An Introduction to Parameterized Complexity

Lecture 1: Fixed-Parameter Tractability

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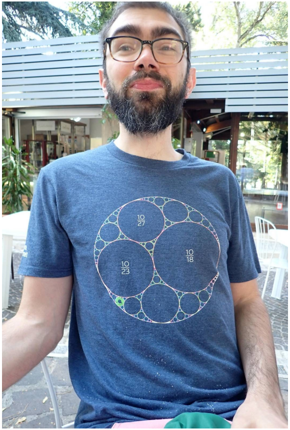
Monday, June 10, 2024

Course overview

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results		**Algorithmic Meta-Theorems**		
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	GDA	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	GDA	GDA
Algorithmic Techniques		*Formal-Method* & *Algorithmic Techniques*		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width			**FPT-Intractability Classes & Hierarchies**
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Course developers



Hugo Gilbert
course 2019/20 (Hugo & Clemens)

CG & Alessandro Aloisio
course 2020/21 (Alessandro & C)

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Motivation

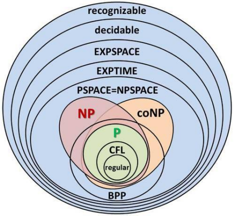
Classical complexity theory

- analyses problems by resource (space or time)
  needed to solve them on a reasonable machine model
- as a function of the input size $n = |x|$ (Hartmanis/Stearns, 1965)

⇒ variety of complexity classes
(P, LOGSPACE, NP, PSPACE, . . .)

⇒ tractable problems
= polynomial-time computable (in P)

⇒ theory of intractability
(reductions, NP completeness)



Drawback

- measures problem size $n = |x|$
  only in terms of input instances x,
  and ignores structural information about instances
- sometimes problems are easier to solve
  for instances if additional structure information is available

Motivation

Classical complexity theory

- analyses problems by resource (space or time) needed to solve them on a reasonable machine model
- as a function of the input size $n = |x|$ (Hartmanis/Stearns, 1965)

$\Rightarrow$ variety of complexity classes (P, LOGSPACE, NP, PSPACE, ...)
$\Rightarrow$ tractable problems = polynomial-time computable (in P)
$\Rightarrow$ theory of intractability (reductions, NP completeness)

Parameterized complexity

- measures complexity also in terms of a parameter $k = \kappa(x)$ that may depend on the input x in an arbitrary way

$\Rightarrow$ fixed-parameter tractable problems
relaxes polynomial time solvability to algorithms whose non-polynomial behavior $f(k) \cdot p(n)$ is restricted by parameter k
$\Rightarrow$ complexity classes (FPT, XP, W[P], W- and A-hierarchies)
$\Rightarrow$ theory of fixed-parameter intractability

Parameterized (versus classical) problems

Definition

A classical (decision) problem is a pair $\langle \Sigma, Q \rangle$ where:

- $Q \subseteq \Sigma^*$ the set of *problem yes-instances* over a finite alphabet Σ

A *parameterized (decision) problem* is a triple $\langle \Sigma, Q, \kappa \rangle$ where:

- $Q \subseteq \Sigma^*$ the set of *problem yes-instances* over a finite alphabet Σ,
- $\kappa : \Sigma^* \to \mathbb{N}$ a function, *the parameterization*.

We regularly shorten $\langle \Sigma, Q, \kappa \rangle$ to a pair $\langle Q, \kappa \rangle$.

Assumption

The parameterization κ can be efficiently computed.

Parameterized problems (examples)

A Parameterized Clique Problem

p-CLIQUE:

Given: a graph G and an integer k,
Question: Does there exists a clique of size k in G?

Parameter: k.

A Parameterized Hitting Set Problem

p-HITTING SET

Given: a universe $U = \{x_1, \ldots, x_n\}$, a collection of sets $\mathcal{S} = (S_1, \ldots, S_m)$ where $S_i \subseteq U$ and an integer k,
Question: Does there exists a set $S \subseteq U$ such that $|S| \leq k$ and $S \cap S_i \neq \varnothing$, $\forall i \in \{1, \ldots, m\}$.

Parameter: $\max |S_i|$.

- NP-hard even if $\max |S_i| = 2$,
- is fixed-parameter tractable.

The art of parameterization

What is a good parameter?

- We should have reasons to believe that the parameter is "small" for some applications.
- It is better if the parameter is intuitive.
- It is better if the parameter is efficiently computable.

There is a hierarchy on parameters.

The art of parameterization

There are many different types of parameters!

- The size of the solution we are looking for.
- The size of some parts of the instance.
  E.g., the number of voters in an election problem.
- Some more structural property of the instance.
  E.g., the diameter of a graph.
- It can be a combination of values, a difference, ...

The art of parameterization

- Graph problems: maximum degree, treewidth, diameter...
- Social choice problems: number of voters, candidates, correlation of preferences...
- Boolean formulas: number of variables, number of clauses...
- Problems on strings: maximum length of a string, size of the alphabet...

Fixed Parameter Tractability (Class FPT)

Definition

A parameterized problem $\langle Q, \kappa \rangle$ is *fixed-parameter tractable* if:

$$\exists f : \mathbb{N} \to \mathbb{N} \text{ computable } \exists p \in \mathbb{N}[X] \text{ polynomial}$$
$$\exists \mathbb{A} \text{ algorithm, takes inputs in } \Sigma^* \text{ and } \forall x \in \Sigma^*$$
$$\big[\mathbb{A} \text{ decides if } x \in Q \text{ in time } \leq f(\kappa(x)) \cdot p(|x|)\big].$$

FPT := complexity class of all fixed-parameter tractable problems.

Assumption for a robust fpt-theory:

κ is polynomially computable, or itself fpt-computable.

Goal in parameterized algorithmics:

⇒ design FPT algorithms,
⇒ try to make both factors $f(\kappa(x))$ and $p(|x|)$ as small as possible.
⇒ or show (if possible) that finding such factors is impossible

Slices of FPT problems are in P

The ℓ*-th slice* of a parameterized problem $\langle Q, \kappa \rangle$:

$$\langle Q, \kappa \rangle_{\ell} := \{ x \in Q \mid \kappa(x) = \ell \} \qquad \text{(as classical problem)}.$$

Proposition

If $\langle Q, \kappa \rangle \in$ FPT, then $\langle Q, \kappa \rangle_{\ell} \in$ P for all $\ell \in \mathbb{N}$.

Proof.

If $\langle Q, \kappa \rangle \in$ FPT, then there are a computable function $f : \mathbb{N} \to \mathbb{N}$, a polynomial p, and an algorithm $\mathbb{A}$ that decides $x \in \Sigma^*$ in running time $\leq f(\kappa(x)) \cdot p(|x|)$ time. This algorithm can also be used to decide the ℓ-th slice in time $\leq f(\ell) \cdot p(|x|)$, which for fixed ℓ is a polynomial. $\square$

A problem not in FPT (unless P = NP)

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If $\langle Q, \kappa \rangle \in$ FPT, then $\langle Q, \kappa \rangle_\ell \in$ P for all $\ell \in \mathbb{N}$.

Application

p-COLORABILITY
Instance: a graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Parameter: k.
Problem: Decide whether $\mathcal{G}$ is k-colorable.

Known: 3-COLORABILITY $\in$ NP-complete (Lovàsz, Stockmeyer, 1973).
Since 3-COLORABILITY = p-COLORABILITY$_3$,
it follows that p-COLORABILITY $\notin$ FPT (unless P = NP).

Slice-wise polynomial problems (Class XP)

Definition

A parameterized problem $\langle Q, \kappa \rangle$ is *slice-wise polynomial* if:

$\exists f, g : \mathbb{N} \to \mathbb{N}$ computable
$\exists \mathbb{A}$ algorithm, takes inputs in Σ^* and $\forall x \in \Sigma^*$
$\left[\mathbb{A} \text{ decides if } x \in Q \text{ in time } \leq f(\kappa(x)) \cdot |x|^{g(\kappa(x))} \right]$.

XP := complexity class of slice-wise polynomial problems.

Slices of XP problems are in P

The *ℓ-th slice* of a parameterized problem $\langle Q, \kappa \rangle$:

$$\langle Q, \kappa \rangle_{\ell} := \{ x \in Q \mid \kappa(x) = \ell \} \qquad \text{(as classical problem)}\,.$$

Proposition

If $\langle Q, \kappa \rangle \in \text{XP}$, then $\langle Q, \kappa \rangle_{\ell} \in \text{P}$ for all $\ell \in \mathbb{N}$.

Proof.

If $\langle Q, \kappa \rangle \in \text{XP}$, then there are a function $f : \mathbb{N} \to \mathbb{N}$ computable, a polynomial p, and an algorithm $\mathbb{A}$ that decides $x \in \Sigma^*$ in running time $\leq f(\kappa(x)) \cdot |x|^{g(\kappa(x))}$ time. This algorithm can be used to decide the ℓ-th slice in time $\leq f(\ell) \cdot |x|^{g(\ell)}$, which for fixed ℓ is a polynomial. $\square$

A problem not in XP (unless P = NP)

The *ℓ-th slice* of a parameterized problem $\langle Q, \kappa \rangle$:

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Aims of the course

1. Acquire a basic notions of parameterized complexity.
2. Obtain an introduction to some techniques to derive FPT or XP results.
3. Obtain an introduction to a variety of techniques to prove algorithmic lower bounds and in particular prove parameterized hardness results.

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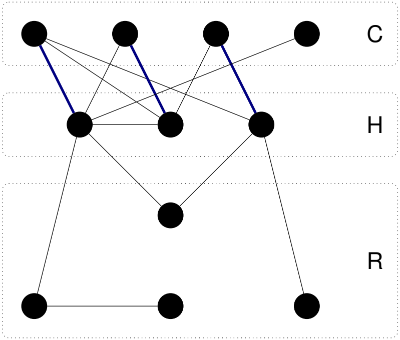
From today's lecture



A **crown decomposition** of a graph G is a partitioning (C, H, R) of $V(G)$, such that:

1. C is nonempty.
2. C is an independent set.
3. H separates C and R.
4. G contains a matching of H into C.

Crown Lemma (⇐ results by Kőnig, Hall)

Let G be a graph with no isolated vertices and with at least $3k + 1$ vertices. There is a polynomial-time algorithm that:

- either finds a matching of size $k + 1$ in G;
- or finds a crown decomposition of G.

Tomorrow

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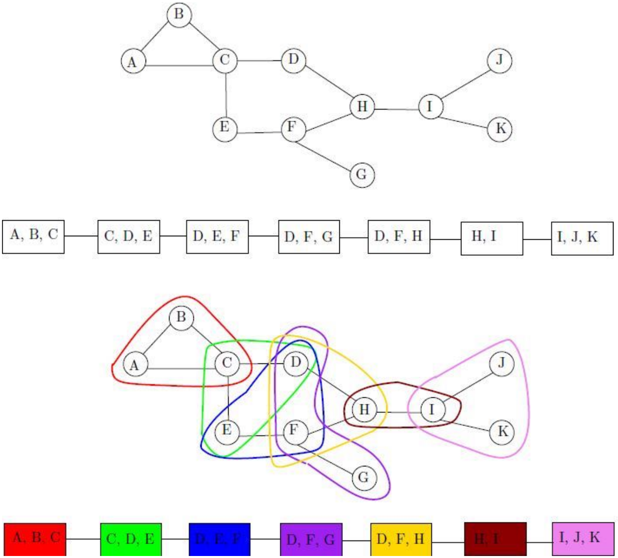
In tomorrow's lecture: a path decomposition of a graph



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In Wednesday's lecture: Monadic second-order logic

$$\psi_3 \;:=\; \exists C_1 \exists C_2 \exists C_3 \Big(\big(\forall x \bigvee_{i=1}^{3} C_i(x)\big) \wedge \forall x \forall y \Big(E(x,y) \to \bigwedge_{i=1}^{3} \neg\big(C_i(x) \wedge C_i(y)\big)\Big)\Big)$$

$\mathcal{A}(\mathcal{G}) \vDash \psi_3 \quad \Longleftrightarrow \quad \mathcal{G}$ has is 3-colorable.

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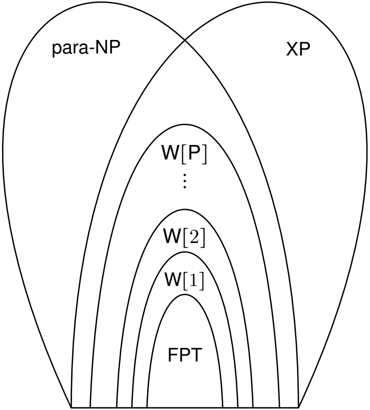
From Friday's lecture: W-Hierarchy

'There is no definite single class that can be viewed as "the parameterized NP". Rather, there is a whole hierarchy of classes playing this role. (Flum, Grohe [FG06])



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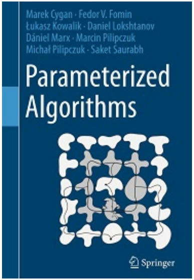
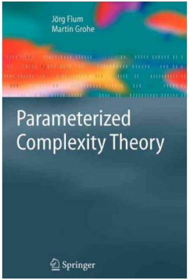
Books





- Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh, *Parameterized Algorithms*, 1st ed., Springer, 2015.
- Jörg Flum and Martin Grohe, *Parameterized Complexity Theory*, Springer, 2006.

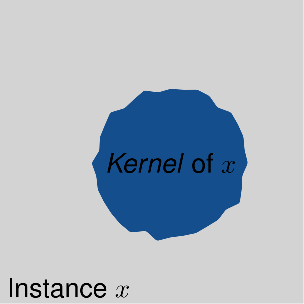
Kernelization

- Idea
- Definition
- Kernel examples for:
  - point line cover problem
  - vertex cover problem
- Kernelization $\Leftrightarrow$ FPT
- Crown lemma and crown decomposition
  - smaller kernel for vertex cover problem
  - kernel for dual colorability problem
- Sunflower lemma
  - kernel for hitting set problem

Kernelization methods (informally)

Kernelization is:

- a systematic study of polynomial-time preprocessing algorithms,
- an important tool in the design of parameterized algorithms.



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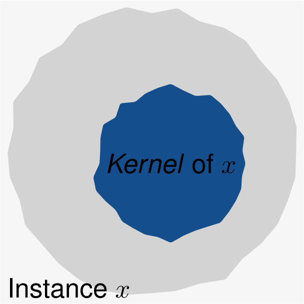

→ Application of rule 1

- Often a collection of efficient preprocessing rules.

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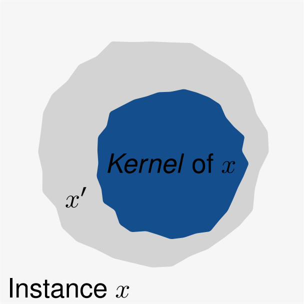

→ Application of rule 1

→ Application of rule 2

- Often a collection of efficient preprocessing rules.
- Transform an instance x into a smaller equivalent instance x'.

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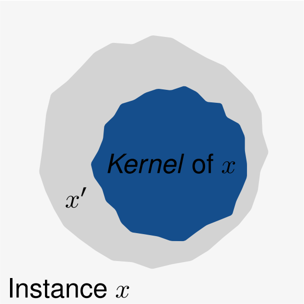

→ Application of rule 1

→ Application of rule 2

- Often a collection of efficient preprocessing rules.
- Transform an instance x into a smaller equivalent instance x'.
- Hopefully, $|x'| \leq g(\kappa(x))$.
→ use a (non-efficient) exact algorithm.

Kernelization (formally)

Definition

Let $\langle Q, \kappa \rangle$ be a parameterized problem over Σ.
A *kernelization* of $\langle Q, \kappa \rangle$ is a function $K : \Sigma^* \to \Sigma^*$ such that:

(K1) For all $x \in \Sigma^*$: $(x \in Q \iff K(x) \in Q)$.

(K2) K is polynomial-time computable.

(K3) There is a computable function $h : \mathbb{N} \to \mathbb{N}$
such that for all $x \in \Sigma^*$: $|K(x)| \leq h(\kappa(x))$.

We say that such a kernelization K is *polynomial* (resp. *linear*)
(and that Q *has a polynomial* (resp. *linear*) kernel)
if the function h is polynomial (resp. linear).

Lemma

If $\langle Q, \kappa \rangle$ admits a kernel and is decidable, then $\langle Q, \kappa \rangle \in$ FPT.

Lemma

If $\langle Q, \kappa \rangle \in$ FPT, the $\langle Q, \kappa \rangle$ admits a kernel.

The (parameterized) Point Line Cover Problem

p-POINT-LINE-COVER:

Given: n points in the plane and an integer k.
Parameter: The integer k.
Question: Do there exist k lines that cover all points?

Rule 1:
If we have a line that hits $k+1$ or more points, **then**:
i) include it in the solution;
ii) remove the points hit by the line;
iii) set $k := k - 1$.

Observation: Let (x, κ) be a yes instance of the p-Point-Line-Cover such that Rule 1 cannot be applied. **Then** $n \leq k^2$ holds.

Rule 2:
If we cannot apply Rule 1, and we have more than k^2 points, **then** say no, and return a trivial no instance.

Proposition

p-POINT-LINE-COVER $\in$ FPT: it admits a kernel of size with k^2 points.

The (parameterized) Vertex Cover Problem

p-VERTEX-COVER:
Given: A graph G, and an integer k.
Parameter: The integer k.
Question: Does there exists a vertex cover of size at most k?

Definition

Let G be a graph and $S \subseteq V(G)$. The set S is called a vertex cover if for every edge of G at least one of its endpoints is in S.

Exercise

Find an $O(k^2)$ kernel for p-VERTEX-COVER.

The (parameterized) Vertex Cover Problem *(Buss kernel)*

Rule 1: If G contains an isolated vertex v, delete v from G.
The new instance is $(G \setminus v, k)$

Rule 2: If there is a vertex v of degree at least $k+1$, then delete v (and its incident edges) from G and decrement the parameter k by 1.
The new instance is $(G \setminus v; k-1)$

Observations

- After exhaustive application of Rule 1 and Rule 2 all vertices have degree between 1 and k.
- If G has maximum degree d, k vertices can cover $\leq k \cdot d$ edges.
- If G has a vertex cover of $\leq k$ vertices after exhaustive application of Rules 1 & 2, then G has $\leq k^2$ edges (and $\leq k^2 + k$ vertices).

The (parameterized) Vertex Cover Problem *(Buss kernel)*

Rule 1: If G contains an isolated vertex v, delete v from G. The new instance is $(G \setminus v, k)$

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Observations

- If G has a vertex cover of $\leq k$ vertices after exhaustive application of Rules 1 & 2, then G has $\leq k^2$ edges (and $\leq k^2 + k$ vertices).

Rule 3: Let (G, k) be an instance to which Rules 1 & 2 are not applicable. If G has $> k^2 + k$ vertices, or $> k^2$ edges, then (G, k) is a no-instance that can be replaced by a trivial no-instance.

Theorem (Samuel Buss)

p-VERTEX-COVER $\in$ FPT, *because it admits a kernel with at most $O(k^2)$ vertices and $O(k^2)$ edges.*

Kernelization ⇒ FPT

Exercise

If $\langle Q, \kappa \rangle$ admits a kernel and is decidable, then $\langle Q, \kappa \rangle \in$ FPT.

Definitions

A *kernelization* of $\langle Q, \kappa \rangle$ is a function $K : \Sigma^* \to \Sigma^*$ such that:

(K1) For all $x \in \Sigma^*$: $(x \in Q \iff K(x) \in Q)$.

(K2) K is polynomial-time computable.

(K3) There is a computable function $h : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \Sigma^*$: $|K(x)| \leq h(\kappa(x))$.

A parameterized problem $\langle Q, \kappa \rangle$ is *fixed-parameter tractable* if:

$\exists f : \mathbb{N} \to \mathbb{N}$ computable $\exists p \in \mathbb{N}[X]$ polynomial
$\exists \mathbb{A}$ algorithm, takes inputs in Σ^* and $\forall x \in \Sigma^*$
$\big[\mathbb{A}$ decides if $x \in Q$ in time $\leq f(\kappa(x)) \cdot p(|x|)\big]$.

FPT := complexity class of all fixed-parameter tractable problems.

Kernelization ⇒ FPT

Lemma

If $\langle Q, \kappa \rangle$ admits a kernel and is decidable, then $\langle Q, \kappa \rangle \in$ FPT.

$\langle Q,\kappa\rangle$ a parameterized problem, $Q \subseteq \Sigma^*$

Definition. $K : \Sigma^* \longrightarrow \Sigma^*$ a kernelization for $\langle Q,\kappa\rangle$ if:

(K1) $\forall x \in \Sigma^* \, (\, x \in Q \iff K(x) \in Q \,)$

(K2) K is poly time computable

(K3) $\exists h : \mathbb{N} \longrightarrow \mathbb{N} \; \forall x \in \Sigma^* \, (\, |K(x)| \leq h(\kappa(x)) \,)$.

Proposition: If $\langle Q,\kappa\rangle$ is decidable, and has kernelization K, then $\langle Q,\kappa\rangle \in$ FPT

Proof. Since $\langle Q,\kappa\rangle$ is decidable, there is an algorithm A that decides instances $x \in \Sigma^*$ in time $\leq f(|x|)$ steps for some computable function $f : \mathbb{N} \to \mathbb{N}$.

Then, assuming a polynomial algorithm A_K for K (time bounded by $p(x)$) we construct an FPT algorithm $A(K)$ for $\langle Q,\kappa\rangle$:

$\boxed{A(K)}$ $\quad \Sigma^* \ni x \xmapsto[A_K]{K} K(x) \in \Sigma^*$

A: $K(x) \notin Q$ → "no" → "no" $(x \notin Q)$ (K1)

A: $K(x) \in Q$ → "yes" → "yes" $(x \in Q)$ (K1)

Running time $A(K) =$

$= \text{time}(A_K) + \text{time}(A(K(x)))$

$= p(|x|) + f(|K(x)|)$ by (K2) by (K3) $\leq h(\kappa(x))$

$= p(|x|) + f(h(\kappa(x)))$

$= (f \circ h)(\kappa(x)) \cdot (1+p)(|x|)$

$= \tilde{f}(\kappa(x)) \cdot \text{poly}(|x|) \in$ FPT.

FPT ⇒ Kernelization

Lemma

If $\langle Q, \kappa \rangle \in$ FPT, then $\langle Q, \kappa \rangle$ admits a kernel.

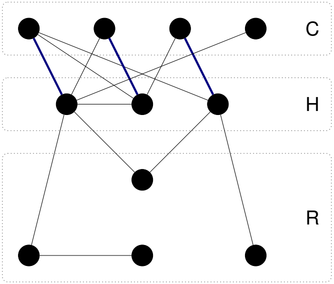
Proof.

Let $\mathbb{A}$ be an algorithm that solves $\langle Q, \kappa \rangle$ in time $f(\kappa(x)) \cdot p(x)$, for all $x \in \Sigma^*$, where $f : \mathbb{N} \to \mathbb{N}$ computable, and $p(n)$ a polynomial. We can assume $p(n) \geq \max\{n, 1\}$ for all $n \in \mathbb{N}$.
If $Q = \varnothing$ or $Q = \Sigma^*$, then we can defined $K(x) := \epsilon$. Otherwise we have $\varnothing \subsetneq Q \subsetneq \Sigma^*$, and we choose some $x_0 \in Q$, and $x_1 \in \Sigma^* \setminus Q$.
We define the polynomial-time computable function $K : \Sigma^* \to \Sigma^*$ by:

$$K(x) := \begin{cases} x_0 & \ldots\ \mathbb{A} \text{ accepts } x \text{ in} \leq p(|x|) \cdot p(|x|) \text{ steps,} \\ x_1 & \ldots\ \mathbb{A} \text{ rejects } x \text{ in} \leq p(|x|) \cdot p(|x|) \text{ steps,} \\ x & \ldots\ \mathbb{A} \text{ does not terminate in} \leq p(|x|) \cdot p(|x|) \text{ steps.} \end{cases}$$

In the last case ($K(x) = x$) we have $p(|x|) \cdot p(|x|) \leq f(\kappa(x)) \cdot p(|x|)$, and hence $|K(x)| = |x| \leq p(|x|) \leq f(\kappa(x))$. Therefore K is a kernel. □

Crown Decomposition and Crown Lemma



A **crown decomposition** of a graph G is a partitioning (C, H, R) of $V(G)$, such that:

1. C is nonempty.
2. C is an independent set.
3. H separates C and R.
4. G contains a matching of H into C.

Crown Lemma ($\Leftarrow$ results by Kőnig, Hall)

Let G be a graph with no isolated vertices and with at least $3k+1$ vertices. There is a polynomial-time algorithm that:

- either finds a matching of size $k+1$ in G;
- or finds a crown decomposition of G.

Exercise

Apply the Crown Lemma to the Vertex Cover Problem.

The (par.) Vertex Cover Problem (smaller kernel)

p-VERTEX-COVER:
Given: A graph G, and an integer k.
Parameter: The integer k.
Question: Does there exists a vertex cover of size at most k?

Rule 1: If G contains an isolated vertex v, delete v from G.
The new instance is $(G - v, k)$

Rule 2: If $|(V(G))| \geq 3k + 1$, apply the Crown Lemma.

- If it returns a matching of size $k + 1$, then conclude that (G, k) is a no-instance
- If it returns a crown decomposition $V(G) = C \cup H \cup R$:
  - Pick the vertices in H in the solution
  - Reduce (G, k) to $(G - H, k - |H|)$
  - Reduce $(G - H, k - |H|)$ to $(G - H - C, k - |H|)$ by using Rule 1 (note that vertices in C are isolated)

Theorem

p-VERTEX-COVER admits a kernel with at most $3k$ vertices.

The (parameterized) Dual-Coloring Problem

p-COLORABILITY:
Given: A graph $G = \langle V, E\rangle$ on n vertices and an integer k.
Parameter: The integer k.
Question: Is G k-colorable?

Definition

Let $k \in \mathbb{N}$. A graph $G = \langle V, E\rangle$ is k-colorable if there is a function $C : V \to \{1, \ldots, k\}$ such that $C(u) \neq C(v)$ for all edges $\{u, v\} \in E$.

Exercise

Obtain a kernel with $O(k)$ vertices using crown decomposition.

The Dual-Coloring Problem

Rule 1: Let $I \subseteq V(G)$ be the isolated vertices. Remove I from G, and color them with one color. The new instance is $(G - I, k)$

Rule 2: Consider graph $\overline{G}(V, \overline{E})$ obtained from G by saying that $e \in \overline{E}$ iff $e \notin E$.
If $|(V(G))| > 3k$, apply the Crown Lemma to $\overline{G}$.

- If it returns a matching of size $k + 1$,
  then conclude that (G, k) is a yes-instance
- If it returns crown decomposition $V(G) = V(\overline{G}) = C \cup H \cup R$:
  - The vertices in H can be saved.
  - Reduce (G, k) to $(G - H - C, k - |H|)$ if $|H| < k$, and otherwise to a yes-instance
  - Note that the vertices in C belong to a clique in $G(V, E)$, that is we need $|C|$ colors, and that we need different colors for R.

Theorem

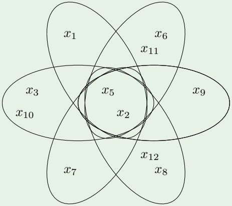
p-DUAL-COLORING *admits a kernel with at most $3k$ vertices.*

Sunflower Lemma

Definition

A sunflower with k petals and a core Y is a collection of sets $S_1, \ldots, S_k$ such that $S_i \cap S_j = Y$ for all $i \neq j$. The sets $S_i \setminus Y$ are petals and they must be non-empty.

A sunflower with 6 petals and a core $Y = \{x_2, x_5\}$.
$S_1 = \{x_2, x_3, x_5, x_{10}\}$
$S_2 = \{x_1, x_2, x_5\}$
$S_3 = \{x_2, x_5, x_6, x_{11}\}$
$\ldots$



Sunflower Lemma (Erdős, Rado)

Let $\mathcal{A}$ be a family of sets (without duplicates) over a universe U such that each set in $\mathcal{A}$ has cardinality $= d$.
If $|\mathcal{A}| > d!(k-1)^d$, then $\mathcal{A}$ contains a sunflower with k petals which can be computed in time polynomial in $|\mathcal{A}|$, $|U|$, and k.

Application to d-Hitting Set

Sunflower Lemma (Erdős, Rado)

Let $\mathcal{A}$ be a family of sets (without duplicates) over a universe U such that each set in $\mathcal{A}$ has cardinality $= d$.
If $|\mathcal{A}| > d!(k-1)^d$, then $\mathcal{A}$ contains a sunflower with k petals which can be computed in time polynomial in $|\mathcal{A}|$, $|U|$, and k.

Parameterized d-Hitting Set Problem

p-d-HITTING-SET:

Given: A family $\mathcal{A}$ of sets over a universe U, where each set has cardinality $\leq d$ and a positive integer k,

Parameter: The integer k.

Question: Does there exists a subset $H \subseteq U$ of size at most k such that H intersects each set in $\mathcal{A}$?

Exercise

Apply the sunflower lemma.

Application to d-Hitting Set

Observation

If $\mathcal{A}$ contains a sunflower $\mathcal{S} = \{S_1, \dots, S_{k+1}\}$ of $k+1$ sets, then every hitting set H of $\mathcal{A}$ with $|H| \leq k$ must intersect the core Y of $\mathcal{S}$. Otherwise it is a no-instance, because H cannot intersect each of the $k+1$ petals $S_i \setminus Y$.

Rule **HS.1**: Let $(U, \mathcal{A}, k)$ be an instance of d-HITTING SET.
Assume that $\mathcal{A}$ contains a sunflower $\mathcal{S} = \{S_1, \dots, S_{k+1}\}$ of cardinality $k+1$ with core Y.
Then return $(U', \mathcal{A}', k)$, where $\mathcal{A}' := (\mathcal{A} \setminus \mathcal{S}) \cup Y$,
$U' := \bigcup \mathcal{A}' = \bigcup_{X \in \mathcal{A}'} X$.

Proof (kernel of p-d-HITTING-SET with $\leq d!k^d d$ sets and $\leq d!k^d d^2$ elements).

If for some $d' \in \{1, ..., d\}$, the number of sets in $\mathcal{A}$ of size $= d'$ is more than $d'!k^{d'}$, then the sunflower lemma yields a sunflower of size $k+1$. Rule **HS.1** applies. By applying this rule exhaustively, we obtain a new family of sets $\mathcal{A}'$ with $\leq d'!k^{d'}$ sets of size $= d'$ for every $d' \in \{1, ..., d\}$. Hence $|\mathcal{A}'| \leq d!k^d d$ and $|U'| = d!k^d d^2$.
If $\varnothing \in \mathcal{A}'$ (a sunflower had an empty core), then it is a no instance. $\square$

Tomorrow

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results		**Algorithmic Meta-Theorems**		
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	GDA	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	GDA	GDA
Algorithmic Techniques		*Formal-Method* & *Algorithmic Techniques*		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width			**FPT-Intractability Classes & Hierarchies**
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies