

Lecture 3: Recursive Functions

Models of Computation

<https://clegra.github.io/moc/moc.html>

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July 9, 2025

Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
<i>intro</i>	<i>classic models</i>			<i>additional models</i>
Introduction to Computability	Machine Models	Recursive Functions	Lambda Calculus	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = Turing-computable, Church's Thesis	λ -terms, β -reduction, λ -definable functions, partial recursive = λ -definable = Turing computable	
	<i>imperative programming</i>	<i>algebraic programming</i>	<i>functional programming</i>	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran
				comparing computational power

Summary

Recursive functions

- ▶ **primitive recursive** functions
- ▶ Gödel–Herbrand(–Kleene) general recursive functions
- ▶ **partial recursive** functions
 - ▶ defined with μ -recursion (unbounded minimisation)
- ▶ Partial recursive functions = Turing computable functions
- ▶ **Church's thesis**
 - ▶ **effectively calculable** functions $\hat{=}$ partial-recursive functions
 - ▶ some debate

Timeline: From logic to computability

- 1900 Hilbert's 23 Problems in mathematics
- 1910/12/13 Russell/Whitehead: Principia Mathematica
- 1928 Hilbert/Ackermann: formulate completeness/decision problems for the predicate calculus (the latter called '[Entscheidungsproblem](#)')
- 1929 Presburger: completeness/decidability of theory of addition on \mathbb{Z}
- 1930 Gödel: completeness theorem of predicate calculus
- 1931 Gödel: incompleteness theorems for first-order arithmetic
- 1932 Church: λ -calculus
- 1933/34 [Herbrand/Gödel: general recursive functions](#)
- 1936 [Church/Kleene: \$\lambda\$ -definable ~ general recursive](#)
[Church Thesis](#): 'effectively calculable' be defined as either
Church shows: the '[Entscheidungsproblem](#)' is unsolvable
Post: [machine model](#); Church's thesis as 'working hypothesis'
- 1937 Turing: convincing analysis of a 'human computer' leading to the '[Turing machine](#)'

Turing-computable (total) functions

Definition

A **total function** $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is **Turing-computable** if there exists a Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \blacksquare, F \rangle$ and a **calculable** coding function $\langle \cdot \rangle : \mathbb{N} \rightarrow \Sigma^*$ such that:

- ▶ for all $n_1, \dots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:

$$q_0 \langle n_1 \rangle \blacksquare \langle n_2 \rangle \blacksquare \dots \blacksquare \langle n_k \rangle \vdash_M^* q \langle f(n_1, \dots, n_k) \rangle$$

Recursive Functions

Functions defined by recursive equations:

like e.g. functions $+, \cdot, (\cdot)' : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, and $(\cdot)! : \mathbb{N} \rightarrow \mathbb{N}$:

$$n + 0 = n$$

$$n \cdot 0 = 0$$

$$n + (m + 1) = (n + m) + 1 \quad n \cdot (m + 1) = n \cdot m + n$$

$$n^0 = 1$$

$$0! = 1$$

$$n^{m+1} = n^m \cdot n$$

$$(n + 1)! = (n + 1) \cdot n!$$

Primitive recursive functions: defined by such equations (termination of the evaluation process guaranteed)

General recursive functions: defined by more general systems of equations

μ -Recursive (partial recursive) functions: extend the primitive recursive functions by a μ -operator that allows to construct partial functions

Róza Péter



Róza Péter (1905–1977)

Primitive recursive functions ($\mathbb{N}^k \rightarrow \mathbb{N}$)

Base functions:

- ▶ $\mathcal{O} : \mathbb{N}^0 = \{\emptyset\} \rightarrow \mathbb{N}$, $\emptyset \mapsto 0$ (0-ary constant-0 function)
- ▶ $\text{succ} : \mathbb{N} \rightarrow \mathbb{N}$, $x \mapsto x + 1$ (successor function)
- ▶ $\pi_i^n : \mathbb{N}^n \rightarrow \mathbb{N}$, $\vec{x} = \langle x_1, \dots, x_n \rangle \mapsto x_i$ (projection function)

Closed under operations:

- ▶ **composition**: if $f : \mathbb{N}^k \rightarrow \mathbb{N}$, and $g_i : \mathbb{N}^n \rightarrow \mathbb{N}$ are prim. rec., then so is $h = f \circ (g_1 \times \dots \times g_k) : \mathbb{N}^n \rightarrow \mathbb{N}$:

$$h(\vec{x}) = f(g_1(\vec{x}), \dots, g_k(\vec{x}))$$
- ▶ **primitive recursion**: if $f : \mathbb{N}^n \rightarrow \mathbb{N}$, $g : \mathbb{N}^{n+2} \rightarrow \mathbb{N}$ are prim. rec., then so is $h = \text{pr}(f; g) : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$:

$$h(\vec{x}, 0) = f(\vec{x})$$

$$h(\vec{x}, y + 1) = g(\vec{x}, h(\vec{x}, y), y)$$

Primitive recursive functions ($\mathbb{N}^n \rightarrow \mathbb{N}^l$)

Base functions:

- ▶ $\mathcal{O} : \mathbb{N}^0 = \{\emptyset\} \rightarrow \mathbb{N}$, $\emptyset \mapsto 0$ (0-ary constant-0 function)
- ▶ $\text{succ} : \mathbb{N} \rightarrow \mathbb{N}$, $x \mapsto x + 1$ (successor function)
- ▶ $\pi_i^n : \mathbb{N}^n \rightarrow \mathbb{N}$, $\vec{x} = \langle x_1, \dots, x_n \rangle \mapsto x_i$ (projection function)
- ▶ for $n > 1$: $\text{id}^n : \mathbb{N}^n \rightarrow \mathbb{N}^n$, $\vec{x} = \langle x_1, \dots, x_n \rangle \mapsto \vec{x}$ (n -ary identity f.)

Closed under operations:

- ▶ **composition**: if $f : \mathbb{N}^{km} \rightarrow \mathbb{N}^l$, and $g_i : \mathbb{N}^n \rightarrow \mathbb{N}^m$ are prim. rec., then so is $h = f \circ (g_1 \times \dots \times g_k) : \mathbb{N}^n \rightarrow \mathbb{N}^l$:

$$h(\vec{x}) = f(g_1(\vec{x}), \dots, g_k(\vec{x}))$$
- ▶ **primitive recursion**: if $f : \mathbb{N}^n \rightarrow \mathbb{N}^l$, $g : \mathbb{N}^{n+l+1} \rightarrow \mathbb{N}^l$ are prim. rec., then so is $h = \text{pr}(f; g) : \mathbb{N}^{n+1} \rightarrow \mathbb{N}^l$:

$$h(\vec{x}, 0) = f(\vec{x})$$

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Primitive recursive functions (exercises)

Exercise

Show that the following functions are primitive recursive:

- ▶ addition
- ▶ constant functions
- ▶ multiplication
- ▶ (positive) sign-function
- ▶ the representing functions $\chi_=_$ and $\chi_<$ for the predicates = and <.

Try-yourself-Examples

Show that the following functions are primitive recursive:

- ▶ exponentiation
- ▶ factorial

Admissible operations for primitive recursive functions

Proposition

① definition by *case distinction*:

$$f(\vec{x}) := \begin{cases} f_1(\vec{x}) & \dots P_1(\vec{x}) \\ f_2(\vec{x}) & \dots \wedge P_2(\vec{x}) \neg P_1(\vec{x}) \\ \dots & \dots \\ f_k(\vec{x}) & \dots \wedge P_k(\vec{x}) \wedge \neg P_{k-1}(\vec{x}) \wedge \dots \neg P_1(\vec{x}) \\ f_{k+1}(\vec{x}) & \dots \wedge \neg P_k(\vec{x}) \wedge \dots \neg P_1(\vec{x}) \end{cases}$$

② definition by *bounded recursion*:

$$\mu_{z \leq y} \cdot [P(x_1, \dots, x_n, z)] := \begin{cases} z & \dots \neg P(x_1, \dots, x_n, i) \text{ for } 0 \leq i < z \leq y, \\ & \text{and } P(x_1, \dots, x_n, z) \\ y + 1 & \dots \neg \exists z. \wedge 0 \leq z \leq y P(x_1, \dots, x_n, z) \end{cases}$$

Properties of primitive recursive functions

Proposition

- 1 *Every primitive recursive function is total.*
- 2 *Every primitive recursive function is Turing-computable.*

Proof.

For (2):

- ▶ the base functions are Turing-computable
- ▶ the Turing-computable functions are closed under the schemes [composition](#) and [primitive recursion](#)



Turing-computable (total) functions

Definition

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- ▶ for all $n_1, \dots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:

$$q_0 \langle n_1 \rangle \blacksquare \langle n_2 \rangle \blacksquare \dots \blacksquare \langle n_k \rangle \vdash_M^* q \langle f(n_1, \dots, n_k) \rangle$$

Features of computationally complete MoC's present?

- ▶ storage (unbounded) ✓
- ▶ control (finite, given) ✓
- ▶ modification ✓
 - of (immediately accessible) stored data
 - of control state
- ▶ conditionals ✓
- ▶ loop (unbounded)
- ▶ stopping condition ✓

Not primitive recursive (I)

Proposition

*There exist calculable/Turing-computable functions that are **not primitive recursive**.*

Proof.

By diagonalisation.



Not primitive recursive (II): Ackermann function



Wilhelm Ackermann (1896–1962)

Not primitive recursive (II): Ackermann function

Ackermann function $A : \mathbb{N}^2 \rightarrow \mathbb{N}$ (simplified version by Rózsa Péter):

$$A(0, x) = \text{Succ}(x)$$

$$A(x + 1, 0) = A(x, \text{Succ}(0))$$

$$A(x + 1, y + 1) = A(x, A(x + 1, y))$$

A is **not** primitive recursive, it grows **too fast**:

$$A(0, n) = n + 1$$

$$A(1, n) = n + 2$$

$$A(2, n) = 2n + 3$$

$$A(3, n) = 2^{n+3} - 2$$

$$A(4, n) = \underbrace{2^{2^{\cdot^{2^{16}}}}}_n - 3$$

...

Not primitive recursive (II): Ackermann function

Ackermann function $A : \mathbb{N}^2 \rightarrow \mathbb{N}$ (simplified version by Rózsa Péter):

$$A(0, x) = \text{Succ}(x)$$

$$A(x + 1, 0) = A(x, \text{Succ}(0))$$

$$A(x + 1, y + 1) = A(x, A(x + 1, y))$$

A grows faster than every primitive recursive function:

Theorem

For every primitive recursive $f : \mathbb{N} \rightarrow \mathbb{N}$ there exists some $i \in \mathbb{N}$ such that $f(i) < A(i, i)$.

Jacques Herbrand



Jacques Herbrand (1908–1931)

Kurt Gödel



Kurt Gödel (1906–1978)

Gödel–Herbrand general recursive function

Defined by systems of recursion equations like that for the Ackermann function:

$$A(0, x) = \text{Succ}(x)$$

$$A(\text{Succ}(x), 0) = A(x, \text{Succ}(0))$$

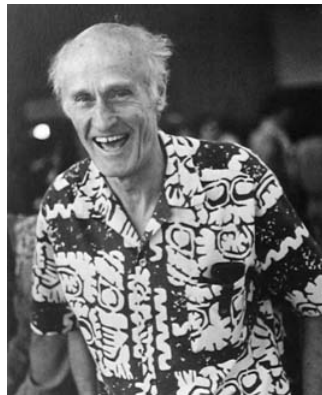
$$A(\text{Succ}(x), \text{Succ}(y)) = A(x, A(\text{Succ}(x), y))$$

Numerals: $\langle 0 \rangle := 0$, and $\langle n \rangle := \underbrace{\text{Succ}(\dots \text{Succ}(0))}_n$ for $n > 1$.

Definition

A function $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is called **general recursive** if it can be defined by (such a) system \mathcal{S} of recursion equations via a function symbol F if for all $n_1, \dots, n_k \in \mathbb{N}$, the expression $F(\langle n_1 \rangle, \dots, \langle n_k \rangle)$ evaluates according to \mathcal{S} to a **unique numeral** $\langle n \rangle$, and such that furthermore: $n = f(n_1, \dots, n_k)$.

Stephen Cole Kleene



Stephen Cole Kleene (1906–1994)

Unbounded minimisation (μ -recursion)

Let $f : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ **total**. Then the partial function defined by:

$$\mu(f) : \mathbb{N}^k \rightarrow \mathbb{N}$$

$$\vec{x} \mapsto \begin{cases} \min\{y \in \mathbb{N} \mid f(\vec{x}, y) = 0\} & \dots \exists y (f(\vec{x}, y) = 0) \\ \uparrow & \dots \text{else} \end{cases}$$

is called the **unbounded minimisation of f** .

Let $f : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ **partial**. Then the partial function $\mu(f)$:

$$\mu(f) : \mathbb{N}^k \rightarrow \mathbb{N}$$

$$\vec{x} \mapsto \begin{cases} z & \dots f(\vec{x}, z) = 0 \wedge \forall y (0 \leq y < z \rightarrow (f(\vec{x}, y) \downarrow \neq 0)) \\ \uparrow & \dots \neg \exists y (f(\vec{x}, y) = 0 \wedge \forall z (0 \leq z < y \rightarrow (f(\vec{x}, z) \downarrow))) \end{cases}$$

is called the **unbounded minimisation of f** .

Partial, and total, recursive functions

Definition

A **partial function** $f : \mathbb{N}^n \rightarrow \mathbb{N}^l$ is called **partial recursive** if it can be specified from base functions (\mathcal{O} , **succ**, π_i^n , and id^n) by successive applications of **composition**, **primitive recursion**, and **unbounded minimisation**.

A partial recursive function is called **(total) recursive** if it is total.

Proposition

Every partial recursive function is Turing-computable.

Primitive recursive

- ▶ storage (unbounded) ✓
- ▶ control (finite, given) ✓
- ▶ modification ✓
 - of (immediately accessible) stored data
 - of control state
- ▶ conditionals ✓
- ▶ loop (unbounded)
- ▶ stopping condition ✓

Turing-computable functions

Definition

- ① A **total function** $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is **Turing-computable** if there exists a Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \blacksquare, F \rangle$ and a **calculable** coding function $\langle \cdot \rangle : \mathbb{N} \rightarrow \Sigma^*$ such that:

 - for all $n_1, \dots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:

$$q_0 \langle n_1 \rangle \blacksquare \langle n_2 \rangle \blacksquare \dots \blacksquare \langle n_k \rangle \vdash_M^* q \langle f(n_1, \dots, n_k) \rangle$$
- ② A **partial function** $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is **Turing-computable** if there exists a Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \blacksquare, F \rangle$ and a **calculable** coding function $\langle \cdot \rangle : \mathbb{N} \rightarrow \Sigma^*$ such that:

 - for all $n_1, \dots, n_k \in \mathbb{N}$:

$$M \text{ accepts } \langle n_1 \rangle \blacksquare \langle n_2 \rangle \blacksquare \dots \blacksquare \langle n_k \rangle \iff f(n_1, \dots, n_k) \downarrow$$
 - for all $n_1, \dots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:

$$f(n_1, \dots, n_k) \downarrow \implies q_0 \langle n_1 \rangle \blacksquare \langle n_2 \rangle \blacksquare \dots \blacksquare \langle n_k \rangle \vdash_M^* q \langle f(n_1, \dots, n_k) \rangle$$

Partial recursive vs. Turing-computable functions

Lemma

Every Turing-computable function is partial recursive.

Proof by [arithmetization](#) of Turing machines, showing:

Theorem (Kleene's normal form theorem)

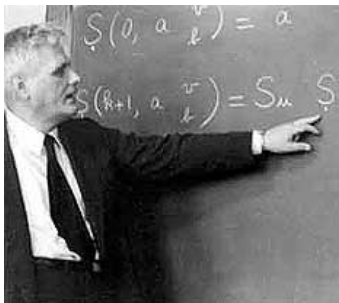
*For every **Turing-computable, partial** function (and hence for every **partial recursive** function) $h : \mathbb{N}^k \rightarrow \mathbb{N}$ there exist **primitive recursive** functions $f : \mathbb{N} \rightarrow \mathbb{N}$ and $g : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ such that:*

$$h(x_1, \dots, x_n) = (f \circ \mu(g))(x_1, \dots, x_n)$$

Theorem

The Turing-computable (partial) functions coincide with the partial recursive functions.

Alonzo Church



Alonzo Church (1903 –1995)

Effectively calculable functions

Alonzo Church (1936):

*“We now define the notion $[\dots]$ of an **effectively calculable** function of positive integers by **identifying it** with the notion of a **recursive** function of positive integers (or a λ -definable function of positive integers). This definition is thought to be justified by the considerations which follow, **so far as positive justification can ever be obtained for the selection of formal definition to correspond to an intuitive notion.**”*

Definition (Church)

For every **total** function $f : \mathbb{N} \rightarrow \mathbb{N}$, and **partial** function $g : \mathbb{N} \rightarrow \mathbb{N}$,

f is **effectively calculable** : \iff f is **recursive**

g is **effectively calculable** : \iff g is **partial-recursive**

Effectively calculable functions

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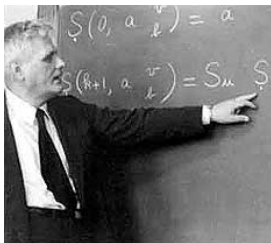
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λ -calculus



Alonzo Church (1903 –1992)

Theorem (Kleene/Church, 1935)

Every λ -definable function is general recursive, and vice versa.

Recommended reading

1 Recursive and primitive-recursive functions:

Chapter 3, The Lambda Calculus of the book:

- ▶ Maribel Fernández [2]: *Models of Computation (An Introduction to Computability Theory)*, Springer-Verlag London, 2009.

Post's 'working hypothesis'

E.L. Post in his 1936 article (Post machines):

*"The writer expects the present formulation to turn out to be logically equivalent to recursiveness in the sense of the Gödel–Church development. However, is not only to present a system of a certain logical potency but also, in its restricted field, of **psychological fidelity**. In the latter sense wider and wider formulations are contemplate. On the other hand, our aim will be to show that all such are logically reducible to formulation 1 [Post machines]. We offer this conclusion at the present moment as a **working hypothesis**. And to our mind such is Church's identification of effective calculability with recursiveness."*

Church on Post's 'working hypothesis'

Alonzo Church in his review (1937) of Post's 1936 article:

"The author proposes a definition of "finite 1-process" which is similar in formulation, and in fact equivalent, to computation by a Turing machine (see the preceding review). He does not, however, regard his formulation as certainly to be identified with effectiveness in the ordinary sense, but takes this identification as a "working hypothesis" in need of continual verification. To this the reviewer would object that effectiveness in the ordinary sense has not been given an exact definition, and hence the working hypothesis in question has not an exact meaning. To define effectiveness as computability by an arbitrary machine, subject to restrictions of finiteness, would seem to be an adequate representation of the ordinary notion, and if this is done the need for a working hypothesis disappears."

Church on Turing's paper

A. Church in his review (1937) of Turing's 1936 article:

"The author proposes as a criterion that an infinite sequence of digits 0 and 1 be "computable" that it shall be possible to devise a computing machine, occupying a finite space and with working parts of finite size, which will write down the sequence to any desired number of terms if allowed to run for a sufficiently long time. As a matter of convenience, certain further restrictions are imposed on the character of the machine, but these are of such a nature as obviously to cause no loss of generality—in particular, a human calculator, provided with pencil and paper and explicit instructions, can be regarded as a kind of Turing machine. It is thus immediately clear that computability, so defined, can be identified with (especially, is no less general than) the notion of effectiveness as it appears in certain mathematical problems [...]."

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References I



Alonzo Church.

An Unsolvable Problem of Elementary Number Theory.

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References II



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[http://www.wolframscience.com/prizes/tm23/
images/Turing.pdf](http://www.wolframscience.com/prizes/tm23/images/Turing.pdf).