Lecture 2: Graph width notions, dynamical programming

An Introduction to Parameterized Complexity

https://clegra.github.io/paracompl/paracompl.html

Clemens Grabmayer

Ph.D. Program, Advanced Period

Gran Sasso Science Institute

L'Aquila, Italy

Tuesday, July 15, 2025

Course overview

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
Algorithmic Techniques		*Formal-Method & Algorithmic Techniques*		
Introduction & basic FPT results	**Notions of bounded graph width**	**Algorithmic Meta-Theorems**	**FPT-Intractability Classes & Hierarchies**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour

Overview

- comparing parameterizations
- dynamical programming on trees, example:
  - WEIGHTED-INDEPENDENT-SET (and VERTEX-COVER)
- path-width
  - example: fpt-algorithm for bounded path-width
- tree-width
  - example: fpt-algorithm for bounded path-width
- fpt-results for other problems, obtained similarly
- other notions of width
  - clique-width
  - using f-width to define:
    - carving-width (and cut-width)
    - branch-width
    - rank-width
- comparing width-notions

Fixed-Parameter tractable

A *parameterized problem* is a triple $\langle Q, \Sigma, \kappa \rangle$ (short: $\langle Q, \kappa \rangle$) where:

- $Q \subseteq \Sigma^*$ is the set of *(classical) problem instances*,
- $\kappa : \Sigma^* \to \mathbb{N}$ is a (general) function, *the parameterization*.

Parameterized problem $\langle Q, \Sigma, \kappa \rangle$
Instance: $x \in \Sigma^*$.
Parameter: $\kappa(x)$.
Problem: Is $x \in Q$?

Fixed-Parameter tractable

A *parameterized problem* is a triple $\langle Q, \Sigma, \kappa \rangle$ (short: $\langle Q, \kappa \rangle$) where:

- ▷ $Q \subseteq \Sigma^*$ is the set of *(classical) problem instances*,
- ▷ $\kappa : \Sigma^* \to \mathbb{N}$ is a (general) function, *the parameterization*.

Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *fixed-parameter tractable* (is in FPT) if:

$$\exists f : \mathbb{N} \to \mathbb{N} \text{ computable } \exists p \in \mathbb{N}[X] \text{ polynomial}$$
$$\exists \mathbb{A} \text{ algorithm, takes inputs in } \Sigma^*$$
$$\forall x \in \Sigma^* \Big[\, \mathbb{A} \text{ decides whether } x \in Q \text{ holds in time } \leq f(\kappa(x)) \cdot p(|x|) \,\Big]$$

†) Assumptions for a robust fpt-theory

$\kappa(x)$ is polynomially computable, or itself fpt-computable: for all $x \in \Sigma^*$ in time $\leq g(\kappa(x)) \cdot q(|x|)$ for g computable, $q \in \mathbb{N}[X]$.

Comparing parameterizations

Definition (computably bounded below)

Let $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ parameterizations.

- $\kappa_1 \succeq \kappa_2 \;:\Longleftrightarrow\; \exists g : \mathbb{N} \to \mathbb{N}$ computable $\forall x \in \Sigma^* \big[\, g(\kappa_1(x)) \geq \kappa_2(x) \,\big]$.
- $\kappa_1 \approx \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \kappa_2 \succeq \kappa_1$.
- $\kappa_1 \succ \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \neg\big(\kappa_2 \succeq \kappa_1\big)$.

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ with $\kappa_1 \succeq \kappa_2$:

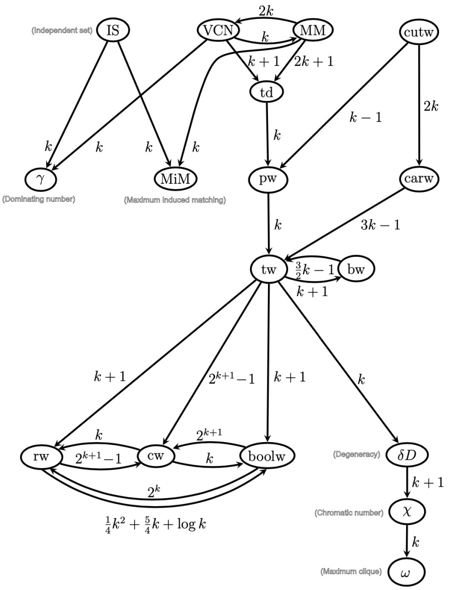
$$\langle Q, \kappa_1 \rangle \in \text{FPT} \quad \Longleftarrow \quad \langle Q, \kappa_2 \rangle \in \text{FPT}$$
$$\langle Q, \kappa_1 \rangle \notin \text{FPT} \quad \Longrightarrow \quad \langle Q, \kappa_2 \rangle \notin \text{FPT}$$

Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \geq wd_2 \;:\Leftrightarrow\; wd_1 \xrightarrow{g} wd_2$$

- FPT-results
  transfer upwards
  (and conversely to $\xrightarrow{g}$)
- ($\notin$ FPT)-results
  transfer downwards
  (and along $\xrightarrow{g}$)



You Always Walk Alone (with your children)

Attività motoria con i figli:

'la possibilità di uscire con i figli minori è consentita a un solo genitore per camminare purché questo avvenga in prossimità della propria abitazione'

(Ministero dell'Interno)

PHYSICAL-DISTANCE-WALKING

Instance: Graph $\mathcal{G} = \langle V, E \rangle$ with V people who want to go for a walk in the next hour in a radius of 200m of their home, and edges in E between them if they live closer than 400m of each other. A number $\ell \in \mathbb{N}$.

Problem: Is it possible that ℓ or more people can go out in the next hour so that everybody walks alone (with their children)?

corresponds to: INDEPENDENT-SET

Weighted Independent Set, and Vertex Cover

Let $\mathcal{G} = \langle V, E \rangle$ a graph. For all $S \subseteq V$:

S is independent set in $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \in E \, (\neg(u \in S \wedge v \in S) \,)$
$\Longleftrightarrow \forall e = \{u, v\} \in E \, (u \notin S \vee v \notin S) \,)$

WEIGHTED-INDEPENDENT-SET
Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.
Problem: What is the max. weight of an independent set of $\mathcal{G}$?

S is a vertex cover of $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \in E \, (u \in S \vee v \in S))$
$\Longleftrightarrow \forall e = \{u, v\} \in E \, (u \notin V \setminus S \vee v \notin V \setminus S) \,)$
$\Longleftrightarrow V \setminus S$ is an independent set of $\mathcal{G}$

VERTEX-COVER
Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and $\ell \in \mathbb{N}$.
Problem: Does $\mathcal{G}$ have a vertex cover of size at most ℓ?

$S \subseteq V$ is *minimal* vertex cover $\Longleftrightarrow$ $V \setminus S$ is *maximal* independent set
Hence: solution of WEIGHTED-INDEPENDENT-SET
$\Longrightarrow$ solution of VERTEX-COVER.

Weighted Ind. Set / Vertex Cover, width-parameterized

p^*-WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

Parameter: path-width / tree-width k.

Problem: What is the max. weight of an independent set of $\mathcal{G}$?

p^*-VERTEX-COVER

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and $\ell \in \mathbb{N}$.

Parameter: path-width / tree-width k.

Problem: Does $\mathcal{G}$ have a vertex cover of size at most ℓ ?

Dynamical programming on trees (example)

WEIGHTED-INDEPENDENT-SET
Instance: A tree $\mathcal{T} = \langle T, F \rangle$, and a weight function $\boldsymbol{w} : T \to \mathbb{R}_0^+$.
Problem: What is the max. weight of an independent set of $\mathcal{T}$?

Obtain a directed tree $\mathcal{T} = \langle T, F, r \rangle$ (pick a root r, orient edges away).

- $A[v] :=$ max. weight of an independent set in subtree $\mathcal{T}_v$ at v,
- $B[v] :=$ max. weight of an ind. set in $\mathcal{T}_v$ that does not contain v.

Computation of $A[v]$ and $B[v]$:

- in leafs: $B[v] = 0\,, \qquad A[v] = \boldsymbol{w}(v)\,.$
- for inner vertices v with children $v_1, \ldots, v_q$:

$$B[v] = \sum_{i=1}^{q} A[v_i]\,, \qquad A[v] = \max\Big\{B[v], \boldsymbol{w}(v) + \sum_{i=1}^{q} B[v_i]\Big\}\,.$$

Solution: value of $A[r]$, can be computed bottom-up in linear time.

Dynamical programming on trees (example)

WEIGHTED-INDEPENDENT-SET

Instance: A tree $\mathcal{T} = \langle T, F \rangle$, and a weight function $w : T \to \mathbb{R}_0^+$.

Problem: What is the max. weight of an independent set of $\mathcal{T}$?

Theorem

On trees with n nodes,

WEIGHTED-INDEPENDENT-SET $\in$ DTIME$(O(n))$.

VERTEX-COVER

Instance: A tree $\mathcal{T} = \langle T, F \rangle$, and $\ell \in \mathbb{N}$.

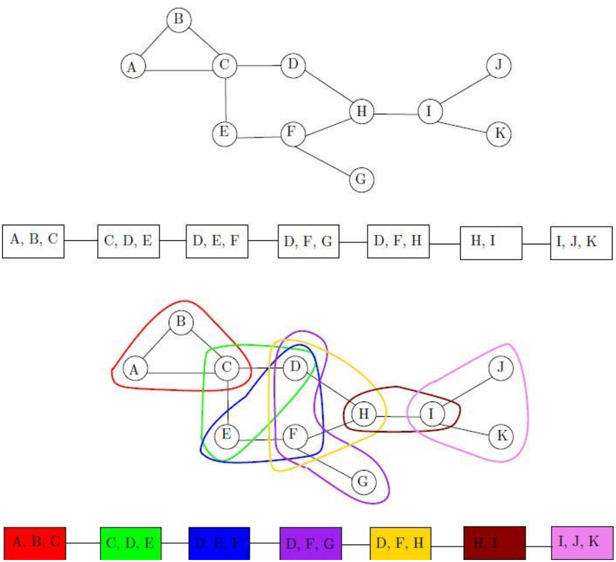
Problem: Does $\mathcal{T}$ have a vertex cover of size at most ℓ?

Corollary

On trees with n nodes,

VERTEX-COVER $\in$ DTIME$(O(n))$.

Path-decomposition (example)



Path decompositions, and path-width

Definition (Robertson–Seymour, 1983)

A *path decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a sequence $\langle B_1, B_2, \ldots, B_r \rangle$ of bags $B_i \subseteq V$ such that:

(P1) $V = \bigcup_{i=1}^{r} B_i$ (every vertex of $\mathcal{G}$ is in some bag).

(P2) $(\forall \{u, v\} \in E)\,(\exists i \in \{1, 2, \ldots, r\})\big[\,\{u, v\} \subseteq B_i\,\big]$
(every edge of $\mathcal{G}$ is realized in some bag).

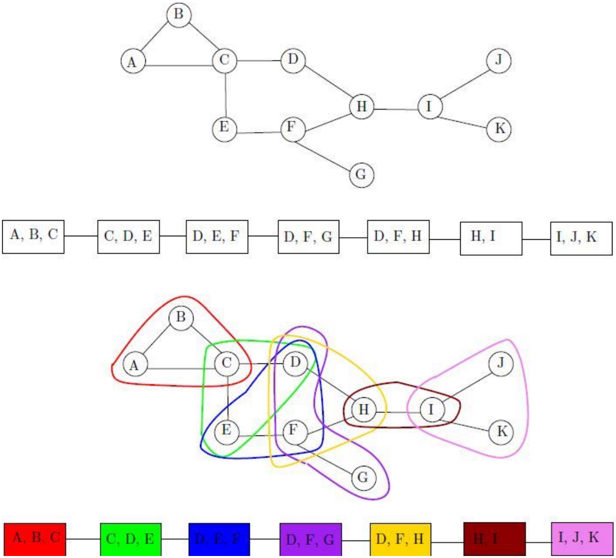
(P3) $(\forall v \in V)\,(\exists i, k \in \{1, \ldots, r\}, i \le k)\big[\,\{j \mid v \in B_j\} = [i, k]\,\big]$
(the list of bags that contains a vertex of $\mathcal{G}$
is $\langle B_i, \ldots, B_k \rangle$ for some interval $[i, k]$)

The *width* of path decomp. $\langle B_1, B_2, \ldots, B_r \rangle$ is $\max\{|B_t| - 1 \mid 1 \le t \le r\}$.

The *path-width* $pw(\mathcal{G})$ of a graph $\mathcal{G} = \langle V, E \rangle$ is defined by:

$pw(\mathcal{G})$:= minimal width of a path decomposition of $\mathcal{G}$.

Path-decomposition (example)



Path decomposition defines separations

Lemma

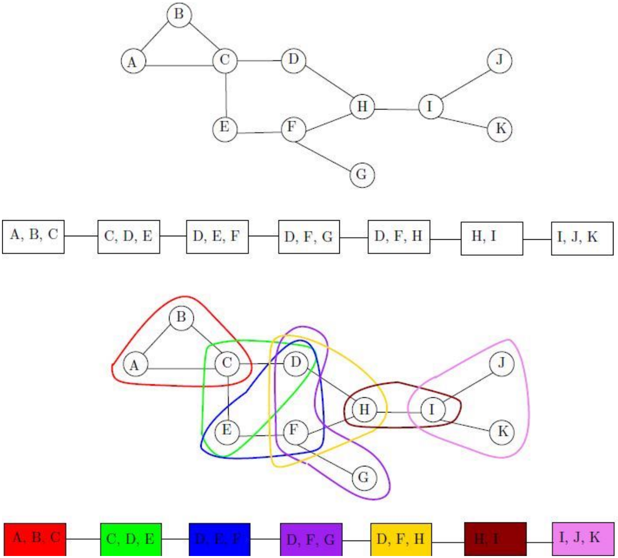
Let $\langle B_1, B_2, \ldots, B_r \rangle$ *be a path decomposition of a graph* $\mathcal{G} = \langle V, E \rangle$.
Then for all $i \in \{1, \ldots, r-1\}$ *it holds:*

- $\langle \bigcup_{j=1}^{i} B_j, \bigcup_{j=i+1}^{r} B_j \rangle$ *is a separation of* $\mathcal{G}$ *with separator* $B_i \cap B_{i+1}$.
- $\partial(\bigcup_{j=1}^{i} B_j) \subseteq B_i \cap B_{i+1}$.

- A pair $\langle A, B \rangle$ of subsets $A, B \subseteq V$ is a *separation* of $\mathcal{G}$ if:
  - $V = A \cup B$
  - there is no edge between $A \setminus B$ and $B \setminus A$.

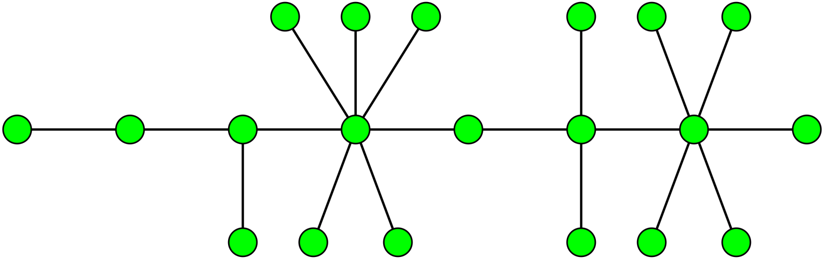
  $A \cap B$ is called the *separator* of a separation $\langle A, B \rangle$, and $|A \cap B|$ is called its *order*.
- The *border (set of border vertices)* $\partial(A)$ *of* a set $A \subseteq V$ of vertices consists of all vertices that have a neighbor in $V \setminus A$. Note that:
  - $\partial(A) = \partial(V \setminus A)$.
  - $\langle A, (V \setminus A) \cup \partial(A) \rangle$ is a separation of $\mathcal{G}$, for all $A \subseteq V$.

Path-decomposition (example)



Caterpillar

Path-width?

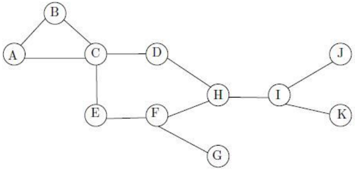


Nice path decomposition

Definition

A *path decomposition* $\langle B_1, B_2, \ldots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if:

- $B_1 = B_r = \varnothing$
- Every index $i > 1$ is either of:
  - introduce index: there is $v \in V$ such that $B_{i+1} = B_i \cup \{v\}$ and $v \notin B_i$,
  - forget index: there is $v \in V$ such that $B_{i+1} = B_i \setminus \{v\}$ and $v \in B_i$.



A, B, C — C, D, E — D, E, F — D, F, G — D, F, H — H, I — I, J, K

Nice path decomposition:

$\varnothing$ — A — A,B — A,B,C — B,C — C — C,D — C,D,E — D,E — D,E,F —

— D,F — D,F,G — D,F — D,F,H — F,H — H — H,I — I — I,J — I,J,K — J,K — K — $\varnothing$

Nice path decomposition

Definition

A *path decomposition* $\langle B_1, B_2, \ldots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if:

- $B_1 = B_r = \varnothing$
- Every index $i > 1$ is either of:
  - introduce index: there is $v \in V$ such that $B_{i+1} = B_i \cup \{v\}$ and $v \notin B_i$,
  - forget index: there is $v \in V$ such that $B_{i+1} = B_i \setminus \{v\}$ and $v \in B_i$.

Lemma

From every path decomposition $\langle B_1, B_2, \ldots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ of width k a nice path decomposition $\langle B'_1, B'_2, \ldots, B'_{r'} \rangle$ of width k can be constructed in time $O(k^2 \cdot \max\{r, n\})$ where $n := |V|$.

A,B,C – C,D,E – D,E,F – D,F,G – D,F,H – H,I – I,J,K

∅ – A – A,B – A,B,C – B,C – C – C,D – C,D,E – D,E – D,E,F –
– D,F – D,F,G – D,F – D,F,H – F,H – H – H,I – I – I,J – I,J,K – J,K – K – ∅

Weighted Independent Set

Let $\mathcal{G} = \langle V, E \rangle$ a graph.

$S \subseteq V$ is independent set in $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \big(\neg(u \in S \wedge v \in S) \big)$.

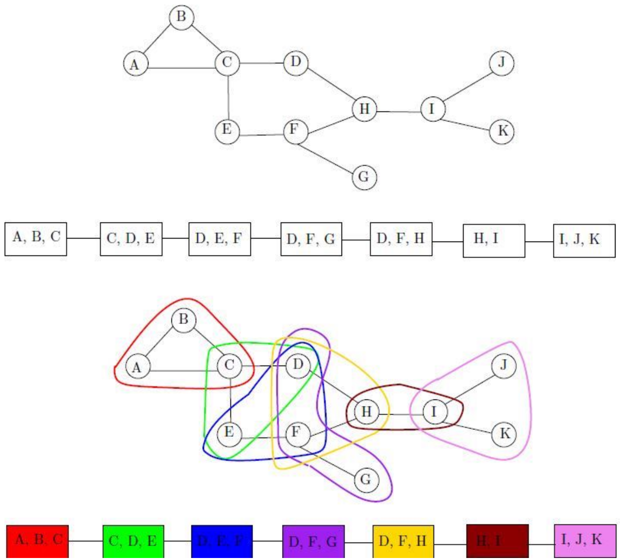
WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

Parameter: path-width k.

Problem: What is the max. weight of an independent set of $\mathcal{G}$?

Path-decomposition (example)



Dyn. programming using path-width (Weigh. Ind. Set)

Let $\langle B_1, \ldots, B_r \rangle$ be a nice path decomposition of $\mathcal{G} = \langle V, E \rangle$.
Then for every $i \in \{1, \ldots, r\}$, and every $S \subseteq B_i$, we define:

$$c[i,S] := \begin{cases} -\infty & \text{if } S \text{ is not independent} \\ \text{maximum possible weight of a set } \hat{S} \text{ such that} & \\ \hat{S} \text{ is independent} \wedge S \subseteq \hat{S} \subseteq V_i = \bigcup_{j=1}^{i} B_j \wedge \hat{S} \cap B_i = S & \\ & \text{if } S \text{ is independent.} \end{cases}$$

Recursive equations for computing $c[i,S]$ for independent S:

- Case $i = 1$: $c[1, \varnothing] = 0$
- Case $i + 1$:
  - $i+1$ introduces v: $B_{i+1} = B_i \cup \{v\}$ and $v \notin B_i$,
  $$c[i+1,S] = \begin{cases} c[i,S] & \text{if } v \notin S, \\ c[i, S \smallsetminus \{v\}] + \boldsymbol{w}(v) & \text{if } v \in S; \end{cases}$$
  - $i+1$ forgets v: $B_{i+1} = B_i \smallsetminus \{v\}$ and $v \in B_i$,
  $$c[i+1,S] = \max\big\{c[i,S],\ c[i, S \cup \{v\}]\big\}.$$

Dyn. programming using path-width (Weigh. Ind. Set)

Let $\langle B_1, \ldots, B_r \rangle$ be a nice path dec. of $\mathcal{G} = \langle V, E \rangle$ of width k.
For every $i \in \{1, \ldots, r\}$, and every independent $S \subseteq B_i$, we define:

$$c[i, S] := \begin{cases} \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent} \wedge S \subseteq \hat{S} \subseteq V_i = \bigcup_{j=1}^{i} B_j \wedge \hat{S} \cap B_i = S \end{cases}$$

Time Complexity: Based on the values of $c[i, S]$, the maximum possible weight of an independent set $S \subseteq V$ can be computed as:

$$= \max_{S \subseteq B_r} c[r, S] \quad = c[r, \varnothing]$$

Then for all $i \in \{1, \ldots, n\}$:

- $|B_i| \le k + 1$,
- $\Rightarrow$ number of values $c[i, S]$ at index i: $2^{|B_i|} = 2^{k+1}$,
- $\Rightarrow$ adjacency/independence check for $S \subseteq B_t$ possible in: $O(k^2)$ using a datastructure computable in time $O(k^{O(1)} \cdot n)$,
- time for comp. a value at i, using map of values at $i - 1$: $\sim O(k)$
- time for comp. all values at i, using values at $i - 1$: $2^{k+1} \cdot O(k^2)$

$\Rightarrow$ the time for computing all values at r:

$$(2^{k+1} \cdot O(k^2)) \cdot r + O(k^{O(1)} \cdot n) \in O(2^k \cdot k^{O(1)} \cdot n), \text{ since } r = 2n.$$

Dynamical programming with path width (example)

Theorem

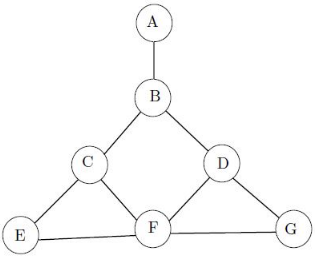
For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and path-width pw$(\mathcal{G}) = k$,
p^*-WEIGHTED-INDEPENDENT-SET $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$.

S is a *minimal* vertex cover
$\iff V \setminus S$ is a *maximal* independent set.

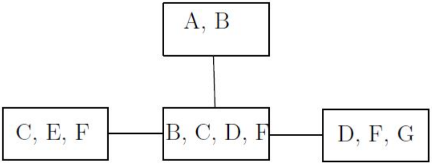
Corollary

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and path-width pw$(\mathcal{G}) = k$,
p^*-VERTEX-COVER $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$.

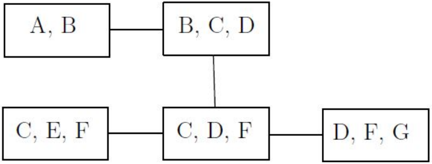
Tree decomposition (example)



The original graph G



A tree-decompostion of width 3



A tree-decompostion of width 2

Tree decompositions, and tree-width

Definition (Bertelé–Brioschi, 1972, Halin, 1976, Robertson–Seymour, 1984)

A *tree decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a pair $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ where $\mathcal{T} = \langle T, F \rangle$ a (undirected, unrooted) tree, and $B_t \subseteq V$ such that:

(T1) $V = \bigcup_{t \in T} B_t$ (every vertex of $\mathcal{G}$ is in some bag).

(T2) $(\forall \{u, v\} \in E)\,(\exists t \in T)\big[\, \{u, v\} \subseteq B_t \,\big]$
(the vertices of every edge of $\mathcal{G}$ are realized in some bag).

(T3) $(\forall v \in V)\big[$ subgraph of $\mathcal{T}$ defd. by $\{t \in T \mid v \in B_t\}$ is connected $\big]$
(the tree vertices (in $\mathcal{T}$) whose bags contain some vertex of $\mathcal{G}$ induce a subgraph of $\mathcal{T}$ that is connected).

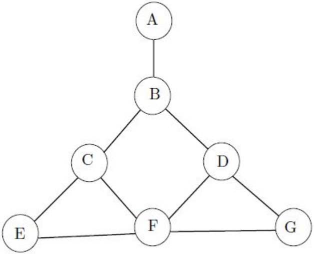
The *width* of a tree decomposition $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ is

$$\max \{|B_t| - 1 \mid t \in T\}\,.$$

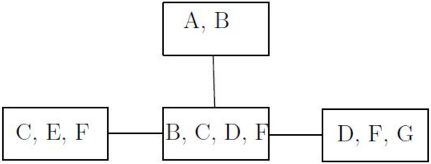
The *tree-width* $tw(\mathcal{G})$ of a graph $\mathcal{G} = \langle V, E \rangle$ is defined by:

$$tw(\mathcal{G}) := \text{minimal width of a tree decomposition of } \mathcal{G}.$$

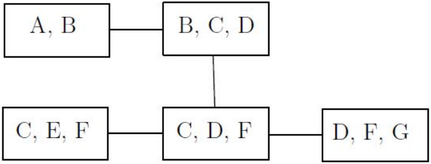
Tree decomposition (example)



The original graph G



A tree-decompostion of width 3



A tree-decompostion of width 2

Tree decomposition defines separations

Lemma

Let $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ be a tree decomposition of a graph $\mathcal{G} = \langle V, E \rangle$. Let $e = \langle a, b \rangle$ be an edge of $\mathcal{T}$. The $\mathcal{T} \smallsetminus e$ is the union of a tree $\mathcal{T}_a$ containing a, and a tree $\mathcal{T}_b$ containing b.
Then for $A := \bigcup_{t \in V(\mathcal{T}_a)} B_t$ and $B := \bigcup_{t \in V(\mathcal{T}_b)} B_t$ it holds:

- *$\langle A, B \rangle$ is a separation of $\mathcal{G}$ with separator $B_a \cap B_b$.*
- *$\partial(A), \partial(B) \subseteq B_a \cap B_b$.*

Recall, for a graph $\mathcal{G} = \langle V, E \rangle$:

- A pair $\langle A, B \rangle$ of subsets $A, B \subseteq V$ is a *separation* of $\mathcal{G}$ if:
  - $V = A \cup B$
  - there is no edge between $A \smallsetminus B$ and $B \smallsetminus A$.

  $A \cap B$ is called the *separator* of a separation $\langle A, B \rangle$,
  and $|A \cap B|$ is called its *order*.
- The *border (vertices) $\partial(A)$ of* a set $A \subseteq V$ of vertices consists of all vertices that have a neighbor in $V \smallsetminus A$.

Computing tree-width

TREE-WIDTH
Instance: A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Problem: Decide whether *tw*$(\mathcal{G}) = k$.

Theorem

TREE-WIDTH *is* NP*-complete.*

p-TREE-WIDTH
Instance: A graph $\mathcal{G} = \langle V, E \rangle$ and $k \in \mathbb{N}$.
Parameter: k.
Problem: Decide whether *tw*$(\mathcal{G}) = k$.

Theorem

p-TREE-WIDTH *is fixed-parameter tractable,*
in time $2^{p(k)} \cdot n$ *where* $n := |V|$.

Nice tree decomposition

Definition

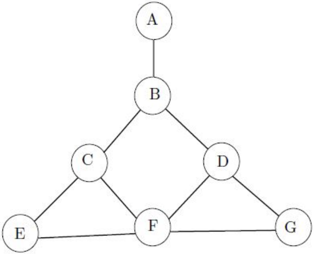
A *tree decomposition* $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ of graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if it is based on the choice of a leaf as *root* r and the parent–children relation away from r such that:

- $B_r = \varnothing$, and $B_\ell = \varnothing$ for every leaf $\ell \in T$.
- Every non-leaf node $t \in T$ is of one of three types:
  - introduce node: t has exactly one child t' such that $B_t = B_{t'} \cup \{v\}$; we say v is introduced at t.
  - forget node: t has exactly one child t' such that $B_t = B_{t'} \setminus \{w\}$ for some $w \in B_{t'}$; we say w is forgotten at t.
  - join node: a node t with two children t_1, t_2 such that $B_t = B_{t_1} = B_{t_2}$.

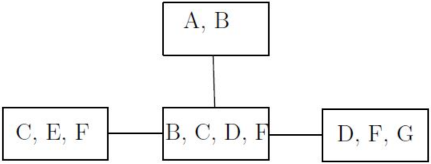
Lemma

From every tree decomposition $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ of width k a nice tree decomposition $\langle \mathcal{T}', \{B'_t\}_{t \in T'} \rangle$ of width k and with $r := |V(\mathcal{T})| \in O(kn)$ vertices can be constructed in time $O(k^2 \cdot \max\{r, n\})$ where $n := |V|$.

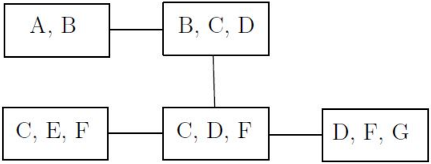
Tree decomposition (example)



The original graph G



A tree-decompostion of width 3



A tree-decompostion of width 2

Weighted Independent Set

Let $\mathcal{G} = \langle V, E \rangle$ a graph.

$S \subseteq V$ is independent set in $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \big(\neg(u \in S \wedge v \in S) \big)$.

WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

Parameter: tree-width k.

Problem: What is the max. weight of an independent set of $\mathcal{G}$?

Dynamical programming using tree-width (example)

For every node t of a nice tree decomposition, and every $S \subseteq B_t$, we define:

$$c[t,S] := \begin{cases} -\infty & \text{if } S \text{ is not independent,} \\ \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent} \wedge S \subseteq \hat{S} \subseteq V_t \wedge \hat{S} \cap B_t = S \\ & \text{if } S \text{ is independent.} \end{cases}$$

Recursive equations for computing $c[t,S]$ for independent S:

- leaf node t: $c[t,\varnothing] = 0$
- introduction node t of vertex v with child t':

$$c[t,S] = \begin{cases} c[t',S] & \text{if } v \notin S \\ c[t',S \setminus \{v\}] + \boldsymbol{w}(v) & \text{otherwise} \end{cases}$$

- forget node t of vertex v with child t':

$$c[t,S] = \max\{c[t',S],\, c[t',S \cup \{v\}]\}$$

- join node t with children t_1 and t_2:

$$c[t,S] = c[t_1,S] + c[t_2,S] - \boldsymbol{w}(S)$$

Dyn. programming using tree-width (Weigh. Ind. Set)

Let $\langle \mathcal{T} = \langle T, F, r \rangle, \{B_t\}_{t \in T} \rangle$ be a nice tree decomposition of $\mathcal{G} = \langle V, E \rangle$ of width k. For every $t \in T$, and every independent $S \subseteq B_t$:

$$c[t,S] := \begin{cases} \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent } \wedge\ S \subseteq \hat{S} \subseteq V_t = \bigcup_{s \in T_t} B_s \ \wedge\ \hat{S} \cap B_i = S \end{cases}$$

Time Complexity: Based on the values of $c[t,S]$, the maximum possible weight of an independent set $S \subseteq V$ can be computed as:

$$= \max_{S \subseteq B_r} c[r,S] \quad = c[r,\varnothing]$$

Then for all $t \in T$:

- $|B_t| \le k+1$,
- $\Rightarrow$ number of values $c[t,S]$ at index t : $2^{|B_t|} = 2^{k+1}$,
- $\Rightarrow$ adjacency/independence check for $S \subseteq B_t$ possible in: $O(k^2)$ using a datastructure computable in time $O(k^{O(1)} \cdot n)$,
- time for comp. a value at t, using map of values at $t-1$: $O(k)$
- time for comp. all values at t, using values at $t-1$: $2^{k+1} \cdot O(k^2)$

$\Rightarrow$ the time for computing all values at the root r :
$(2^{k+1} \cdot O(k^2)) \cdot |T| + O(k^{O(1)} \cdot n) \in O(2^k \cdot k^{O(1)} \cdot n)$, since $|T| \in O(k \cdot n)$.

Dynamical programming with tree width (example)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ *with* $|V| = n$ *and tree-width* tw$(\mathcal{G}) = k$,

p^*-WEIGHTED-INDEPENDENT-SET $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$.

S is a *minimal* vertex cover

$\iff V \setminus S$ is a *maximal* independent set.

Corollary

For every graph $\mathcal{G} = \langle V, E \rangle$ *with* $|V| = n$ *and tree-width* tw$(\mathcal{G}) = k$,

p^*-VERTEX-COVER $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$.

Dyn. programming with tree-width: general strategy

We consider problem P for graphs $\mathcal{G} = \langle V, E \rangle$ of size n and nice tree decompositions $\langle \mathcal{T} = \langle T, F, r \rangle, \{B_t\}_{t \in T} \rangle$ of tree width k.

- Formulate a family of properties that can be restricted to subtrees of $\mathcal{T}$ such that
  - a solution of P can be obtained from the properties at the root of $\mathcal{T}$.
- Find recursion equations for bottom-up evaluation on $\mathcal{T}$.
- Prove correctness of these recursion equations by showing two inequalities for each type of node:
  - one relating an optimum solution for the node to some solutions for its children,
  - one relating optimum solutions for a node's children to a solution for the node.
- Obtain an estimate of the time needed to compute the properties in a node t depending on n and k.
- Sum up the time needed to compute the solution(s) at root r of $\mathcal{T}$.
- Add time needed to obtain the solution of P from properties at r.

Dynamical programming: similar results (I)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and tw$(\mathcal{G}) = k$,

- p^*-VERTEX-COVER, INDEPENDENT-SET $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$,
- p^*-DOMINATING-SET $\in$ DTIME$(4^k \cdot k^{O(1)} \cdot n)$,
- p^*-ODD CYCLE TRAVERSAL $\in$ DTIME$(3^k \cdot k^{O(1)} \cdot n)$,
- p^*-MAXCUT $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$,
- p^*-q-COLORABILITY $\in$ DTIME$(q^k \cdot k^{O(1)} \cdot n)$.

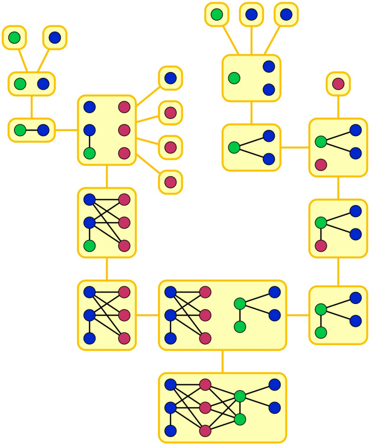
Dynamical programming: similar results (II)

Theorem

For every graph $\mathcal{G} = \langle V, E\rangle$ with $|V| = n$ and tw($\mathcal{G}$) $= k$, the following problems are in DTIME$(k^{O(k)} \cdot n)$ *:*

- p^*-STEINER-TREE,
- p^*-FEEDBACK-VERTEX-SET,
- p^*-HAMILTONIAN-PATH *and* p^*-LONGEST-PATH,
- p^*-HAMILTONIAN-CYCLE *and* p^*-LONGEST-CYCLE,
- p^*-CHROMATIC-NUMBER,
- p^*-CYCLE-PACKING,
- p^*-CONNECTED-VERTEX-COVER,
- p^*-CONNECTED-FEEDBACK-VERTEX-SET.

Clique width (example)



Clique-Width

For $k \in \mathbb{N}$, the *k-expressions* are defined by:

$$\varphi, \varphi_1, \varphi_2 \;::=\; \boldsymbol{i} \mid \text{edge}_{i-j}(\varphi) \mid \text{recolor}_{i \to j}(\varphi) \mid (\varphi_1 \oplus \varphi_2)$$

for $i, j \in [k]$ with $i \neq j$. k-expressions φ *generate* graphs $\mathcal{G}(\varphi)$:

- $\mathcal{G}(\boldsymbol{i})$ is the graph with a single vertex of color i.
- $\mathcal{G}(\text{edge}_{i-j}(\varphi))$ results from $\mathcal{G}(\varphi)$ by adding edges between every vertex of color i and every vertex of color j.
- $\mathcal{G}(\text{recolor}_{i \to j}(\varphi))$ results from $\mathcal{G}(\varphi)$ by recoloring every vertex of color i by color j.
- $\mathcal{G}(\varphi_1 \oplus \varphi_2)$ is the disjoint union of $\mathcal{G}(\varphi_1)$ and $\mathcal{G}(\varphi_2)$.

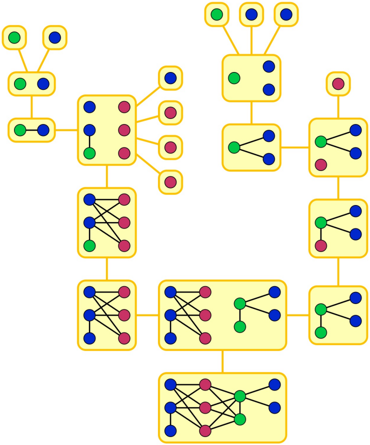
Definition (Courcelle, Engelfriet, Rozenberg, 1993, [2])

The *clique-width $clw(\mathcal{G})$* of $\mathcal{G} = \langle V, E \rangle$ is defined by:

$$clw(\mathcal{G}) := \text{the least } k \in \mathbb{N} \text{ such that, for some } k\text{-expression } \varphi,$$
$$\mathcal{G} = \mathcal{G}(\varphi) \text{ (when removing colors)}$$

Clique width (example)

Building a graph $\mathcal{G}$ of clique-width *clw*$(\mathcal{G}) = 3$:



Clique-Width (examples, properties, computability)

Example

- The class of cliques has clique-width 2.
- The class of stars has clique-width 2.
- The class of trees has clique-width 3.
- The class of $n \times n$ grids has clique-width $\Theta(n)$.

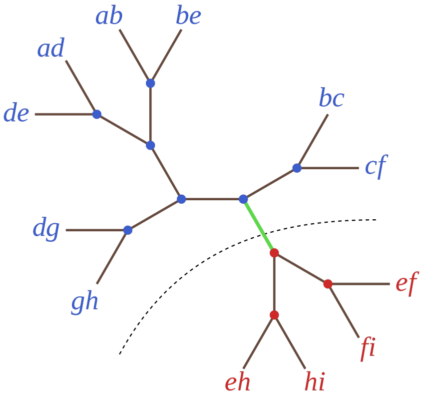
- subgraphs/induced subgraphs:
  - clique-width is preserved under taking induced subgraphs,
  - clique-width is not preserved under taking subgraphs (e.g. minors).
- *clw* $\prec$ *tw*:
  - *clw* $\leq$ *tw*: *clw*$(\mathcal{G}) \leq 3 \cdot 2^{tw(\mathcal{G})-1}$
  - $\neg$(*tw* $\leq$ *clw*): for example, *clw*$(K_n) = 2$, and *tw*$(K_n) = n - 1$.
- Deciding whether *clw*$(\mathcal{G}) \leq k$ is NP-hard. With parameter k it is in XP (slice-wise polynomial), but unknown to be in FPT.
- Every graph property expressible in MSO (monadic second-order logic) can be decided in linear time w.r.t. the graph's clique-width.

f-Width (of sets)

By a *cut function* or a *connectivity function* we mean a function $f : 2^U \to \mathbb{R}_0^+$ such that:

$$f \text{ is } \textit{symmetric} :\Longleftrightarrow \forall X \subseteq U \left[f(X) = f(U \setminus X) \right];$$

$$f \text{ is } \textit{fair} :\Longleftrightarrow f(\varnothing) = f(U) = 0\,.$$



Branch-Width

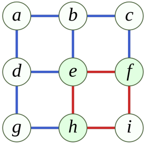
Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph. The *border (vertices)* of a set $X \subseteq E$ of edges is defined by:

$$\partial(X) := \big\{ v \in V \mid \exists e_1 \in X \; \exists e_2 \in E \smallsetminus X \; \big[v \text{ is incident to } e_1 \text{ and } e_2 \big] \big\}$$

The *branch-width bw*$(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is defined as

$$\mathit{bw}(\mathcal{G}) := w_f(E) \quad \text{for} \quad f : 2^E \to \mathbb{R}_0^+,\; X \mapsto |\partial(X)|$$



Rank-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.
For $X \subseteq V$ we define the $GF(2)$-matrix:

$$B_{\mathcal{G}}(X) := (b_{x,y})_{x \in X, y \in V \setminus X}\,, \text{ where, for all } x \in X, y \in V \setminus X\text{:}$$
$$b_{x,y} = 1 \iff \{x, y\} \in E\,.$$

($B_{\mathcal{G}}(X)$ is the adjacency matrix of the bipartite graph induced by $\mathcal{G}$ between X and $V \setminus X$.)

The *rank-width rw*($\mathcal{G}$) of a graph $\mathcal{G} = \langle G, E \rangle$ is:

$$\textit{rw}(\mathcal{G}) := w_{\rho_{\mathcal{G}}}(E) \quad \text{for} \quad \rho_{\mathcal{G}} : 2^V \to \mathbb{N}_0,\ X \mapsto \text{rank of } B_{\mathcal{G}}(X)$$

Properties

- *rw*($\mathcal{G}$) $\le$ *tw*($\mathcal{G}$).
- tree-width cannot be bounded functionally by rank-width: *rw*(K_n) $= 1$, but *tw*(K_n) $= n - 1$.

Carving-Width and Cut-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.
For $X \subseteq V$ the *edge-cut* of X is:

$$\textit{cut}_{\mathcal{G}}(X) := \{e = \{u, v\} \in E \mid u \in X, v \in V \setminus X\} .$$

The *carving-width* $\textit{carw}(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is:

$$\textit{carw}(\mathcal{G}) := w_{\textit{cut}}(E) \quad \text{for} \quad \textit{cut} : 2^V \to \mathbb{N}_0,\ X \mapsto |\textit{cut}_{\mathcal{G}}(X)| .$$

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph with $n = |V|$.
For a permutation $\pi : \{1, \ldots, n\} \to V$ on V we define:

$$\textit{width}(\pi) := \max_{1 \le i \le n} \textit{cut}_{\mathcal{G}}(\{\pi(j) \mid 1 \le j \le i\}) .$$

The *cut-width* $\textit{cutw}(\mathcal{G})$ of $\mathcal{G}$ is:

$$\textit{cutw}(\mathcal{G}) := \min_{\pi \text{ perm. of } V} \textit{width}(\pi) .$$

Coverage in Multi-Interface Networks

CMI(p) (for $p \in \mathbb{N}$)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, $W : V \to 2^{\{1,\dots,a\}}$ available-interface allocation, $c : \{1, \dots, a\} \to \mathbb{R}^+$ interface cost function.

Solution: An allocation $W_A : V \to 2^{\{1,\dots,a\}}$ of active interfaces covering $\mathcal{G}$ such that $W_A(v) \subseteq W(v)$, and $|W_A(v)| \leq p$ for all $v \in V$, if possible; otherwise, a negative answer.

Problem: Obtain, if possible, a minimal solution with respect to the total cost of the interfaces that are activated, that is, $c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i)$.

Coverage in Multi-Interface Networks (parameterized)

Theorem

CMI$(2) \in$ NP-complete*, also for graphs with max. node degree* ≥ 4*.*

p^*-*CMI*(p) (for $p \in \mathbb{N}$)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, $W : V \to 2^{\{1,\ldots,a\}}$ available-interface allocation, $c : \{1, \ldots, a\} \to \mathbb{R}^+$ interface cost function.

Parameter: path-width / carving-width k

Problem: Obtain, if possible, a minimal solution with respect to the total cost of the interfaces that are activated, that is, $c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i)$.

Theorem (Aloisio, Navarra, 2020, [1])

- *For path-width pw*$(\mathcal{G}) = k$*,*
  p^*-*CMI*$(2) \in \mathrm{DTIME}(n \cdot (a + \binom{a}{2})^{k+1})$*.*
- *For carving-width carw*$(\mathcal{G}) = k$*,* p^*-*CMI*$(2) \in \mathrm{DTIME}(n \cdot a^{4k})$*.*

Coverage in Multi-Interface Networks (parameterized)

Theorem (Aloisio, Navarra, 2020, [1])

- *For path-width $pw(\mathcal{G}) = k$,*
  p^*-*CMI*$(2) \in \mathrm{DTIME}(n \cdot (a + \binom{a}{2})^{k+1})$.
- *For carving-width $carw(\mathcal{G}) = k$, p^*-CMI$(2) \in \mathrm{DTIME}(n \cdot a^{4k})$.*

$(p^*)'$-*CMI*(p) (for $p \in \mathbb{N}$)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, $W : V \to 2^{\{1,\ldots,a\}}$ available-interface allocation, $c : \{1, \ldots, a\} \to \mathbb{R}^+$ interface cost function.

Parameter: $a +$ (path-width / carving-width k)

Problem: Obtain, if possible, a minimal solution with respect to the total cost of the interfaces that are activated, that is, $c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i)$.

Corollary

$(p^*)'$-*CMI*$(p) \in$ FPT.

Comparing parameterizations

Definition (computably bounded)

Let $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ parameterizations.

- $\kappa_1 \succeq \kappa_2 \;:\Longleftrightarrow\; \exists g : \mathbb{N} \to \mathbb{N}$ computable $\forall x \in \Sigma^* \big[\, g(\kappa_1(x)) \geq \kappa_2(x) \,\big]$.
- $\kappa_1 \approx \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \kappa_2 \succeq \kappa_1$.
- $\kappa_1 \succ \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \neg(\kappa_2 \succeq \kappa_1)$.

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ with $\kappa_1 \succeq \kappa_2$:

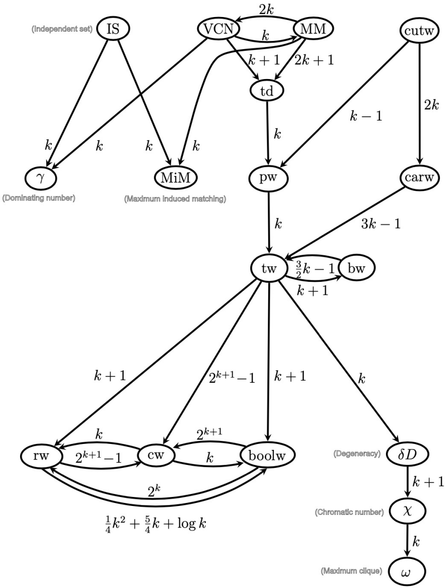
$$\langle Q, \kappa_1 \rangle \in \mathrm{FPT} \quad \Longleftarrow \quad \langle Q, \kappa_2 \rangle \in \mathrm{FPT}$$
$$\langle Q, \kappa_1 \rangle \notin \mathrm{FPT} \quad \Longrightarrow \quad \langle Q, \kappa_2 \rangle \notin \mathrm{FPT}$$

Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \geq wd_2 \; : \Leftarrow \; wd_1 \xrightarrow{g(k)} wd_2$$

- FPT-results
  transfer upwards
  (and conversely to $\xrightarrow{g}$)
- ($\notin$ FPT)-results
  transfer downwards
  (and along $\xrightarrow{g}$)



Summary

- comparing parameterizations
- dynamical programming on trees, example:
  - WEIGHTED-INDEPENDENT-SET (and VERTEX-COVER)
- path-width
  - example: fpt-algorithm for bounded path-width
- tree-width
  - example: fpt-algorithm for bounded path-width
- fpt-results for other problems, obtained similarly
- other notions of width
  - clique-width
  - using f-width to define:
    - carving-width (and cut-width)
    - branch-width
    - rank-width
- example problem: coverage in multi-interface networks
- comparing width-notions

Wednesday

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
Algorithmic Techniques		*Formal-Method* & *Algorithmic Techniques*		
Introduction & basic FPT results	**Notions of bounded graph width**	**Algorithmic Meta-Theorems**	**FPT-Intractability Classes & Hierarchies**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour

Thursday

- recalling notions from logic:
  - propositional, and first-order logic
  - monadic second-order logic (MSO)
- Courcelle's Theorem: obtaining FPT-results by
  - model-checking of MSO-properties
    on graphs and structures of bounded tree-/clique-width

References I

Alessandro Aloisio and Alfredo Navarra.
Constrained connectivity in bounded x-width multi-interface networks.
Algorithms, 13(2), 2020.

Bruno Courcelle, Joost Engelfriet, and Grzegorz Rozenberg.
Handle-rewriting hypergraph grammars.
Journal of Computer and System Sciences, 46(2):218 – 270, 1993.

Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh.
Parameterized Algorithms.
Springer, 1st edition, 2015.

References II

Jörg Flum and Martin Grohe.
Parameterized Complexity Theory.
Springer, 2006.

Róbert Sásak.
Comparing 17 graph parameters.
Master's thesis, University of Bergen, Norway, 2010.