Lecture 2: Graph width notions, dynamical programming

An Introduction to Parameterized Complexity

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Ph.D. Program, Advanced Period

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L'Aquila, Italy

Tuesday, June 17, 2025

Course overview

Monday, June 16 10.30 – 12.30	Tuesday, June 17 10.30 – 12.30	Wednesday, June 18	Thursday, June 19 10.30 – 12.30	Friday, June 20
Algorithmic Techniques			*Formal-Method* & *Algorithmic Techniques*	
Introduction & basic FPT results	**Notions of bounded graph width**		**Algorithmic Meta-Theorems**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width		1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	(Fair Division)
				14.30 – 16.30
				FPT-Intractability Classes & Hierarchies
			(Fair Division)	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Overview

- comparing parameterizations

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- dynamical programming on trees, example:
  - WEIGHTED-INDEPENDENT-SET (and VERTEX-COVER)
- path-width
  - example: fpt-algorithm for bounded path-width
- tree-width
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- fpt-results for other problems, obtained similarly
- other notions of width
  - clique-width
  - using f-width to define:
    - carving-width (and cut-width)
    - branch-width
    - rank-width

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- comparing width-notions

Fixed-Parameter tractable

A *parameterized problem* is a triple $\langle Q, \Sigma, \kappa \rangle$ (short: $\langle Q, \kappa \rangle$) where:

- ▷ $Q \subseteq \Sigma^*$ is the set of *(classical) problem instances*,
- ▷ $\kappa : \Sigma^* \to \mathbb{N}$ is a (general) function, *the parameterization*.

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Parameterized problem $\langle Q, \Sigma, \kappa \rangle$

Instance: $x \in \Sigma^*$.

Parameter: $\kappa(x)$.

Problem: Is $x \in Q$?

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Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *fixed-parameter tractable* (is in FPT) if:

$\exists f : \mathbb{N} \to \mathbb{N}$ computable $\exists p \in \mathbb{N}[X]$ polynomial

$\exists \mathbb{A}$ algorithm, takes inputs in Σ^*

$\forall x \in \Sigma^* [\ \mathbb{A}$ decides whether $x \in Q$ holds in time $\leq f(\kappa(x)) \cdot p(|x|)\ ]$

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$$\exists \mathbb{A} \text{ algorithm, takes inputs in } \Sigma^*$$
$$\forall x \in \Sigma^* \big[\, \mathbb{A} \text{ decides whether } x \in Q \text{ holds in time} \le f(\kappa(x)) \cdot p(|x|) \,\big]$$

†) Assumptions for a robust fpt-theory

$\kappa(x)$ is polynomially computable, or itself fpt-computable: for all $x \in \Sigma^*$ in time $\le g(\kappa(x)) \cdot q(|x|)$ for g computable, $q \in \mathbb{N}[X]$.

Comparing parameterizations

Definition (computably bounded below)

Let $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ parameterizations.

- $\kappa_1 \succeq \kappa_2 \;:\Longleftrightarrow\; \exists g : \mathbb{N} \to \mathbb{N}$ computable $\forall x \in \Sigma^* \big[\, g(\kappa_1(x)) \geq \kappa_2(x) \,\big]$.
- $\kappa_1 \approx \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \kappa_2 \succeq \kappa_1$.
- $\kappa_1 \succ \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \neg\big(\kappa_2 \succeq \kappa_1\big)$.

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- $\kappa_1 > \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \geq \kappa_2 \;\wedge\; \neg\big(\kappa_2 \geq \kappa_1\big)$.

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ with $\kappa_1 \geq \kappa_2$:

$$\langle Q, \kappa_1 \rangle \in \mathrm{FPT} \quad \Longleftarrow \quad \langle Q, \kappa_2 \rangle \in \mathrm{FPT}$$

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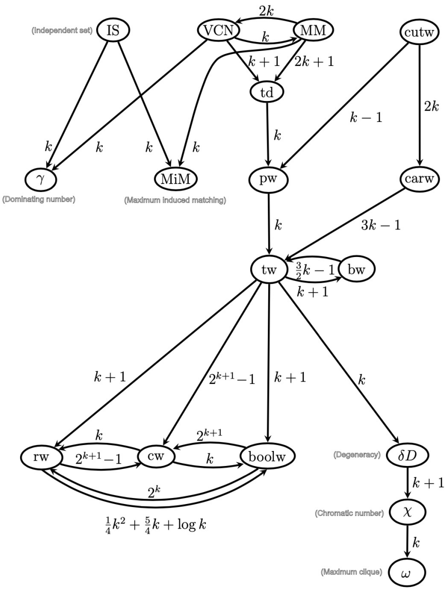
$$\langle Q, \kappa_1\rangle \in \text{FPT} \quad \Longleftarrow \quad \langle Q, \kappa_2\rangle \in \text{FPT}$$
$$\langle Q, \kappa_1\rangle \notin \text{FPT} \quad \Longrightarrow \quad \langle Q, \kappa_2\rangle \notin \text{FPT}$$

Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \succeq wd_2 \;:\Leftrightarrow\; wd_1 \xrightarrow{g} wd_2$$



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  transfer upwards
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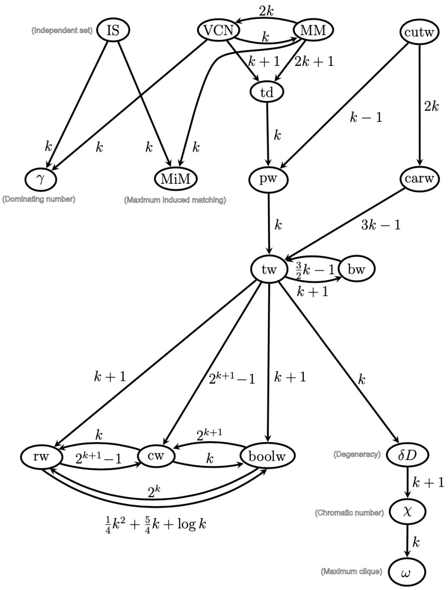

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- FPT-results
  transfer upwards
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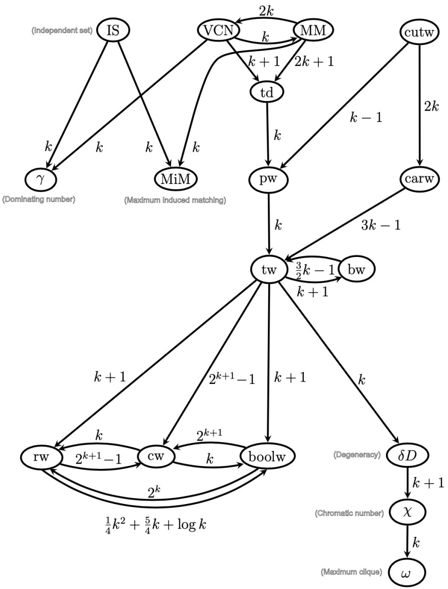
- ($\notin$ FPT)-results
  transfer downwards
  (and along $\xrightarrow{g}$)



You Always Walk Alone (with your children)

Attività motoria con i figli:

'la possibilità di uscire con i figli minori è consentita a un solo genitore per camminare purché questo avvenga in prossimità della propria abitazione'

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PHYSICAL-DISTANCE-WALKING

Instance: Graph $\mathcal{G} = \langle V, E\rangle$ with V people who want to go for a walk in the next hour in a radius of 200m of their home, and edges in E between them if they live closer than 400m of each other. A number $\ell \in \mathbb{N}$.

Problem:

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corresponds to: INDEPENDENT-SET

Weighted Independent Set, and Vertex Cover

Let $\mathcal{G} = \langle V, E\rangle$ a graph. For all $S \subseteq V$:

S is independent set in $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \in E\,(\,\neg(u \in S \wedge v \in S)\,)$
$\Longleftrightarrow \forall e = \{u, v\} \in E\,(\,u \notin S \vee v \notin S)\,)$

WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E\rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

Problem: What is the max. weight of an independent set of $\mathcal{G}$?

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$S \subseteq V$ is *minimal* vertex cover $\Longleftrightarrow$ $V \smallsetminus S$ is *maximal* independent set
Hence: solution of WEIGHTED-INDEPENDENT-SET
$\Longrightarrow$ solution of VERTEX-COVER.

Weighted Ind. Set / Vertex Cover, width-parameterized

p^*-WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

Parameter: path-width / tree-width k.

Problem: What is the max. weight of an independent set of $\mathcal{G}$?

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Dynamical programming on trees (example)

WEIGHTED-INDEPENDENT-SET

Instance: A tree $\mathcal{T} = \langle T, F \rangle$, and a weight function $w : T \to \mathbb{R}_0^+$.

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- in leafs: $B[v] = 0$, $A[v] = \boldsymbol{w}(v)$.
- for inner vertices v with children $v_1, \ldots, v_q$:

$$B[v] = \sum_{i=1}^{q} A[v_i],$$

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Solution: value of $A[r]$, can be computed bottom-up in linear time.

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Theorem

On trees with n nodes,

WEIGHTED-INDEPENDENT-SET $\in$ DTIME$(O(n))$.

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Corollary

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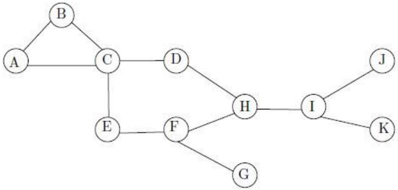
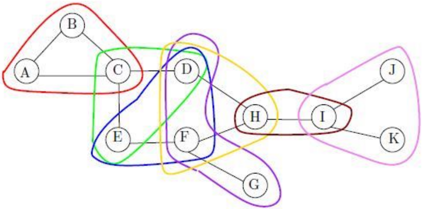
Path-decomposition (example)





Path decompositions, and path-width

Definition (Robertson–Seymour, 1983)

A *path decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a sequence $\langle B_1, B_2, \ldots, B_r \rangle$ of bags $B_i \subseteq V$ such that:

(P1) $V = \bigcup_{i=1}^{r} B_i$ (every vertex of $\mathcal{G}$ is in some bag).

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(P2) $(\forall \{u, v\} \in E) (\exists i \in \{1, 2, \ldots, r\}) \big[\{u, v\} \subseteq B_i \big]$
(every edge of $\mathcal{G}$ is realized in some bag).

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(P3) $(\forall v \in V)\, (\exists i, k \in \{1, \ldots, r\}\,,\, i \leq k) \big[\, \{j \mid v \in B_j\} = [i, k] \,\big]$
(the list of bags that contains a vertex of $\mathcal{G}$ is $\langle B_i, \ldots, B_k \rangle$ for some interval $[i, k]$)

Path decompositions, and path-width

Definition (Robertson–Seymour, 1983)

A *path decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a sequence $\langle B_1, B_2, \ldots, B_r \rangle$ of bags $B_i \subseteq V$ such that:

(P1) $V = \bigcup_{i=1}^{r} B_i$ (every vertex of $\mathcal{G}$ is in some bag).

(P2) $(\forall \{u, v\} \in E) (\exists i \in \{1, 2, \ldots, r\}) [\, \{u, v\} \subseteq B_i \,]$
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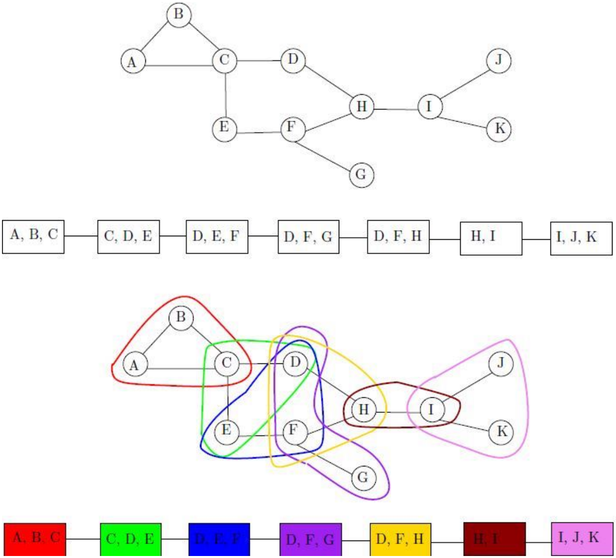
The *path-width* $pw(\mathcal{G})$ of a graph $\mathcal{G} = \langle V, E \rangle$ is defined by:

$pw(\mathcal{G})$:= minimal width of a path decomposition of $\mathcal{G}$.

Path-decomposition (example)



Path decomposition defines separations

Lemma

Let $\langle B_1, B_2, \ldots, B_r \rangle$ *be a path decomposition of a graph* $\mathcal{G} = \langle V, E \rangle$. *Then for all* $i \in \{1, \ldots, r-1\}$ *it holds:*

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  $A \cap B$ is called the *separator* of a separation $\langle A, B \rangle$, and $|A \cap B|$ is called its *order*.

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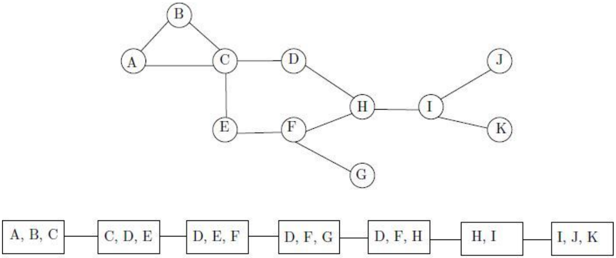
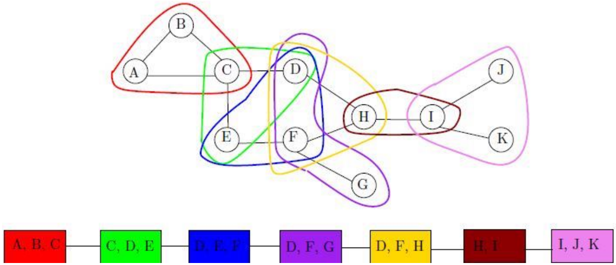
- The *border (set of border vertices)* $\partial(A)$ *of* a set $A \subseteq V$ of vertices consists of all vertices that have a neighbor in $V \setminus A$. Note that:
  - $\partial(A) = \partial(V \setminus A)$.
  - $\langle A, (V \setminus A) \cup \partial(A) \rangle$ is a separation of $\mathcal{G}$, for all $A \subseteq V$.

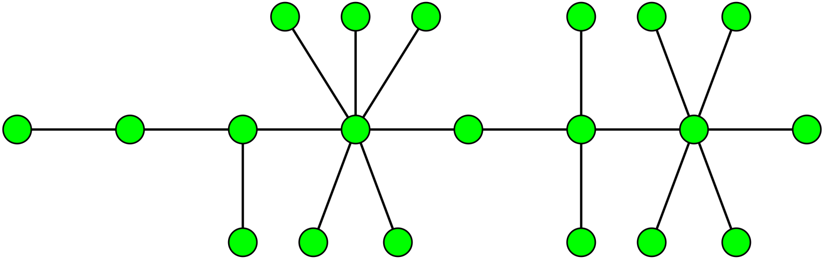
Path-decomposition (example)





Caterpillar

Path-width?



Nice path decomposition

Definition

A *path decomposition* $\langle B_1, B_2, \ldots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if:

- $B_1 = B_r = \varnothing$
- Every index $i > 1$ is either of:
  - introduce index: there is $v \in V$ such that $B_{i+1} = B_i \cup \{v\}$ and $v \notin B_i$,
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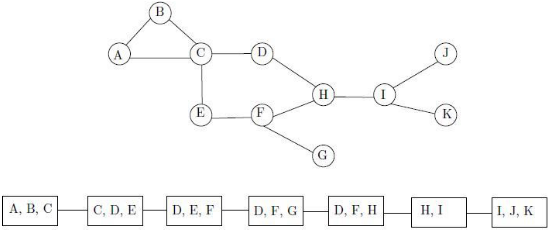

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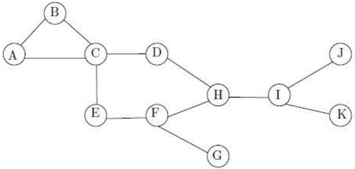

[A, B, C] — [C, D, E] — [D, E, F] — [D, F, G] — [D, F, H] — [H, I] — [I, J, K]

Nice path decomposition:

[∅] — [A] — [A,B] — [A,B,C] — [B,C] — [C] — [C,D] — [C,D,E] — [D,E] — [D,E,F] —

— [D,F] — [D,F,G] — [D,F] — [D,F,H] — [F,H] — [H] — [H,I] — [I] — [I,J] — [I,J,K] — [J,K] — [K] — [∅]

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Lemma

From every path decomposition $\langle B_1, B_2, \ldots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ of width k a nice path decomposition $\langle B'_1, B'_2, \ldots, B'_{r'} \rangle$ of width k can be constructed in time $O(k^2 \cdot \max\{r, n\})$ where $n := |V|$.

A,B,C — C,D,E — D,E,F — D,F,G — D,F,H — H,I — I,J,K

∅ — A — A,B — A,B,C — B,C — C — C,D — C,D,E — D,E — D,E,F —

— D,F — D,F,G — D,F — D,F,H — F,H — H — H,I — I — I,J — I,J,K — J,K — K — ∅

Weighted Independent Set

Let $\mathcal{G} = \langle V, E \rangle$ a graph.

$S \subseteq V$ is independent set in $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \big(\neg(u \in S \wedge v \in S) \big)$.

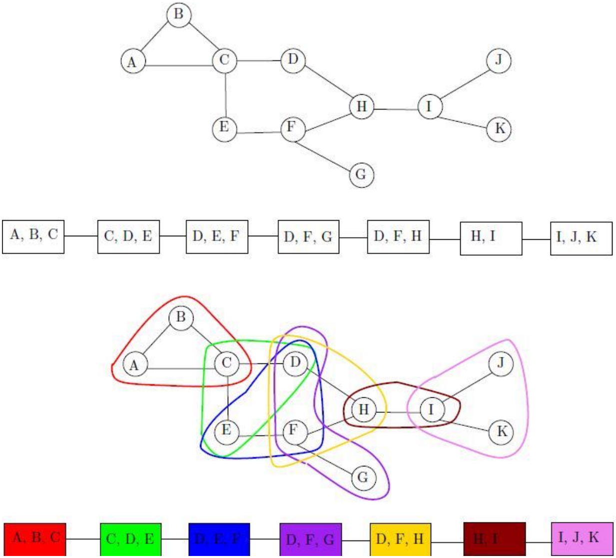
WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

Parameter: path-width k.

Problem: What is the max. weight of an independent set of $\mathcal{G}$?

Path-decomposition (example)



Dyn. programming using path-width (Weigh. Ind. Set)

Let $\langle B_1, \ldots, B_r \rangle$ be a nice path decomposition of $\mathcal{G} = \langle V, E \rangle$.
Then for every $i \in \{1, \ldots, r\}$, and every $S \subseteq B_i$, we define:

$$c[i,S] := \begin{cases} -\infty & \text{if } S \text{ is not independent} \\ \text{maximum possible weight of a set } \hat{S} \text{ such that} & \\ \hat{S} \text{ is independent} \wedge S \subseteq \hat{S} \subseteq V_i = \bigcup_{j=1}^{i} B_j \wedge \hat{S} \cap B_i = S & \\ & \text{if } S \text{ is independent.} \end{cases}$$

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- Case $i = 1$: $c[1,\varnothing] = 0$

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For every $i \in \{1, \ldots, r\}$, and every independent $S \subseteq B_i$, we define:

$$c[i,S] := \begin{cases} \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent } \wedge\ S \subseteq \hat{S} \subseteq V_i = \bigcup_{j=1}^{i} B_j\ \wedge\ \hat{S} \cap B_i = S \end{cases}$$

Time Complexity: Based on the values of $c[i,S]$, the maximum possible weight of an independent set $S \subseteq V$ can be computed as:

$$= \max_{S \subseteq B_r} c[r,S] \quad = c[r, \varnothing]$$

Then for all $i \in \{1, \ldots, n\}$:

- $|B_i| \le k + 1$,
- $\Rightarrow$ number of values $c[i,S]$ at index i : $2^{|B_i|} = 2^{k+1}$,
- $\Rightarrow$ adjacency/independence check for $S \subseteq B_t$ possible in: $O(k^2)$ using a datastructure computable in time $O(k^{O(1)} \cdot n)$,
- time for comp. a value at i, using map of values at $i-1$: $\sim O(k)$
- time for comp. all values at i, using values at $i-1$: $2^{k+1} \cdot O(k^2)$

$\Rightarrow$ the time for computing all values at r :

$$(2^{k+1} \cdot O(k^2)) \cdot r + O(k^{O(1)} \cdot n) \in O(2^k \cdot k^{O(1)} \cdot n), \text{ since } r = 2n.$$

Dynamical programming with path width (example)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and path-width pw$(\mathcal{G}) = k$,

p^*-WEIGHTED-INDEPENDENT-SET $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$.

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Corollary

For every graph $\mathcal{G} = \langle V, E \rangle$ *with* $|V| = n$ *and path-width pw*$(\mathcal{G}) = k$,

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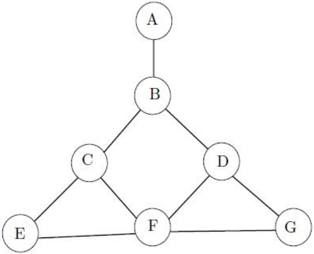
Tree decomposition (example)



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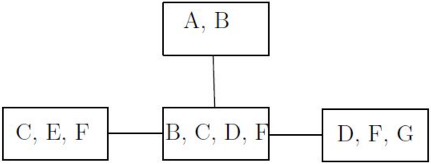
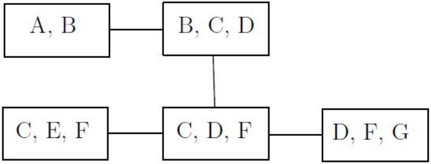


A tree-decompostion of width 3



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Tree decompositions, and tree-width

Definition (Bertelé–Brioschi, 1972, Halin, 1976, Robertson–Seymour, 1984)

A *tree decomposition* of a graph $\mathcal{G} = \langle V, E\rangle$ is a pair $\langle \mathcal{T}, \{B_t\}_{t\in T}\rangle$ where $\mathcal{T} = \langle T, F\rangle$ a (undirected, unrooted) tree, and $B_t \subseteq V$ such that:

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The *tree-width* $tw(\mathcal{G})$ of a graph $\mathcal{G} = \langle V, E\rangle$ is defined by:

$$tw(\mathcal{G}) := \text{minimal width of a tree decomposition of } \mathcal{G}.$$

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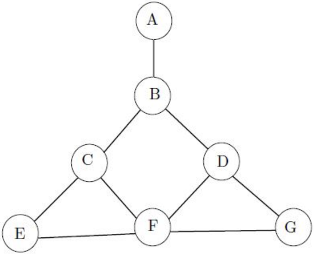

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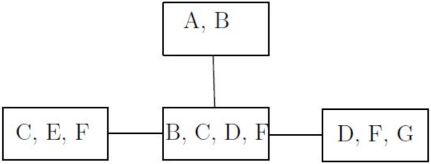

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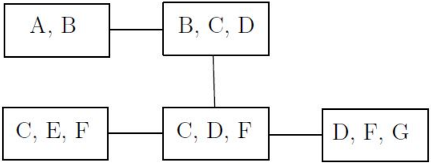

A tree-decompostion of width 2

Tree decomposition defines separations

Lemma

Let $\langle \mathcal{T}, \{B_t\}_{t\in T}\rangle$ be a tree decomposition of a graph $\mathcal{G} = \langle V, E\rangle$.
Let $e = \langle a, b\rangle$ be an edge of $\mathcal{T}$. The $\mathcal{T} \smallsetminus e$ is the union of a tree $\mathcal{T}_a$ containing a, and a tree $\mathcal{T}_b$ containing b.
Then for $A := \bigcup_{t\in V(\mathcal{T}_a)} B_t$ and $B := \bigcup_{t\in V(\mathcal{T}_b)} B_t$ it holds:

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Recall, for a graph $\mathcal{G} = \langle V, E\rangle$:

- A pair $\langle A, B\rangle$ of subsets $A, B \subseteq V$ is a *separation* of $\mathcal{G}$ if:
  - $V = A \cup B$
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  $A \cap B$ is called the *separator* of a separation $\langle A, B\rangle$, and $|A \cap B|$ is called its *order*.

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  and $|A \cap B|$ is called its *order*.
- The *border (vertices) $\partial(A)$ of* a set $A \subseteq V$ of vertices consists of all vertices that have a neighbor in $V \setminus A$.

Computing tree-width

TREE-WIDTH
Instance: A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Problem: Decide whether $tw(\mathcal{G}) = k$.

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Theorem

p-TREE-WIDTH *is fixed-parameter tractable,*
in time $2^{p(k)} \cdot n$ *where* $n := |V|$.

Nice tree decomposition

Definition

A *tree decomposition* $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ of graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if it is based on the choice of a leaf as *root* r and the parent–children relation away from r such that:

- $B_r = \varnothing$, and $B_\ell = \varnothing$ for every leaf $\ell \in T$.
- Every non-leaf node $t \in T$ is of one of three types:
  - introduce node: t has exactly one child t' such that $B_t = B_{t'} \cup \{v\}$; we say v is introduced at t.
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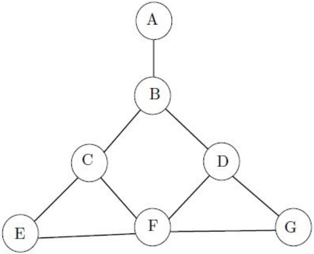
Lemma

From every tree decomposition $\langle \mathcal{T}, \{B_t\}_{t\in T}\rangle$ of a graph $\mathcal{G} = \langle V, E\rangle$ of width k a nice tree decomposition $\langle \mathcal{T}', \{B'_t\}_{t\in T'}\rangle$ of width k and with $r := |V(\mathcal{T})| \in O(kn)$ vertices can be constructed in time $O(k^2 \cdot \max\{r, n\})$ where $n := |V|$.

Tree decomposition (example)



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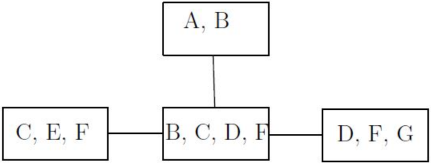

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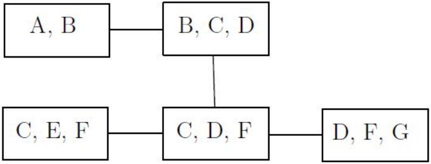

A tree-decompostion of width 2

Weighted Independent Set

Let $\mathcal{G} = \langle V, E \rangle$ a graph.

$S \subseteq V$ is independent set in $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \big(\neg(u \in S \wedge v \in S) \big)$.

WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

Parameter: tree-width k.

Problem: What is the max. weight of an independent set of $\mathcal{G}$?

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For every node t of a nice tree decomposition, and every $S \subseteq B_t$, we define:

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$\Rightarrow$ the time for computing all values at the root r :
$(2^{k+1} \cdot O(k^2)) \cdot |T| + O(k^{O(1)} \cdot n) \in O(2^k \cdot k^{O(1)} \cdot n)$, since $|T| \in O(k \cdot n)$.

Dynamical programming with tree width (example)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and tree-width tw$(\mathcal{G}) = k$,

$$p^*\text{-WEIGHTED-INDEPENDENT-SET} \in \mathrm{DTIME}(2^k \cdot k^{O(1)} \cdot n).$$

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Corollary

For every graph $\mathcal{G} = \langle V, E\rangle$ with $|V| = n$ and tree-width tw($\mathcal{G}$) = k,

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Dyn. programming with tree-width: general strategy

We consider problem P for graphs $\mathcal{G} = \langle V, E \rangle$ of size n and nice tree decompositions $\langle \mathcal{T} = \langle T, F, r \rangle, \{B_t\}_{t \in T} \rangle$ of tree width k.

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- Obtain an estimate of the time needed to compute the properties in a node t depending on n and k.
- Sum up the time needed to compute the solution(s) at root r of $\mathcal{T}$.
- Add time needed to obtain the solution of P from properties at r.

Dynamical programming: similar results (I)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ *with* $|V| = n$ *and tw*$(\mathcal{G}) = k$*,*

- p^*-VERTEX-COVER, INDEPENDENT-SET $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$,
- p^*-DOMINATING-SET $\in$ DTIME$(4^k \cdot k^{O(1)} \cdot n)$,
- p^*-ODD CYCLE TRAVERSAL $\in$ DTIME$(3^k \cdot k^{O(1)} \cdot n)$,
- p^*-MAXCUT $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$,
- p^*-q-COLORABILITY $\in$ DTIME$(q^k \cdot k^{O(1)} \cdot n)$.

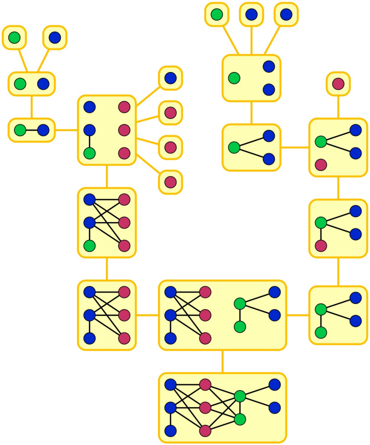
Dynamical programming: similar results (II)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and tw$(\mathcal{G}) = k$, the following problems are in DTIME$(k^{O(k)} \cdot n)$ *:*

- p^*-STEINER-TREE,
- p^*-FEEDBACK-VERTEX-SET,
- p^*-HAMILTONIAN-PATH *and* p^*-LONGEST-PATH,
- p^*-HAMILTONIAN-CYCLE *and* p^*-LONGEST-CYCLE,
- p^*-CHROMATIC-NUMBER,
- p^*-CYCLE-PACKING,
- p^*-CONNECTED-VERTEX-COVER,
- p^*-CONNECTED-FEEDBACK-VERTEX-SET.

Clique width (example)



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For $k \in \mathbb{N}$, the *k-expressions* are defined by:

$$\varphi, \varphi_1, \varphi_2 ::= \boldsymbol{i} \mid \mathsf{edge}_{i-j}(\varphi) \mid \mathsf{recolor}_{i\to j}(\varphi) \mid (\varphi_1 \oplus \varphi_2)$$

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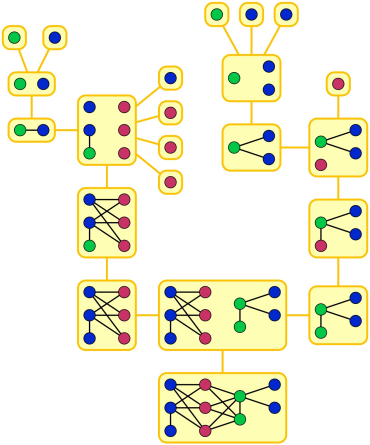
Definition (Courcelle, Engelfriet, Rozenberg, 1993, [2])

The *clique-width clw*$(\mathcal{G})$ of $\mathcal{G} = \langle V, E \rangle$ is defined by:

clw$(\mathcal{G})$:= the least $k \in \mathbb{N}$ such that, for some k-expression φ, $\mathcal{G} = \mathcal{G}(\varphi)$ (when removing colors)

Clique width (example)

Building a graph $\mathcal{G}$ of clique-width *clw*$(\mathcal{G}) = 3$:



Clique-Width (examples, properties, computability)

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- Every graph property expressible in MSO (monadic second-order logic) can be decided in linear time w.r.t. the graph's clique-width.

f-Width (of sets)

By a *cut function* or a *connectivity function* we mean a function $f : 2^U \to \mathbb{R}_0^+$ such that:

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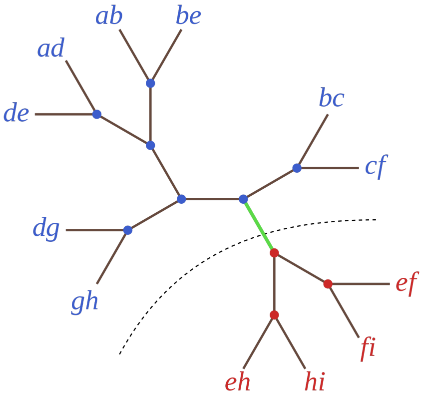

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The *width* of an edge $e \in T$ (with respect to f) is $f(X_e) = f(Y_e)$. The *width of $\langle \mathcal{T}, \eta \rangle$ w.r.t. f* is the maximum width over the edges of $\mathcal{T}$.

f-Width (of sets)

By a *cut function* or a *connectivity function* we mean a function $f : 2^U \to \mathbb{R}_0^+$ such that:

$$f \text{ is } \textit{symmetric} :\Longleftrightarrow \forall X \subseteq U \left[f(X) = f(U \smallsetminus X) \right];$$

$$f \text{ is } \textit{fair} :\Longleftrightarrow f(\varnothing) = f(U) = 0\,.$$

Definition

Let U be a set, $f : 2^U \to \mathbb{R}_0^+$ a cut function.
A *branch decomposition* of U is a pair $\langle \mathcal{T}, \eta \rangle$ where:

- ▷ $\mathcal{T} = \langle T, F \rangle$ a tree.
- ▷ $\eta : U \to \textit{Leafs}(\mathcal{T})$ a bijective function.

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The f*-width* $w_f(U)$ of U is defined as:

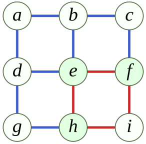
$$w_f(U) := \underline{\text{minimum}}\ \text{width of branch decomp's of } U \text{ w.r.t. } f.$$

Branch-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph. The *border (vertices)* of a set $X \subseteq E$ of edges is defined by:

$$\partial(X) := \left\{ v \in V \mid \exists e_1 \in X \; \exists e_2 \in E \setminus X \; \left[v \text{ is incident to } e_1 \text{ and } e_2 \right] \right\}$$



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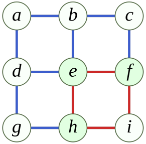
$$bw(\mathcal{G}) := w_f(E) \quad \text{for} \quad f : 2^E \to \mathbb{R}_0^+,\ X \mapsto |\partial(X)|$$



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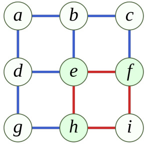

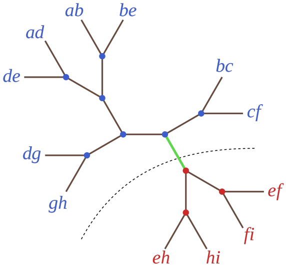

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$$\textit{bw}(\mathcal{G}) := w_f(E) \quad \text{for} \quad f : 2^E \to \mathbb{R}_0^+,\; X \mapsto |\partial(X)|$$

Proposition

bw$(\mathcal{G}) \approx$ *tw*$(\mathcal{G})$, for every graph; more precisely:

$$\textit{bw}(\mathcal{G}) \le \textit{tw}(\mathcal{G}) + 1 \le \tfrac{3}{2} \cdot \textit{bw}(\mathcal{G}) \,.$$

Rank-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.
For $X \subseteq V$ we define the $GF(2)$-matrix:

$$B_{\mathcal{G}}(X) := (b_{x,y})_{x \in X, y \in V \setminus X}\,, \text{ where, for all } x \in X, y \in V \setminus X\text{:}$$
$$b_{x,y} = 1 \iff \{x, y\} \in E\,.$$

($B_{\mathcal{G}}(X)$ is the adjacency matrix of the bipartite graph induced by $\mathcal{G}$ between X and $V \setminus X$.)

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The *rank-width rw*($\mathcal{G}$) of a graph $\mathcal{G} = \langle G, E \rangle$ is:

$$\mathit{rw}(\mathcal{G}) := w_{\rho_{\mathcal{G}}}(E) \quad \text{for} \quad \rho_{\mathcal{G}} : 2^V \to \mathbb{N}_0,\ X \mapsto \text{rank of } B_{\mathcal{G}}(X)$$

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Properties

- *rw*$(\mathcal{G}) \le$ *tw*$(\mathcal{G})$.
- tree-width cannot be bounded functionally by rank-width: *rw*$(K_n) = 1$, but *tw*$(K_n) = n - 1$.

Carving-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.
For $X \subseteq V$ the *edge-cut* of X is:

$$\mathit{cut}_{\mathcal{G}}(X) := \{e = \{u, v\} \in E \mid u \in X, v \in V \setminus X\} .$$

The *carving-width* $\mathit{carw}(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is:

$$\mathit{carw}(\mathcal{G}) := w_{\mathit{cut}}(E) \quad \text{for} \quad \mathit{cut} : 2^V \to \mathbb{N}_0,\ X \mapsto |\mathit{cut}_{\mathcal{G}}(X)| .$$

Carving-Width and Cut-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.
For $X \subseteq V$ the *edge-cut* of X is:

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The *carving-width* $\textit{carw}(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is:

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Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph with $n = |V|$.
For a permutation $\pi : \{1, \ldots, n\} \to V$ on V we define:

$$\textit{width}(\pi) := \max_{1 \le i \le n} \textit{cut}_{\mathcal{G}}(\{\pi(j) \mid 1 \le j \le i\}) .$$

The *cut-width* $\textit{cutw}(\mathcal{G})$ of $\mathcal{G}$ is:

$$\textit{cutw}(\mathcal{G}) := \min_{\pi \text{ perm. of } V} \textit{width}(\pi) .$$

Coverage in Multi-Interface Networks

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CMI(p) (for $p \in \mathbb{N}$)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, $W : V \to 2^{\{1,\dots,a\}}$ available-interface allocation, $c : \{1, \dots, a\} \to \mathbb{R}^{+}$ interface cost function.

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Problem: Obtain, if possible, a minimal solution with respect to the total cost of the interfaces that are activated, that is, $c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i)$.

Coverage in Multi-Interface Networks

Theorem

CMI$(2) \in$ NP-complete, *also for graphs with max. node degree* ≥ 4.

Coverage in Multi-Interface Networks (parameterized)

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p^*-*CMI*(p) (for $p \in \mathbb{N}$)

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Theorem (Aloisio, Navarra, 2020, [1])

- *For path-width pw*$(\mathcal{G}) = k$,
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Corollary

$(p^*)'$*-CMI*$(p) \in$ FPT.

Comparing parameterizations

Definition (computably bounded)

Let $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ parameterizations.

- $\kappa_1 \succeq \kappa_2 \;:\Longleftrightarrow\; \exists g : \mathbb{N} \to \mathbb{N}$ computable $\forall x \in \Sigma^* \big[\, g(\kappa_1(x)) \geq \kappa_2(x) \,\big]$.
- $\kappa_1 \approx \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \kappa_2 \succeq \kappa_1$.
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Proposition

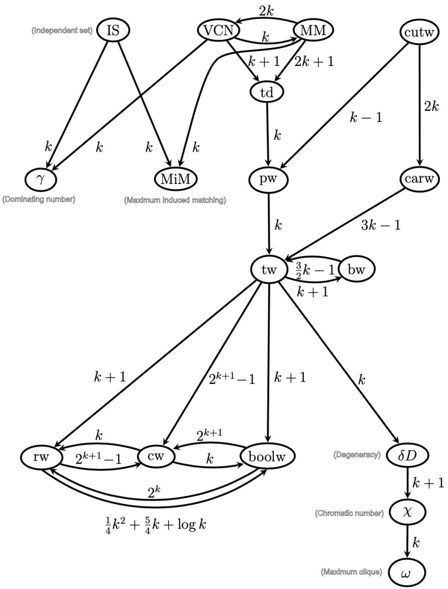
For all parameterized problems $\langle Q, \kappa_1\rangle$ and $\langle Q, \kappa_2\rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ with $\kappa_1 \succeq \kappa_2$:

$$\langle Q, \kappa_1\rangle \in \mathrm{FPT} \quad \Longleftarrow \quad \langle Q, \kappa_2\rangle \in \mathrm{FPT}$$
$$\langle Q, \kappa_1\rangle \notin \mathrm{FPT} \quad \Longrightarrow \quad \langle Q, \kappa_2\rangle \notin \mathrm{FPT}$$

Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \succeq wd_2 \;:\Leftarrow\; wd_1 \xrightarrow{g(k)} wd_2$$



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- FPT-results
  transfer upwards
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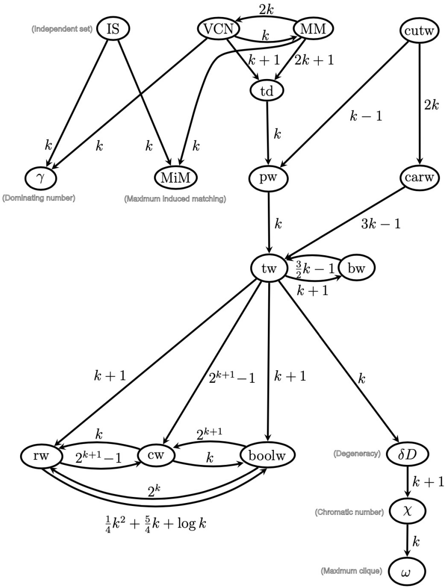

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  transfer upwards
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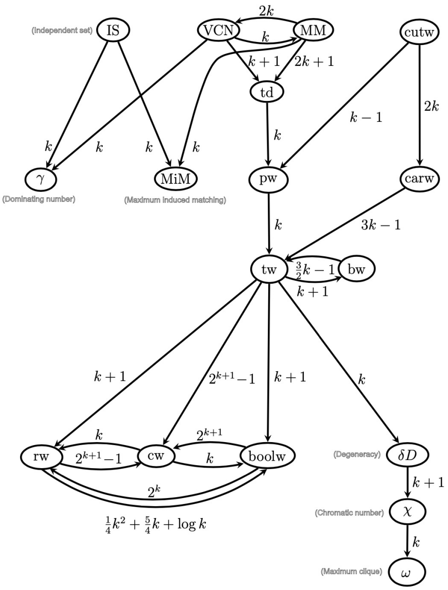
- ($\notin$ FPT)-results
  transfer downwards
  (and along $\xrightarrow{g}$)



Summary

- comparing parameterizations
- dynamical programming on trees, example:
  - WEIGHTED-INDEPENDENT-SET (and VERTEX-COVER)
- path-width
  - example: fpt-algorithm for bounded path-width
- tree-width
  - example: fpt-algorithm for bounded path-width
- fpt-results for other problems, obtained similarly
- other notions of width
  - clique-width
  - using f-width to define:
    - carving-width (and cut-width)
    - branch-width
    - rank-width
- example problem: coverage in multi-interface networks
- comparing width-notions

Course overview

Monday, June 16 10.30 – 12.30	Tuesday, June 17 10.30 – 12.30	Wednesday, June 18	Thursday, June 19 10.30 – 12.30	Friday, June 20
Algorithmic Techniques			*Formal-Method* & *Algorithmic Techniques*	
Introduction & basic FPT results	**Notions of bounded graph width**		**Algorithmic Meta-Theorems**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width		1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	(Fair Division)
				14.30 – 16.30
				FPT-Intractability Classes & Hierarchies
			(Fair Division)	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Thursday

- recalling notions from logic:
  - propositional, and first-order logic
  - monadic second-order logic (MSO)
- Courcelle's Theorem: obtaining FPT-results by
  - model-checking of MSO-properties
    on graphs and structures of bounded tree-/clique-width

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