

Lecture 1: Introduction to Computability

Models of Computation

<https://clegra.github.io/moc/Novi-Sad.html>

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Novi Sad, Serbia

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Course overview

<i>intro</i>	<i>classic models</i>			<i>additional models</i>
Introduction to Computability	Machine Models	Recursive Functions	Lambda Calculus	Three more Models of Computation
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ -terms, β -reduction, λ -definable functions, partial recursive = λ -definable = Turing computable	Post's Correspondence Problem, Interaction-Nets, Fractran
	<i>imperative programming</i>	<i>algebraic programming</i>	<i>functional programming</i>	

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- ▶ What is computation?
 - ▶ questions where the answer may depend on computation
 - ▶ algorithm examples
 - ▶ unsolvable problems

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- ▶ fields for which models of computation are important

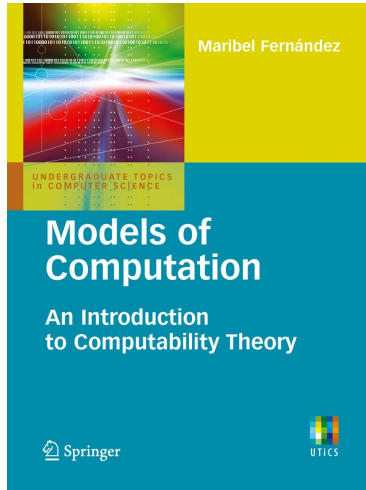
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Book



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A: Yes, if the truth table for ϕ contains (in the row for ϕ) only "T"; no otherwise.

(Comput.) Yes-or-no-questions / Decision problems

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A **decision method for A in E** is a method by which, given an element $a \in E$, we can **decide** in a **finite number** of **steps** whether or not $a \in A$.

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The decision problem for A in E is **solvable** (the set A in E is **(effectively) calculable**) if there exists a decision method for A in E .

(Comput.) What-questions / Computation Problems

Example

Computing the greatest common divisor

Instance: a pair $\langle a, b \rangle$ of numbers $a, b \in \mathbb{N}$ with $a, b > 0$.

Question: What is $\gcd(a, b)$, the greatest common divisor of a and b ?

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Suppose $F : A \rightarrow B$ is a mapping, where the elements of A, B are finitely describable objects.

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A mapping F is **calculable** if there exists a computation method for F .

Representing function

Let $P(a_1, \dots, a_n)$ be an n -ary number-theoretic predicate.

The **representing function** f of P :

$$f(a_1, \dots, a_n) := \begin{cases} 1 & \dots P(a_1, \dots, a_n) \text{ is true} \\ 0 & \dots P(a_1, \dots, a_n) \text{ is false} \end{cases}$$

Hence:

A **decision procedure** can be handled as a **computation procedure** f by taking '0' for 'yes', and '1' for 'no'.

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- Similar for a **decision methods**.

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Due to $3 \mid 12$ and $(*)$ we conclude:

A: **Yes.** (Infinitely many solutions, e.g. $x = 4$ and $y = -8$.)

Not effectively calculable

Examples (Shoenfield)

- ▶ methods that involve chance procedures: tossing a coin

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- ▶ methods that require (unformalised, unmechanised) insight

Effectively calculable?

Example

Hilbert's 10th Problem

Instance: An equation $p(x_1, \dots, x_n) = 0$, where
 p a polynomial with integer coefficients.

Question: Is the equation solvable for $x_1, \dots, x_n \in \mathbb{Z}$?

Instances based on quadratic polynomials are of the form
 $ax^2 + bxy + cy^2 + dx + ey + f = 0$ with $a, b, c, d, e, f \in \mathbb{Z}$.

Effectively calculable? – No!

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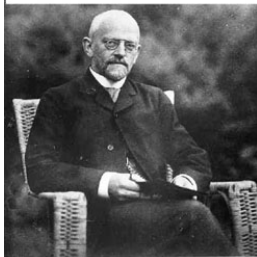
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Theorem (Matijasevic, 1970)

Hilbert's 10th Problem is unsolvable.

David Hilbert (1862–1943)



Hilbert

Problem (Entscheidungsproblem, 1928)

Is there a method for deciding, given a formula ϕ of the predicate calculus, whether or not ϕ is a tautology?

Timeline: From logic to computability

1900

Hilbert's 23 Problems in mathematics

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- 1937 Post: machine model; Church's thesis as 'working hypothesis'
Turing: convincing analysis of a 'human computer'
leading to the 'Turing machine'

Calculable functions?

Questions/Exercises

- 1 Suppose $P(a, b)$ is a calculable predicate.
Why does $(\exists x)P(a, x)$ not have to be calculable?

Calculable functions?

Questions/Exercises

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② Let $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$n \mapsto \begin{cases} 0 & \dots n = 0 \text{ \& Goldbach's conjecture is false} \\ 1 & \dots n = 0 \text{ \& Goldbach's conjecture is true} \\ n + 1 & \dots n > 0 \end{cases}$$

Is f calculable?

Calculable functions?

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- 1 Suppose $P(a, b)$ is a calculable predicate.
Why does $(\exists x)P(a, x)$ not have to be calculable?

- 2 Let $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$n \mapsto \begin{cases} 0 & \dots n = 0 \text{ \& Goldbach's conjecture is false} \\ 1 & \dots n = 0 \text{ \& Goldbach's conjecture is true} \\ n + 1 & \dots n > 0 \end{cases}$$

Is f calculable?

- 3 Can computation problems for mappings $F : \mathbb{N}^n \rightarrow \mathbb{N}^m$ always be represented by decision problems?

Some Models of Computation

machine model	mathematical model	sort
Turing machine Post machine register machine	Combinatory Logic λ -calculus Herbrand–Gödel recursive functions partial-recursive/ μ -recursive functions Post canonical system (tag system) Post's Correspondence Problem Markov algorithms Lindenmayer systems	<i>classical</i>
	Fractran	<i>less well known</i>
cellular automata neural networks	term rewrite systems interaction nets logic-based models of computation concurrency and process algebra ς -calculus evolutionary programming/genetic algorithms abstract state machines	<i>modern</i>
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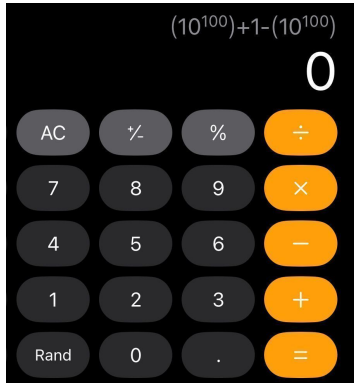
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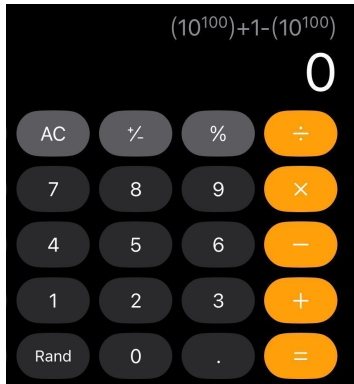
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Example MoC relevance: Calculator (1/5)

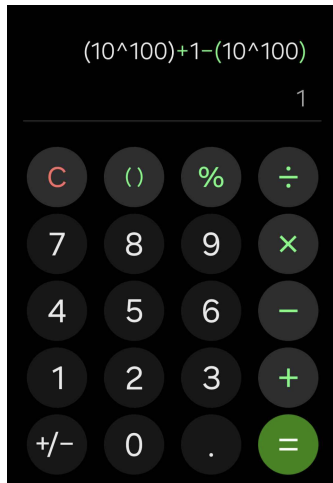


iOS

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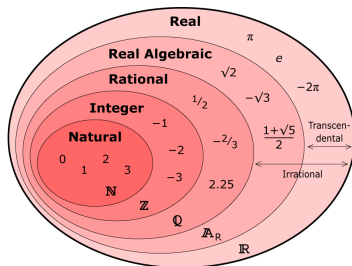


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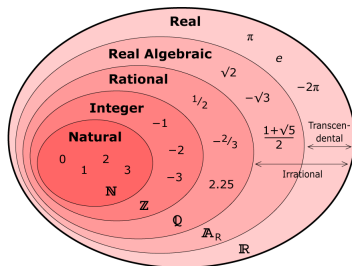
Android

Calculator (2/5): constructive real numbers

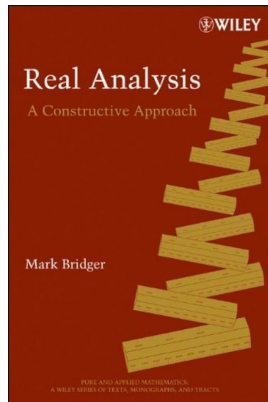


subclasses of real numbers \mathbb{R}

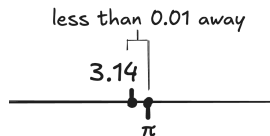
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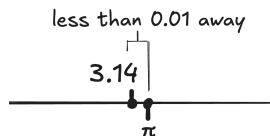


Calculator (3/5): constructive real numbers



approximating π within 0.01

Calculator (3/5): constructive real numbers



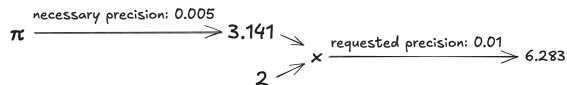
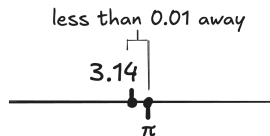
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Definition

A real number $x \in \mathbb{R}$ is **constructive** if:

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Calculator (3/5): constructive real numbers



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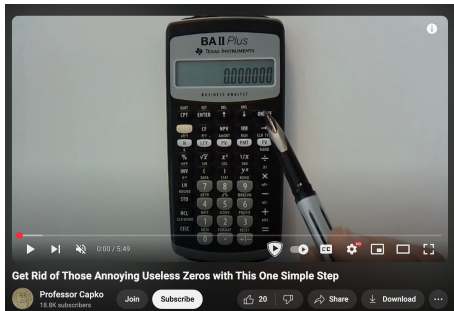
approximating 2π within 0.01

Definition

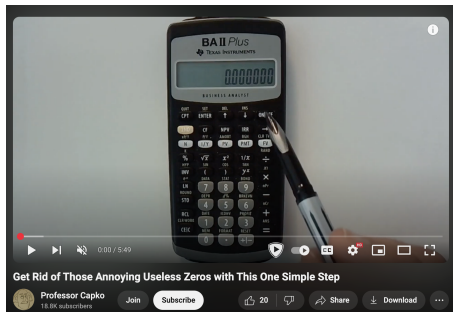
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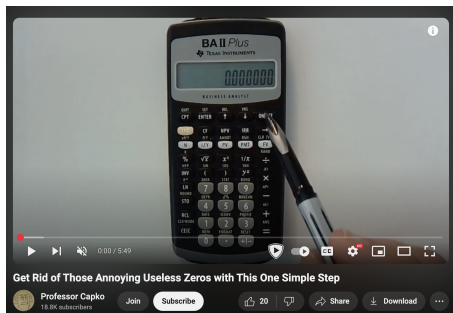


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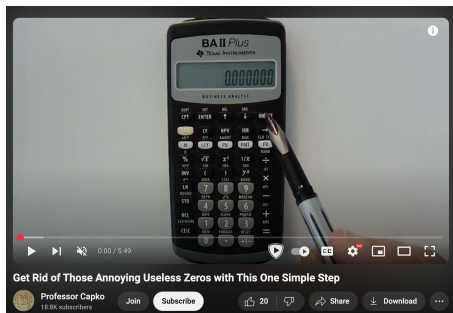
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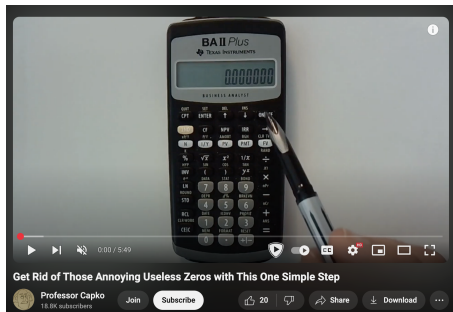
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Calculator (4/5): constructive real numbers



Undecidable problem

Article Talk



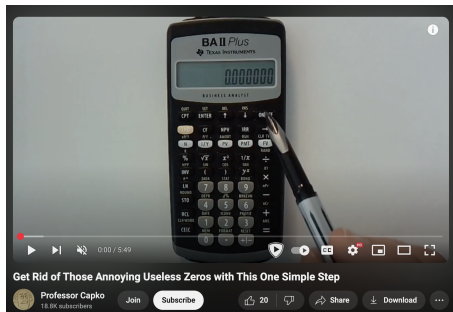
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(July 2019)

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Calculator (5/5): Böhm's full precision calculator



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- ▶ uses products of:
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- ▶ Credits: tech-blogger [Chad Nauseam](#) ([link](#)) for post
"A calculator app? Anyone could make that." ([link](#)) [2].

Some fields in which MoC's are important (I)

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Rewriting

- ▶ **study in a systematic way** the operational and denotational aspects of MoC's like λ -calculus, CL, string rewriting, term rewriting, interaction nets

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Linguistics

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Recommended reading

- 1 Post machine: Page 1 + first paragraph on page 2 of:
 - ▶ Emil Post: *Finite Combinatory Processes – Formulation 1*, Journal of Symbolic Logic (1936), [3], <https://www.wolframscience.com/prizes/tm23/images/Post.pdf>.

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- ② **Turing machine motivation:** Turing's analysis of a human computer:
Part I of Section 9, pp. 249–252 of:
 - ▶ Alan M. Turing's: *On computable numbers, with an application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society (1936), [4], <http://www.wolframscience.com/prizes/tm23/images/Turing.pdf>.

Course overview

<i>intro</i>	<i>classic models</i>			<i>additional models</i>
Introduction to Computability	Machine Models	Recursive Functions	Lambda Calculus	Three more Models of Computation
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ -terms, β -reduction, λ -definable functions, partial recursive = λ -definable = Turing computable	Post's Correspondence Problem, Interaction-Nets, Fractran
	<i>imperative programming</i>	<i>algebraic programming</i>	<i>functional programming</i>	

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