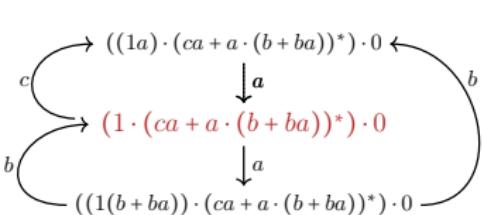


The Graph Structure of Process Interpretations of Regular Expressions

Clemens Grabmayer

<https://clegra.github.io>

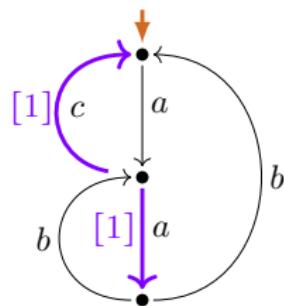


Department of Computer Science



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L'Aquila, Italy



Computer Science Seminar
University of Novi Sad
March 2026

Overview

- ▶ regular expressions (with unary/binary star, under-star-1-free (\ast/\dagger))
- ▶ Milner's process interpretation P /semantics $\llbracket \cdot \rrbracket_P$
 - ▶ P -/ $\llbracket \cdot \rrbracket_P$ -expressible graphs (\leadsto expressibility question)
 - ▶ axioms for $\llbracket \cdot \rrbracket_P$ -identity (\leadsto completeness question)
- ▶ loop existence and elimination (LEE)
 - ▶ defined by loop elimination rewrite system, its completion
 - ▶ describes interpretations of (\ast/\dagger) reg. expr.s (extraction possible)
 - ▶ LEE-witnesses: labelings of process graphs with LEE
 - ▶ LEE is preserved under bisimulation collapse (stepwise collapse)
- ▶ 1-LEE = sharing via 1-transitions facilitates LEE
- ▶ LEE/1-LEE characterize image of P^\bullet (restricted/unrestricted)
 - ▶ where P^\bullet a compact (sharing-increased) refinement of P
- ▶ outlook on work-to-do

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 - ▶ LEE-witnesses: labelings of process graphs with LEE
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 - ▶ describes interpretations of all reg. expr.s (extraction possible)
 - ▶ not preserved under bisimulation collapse (approximation possible)
- ▶ LEE/1-LEE characterize image of P^\bullet (restricted/unrestricted)
 - ▶ where P^\bullet a compact (sharing-increased) refinement of P
 - ▶ via refined extraction using LEE/1-LEE
- ▶ outlook on work-to-do

Regular Expressions

Definition (*~ Copi–Elgot–Wright, 1958*)

Regular expressions over alphabet A with unary Kleene star:

$e, e_1, e_2 ::= \mathbf{0} \mid a \mid e_1 + e_2 \mid e_1 \cdot e_2 \mid e^*$ (for $a \in A$).

- ▶ symbol **0** instead of \emptyset , symbol **1** instead of $\{\epsilon\}$

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Regular Expressions

Definition (~ Kleene, 1951, ~ Copi–Elgot–Wright, 1958)

Regular expressions over alphabet A with unary / binary Kleene star:

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$$e, e_1, e_2 ::= \mathbf{0} \mid \mathbf{1} \mid \mathbf{a} \mid e_1 + e_2 \mid e_1 \cdot e_2 \mid e_1^{\circledast} e_2 \quad (\text{for } \mathbf{a} \in A).$$

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- ▶ with binary star $^{\circledast}$: **1** is **not** definable (in its absence)

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Regular Expressions (1-free)

Definition (~ Kleene, 1951, ~ Copi–Elgot–Wright, 1958)

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Definition (for process interpretation)

1-free regular expressions over alphabet A with binary Kleene star:

$$f, f_1, f_2 ::= \mathbf{0} \mid \mathbf{a} \mid f_1 + f_2 \mid f_1 \cdot f_2 \mid f_1^{\circledast} f_2 \quad (\text{for } \mathbf{a} \in A).$$

Regular Expressions (1-free)

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Definition (for process interpretation)

1-free regular expressions over alphabet A with unary/binary Kleene star:

$$f, f_1, f_2 ::= \mathbf{0} \mid \mathbf{a} \mid f_1 + f_2 \mid f_1 \cdot f_2 \mid (f_1^*) \cdot f_2 \quad (\text{for } \mathbf{a} \in A),$$

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Regular Expressions (under-star-/1-free)

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Regular expressions over alphabet A with unary / binary Kleene star:

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$$e, e_1, e_2 ::= \mathbf{0} \mid \mathbf{1} \mid \mathbf{a} \mid e_1 + e_2 \mid e_1 \cdot e_2 \mid e_1^{\circledast} e_2 \quad (\text{for } \mathbf{a} \in A).$$

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Definition (for process interpretation)

The set $RExp^{(+)}(A)$ of **1-free** regular expressions over A is defined by:

$$f, f_1, f_2 ::= \mathbf{0} \mid \mathbf{a} \mid f_1 + f_2 \mid f_1 \cdot f_2 \mid f_1^* \cdot f_2 \quad (\text{for } \mathbf{a} \in A),$$

the set $RExp^{(*/+)}(A)$ of **under-star-1-free** regular expressions over A by:

$$uf, uf_1, uf_2 ::= \mathbf{0} \mid \mathbf{1} \mid \mathbf{a} \mid uf_1 + uf_2 \mid uf_1 \cdot uf_2 \mid f^* \quad (\text{for } \mathbf{a} \in A).$$

Process interpretation P of regular expressions (Milner, 1984)

0 \xrightarrow{P} deadlock δ , no termination

1 \xrightarrow{P} empty-step process ϵ , then terminate

a \xrightarrow{P} atomic action a , then terminate

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$e_1 + e_2 \xrightarrow{P}$ (*choice*) execute $P(e_1)$ or $P(e_2)$

$e_1 \cdot e_2 \xrightarrow{P}$ (*sequentialization*) execute $P(e_1)$, then $P(e_2)$

$e^* \xrightarrow{P}$ (*iteration*) repeat (terminate or execute $P(e)$)

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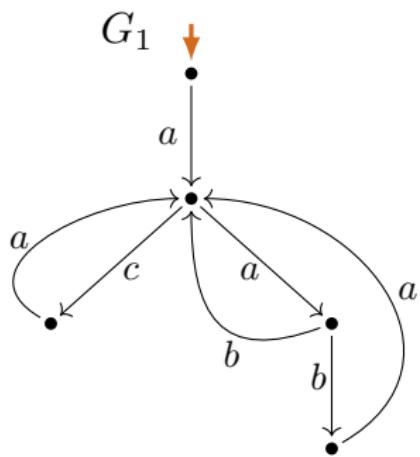
$e_1 + e_2 \xrightarrow{P}$ (choice) execute $P(e_1)$ or $P(e_2)$

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$e^* \xrightarrow{P}$ (iteration) repeat (terminate or execute $P(e)$)

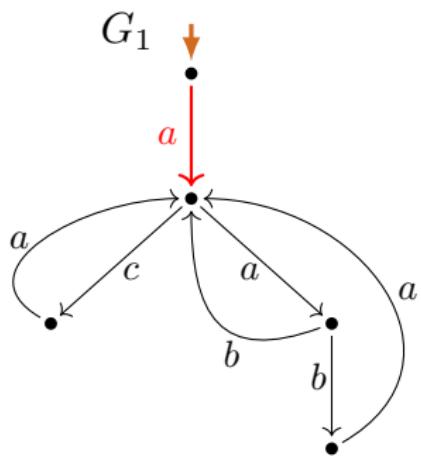
$\llbracket e \rrbracket_P := [P(e)]_{\underline{\equiv}}$ (bisimilarity equivalence class of process $P(e)$)

P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (example, informally)



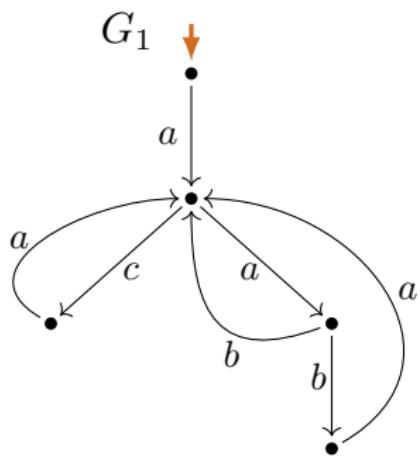
$$P\left(\overbrace{(a \cdot (c \cdot a + a \cdot (b + b \cdot a))^*) \cdot 0}^f\right)$$

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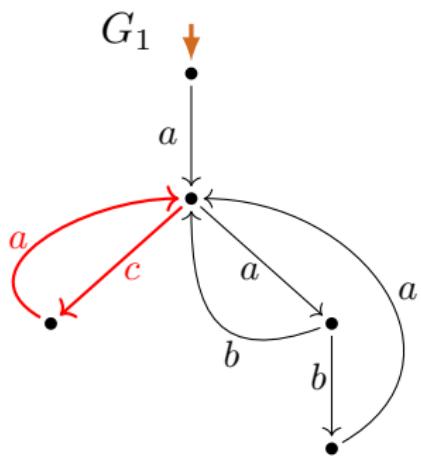
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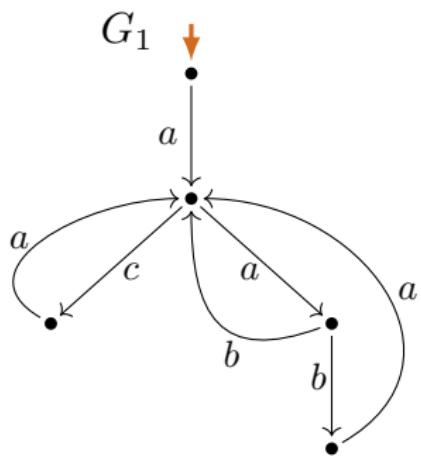
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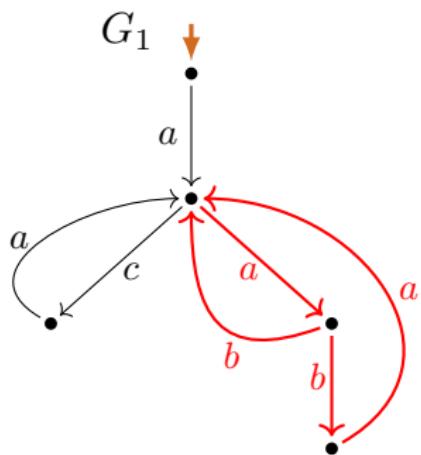
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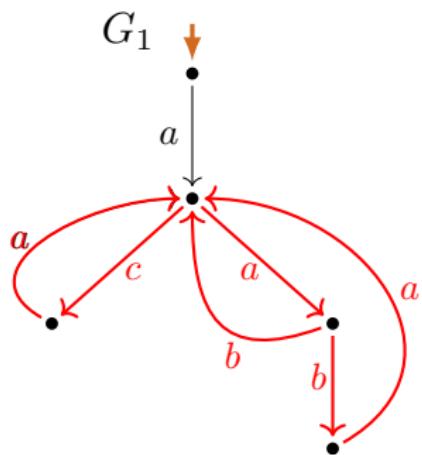
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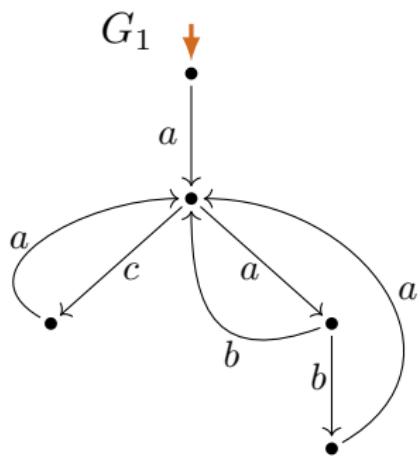
$$P\left(\overbrace{(a \cdot (c \cdot a + \textcolor{red}{a} \cdot (b + b \cdot a))^*) \cdot 0}^f\right)$$

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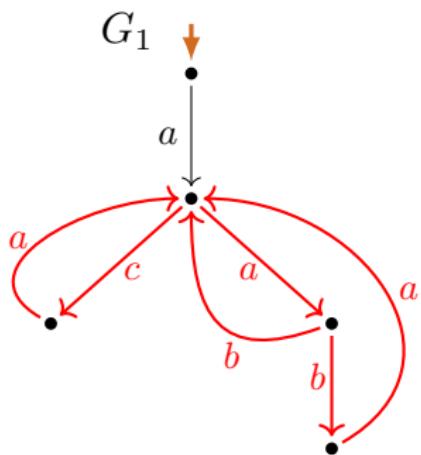
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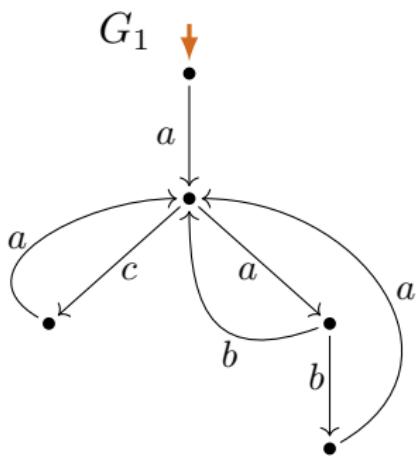
$$\begin{aligned} & P\left(\overbrace{(a \cdot (c \cdot a + a \cdot (b + b \cdot a))^*) \cdot 0}^f \right) \\ & P\left(a \cdot (c \cdot a + a \cdot (b + b \cdot a))^{\otimes 0} \right) \end{aligned}$$

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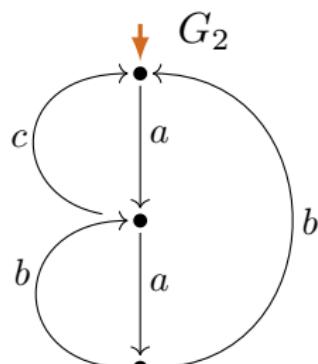
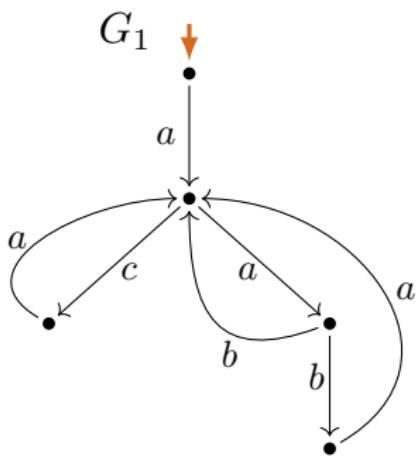
$\textcolor{violet}{P}$ -expressibility and $\llbracket \cdot \rrbracket_{\textcolor{violet}{P}}$ -expressibility (example, informally)



$$\begin{aligned} & \overbrace{\textcolor{violet}{P}\left((a \cdot (c \cdot a + a \cdot (b + b \cdot a))^*) \cdot 0 \right)}^{\textcolor{red}{f}} \\ & \textcolor{violet}{P}\left(a \cdot (c \cdot a + a \cdot (b + b \cdot a))^{\otimes 0} \right) \end{aligned}$$

$$G_1 \in \llbracket \textcolor{brown}{f} \rrbracket_{\textcolor{violet}{P}}$$

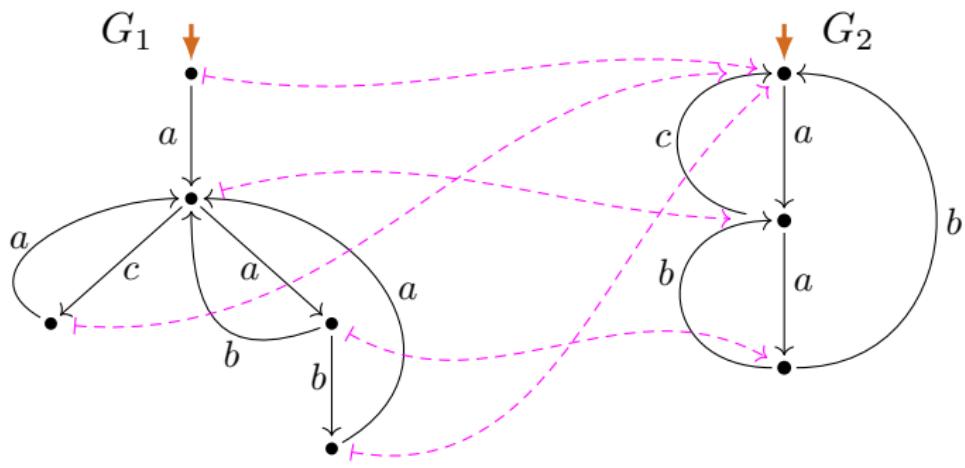
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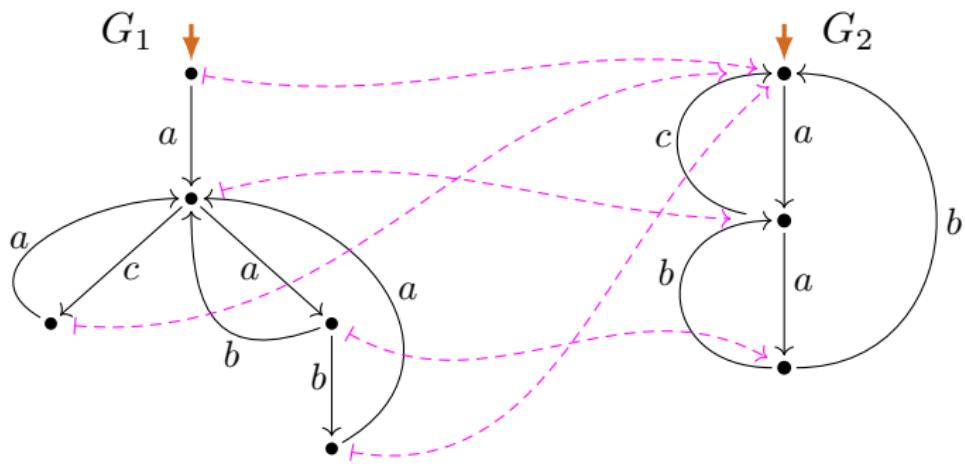
P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (example, informally)



$$\begin{aligned} & \overbrace{P\left((a \cdot (c \cdot a + a \cdot (b + b \cdot a))^*) \cdot 0 \right)}^f \\ & P\left(a \cdot (c \cdot a + a \cdot (b + b \cdot a))^{\otimes 0} \right) \end{aligned}$$

$$G_1 \in \llbracket f \rrbracket_P$$

$\textcolor{violet}{P}$ -expressibility and $\llbracket \cdot \rrbracket_{\textcolor{violet}{P}}$ -expressibility (example, informally)



$$\textcolor{violet}{P} \left(\overbrace{(a \cdot (c \cdot a + a \cdot (b + b \cdot a))^*) \cdot 0}^{\textcolor{brown}{f}} \right)$$

$$\textcolor{violet}{P} \left(a \cdot (c \cdot a + a \cdot (b + b \cdot a))^* 0 \right)$$

$$G_2 \in \llbracket \textcolor{brown}{f} \rrbracket_{\textcolor{violet}{P}}$$

$$G_1 \in \llbracket \textcolor{brown}{f} \rrbracket_{\textcolor{violet}{P}}$$

Process interpretation P (formally)

Definition (Transition system specification \mathcal{T})

$$\frac{}{a \xrightarrow{a} 1} \qquad \frac{e_i \xrightarrow{a} e'_i}{e_1 + e_2 \xrightarrow{a} e'_i} \quad (i \in \{1, 2\})$$

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 \\
 \frac{}{a \xrightarrow{a} \mathbf{1}} \qquad \frac{e_i \xrightarrow{a} e'_i}{e_1 + e_2 \xrightarrow{a} e'_i} \ (i \in \{1, 2\}) \\
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 \end{array}$$

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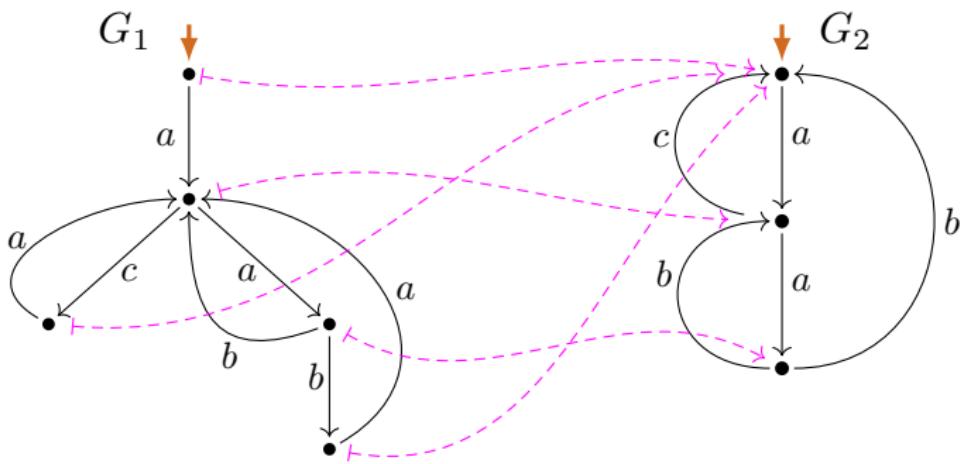
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 \end{array}$$

Definition

The process (graph) interpretation $P(e)$ of a regular expression e :

$P(e) :=$ labeled transition graph generated by e by derivations in \mathcal{T} .

P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (example, informally)

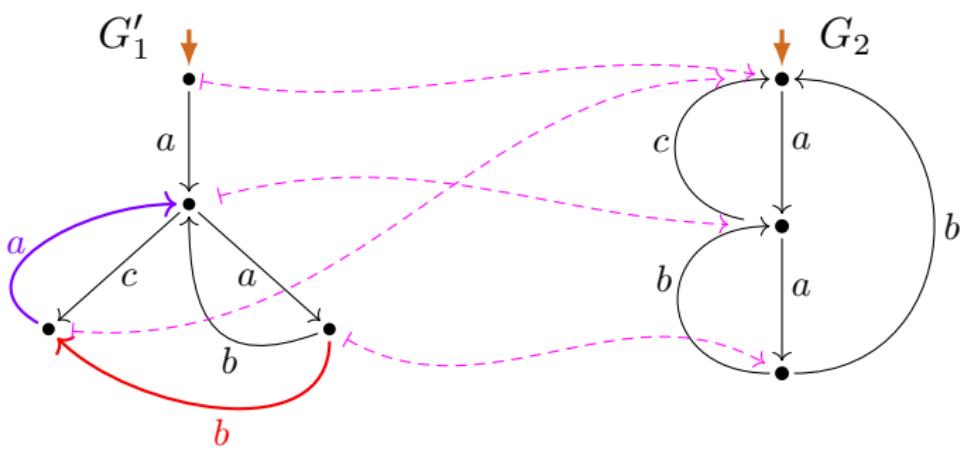


$$P\left(\overbrace{a \cdot ((c \cdot a + a \cdot (b + b \cdot a)^*) \cdot 0)}^f\right)$$

$$G_1 \in \llbracket f \rrbracket_P$$

$$G_2 \in \llbracket f \rrbracket_P$$

P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (example, formally)

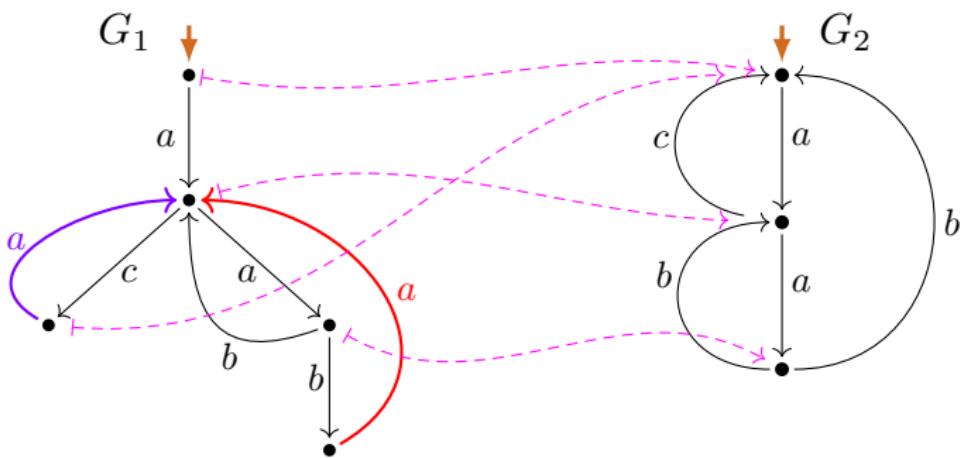


$$P\left(\overbrace{a \cdot ((c \cdot a + a \cdot (b + b \cdot a)^*) \cdot 0)}^f\right)$$

$$G'_1 \in \llbracket f \rrbracket_P$$

$$G_2 \in \llbracket f \rrbracket_P$$

P -expressibility and $\|\cdot\|_P$ -expressibility (example, formally)

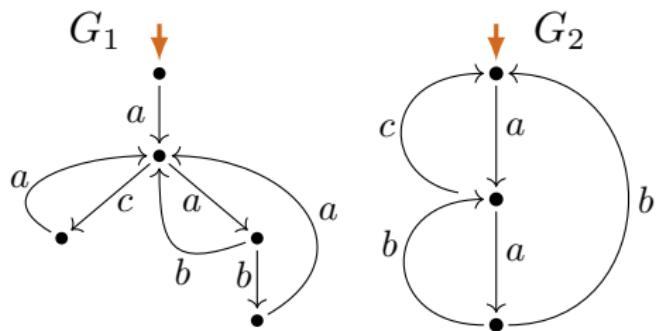


$$P\left(\overbrace{a \cdot ((c \cdot a + a \cdot (b + b \cdot (a + a))^*) \cdot 0)}^{\textcolor{brown}{f}} \right)$$

$$G_1 \in \llbracket f \rrbracket_P$$

$$G_2 \in \llbracket f \rrbracket_P$$

P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (examples)

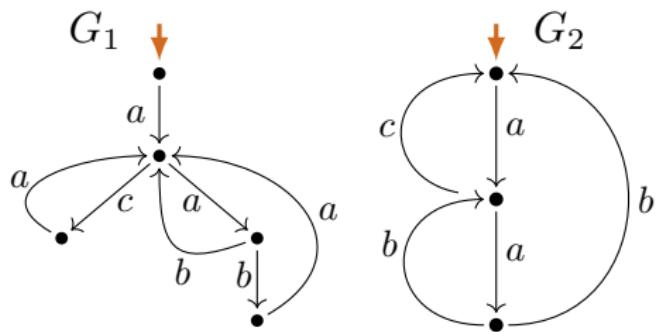


P -expressible

$\llbracket \cdot \rrbracket_P$ -expressible

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P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (examples)



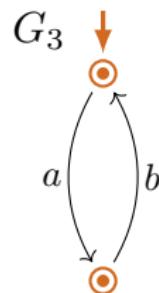
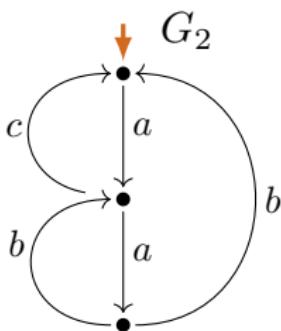
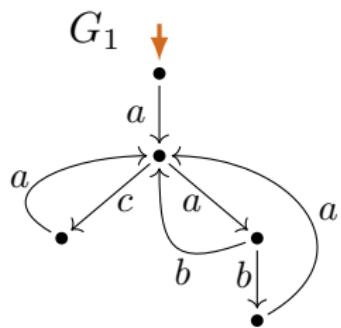
P -expressible

?

$\llbracket \cdot \rrbracket_P$ -expressible

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P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (examples)



P -expressible

?

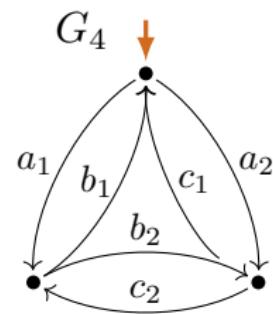
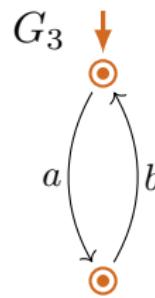
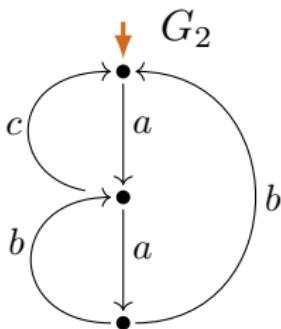
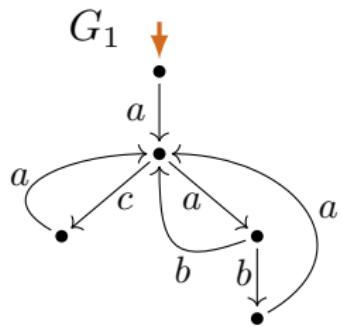
not P -expressible

$\llbracket \cdot \rrbracket_P$ -expressible

$\llbracket \cdot \rrbracket_P$ -expressible

not $\llbracket \cdot \rrbracket_P$ -expressible

P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility (examples)



P -expressible

?

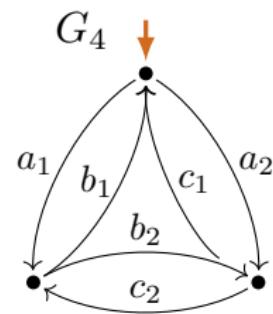
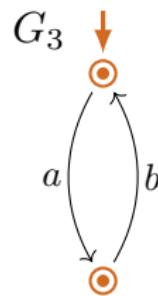
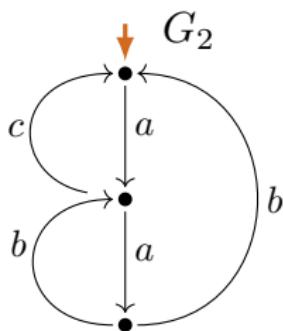
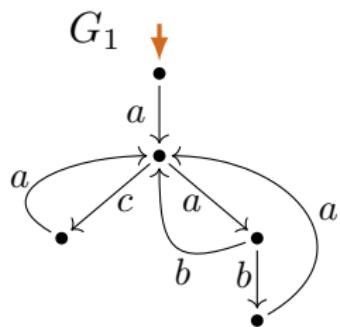
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P -expressible

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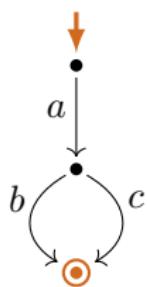
$\llbracket \cdot \rrbracket_P$ -expressible

not $\llbracket \cdot \rrbracket_P$ -expressible

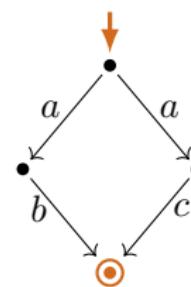
Q2: How can P -expressibility and $\llbracket \cdot \rrbracket_P$ -expressibility be characterized?

Process semantics equality $=_{\llbracket \cdot \rrbracket_P}$

- Fewer identities hold for $=_{\llbracket \cdot \rrbracket_P}$ than for $=_{\llbracket \cdot \rrbracket_L}$:



$$P(a \cdot (b + c))$$



$$P(a \cdot b + a \cdot c)$$

Process semantics equality $=_{\llbracket \cdot \rrbracket_P}$

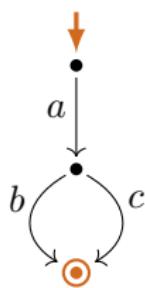
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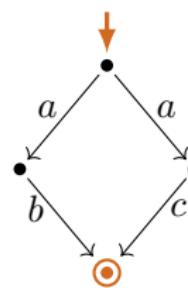
$$P(a \cdot (b + c)) \quad \cancel{\Leftarrow} \quad P(a \cdot b + a \cdot c)$$

Process semantics equality $=_{\llbracket \cdot \rrbracket_P}$

- Fewer identities hold for $=_{\llbracket \cdot \rrbracket_P}$ than for $=_{\llbracket \cdot \rrbracket_L}$:



$\neq_{\llbracket \cdot \rrbracket_P}$



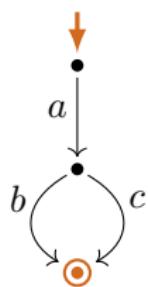
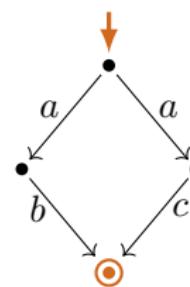
$$a \cdot (b + c)$$

$$\neq_{\llbracket \cdot \rrbracket_P}$$

$$a \cdot b + a \cdot c$$

Process semantics equality $=_{\llbracket \cdot \rrbracket_P}$

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 $=_{\llbracket \cdot \rrbracket_L}$ 

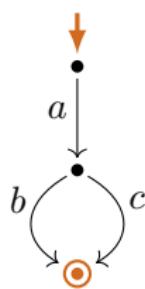
$$a \cdot (b + c)$$

$$=_{\llbracket \cdot \rrbracket_L}$$

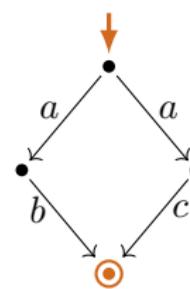
$$a \cdot b + a \cdot c$$

Process semantics equality $=_{\llbracket \cdot \rrbracket_P}$

- Fewer identities hold for $=_{\llbracket \cdot \rrbracket_P}$ than for $=_{\llbracket \cdot \rrbracket_L}$: $=_{\llbracket \cdot \rrbracket_P} \subsetneq =_{\llbracket \cdot \rrbracket_L}$.



$\neq_{\llbracket \cdot \rrbracket_P}$



$$a \cdot (b + c)$$

$$\neq_{\llbracket \cdot \rrbracket_P}$$

$$a \cdot b + a \cdot c$$

Milner's proof system Mil

Axioms:

$$(A1) \quad e + (f + g) = (e + f) + g$$

$$(A2) \quad e + 0 = e$$

$$(A3) \quad e + f = f + e$$

$$(A4) \quad e + e = e$$

$$(A5) \quad e \cdot (f \cdot g) = (e \cdot f) \cdot g$$

$$(A6) \quad (e + f) \cdot g = e \cdot g + f \cdot g$$

$$(A7) \quad e = 1 \cdot e$$

$$(A8) \quad e = e \cdot 1$$

$$(A9) \quad 0 = 0 \cdot e$$

$$(A10) \quad e^* = 1 + e \cdot e^*$$

$$(A11) \quad e^* = (1 + e)^*$$

But: $e \cdot (f + g) \neq e \cdot f + e \cdot g$

But: $e \cdot 0 \neq 0$

Inference rules: rules of equational logic plus

$$\frac{e = f \cdot e + g}{e = f^* \cdot g} \text{ RSP}^* \text{ (if } f \text{ does not terminate immediately)}$$

Milner's Question (Q1)

Is Mil complete with respect to $=_{[\cdot]_P}$? (Does $e =_{[\cdot]_P} f \implies e =_{\text{Mil}} f$ hold?)

Milner's questions

(Q1) Complete axiomatization:

Is the proof system Mil complete for $=_{[\cdot]_P}$?

(Q2) $[\cdot]_P$ -Expressibility:

What structural property characterizes process graphs that are $[\cdot]_P$ -expressible ?

Milner's questions

(Q1) Complete axiomatization:

Is the proof system Mil complete for $\llbracket \cdot \rrbracket_P$?

(Q2) $\llbracket \cdot \rrbracket_P$ -Expressibility:

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- ▶ is decidable (Baeten/Corradini/G, 2007)

Milner's questions

(Q1) Complete axiomatization:

Is the proof system Mil complete for $=_{[\cdot]_P}$?

(Q2) $[\cdot]_P$ -Expressibility:

What structural property characterizes process graphs that are $[\cdot]_P$ -expressible ?

- ▶ is decidable (Baeten/Corradini/G, 2007)
- ▶ partial new answer (G/Fokkink, 2020):
 - ▶ bisimulation collapse has loop existence & elimination property (LEE) if expressible by under-star-1-free regular expression

Milner's questions

(Q1) Complete axiomatization:

Is the proof system Mil complete for $\llbracket \cdot \rrbracket_P$?

- ▶ series of partial completeness results for:
 - ▶ exitless iterations (Fokkink, 1998)
 - ▶ with a stronger fixed-point rule (G, 2006)
 - ▶ under-star 1-free, and without 0 (Corradini/de Nicola/Labellà, 2004)
 - ▶ with 0 but under-star-1-free (G/Fokkink, 2020)

(Q2) $\llbracket \cdot \rrbracket_P$ -Expressibility:

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Milner's questions

(Q1) Complete axiomatization:

Is the proof system Mil complete for $\llbracket \cdot \rrbracket_P$?

- ▶ Yes! (G, 2022, proof summary, employing LEE and crystallization)
- ▶ series of partial completeness results for:
 - ▶ exitless iterations (Fokkink, 1998)
 - ▶ with a stronger fixed-point rule (G, 2006)
 - ▶ under-star 1-free, and without 0 (Corradini/de Nicola/Labellà, 2004)
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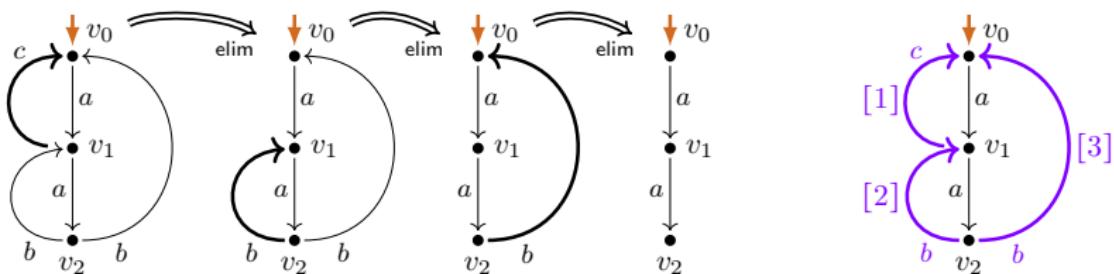
Question (Q2) specialized

(Q2)₀ P -Expressibility and P - $(*/\perp)$ -Expressibility:

What *structural property* characterizes:

- ▶ process graphs that are P -expressible ?
(... in the *image of P ?*)
- ▶ process graphs that are P -expressible by $(*/\perp)$ regular expressions?
(... in the *image of $(*/\perp)$ expressions under P ?*)

Loop Existence and Elimination (LEE)



Loop graphs (interpretations of innermost iterations without 1)

Definition

A process graph is a **loop graph** if:

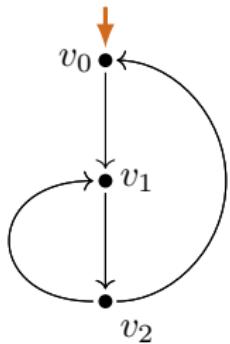
- (L1) There is an infinite path from the **start vertex**.
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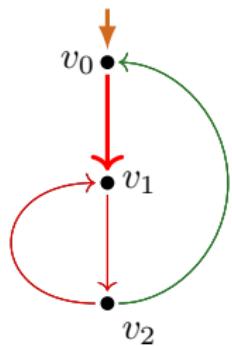


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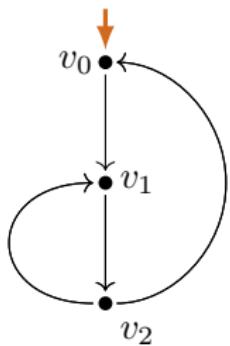
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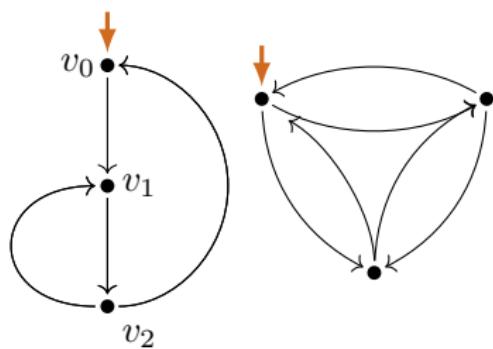
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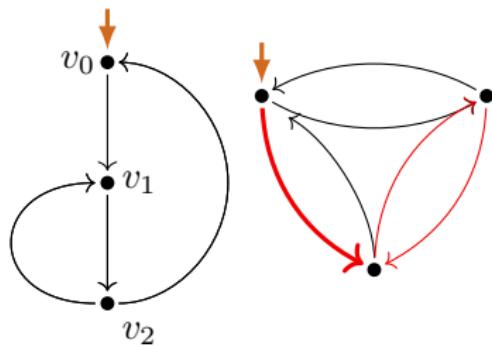
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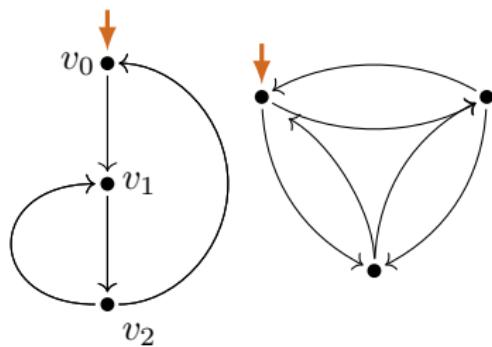
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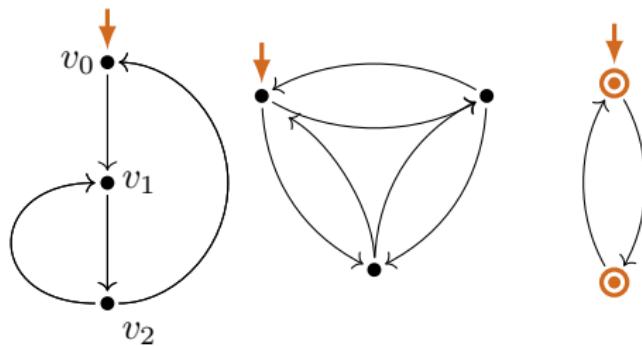
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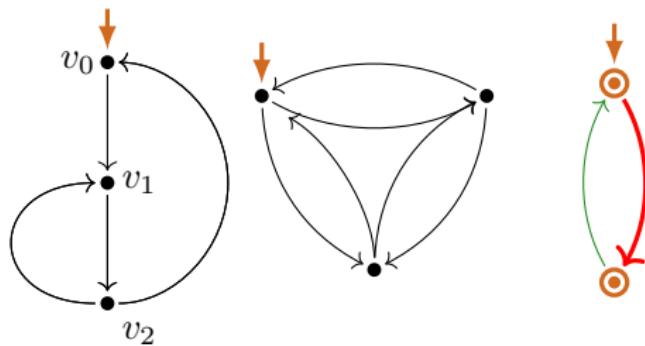
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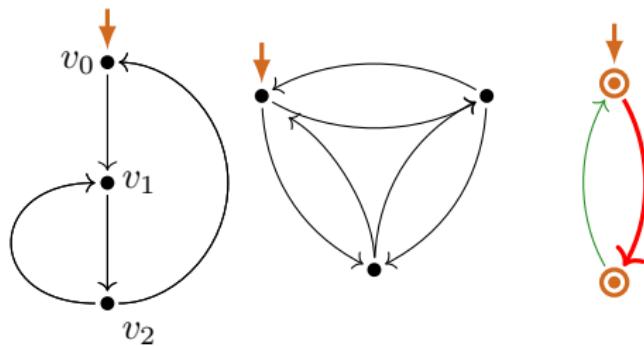
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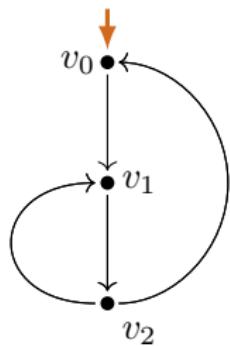
(L1),(L2),**(L3)**

Loop graphs (interpretations of innermost iterations without 1)

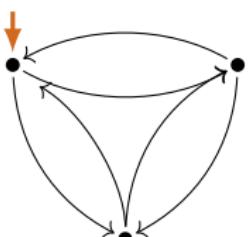
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(L2)



(L1),(L2),
(L3)

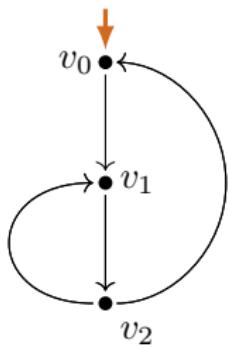


Loop graphs (interpretations of innermost iterations without 1)

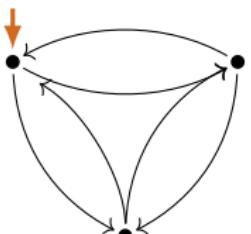
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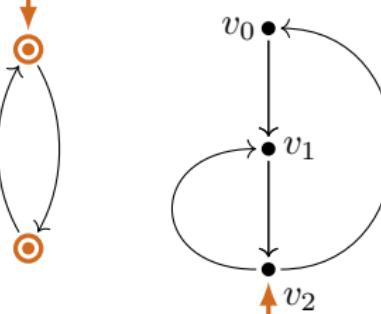
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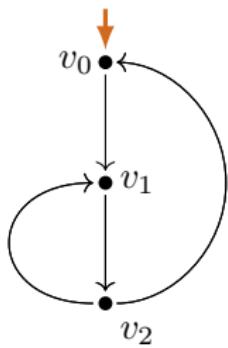


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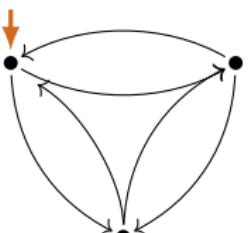
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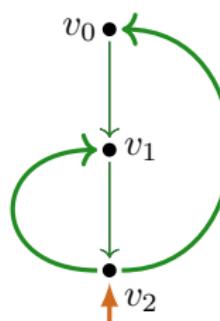
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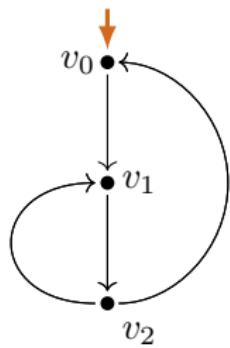


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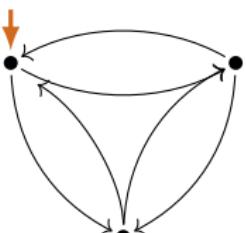
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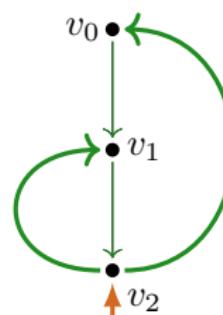
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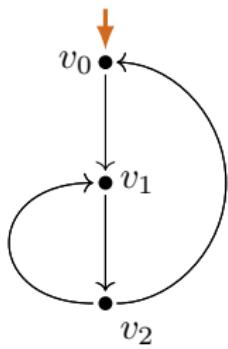
loop chart

Loop graphs (interpretations of innermost iterations without 1)

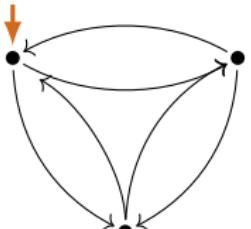
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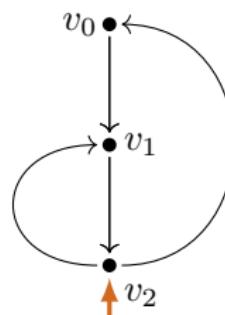
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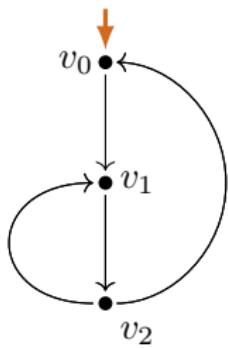
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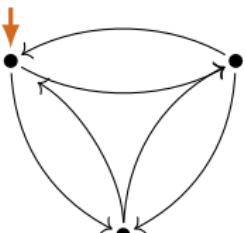
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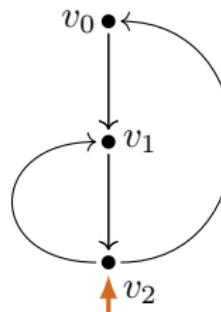
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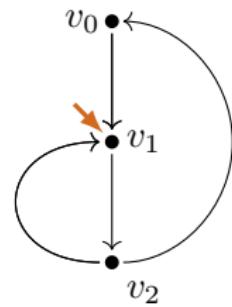
(L1),**(L2)**



(L1),(L2),**(L3)**



loop chart

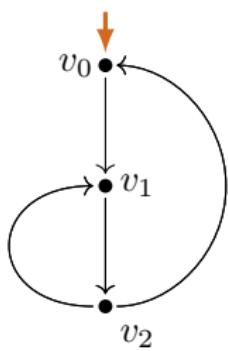


Loop graphs (interpretations of innermost iterations without 1)

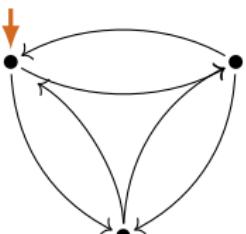
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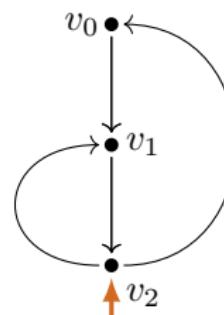
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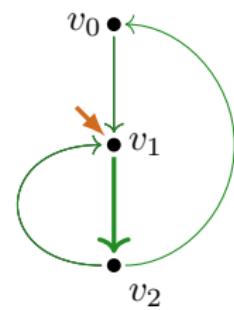
(L1),
(L2)



(L1),(L2),
(L3)



loop chart

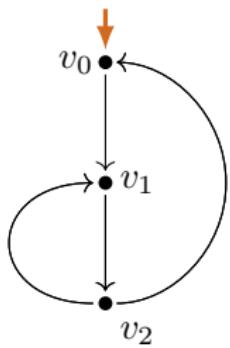


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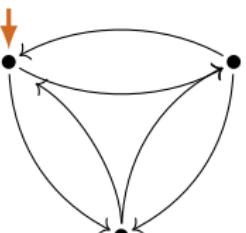
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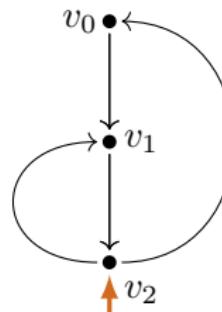
- (L1) There is an infinite path from the **start vertex**.
- (L2) Every infinite path from the **start vertex** returns to it.
- (L3) Termination is **only** possible at the **start vertex**.



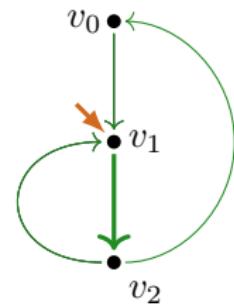
(L1),
(L2)



(L1),(L2),
(L3)



loop chart



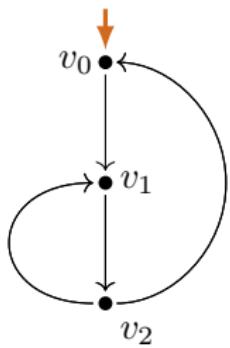
loop chart

Loop graphs (interpretations of innermost iterations without 1)

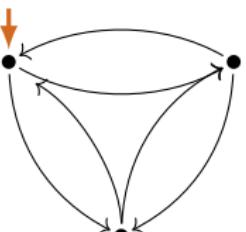
Definition

A process graph is a **loop graph** if:

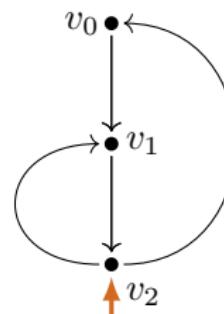
- (L1) There is an infinite path from the **start vertex**.
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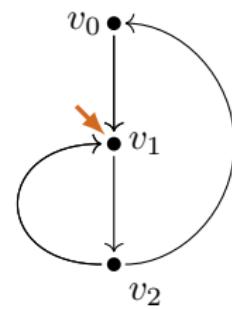
(L1),(L2)



(L1),(L2),(L3)



loop chart



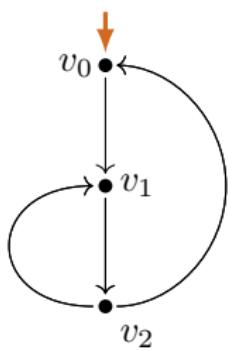
loop chart

Loop graphs (interpretations of innermost iterations without 1)

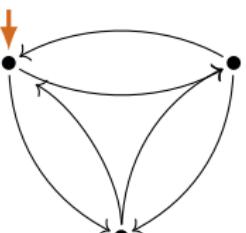
Definition

A process graph is a **loop graph** if:

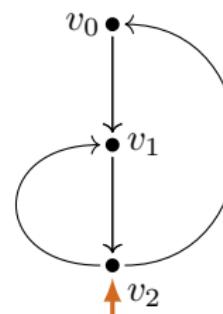
- (L1) There is an infinite path from the **start vertex**.
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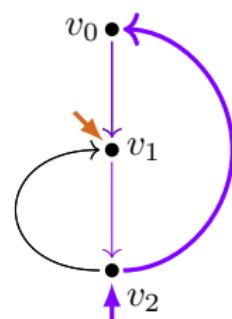
(L1),(L2)



(L1),(L2),(L3)

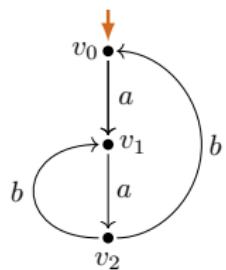


loop chart

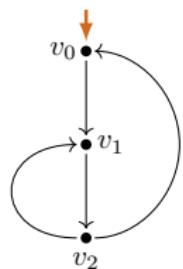


loop subchart

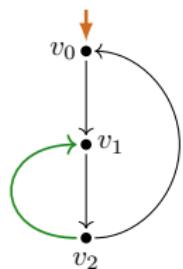
Loop existence and elimination (example 1)



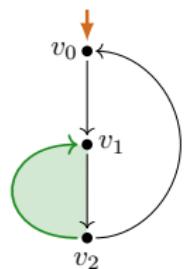
Loop existence and elimination (example 1)



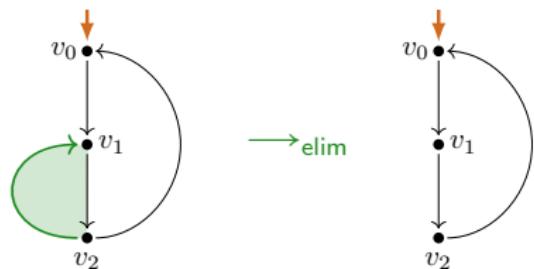
Loop existence and elimination (example 1)



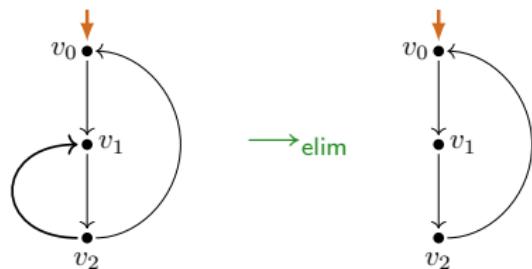
Loop existence and elimination (example 1)



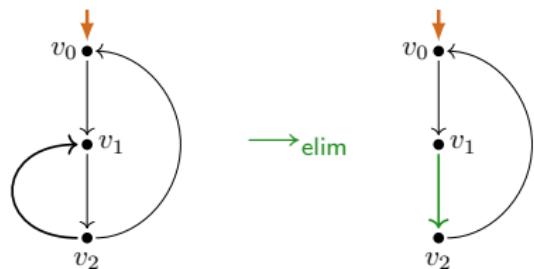
Loop existence and elimination (example 1)



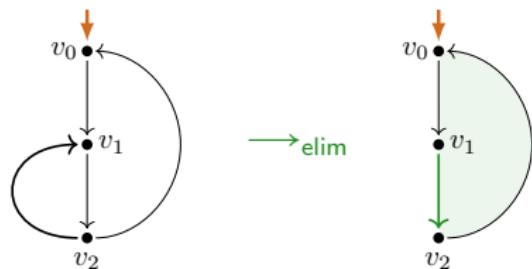
Loop existence and elimination (example 1)



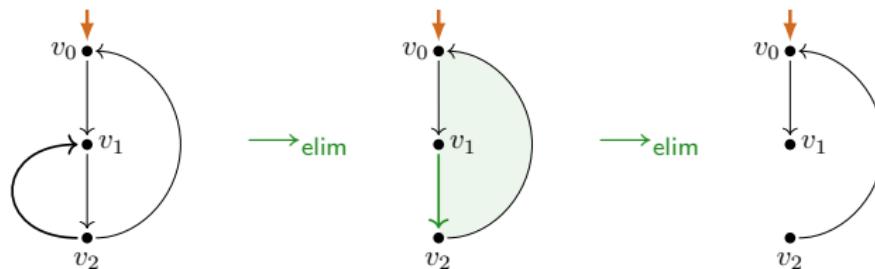
Loop existence and elimination (example 1)



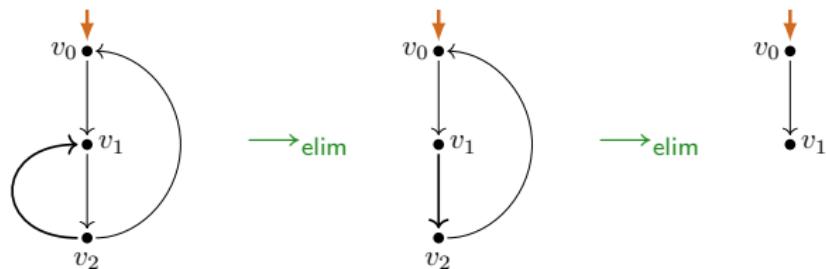
Loop existence and elimination (example 1)



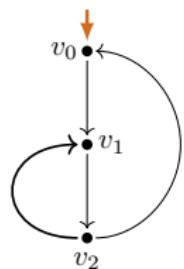
Loop existence and elimination (example 1)



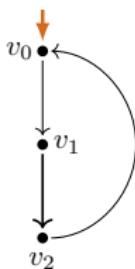
Loop existence and elimination (example 1)



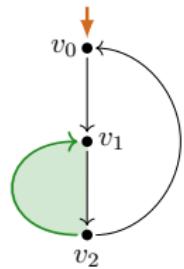
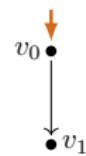
Loop existence and elimination (example 1)



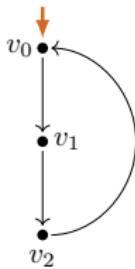
→ elim



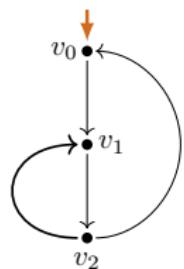
→ elim



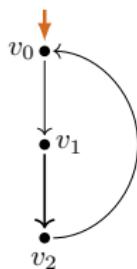
→ elim



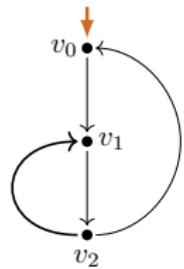
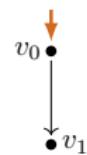
Loop existence and elimination (example 1)



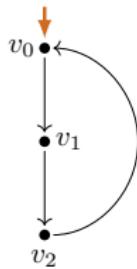
→ elim



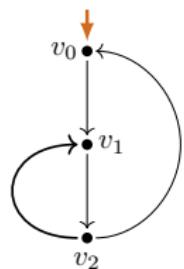
→ elim



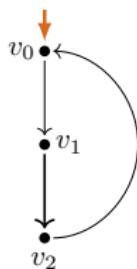
→ elim



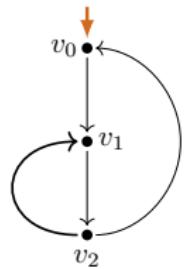
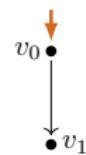
Loop existence and elimination (example 1)



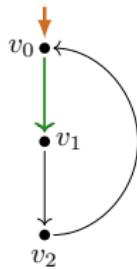
→ elim



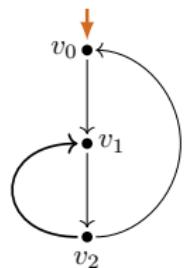
→ elim



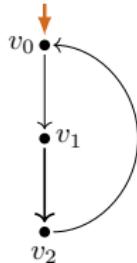
→ elim



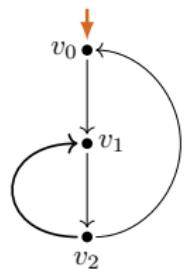
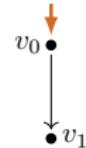
Loop existence and elimination (example 1)



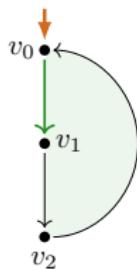
→ elim



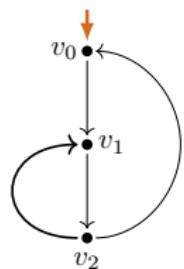
→ elim



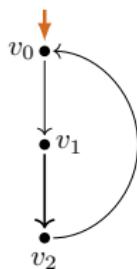
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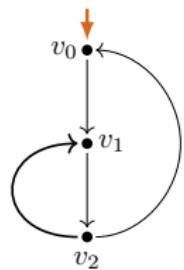
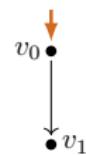
Loop existence and elimination (example 1)



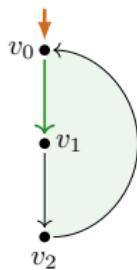
→ elim



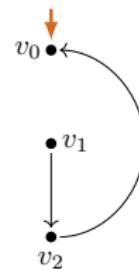
→ elim



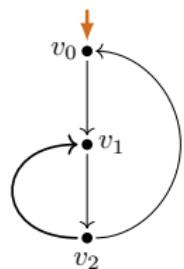
→ elim



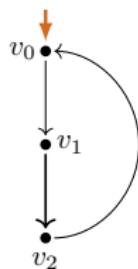
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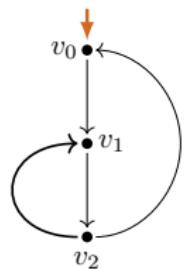
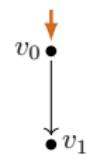
Loop existence and elimination (example 1)



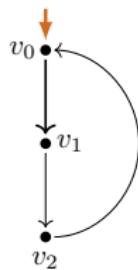
→ elim



→ elim



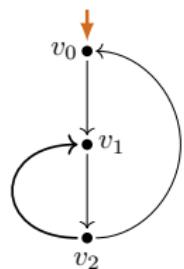
→ elim



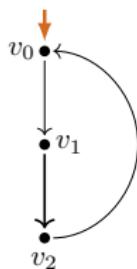
→ elim



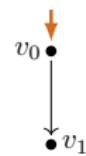
Loop existence and elimination (example 1)



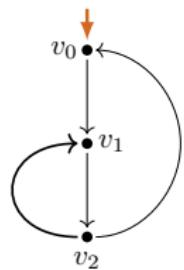
→ elim



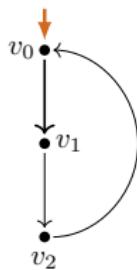
→ elim



→ prune



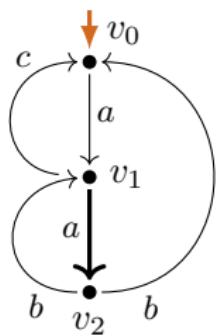
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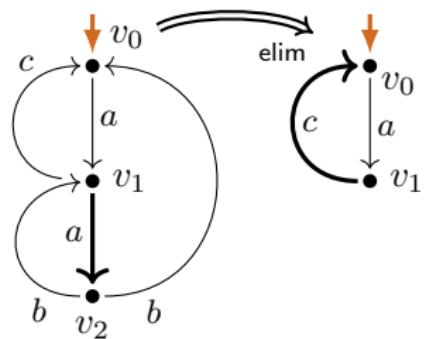
→ elim



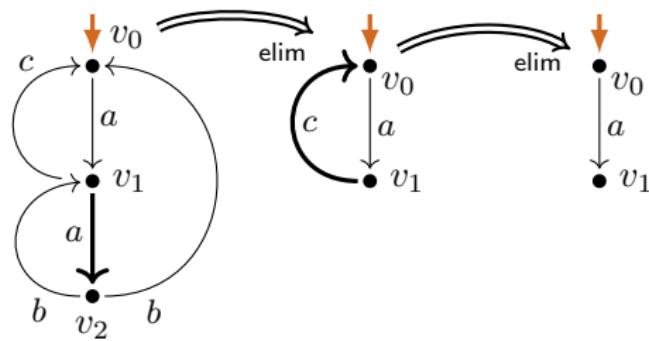
Loop existence and elimination (example 2)



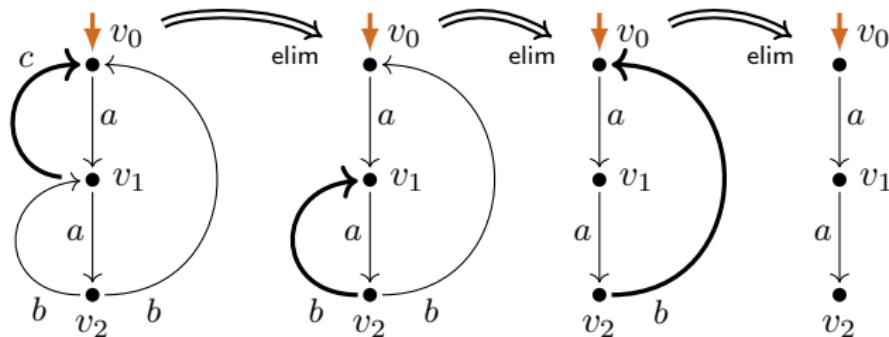
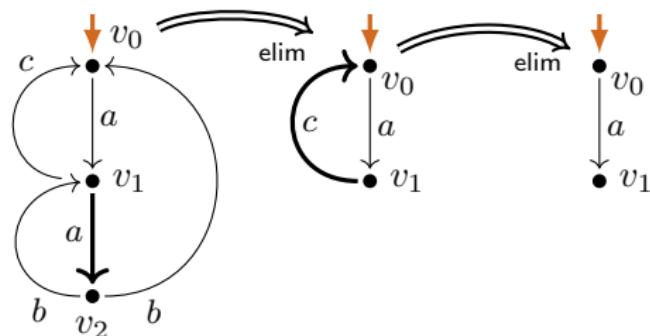
Loop existence and elimination (example 2)



Loop existence and elimination (example 2)



Loop existence and elimination (example 2)



LEE

Definition

A chart \mathcal{C} satisfies LEE (*loop existence and elimination*) if:

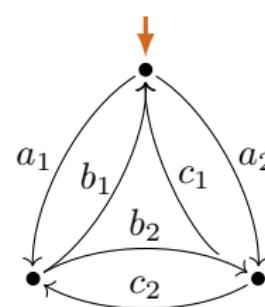
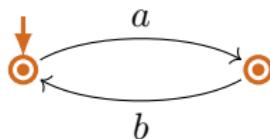
$$\exists \mathcal{C}_0 (\mathcal{C} \xrightarrow{*_{\text{elim}}} \mathcal{C}_0 \not\rightarrow_{\text{elim}} \wedge \mathcal{C}_0 \text{ permits no infinite path}).$$

LEE

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A chart \mathcal{C} satisfies LEE (*loop existence and elimination*) if:

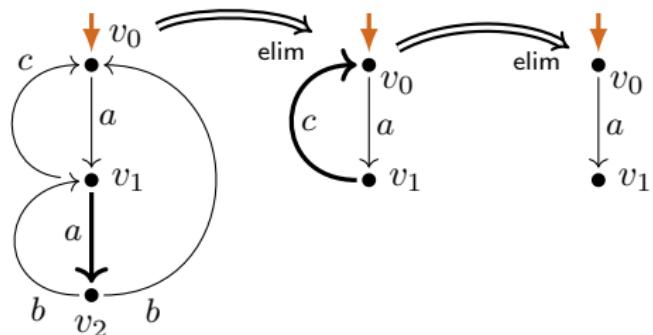
$$\exists \mathcal{C}_0 (\mathcal{C} \xrightarrow{*_{\text{elim}}} \mathcal{C}_0 \not\rightarrow_{\text{elim}} \wedge \mathcal{C}_0 \text{ permits no infinite path}).$$



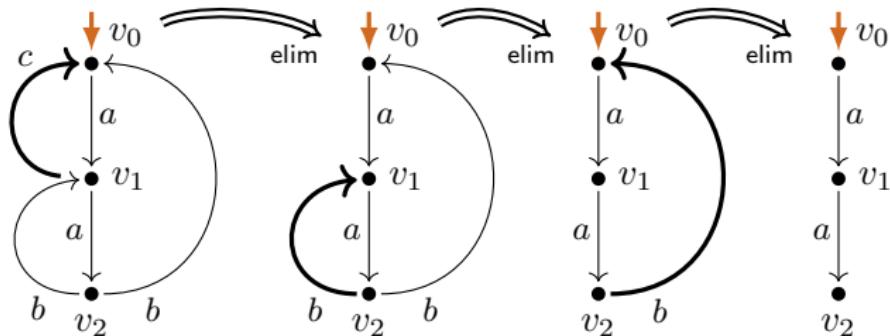
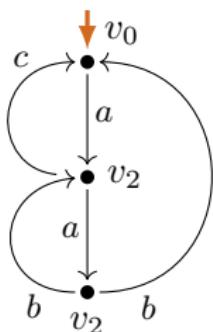
LEE

LEE

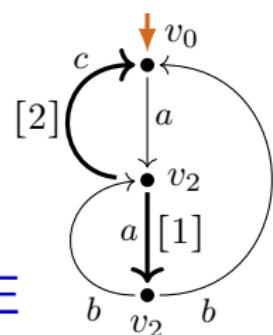
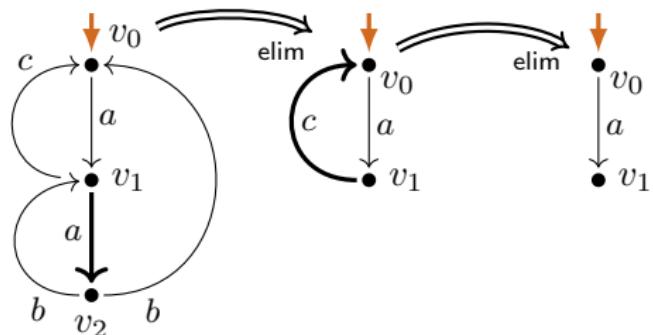
LEE



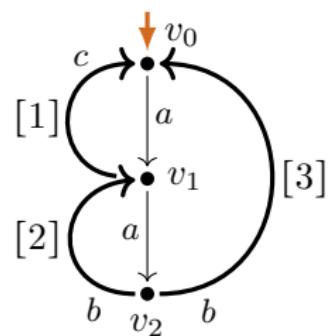
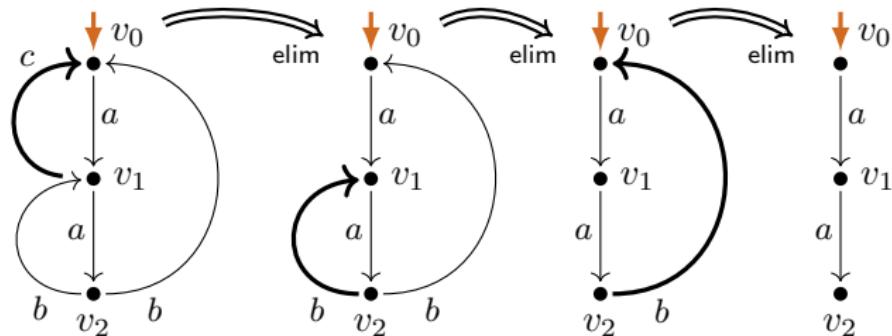
LEE



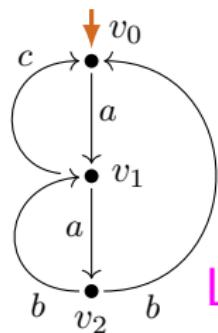
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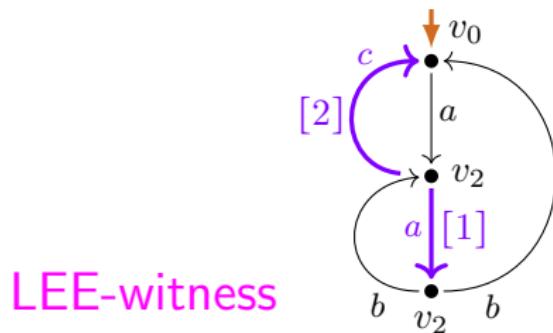
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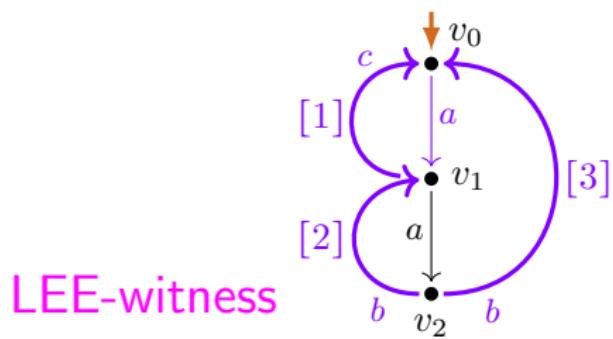
LEE



LEE-graph

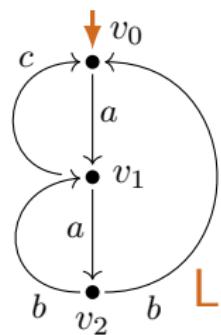


LEE-witness



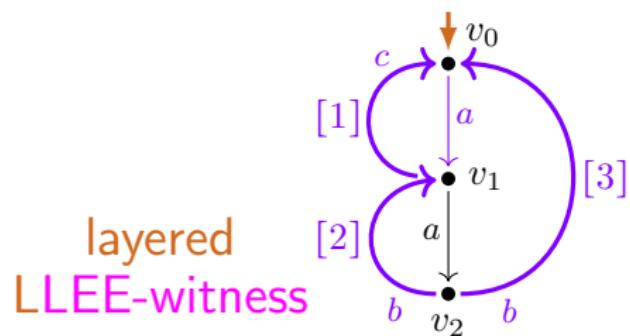
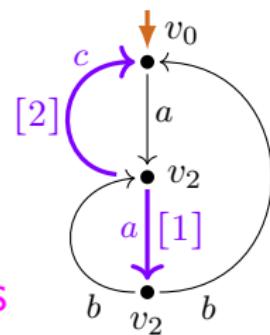
LEE-witness

Layered LEE



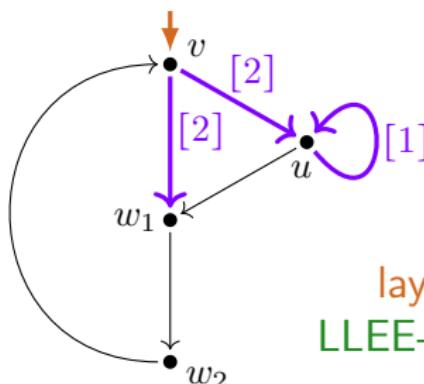
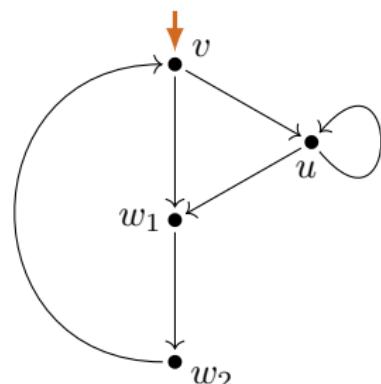
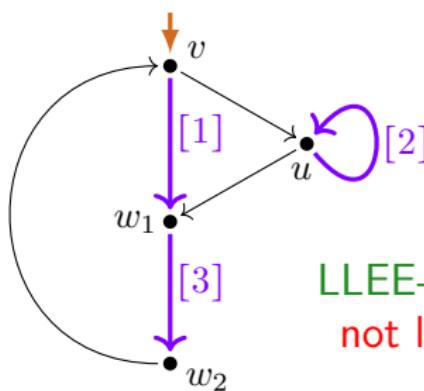
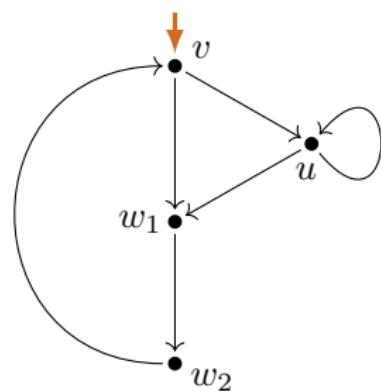
LLEE-graph

layered
LLEE-witness

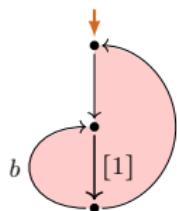


layered
LLEE-witness

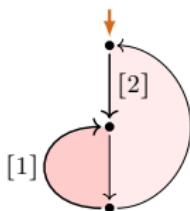
Layered LEE-witness (LLEE-witness)



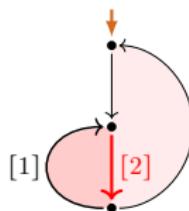
7 LEE-witnesses



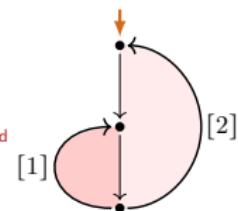
layered



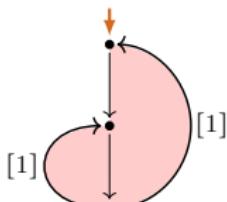
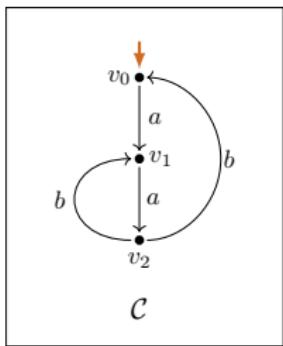
layered



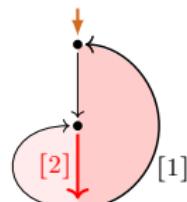
not layered



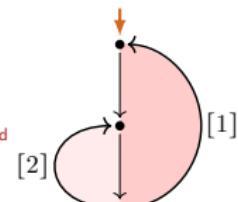
layered



layered



not layered



layered

Loop elimination: properties

$\longrightarrow_{\text{elim}}$: eliminate a transition-induced loop by:

- ▶ removing the loop-entry transition(s)
- ▶ garbage collection

$\longrightarrow_{\text{prune}}$: remove a transition to a deadlocking state

Lemma

(i) $\longrightarrow_{\text{elim}} \cup \longrightarrow_{\text{prune}}$ is terminating.

Loop elimination: properties

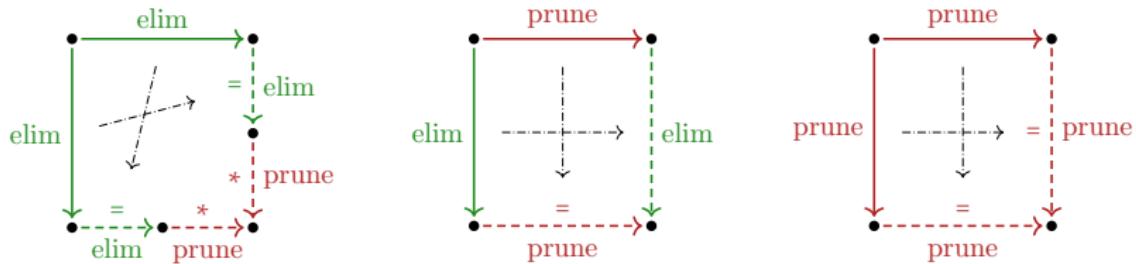
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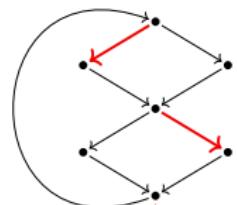
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Lemma

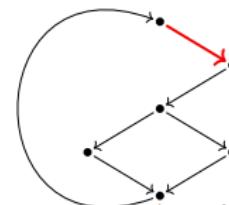
- (i) $\rightarrow_{\text{elim}} \cup \rightarrow_{\text{prune}}$ is terminating.
- (ii) $\rightarrow_{\text{elim}} \cup \rightarrow_{\text{prune}}$ is decreasing [Van Oostrom, de Bruijn]



'Critical pair': bi-loop elimination

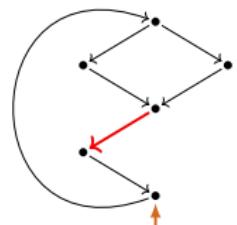


elim

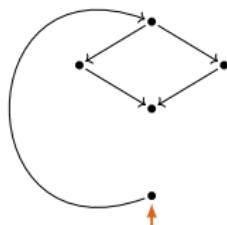


elim

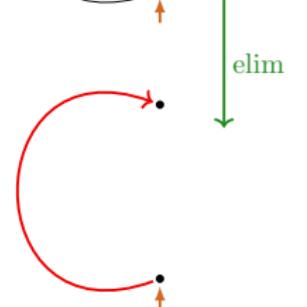
elim



elim



*
prune



*
prune

Loop elimination, and properties

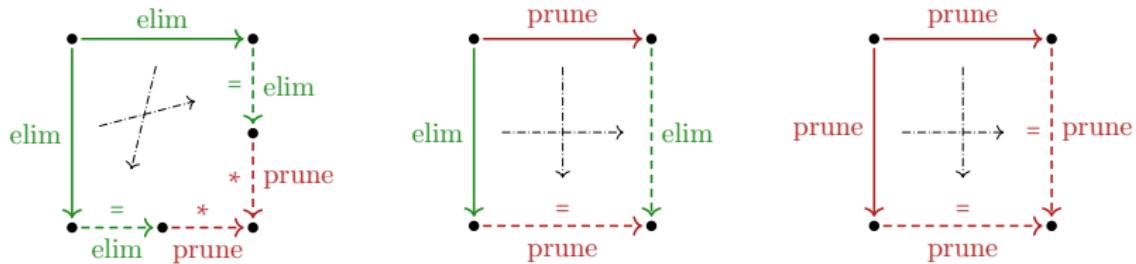
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Loop elimination, and properties

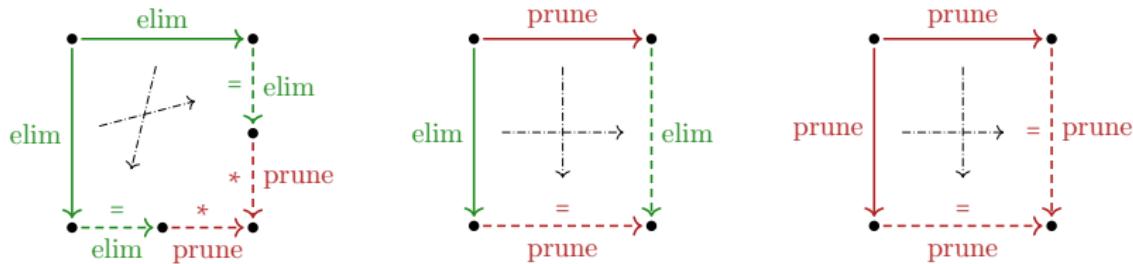
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- (iii) $\rightarrow_{\text{elim}} \cup \rightarrow_{\text{prune}}$ is confluent.



Structure property LEE

$$\text{LEE}(\textcolor{blue}{G}) : \iff \exists G_0 \left(\textcolor{blue}{G} \xrightarrow{\text{elim}}^* G_0 \not\rightarrow_{\text{elim}} \wedge G_0 \text{ has no infinite trace} \right).$$

Lemma (by using termination and confluence)

For every process graph $\textcolor{blue}{G}$ the following are equivalent:

- (i) $\text{LEE}(\textcolor{blue}{G})$.
- (ii) *There is an $\xrightarrow{\text{elim}}$ normal form without an infinite trace.*

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Theorem (efficient decidability)

The problem of deciding $\text{LEE}(\textcolor{blue}{G})$ for process graphs $\textcolor{blue}{G}$ is in PTIME.

Interpretation/extraction correspondences with LEE

(\Leftarrow G/Fokkink 2020, G 2021)

(Int)_P^(*/+): *P*-(*/+) -expressible graphs have the **structural property LEE**.

Process interpretations $P(e)$ of $(*/\pm)$ regular expressions e are finite process graphs that satisfy LEE.

(Extr)_P: LEE implies $\llbracket \cdot \rrbracket_P$ -expressibility

From every finite process graph G with LEE a regular expression e can be extracted such that $G \xrightarrow{\sim} P(e)$.

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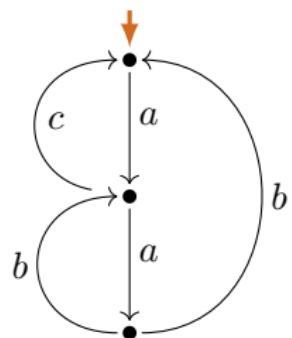
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(Coll): LEE is preserved under collapse

The class of finite process graphs with LEE is closed under bisimulation collapse.

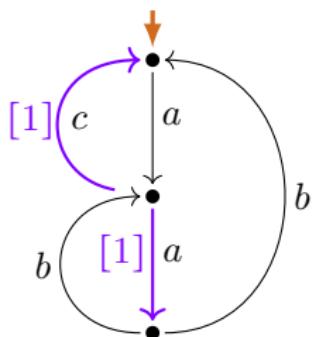
Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

G_2



Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

\widehat{G}_2



Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

\widehat{G}_2



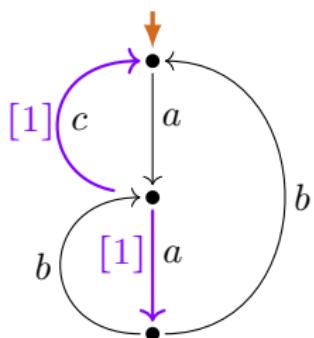
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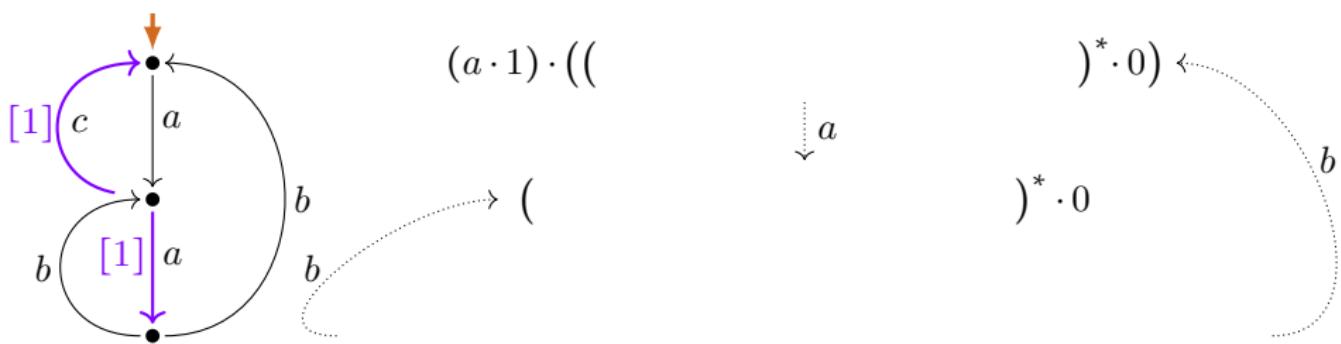


$$(a \cdot 1) \cdot ((\quad)^* \cdot 0)$$

()
 ↓
 a
)^* · 0

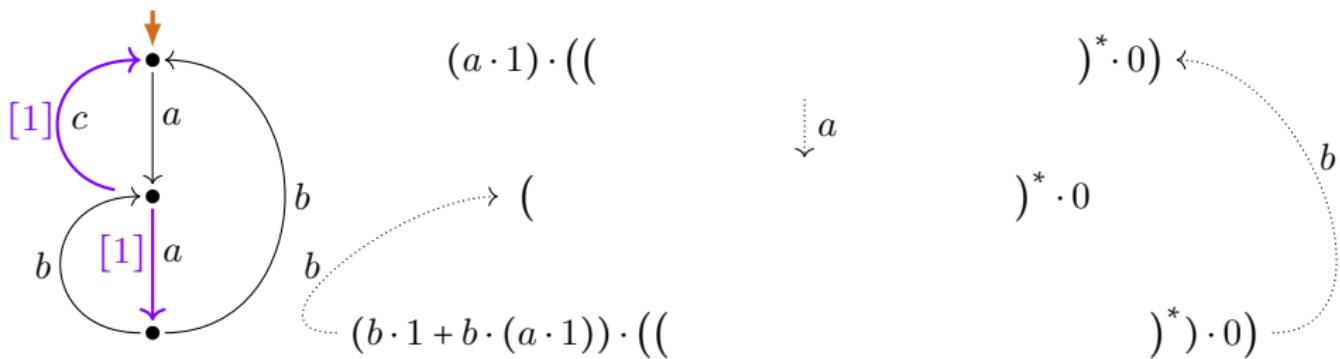
Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

\widehat{G}_2



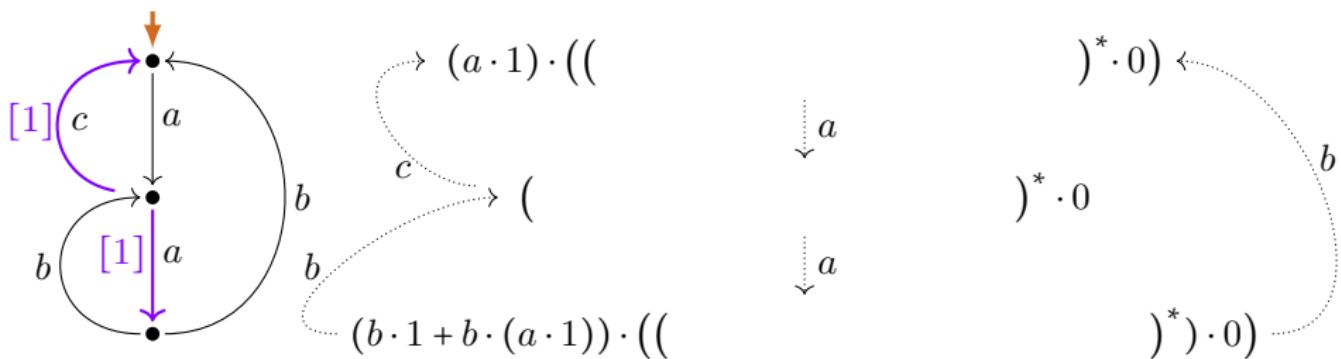
Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

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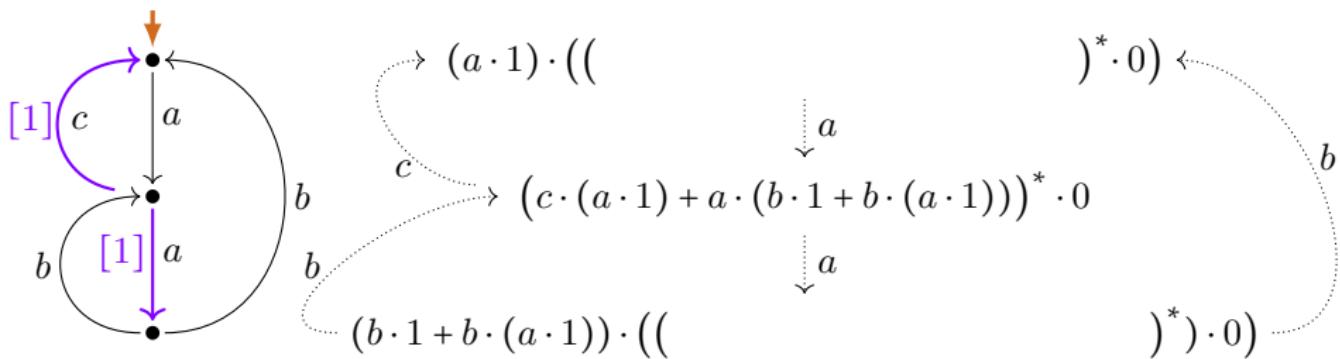
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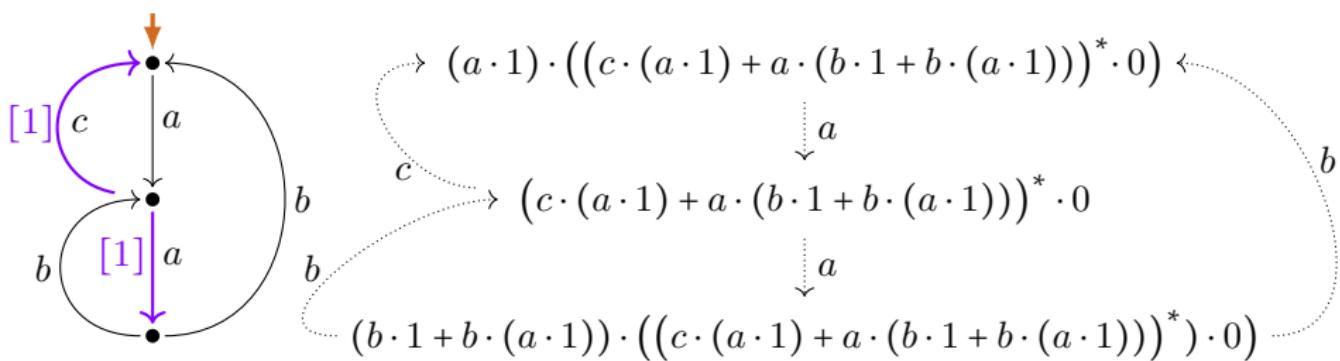
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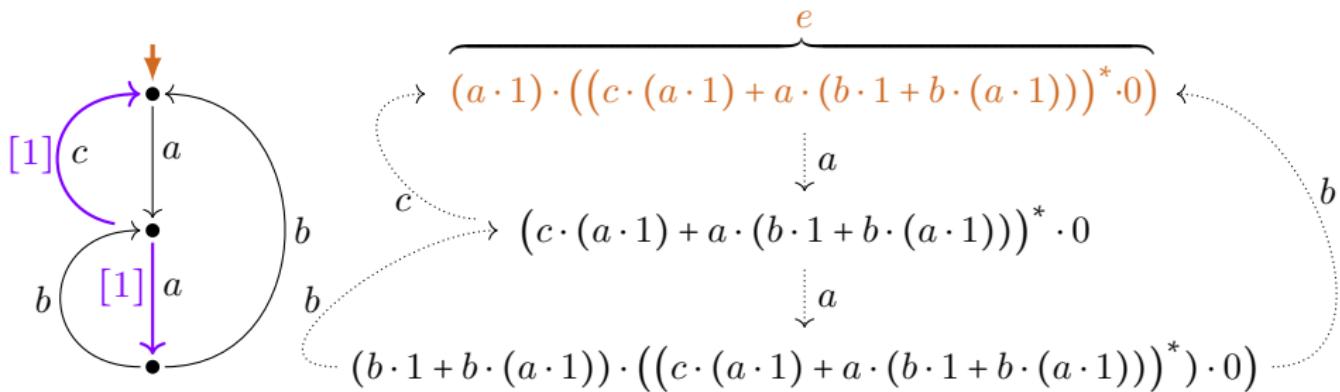
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\widehat{G}_2

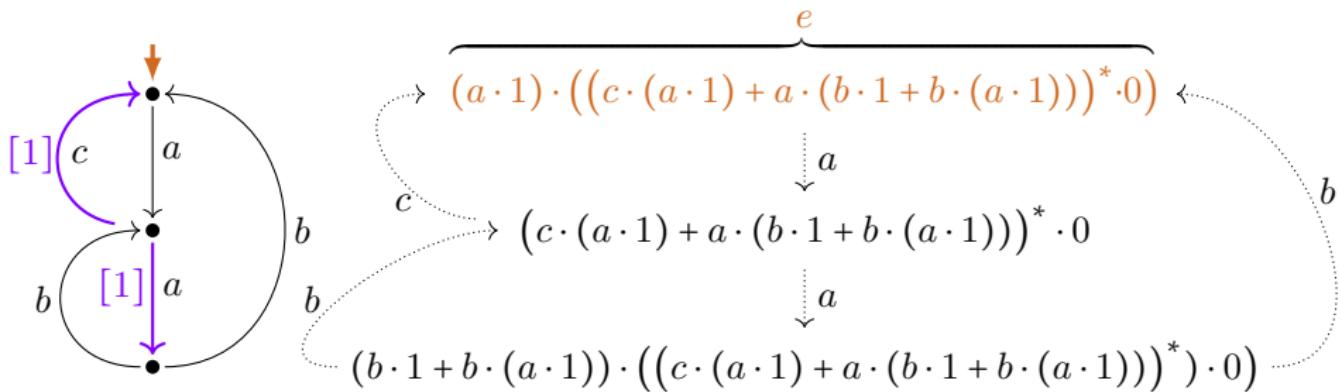


Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

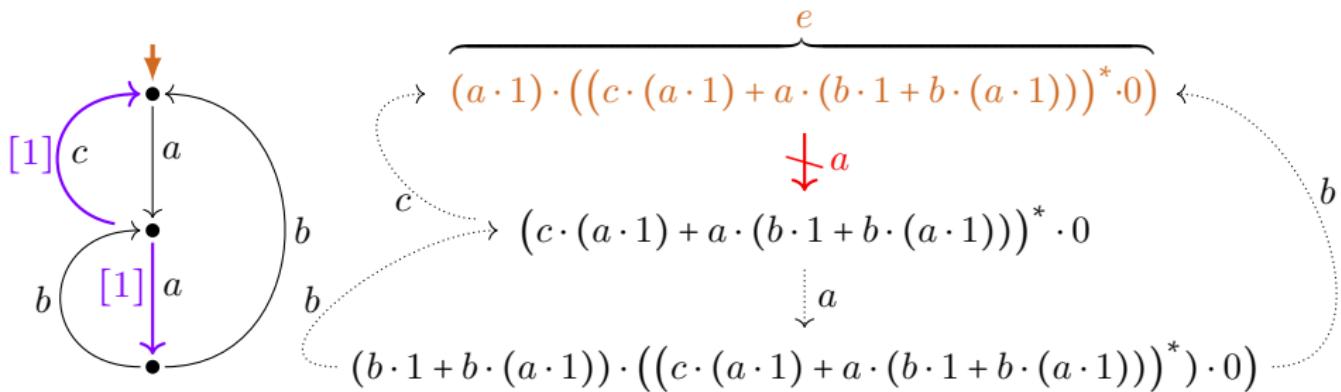
\widehat{G}_2



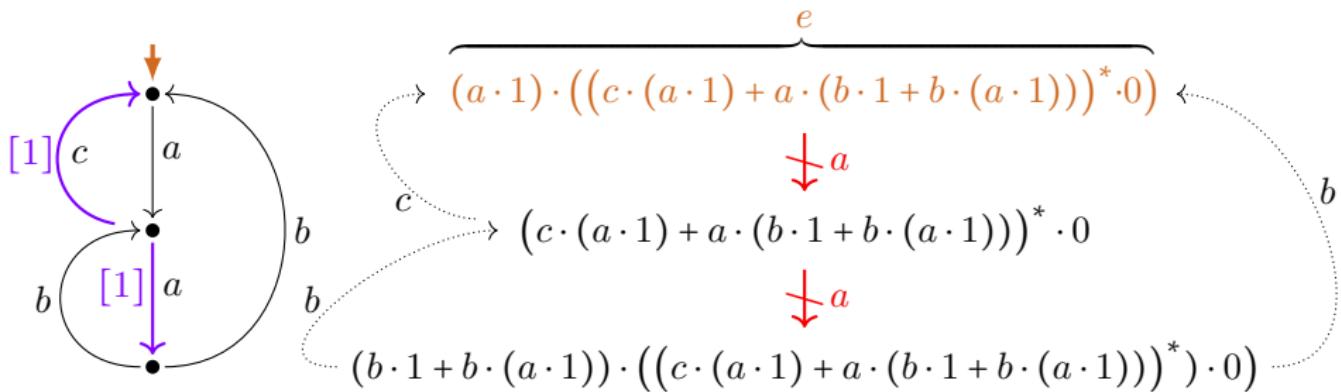
Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

 $\widehat{G_2}$ $P(e) \rightarrow G_2$ 

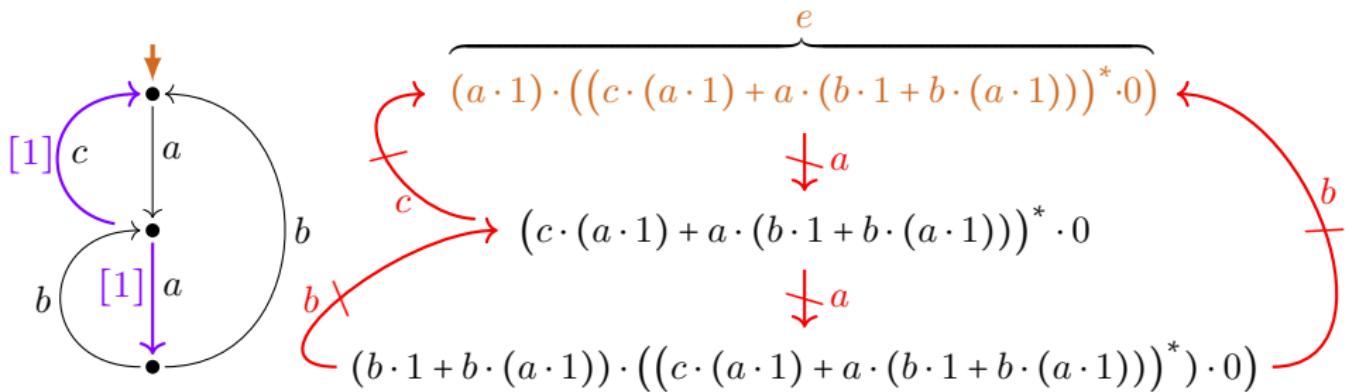
Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

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Expression extraction using LLEE (G/Fokkink 2020, G 2021/22)

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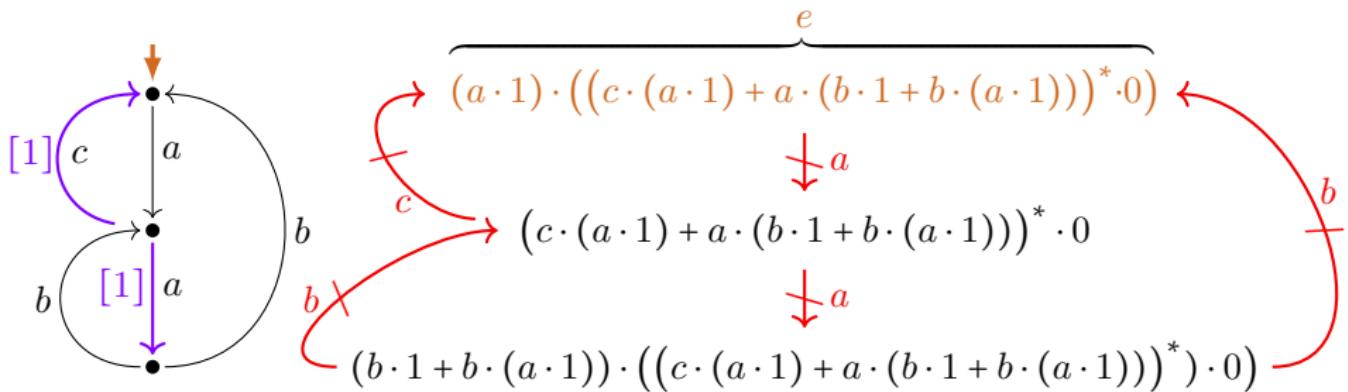
 \widehat{G}_2 $P(e) \rightarrow G_2$ 

Expression extraction using LLEE

(G/Fokkink 2020, G 2021/22)

\widehat{G}_2

$P(e) \supseteq G_2 \not\subseteq P(e)$

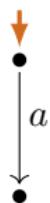


Interpretation of extracted expression

 G'_2 $P(e) = G'_2$ 

$$\overbrace{(a \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)}^{e}$$

Interpretation of extracted expression

 G'_2 $P(\textcolor{brown}{e}) = G'_2$ 

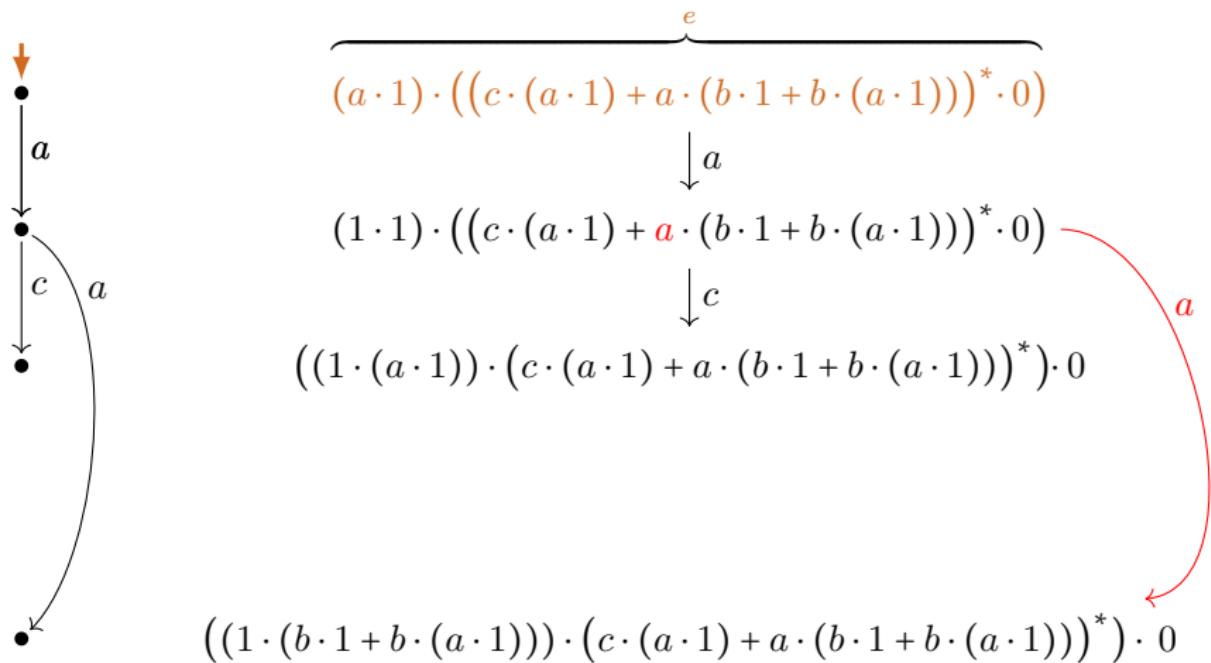
$$\overbrace{(\textcolor{red}{a} \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)}^{e}$$
$$(1 \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)$$

Interpretation of extracted expression

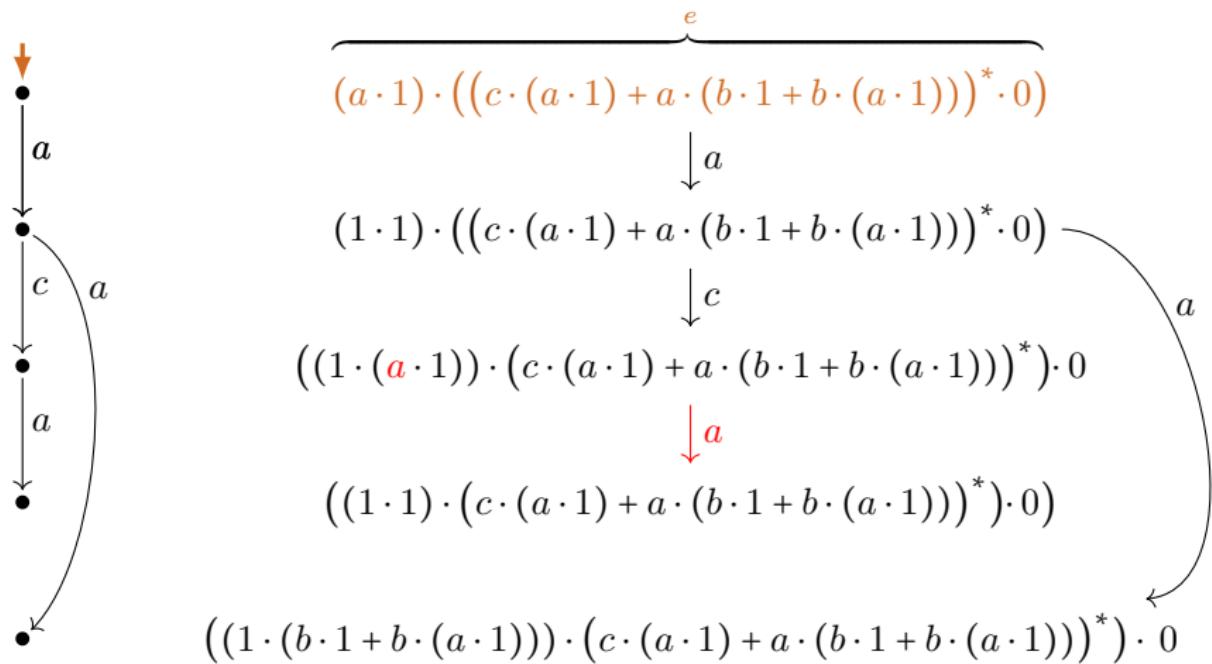
 G'_2 $P(e) = G'_2$ 

$$\overbrace{(a \cdot 1) \cdot ((c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)}^e$$
$$(1 \cdot 1) \cdot ((\cancel{c} \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^* \cdot 0)$$
$$((1 \cdot (a \cdot 1)) \cdot (c \cdot (a \cdot 1) + a \cdot (b \cdot 1 + b \cdot (a \cdot 1)))^*) \cdot 0$$

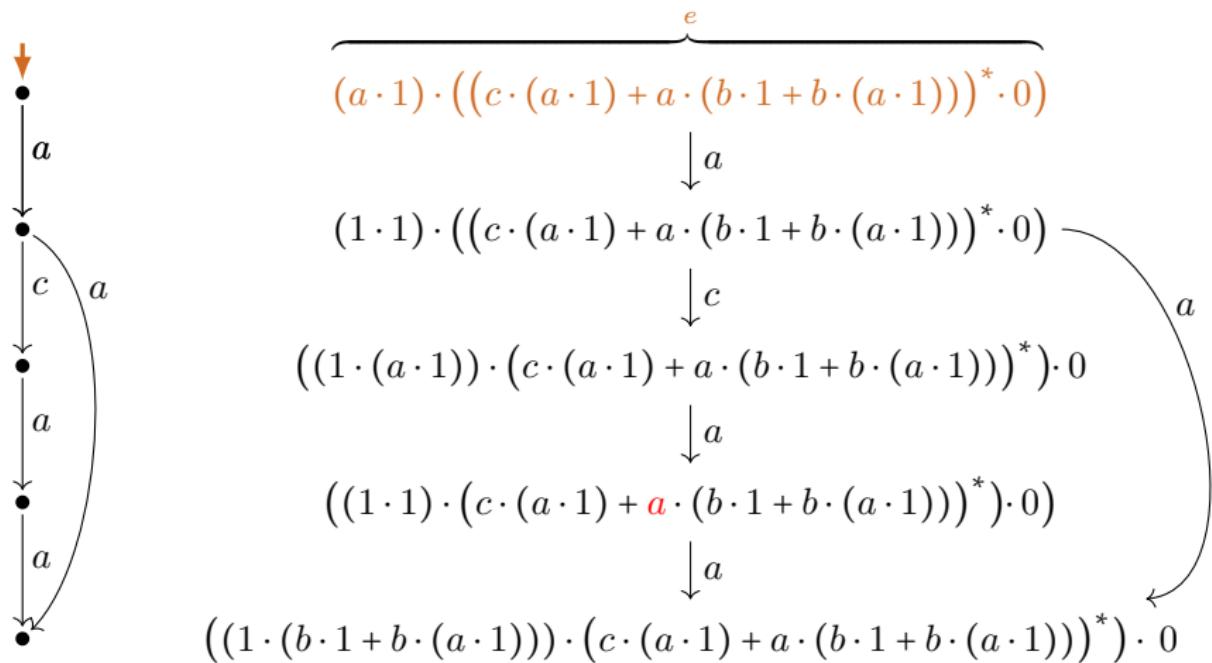
Interpretation of extracted expression

 G'_2 $P(e) = G'_2$ 

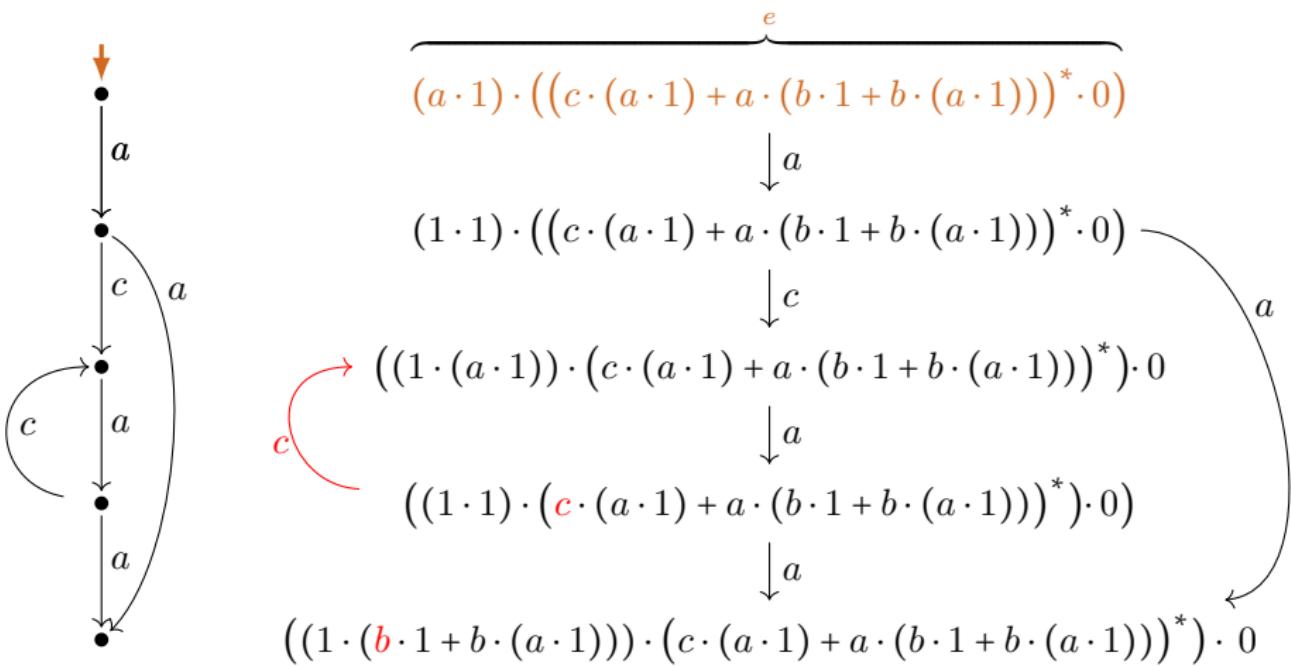
Interpretation of extracted expression

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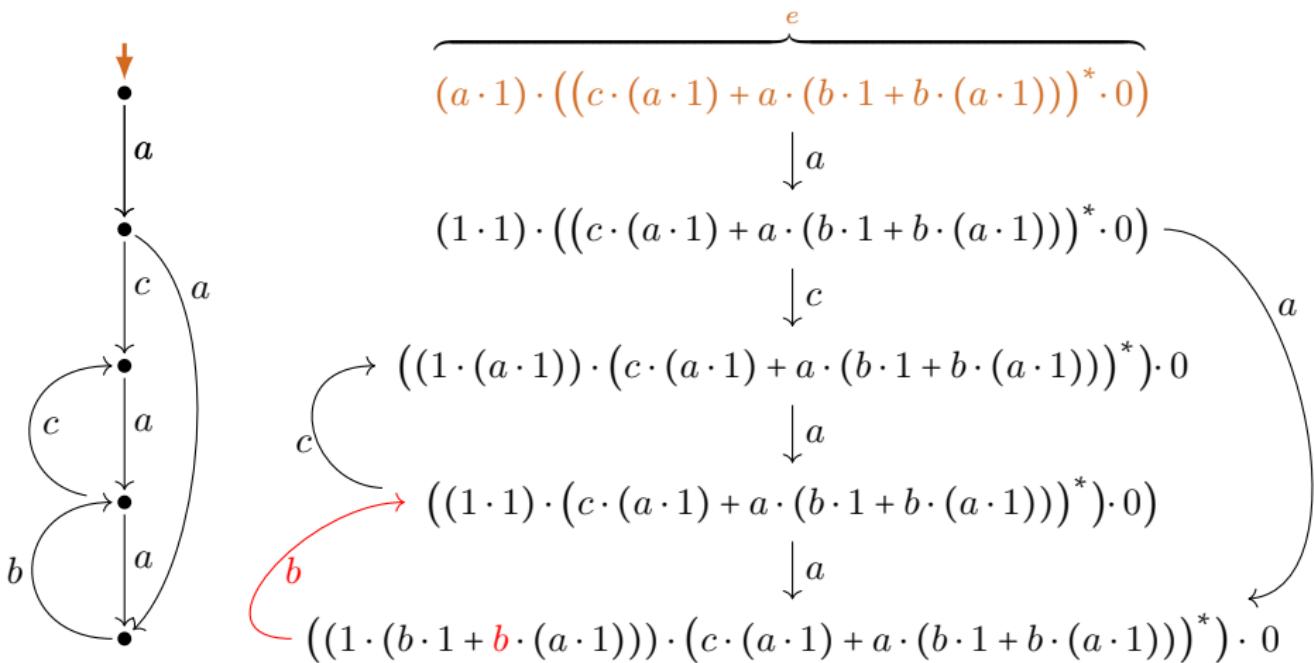
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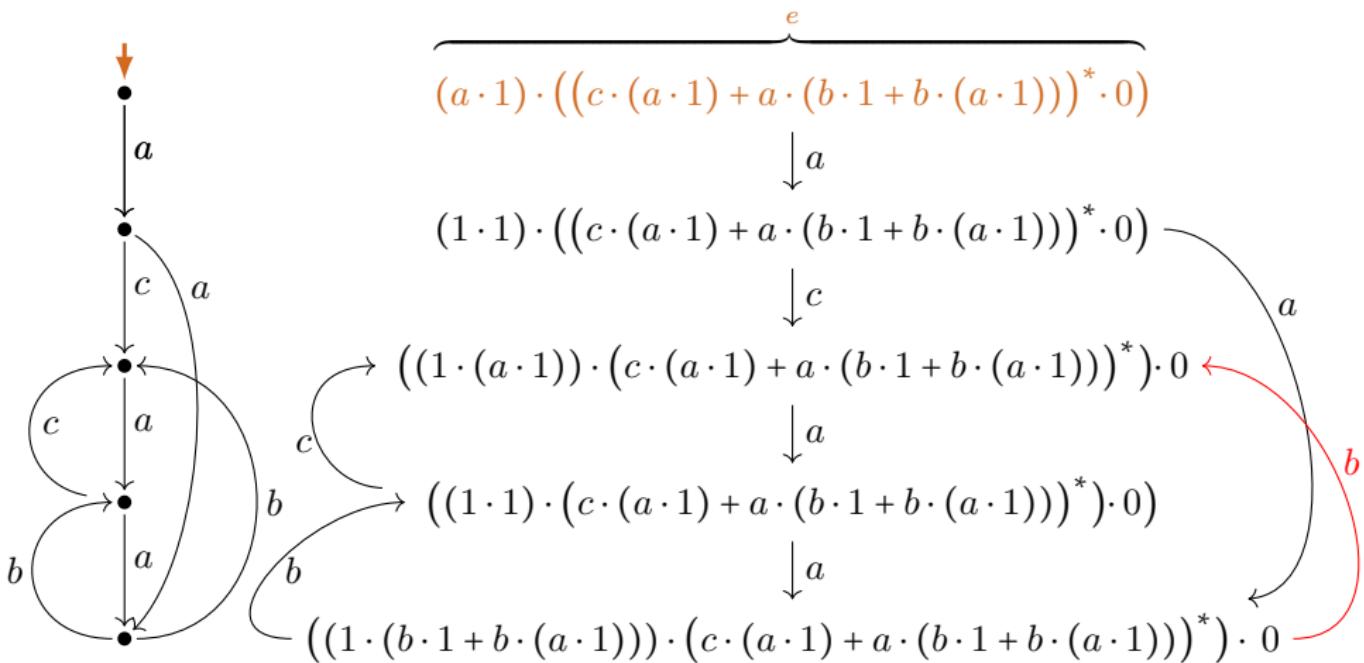
Interpretation of extracted expression

 G'_2 $P(e) = G'_2$ 

Interpretation of extracted expression

$$G'_2$$

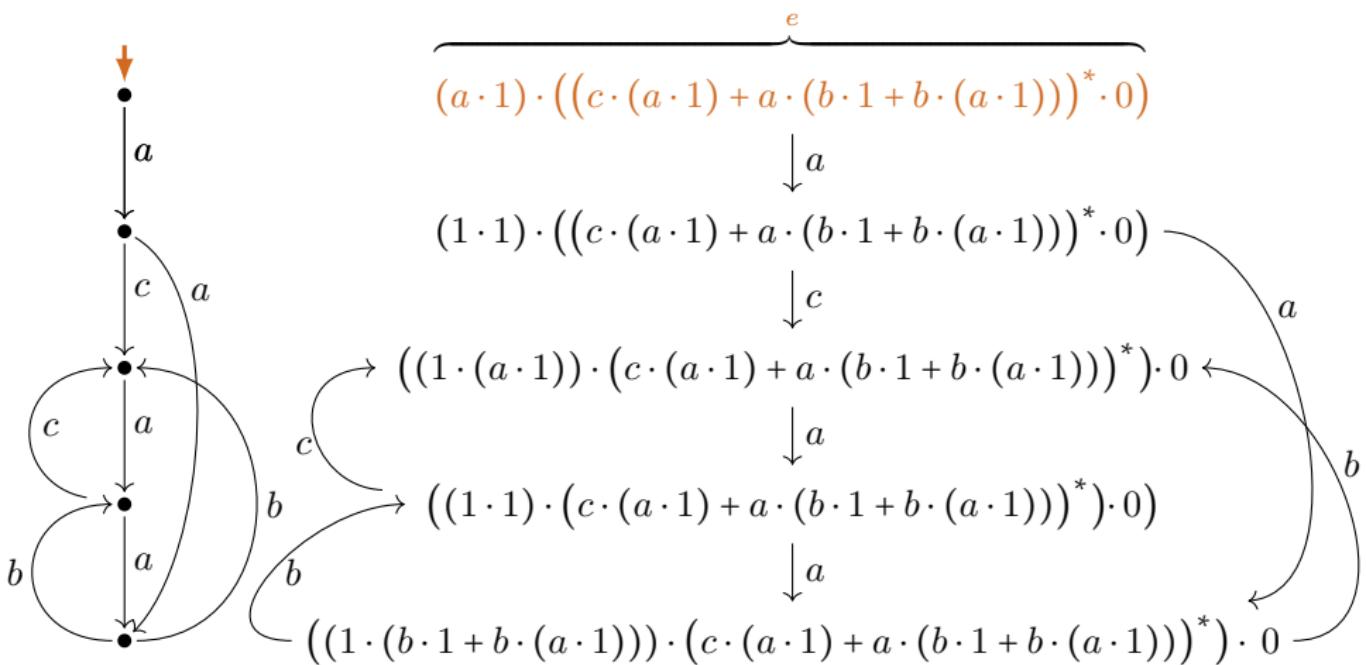
$$P(e) = G'_2$$



Interpretation of extracted expression

 G'_2

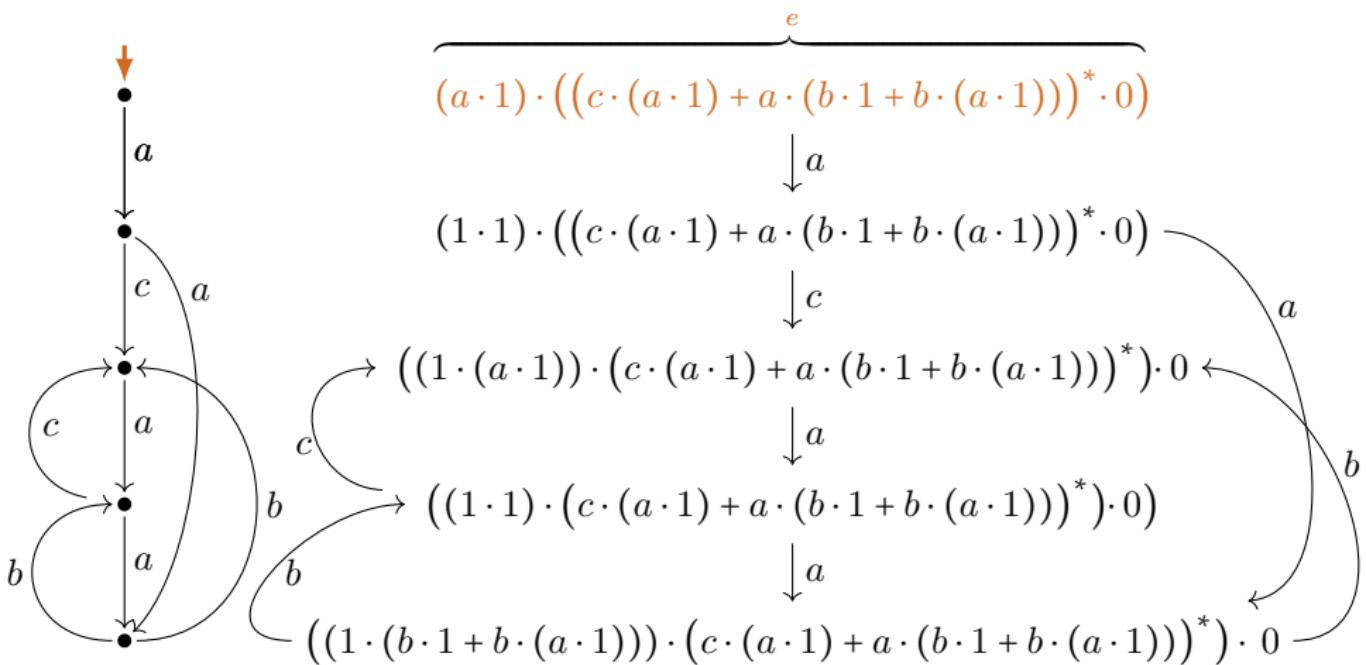
$$\textcolor{green}{P}(\textcolor{brown}{e}) = G'_2 \xrightarrow{\textcolor{magenta}{G_2}} G_2$$



Interpretation of extracted expression

 G'_2

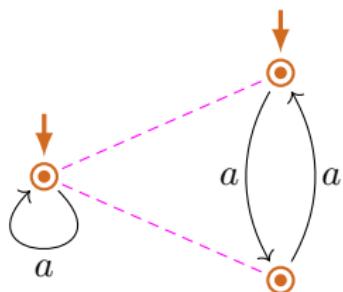
$$P(e) = G'_2 \xrightarrow{e} G_2 \not\cong G'_2$$



LEE under bisimulation

Observation

- ▶ LEE is **not** invariant under bisimulation.



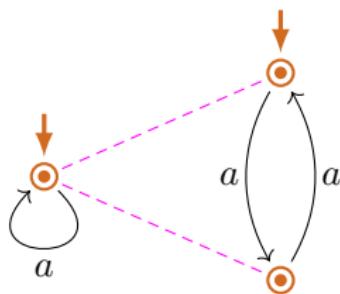
LEE

\neg LEE

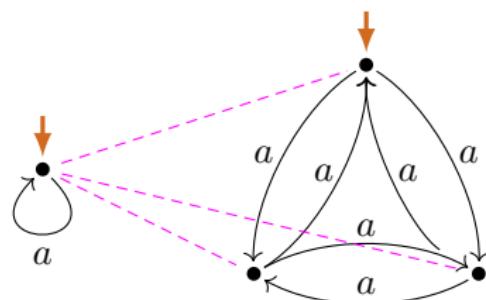
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LEE

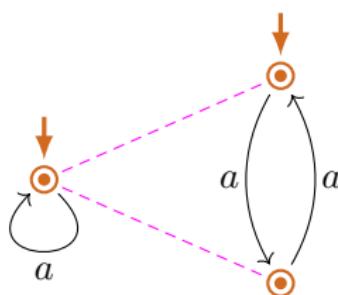


LEE

LEE under bisimulation

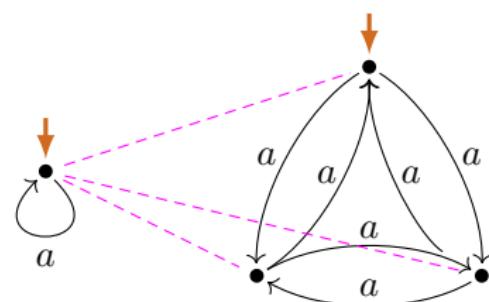
Observation

- ▶ LEE is **not** invariant under bisimulation.
- ▶ LEE is **not** preserved by converse functional bisimulation.



LEE

\neg LEE



LEE

\neg LEE

LEE under functional bisimulation

Lemma

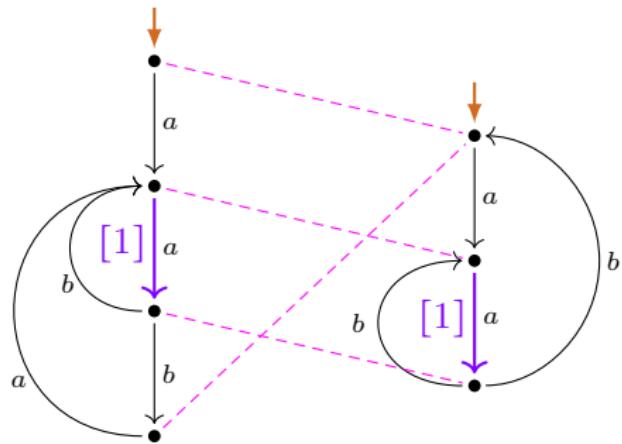
(i) LEE is preserved by *functional bisimulations*:

$$\text{LEE}(G_1) \wedge G_1 \sqsupseteq G_2 \implies \text{LEE}(G_2).$$

Proof (Idea).

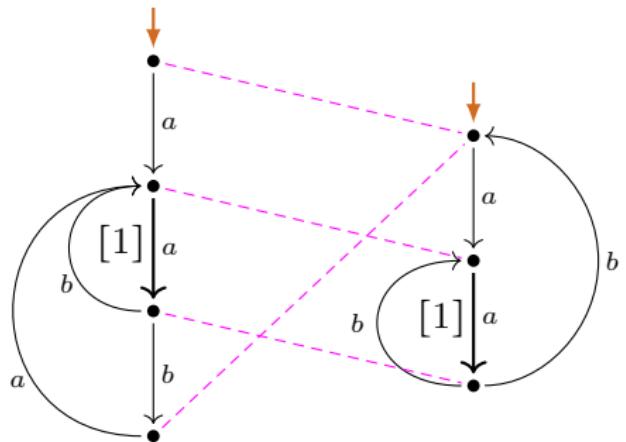
Use loop elimination in G_1 to carry out loop elimination in G_2 .

Collapsing LEE-witnesses

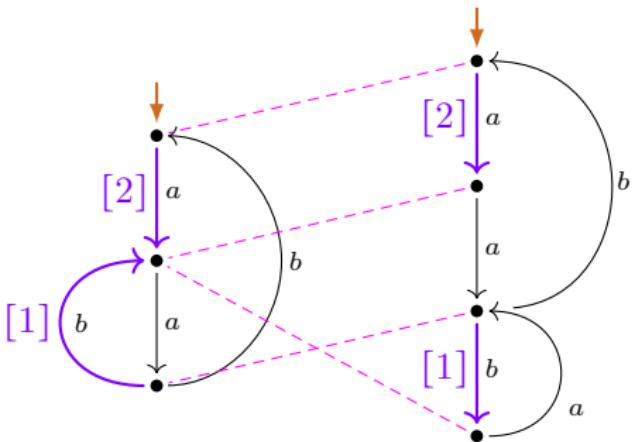


$$P(a(a(b+ba))^* \cdot 0)$$

Collapsing LEE-witnesses



$$\textcolor{green}{P}(a(a(b+ba))^* \cdot 0)$$



$$\textcolor{green}{P}((aa(ba))^* \cdot b)^* \cdot 0)$$

LEE under functional bisimulation / bisimulation collapse

Lemma

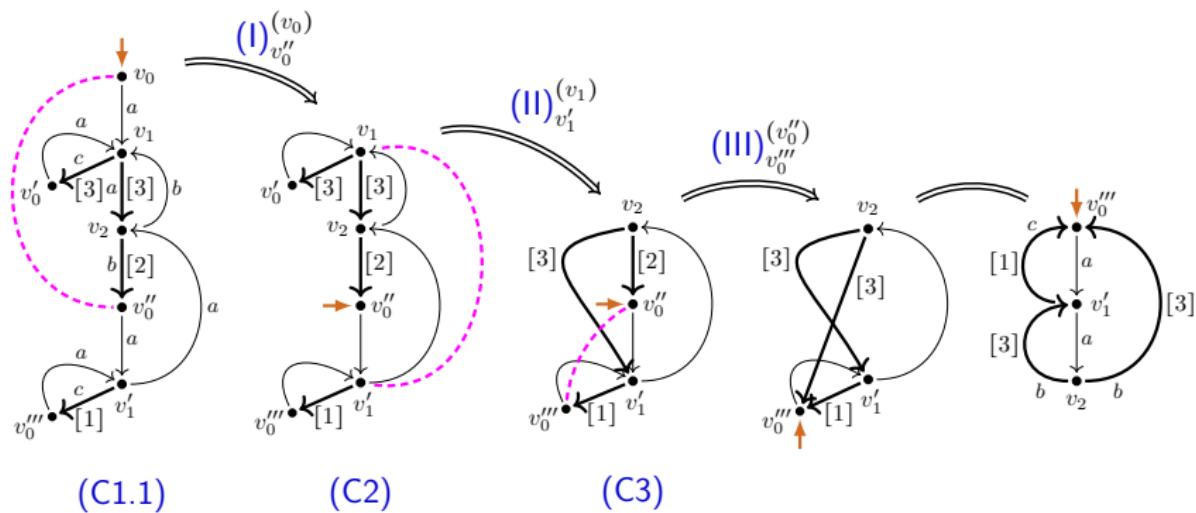
(i) LEE is preserved by *functional bisimulations*:

$$\text{LEE}(\textcolor{violet}{G}_1) \wedge \textcolor{violet}{G}_1 \not\sim \textcolor{violet}{G}_2 \implies \text{LEE}(\textcolor{violet}{G}_2).$$

(ii) LEE is preserved from a process graph to its *bisimulation collapse*:

$$\text{LEE}(\textcolor{violet}{G}) \wedge \textcolor{violet}{G} \text{ has bisimulation collapse } \textcolor{brown}{C} \implies \text{LEE}(\textcolor{brown}{C}).$$

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20) (no 1-transitions!)



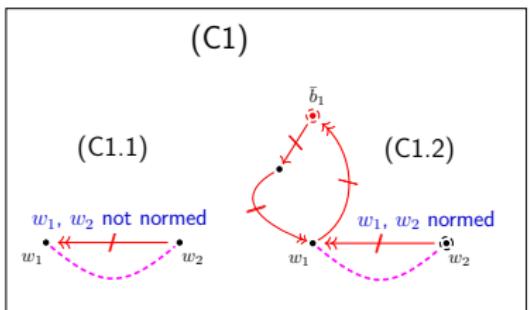
Lemma

The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

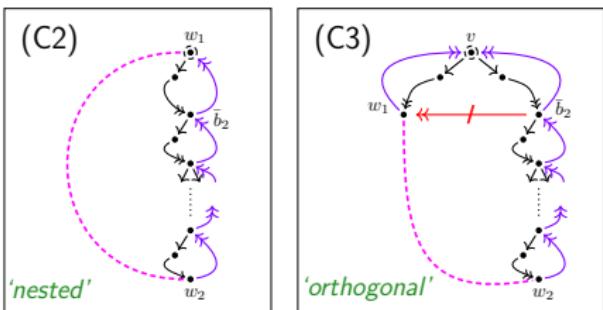
Reduced bisimilarity redundancies in LLEE-graphs (no 1-trans.!)

(G/Fokkink, LICS'20)

w_1, w_2 in different scc's



w_1, w_2 in the same scc



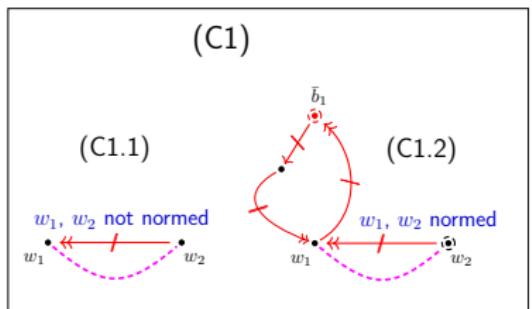
Lemma

Every *not collapsed* LLEE-graph contains bisimilar vertices $w_1 \neq w_2$ of kind (C1), (C2), or (C3) (a *reduced bisimilarity redundancy* $\langle w_1, w_2 \rangle$):

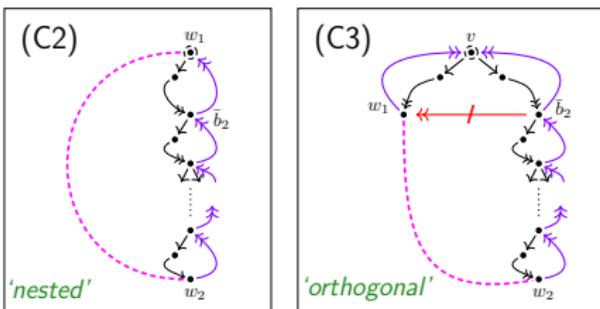
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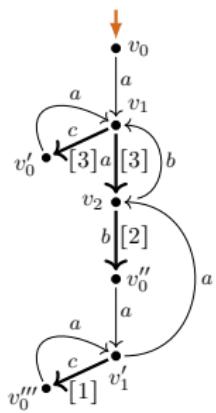
Lemma

Every *reduced bisimilarity redundancy* in a LLEE-graph can be eliminated LLEE-preservingly.

LLEE-preserving collapse of LLEE-charts

(G/Fokkink, LICS'20)

(no 1-transitions!)



(C1.1)

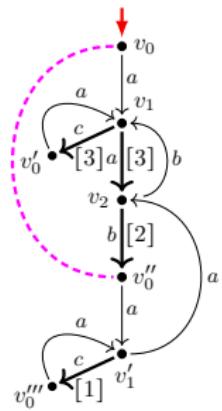
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LLEE-preserving collapse of LLEE-charts

(G/Fokkink, LICS'20)

(no 1-transitions!)

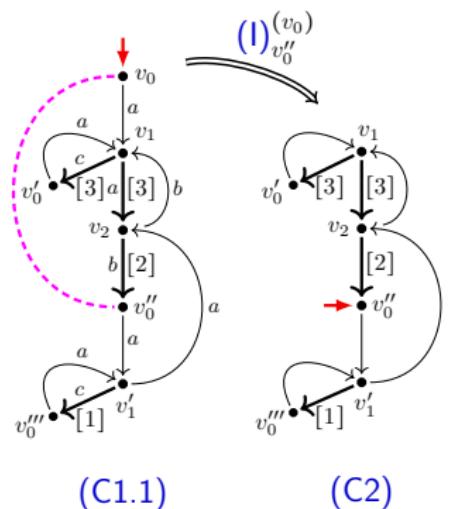


(C1.1)

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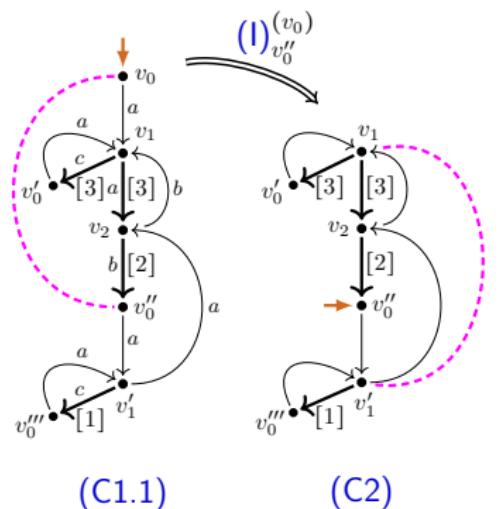
LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20) (no 1-transitions!)



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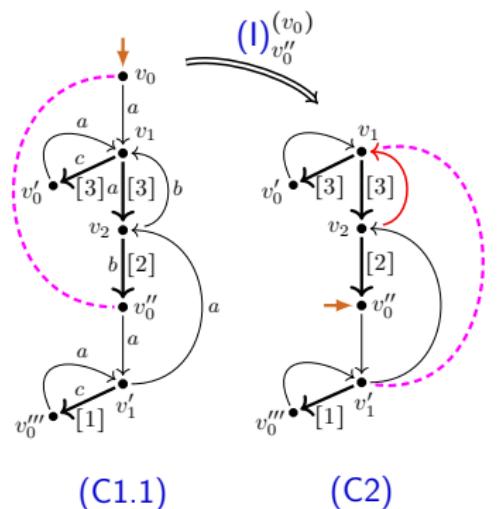


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(C1.1)

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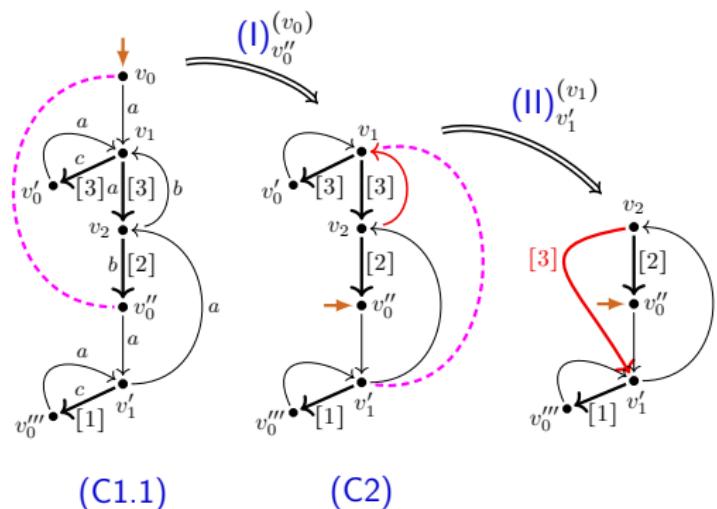
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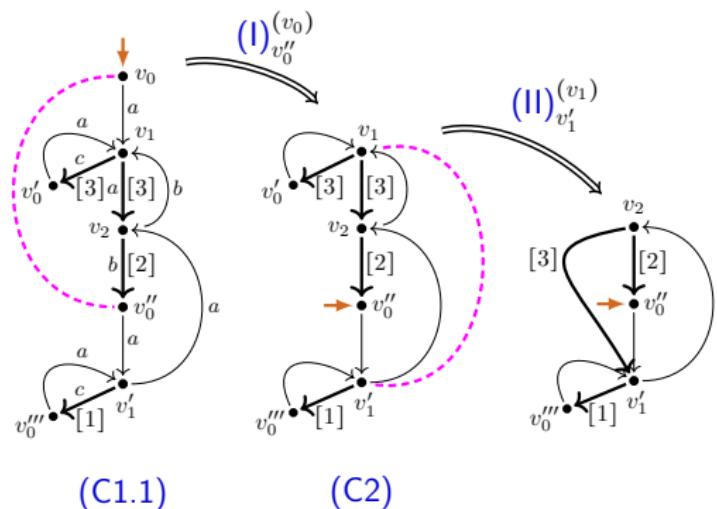
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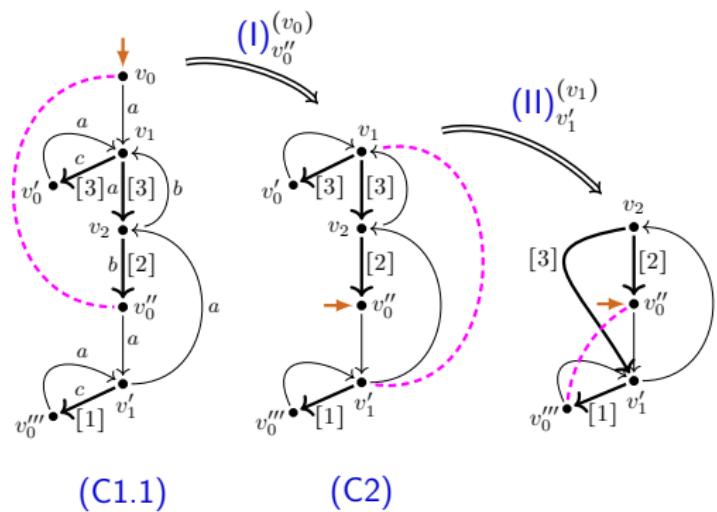


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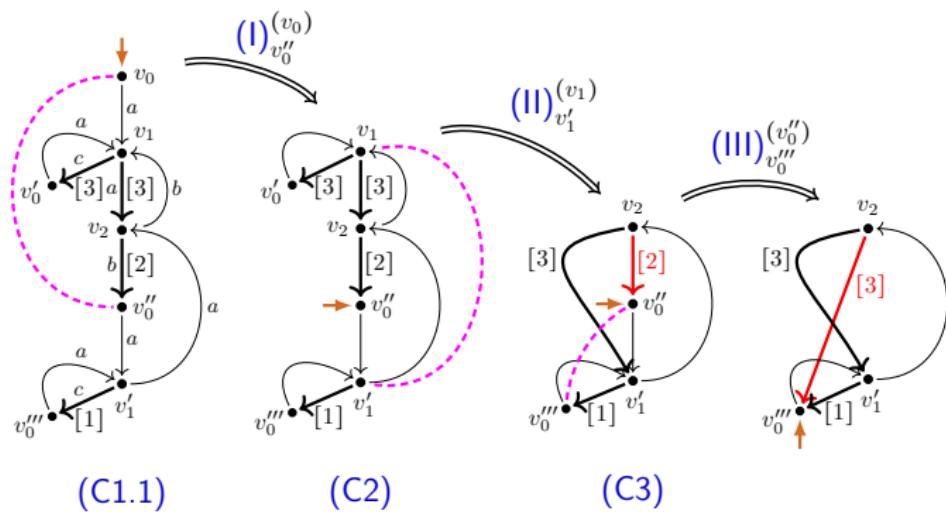
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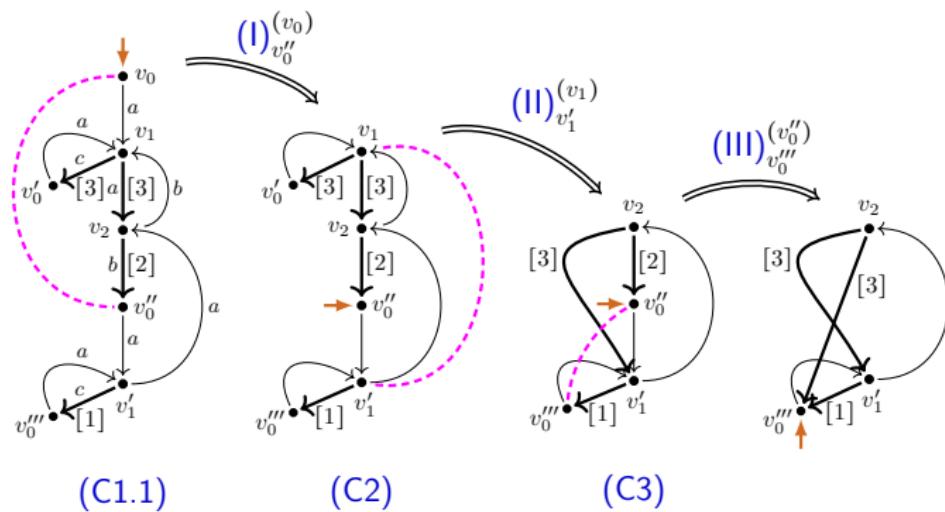
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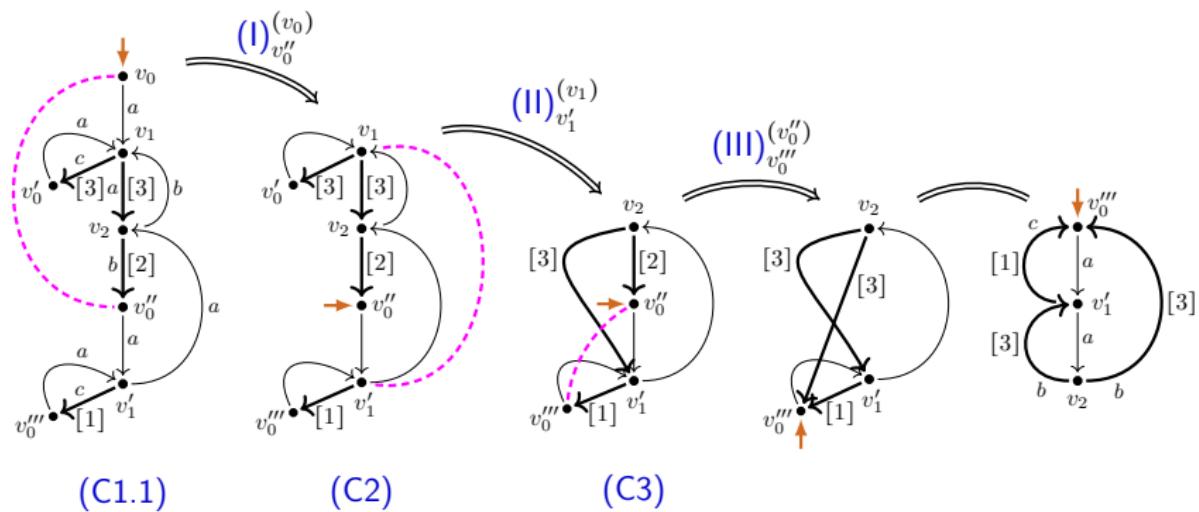
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Properties of LEE-charts

Theorem (\Leftarrow G/Fokkink, 2020)

A process graph G

is $\llbracket \cdot \rrbracket_P$ -expressible by an under-star-1-free regular expression

(i.e. P -expressible modulo bisimilarity by an $(\perp \backslash *)$ reg. expr.)

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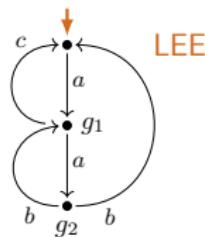
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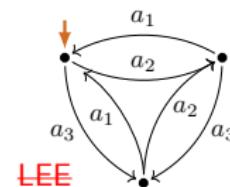
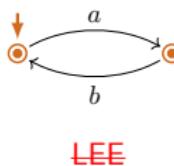
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Hence $[\cdot]_P$ -expressible | not $[\cdot]_P$ -expressible by 1-free regular expressions:



|



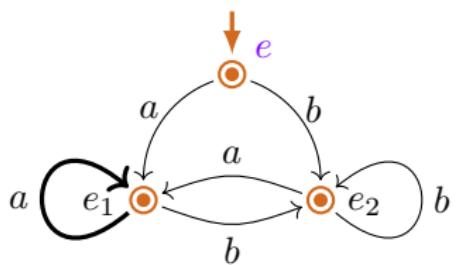
1-LEE

$\hat{=}$ sharing via 1-transitions facilitates LEE

Failure of LEE in general (example)

$(a^* \cdot b^*)^*$ not $(*/\pm)!$

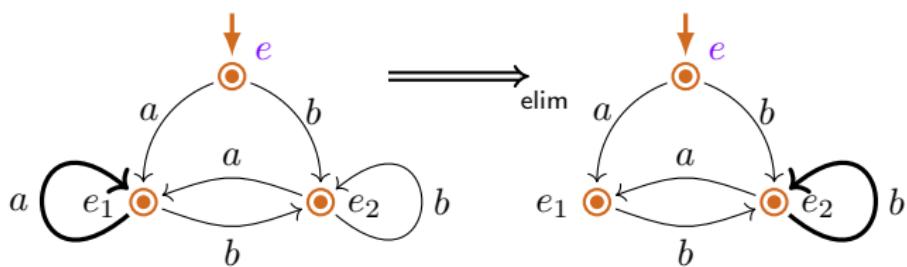
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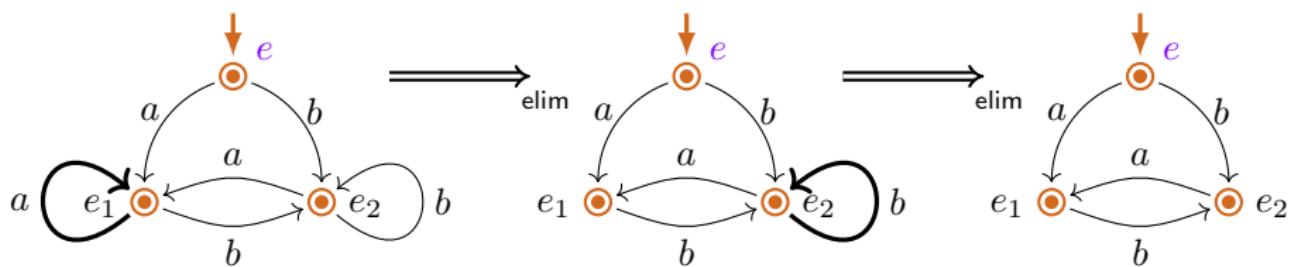
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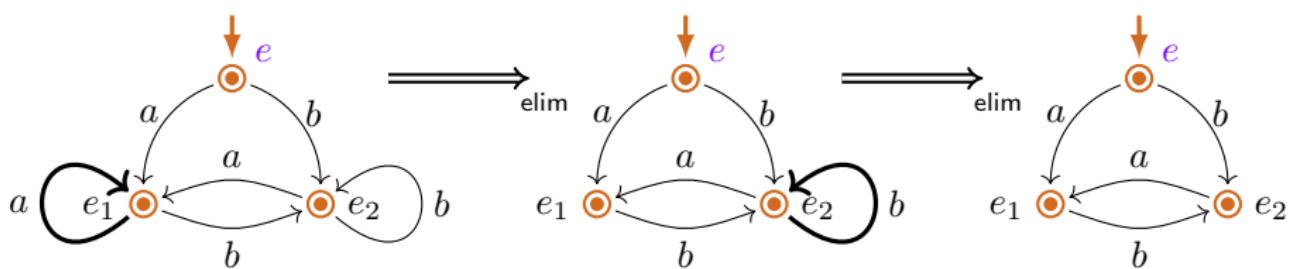
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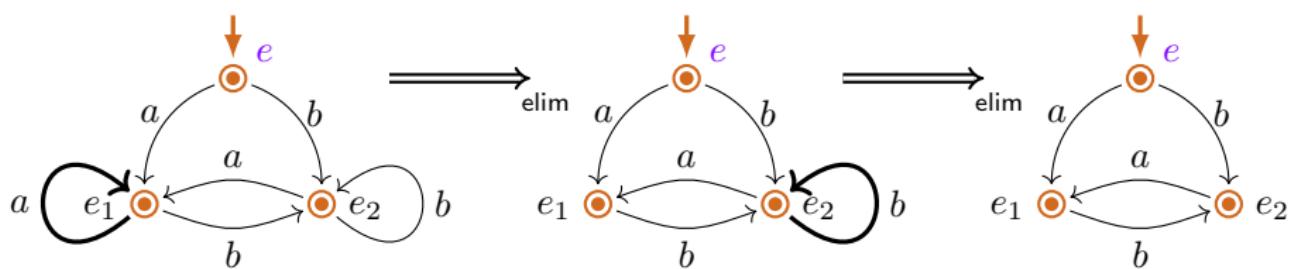


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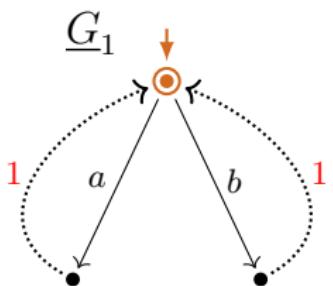
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1-Graphs and induced graphs

Definition

$\xrightarrow{1} \dots \xrightarrow{1} \xrightarrow{a}$ \equiv $\xrightarrow{(a]}$ induced a -transitions, for $a \in A$

$\xrightarrow{1} \dots \xrightarrow{1} \Downarrow$ \equiv $\Downarrow^{(1)}$ induced termination.

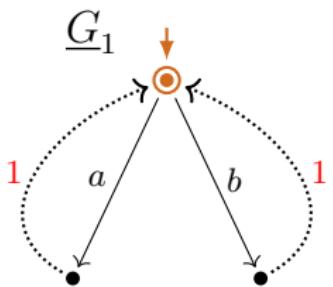


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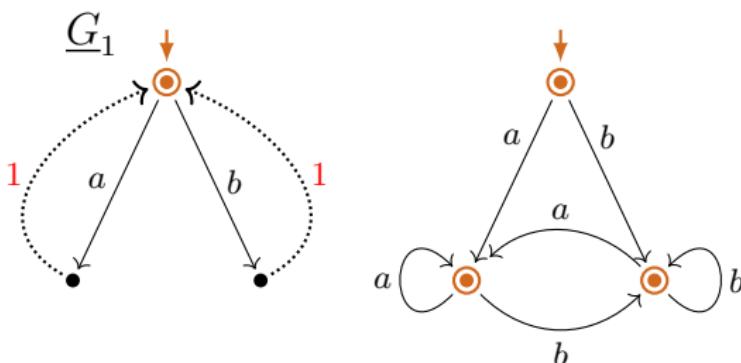
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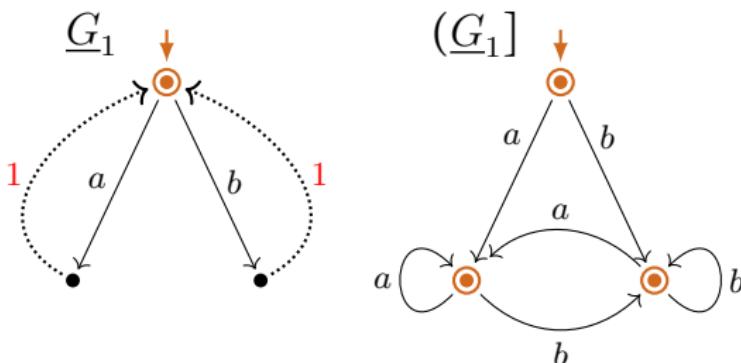
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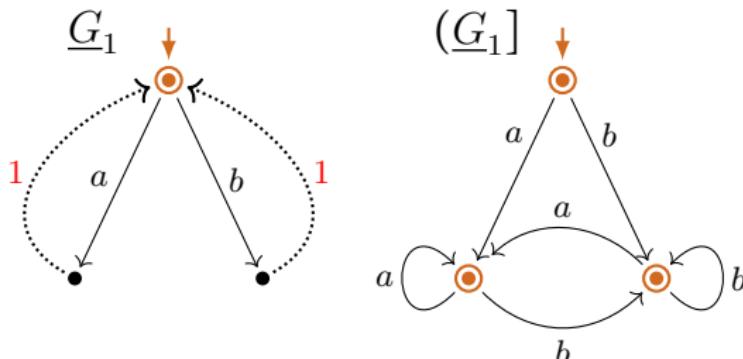
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The induced (process) graph of a 1-graph $\underline{G} = \langle V, A, \textcolor{red}{1}, \textcolor{brown}{v_s}, \rightarrow, \Downarrow \rangle$ is:

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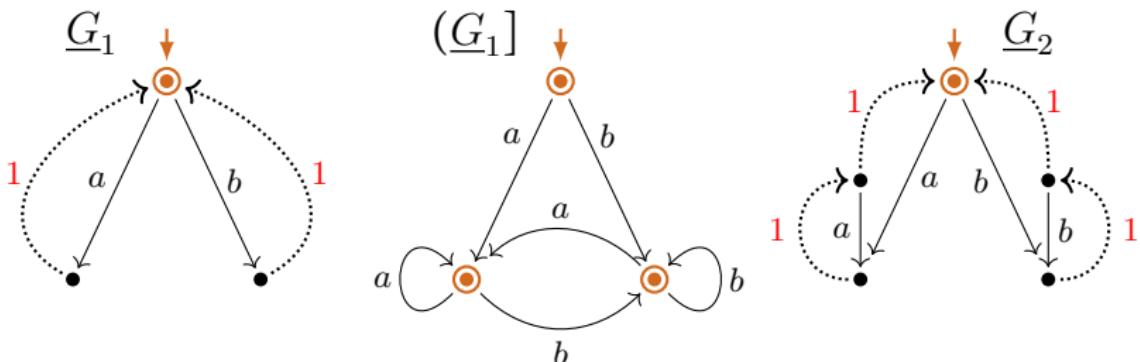
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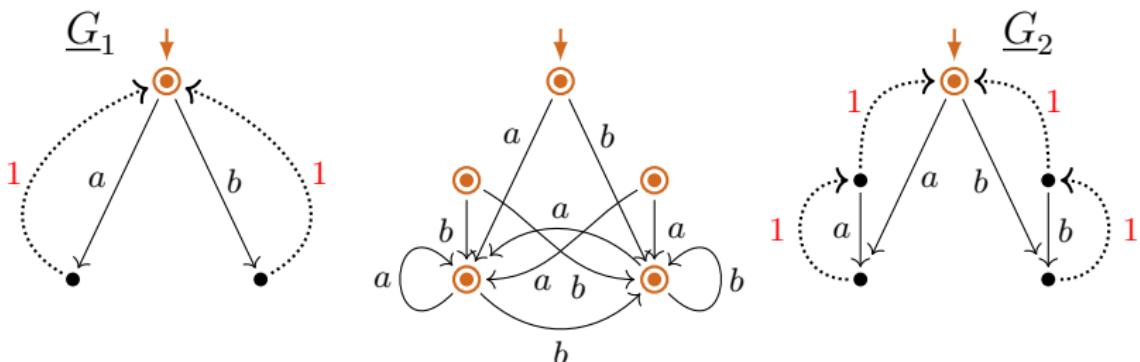
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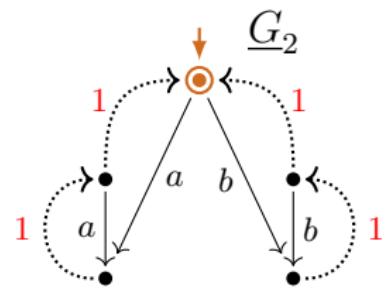
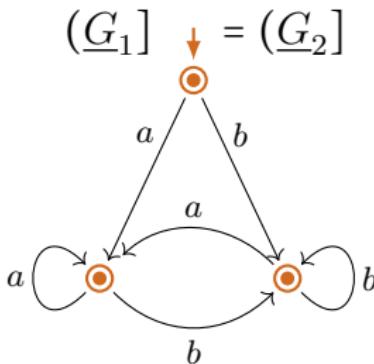
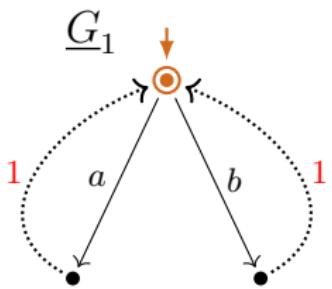
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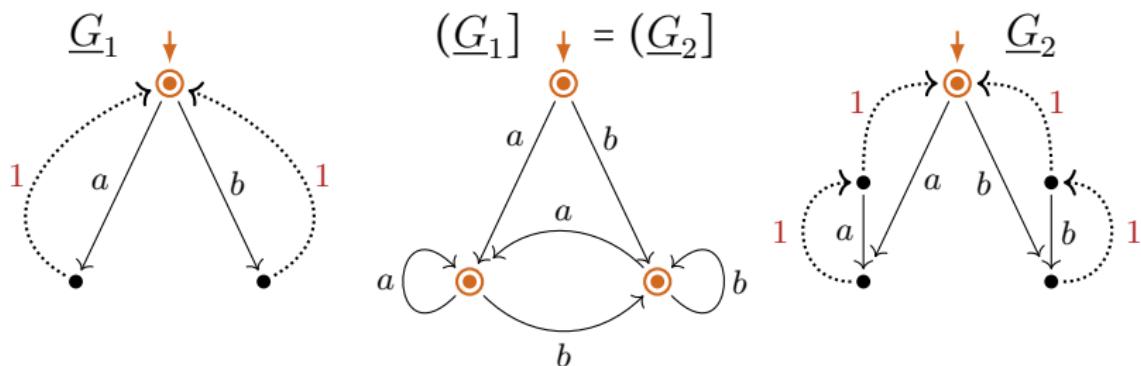
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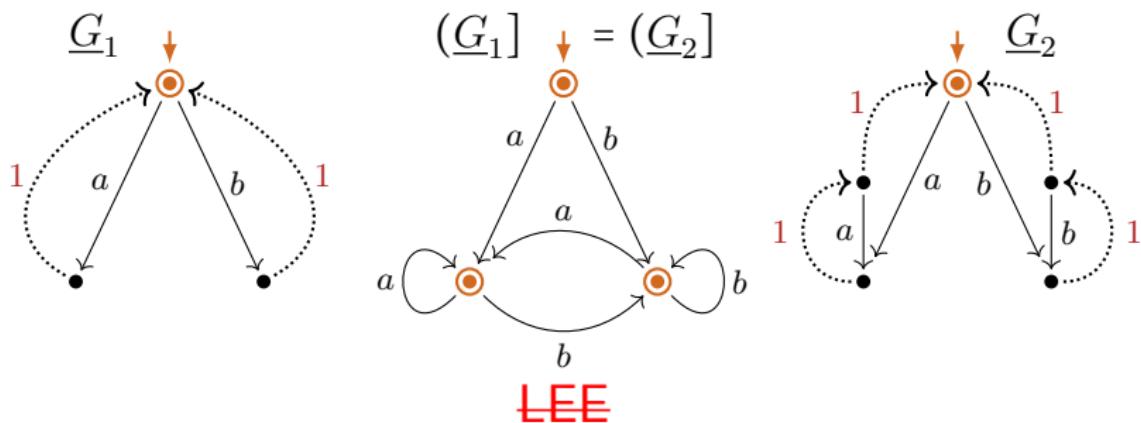


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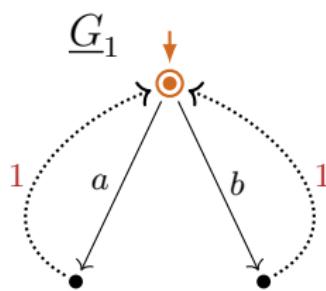


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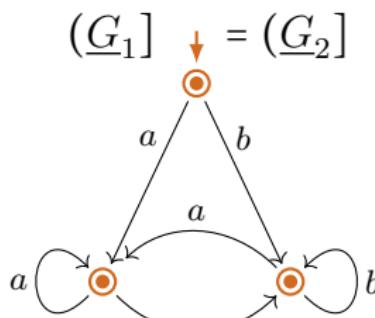
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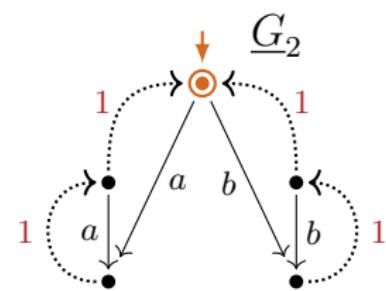
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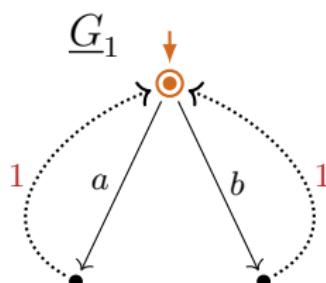


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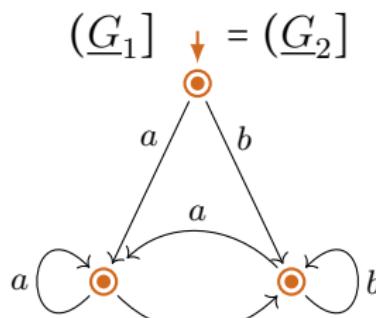
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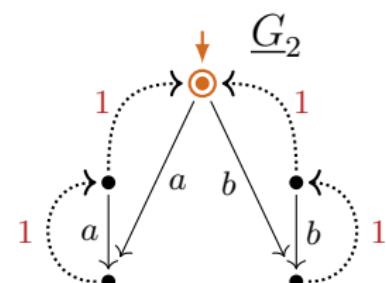
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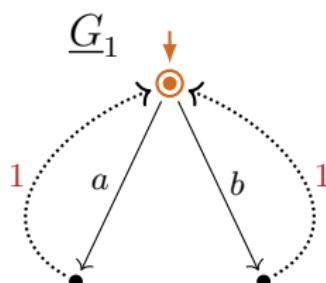
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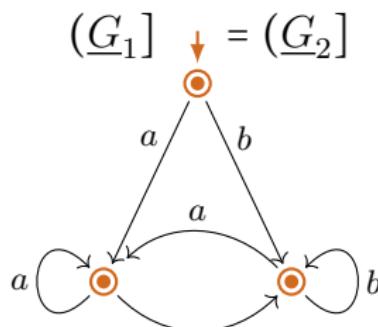
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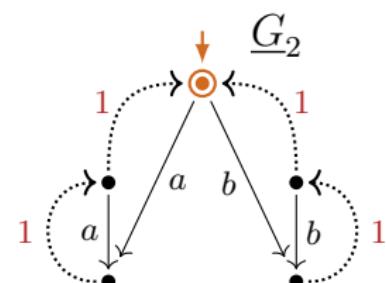
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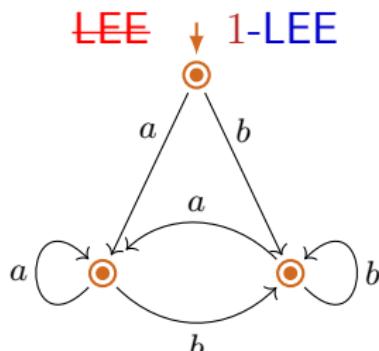
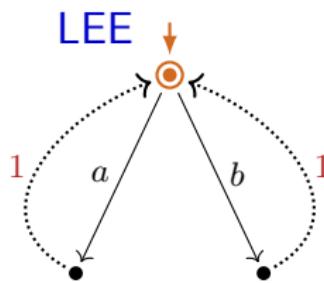
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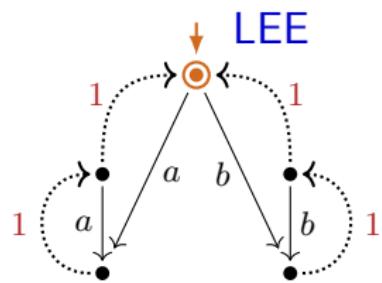
1-LEE

1-LEE holds for process interpretations



$$P((a^* \cdot b^*)^*)$$

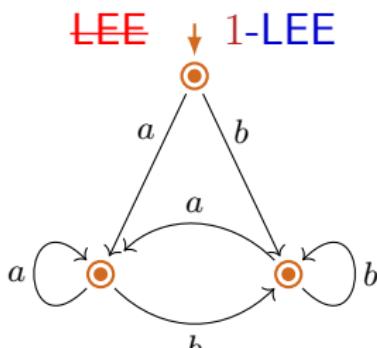
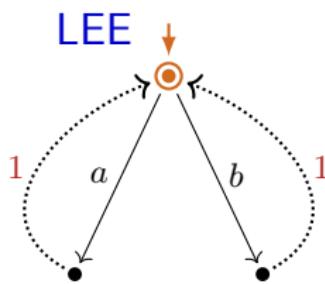
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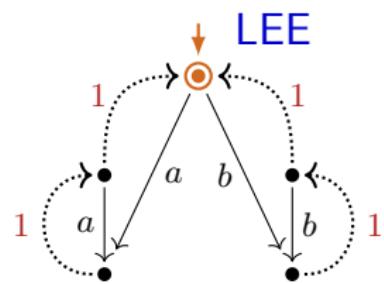
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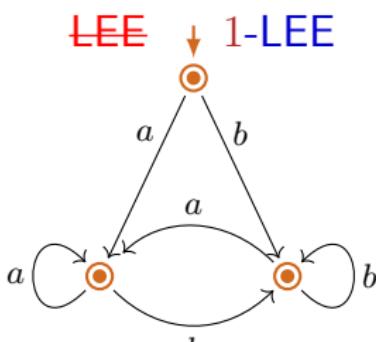
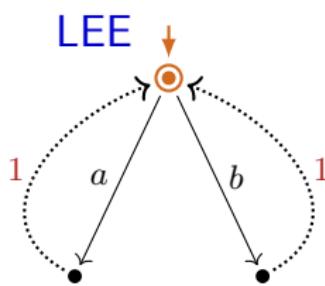
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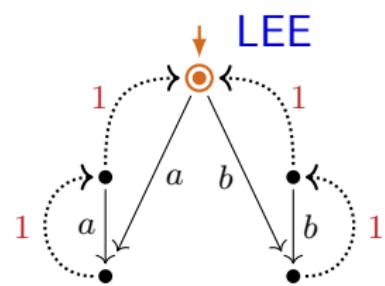
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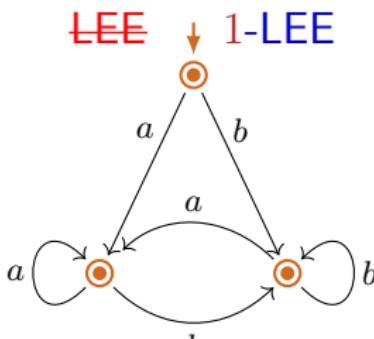
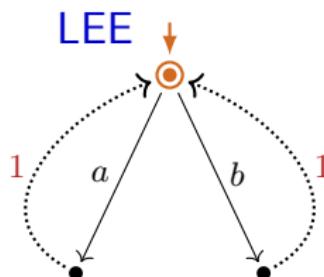
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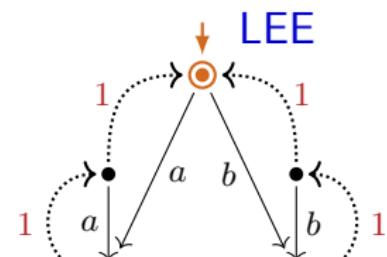
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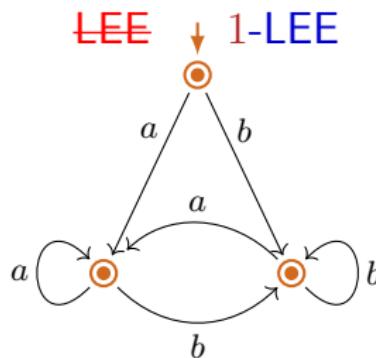
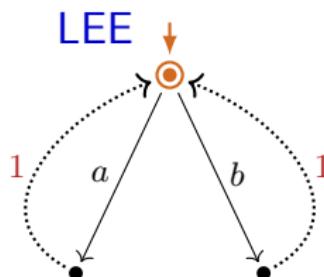
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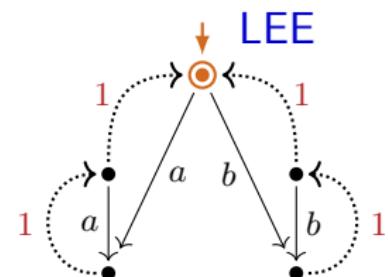
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Interpretation/extraction correspondences with 1-LEE

(\Leftarrow G 2021/22/23)

(Int)_P: *P-expressible graphs have the structural property 1-LEE*

Process interpretations $P(e)$ of regular expressions e
are finite process graphs that satisfy 1-LEE.

(Extr)_P: 1-LEE implies $\llbracket \cdot \rrbracket_P$ -expressibility

From every finite 1-process-graph G with 1-LEE
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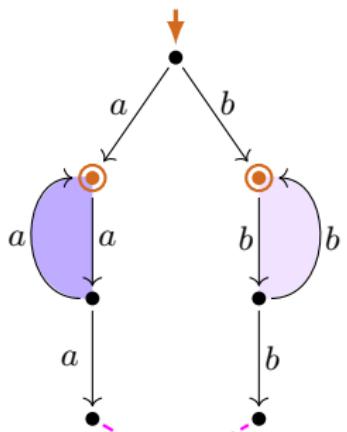
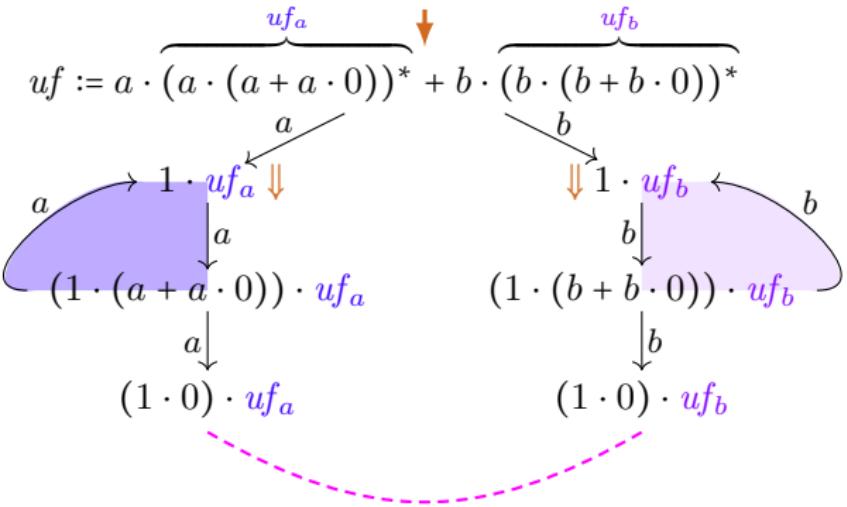
(Coll): 1-LEE is not preserved under collapse

The class of finite process graphs with 1-LEE
is not closed under bisimulation collapse.

1-LEE/ LEE characterize

the un-/restricted image of compact version P^\bullet of P

Image of P is **not closed** under bisimulation collapse not even for $(*/\perp)$ regular expressions (example)

 $P(uf)$  $P(uf)$ 

Compact process interpretation P^\bullet

Definition (Transition system specification \mathcal{T})

$$\begin{array}{c}
 \frac{}{\mathbf{1} \Downarrow} \qquad \frac{e_i \Downarrow}{(e_1 + e_2) \Downarrow} \ (i \in \{1, 2\}) \qquad \frac{e_1 \Downarrow \quad e_2 \Downarrow}{(e_1 \cdot e_2) \Downarrow} \qquad \frac{}{(e^*) \Downarrow} \\
 \\
 \frac{}{a \xrightarrow{a} \mathbf{1}} \qquad \frac{e_i \xrightarrow{a} e'_i}{e_1 + e_2 \xrightarrow{a} e'_i} \ (i \in \{1, 2\}) \\
 \\
 \frac{e_1 \xrightarrow{a} e'_1}{e_1 \cdot e_2 \xrightarrow{a} e'_1 \cdot e_2} \qquad \frac{e_1 \Downarrow \quad e_2 \xrightarrow{a} e'_2}{e_1 \cdot e_2 \xrightarrow{a} e'_2} \qquad \frac{e \xrightarrow{a} e'}{e^* \xrightarrow{a} e' \cdot e^*}
 \end{array}$$

Compact process interpretation P^\bullet

Definition (Transition system specification \mathcal{T})

$$\frac{e_1 \xrightarrow{a} e'_1}{e_1 \cdot e_2 \xrightarrow{a} e'_1 \cdot e_2}$$

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Compact process interpretation P^\bullet

Definition (Transition system specification \mathcal{T}^\bullet , changed rules w.r.t. \mathcal{T})

$$\frac{e_1 \xrightarrow{a} e'_1}{e_1 \cdot e_2 \xrightarrow{a} e'_1 \cdot e_2} \text{ (if } e'_1 \text{ is normed)}$$

$$\frac{e \xrightarrow{a} e'}{e^* \xrightarrow{a} e' \cdot e^*} \text{ (if } e' \text{ is normed)}$$

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Definition (Transition system specification \mathcal{T}^\bullet , changed rules w.r.t. \mathcal{T})

$$\frac{e_1 \xrightarrow{a} e'_1}{e_1 \cdot e_2 \xrightarrow{a} e'_1 \cdot e_2} \text{ (if } e'_1 \text{ is normed)} \quad \frac{e_1 \xrightarrow{a} e'_1}{e_1 \cdot e_2 \xrightarrow{a} e'_1} \text{ (if } e'_1 \text{ is not normed)}$$
$$\frac{e \xrightarrow{a} e'}{e^* \xrightarrow{a} e' \cdot e^*} \text{ (if } e' \text{ is normed)} \quad \frac{e \xrightarrow{a} e'}{e^* \xrightarrow{a} e'} \text{ (if } e' \text{ is not normed)}$$

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Definition

The compact process (graph) interpretation $P^\bullet(e)$ of a reg. expr's e :

$P^\bullet(e) :=$ labeled transition graph generated by e by derivations in \mathcal{T}^\bullet .

Compact process interpretation P^\bullet

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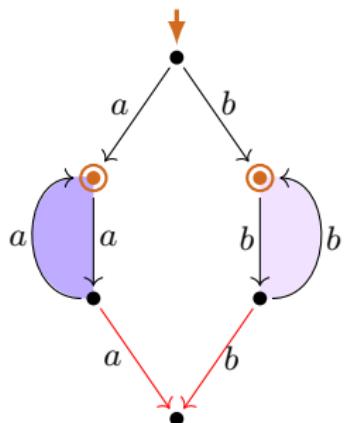
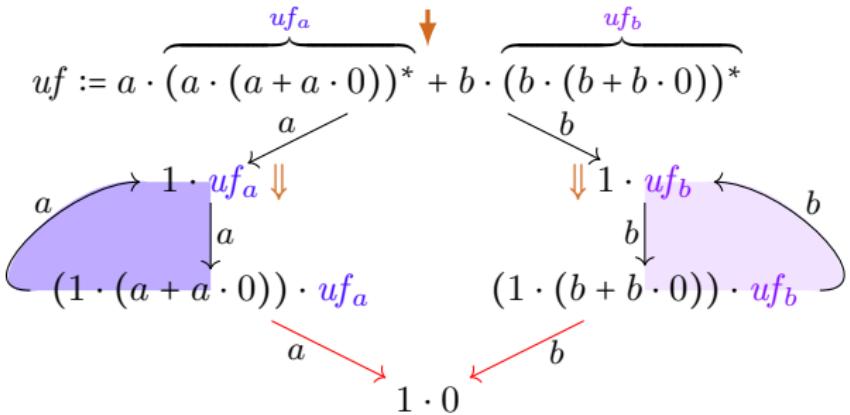
$P^\bullet(e) :=$ labeled transition graph generated by e by derivations in \mathcal{T}^\bullet .

Lemma (P^\bullet increases sharing; P^\bullet, P have same bisimulation semantics)

- (i) $P(e) \supseteq P^\bullet(e)$ for all regular expressions e .
- (ii) (G is $[\cdot]_{P^\bullet}$ -expressible \iff G is $[\cdot]_P$ -expressible) for all graphs G .

Image of P restricted to $(*/\perp)$ regular expressions

... contains all of its bisimulation collapses (example)

 $P^\bullet(uf)$

 $P^\bullet(uf)$


Interpretation correspondence of P^\bullet with LEE

(Int) $_{P^\bullet}^{(*/+)}$: By under-star-1-free expressions P^\bullet -expressible graphs satisfy LEE:

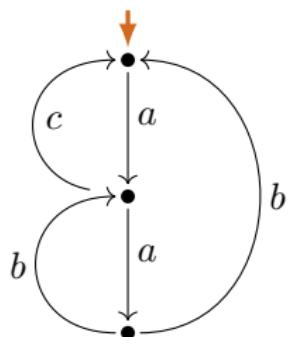
Compact process interpretations $P^\bullet(uf)$
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(Ext) $_{P^\bullet}^{(*/+)}$: LEE implies $\llbracket \cdot \rrbracket_{P^\bullet}$ -expressibility by under-star-1-free reg. expr's:

From every finite process graph G with LEE
an under-star-1-free regular expression uf can be extracted
such that $G \rightarrow P(uf)$.

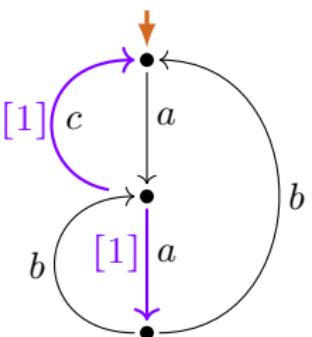
Refined extraction expression (example)

G_2



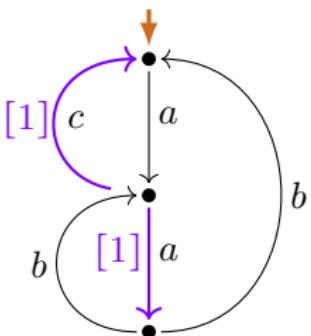
Refined extraction expression (example)

\widehat{G}_2



Refined extraction expression (example)

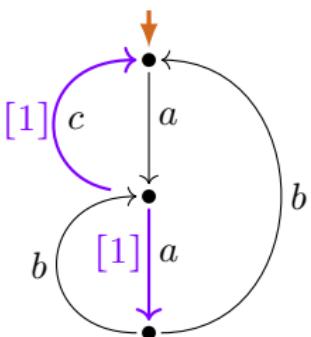
\widehat{G}_2



$$(1 \cdot (\quad)^*) \cdot 0$$

Refined extraction expression (example)

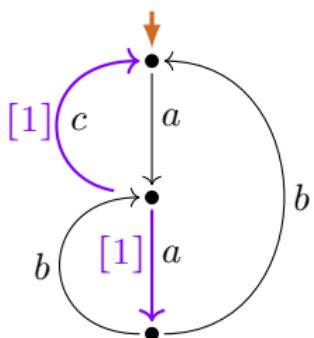
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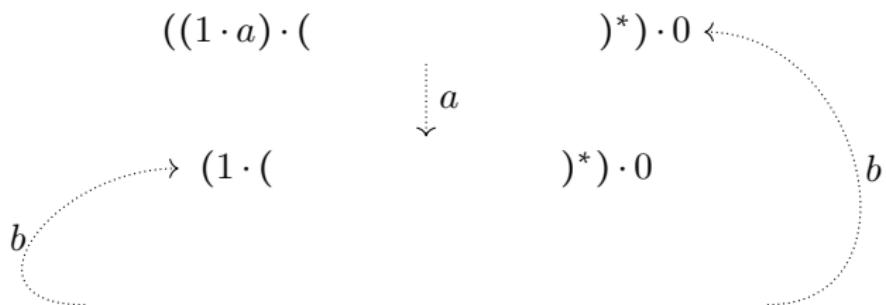
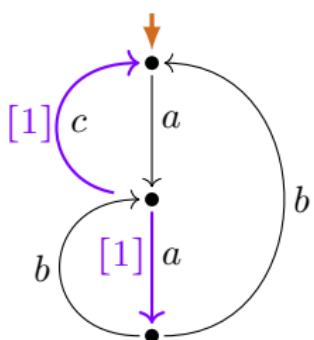
\widehat{G}_2



$$\begin{aligned} & ((1 \cdot a) \cdot (\quad \quad \quad)^*) \cdot 0 \\ & \quad \quad \quad \downarrow a \\ & (1 \cdot (\quad \quad \quad)^*) \cdot 0 \end{aligned}$$

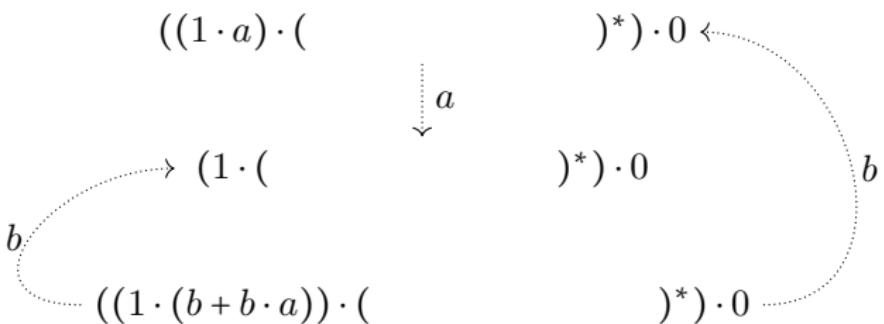
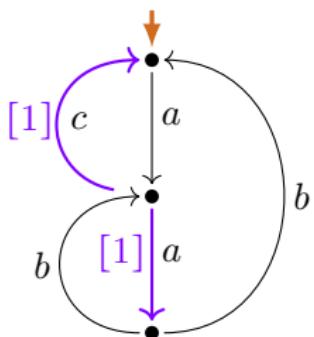
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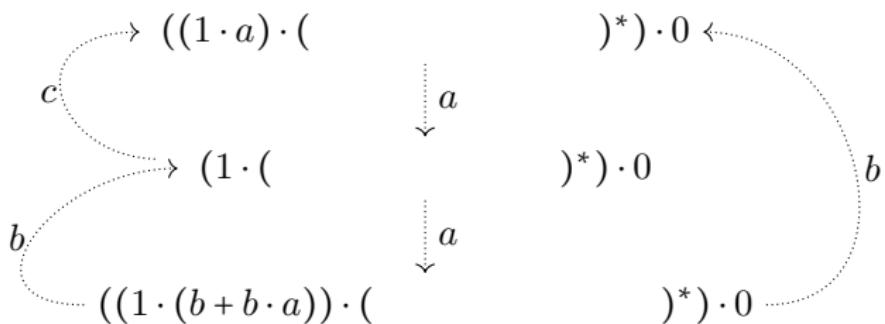
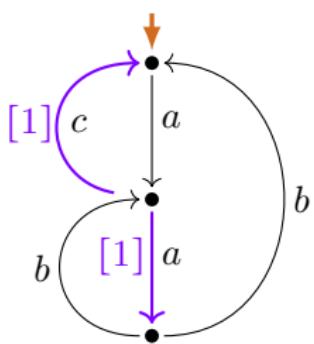
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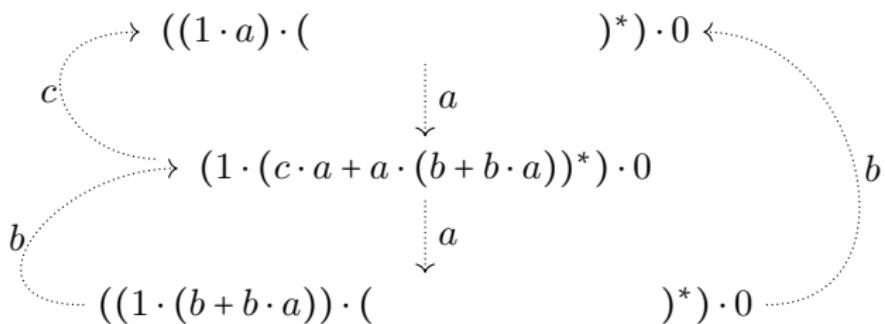
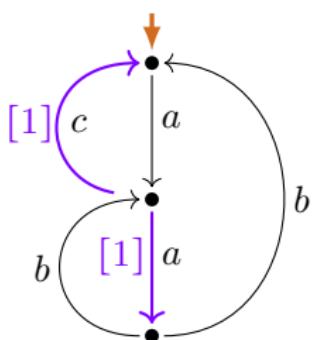
Refined extraction expression (example)

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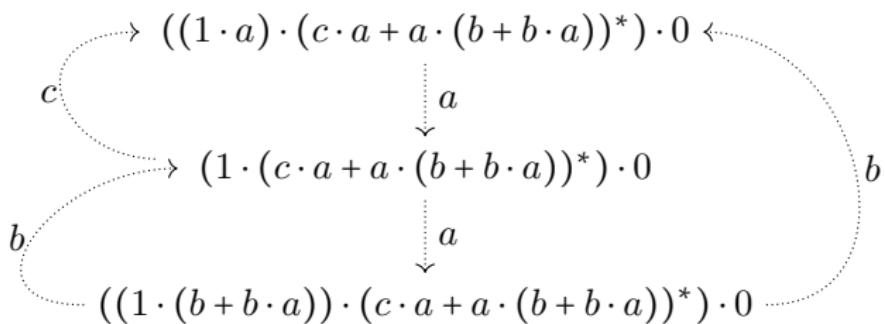
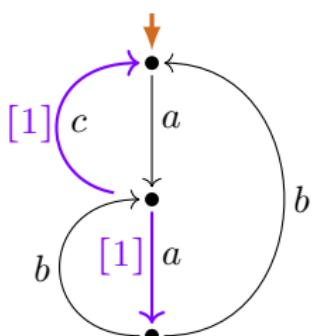
Refined extraction expression (example)

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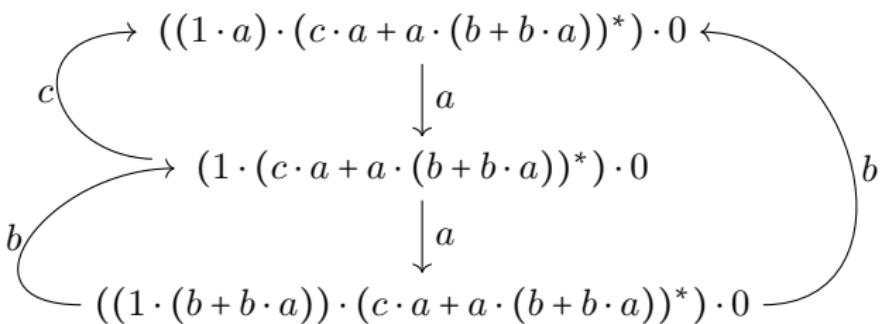
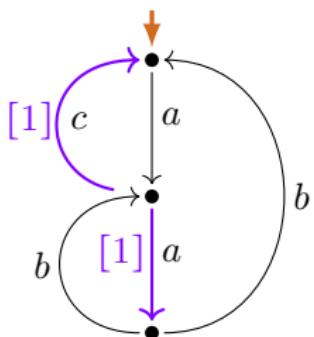
Refined extraction expression (example)

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Refined extraction expression (example)

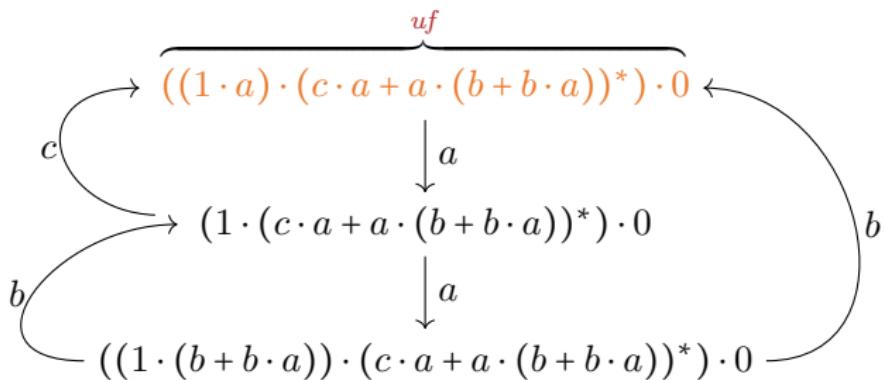
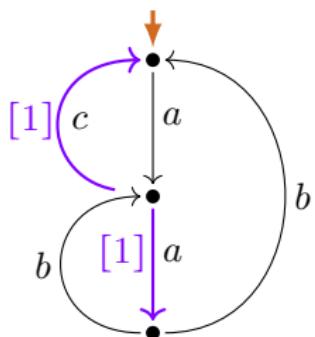
\widehat{G}_2



Refined extraction expression (example)

G₂

$$P^\bullet(uf) = P(uf) \simeq G_2$$



Interpretation/extraction correspondences of P^\bullet with LEE

(Int) $_{P^\bullet}^{(*/+)}$: By under-star-1-free expressions P^\bullet -expressible graphs satisfy LEE:

Compact process interpretations $P^\bullet(uf)$
of under-star-1-free regular expressions uf
are finite process graphs that satisfy LEE.

(Ext) $_{P^\bullet}^{(*/+)}$: LEE implies $\llbracket \cdot \rrbracket_{P^\bullet}$ -expressibility by under-star-1-free reg. expr's:

From every finite process graph G with LEE
an under-star-1-free regular expression uf can be extracted
such that $G \succeq P^\bullet(uf)$.

From every finite collapsed process graph G with LEE
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(ImColl) $_{P^\bullet}^{(*/+)}$: The image of P^\bullet ,
restricted to under-star-1-free regular expressions,
is closed under bisimulation collapse.

Interpretation/extraction correspondences of P^\bullet with 1-LEE

(Int) $_{P^\bullet}$: P^\bullet -expressible graphs satisfy 1-LEE:

Compact process interpretations $P^\bullet(e)$ of regular expressions e are finite process graphs that satisfy 1-LEE.

(Extr) $_{P^\bullet}$: LEE implies $\llbracket \cdot \rrbracket_{P^\bullet}$ -expressibility:

From every finite process graph G with 1-LEE
a regular expression e can be extracted
such that $G \xrightarrow{\sim} P^\bullet(e)$.

From every finite collapsed process graph G with 1-LEE
a regular expression e can be extracted
such that $G \simeq P^\bullet(e)$.

Interpretation/extraction correspondences of P^\bullet with 1-LEE

(Int) _{P^\bullet} : P^\bullet -expressible graphs satisfy 1-LEE:

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From every finite collapsed process graph G with 1-LEE
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such that $G \simeq P^\bullet(e)$.

(ImColl) _{P^\bullet} : The image of P^\bullet is not closed under bisimulation collapse .

$$\text{LEE} \stackrel{\wedge}{=} \text{image of } P^\bullet|_{R\text{Exp}^{(\ast/\pm)}}$$

Theorem

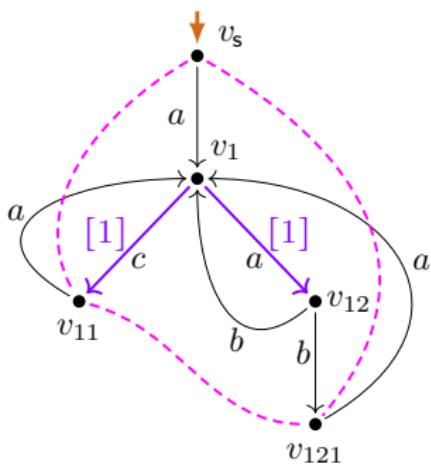
For every process graph G TFAE:

- (i) $\text{LEE}(G)$.
- (ii) G is P^\bullet -expressible by an (\ast/\pm) regular expression
(i.e. $G \simeq P^\bullet(e)$ for some $e \in R\text{Exp}^{(\ast/\pm)}$).
- (iii) G is isomorphic to a graph in the image of P^\bullet on (\ast/\pm) reg. expr's
(i.e. $G \simeq G'$ for some $G' \in \text{im}(P^\bullet|_{R\text{Exp}^{(\ast/\pm)}})$).

Adapted (refined) extraction from LLEE-graph

$$G_1 / \widehat{G}_1$$

$$P^\bullet(uf) = P(uf) \simeq G_1$$



$$\begin{aligned}
 & (1 \cdot (a + (a + a))) \cdot ((c \cdot a + a \cdot (b + b \cdot (a + a)))^* \cdot 0) =: uf \\
 & \downarrow a \\
 & (1 \cdot (c \cdot a + a \cdot (b + b \cdot (a + a)))^*) \cdot 0 \\
 & \quad \swarrow c \quad \searrow a \\
 & ((1 \cdot a) \cdot (\dots)^*) \cdot 0 \quad ((1 \cdot (b + b \cdot (a + a))) \cdot (\dots)^*) \cdot 0 \\
 & \quad \uparrow b \quad \downarrow b \\
 & ((1 \cdot (a + a)) \cdot (\dots)^*) \cdot 0
 \end{aligned}$$

1-LEE $\stackrel{\wedge}{=}$ image of P^\bullet

Theorem

For every process graph G TFAE:

- (i) $1\text{-LEE}(G)$
(i.e. $G = (\underline{G})$ for some 1-transition-process-graph \underline{G} with $\text{LEE}(\underline{G})$).
- (ii) G is P^\bullet -expressible by a regular expression
(i.e. $G \simeq P^\bullet(e)$ for some $e \in RExp$).
- (iii) G is isomorphic to a graph in the image of P^\bullet
(i.e. $G \simeq G'$ for some $G' \in im(P^\bullet)$).

Summary

- ▶ Characterizations of the image of P^\bullet (refinement of P):
 - ▶ LEE \triangleq image of $P^\bullet|_{RExp^{(*/+)}}$ \supsetneqq image of $P|_{RExp^{(*/+)}}$
 - ▶ 1-LEE \triangleq image of P^\bullet \supsetneqq image of P

Summary

- ▶ process interpretation P /semantics $\llbracket \cdot \rrbracket_P$ of regular expressions
 - ▶ expressibility and completeness questions
- ▶ loop existence and elimination (LEE)
 - ▶ loop elimination rewrite system can be completed
 - ▶ interpretation/extraction correspondences with $(*/\perp)$ reg. expr.s
 - ▶ LEE-witnesses: labelings of graphs with LEE
 - ▶ stepwise LEE-preserving bisimulation collapse
- ▶ 1-LEE = sharing via 1-transitions facilitates LEE
 - ▶ interpretation/extraction correspondences with all regular expressions
 - ▶ not preserved under bisim. collapse (approximation possible)
- ▶ Characterizations of the image of P^\bullet (refinement of P):
 - ▶ LEE \triangleq image of $P^\bullet|_{RExp^{(*/\perp)}}$ $\not\supseteq$ image of $P|_{RExp^{(*/\perp)}}$
 - ▶ 1-LEE \triangleq image of P^\bullet $\not\supseteq$ image of P
- ▶ outlook on work-to-do

My next aims

Completeness problem, solution:

- A1:** graph structure of regular expression processes (LEE/1-LEE)
- A2:** motivation of crystallization
- A4:** details of crystallization procedure,
and completeness of Milner's proof system

Expressibility problem

- A3:** LEE is decidable in polynomial time.
- Q:** Is 1-LEE decidable in polynomial time?
- P:** Is expressibility by a regular expression, for a finite process graph,
decidable in polynomial time/fixed-parameter tractable time?

Resources

- ▶ Slides/abstract on clegra.github.io
 - ▶ slides: [.../lf/IFIP-1_6-2024.pdf](https://clegra.github.io/lf/IFIP-1_6-2024.pdf)
 - ▶ abstract: [.../lf/abstract-IFIP-1_6-2024.pdf](https://clegra.github.io/lf/abstract-IFIP-1_6-2024.pdf)
- ▶ CG: Closing the Image of the Process Interpretation
of 1-Free Regular Expressions Under Bisimulation Collapse
 - ▶ TERMGRAPH 2024, [extended abstract](#).
- ▶ CG: The Image of the Process Interpretation of Regular Expressions
is Not Closed under Bisimulation Collapse,
 - ▶ arXiv:2303.08553, 2021/2023.
- ▶ CG: Milner's Proof System for
Regular Expressions Modulo Bisimilarity is Complete,
 - ▶ LICS 2022, [arXiv:2209.12188](#), [poster](#).
- ▶ CG, Wan Fokkink: A Complete Proof System for
1-Free Regular Expressions Modulo Bisimilarity,
 - ▶ LICS 2020, [arXiv:2004.12740](#), [video on youtube](#).
- ▶ CG: Modeling Terms by Graphs with Structure Constraints,
 - ▶ TERMGRAPH 2018, [EPTCS 288](#), [arXiv:1902.02010](#).

Language semantics $\llbracket \cdot \rrbracket_L$ of reg. expr's (Copi–Elgot–Wright, 1958)

0 \xrightarrow{L} empty language \emptyset

1 \xrightarrow{L} $\{\epsilon\}$ (ϵ the empty word)

a \xrightarrow{L} $\{a\}$

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$e_1 + e_2$ \xrightarrow{L} union of $L(e_1)$ and $L(e_2)$

$e_1 \cdot e_2$ \xrightarrow{L} element-wise concatenation of $L(e_1)$ and $L(e_2)$

e^* \xrightarrow{L} set of words formed by concatenating words in $L(e)$,
and adding the empty word ϵ

Language semantics $\llbracket \cdot \rrbracket_L$ of reg. expr's (Copi–Elgot–Wright, 1958)

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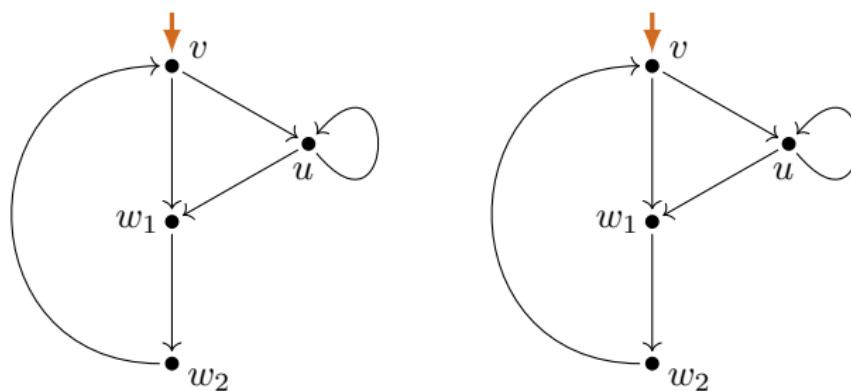
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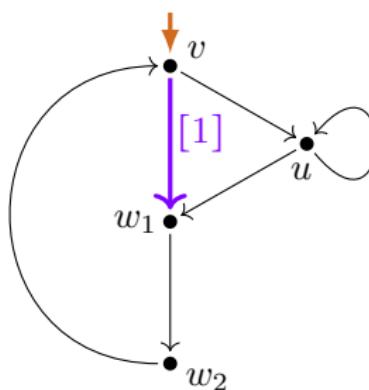
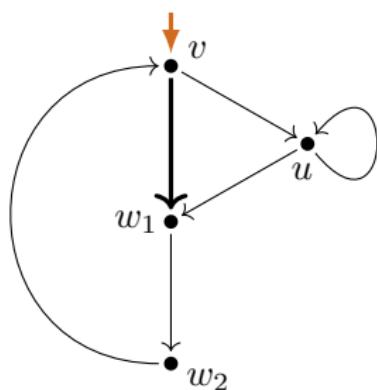
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$\llbracket e \rrbracket_L := L(e)$ (language defined by e)

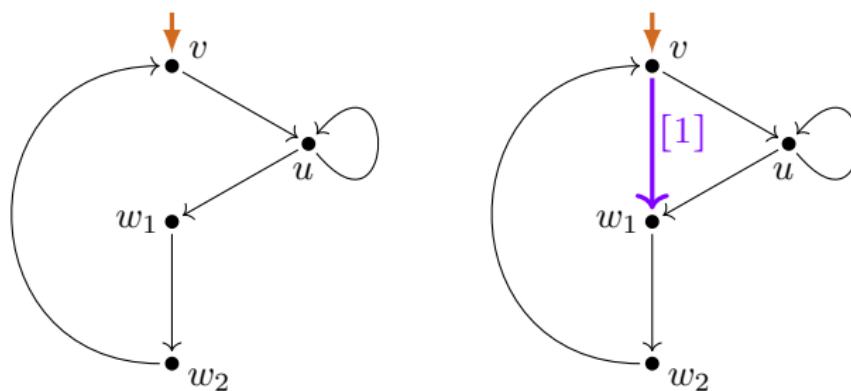
Layered LEE-witness (LLEE-witness)



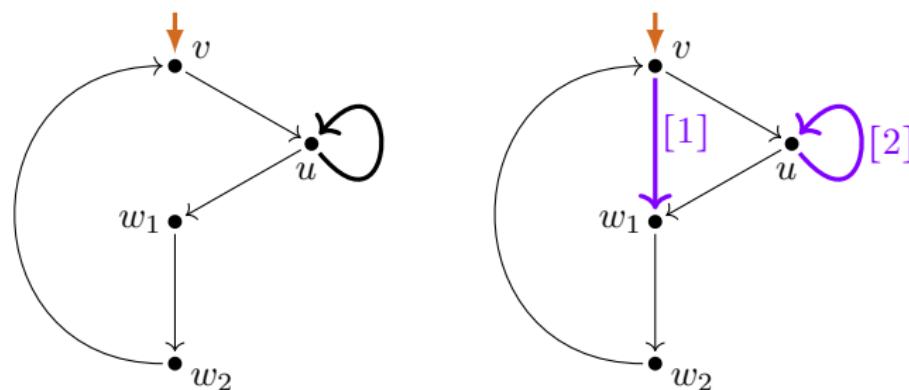
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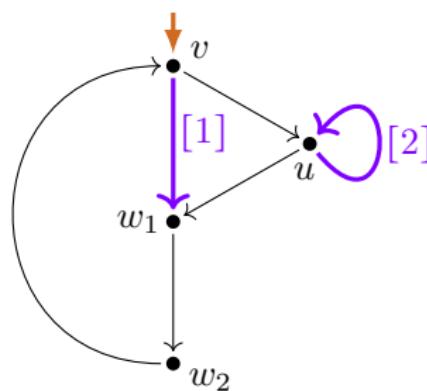
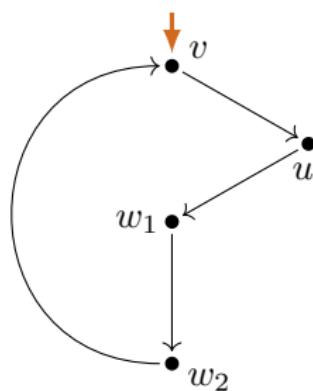
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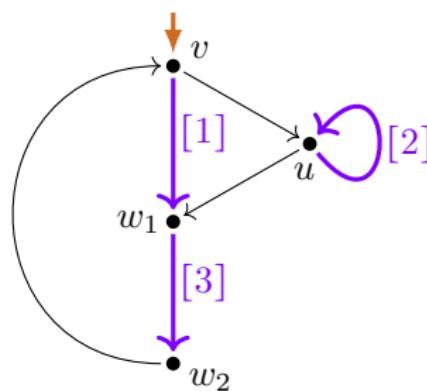
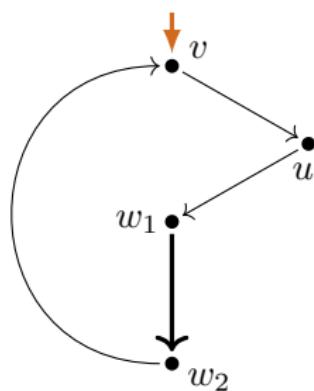
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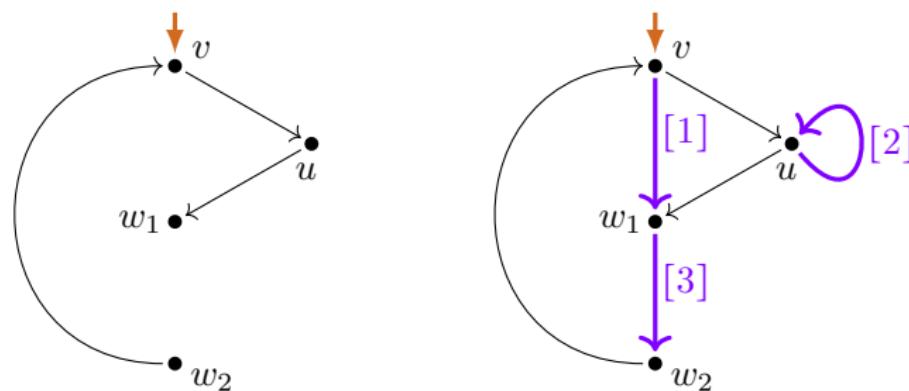
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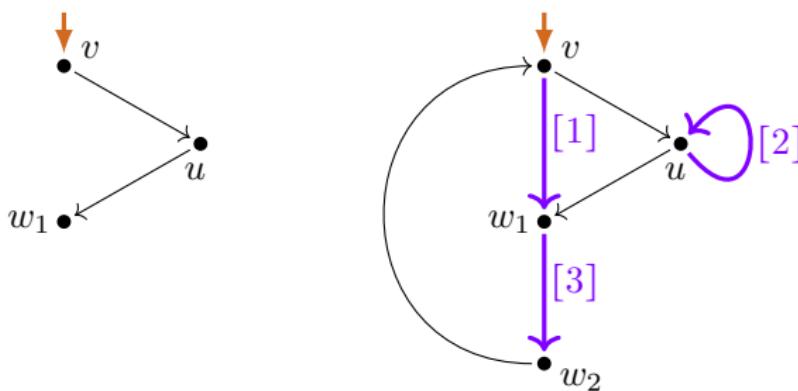
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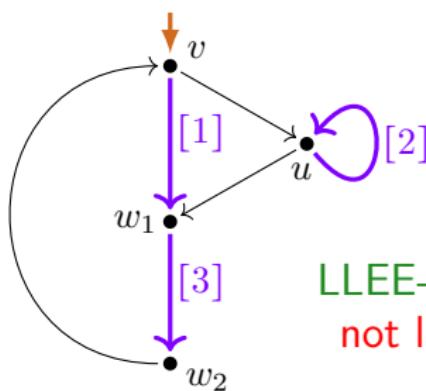
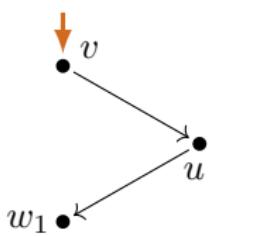
Layered LEE-witness (LLEE-witness)



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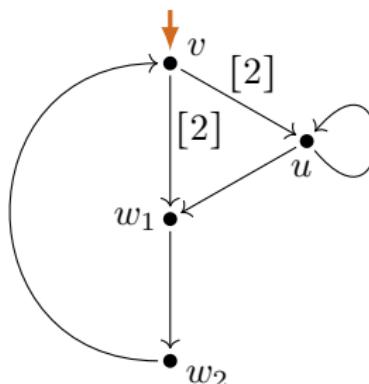
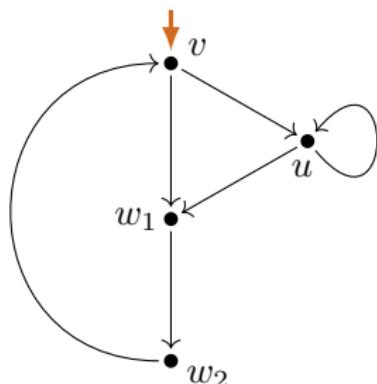
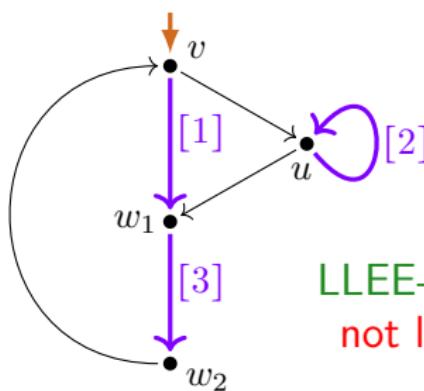
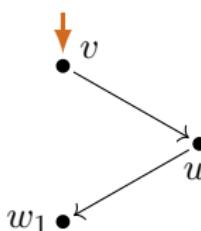


Layered LEE-witness (LLEE-witness)

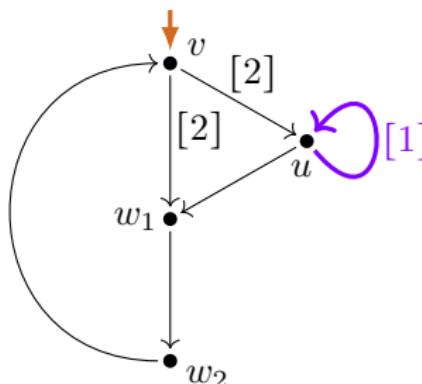
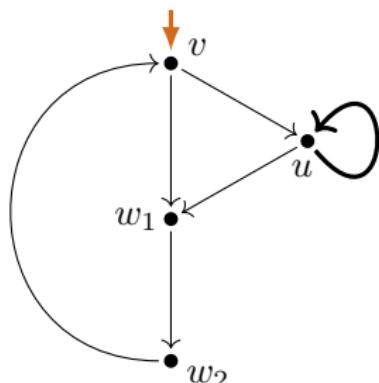
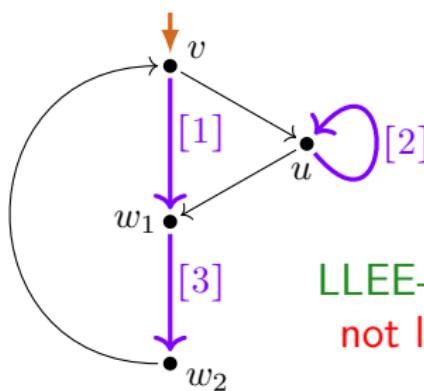
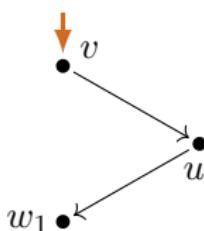


LLEE-witness
not layered

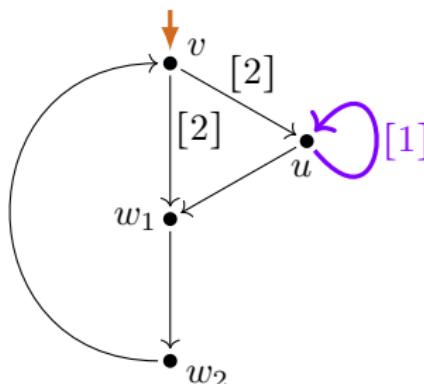
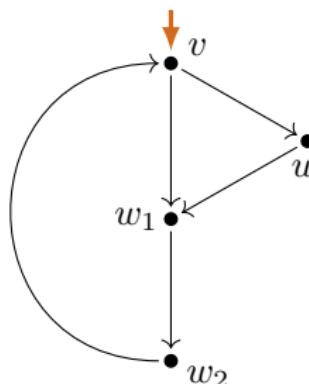
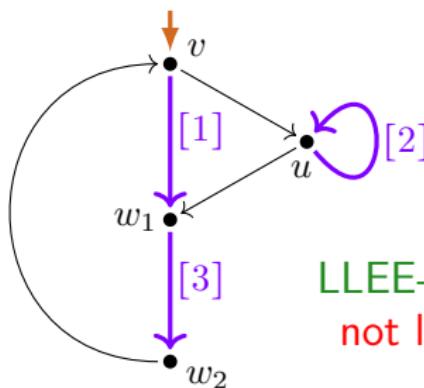
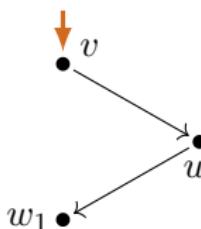
Layered LEE-witness (LLEE-witness)



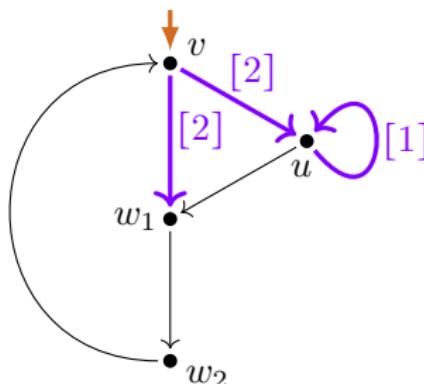
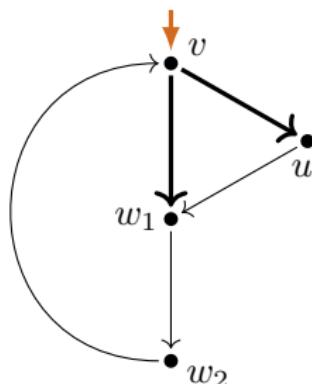
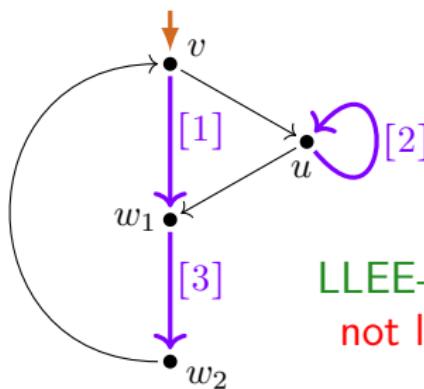
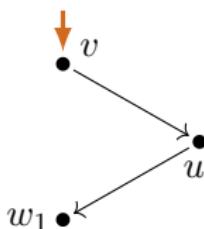
Layered LEE-witness (LLEE-witness)



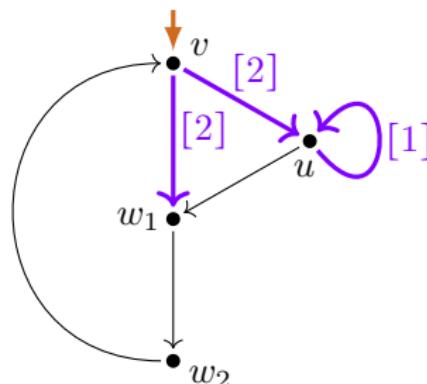
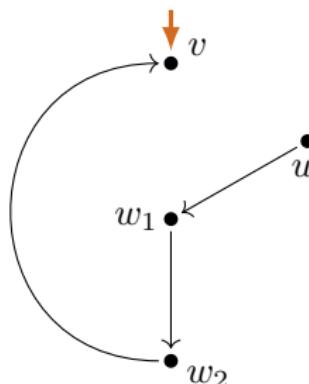
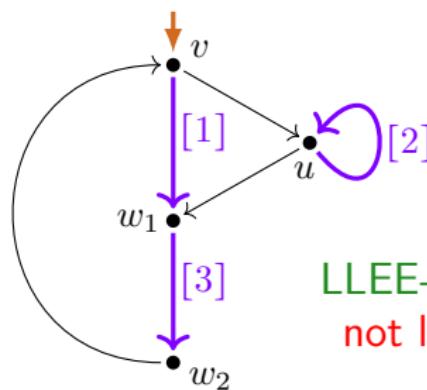
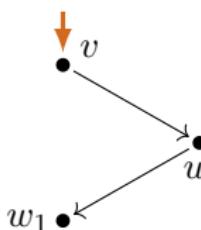
Layered LEE-witness (LLEE-witness)



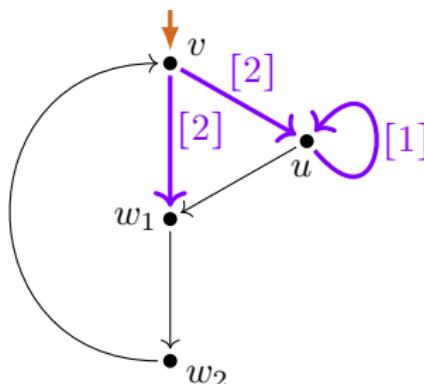
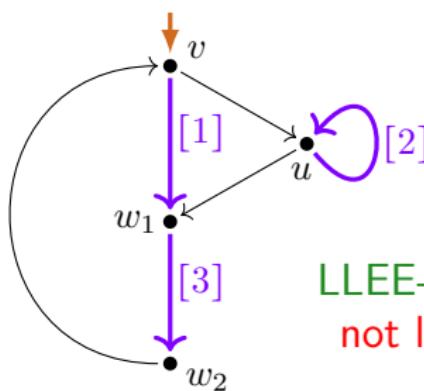
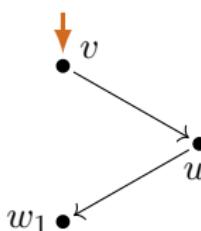
Layered LEE-witness (LLEE-witness)



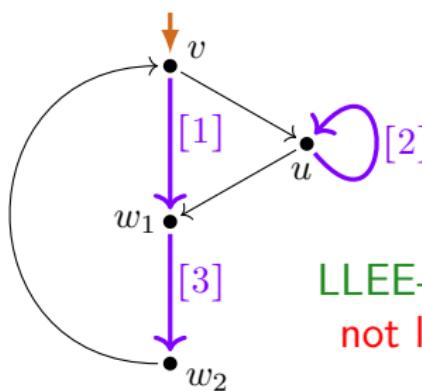
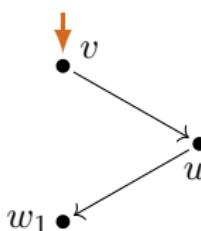
Layered LEE-witness (LLEE-witness)



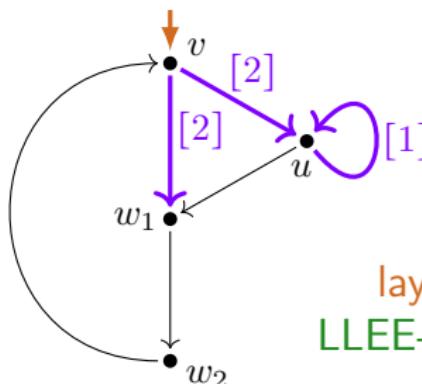
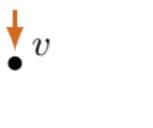
Layered LEE-witness (LLEE-witness)



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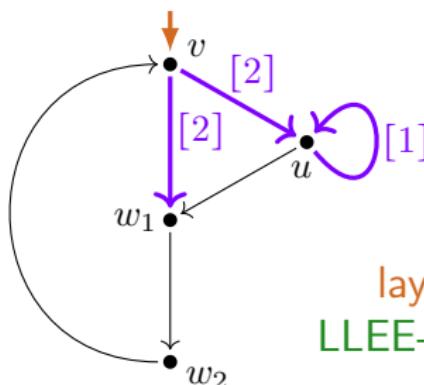
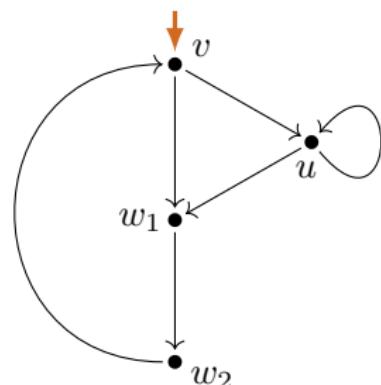
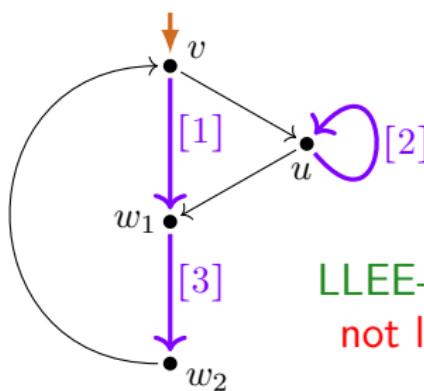
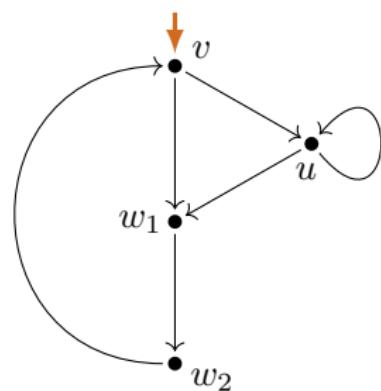


LLEE-witness
not layered

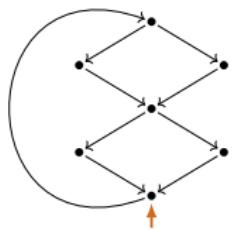


layered
LLEE-witness

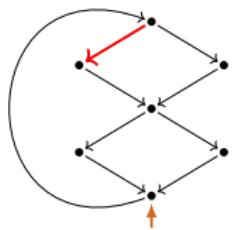
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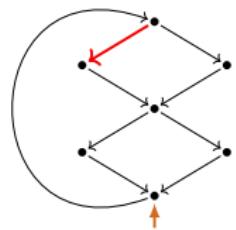
‘Critical pair’: bi-loop elimination



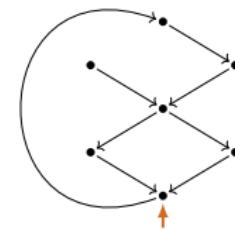
'Critical pair': bi-loop elimination



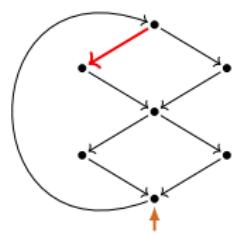
'Critical pair': bi-loop elimination



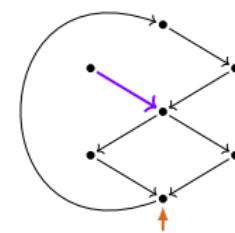
elim →



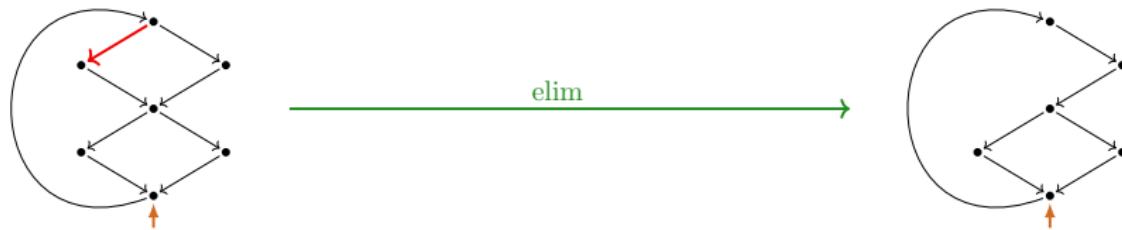
'Critical pair': bi-loop elimination



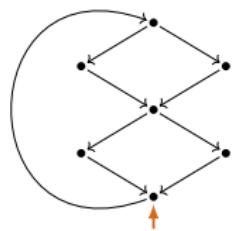
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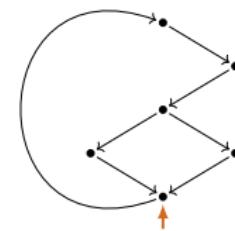
'Critical pair': bi-loop elimination



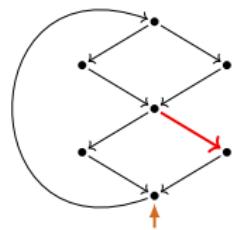
'Critical pair': bi-loop elimination



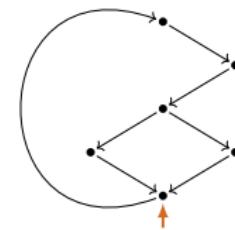
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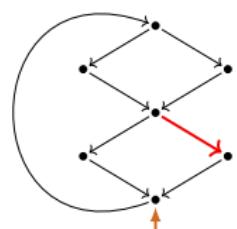
'Critical pair': bi-loop elimination



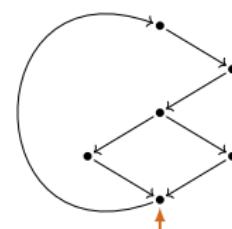
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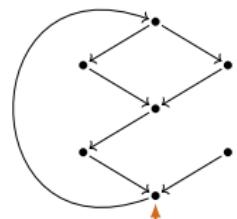
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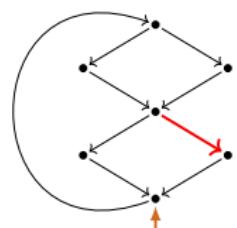
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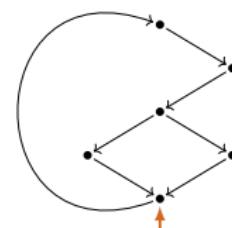
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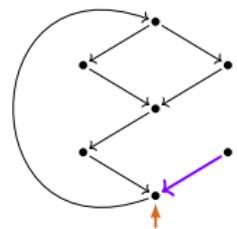
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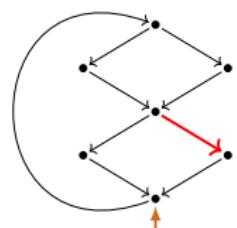
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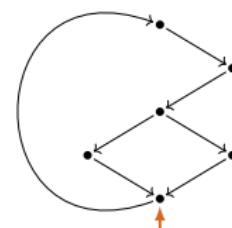
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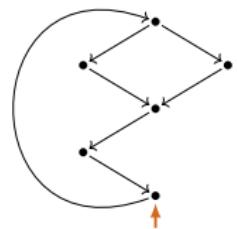
'Critical pair': bi-loop elimination



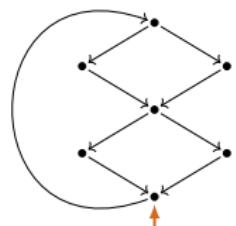
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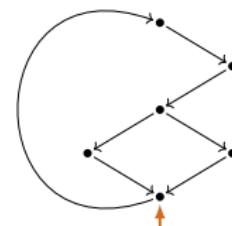
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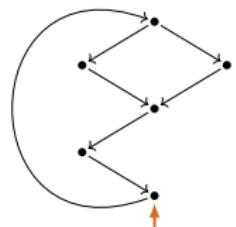
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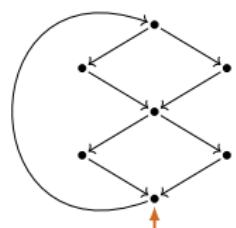
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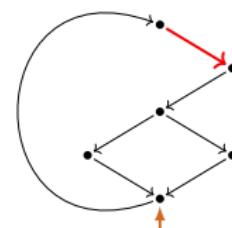
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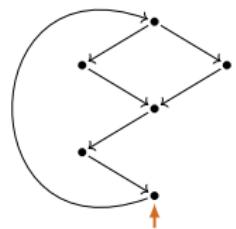
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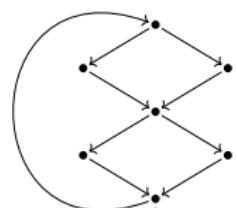
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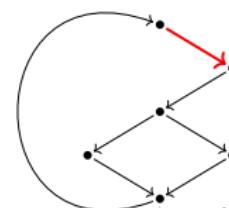
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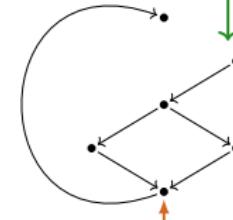
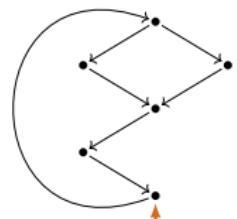


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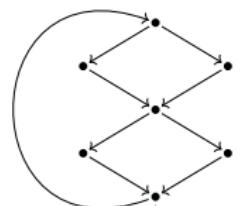


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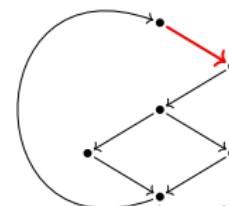
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'Critical pair': bi-loop elimination

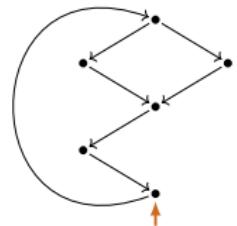


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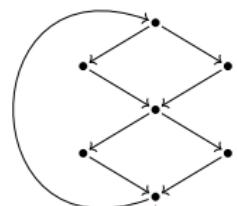


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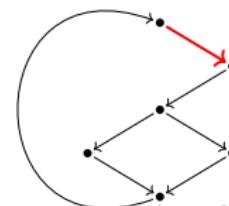
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'Critical pair': bi-loop elimination

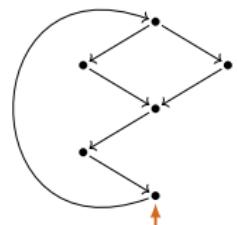


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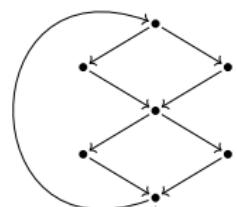


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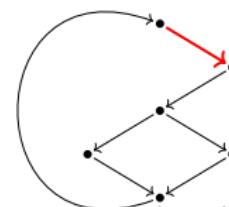
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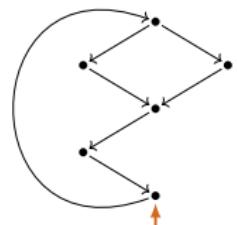


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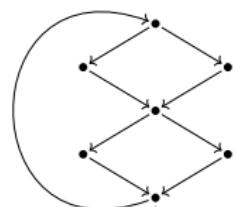


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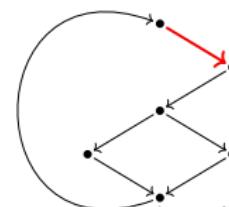
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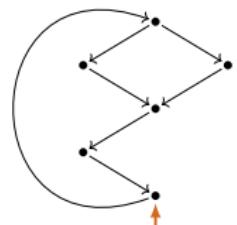


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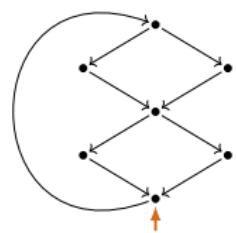


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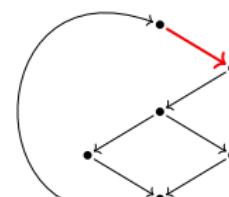
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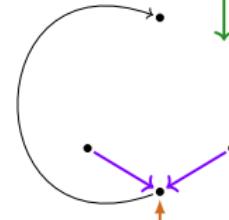
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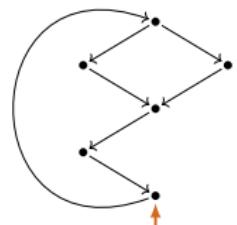
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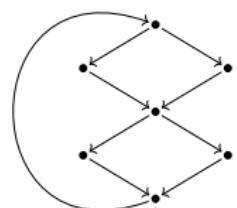
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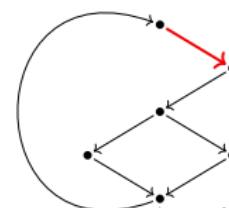
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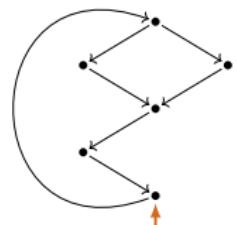


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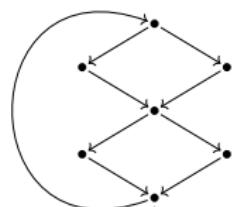


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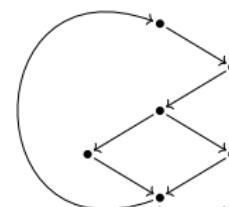
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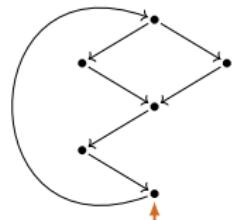


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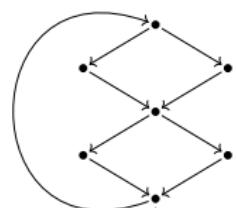


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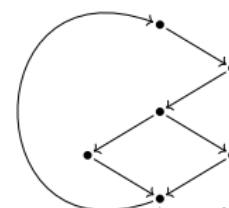
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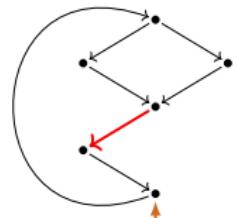
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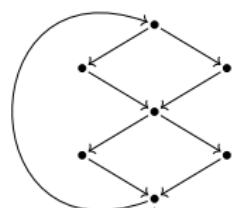
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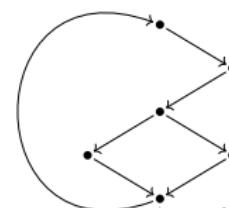
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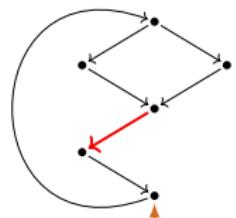
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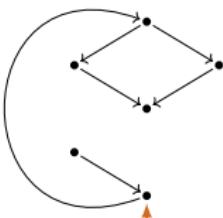
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elim

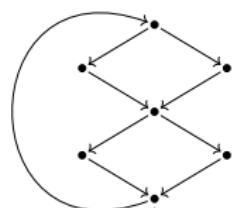
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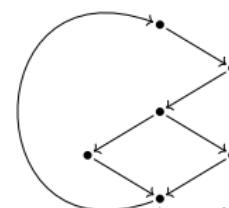
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'Critical pair': bi-loop elimination

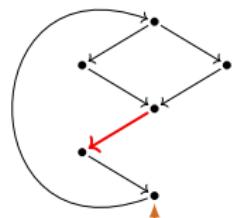


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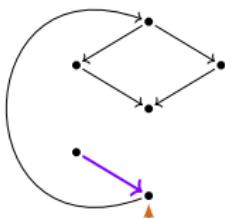


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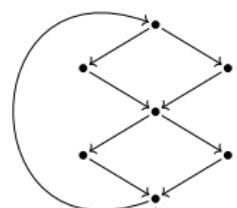
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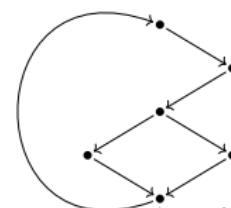
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'Critical pair': bi-loop elimination

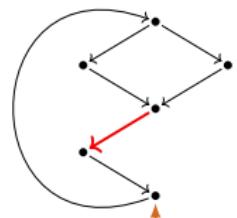


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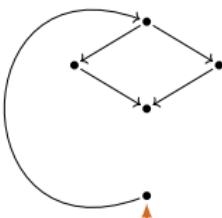


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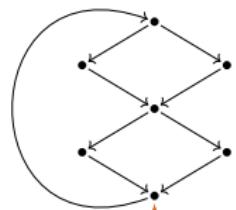
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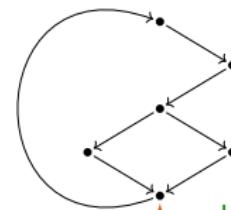
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'Critical pair': bi-loop elimination

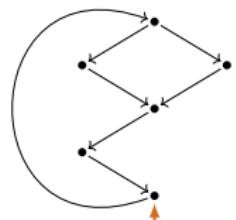


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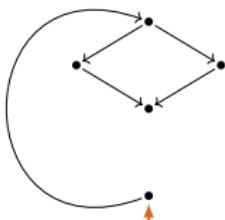


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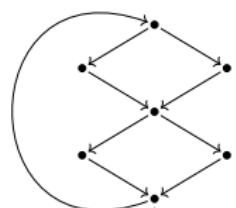
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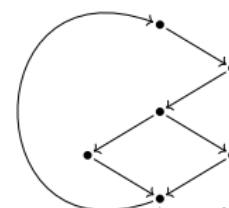
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'Critical pair': bi-loop elimination

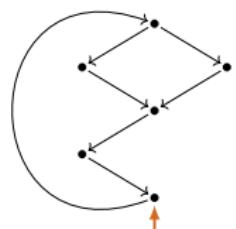


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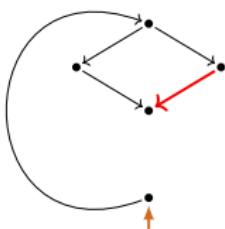


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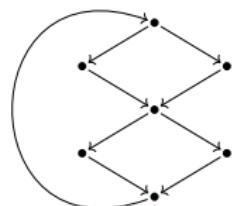
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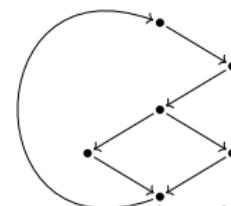
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'Critical pair': bi-loop elimination

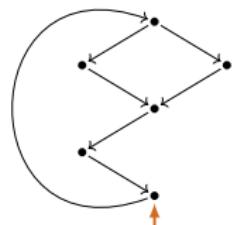


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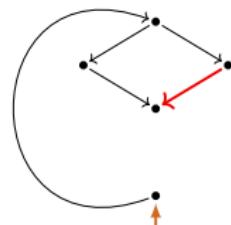


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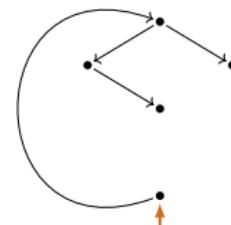
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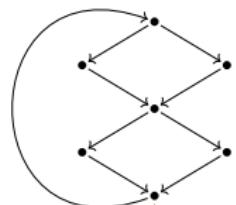
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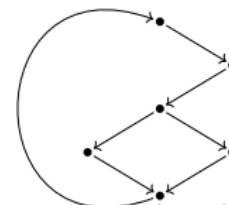
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'Critical pair': bi-loop elimination

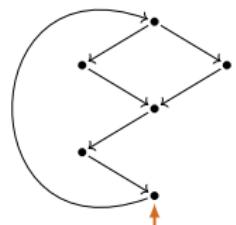


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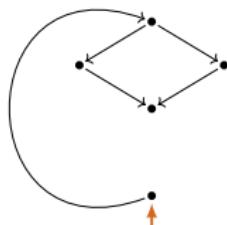


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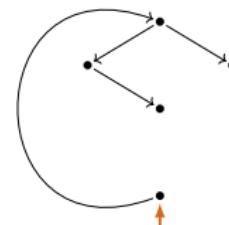
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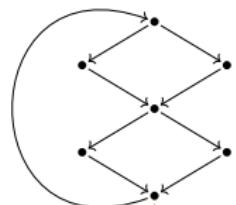
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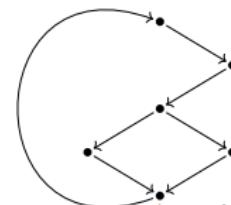
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'Critical pair': bi-loop elimination

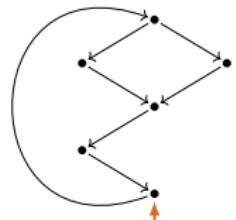


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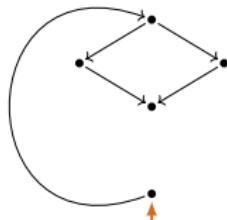


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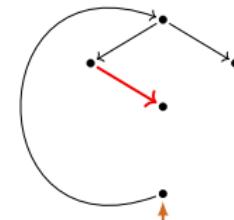
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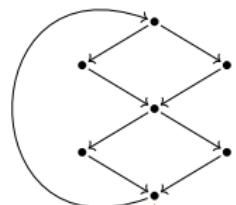
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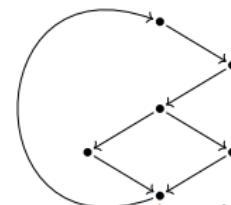
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'Critical pair': bi-loop elimination

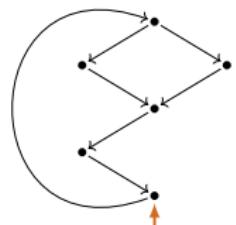


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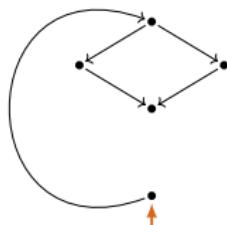


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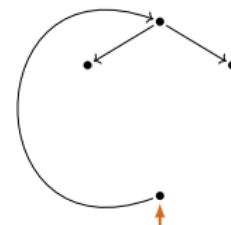
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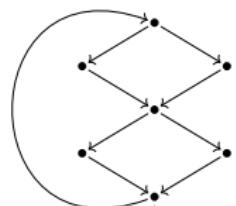
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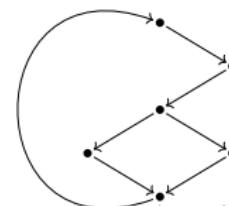
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'Critical pair': bi-loop elimination

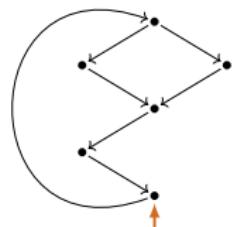


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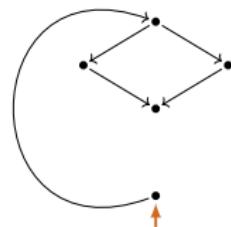


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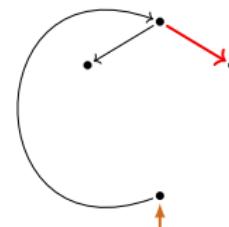
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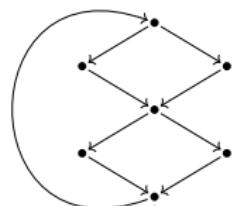
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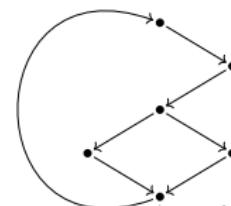
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'Critical pair': bi-loop elimination

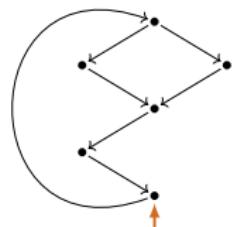


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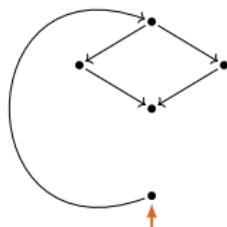


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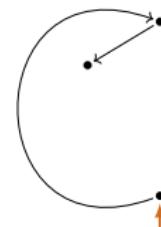
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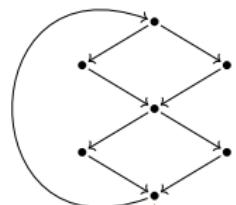
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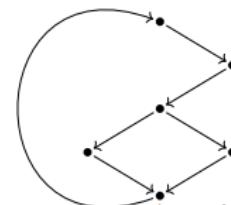
$\xrightarrow{* \text{ prune}}$



'Critical pair': bi-loop elimination

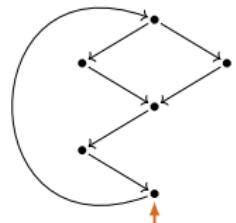


$\xrightarrow{\text{elim}}$

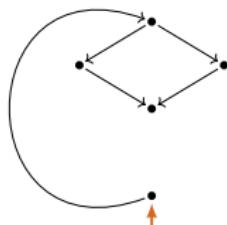


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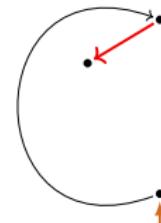
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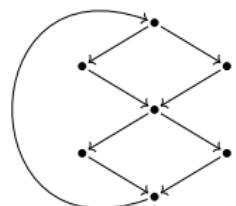
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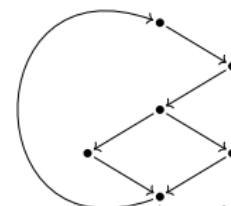
$\xrightarrow{* \text{ prune}}$



'Critical pair': bi-loop elimination

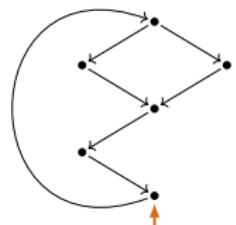


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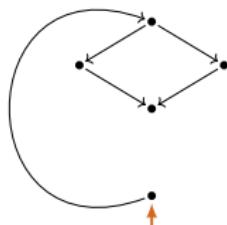


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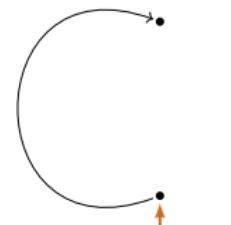
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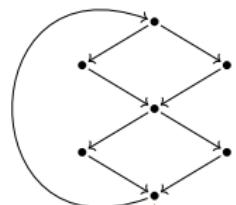
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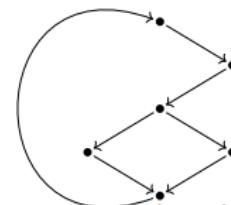
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'Critical pair': bi-loop elimination

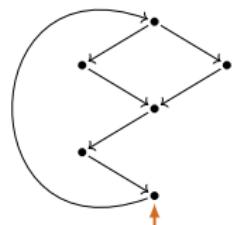


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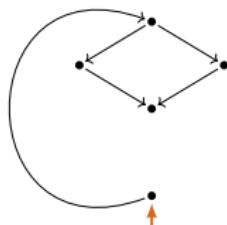


$\downarrow \text{elim}$

$\downarrow \text{elim}$



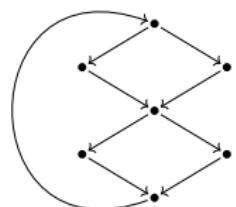
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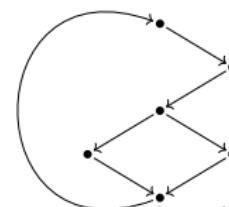
$\xrightarrow{* \text{ prune}}$



'Critical pair': bi-loop elimination



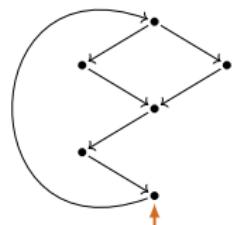
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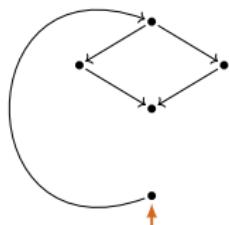
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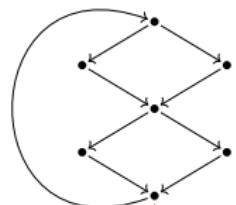


elim →

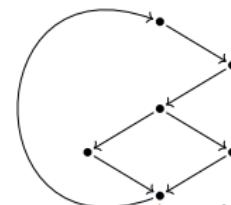


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prune →

'Critical pair': bi-loop elimination

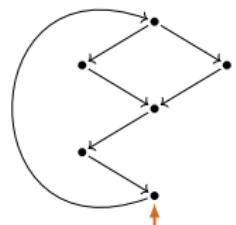


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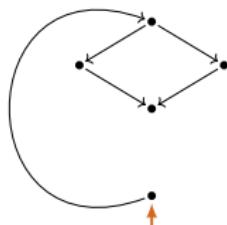


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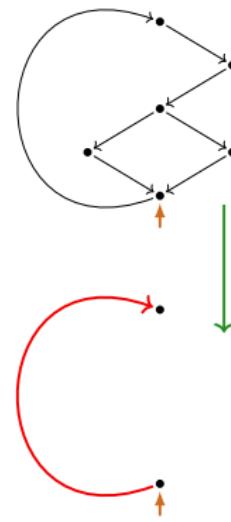
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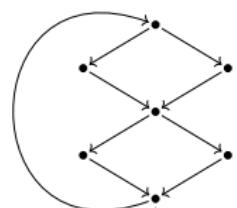
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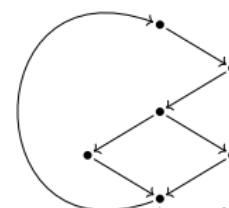
$\xrightarrow{* \text{ prune}}$



'Critical pair': bi-loop elimination

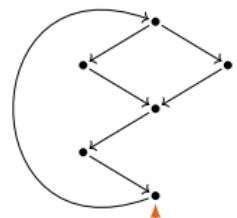


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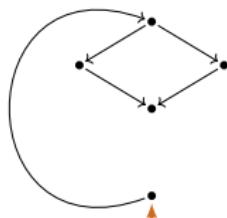


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elim



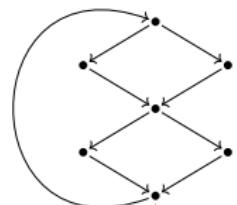
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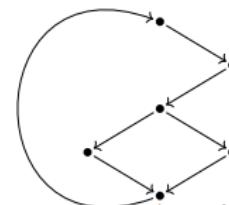
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prune

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'Critical pair': bi-loop elimination

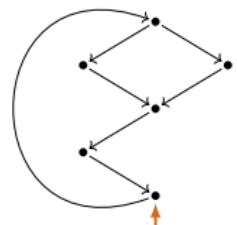


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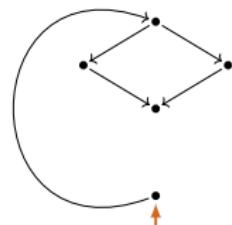


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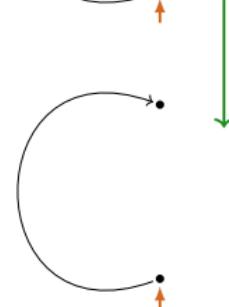
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elim

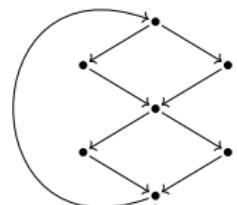


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prune

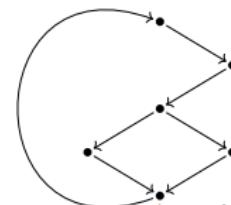


prune

'Critical pair': bi-loop elimination

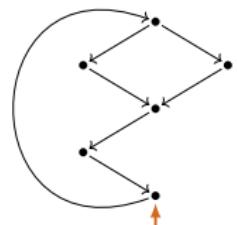


$\xrightarrow{\text{elim}}$

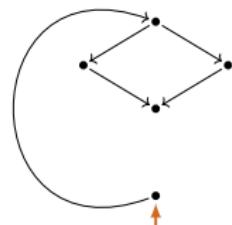


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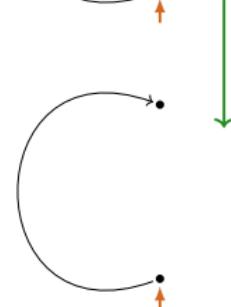
elim



$\xrightarrow{\text{elim}}$



$\xrightarrow{* \text{ prune}}$



* prune