Lecture 2: Machine Models, Basic Computability Theory

Models of Computation

https://clegra.github.io/moc/moc.html

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Gran Sasso Science Institute
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Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

Overview

- exercise calculable predicate

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- exercise calculable predicate
- Post machine
- Turing machine
  - Turing's analysis of computations done by (human) computers
  - formal definition
  - video

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- Elementary recursion theory
  - an unsolvable problem
  - Halting problem
  - recursively enumerable, and recursive sets
  - universal language
  - Chomsky hierarchy

Calculable functions?

Questions/Exercises

1. Suppose $P(a,b)$ is a calculable predicate.
   Why does $(\exists x)P(a,x)$ not have to be calculable?

2. Let $f : \mathbb{N} \to \mathbb{N}$ defined by

$$n \longmapsto \begin{cases} 0 & \ldots n = 0 \ \&\ \text{Goldbach's conjecture is false} \\ 1 & \ldots n = 0 \ \&\ \text{Goldbach's conjecture is true} \\ n+1 & \ldots n > 0 \end{cases}$$

   Is f calculable?

3. Can computation problems for mappings $F : \mathbb{N}^n \to \mathbb{N}^m$ always be represented by decision problems?

Calculable functions?

Questions/Exercises

1. Suppose $P(a,b)$ is a calculable predicate.
   Why does $(\exists x)P(a,x)$ not have to be calculable?

(Comput.) Yes-or-no-questions / Decision problems

Suppose $A \subseteq E$, where E a set of finitely describable objects.

A decision method for A in E is a method by which, given an element $a \in E$, we can decide in a finite number of steps whether or not $a \in A$.

(Comput.) Yes-or-no-questions / Decision problems

Suppose $A \subseteq E$, where E a set of finitely describable objects.

A decision method for A in E is a method by which, given an element $a \in E$, we can decide in a finite number of steps whether or not $a \in A$.

The decision problem for A in E is solvable (the set A in E is (effectively) calculable) if there exists a decision method for A in E.

Reading recommended (for today)

1. Post machine: Page 1 + first paragraph on page 2 of:
   - Emil Post: *Finite Combinatory Processes – Formulation 1*, Journal of Symbolic Logic (1936), [2].

2. Turing machine motivation:
   Turing's analysis of a human computer:
   Part I of Section 9, pp. 249–252 of:
   - Alan M. Turing's: *On computable numbers, with an application to the Entscheidungsproblem*', Proceedings of the London Mathematical Society (1936), [3].

Emil Post

Emil Leon Post (1897–1954)

Post about . . .

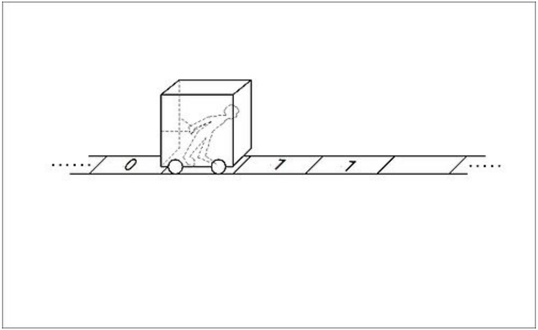
. . . a result of his from 1921 similar to the Incompleteness Theorem:

Theorem (Gödel, 1931 (paraphrased here))

Every axiomatisable, consistent first-order-logic system of number theory is incomplete: it contains true, but unprovable formulas.

> *"For full generality a complete analysis would have to be given of all possible ways in which the human mind could set up finite processes for generating sequences."*

Post machine (1936)



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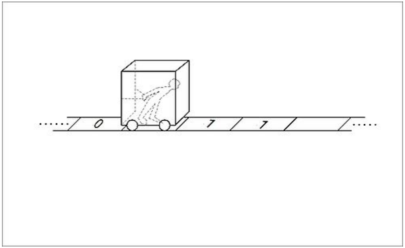

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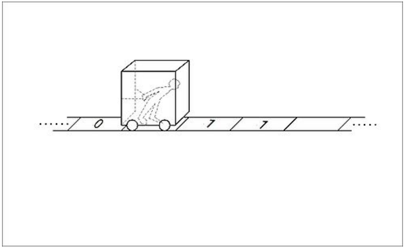

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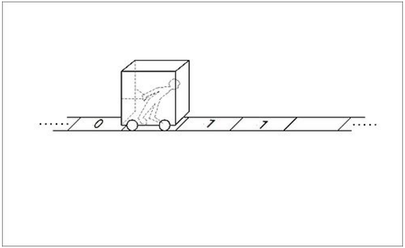

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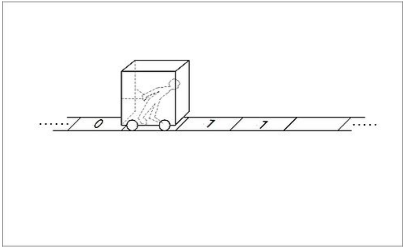

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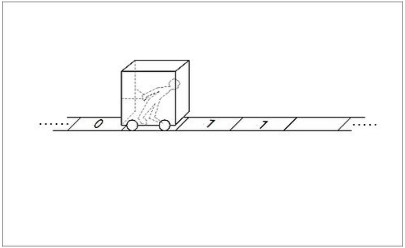

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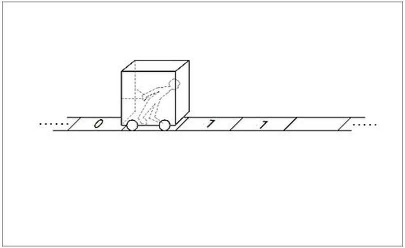

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(e) Determining whether the box he is in, is or is not marked.”

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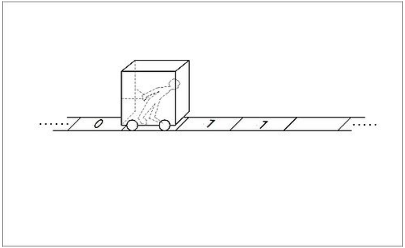

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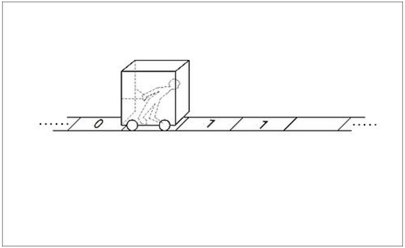

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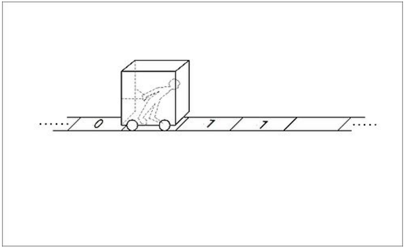

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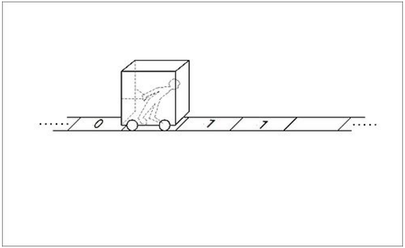

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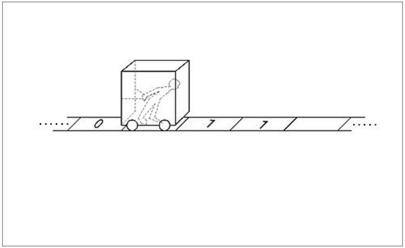

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  - Perform operation (e) and according as the answer is yes or no correspondingly follow direction j_i' or j_i''.
  - Stop.

Exercise

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Construct a Post machine that adds one to a natural number in unary representation.

Typical features of ‘computationally complete’ MoC’s

- storage (unbounded)

(Credits due to: Vincent van Oostrom)

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- stopping condition

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Turing computability

Alan Turing (1912 –1954)

Turing's analysis of a human 'computer'

Section 9 in Turing's 1937 paper 'On computable numbers, with an application to the Entscheidungsproblem' [3].

A direct appeal to intuition in analysing human computation:

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- bound B on the number of symbols/squares that the computer can observe at any moment
- number of 'states of mind' of the computer is finite

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  - A change (b) of observed square, together with a possible change of state of mind.

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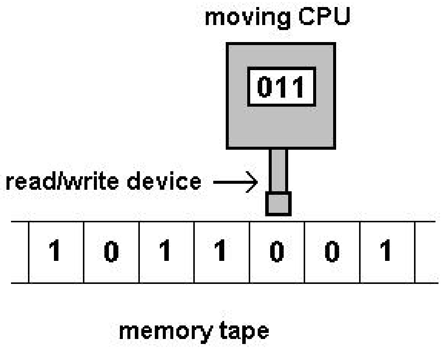
"It is my contention that these operations include all those which are used in the computation of a number."

Turing machine



Church–Turing Thesis

Thesis (Church–Turing, 1937)

Every effectively calculable function is computable by a Turing-machine.

Turing machine: formal definition

Definition

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- $\delta : (Q \smallsetminus F) \times \Gamma \rightharpoonup Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ is a partial function, called the transition function;
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Turing machine: definition notions

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Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \cancel{b}, F \rangle$ be a Turing machine.

A configuration of M is elements $w_1 q w_2 \in \Gamma^* \times Q \times \Gamma^*$ such that the first letter in w_1 and the last letter in w_2 are different from $\cancel{b}$.

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- M accepts w if $q_0 w \vdash_M^* uqv$ for some accepting config. uqv.

$L(M) := \{ w \in \Sigma^* \mid M \text{ accepts } w \}$ is the language accepted by M.

Recursively enumerable/recursive languages

Definition

Let $L \subseteq \Sigma^*$ a language.

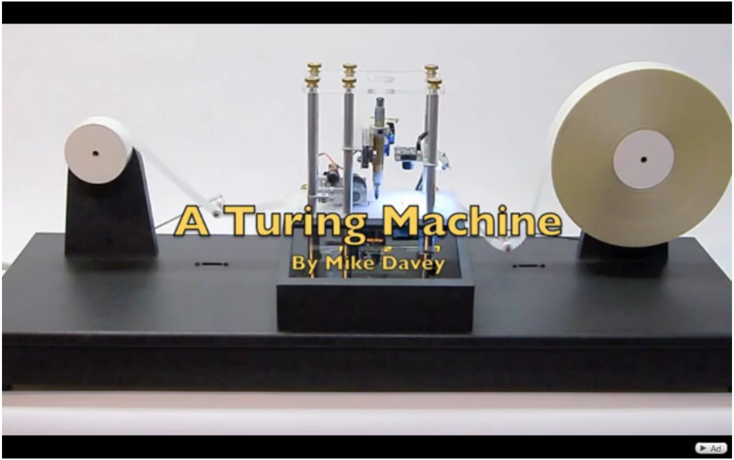
L is called recursively enumerable if

- $L = L(M)$ for some Turing machine M with input symbols Σ.

L is called recursive if

- there is a Turing machine M with input symbols Σ such that
  - $L = L(M)$
  - M halts on all of its inputs.

Mike Davey's Turing machine (link)



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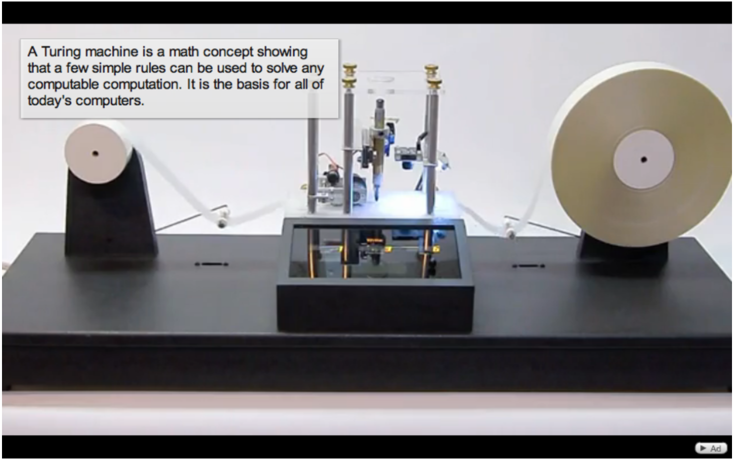

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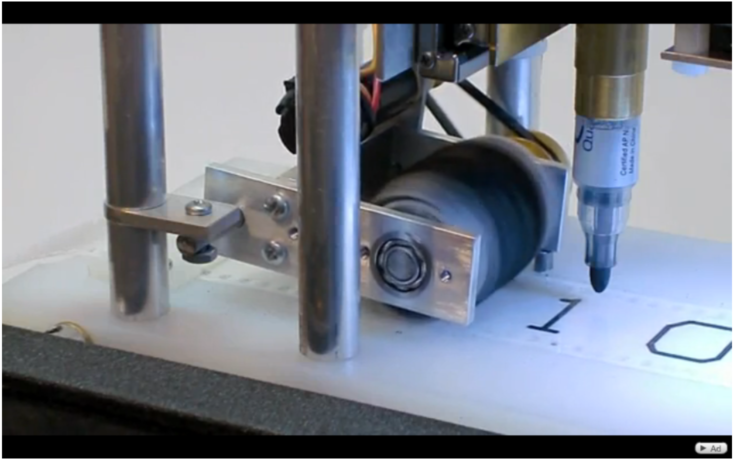

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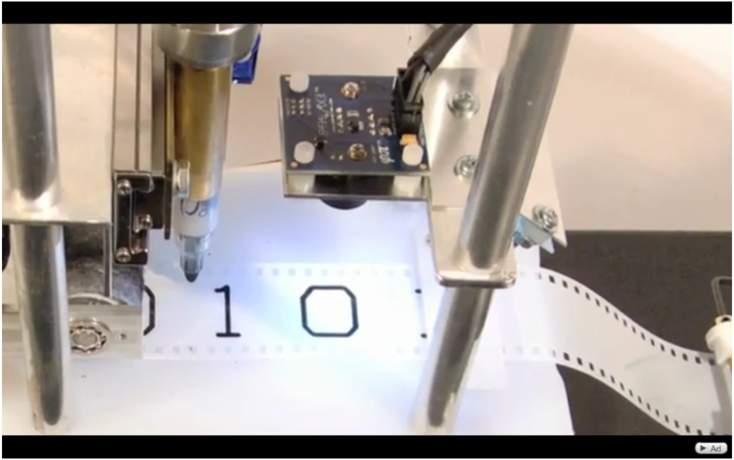

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Exercises

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Exercise

Construct a Turing machine that, if started on the empty tape, writes the sequence

$$0101101110111101111110\ldots$$

on the tape, but does not halt.

(Compare your machine with Turing's machine for this purpose.)

Typical features of ‘computationally complete’ MoC’s

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- control (finite, given)
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  - of (immediately accessible) stored data
  - of control state
- conditionals
- loop (unbounded)
- stopping condition

Variants of Turing machines

- TM's with semi-infinite tapes (infinite in only one direction)
- TM's with multiple tapes
  - Input/Output Turing machines (with input- and output tapes)
- non-deterministic TM's: $\delta \subseteq ((Q \times \Gamma) \times (Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}))$
- tape-bounded TM's (by $f(n)$ for inputs of length n)
- oracle Turing machines
- Turing machines with advice
- alternating Turing machines
- …
- interactive/reactive TM's

Elementary Recursion Theory

An unsolvable problem

The diagonalisation language:

$$L_d := \{w \mid w = \langle M \rangle, w \notin L(M)\} = \{\langle M \rangle \mid \langle M \rangle \notin L(M)\}$$

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Instance: w a binary word.
Question: Does $w \in L_d$ hold? (Does Tm. M with $\langle M \rangle = w$ accept w?)

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Theorem

There exist unsolvable decision problems.

Exercise: Halting Problem

Exercise

Try to adapt the diagonalisation argument to show that for the Halting Problem

$$H = \{ \langle \langle M \rangle , w \rangle \mid M \text{ halts on input } w \}$$

it holds:

- H is not recursive

and show that:

- H is recursively enumerable

Properties of r.e./recursive sets (I)

For $L \subseteq \Sigma^*$, $\bar{L} := \Sigma^* \setminus L$ is called the complement of L.

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If L is recursive, then $\bar{L}$ is recursive.

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First idea: Swap the accepting states of M with the non-accepting states of M in which computations may halt.

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M is modified as follows to obtain $\bar{M}$:

- the accepting states of M are made non-accepting in $\bar{M}$.
- $\bar{M}$ has a new accepting state r.
- for each $q \in Q$ and tape symbol $s \in \Gamma$ such that $\delta_M(q,s)$ is undefined, add the transition $\delta_{\bar{M}}(q,s) = \langle r, s, R \rangle$.

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It follows that $\bar{L} = L(\bar{M})$, and that $\bar{M}$ halts on all inputs. □

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Let M_1 and M_2 be Tm's such that $L = L(M_1)$ and $\bar{L} = L(M_2)$.

To decide, for a given $w \in \Sigma^*$, whether $w \in L$, build a Tm M that executes M_1 and M_2 on w in parallel, and such that:

- if M_1 accepts w, then also M accepts w.
- if M_2 accepts w, then also M halts, but does not accept w.

Hence M accepts w iff $w \in L(M_1) = L$. Thus $L(M) = L$.

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Since for all w, either $w \in L$ or $w \in \bar{L}$, it follows that either M_1 or M_2 halts on w, and hence M halts on all inputs.

Hence $L = L(M)$ is recursive.

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L_u is recursively enumerable, but not recursive.

Proof.

- L_u is r.e.: $L_u = L(M_u)$ for an universal machine M_u.
- L_u is not recursive:

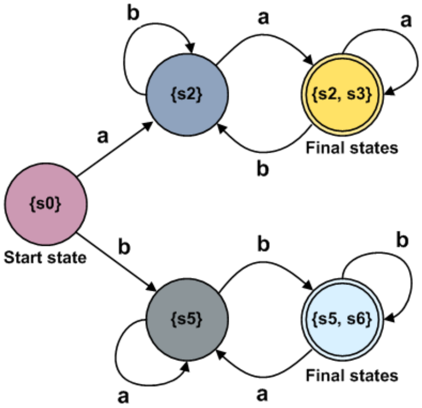
  Suppose that L_u is recursive. Then $\bar{L_u}$ is recursive, and hence there exists a Tm. M such that $\bar{L_u} = L(M)$.

  M can be used to build a Tm. M' that accepts the diagonalisation language L_d, entailing $L_u = L(M')$.

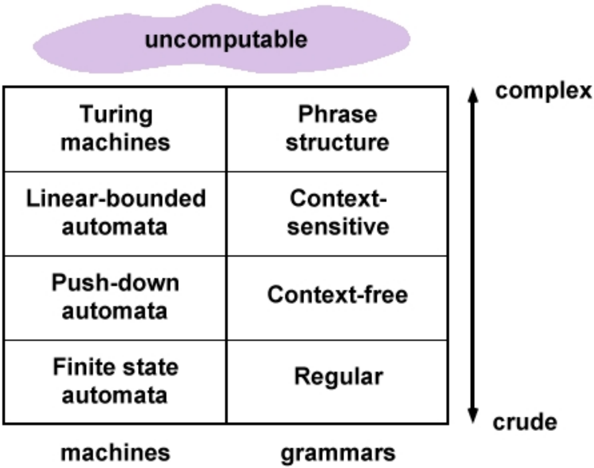
  [picture of M' to be given]

  But then L_u would actually be r.e., in contradiction with what we proved before.

Finite-state automaton



Formal-languages Chomsky hierarchy



Summary

- Post machine
- Turing machine
  - Turing's analysis of computations done by (human) computers
  - formal definition
  - video
- Elementary recursion theory
  - an unsolvable problem
  - Halting problem
  - recursively enumerable, and recursive sets
  - universal language
  - Chomsky hierarchy

Recommended reading

1. Recursive and primitive-recursive functions:
   Chapter 4, Recursive Functions of the book:
   - Maribel Fernández [1]: *Models of Computation (An Introduction to Computability Theory)*, Springer-Verlag London, 2009.

Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

References

- Maribel Fernández.
  Models of Computation (An Introduction to Computability Theory).
  Springer, Dordrecht Heidelberg London New York, 2009.

- Emil Leon Post.
  Finite Combinatory Processes – Formulation 1.
  Journal of Symbolic Logic, 1(3):103–105, 1936.
  https://www.wolframscience.com/prizes/tm23/images/Post.pdf.

- Alan M. Turing.
  On Computable Numbers, with an Application to the Entscheidungsproblem.
  Proceedings of the London Mathematical Society, 42(2):230–265, 1936.
  http://www.wolframscience.com/prizes/tm23/images/Turing.pdf.