

An Introduction to Parameterized Complexity

Lecture 1: Fixed-Parameter Tractability

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Ph.D. Program, Advanced Period

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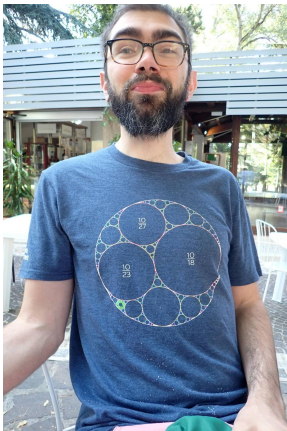
L'Aquila, Italy

Monday, June 10, 2024

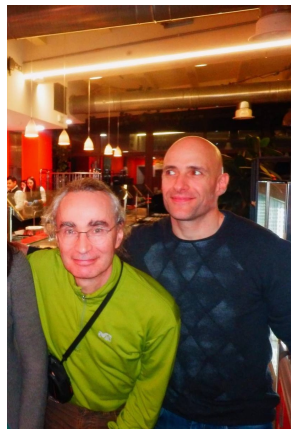
Course overview

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results motivation for FPT kernelization, Crown Lemma, Sunflower Lemma		Algorithmic Meta-Theorems 1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width		
	GDA		GDA	GDA
<i>Algorithmic Techniques</i>		<i>Formal-Method & Algorithmic Techniques</i>		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width			FPT-Intractability Classes & Hierarchies
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Course developers



Hugo Gilbert
course 2019/20 (Hugo & Clemens)



CG & Alessandro Aloisio
course 2020/21 (Alessandro & C)

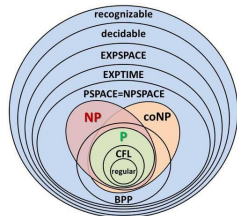
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Motivation

Classical complexity theory

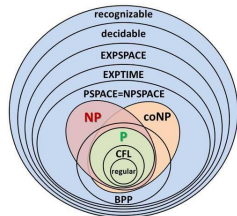
- ▶ analyses problems by **resource** (**space** or **time**)
needed to solve them on a **reasonable machine model**
- ▶ as a function of the **input size** $n = |x|$ (Hartmanis/Stearns, 1965)
- ⇒ variety of **complexity classes**
(P, LOGSPACE, NP, PSPACE, ...)
- ⇒ **tractable problems**
= polynomial-time computable (in P)
- ⇒ **theory of intractability**
(reductions, NP completeness)



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Drawback

- ▶ measures problem size $n = |x|$
only in terms of input instances x ,
and **ignores structural information** about instances
- ▶ sometimes problems are **easier to solve**
for instances if additional structure information is available

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Parameterized complexity

- ▶ measures complexity also in terms of a parameter $k = \kappa(x)$
that may depend on the input x in an arbitrary way
- ⇒ **fixed-parameter tractable problems**
relaxes polynomial time solvability to algorithms whose
non-polynomial behavior $f(k) \cdot p(n)$ is restricted by parameter k
- ⇒ complexity classes (FPT, XP, W[P], W- and A-hierarchies)
- ⇒ **theory of fixed-parameter intractability**

Parameterized (versus classical) problems

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Assumption

The parameterization κ can be **efficiently** computed.

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The **size** of an instance $\langle x, \kappa(x) \rangle$ of $\langle Q, \kappa \rangle$ is

$$|\langle x, \kappa(x) \rangle| = |x| + \kappa(x).$$

Parameterized problems (examples)

A Parameterized Clique Problem

p-CLIQUE:

Given: a graph G and an integer k ,

Question: Does there exist a clique of size k in G ?

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Question: Does there exists a set $S \subseteq U$ such that $|S| \leq k$ and $S \cap S_i \neq \emptyset, \forall i \in \{1, \dots, m\}$.

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- ▶ is **fixed-parameter tractable**.

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There is a hierarchy on parameters.

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- ▶ Some more **structural property** of the instance.
E.g., the diameter of a graph.
- ▶ It can be a **combination** of values, a **difference**, ...

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- ▶ **Social choice problems**: number of voters, candidates, correlation of preferences...
- ▶ **Boolean formulas**: number of variables, number of clauses...
- ▶ **Problems on strings**: maximum length of a string, size of the alphabet...

Fixed Parameter Tractability (Class FPT)

Definition

A parameterized problem $\langle Q, \kappa \rangle$ is *fixed-parameter tractable* if:

$\exists f : \mathbb{N} \rightarrow \mathbb{N}$ computable $\exists p \in \mathbb{N}[X]$ polynomial
 $\exists \mathbb{A}$ algorithm, takes inputs in Σ^* and $\forall x \in \Sigma^*$
 $\left[\mathbb{A} \text{ decides if } x \in Q \text{ in time } \leq f(\kappa(x)) \cdot p(|x|) \right].$

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Goal in parameterized algorithmics:

- \Rightarrow design FPT algorithms,
- \Rightarrow try to make both factors $f(\kappa(x))$ and $p(|x|)$ as small as possible.
- \Rightarrow or show (if possible) that finding such factors is impossible

Slices of FPT problems are in P

The ℓ -th slice of a parameterized problem $\langle Q, \kappa \rangle$:

$$\langle Q, \kappa \rangle_{\ell} := \{x \in Q \mid \kappa(x) = \ell\} \quad (\text{as classical problem}).$$

Proposition

If $\langle Q, \kappa \rangle \in \text{FPT}$, then $\langle Q, \kappa \rangle_{\ell} \in \text{P}$ for all $\ell \in \mathbb{N}$.

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A problem not in FPT (unless $P = NP$)

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Application

p -COLORABILITY

Instance: a graph \mathcal{G} and $k \in \mathbb{N}$.

Parameter: k .

Problem: Decide whether \mathcal{G} is k -colorable.

Known: 3-COLORABILITY \in NP-complete (Lovász, Stockmeyer, 1973).

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Slice-wise polynomial problems (Class XP)

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Aims of the course

- 1 Acquire a **basic notions** of parameterized complexity.
- 2 Obtain an introduction to some techniques to derive **FPT or XP results**.
- 3 Obtain an introduction to a variety of techniques to prove **algorithmic lower bounds** and in particular prove **parameterized hardness results**.

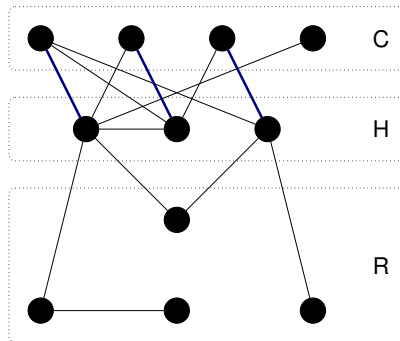
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From today's lecture



A **crown decomposition** of a graph G is a partitioning (C, H, R) of $V(G)$, such that:

- 1 C is nonempty.
- 2 C is an independent set.
- 3 H separates C and R .
- 4 G contains a matching of H into C .

Crown Lemma (\Leftarrow results by König, Hall)

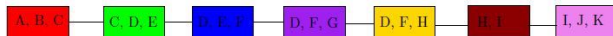
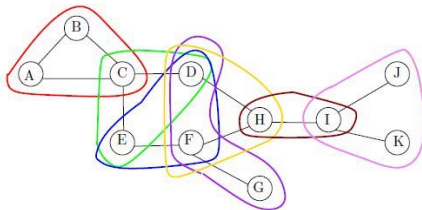
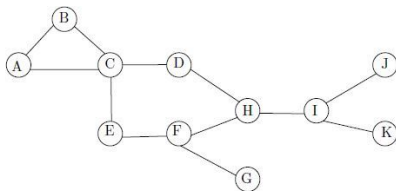
Let G be a graph with no isolated vertices and with at least $3k + 1$ vertices. There is a polynomial-time algorithm that:

- ▶ either finds a matching of size $k + 1$ in G ;
- ▶ or finds a crown decomposition of G .

Tomorrow

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results motivation for FPT kernelization, Crown Lemma, Sunflower Lemma		Algorithmic Meta-Theorems 1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width		
<i>Algorithmic Techniques</i>		<i>Formal-Method & Algorithmic Techniques</i>		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths			FPT-Intractability Classes & Hierarchies motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

In tomorrow's lecture: a path decomposition of a graph



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In Wednesday's lecture: Monadic second-order logic

$$\psi_3 := \exists C_1 \exists C_2 \exists C_3 \left(\left(\forall x \bigvee_{i=1}^3 C_i(x) \right) \right. \\ \left. \wedge \forall x \forall y \left(E(x, y) \rightarrow \bigwedge_{i=1}^3 \neg (C_i(x) \wedge C_i(y)) \right) \right)$$

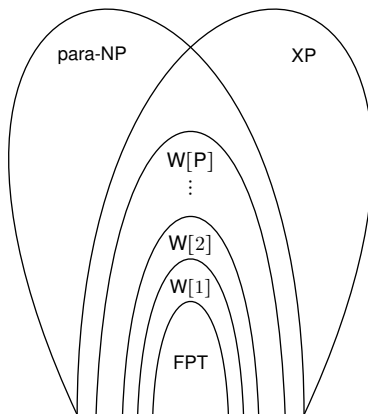
$$\mathcal{A}(\mathcal{G}) \models \psi_3 \iff \mathcal{G} \text{ has is } 3\text{-colorable.}$$

Friday

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From Friday's lecture: W-Hierarchy

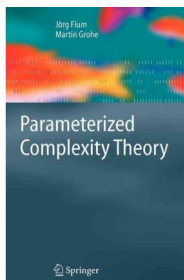
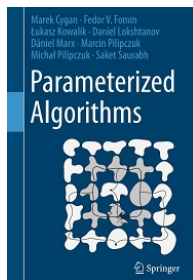
‘There is no definite single class that can be viewed as “the parameterized NP”. Rather, there is a whole hierarchy of classes playing this role. (Flum, Grohe [FG06])



Course overview

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Books



Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh, *Parameterized Algorithms*, 1st ed., Springer, 2015.



Jörg Flum and Martin Grohe, *Parameterized Complexity Theory*, Springer, 2006.

Kernelization

- ▶ Idea
- ▶ Definition

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- ▶ Kernel examples for:
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 - ▶ smaller kernel for vertex cover problem
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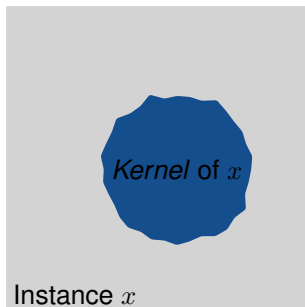
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 - ▶ kernel for hitting set problem

Kernelization methods (informally)

Kernelization is:

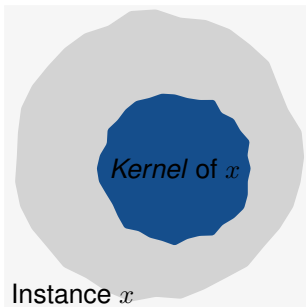
- ▶ a systematic study of polynomial-time preprocessing algorithms,
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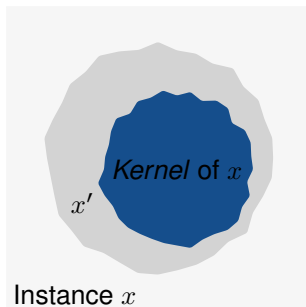
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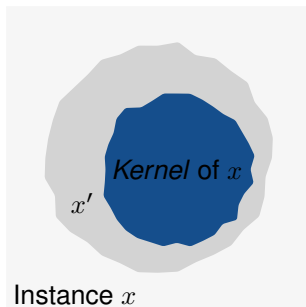
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- ▶ Often a collection of efficient **preprocessing rules**.
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→ Application of rule 1

→ Application of rule 2

- ▶ Often a collection of efficient preprocessing rules.
- ▶ Transform an instance x into a smaller equivalent instance x' .
- ▶ Hopefully, $|x'| \leq g(\kappa(x))$.
→ use a (non-efficient) exact algorithm.

Kernelization (formally)

Definition

Let $\langle Q, \kappa \rangle$ be a parameterized problem over Σ .

A **kernelization** of $\langle Q, \kappa \rangle$ is a function $K: \Sigma^* \rightarrow \Sigma^*$ such that:

- ▶ K is **polynomial-time computable**
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 - ▶ $(x \in Q \iff K(x) \in Q)$,
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We say that such a kernelization K is **polynomial** (resp. **linear**)
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Given: n points in the plane and an integer k ,

Parameter: The integer k .

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Proposition

p-POINT-LINE-COVER \in **FPT**: it admits a kernel of size with k^2 points.

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Given: A graph G .

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Let G be a graph and $S \subseteq V(G)$. The set S is called a **vertex cover** if for every edge of G at least one of its endpoints is in S .

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Exercise

Find an $O(k^2)$ kernel for p-VERTEX-COVER.

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Theorem (Samuel Buss)

p-VERTEX-COVER \in **FPT**, because it admits a kernel with at most $O(k^2)$ vertices and $O(k^2)$ edges.

Kernelization \Rightarrow FPT

Exercise

If $\langle Q, \kappa \rangle$ admits a kernel and is decidable, then $\langle Q, \kappa \rangle \in \text{FPT}$.

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A parameterized problem $\langle Q, \kappa \rangle$ is *fixed-parameter tractable* if:

$$\begin{aligned} &\exists f: \mathbb{N} \rightarrow \mathbb{N} \text{ computable } \exists p \in \mathbb{N}[X] \text{ polynomial} \\ &\quad \exists \text{ algorithm, takes inputs in } \Sigma^* \text{ and } \forall x \in \Sigma^* \\ &\quad \left[\text{algorithm decides if } x \in Q \text{ in time } \leq f(\kappa(x)) \cdot p(|x|) \right]. \end{aligned}$$

FPT := complexity class of all fixed-parameter tractable problems.

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$\langle Q, \kappa \rangle$ a parameterized problem, $Q \subseteq \Sigma^*$

Definition: $K: \Sigma^* \rightarrow \Sigma^*$ a kernelization for $\langle Q, \kappa \rangle$ if:

(K1) $\forall x \in \Sigma^* (x \in Q \Leftrightarrow K(x) \in Q)$

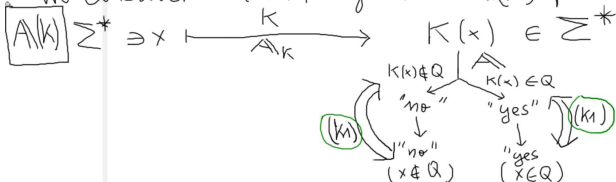
(K2) K is poly time computable

(K3) $\exists h: \mathbb{N} \rightarrow \mathbb{N} \forall x \in \Sigma^* (|K(x)| \leq h(\kappa(x)))$.

Proposition: If $\langle Q, \kappa \rangle$ is decidable, and has kernelization K , then $\langle Q, \kappa \rangle \in \text{FPT}$

Proof. Since $\langle Q, \kappa \rangle$ is decidable, there is an algorithm A that decides instances $x \in \Sigma^*$ in time $\leq f(|x|)$ steps for some computable function $f: \mathbb{N} \rightarrow \mathbb{N}$.

Then, assuming a polynomial algorithm A_κ for κ (time bounded by $P(x)$) we construct an FPT algorithm $A(K)$ for $\langle Q, \kappa \rangle$:



$$\begin{aligned}
 \text{Running time } A(K) &= \\
 &= \text{time}(A_\kappa) + \text{time}(A(K(x))) \\
 &= p(|x|) + f(|K(x)|) \\
 &\quad \text{by (K2) by (K3) } \leq h(\kappa(x)) \\
 &= p(|x|) + f(h(\kappa(x))) \\
 &= (f \circ h)(\kappa(x)) \cdot (1+p)(|x|) \\
 &= f(\kappa(x)) \cdot \text{poly}(|x|) \in \text{FPT}.
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*If $\langle Q, \kappa \rangle \in \text{FPT}$, then $\langle Q, \kappa \rangle$ admits a **kernel**.*

Proof.

Let \mathbb{A} be an algorithm that solves $\langle Q, \kappa \rangle$ in time $f(\kappa(x)) \cdot p(x)$, for all $x \in \Sigma^*$, where $f : \mathbb{N} \rightarrow \mathbb{N}$ computable, and $p(n)$ a polynomial.

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We define the **polynomial-time computable function** $K : \Sigma^* \rightarrow \Sigma^*$ by:

$$K(x) := \begin{cases} x_0 & \dots \mathbb{A} \text{ accepts } x \text{ in } \leq p(|x|) \cdot p(|x|) \text{ steps,} \\ x_1 & \dots \mathbb{A} \text{ rejects } x \text{ in } \leq p(|x|) \cdot p(|x|) \text{ steps,} \\ x & \dots \mathbb{A} \text{ does not terminate in } \leq p(|x|) \cdot p(|x|) \text{ steps.} \end{cases}$$

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If $Q = \emptyset$ or $Q = \Sigma^*$, then we can defined $K(x) := \epsilon$. Otherwise we have $\emptyset \subsetneq Q \subsetneq \Sigma^*$, and we choose some $x_0 \in Q$, and $x_1 \in \Sigma^* \setminus Q$.

We define the **polynomial-time computable function** $K : \Sigma^* \rightarrow \Sigma^*$ by:

$$K(x) := \begin{cases} x_0 & \dots \mathbb{A} \text{ accepts } x \text{ in } \leq p(|x|) \cdot p(|x|) \text{ steps,} \\ x_1 & \dots \mathbb{A} \text{ rejects } x \text{ in } \leq p(|x|) \cdot p(|x|) \text{ steps,} \\ x & \dots \mathbb{A} \text{ does not terminate in } \leq p(|x|) \cdot p(|x|) \text{ steps.} \end{cases}$$

In the last case ($K(x) = x$) we have $p(|x|) \cdot p(|x|) \leq f(\kappa(x)) \cdot p(|x|)$,

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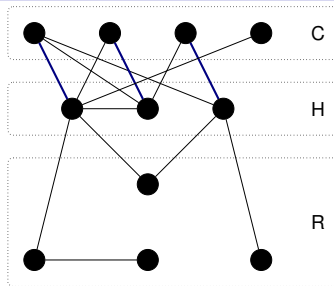
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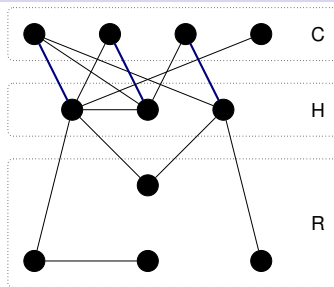
Crown Decomposition and Crown Lemma



A **crown decomposition** of a graph G is a partitioning (C, H, R) of $V(G)$, such that:

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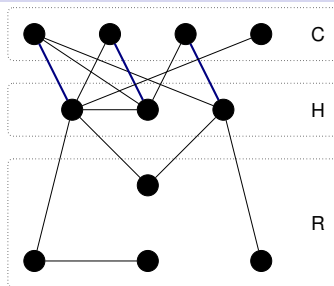
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Crown Lemma (\Leftarrow results by König, Hall)

Let G be a graph with no isolated vertices and with at least $3k + 1$ vertices. There is a polynomial-time algorithm that:

- ▶ either finds a matching of size $k + 1$ in G ;
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Exercise

Apply the Crown Lemma to the Vertex Cover Problem.

The (par.) Vertex Cover Problem (smaller kernel)

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Theorem

p-VERTEX-COVER admits a kernel with at most $3k$ vertices.

The (parameterized) Dual-Coloring Problem

p-COLORABILITY:

Given: A graph $G = \langle V, E \rangle$ on n vertices and an integer k .

Parameter: The integer k .

Question: Is G k -colorable?

Definition

Let $k \in \mathbb{N}$. A graph $G = \langle V, E \rangle$ is k -colorable if there is a function $C : V \rightarrow \{1, \dots, k\}$ such that $C(u) \neq C(v)$ for all edges $\{u, v\} \in E$.

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Exercise

Obtain a kernel with $O(k)$ vertices using crown decomposition.

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If $|V(G)| > 3k$, apply the Crown Lemma to \overline{G} .

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Theorem

p -DUAL-COLORING admits a kernel with at most $3k$ vertices.

Sunflower Lemma

Definition

A **sunflower** with k **petals** and a **core** Y is a collection of sets S_1, \dots, S_k such that $S_i \cap S_j = Y$ for all $i \neq j$. The sets $S_i \setminus Y$ are petals and they must be non-empty.

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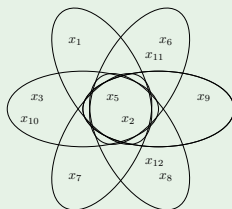
A sunflower with 6 petals and a core $Y = \{x_2, x_5\}$.

$$S_1 = \{x_2, x_3, x_5, x_{10}\}$$

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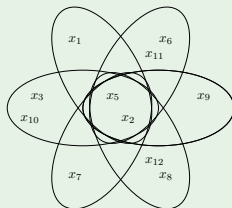
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Sunflower Lemma (Erdős, Rado)

Let \mathcal{A} be a family of sets (without duplicates) over a universe U such that each set in \mathcal{A} has cardinality $= d$.

If $|\mathcal{A}| > d!(k-1)^d$, then \mathcal{A} contains a sunflower with k petals which can be computed in time polynomial in $|\mathcal{A}|$, $|U|$, and k .

Application to d -Hitting Set

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Parameterized d -Hitting Set Problem

p-d-HITTING-SET:

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Question: Does there exist a subset $H \subseteq U$ of size at most k such that H intersects each set in \mathcal{A} ?

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Exercise

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Theorem

p - d -HITTING-SET has a kernel with $\leq d!k^d d$ sets & $\leq d!k^d d^2$ elements.

Application to d -Hitting Set

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If \mathcal{A} contains a sunflower $\mathcal{S} = \{S_1, \dots, S_{k+1}\}$ of $k+1$ sets, then every hitting set H of \mathcal{A} with $|H| \leq k$ must intersect the core Y of \mathcal{S} . Otherwise it is a **no-instance**, because H cannot intersect each of the $k+1$ petals $S_i \setminus Y$.

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Rule HS.1: Let (U, \mathcal{A}, k) be an instance of d -HITTING SET.

Assume that \mathcal{A} contains a sunflower $\mathcal{S} = \{S_1, \dots, S_{k+1}\}$ of cardinality $k+1$ with core Y .

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Proof (kernel of p - d -HITTING-SET with $\leq d!k^d d$ sets and $\leq d!k^d d^2$ elements).

If for some $d' \in \{1, \dots, d\}$, the number of sets in \mathcal{A} of size $= d'$ is more than $d'!k^{d'}$, then the sunflower lemma yields a sunflower of size $k+1$.

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Application to d -Hitting Set

Observation

If \mathcal{A} contains a sunflower $\mathcal{S} = \{S_1, \dots, S_{k+1}\}$ of $k+1$ sets, then every hitting set H of \mathcal{A} with $|H| \leq k$ must intersect the core Y of \mathcal{S} . Otherwise it is a **no-instance**, because H cannot intersect each of the $k+1$ petals $S_i \setminus Y$.

Rule HS.1: Let (U, \mathcal{A}, k) be an instance of d -HITTING SET.

Assume that \mathcal{A} contains a sunflower $\mathcal{S} = \{S_1, \dots, S_{k+1}\}$ of cardinality $k+1$ with core Y .

Then return (U', \mathcal{A}', k) , where $\mathcal{A}' := (\mathcal{A} \setminus \mathcal{S}) \cup Y$,
 $U' := \bigcup \mathcal{A}' = \bigcup_{X \in \mathcal{A}'} X$.

Proof (kernel of p - d -HITTING-SET with $\leq d!k^d d$ sets and $\leq d!k^d d^2$ elements).

If for some $d' \in \{1, \dots, d\}$, the number of sets in \mathcal{A} of size $= d'$ is more than $d'!k^{d'}$, then the sunflower lemma yields a sunflower of size $k+1$.

Rule **HS.1** applies. By applying this rule exhaustively, we obtain a new family of sets \mathcal{A}' with $\leq d'!k^{d'}$ sets of size $= d'$ for every $d' \in \{1, \dots, d\}$. Hence $|\mathcal{A}'| \leq d!k^d d$ and $|U'| = d!k^d d^2$.

If $\emptyset \in \mathcal{A}'$ (a sunflower had an empty core), then it is a **no instance**. \square

Tomorrow

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results motivation for FPT kernelization, Crown Lemma, Sunflower Lemma		Algorithmic Meta-Theorems 1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width		
	GDA		GDA	GDA
<i>Algorithmic Techniques</i>		<i>Formal-Method & Algorithmic Techniques</i>		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width			FPT-Intractability Classes & Hierarchies
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slice-wise polynomial), W- and A-Hierarchies, placing problems on these hierarchies