From Compactifying Lambda-Letrec Terms to Recognizing Regular-Expression Processes

(Extended Abstract and Literature)

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As a supplement to my talk at the workshop, this extended abstract motivates and summarizes my work with co-authors on problems in two separate areas: first, in the λ-calculus with letrec, a universal model of computation, and second, on Milner's process interpretation of regular expression, a proper subclass of the finite-state processes. The aim of my talk was to motivate a transferal of ideas for workable concepts of structure-constrained graphs: from the problem of finding compact graph representations for terms in the λ-calculus with letrec to the problem of recognizing finite process graphs that can be expressed by regular expressions. In both cases the construction of structure-constrained graphs was expedient in order to enable to go back and forth easily between, in the first case, λ-terms and term graphs, and in the second case, regular expressions and process graphs.

The main focus is on providing pointers to my work with co-authors, in both areas separately. A secondary focus is on explaining directions of my present projects, and describing research questions of possibly general interest that have developed out of my work in these two areas.

1 Introduction

The purpose of this extended abstract is to supplement my talk at the workshop [3] with a brief description of my work with co-authors in two areas, including ample references. While my workshop-presentation covered similar topics as my talk [1] at TERMGRAPH 2018, and while the proceedings article [2] for that workshop remains a useful resource, this article is a rewritten account with a detailed update on results that have been obtained in the meantime, and with an outlook on remaining challenging problems.

My talk [3] at the workshop aimed at motivating a fruitful transferal of ideas between two areas on which I worked in the (a bit removed, and more recent) past: λ-calculus, and the implementation of functional programming languages (2009–2014), and the process theory of finite-state processes (from 2016). My intention was to show, informally and supported by many pictures: How a solution to the problem of finding adequate graph representations for terms in the λ-calculus with letrec, a universal model of computation, turned out to be very helpful in understanding process graphs that can be expressed by regular expressions (via Milner's process interpretation), a proper subclass of finite-state processes.

In both cases the definition of an adequate notion of structure-constrained (term or process) graph was the key to solve a specific practical, and respectively theoretical problem. It was central that the structure-constrained graphs facilitate to go back and forth easily between, on the one hand, terms in the λ-calculus with letrec and term graphs, and on the other hand, regular expressions and process graphs. The graph representations respect the appertaining operational semantics, but were conceived with specific purposes in mind: to optimize functional programs in the Lambda Calculus with letrec; and

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respectively, to reason with process graphs denoted by regular expressions, and to decide recognizability of these graphs. For a detailed comparison of the similarities and differences of the structure-constrained graphs as defined in the term graph semantics of terms in the λ-calculus with letrec (see Section 2), and the process (graph) semantics of regular expressions (see Section 3), we want to refer to Section 4 of [2].

Section 2 summarizes work by Jan Rochel and myself that led us to the definition, and efficient implementation of maximal sharing for the higher-order terms in the λ-calculus with letrec. Specifically we formulated a representation-pipeline: Higher-order terms can be represented by, appropriately defined, higher-order term graphs, then these can be encoded as first-order term graphs, and subsequently those can in turn be represented as deterministic finite-state automata (DFAs). Via these correspondences and DFA minimization, maximal shared forms of higher-order terms can be computed.

Section 3 gives an overview of my work, in crucial parts done together with Wan Fokkink, on two non-trivial problems concerning the process semantics of regular expression. In Milner's process semantics, regular expressions are interpreted as nondeterministic finite-state automata (NFAs) whose equality is studied modulo bisimulation. Unlike for the standard language interpretation, not every NFA can be expressed by a regular expression. This raised a non-trivial recognition (or expression) problem, which was formulated by Milner (1984) next to a completeness problem for an equational proof system. In Section 3 I report on the crucial steps that have led me to a solution of the completeness problem.

Finally Section 4 reports on my present projects, and lists research questions that have developed out of my work in these two areas.

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[1] Clemens Grabmayer (2018): *Modeling Terms by Graphs with Structure Constraints (Two Illustrations)*. Invited talk at the FSCD/FLoC Workshop *TERMGRAPH 2018*, Oxford, UK, July 7. Slides are available at `https://clegra.github.io/lf/TERMGRAPH-2018-invited-talk.pdf`.

[2] Clemens Grabmayer (2019): *Modeling Terms by Graphs with Structure Constraints (Two Illustrations)*. In Maribel Fernández & Ian Mackie, editors: *Proceedings Tenth International Workshop on Computing with Terms and Graphs, TERMGRAPH@FSCD 2018, Oxford, UK, 7th July 2018*, *EPTCS* 288, pp. 1–13, doi:10.4204/EPTCS.288.1.

[3] Clemens Grabmayer (2023): *From Compactifying Lambda-Letrec Terms to Recognizing Regular-Expression Processes*. Invited talk at the 13th International Workshop on *Developments in Computational Models*, affiliated with the conference FSCD, Sapienza Università, Rome, July 2. Slides available at `https://clegra.github.io/lf/DCM-2023-invited-talk.pdf`.

2 Compactifying Lambda-Letrec Terms

This section gives an overview about work that Jan Rochel and I did in the framework of the NWO-project Realizing Optimal Sharing (ROS) at Utrecht University (2009–2014).[1] It eventually led us to the definition and practical implementation of maximal sharing for terms in the λ-calculus with letrec, the Core language for the compilation of functional programming languages.

We started with the intention to study phenomena that arise practically for optimal-sharing implementations of the λ-calculus (by graph-transformation schemes by Lamping [18], and Kathail [17], and later interaction-net formalizations by Gonthier, Abadi, Lèvy [6], and also van Oostrom, van de Looij,

[1] This project was headed jointly by Vincent van Oostrom (rewriting and λ-calculus) and Doitse Swierstra (implementation of functional languages). The project was concluded successfully in June 2016 with Jan Rochel's defense of his thesis [22].

Zwitserlood [20]), which are implementations of optimal or parallel β-reduction (due to Lèvy [19]). For this purpose Rochel wrote an impressive visualization and animation tool [21] for transforming graphs by reducing graph-rewrite redexes per mouse-click. It produces beautifully rendered graphs that slowly float over the screen like bacteria in a liquid under a microscope. This animation tool provided us with much room for experimentation. We first tried to understand whether optimal implementations could render the so-called static-argument transformation unnecessary. When we could not establish that, we first tried to understand in how far the static-argument transformation changes the evaluation of programs with respect to usual scope-preserving graph evaluation. As a consequence, we partly turned our attention away from optimal evaluation (in the hope that we would later come back to it with a better understanding).

We started by generalizing the static-argument transformation to more general optimizations.

Parameter-dropping optimization transformations

In [23] we described an optimization transformation for the compilation of functional programs that drops parameters that are passed along unchanged between a number of recursive functions from the definitions of these functions. We used higher-order rewrite rules to describe this generalization of the static-argument transformation that permits the avoidance of repetitive evaluation patters [23]. We discovered later a close connection with Lambda Dropping due to Danvy and Schultz [5].

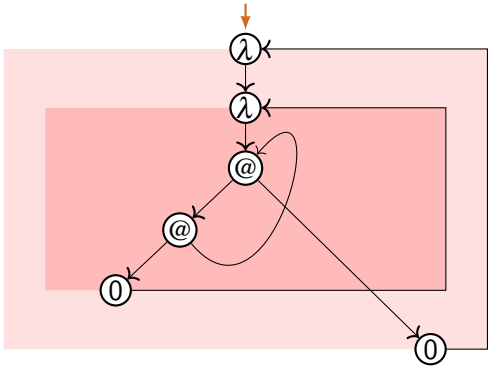
Realizing that we had moved on to terrain for which a strong theory had already been established, we set ourselves more ambitious goals: First, to understand formally and conceptually the relationship between terms in the λ-calculus with letrec ($\lambda_{\mathsf{letrec}}$) and the infinite λ-terms they represent (in λ^{∞}, the infinitary λ-calculus). Second, to find term graph representations of $\lambda_{\mathsf{letrec}}$-terms that are preserved under homomorphism (functional bisimularity). Finally third, we wanted to use possible answers for these two points to define maximally-shared representations of arbitrary $\lambda_{\mathsf{letrec}}$-terms. Below we report on our results concerning these three goals.

1. Expressibility of infinite λ-terms by terms in $\lambda_{\mathsf{letrec}}$ (and in λ_{μ}).

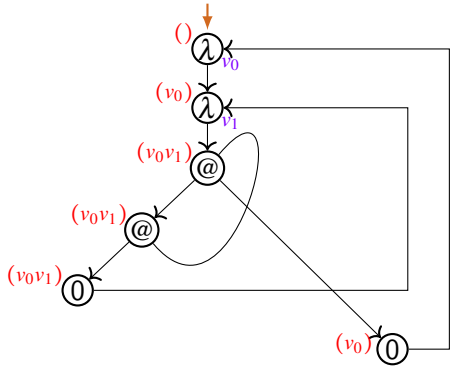
   We studied the question: Which infinite λ-terms are (infinite) unfoldings of terms in $\lambda_{\mathsf{letrec}}$, the λ-calculus with letrec, or (equivalent, but formally easier) in λ_{μ}, the λ-calculus with μ? Clearly, such infinite λ-terms have to be *regular* in the sense that their syntax-trees have only finitely many subtrees modulo α-conversion. However, while regularity is necessary for expressibility by a $\lambda_{\mathsf{letrec}}$-term under infinite unfolding, it is not sufficient. What is missing is, intuitively, that the abstraction scopes in regular infinite λ-terms are not infinitely entangled. We formulated this requirement in two different ways: that the infinite (regular) λ-term in question (i) has only finitely many 'generated subterms' that are generated by a certain decomposition rewrite system that uses eager scope closure, (ii) does not contain infinite 'binding–capturing chains'. Both conditions delineate the *strongly regular* infinite λ-terms among the regular ones. For this concept we showed that an infinite λ-term M is the unfolding of a term in $\lambda_{\mathsf{letrec}}$ (resp. a term in λ_{μ}) if and only if M is strongly regular. For $\lambda_{\mathsf{letrec}}$-expressibility we showed that in [9], and for λ_{μ}-expressibility in [11, 12]; slides with many suggestive illustrations can be found in [7].

   Part of [9], and described separately in [24], is a non-trivial proof of confluence of a higher-order rewriting system that defines the unfolding semantics for $\lambda_{\mathsf{letrec}}$-terms. Furthermore in [10] we showed confluence of let-floating operations on $\lambda_{\mathsf{letrec}}$-terms, obtaining a unique-normal-form result for let-floating, by using a higher-order rewriting system for the formalization of let-floating.

2. Term graph representations of cyclic λ-terms.

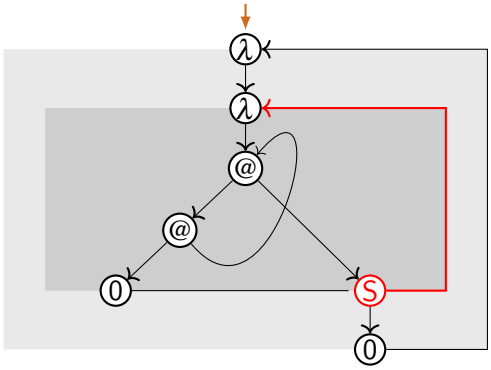


λ-*higher-order term graph*, version 1:
term graph with scope sets
as structure constraints

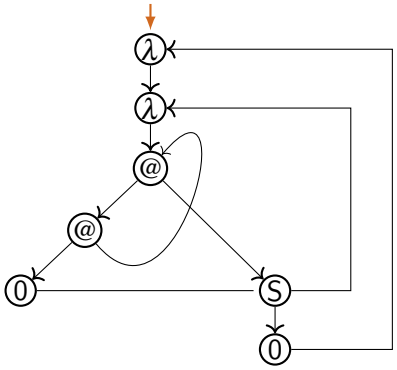


λ-*higher-order term graph*, version 2:
term graph with abstraction prefix function
as structure constraint

Figure 1: Translation of the $\lambda_{\mathsf{letrec}}$-term $L_0 := \lambda x.\lambda f.\mathsf{let}\ r = f\,r\,x\ \mathsf{in}\ r$ into a λ-higher-order term graph with scope sets à la Blom (left), and a λ-h-o term graph $[\![L_0]\!]_{\mathcal{H}}$ with an abstraction-prefix function (right). Note that the inner scope has been chosen minimally here, applying eager scope closure. (Non-eager scope λ-ho-term-graphs can be defined as well, but are not expedient for maximal sharing.)



first-order term graph with
scope vertices with backlinks (+ scope sets)



λ-*term graph*: first-order term graph
with scope vertices with backlinks

Figure 2: Translation of the $\lambda_{\mathsf{letrec}}$-term $L_0 := \lambda x.\lambda f.\mathsf{let}\ r = f\,r\,x\ \mathsf{in}\ r$ into a λ-term-graph $[\![L_0]\!]_{\mathcal{T}}$ by adding a scope vertex delimiting the inner scope to the λ-higher-order term graphs in Fig. 1, and by then dropping the scope sets (which now can be reconstructed as well as a corresponding abstraction prefix function). While the backlink from the left variable vertex to its binding abstraction vertex is drawn suggestively along the scope border, it does not target the scope-delimiting vertex, but continues invisibly below the backlink of that scope-delimiting vertex onwards to the commonly targeted abstraction vertex. (While not relevant for maximal sharing, relaxing the condition of eager scope closure for λ-ho-term-graphs can be dealt with by an adapted encoding as first-order term graph.)

In [13, 16] we systematically investigated a range of natural options for faithfully representing the cyclic λ-terms in $\lambda_{\mathsf{letrec}}$ by higher-order term graphs (first-order term graphs with additional features that describe scopes), and by first-order term graphs (with specific scope-delimiting vertices).

As the result of this analysis we arrived at a natural class of higher-order term graphs (of 'λ-ho-term-graphs', see below) that can be implemented faithfully as first-order term graphs ('λ-term-graphs', see below). The basis of the higher-order term graphs (as well as of their first-order implementations) for representing $\lambda_{\mathsf{letrec}}$-terms are first-order term graphs with three different kinds of vertex labels:

- unary symbols λ for abstraction vertices,
- binary symbols @ for application vertices, and
- unary symbols 0 for nameless variable vertices that enable backlinks to the binding abstraction vertices.

The first-order λ-term-graphs also permit:

- binary symbols S for scope-delimiting vertices that facilitate backlinks to abstraction vertices.

With this preparations we can now explain the higher-order λ-ho-term-graphs and first-order λ-term-graphs in more detail. For the precise definitions and statements we refer to [13, 16, 14].

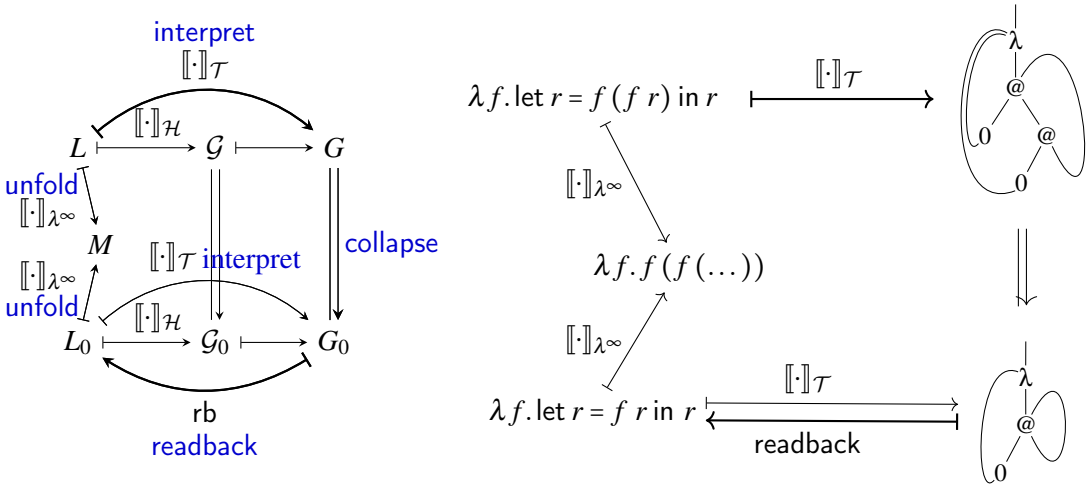
λ-ho-term-graphs appear in two versions:

λ-ho-term-graphs with scope-sets are extensions of first-order term graphs with vertex labels λ, @, and 0 by adding, to each abstraction vertex w, a scope set that consists of all vertices in the scope of w. The scope sets of abstraction vertices in a λ-ho-term-graph satisfy a number of conditions that safeguard that (i) scopes are nested, (ii) scopes arise by eager scope closure, and (iii) each variable vertex is contained in the scope of the abstraction vertex to which its backlink points to. In this way, scope sets aggregate scope information that is available locally at the abstraction vertices.
λ-term-graphs with scope sets are an adaptation of Blom's of higher-order term graphs with scope sets [4] to representing the cyclic λ-terms in $\lambda_{\mathsf{letrec}}$ (and the strongly regular infinite λ-terms in λ^{∞}). For an example, see Figure 1 on the left for the translation of a (variant) fixed-point combinator into a λ-ho-term-graph with (eager-scope) scope sets.

λ-ho-term-graphs with abstraction-prefix function are extensions of first-order term graphs with vertex labels λ, @, and 0 by adding an abstraction prefix function: That function assigns, to each vertex w, an abstraction prefix $(v_1 \dots v_n)$ consisting of a word of abstraction vertices that lists those abstractions (from the top down) for which w is in their 'extended scope' (transitive closure of scope relation) as obtained by eager scope closure. Abstraction prefixes aggregate scope information that then is locally available at individual vertices.
See Figure 1 on the right for the translation of a (variant) fixed-point combinator into a λ-ho-term-graph with abstraction prefixes (obtained by eager scope closure).

In both versions of λ-ho-term-graph, the added constraints guarantee that each variable vertex (with label 0) has a backlink to the binding λ-abstraction vertex. A bijective correspondence can be shown to exist between both versions of λ-ho-term-graphs (see [13, 16]).

(λ-term-graphs) are first-order term graphs that represent λ-ho-term-graphs of both kinds as above. Scopes are delimited, again by using eager scope closure, by scope-delimiting vertices (with label S) that have backlinks to the abstraction vertex whose scope is closed. Variable vertices (with label 0) have backlinks to the binding λ-abstraction vertex.
See Figure 2 for the encoding of the λ-ho-term-graphs in Figure 1 into a λ-term-graph. For this purpose a scope-delimiter vertex with label S is used to represent the (eager) closure of the inner scope.



(1) term graph interpretations $[\![\cdot]\!]$. of $\lambda_{\mathsf{letrec}}$-term L as:
   a. higher-order term graph $\mathcal{G} = [\![L]\!]_{\mathcal{H}}$
   b. first-order term graph $G = [\![L]\!]_{\mathcal{T}}$

(2) bisimulation collapse $\Downarrow$ of first-order term graph G with as result G_0

(3) readback rb of first-order term graph G_0 yielding $\lambda_{\mathsf{letrec}}$-term $L_0 = \mathsf{rb}(G_0)$.

Figure 3: Schematic representation of the maximal sharing method, and its application to a toy example: Maximal sharing of a $\lambda_{\mathsf{letrec}}$-term L proceeds via three steps: (1) interpretation of L as a λ-term-graph $G = [\![L]\!]_{\mathcal{T}}$, (2) collapse of G via bisimilarity to λ-term-graph G_0, and (3) readback of $\lambda_{\mathsf{letrec}}$-term L_0 from G_0. On the top right these steps are illustrated for a redundant $\lambda_{\mathsf{letrec}}$-term formulation of a fixed-point combinator, yielding an efficient representation of fixed-point combinator as $\lambda_{\mathsf{letrec}}$-term.

The conditions underlying λ-ho-term-graphs and λ-term-graphs (see [13, 16]) guarantee that they represent finite or infinite closed λ-terms; that is, they do not contain meaningless parts. Both λ-ho-term-graphs (all two versions) and λ-term-graphs induce appropriate concepts of homomorphism (functional bisimulation) and bisimulation. Homomorphisms increase sharing, and introduce a sharing (partial) order. Bisimulations preserve the unfolding semantics (as do homomorphisms). We established in [13, 16] a bijective correspondence between λ-ho-term-graphs and λ-term-graphs that preserves and reflects homomorphisms, and hence the sharing (partial) order. These results form the basis of the maximal-sharing method, see below.

The property that is of the most central importance for the maximal-sharing method is that homomorphisms (functional bisimulations) between first-order term graphs preserve λ-term-graphs: if G_1 is a λ-term-graph, and $G_1 \trianglerighteq G_2$ for a term graph G_2 (there is a homomorphism from G_1 to G_2), then also G_2 is a λ-term-graph. For this property to hold, eager scope closure is crucial.[2]

[2]While that is not relevant for the maximal-sharing method (for which use of eager scope closure is essential), we mention as an aside that this restriction can be circumnavigated: a generalization of the preservation property can also be shown for a different kind of encoding of (also non-eager-scope) λ-ho-term-graphs into first-order term graphs (see Remark 7.10 in [16]).

3. Maximal sharing in $\lambda_{\mathsf{letrec}}$.

For defining maximally shared versions of terms in $\lambda_{\mathsf{letrec}}$ in a natural way we defined a 'representation pipeline' in [14, 15] (see Figure 3 for a suggestive illustration): First we linked $\lambda_{\mathsf{letrec}}$-terms by an interpretation function $[\![\cdot]\!]_{\mathcal{H}}$ to the class $\mathcal{H}$ of λ-ho-term-graphs that we formulated earlier in [13, 16]. Then we extended $[\![\cdot]\!]_{\mathcal{H}}$ by using the representation of λ-ho-term-graphs as λ-term-graphs (first-order term graphs) from [13, 16] to define an interpretation function $[\![\cdot]\!]_{\mathcal{T}}$ of $\lambda_{\mathsf{letrec}}$-terms to the class $\mathcal{T}$ of λ-term-graphs. For this representation pipeline we showed that unfolding equivalence $=_{[\![\cdot]\!]_{\lambda^\infty}}$ of $\lambda_{\mathsf{letrec}}$-terms is faithfully represented by bisimulation equivalence $\leftrightarrows$ on λ-ho-term-graphs, and equally, by bisimulation equivalence $\leftrightarrows$ on λ-term-graphs.

Then we defined a readback operation rb on λ-term-graphs in the class $\mathcal{T}$ (see also in Figure 3) with the property that the interpretation operation $[\![\cdot]\!]_{\mathcal{T}}$ is a left-inverse of rb on $\mathcal{T}$:

$$[\![\cdot]\!]_{\mathcal{T}} \circ \mathsf{rb} = \mathrm{id}_{\mathcal{T}} \quad \text{(modulo isomorphism)}.$$

These three operations facilitate to compute, for any given $\lambda_{\mathsf{letrec}}$-term L, a maximally shared form L_0, by the following three-step procedure (see Figure 3):

(interpret) from L its interpretation $[\![L]\!]_{\mathcal{T}}$ as λ-term-graph is obtained,

(collapse) from the λ-term-graph $[\![L]\!]_{\mathcal{T}}$ its bisimulation collapse G_0 is computed, which is again a λ-term-graph in $\mathcal{T}$ (due to preservation of λ-term-graphs along functional bisimulations),

(readback) from the collapsed λ-term-graph G_0 its readback $\mathsf{rb}(G_0)$ is computed, thereby obtaining the term $L_0 := \mathsf{rb}(G_0)$ as a maximally shared from L with $L_0 =_{[\![\cdot]\!]_{\lambda^\infty}} L$ (and hence so that L_0 has the same infinite unfolding as L).

This procedure permits an efficient implementation. We could derive its complexity as (at about) quadratic in the size of the input $\lambda_{\mathsf{letrec}}$-term.

See Figure 4 for an example of the collapse step on the λ-term-graph interpretation $[\![L]\!]_{\mathcal{T}}$ of an inefficient version L of a (slight variation of a) fixed-point combinator to obtain the λ-term-graph interpretation $[\![L_0]\!]_{\mathcal{H}}$ that obtains a more efficient and compact version L_0 of such a combinator.

A straightforward adaptation of this procedure permits to obtain also an efficient algorithm for deciding unfolding-semantics equality $=_{[\![\cdot]\!]_{\lambda^\infty}}$ of any two given $\lambda_{\mathsf{letrec}}$-terms L_1 and L_2 by the following two-step procedure:

(interpret) obtain the λ-term-graph interpretations $G_1 := [\![L_1]\!]_{\mathcal{T}}$ of L_1 and $G_2 := [\![L_2]\!]_{\mathcal{T}}$ of L_2;

(check-bisim) check bisimilarity of G_1 and G_2; if $G_1 \leftrightarrows G_2$ holds, conclude that $L_1 =_{[\![\cdot]\!]_{\lambda^\infty}} L_2$ holds (that is, L_1 and L_2 have the same infinite unfolding), otherwise $L_1 \neq_{[\![\cdot]\!]_{\lambda^\infty}} L_2$ holds.

We implemented both the maximal-sharing method and the decision procedure for unfolding equivalence $=_{[\![\cdot]\!]_{\lambda^\infty}}$ by a prototype implementation [25] that is available on Haskell's Hackage platform. For the efficient implementation of these methods we extended the representation pipeline from λ-term-graphs further to λ-DFAs, by which we mean representations of $\lambda_{\mathsf{letrec}}$-terms as deterministic finite-state automata. In this way, unfolding equivalence $=_{[\![\cdot]\!]_{\lambda^\infty}}$ of $\lambda_{\mathsf{letrec}}$-terms is represented as language equivalence of λ-DFAs, and so we could use for the implementation [25] that bisimulation collapse of λ-term-graphs is faithfully represented by state minimization of λ-DFAs.

At the end of this section I want to mention a concept that Vincent van Oostrom suggested after seeing the concept of λ-term-graphs in Jan Rochel's thesis [22]: the concept of 'nested term graphs'.

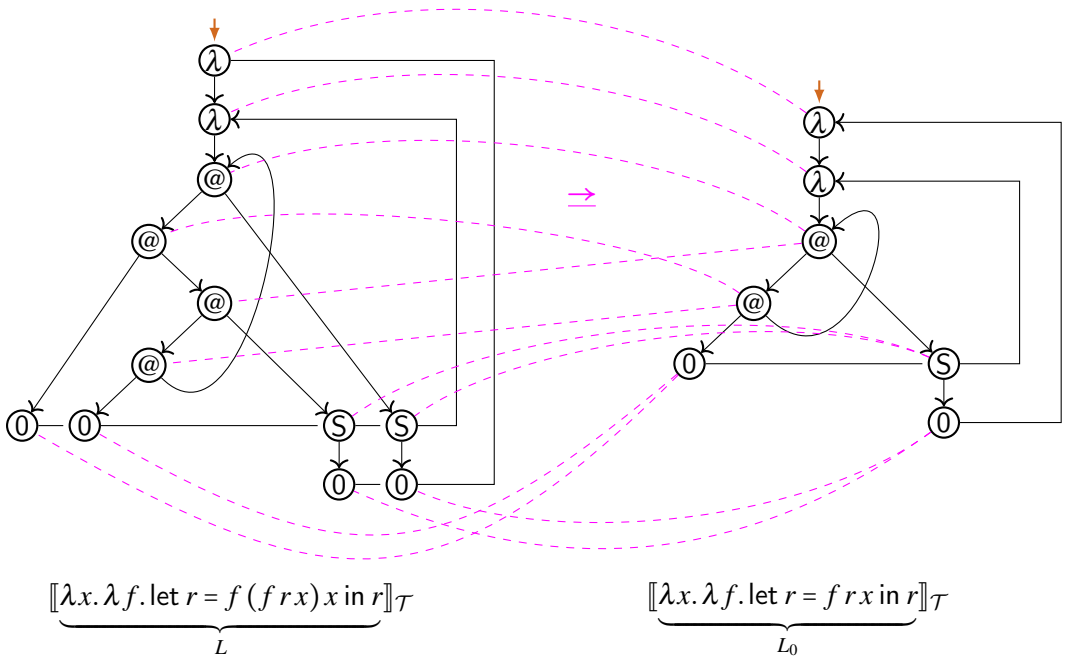


Figure 4: Compactification of the $\lambda_{\mathsf{letrec}}$-term L, a redundant form of a variant fixed-point combinator (compare with the forms in Figure 3), to the more compact $\lambda_{\mathsf{letrec}}$-term L_0. The λ-term-graph interpretations $[\![L]\!]_{\mathcal{T}}$ and $[\![L_0]\!]_{\mathcal{T}}$ of the $\lambda_{\mathsf{letrec}}$ terms L and L_0 are bisimilar. Indeed the links ---- form a functional bisimulation $\twoheadrightarrow$ from $[\![L]\!]_{\mathcal{T}}$ to $[\![L_0]\!]_{\mathcal{T}}$, of which the λ-term-graph $[\![L_0]\!]_{\mathcal{T}}$ is in bisimulation-collapsed form.

Nested Term Graphs

Motivated by the results on term graph representations and maximal sharing for $\lambda_{\mathsf{letrec}}$-terms, Vincent van Oostrom and I formulated a concept of nested term graph [8]. Instead of describing scopes by additional features like scope sets or an abstraction-prefix function in order to define constraints that guarantee that scopes are nested, we introduced 'nesting' itself as a structuring concept. This means that we permitted nesting of first-order term graphs into vertices of other first-order term graphs. In this manner, well-foundedly nested first-order term graphs can be defined by induction. We studied the behavioral semantics of nested term graphs in [8], and also showed, in analogy with the faithful encoding of λ-ho-term-graphs as λ-term-graphs, that nested term graphs can be encoded by first-order term graphs faithfully (in the sense of preserving the respective unfolding semantics).

Nested term graphs not only provide a natural formalization the maximal-sharing method developed in [13, 14], but they make it much more broadly applicable, also outside of Lambda Calculus.

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$$\frac{}{1\Downarrow} \qquad \frac{e_i\Downarrow}{(e_1+e_2)\Downarrow}\,(i\in\{1,2\}) \qquad \frac{e_1\Downarrow \qquad e_2\Downarrow}{(e_1\cdot e_2)\Downarrow} \qquad \frac{}{(e^*)\Downarrow}$$

$$\frac{}{a \xrightarrow{a} 1} \qquad \frac{e_i \xrightarrow{a} e_i'}{e_1+e_2 \xrightarrow{a} e_i'}\,(i\in\{1,2\}) \qquad \frac{e_1 \xrightarrow{a} e_1'}{e_1\cdot e_2 \xrightarrow{a} e_1'\cdot e_2} \qquad \frac{e_1\Downarrow \qquad e_2 \xrightarrow{a} e_2'}{e_1\cdot e_2 \xrightarrow{a} e_2'} \qquad \frac{e \xrightarrow{a} e'}{e^* \xrightarrow{a} e'\cdot e^*}$$

Figure 5: Transition system specification $\mathcal{T}$ for computations enabled by regular expressions.

graph-rewriting, graph-rewriting-layout, graph-rewriting-gl, graph-rewriting-ski, graph-rewriting-trs, graph-rewriting-lambdascope, maxsharing. The rewrite system uses Stewart's port graph grammars (PGGs) [56].

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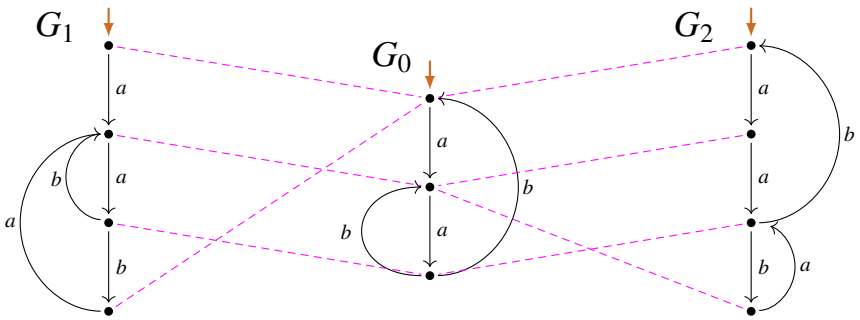
3 Proving Bisimilarity between Regular-Expression Processes

This section motivates, summarizes, and provides references to my work on Milner's process semantics of regular expressions [52]. An important part of it (leading to [49, 50]) was done in close collaboration with Wan Fokkink (2015–2020) who had stimulated me to work on Milner's question already in 2005. While this section focuses on my work on Milner's axiomatization questions (see **(A)** below), my current work on the expressibility question (see **(E)** below) will be mentioned in Section 4.

Milner introduced a process semantics $[\![\cdot]\!]_P$ in [52] for regular expressions (conceived by Kleene [51]) that refines the standard language semantics $[\![\cdot]\!]_L$ (defined by Copi, Elgot, Wright [32]). For regular expressions e that are constructed from constants 0, 1, letters over a given set A with the binary operators + and $\cdot$, and the unary operator $(\cdot)^*$, Milner first defined a process interpretation $P(e)$ that can informally be described as follows: 0 is interpreted as a deadlocking process without any observable behavior, 1 as a process that terminates successfully immediately, letters from the set A stand for atomic actions that lead to successful termination; the binary operators + and $\cdot$ are interpreted as the operations of choice and concatenation of two processes, respectively, and the unary star operator $(\cdot)^*$ is interpreted as the operation of unbounded iteration of a process, but with the option to terminate successfully before each iteration.

Milner formalized this process interpretation in [52] as process graphs that are defined by induction on the structure of regular expressions. But soon afterwards a formal definition by means of a transition system specification (TSS) that defines a labeled transition system (LTS) became more common. The TSS $\mathcal{T}$ in Figure 5 defines, via derivations that it permits from its axioms, labeled transitions $\xrightarrow{a}$ for actions a that occur in a regular expressions, and immediate successful termination via the unary predicate



$P(a\cdot((a\cdot(b+b\cdot a))^*\cdot 0))$	$? \in im(P(\cdot))\ ?$	$P((1\cdot(((a\cdot a)\cdot(b\cdot a)^*)\cdot b)^*)\cdot 0)$
P-expressible, hence $[\![\cdot]\!]_P$-expr.	$[\![\cdot]\!]_P$-expressible	P-expressible, hence $[\![\cdot]\!]_P$-expr.
$a\cdot((a\cdot(b+b\cdot a))^*\cdot 0)$	$=_{[\![\cdot]\!]_P}$	$(1\cdot(((a\cdot a)\cdot(b\cdot a)^*)\cdot b)^*)\cdot 0$

Figure 6: Two process graphs G_1 and G_2 that are P-expressible, and hence $[\![\cdot]\!]_P$-expressible, because they are the process interpretations of regular expressions as indicated. G_1 and G_2 are bisimilar via bisimulations that are drawn as links - - - to their joint bisimulation collapse G_0 (of which P-expressibility is at first unclear). It follows that also G_0 is $[\![\cdot]\!]_P$-expressible, and that process semantics equality holds between the regular expressions with interpretations G_1 and G_2, respectively. In this example G_0 is actually also in the image of $P(\cdot)$, hence P-expressible, as witnessed for example by $G_0 = P(((1\cdot a)\cdot(a\cdot(b+b\cdot a))^*)\cdot 0)$.

$\Downarrow$. The process interpretation $P(e)$ of a regular expression e is then defined as the sub-LTS that is induced by e in the LTS on regular expressions that is defined via derivability in $\mathcal{T}$. See Figure 6 for suggestive examples of (bisimilar) process interpretations of two simple regular expressions. In process graph illustrations there and later we indicate the start vertex by a brown arrow ➙, and the property of a vertex v to permit immediate successful termination by emphasizing v in brown as ◉ including an boldface ring.

It is interesting to note that the so-defined process interpretation of regular expressions corresponds directly to non-deterministic finite-state automata (NFAs) that are defined via iterations of Antimirov's partial derivatives [27].[3]

Based on the process interpretation $P(\cdot)$, Milner then defined the process semantics of a regular expression e as $[\![e]\!]_P := [P(e)]_{\leftrightarrows}$ where $[P(e)]_{\leftrightarrows}$ is the equivalence class of $P(e)$ with respect to bisimilarity $\leftrightarrows$. In analogy to how language-semantics equality $=_{[\![\cdot]\!]_L}$ of regular expressions is defined from the language semantics $[\![\cdot]\!]_L$ (namely as $e =_{[\![\cdot]\!]_L} f$ if $L(e) = [\![e]\!]_L = [\![f]\!]_L = L(e)$, for all regular expressions e and f, where $L(g)$ is the language defined by a regular expression g) Milner was then interested in process-semantics equality $=_{[\![\cdot]\!]_P}$ that is defined, for all regular expressions e and f by:

$$\begin{aligned} e =_{[\![\cdot]\!]_P} f \ &:\Longleftrightarrow\ [\![e]\!]_P = [\![f]\!]_P \\ &\Longleftrightarrow\ P(e) \leftrightarrows P(f)\,. \end{aligned}$$

As the process interpretations of the regular expressions in Figure 7 are bisimilar, it follows that these regular expressions are linked by $=_{[\![\cdot]\!]_P}$.

Milner realized in [52] that the process semantics $[\![\cdot]\!]_P$ of regular expressions differs from the language semantics $[\![\cdot]\!]_L$ in at least two respects: first, $[\![\cdot]\!]_P$ is incomplete, and second, process-semantics

[3] Antimirov did not have a process semantics in mind, but he had set out to define, for every regular expression e, an NFA that is typically smaller than the deterministic automaton (DFA) as usually associated with e in automata and language theory.

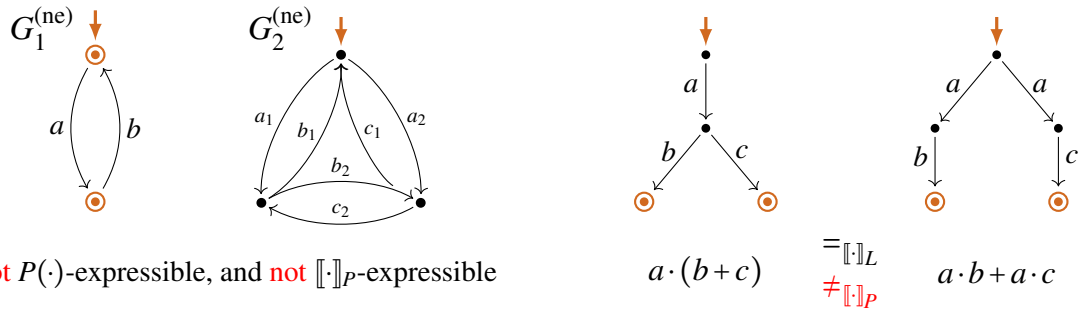


Figure 7: On the left: Two process graphs that are neither $P(\cdot)$-expressible (that is, not in the image of the process interpretation P) nor $[\![\cdot]\!]_P$-expressible (that is, not bisimilar to the process interpretation of any regular expression). On the right: two regular expressions with the same language semantics (associated language) but different process semantics, since the process interpretations are not bisimilar; therefore right-distributivity does not hold for $=_{[\![\cdot]\!]_P}$, which entails that fewer identities hold for $=_{[\![\cdot]\!]_P}$ than for $=_{[\![\cdot]\!]_L}$.

equality $=_{[\![\cdot]\!]_P}$ satisfies fewer identities than language-semantics equality $=_{[\![\cdot]\!]_L}$.

We start by explaining incompleteness of $[\![\cdot]\!]_P$. Language semantics $[\![\cdot]\!]_L$ is complete in the following sense: every language that can be accepted by a finite-state automaton (a regular language) is the language that is defined by a regular expression; that is every regular language is $[\![\cdot]\!]_L$-expressible. However, a comparable statement does not hold for the process interpretation. We call a finite process graph $[\![\cdot]\!]_P$-*expressible* if it is bisimilar to a *P-expressible* process graph, by which we mean the process interpretation of some regular expression (and hence a graph in the image of $P(\cdot)$. While it is easier to argue that not every finite process graph is P-expressible, there are also finite process graphs that are not $[\![\cdot]\!]_P$-expressible, either. Milner proved in [52] that the process graph $G_2^{(ne)}$ in Figure 7 not only is not P-expressible, but that it is also not $[\![\cdot]\!]_P$-expressible. He also conjectured that also $G_1^{(ne)}$ in Figure 7 is not $[\![\cdot]\!]_P$-expressible; this was later shown by Bosscher [31].

Milner also noticed in [52] that some identities that hold for language-semantics equality $=_{[\![\cdot]\!]_L}$ are not true any longer for process semantics equality $=_{[\![\cdot]\!]_P}$. Most notably this is the case for right-distributivity $e \cdot (f+g) = e \cdot f + e \cdot g$, which is violated just as for the comparison of process terms via bisimilarity; see the well-known counterexample in Figure 7. The language-semantics identity $e \cdot 0 = 0$ is also violated in the process semantics. In order to define a natural sound adaptation (that we here designated by) Mil, see Figure 8, of the complete axiom systems for $=_{[\![\cdot]\!]_L}$ by Aanderaa [26] and Salomaa [53], Milner dropped these two identities from Aanderaa's system, but added the sound identity $0 \cdot e = 0$.

These two pecularities of the process semantics led Milner to formulating two questions concerning recognizability of expressible process graphs, and axiomatizability of process-semantics equality:

(E) How can $[\![\cdot]\!]_P$-expressible process graphs be characterized structurally, that is, those finite process graphs that are bisimilar to process interpretations of regular expressions?

(A) Is the natural adaptation Mil to process-semantics equality $=_{[\![\cdot]\!]_P}$ (see Figure 8 for Mil) of Salomaa's and Aanderaa's complete proof systems for language-semantics equality $=_{[\![\cdot]\!]_L}$ complete for $=_{[\![\cdot]\!]_P}$?

The expressibility question **(E)** seems to have received only limited attention at first. The reason may have been because it asks for a structural property of (the $[\![\cdot]\!]_P$-expressible) process graphs that is invariant under bisimilarity. This is a difficult aim, because bisimulations can significantly distort the topological structure of labeled transition graphs. Two variants of **(E)** have been solved after some time:

(A1) $e+(f+g)=(e+f)+g$
(A2) $e+0=e$
(A3) $e+f=f+e$
(A4) $e+e=e$
(A5) $e\cdot(f\cdot g)=(e\cdot f)\cdot g$
(A6) $(e+f)\cdot g=e\cdot g+f\cdot g$
(A7) $e=1\cdot e$
(A8) $e=e\cdot 1$
(A9) $0=0\cdot e$
(A10) $e^*=1+e\cdot e^*$
(A11) $e^*=(1+e)^*$

$$\frac{e=f\cdot e+g}{e=f^*\cdot g}\ \mathrm{RSP}^*\ (\text{if } f\not\Downarrow)$$

Figure 8: Milner's equational proof system Mil for process semantics equality $=_{[\![\cdot]\!]_P}$ of regular expressions with the fixed-point rule RSP* in addition to the (not shown) basic rules for reasoning with equations (which guarantee that derivability in Mil is a congruence relation). From Mil the complete proof system for language equivalence $=_{[\![\cdot]\!]_L}$ due to Aanderaa arises by adding the axioms $e\cdot(f+g)=e\cdot f+e\cdot g$ and $e\cdot 0=0$ (which are not sound for $=_{[\![\cdot]\!]_P}$) and by dropping (A9) (which then is derivable).

First, the question for a natural sufficient condition for $[\![\cdot]\!]_P$-expressibility of process graphs was answered by Baeten and Corradini in [28] by the definition of process graphs that satisfy 'well-behaved' recursive specifications. Second, the question of whether $[\![\cdot]\!]_P$-expressibility of finite process graphs is decidable was answered by Baeten, Corradini, and myself in [29] by giving a decision procedure (unfortunately it is highly super-exponential) that is based on minimizing well-behaved specifications under bisimilarity.

For the axiomatization problem **(A)** at first only a string of partial results have been obtained. In particular Milner's proof system Mil has initially been shown to be complete for $=_{[\![\cdot]\!]_P}$ for the following subclasses of regular expressions:

(a) without 0 and 1, but with binary star iteration $e_1{}^{\circledast}e_2$ with iteration-part e_1 and exit-part e_2 instead of unary star (Fokkink and Zantema, 1994, [36]),

(b) with 0, and with iterations restricted to exit-less ones $(\cdot)^*\cdot 0$ in absence of 1 (Fokkink, 1997, [35]) and in the presence of 1 (Fokkink, 1996 [34]),

(c) without 0, and with restricted occurrences of 1 (Corradini, De Nicola, and Labella, 2002 [33]),

(d) 1-*free* expressions formed with 0, without 1, but with binary iteration $\circledast$ (G, Fokkink, 2020, [49, 50], also showing the completeness of a proof system by Bergstra, Bethke, and Ponse [30]).

While the maximal subclasses in (c) and (d) are incomparable, these results can be joined to apply to an encompassing class that is still a proper subclass of the regular expressions, see [49]. Independently of these partial results concerning completeness of Milner's system Mil for subclasses of regular expressions, I noticed in [37] that from Mil a proof system that is complete for $=_{[\![\cdot]\!]_P}$ arises when the single-equation fixed-point rule RSP* is replaced by a unique-solvability principle USP for systems of guarded equations. Also in [37] I formulated a coinductively motivated proof system for process-semantics equality $=_{[\![\cdot]\!]_P}$ that utilizes Antimirov's partial derivatives [27] of regular expressions.

The principal new idea that facilitated the partial completeness result (d) in [49, 50] of Mil for 1-free regular expressions consisted in formulating a natural structural condition that is sufficient (but not necessary) for $[\![\cdot]\!]_P$-expressibility of process graphs: the *Loop Existence and Elimination Condition* LEE, and its layered form LLEE. This condition is based on the concept of 'loop (process) graph', and an elimination process of 'loop subgraphs' from a given process graph. A process graph G is said to have the property LEE if the non-deterministic iterative procedure, started on G, of repeatedly eliminating loop subgraphs is able to obtain a process graph without an infinite behavior (that is, a graph without infinite paths and traces). We explain the definitions in some more detail below, and provide examples.

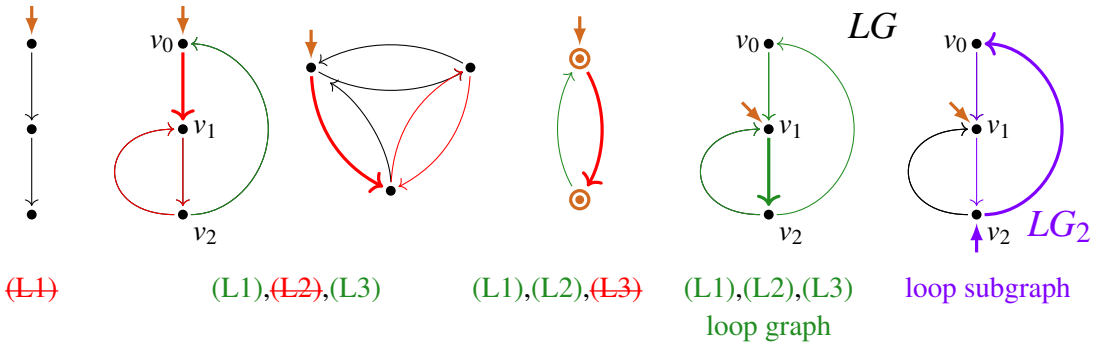


Figure 9: Four process graphs (action labels ignored) that violate at least one loop graph condition (LG1), (LG2), or (LG3), and a loop graph LG with one of its loop subgraph LG_2.

A process graph LG is called a *loop (process) graph* if it satisfies the following three conditions:

(LG1) There is an infinite trace from the start vertex of LG.

(LG2) Every infinite trace from the start vertex v_s of LG returns to v_s.

(LG3) Immediate successful termination is only possible at the start vertex of LG.

In such a loop graph LG, the transitions from the start vertex are called *loop-entry transitions*, and all other transitions are called *loop-body transitions*. By a *loop subgraph* of a process graph G we mean a graph LG such that with respect to a vertex v of G, and a non-empty set T of transitions of G that depart from v the following three conditions are satisfied:

(LSG1) LG is a subgraph of G with start vertex v (which may be different from the start vertex v_s of G).

(LSG2) LG is generated by the transitions T from v in the following sense: LG contains all vertices and transitions of G that are reachable on traces that start from v via transitions in T, and continue onward until v is reached again for the first time.

(LSG3) LG is a loop graph.

In accordance with the stipulation for loop graphs, in such a loop subgraph LG the transitions in T are called *loop-entry* transitions of LG, and all others *loop-body* transitions of LG. In Figure 9 we have gathered, on the left, four examples of process graphs (with action labels ignored) that are *not* loop graphs: each of them violates one of the conditions (LG1), (LG2), or (LG3). The paths in red indicate violations of (LG2), and (LG3), respectively, where the thicker arrows from the start vertex indicate transitions that would need to be (but are not) loop-entry transitions. However, the loop subgraph LG_2 in Figure 9 is indeed a loop graph.

Based on these concepts, elimination of loop subgraphs is then defined as follows. We say that G' is the result of *eliminating a loop subgraph* LG with set T of loop-entry transitions *from* a process graph G, and denote such an elimination step by $G \Rightarrow_{\text{elim}} G'$, if G' results from G by first removing the transitions in T and by then applying garbage collection of vertices and transitions that have become unreachable from the start vertex of G due to the transition removals. See Figure 10 for an example of three loop elimination steps. As for non-examples, note that neither of two not $[\![\cdot]\!]_P$-expressible graphs $G_1^{(\text{ne})}$ and $G_2^{(\text{ne})}$ in Figure 7 are loop graphs, nor do they contain loop subgraphs; hence neither of $G_1^{(\text{ne})}$ and $G_2^{(\text{ne})}$ permits a loop-elimination step.

We say that a process graph G *has the property* LEE (resp. *has the property* LLEE (*layered* LEE)) if there is a finite sequence of loop-elimination steps $G \Rightarrow^*_{\text{elim}} G'$ from G such that the resulting graph G'

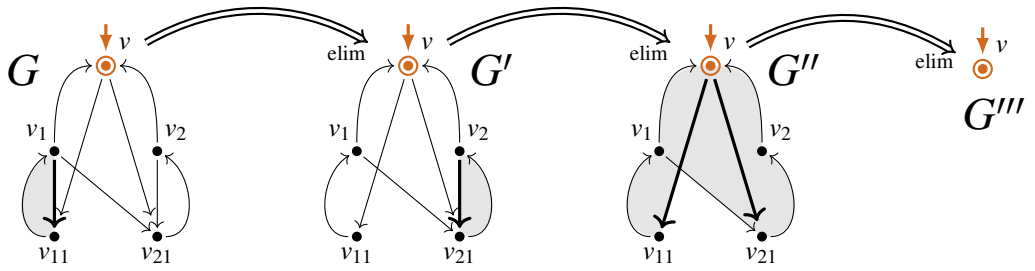


Figure 10: Example of successful loop elimination from the process graph G: three elimination steps of loop subcharts, which are represented as shaded gray areas, lead to the process graph G''' without infinite behaviour. These steps witness that G satisfies the properties LEE and LLEE (as well as do G', G'', G''').

does not permit an infinite trace (and resp., if additionally during the elimination steps in $G \Rightarrow^*_{\mathrm{elim}} G'$ it never happens that a transition is removed that was a loop-body transitions of a loop subgraph that was eliminated in an earlier step). It can be shown that although the property LLEE is a formally stronger requirement than the property LEE, which often helps to simplify proofs, both properties are equivalent. See Figure 10 for an example of a process graph G with the properties LEE and LLEE as is witnessed there by a sequence of three loop elimination steps that lead to graph G''' without infinite traces. The not $[\![\cdot]\!]_P$-expressible graphs $G_1^{(\mathrm{ne})}$ and $G_2^{(\mathrm{ne})}$ in Figure 7 do not satisfy LLEE and LEE, since loop elimination is not successful on them: they do not enable loop-elimination steps, but facilitate infinite traces.

The reason why the definition of the properties LEE and LLEE has facilitated progress concerning the problem **(A)** was that they define manageable conditions that could be used for proofs about process graphs that are linked by functional bisimulations. Specifically for obtaining the partial result (d) in [49, 50] it was crucial that we could prove the following facts:

(I)$_1$ Process interpretations of 1-free regular expressions satisfy LLEE (see [50, 49]).

(E)$_1$ Finite process graphs with LLEE are $[\![\cdot]\!]_P$-expressible, by 1-free regular expressions (see [50, 49]).

(C) LLEE is preserved along functional bisimilarity, and consequently, also by the operation of bisimulation collapse (see [50, 49]).

Additionally, the property LLEE permitted me to formulate a coinductive version cMil of Milner's system Mil that also permits cyclic derivations of the form of process graphs with the property LLEE, see [40, 39, 47]. The system cMil could be viewed as being located proof-theoretically half-way in between Mil and bisimulations between process interpretation. As such it could be expected to form a natural beachhead for a completeness proof of Mil.

These results raised my hope that the argumentation could be extended quite directly to the full set of regular expressions (including 1 and with unary iteration instead of binary iteration) as well as to process graphs with 1-transitions and with the property LEE. While the generalization of **(I)**$_1$ to all regular expressions does not hold, this obstacle could be overcome by defining a refined process interpretation with the desired property. Together with a rather straightforward generalization of **(E)**$_1$ we obtained:

~~**(I)**~~ The process interpretation $P(e)$ of a regular expression e does not always satisfy LLEE (nor LEE) (see [38, 42]).

(RI)$_1$ There is a refined process interpretation $\underline{P}(\cdot)$ that produces finite process graphs with 1-transitions such that, for every regular expression e, $\underline{P}(e)$ satisfies LLEE, $\underline{P}(e)$ is a refinement of $P(e)$ by sharing transitions by means of added 1-transitions, and $\underline{P}(e) \leftrightarrows P(e)$, that is, $\underline{P}(e)$ is bisimilar to $P(e)$ when 1-transitions are interpreted as empty steps (see [48], and a slightly weaker statement in [38, 42]).

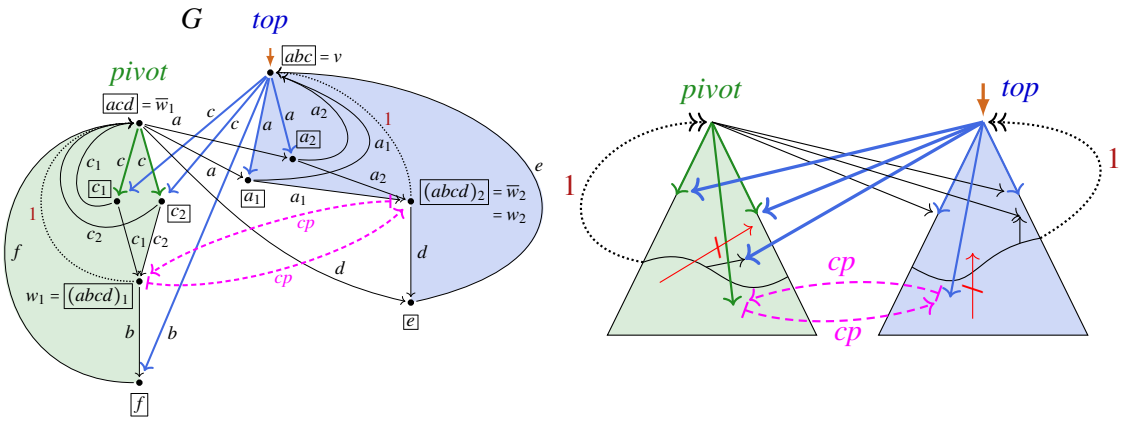


Figure 11: On the left: a finite process graph G with 1-transitions (drawn dotted, representing empty steps) that satisfies LLEE, but cannot be minimized under bisimilarity while preserving LLEE. It is a prototypical example of a twin-crystal. As such it consists of two interlinked parts, the top-part and the pivot-part, which by themselves are bisimulation collapsed, but contain vertices that have bisimilar counterparts in the opposite part of the twin-crystal. The self-inverse counterpart function *cp* links bisimilar vertices in the two parts. On the right: schematic illustration of a twin-crystal with suggestive drawing of its top-part and its pivot-part, together with interconnecting proper transitions from *top* and *pivot*.

(E)$_1$ Finite process graphs with 1-transitions and with LLEE are $[\![\cdot]\!]_P$-expressible. (See [39, 40, 47].)

However, critically, a direct generalization of our argument broke down dramatically due to the fact that the collapse statement **(C)** did not generalize to process graphs with LLEE that contain 1-transitions:

~~**(C)**$_1$~~ LLEE is not preserved under bisimulation collapse of process graphs with 1-transitions. A counterexample holds for the process graph G on the left in Figure 11. (See also [44, 45].)

Therefore the proof strategy we used in [49, 50] for showing completeness of Mil for 1-free regular expressions, turned out not to work for showing completeness of Mil for the full class of regular expressions. At the very least it was in need of a substantial refinement.

What came to my rescue here was that the counterexample for LLEE-preserving collapse of process graphs with 1-transitions and LLEE, the graph G in 11, is of a specific symmetric form. It is a *twin-crystal*, a process graph with 1-transitions and with LLEE that is near-collapsed in the sense that bisimilar vertices appear only as pairs. More precisely, twin-crystals are process graphs with 1-transitions and with LLEE that consist of a single strongly connected component (scc), and of two parts, the top-part and the pivot-part (see in Figure 11 on the right). Each part by itself is bisimulation collapsed, and hence any two bisimilar vertices in the twin-crystal must occur in different of the top and pivot parts, and are linked by a self-inverse (partial) counterpart function. Process graphs with 1-transitions and with LLEE that are collapsed apart from within scc's, and in which all scc's are either collapsed or twin-crystals, we called *crystallized*. For this concept it was possible to show:

(NC)$_1$ Every finite process graph with 1-transitions and with LLEE can be minimized under bisimilarity to obtain a crystallized process graph (see [44, 45, 46].)

This statement is based on an effective *crystallization procedure* of process graphs with LLEE and with 1-transitions: it minimizes all scc's of the graph either to twin-crystals or collapsed parts of the graph, and also guarantees that the resulting graph is collapsed apart from within those scc's that are twin-crystals. The symmetric structure of twin-crystals can then be used to show that self-bisimulations of crystallized

process graphs are of a particularly easy kind, which can be assembled from bisimulation slices that act on the twin-crystal-scc's [41]. This result on crystallized versions of process interpretations permitted me to adapt the proof strategy that Fokkink and I had used previously to also show completeness of Mil for $=_{[\![\cdot]\!]_P}$ on the full class of regular expressions, see [44, 45], and the poster [46].

There is now much hope that the crystallization technique that we developed for solving the axiomatization question **(A)** may turn out to facilitate also significant improvements for answers to the expressibility question **(E)**. We return to the expressibility question **(E)** at the end of the next section.

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4 Current and Future Work

This section touches on my present research, and lists as well as briefly motivates some research questions and projects that have developed out of the work summarized in the past two sections. This is

organized in two subsections that refer to the topics of Section 2 and Section 3, respectively.

4.1 Maximal Sharing at Run Time

Apart from using the maximal-sharing method for functional programs as a static-analysis based optimization transformation during compilation, one of the ideas for applications that Rochel and I gathered in [14] was that maximal sharing could be used as an optimization transformation also repeatedly at run-time. Making that idea fruitful, however, requires that representations of programs that are used in graph evaluators can be linked closely with λ-term-graph representations of $\lambda_{\mathsf{letrec}}$-terms on which the maximal-sharing method operates. This is necessary because graph evaluators in implementations of functional languages typically use supercombinator representations of $\lambda_{\mathsf{letrec}}$-terms, and much computational overhead is to be expected in transformations to and from λ-term-graphs. Yet any such overhead is highly undesirable during program execution. Now supercombinator reduction as carried out by graph evaluators intuitively corresponds to *scope-sharing* forms of β-reduction.[4] And so, since λ-term-graphs contain neatly described scopes of λ-abstractions, the implementation of a scope-sharing form of evaluation on λ-term-graphs is conceivable. These considerations lead me to the following research question.

Research Question 1. *Coupling of maximal sharing with evaluation, generally, and more specifically:*

(i) Can the maximal-sharing method for terms in the λ-calculus with letrec *be coupled naturally with an efficient evaluation method (such as a standard graph-evaluation implementation)?*

(ii) Do λ-term-graphs (which represent $\lambda_{\mathsf{letrec}}$-terms) permit a representation as interaction nets or as port graphs [56] for which a form of β-reduction can be defined that is able to preserve, by adequately chosen multi-steps of interactions, scopes and also λ-term-graph form?

In communication after the workshop Ian Mackie pointed me to his interaction-net based implementation [54, 55] of an evaluation method for the λ-calculus. I am grateful for this reference, first, because this interaction-net representation of λ-terms bears a close resemblance with λ-term-graphs, and second, because it provides a mechanism for implementing scope-preserving forms of β-reduction. Nevertheless it remains a challenging question to relate the two formalisms (λ-term-graphs and interaction-net representations of λ-terms in [54]) closely together. Yet an interaction-net representation of λ-term-graphs close to the representation of λ-terms as used in [54] seems to me to be a plausible and promising in-road for approaching part (ii) of Research Question 1.

4.2 Crystallization: Proof Verification, and Application to the Expressibility Problem

Currently I am writing two articles that will provide the details of the completeness proof of Milner's proof system Mil. The first article is going to explain the motivation of the crystallization process for process interpretations of regular expressions: a limit to the minimization under bisimulation of process graphs that are expressible by a regular expression. This limit will be established specifically for the process graph G in Figure 11 with 1-transitions. The second article will detail the crystallization procedure by which process graphs with the property LLEE (which are $[\![\cdot]\!]_P$-expressible) are minimized under bisimulation to obtain process graphs with LLEE that are close to their bisimulation collapse. This central result will then be used, as explained in [44], to show that Milner's proof system Mil is complete with respect to process semantics equality $=_{[\![\cdot]\!]_P}$.

[4] Note that scope-sharing is distinct from the *context-sharing* forms of graph reduction on which implementations of *parallel* or *optimal β-reduction* are based.

The details of this completeness proof can be explained with clear conceptual concepts and with convincing details. answering Milner's question **(A)** positively. However, a verification of the crystallization procedure and the completeness proof of Mil with respect to $=_{[\![\cdot]\!]_P}$ forms an important goal for me.

Research Project 2. *Formalization of the proofs for crystallization, and completeness of* Mil*:*

(a) Develop formalizations of structure constraints for process graphs in order to verify the correctness of the crystallization procedure for process graphs with LEE *by a proof assistant.*

(b) Use the correctness proof of crystallization to verify the completeness proof of Milner's proof system Mil *by a proof assistant.*

Separately I am working out a proof of the fact that the loop existence and elimination property LEE (and equivalently LLEE) can be decided in polynomial time. For this result the observation is crucial that loop elimination $\Rightarrow_{\text{elim}}$ can be completed to obtain a confluent rewrite system (which is obviously terminating). As a consequence of the efficient decidability of LLEE it will follow that the restriction of the expressibility problem **(E)** to expressibility by regular expressions that are 1-*free under star* (but with unary iteration) can be solved efficiently. This is because the methods and results in [49, 50] permit to show that a finite process graph G is $[\![\cdot]\!]_P$-expressible by a regular expression that is 1-free under star if and only if the bisimulation collapse of G satisfies LLEE. Then it follows that expressibility of finite process graphs by regular expressions that are 1-free under star can be decided in polynomial time.

The crystallization procedure that we use in the completeness proof of Mil with respect to $=_{[\![\cdot]\!]_P}$ suggests that an extension of this characterization statement to one for $[\![\cdot]\!]_P$-expressibility in full generality is conceivable. We formulate that as our final research question.

Research Question 3. *Is the problem of whether a finite process graph is $[\![\cdot]\!]_P$-expressible efficiently decidable? That is, is there a polynomial decision algorithm for it? Or is $[\![\cdot]\!]_P$-expressibility at least fixed-parameter tractable (in* FPT*) for interesting parameterizations?*

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