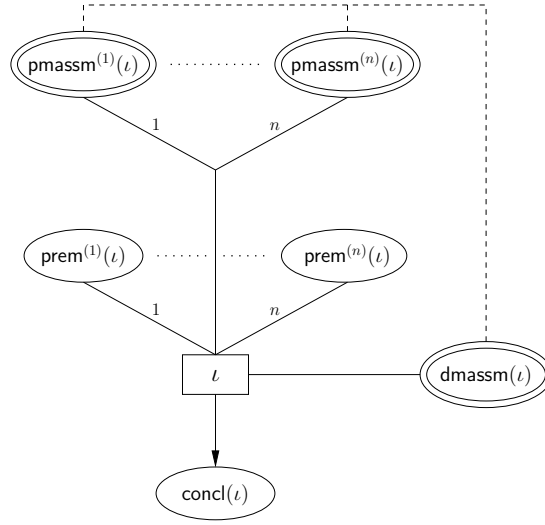


Figure 3: Visualisation as a hypergraph hyperedge of an instance ι with arity $\text{arity}(\iota) = n$ of an ANDS-rule.

conclusion, but its description frequently also involves, per premise, a set of assumptions that has to be present, and a number of assumptions that may be, or in fact are, discharged at ι . Instead of assumption formulas, assumptions in natural-deduction systems are often formalised as formulas with decorating markers that are used to single out those marked assumptions which are discharged at an instance.

Figure 3 contains an illustration of an instance of an unnamed rule for an ‘abstract natural-deduction system’ as defined in [4]: apart from premise and conclusion functions, unnamed ANDS-rules also contain functions **pmassm** and **dmassm** that map instances to their sequences of *present marked assumptions* per premise, and to their *discharged marked assumptions*, respectively.

In a sequent-style representation, a typical instance ι of an ANDS-rule, one with $\text{arity}(\iota) = n$ as depicted in Figure 3, can be written in the following form:

$$\frac{\text{pmassm}^{(1)}(\iota) \Rightarrow \text{prem}^{(1)}(\iota) \quad \dots \quad \text{pmassm}^{(n)}(\iota) \Rightarrow \text{prem}^{(n)}(\iota)}{(\bigcup_{i=1}^n \text{pmassm}^{(i)}(\iota)) \setminus \text{dmassm}(\iota) \Rightarrow \text{concl}(\iota)} \quad (5)$$

For the precise formal definitions of rule derivability and admissibility in ANDSs some care is needed. The situation is considerably less straightforward than in APHSs, and I refer to [4] for the details. I want to mention, however, that the definitions of admissibility and derivability of ANDS-rules in ANDSs can be obtained in the following way: (i) by considering sequent-style representations of ANDS-rules with instances of the form (5) as APHS-rules on sequents as formulas, and by considering sequent-style representations of entire ANDSs as

APHSs with sequents as their formulas, (ii) by applying the definitions of rule derivability and admissibility in APHSs to these APHS-rules and APHSs, and (iii) by transferring the resulting conditions back to ANDSs. That is to say, rule derivability and admissibility in ANDSs can be defined by applying the definitions in APHSs of rule derivability and admissibility for APHS-rules on sequents that represent ANDS-rules with respect to APHSs (with sequents as formulas) that represent ANDSs.

In my thesis, the definition of rule derivability and admissibility for natural-deduction systems proved to be useful in the manner indicated in the Section 1. In essentially all relevant cases of rules R of the Amadio–Cardelli system AC whose status in BH earlier seemed doubtful to me, I succeeded in proving that R is admissible, but not derivable in the Brandt–Henglein system BH; and consequently, that instances of R cannot just be simulated in BH by mimicking derivations, but have to be eliminated in another, likely more complicated, way. Having earlier obtained elimination procedures for instances of such rules R from BH-derivations, I at least obtained some certainty that I had not overlooked an obvious way to mimic R -instances by derivations in BH.

7 Summary, a research idea, and thanks

Summary. By work for my Ph.D. thesis on proof-theoretic interpretations into each other of proof systems for recursive types I became aware of the general relevance of rule derivability and admissibility for interpretational proof theory. Due to the fact that rule derivability and admissibility are usually only defined for concrete proof systems, and since the definitions of these notions in natural-deduction style systems had not been clear to me from the outset, I started a study of general properties of these notions in abstractly viewed proof systems (both Hilbert and natural-deduction style). Formal definitions of abstract notions of inference rule such as extensional rule descriptions, on which I had based this study in my early attempts, were criticized by Roel as inadequate and, most of all, as conceptually unsatisfiable. Stimulated by Roel’s remarks and his suggestion to try to draw inspiration from the concept ‘abstract rewriting system’ in rewriting theory, I formulated the frameworks of APHSs and ANDSs for abstract Hilbert-style and natural-deduction style proof systems. In these systems, rules are treated as sets of (abstract) inference steps rather than as mere relations between formulas, thereby keeping some information about how concrete rules are defined intensionally. In the sequel I used the concepts of APHSs and ANDSs to state general results about derivability and admissibility of rules in pure Hilbert systems, and in natural-deduction systems.

A research idea. The frameworks of APHSs and ANDSs are based on abstract concepts of rules that do not make use (at least not a priori) of special assumptions on the formula language. While studying rule derivability and admissibility in these kinds of abstract proof systems was sufficient for my particular purposes, I think that it would also be fruitful to carry out a similar investigation

in abstract proof systems that are based on syntactic rule concepts, perhaps formalised within a ‘logical framework’ such as LF. In particular, it would be interesting to see what kind of additional results about derivability and admissibility become possible once the syntactic manners in which inference rules are usually defined is exploited systematically. In the setting of proof systems that are formalised within a logical framework, an idea for research would be to view elimination procedures for inference rules as higher-order rewriting systems, and to try to take advantage of the well-developed theory for higher-order rewriting.

Thanks. I want to conclude by expressing my gratitude to Roel for his critical remarks about my earlier use of extensional rule descriptions, his reference to abstract rewriting systems, and the help he gave to me in regular discussions about the notions of abstract proof systems during my last year as an *AIO* at the Vrije Universiteit, Amsterdam.

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