Lecture 2: Graph width notions, dynamical programming

An Introduction to Parameterized Complexity

Clemens Grabmayer

Ph.D. Program, Advanced Period
Gran Sasso Science Institute
L'Aquila, Italy

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Course overview

Monday, June 16 10.30 – 12.30	Tuesday, June 17 10.30 – 12.30	Wednesday, June 18	Thursday, June 19 10.30 – 12.30	Friday, June 20
Algorithmic Techniques			*Formal-Method* & *Algorithmic Techniques*	
Introduction & basic FPT results	**Notions of bounded graph width**		**Algorithmic Meta-Theorems**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width		1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	(Fair Division)
				14.30 – 16.30
				FPT-Intractability Classes & Hierarchies
			(Fair Division)	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Overview

- comparing parameterizations
- dynamical programming on trees, example:
  - WEIGHTED-INDEPENDENT-SET (and VERTEX-COVER)
- path-width
  - example: fpt-algorithm for bounded path-width
- tree-width
  - example: fpt-algorithm for bounded path-width
- fpt-results for other problems, obtained similarly
- other notions of width
  - clique-width
  - using f-width to define:
    - carving-width (and cut-width)
    - branch-width
    - rank-width
- comparing width-notions

Fixed-Parameter tractable

A *parameterized problem* is a triple $\langle Q, \Sigma, \kappa \rangle$ (short: $\langle Q, \kappa \rangle$) where:

- $Q \subseteq \Sigma^*$ is the set of *(classical) problem instances*,
- $\kappa : \Sigma^* \to \mathbb{N}$ is a (general) function, *the parameterization*.

Parameterized problem $\langle Q, \Sigma, \kappa \rangle$
Instance: $x \in \Sigma^*$.
Parameter: $\kappa(x)$.
Problem: Is $x \in Q$?

Fixed-Parameter tractable

A *parameterized problem* is a triple $\langle Q, \Sigma, \kappa \rangle$ (short: $\langle Q, \kappa \rangle$) where:

- $\triangleright$ $Q \subseteq \Sigma^*$ is the set of *(classical) problem instances*,
- $\triangleright$ $\kappa : \Sigma^* \to \mathbb{N}$ is a (general) function, *the parameterization*.

Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *fixed-parameter tractable* (is in FPT) if:

$$\exists f : \mathbb{N} \to \mathbb{N} \text{ computable } \exists p \in \mathbb{N}[X] \text{ polynomial}$$
$$\exists \mathbb{A} \text{ algorithm, takes inputs in } \Sigma^*$$
$$\forall x \in \Sigma^* \Big[\mathbb{A} \text{ decides whether } x \in Q \text{ holds in time } \leq f(\kappa(x)) \cdot p(|x|) \Big]$$

†) Assumptions for a robust fpt-theory

$\kappa(x)$ is polynomially computable, or itself fpt-computable: for all $x \in \Sigma^*$ in time $\leq g(\kappa(x)) \cdot q(|x|)$ for g computable, $q \in \mathbb{N}[X]$.

Comparing parameterizations

Definition (computably bounded below)

Let $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ parameterizations.

- $\kappa_1 \succeq \kappa_2 \;:\Longleftrightarrow\; \exists g : \mathbb{N} \to \mathbb{N}$ computable $\forall x \in \Sigma^* \big[\, g(\kappa_1(x)) \geq \kappa_2(x) \,\big]$.
- $\kappa_1 \approx \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \kappa_2 \succeq \kappa_1$.
- $\kappa_1 \succ \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \neg\big(\kappa_2 \succeq \kappa_1\big)$.

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ with $\kappa_1 \succeq \kappa_2$:

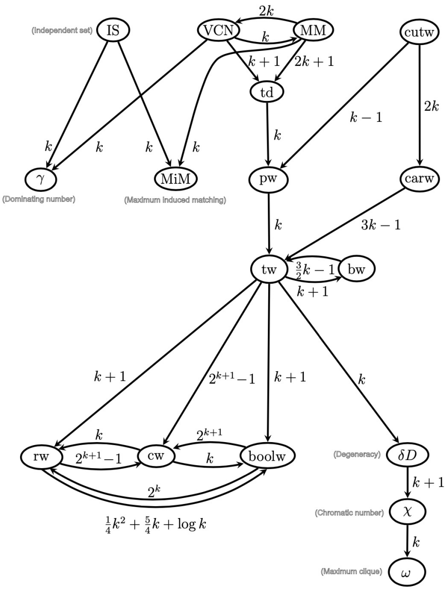
$$\begin{aligned} \langle Q, \kappa_1 \rangle \in \text{FPT} \quad &\Longleftarrow \quad \langle Q, \kappa_2 \rangle \in \text{FPT} \\ \langle Q, \kappa_1 \rangle \notin \text{FPT} \quad &\Longrightarrow \quad \langle Q, \kappa_2 \rangle \notin \text{FPT} \end{aligned}$$

Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \geq wd_2 \;:\Leftrightarrow\; wd_1 \xrightarrow{g} wd_2$$

- FPT-results
  transfer upwards
  (and conversely to $\xrightarrow{g}$)
- ($\notin$ FPT)-results
  transfer downwards
  (and along $\xrightarrow{g}$)



You Always Walk Alone (with your children)

Attività motoria con i figli:

'la possibilità di uscire con i figli minori è consentita a un solo genitore per camminare purché questo avvenga in prossimità della propria abitazione'

(Ministero dell'Interno)

PHYSICAL-DISTANCE-WALKING

Instance: Graph $\mathcal{G} = \langle V, E \rangle$ with V people who want to go for a walk in the next hour in a radius of 200m of their home, and edges in E between them if they live closer than 400m of each other. A number $\ell \in \mathbb{N}$.

Problem: Is it possible that ℓ or more people can go out in the next hour so that everybody walks alone (with their children)?

corresponds to: INDEPENDENT-SET

Weighted Independent Set, and Vertex Cover

Let $\mathcal{G} = \langle V, E \rangle$ a graph. For all $S \subseteq V$:

S is independent set in $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \in E\, (\, \neg(u \in S \wedge v \in S)\,)$
$\Longleftrightarrow \forall e = \{u, v\} \in E\, (\, u \notin S \vee v \notin S)\,)$

WEIGHTED-INDEPENDENT-SET
Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}^{+}_{0}$.
Problem: What is the max. weight of an independent set of $\mathcal{G}$?

S is a vertex cover of $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \in E\, (\, u \in S \vee v \in S))$
$\Longleftrightarrow \forall e = \{u, v\} \in E\, (\, u \notin V \smallsetminus S \vee v \notin V \smallsetminus S)\,)$
$\Longleftrightarrow V \smallsetminus S$ is an independent set of $\mathcal{G}$

VERTEX-COVER
Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and $\ell \in \mathbb{N}$.
Problem: Does $\mathcal{G}$ have a vertex cover of size at most ℓ?

$S \subseteq V$ is *minimal* vertex cover $\Longleftrightarrow$ $V \smallsetminus S$ is *maximal* independent set
Hence: solution of WEIGHTED-INDEPENDENT-SET
$\Longrightarrow$ solution of VERTEX-COVER.

Weighted Ind. Set / Vertex Cover, width-parameterized

p^*-WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

Parameter: path-width / tree-width k.

Problem: What is the max. weight of an independent set of $\mathcal{G}$?

p^*-VERTEX-COVER

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and $\ell \in \mathbb{N}$.

Parameter: path-width / tree-width k.

Problem: Does $\mathcal{G}$ have a vertex cover of size at most ℓ ?

Dynamical programming on trees (example)

WEIGHTED-INDEPENDENT-SET

Instance: A tree $\mathcal{T} = \langle T, F \rangle$, and a weight function $\boldsymbol{w} : T \to \mathbb{R}_0^+$.

Problem: What is the max. weight of an independent set of $\mathcal{T}$?

Obtain a directed tree $\mathcal{T} = \langle T, F, r \rangle$ (pick a root r, orient edges away).

- $A[v] :=$ max. weight of an independent set in subtree $\mathcal{T}_v$ at v,
- $B[v] :=$ max. weight of an ind. set in $\mathcal{T}_v$ that does not contain v.

Computation of $A[v]$ and $B[v]$:

- in leafs: $B[v] = 0\,,\qquad A[v] = \boldsymbol{w}(v)\,.$
- for inner vertices v with children $v_1, \ldots, v_q$:

$$B[v] = \sum_{i=1}^{q} A[v_i]\,, \qquad A[v] = \max\Big\{B[v], \boldsymbol{w}(v) + \sum_{i=1}^{q} B[v_i]\Big\}\,.$$

Solution: value of $A[r]$, can be computed bottom-up in linear time.

Dynamical programming on trees (example)

WEIGHTED-INDEPENDENT-SET
Instance: A tree $\mathcal{T} = \langle T, F \rangle$, and a weight function $w : T \to \mathbb{R}_0^+$.
Problem: What is the max. weight of an independent set of $\mathcal{T}$?

Theorem

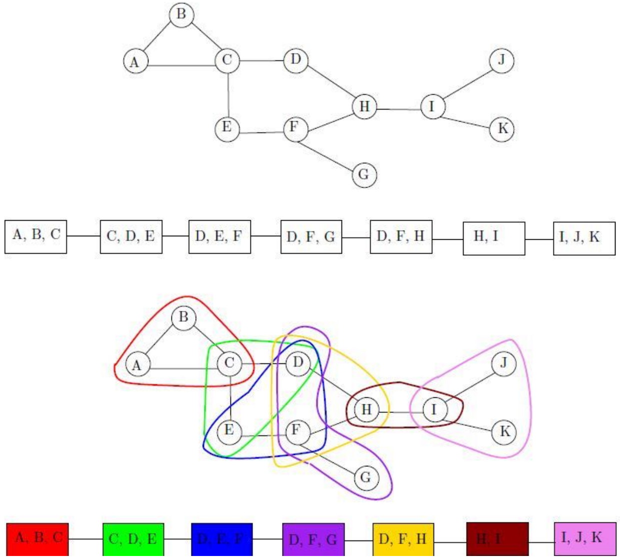
On trees with n nodes,
WEIGHTED-INDEPENDENT-SET $\in$ DTIME$(O(n))$.

VERTEX-COVER
Instance: A tree $\mathcal{T} = \langle T, F \rangle$, and $\ell \in \mathbb{N}$.
Problem: Does $\mathcal{T}$ have a vertex cover of size at most ℓ ?

Corollary

On trees with n nodes,
VERTEX-COVER $\in$ DTIME$(O(n))$.

Path-decomposition (example)



Path decompositions, and path-width

Definition (Robertson–Seymour, 1983)

A *path decomposition* of a graph $\mathcal{G} = \langle V, E\rangle$ is a sequence $\langle B_1, B_2, \ldots, B_r\rangle$ of bags $B_i \subseteq V$ such that:

(P1) $V = \bigcup_{i=1}^{r} B_i$ (every vertex of $\mathcal{G}$ is in some bag).

(P2) $(\forall \{u, v\} \in E)\,(\exists i \in \{1, 2, \ldots, r\})\big[\,\{u, v\} \subseteq B_i\,\big]$
(every edge of $\mathcal{G}$ is realized in some bag).

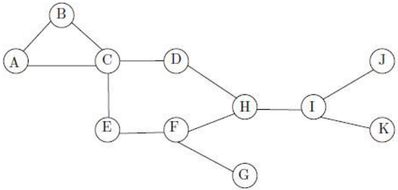
(P3) $(\forall v \in V)\,(\exists i, k \in \{1, \ldots, r\}\,,\, i \le k)\big[\,\{j \mid v \in B_j\} = [i, k]\,\big]$
(the list of bags that contains a vertex of $\mathcal{G}$
is $\langle B_i, \ldots, B_k\rangle$ for some interval $[i, k]$)

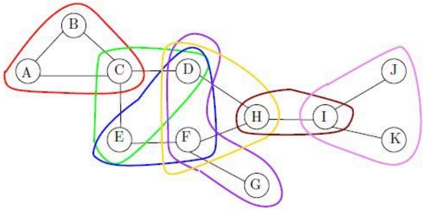
The *width* of path decomp. $\langle B_1, B_2, \ldots, B_r\rangle$ is $\max\{|B_t| - 1 \mid 1 \le t \le r\}$.

The *path-width* $pw(\mathcal{G})$ of a graph $\mathcal{G} = \langle V, E\rangle$ is defined by:

$pw(\mathcal{G})$:= minimal width of a path decomposition of $\mathcal{G}$.

Path-decomposition (example)





Path decomposition defines separations

Lemma

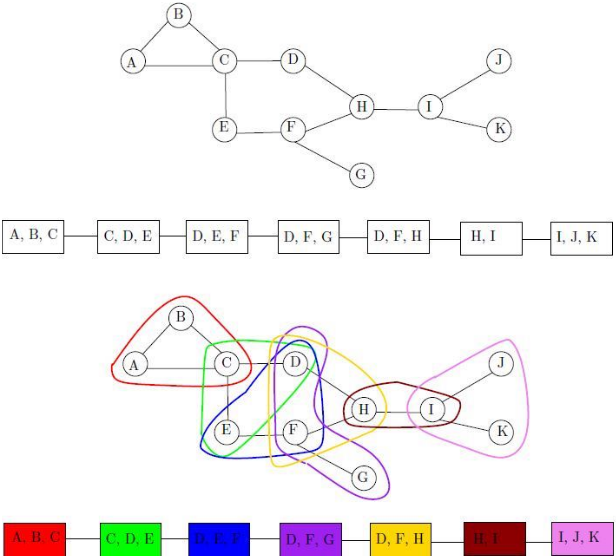
Let $\langle B_1, B_2, \ldots, B_r \rangle$ *be a path decomposition of a graph* $\mathcal{G} = \langle V, E \rangle$*.*
Then for all $i \in \{1, \ldots, r-1\}$ *it holds:*

- $\langle \bigcup_{j=1}^{i} B_j, \bigcup_{j=i+1}^{r} B_j \rangle$ *is a separation of* $\mathcal{G}$ *with separator* $B_i \cap B_{i+1}$*.*
- $\partial(\bigcup_{j=1}^{i} B_j) \subseteq B_i \cap B_{i+1}$*.*

- A pair $\langle A, B \rangle$ of subsets $A, B \subseteq V$ is a *separation* of $\mathcal{G}$ if:
  - $V = A \cup B$
  - there is no edge between $A \smallsetminus B$ and $B \smallsetminus A$.

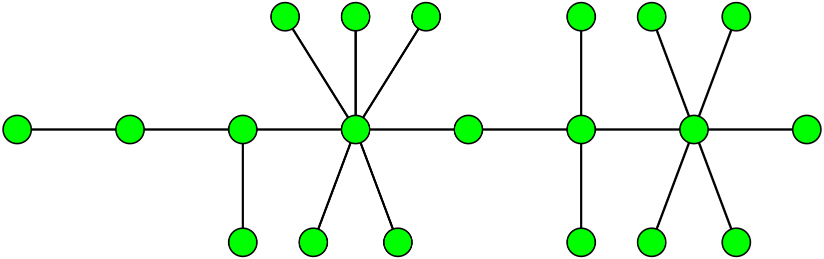
  $A \cap B$ is called the *separator* of a separation $\langle A, B \rangle$,
  and $|A \cap B|$ is called its *order*.
- The *border (set of border vertices)* $\partial(A)$ *of* a set $A \subseteq V$ of vertices consists of all vertices that have a neighbor in $V \smallsetminus A$.
  Note that:
  - $\partial(A) = \partial(V \smallsetminus A)$.
  - $\langle A, (V \smallsetminus A) \cup \partial(A) \rangle$ is a separation of $\mathcal{G}$, for all $A \subseteq V$.

Path-decomposition (example)



Caterpillar

Path-width?

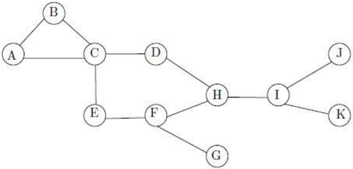


Nice path decomposition

Definition

A *path decomposition* $\langle B_1, B_2, \ldots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if:

- $B_1 = B_r = \varnothing$
- Every index $i > 1$ is either of:
  - introduce index: there is $v \in V$ such that $B_{i+1} = B_i \cup \{v\}$ and $v \notin B_i$,
  - forget index: there is $v \in V$ such that $B_{i+1} = B_i \setminus \{v\}$ and $v \in B_i$.



A, B, C — C, D, E — D, E, F — D, F, G — D, F, H — H, I — I, J, K

Nice path decomposition:

∅ — A — A,B — A,B,C — B,C — C — C,D — C,D,E — D,E — D,E,F —

— D,F — D,F,G — D,F — D,F,H — F,H — H — H,I — I — I,J — I,J,K — J,K — K — ∅

Nice path decomposition

Definition

A *path decomposition* $\langle B_1, B_2, \ldots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if:

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  - forget index: there is $v \in V$ such that $B_{i+1} = B_i \setminus \{v\}$ and $v \in B_i$.

Lemma

From every path decomposition $\langle B_1, B_2, \ldots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ of width k a nice path decomposition $\langle B'_1, B'_2, \ldots, B'_{r'} \rangle$ of width k can be constructed in time $O(k^2 \cdot \max\{r, n\})$ where $n := |V|$.

A,B,C – C,D,E – D,E,F – D,F,G – D,F,H – H,I – I,J,K

∅ – A – A,B – A,B,C – B,C – C – C,D – C,D,E – D,E – D,E,F –

– D,F – D,F,G – D,F – D,F,H – F,H – H – H,I – I – I,J – I,J,K – J,K – K – ∅

Weighted Independent Set

Let $\mathcal{G} = \langle V, E \rangle$ a graph.

$S \subseteq V$ is independent set in $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \big(\neg(u \in S \wedge v \in S) \big)$.

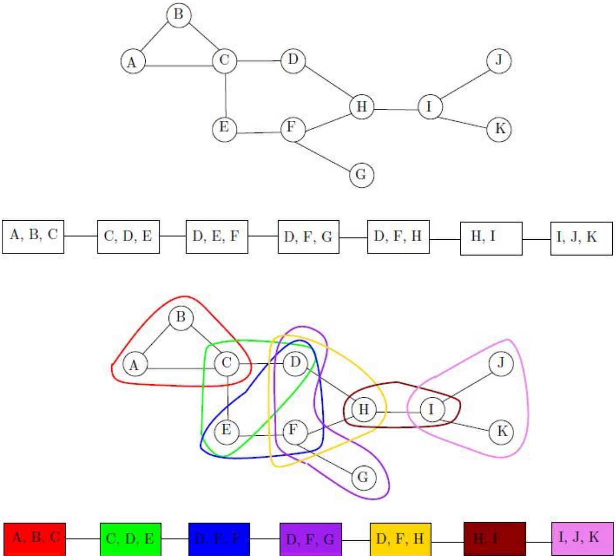
WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

Parameter: path-width k.

Problem: What is the max. weight of an independent set of $\mathcal{G}$?

Path-decomposition (example)



Dyn. programming using path-width (Weigh. Ind. Set)

Let $\langle B_1, \ldots, B_r \rangle$ be a nice path decomposition of $\mathcal{G} = \langle V, E \rangle$.
Then for every $i \in \{1, \ldots, r\}$, and every $S \subseteq B_i$, we define:

$$c[i,S] := \begin{cases} -\infty & \text{if } S \text{ is not independent} \\ \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent } \wedge\ S \subseteq \hat{S} \subseteq V_i = \bigcup_{j=1}^{i} B_j \ \wedge\ \hat{S} \cap B_i = S \\ & \text{if } S \text{ is independent.} \end{cases}$$

Recursive equations for computing $c[i,S]$ for independent S:

- Case $i = 1$: $\quad c[1, \varnothing] = 0$
- Case $i+1$:
  - $i+1$ introduces v: $\quad B_{i+1} = B_i \cup \{v\}$ and $v \notin B_i$,
    $$c[i+1, S] = \begin{cases} c[i,S] & \text{if } v \notin S, \\ c[i, S \setminus \{v\}] + \boldsymbol{w}(v) & \text{if } v \in S; \end{cases}$$
  - $i+1$ forgets v: $\quad B_{i+1} = B_i \setminus \{v\}$ and $v \in B_i$,
    $$c[i+1, S] = \max\big\{c[i,S],\, c[i, S \cup \{v\}]\big\}\,.$$

Dyn. programming using path-width (Weigh. Ind. Set)

Let $\langle B_1, \ldots, B_r \rangle$ be a nice path dec. of $\mathcal{G} = \langle V, E \rangle$ of width k.
For every $i \in \{1, \ldots, r\}$, and every independent $S \subseteq B_i$, we define:

$$c[i, S] := \begin{cases} \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent } \wedge\ S \subseteq \hat{S} \subseteq V_i = \bigcup_{j=1}^{i} B_j\ \wedge\ \hat{S} \cap B_i = S \end{cases}$$

Time Complexity: Based on the values of $c[i, S]$, the maximum possible weight of an independent set $S \subseteq V$ can be computed as:

$$= \max_{S \subseteq B_r} c[r, S] \quad = c[r, \varnothing]$$

Then for all $i \in \{1, \ldots, n\}$:

- $|B_i| \leq k + 1$,
- $\Rightarrow$ number of values $c[i, S]$ at index i : $2^{|B_i|} = 2^{k+1}$,
- $\Rightarrow$ adjacency/independence check for $S \subseteq B_t$ possible in: $O(k^2)$ using a datastructure computable in time $O(k^{O(1)} \cdot n)$,
- time for comp. a value at i, using map of values at $i - 1$: $\sim O(k)$
- time for comp. all values at i, using values at $i - 1$: $2^{k+1} \cdot O(k^2)$

$\Rightarrow$ the time for computing all values at r :

$$(2^{k+1} \cdot O(k^2)) \cdot r + O(k^{O(1)} \cdot n) \in O(2^k \cdot k^{O(1)} \cdot n), \text{ since } r = 2n.$$

Dynamical programming with path width (example)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ *with* $|V| = n$ *and path-width pw*$(\mathcal{G}) = k$,

p^*-WEIGHTED-INDEPENDENT-SET $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$.

S is a *minimal* vertex cover

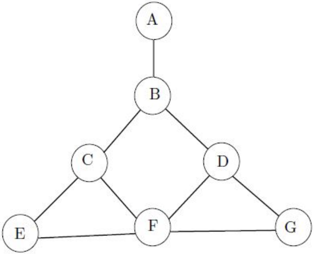
$\iff V \setminus S$ is a *maximal* independent set.

Corollary

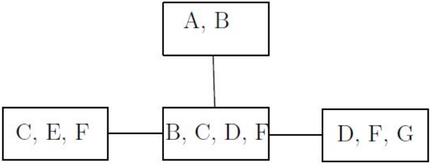
For every graph $\mathcal{G} = \langle V, E \rangle$ *with* $|V| = n$ *and path-width pw*$(\mathcal{G}) = k$,

p^*-VERTEX-COVER $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$.

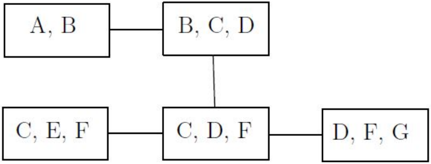
Tree decomposition (example)



The original graph G



A tree-decompostion of width 3



A tree-decompostion of width 2

Tree decompositions, and tree-width

Definition (Bertelé–Brioschi, 1972, Halin, 1976, Robertson–Seymour, 1984)

A *tree decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a pair $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ where $\mathcal{T} = \langle T, F \rangle$ a (undirected, unrooted) tree, and $B_t \subseteq V$ such that:

(T1) $V = \bigcup_{t \in T} B_t$ (every vertex of $\mathcal{G}$ is in some bag).

(T2) $(\forall \{u, v\} \in E)\, (\exists t \in T) \big[\, \{u, v\} \subseteq B_t \,\big]$
(the vertices of every edge of $\mathcal{G}$ are realized in some bag).

(T3) $(\forall v \in V) \big[$ subgraph of $\mathcal{T}$ defd. by $\{t \in T \mid v \in B_t\}$ is connected $\big]$
(the tree vertices (in $\mathcal{T}$) whose bags contain some vertex of $\mathcal{G}$ induce a subgraph of $\mathcal{T}$ that is connected).

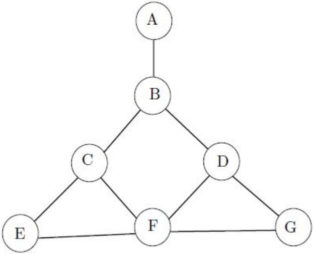
The *width* of a tree decomposition $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ is

$$\max \{ |B_t| - 1 \mid t \in T \} .$$

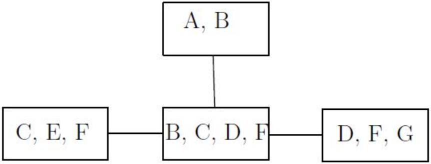
The *tree-width* $tw(\mathcal{G})$ of a graph $\mathcal{G} = \langle V, E \rangle$ is defined by:

$tw(\mathcal{G})$:= minimal width of a tree decomposition of $\mathcal{G}$.

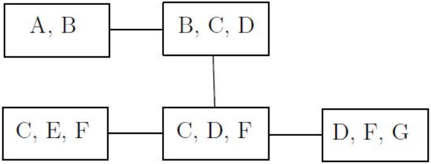
Tree decomposition (example)



The original graph G



A tree-decompostion of width 3



A tree-decompostion of width 2

Tree decomposition defines separations

Lemma

Let $\langle \mathcal{T}, \{B_t\}_{t\in T}\rangle$ be a tree decomposition of a graph $\mathcal{G} = \langle V, E\rangle$.
Let $e = \langle a, b\rangle$ be an edge of $\mathcal{T}$. The $\mathcal{T} \smallsetminus e$ is the union of a tree $\mathcal{T}_a$ containing a, and a tree $\mathcal{T}_b$ containing b.
Then for $A := \bigcup_{t\in V(\mathcal{T}_a)} B_t$ and $B := \bigcup_{t\in V(\mathcal{T}_b)} B_t$ it holds:

- *$\langle A, B\rangle$ is a separation of $\mathcal{G}$ with separator $B_a \cap B_b$.*
- *$\partial(A), \partial(B) \subseteq B_a \cap B_b$.*

Recall, for a graph $\mathcal{G} = \langle V, E\rangle$:

- A pair $\langle A, B\rangle$ of subsets $A, B \subseteq V$ is a *separation* of $\mathcal{G}$ if:
  - $V = A \cup B$
  - there is no edge between $A \smallsetminus B$ and $B \smallsetminus A$.

  $A \cap B$ is called the *separator* of a separation $\langle A, B\rangle$, and $|A \cap B|$ is called its *order*.
- The *border (vertices)* $\partial(A)$ *of* a set $A \subseteq V$ of vertices consists of all vertices that have a neighbor in $V \smallsetminus A$.

Computing tree-width

TREE-WIDTH
Instance: A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Problem: Decide whether *tw*$(\mathcal{G}) = k$.

Theorem

TREE-WIDTH *is* NP*-complete.*

p-TREE-WIDTH
Instance: A graph $\mathcal{G} = \langle V, E \rangle$ and $k \in \mathbb{N}$.
Parameter: k.
Problem: Decide whether *tw*$(\mathcal{G}) = k$.

Theorem

p-TREE-WIDTH *is fixed-parameter tractable, in time* $2^{p(k)} \cdot n$ *where* $n := |V|$.

Nice tree decomposition

Definition

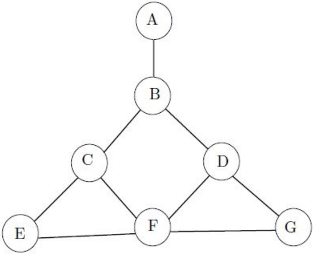
A *tree decomposition* $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ of graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if it is based on the choice of a leaf as *root* r and the parent–children relation away from r such that:

- $B_r = \varnothing$, and $B_\ell = \varnothing$ for every leaf $\ell \in T$.
- Every non-leaf node $t \in T$ is of one of three types:
  - introduce node: t has exactly one child t' such that $B_t = B_{t'} \cup \{v\}$; we say v is introduced at t.
  - forget node: t has exactly one child t' such that $B_t = B_{t'} \setminus \{w\}$ for some $w \in B_{t'}$; we say w is forgotten at t.
  - join node: a node t with two children t_1, t_2 such that $B_t = B_{t_1} = B_{t_2}$.

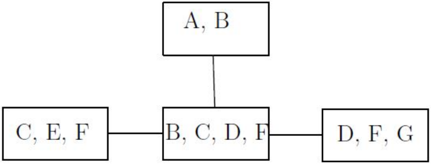
Lemma

From every tree decomposition $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ of width k a nice tree decomposition $\langle \mathcal{T}', \{B'_t\}_{t \in T'} \rangle$ of width k and with $r := |V(\mathcal{T})| \in O(kn)$ vertices can be constructed in time $O(k^2 \cdot \max\{r, n\})$ where $n := |V|$.

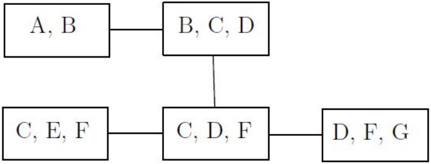
Tree decomposition (example)



The original graph G



A tree-decompostion of width 3



A tree-decompostion of width 2

Weighted Independent Set

Let $\mathcal{G} = \langle V, E \rangle$ a graph.

$S \subseteq V$ is independent set in $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \big(\neg(u \in S \wedge v \in S) \big)$.

WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

Parameter: tree-width k.

Problem: What is the max. weight of an independent set of $\mathcal{G}$?

Dynamical programming using tree-width (example)

For every node t of a nice tree decomposition, and every $S \subseteq B_t$, we define:

$$c[t,S] := \begin{cases} -\infty & \text{if } S \text{ is not independent,} \\ \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent} \wedge S \subseteq \hat{S} \subseteq V_t \wedge \hat{S} \cap B_t = S \\ & \text{if } S \text{ is independent.} \end{cases}$$

Recursive equations for computing $c[t,S]$ for independent S:

- leaf node t: $c[t,\varnothing] = 0$
- introduction node t of vertex v with child t':

$$c[t,S] = \begin{cases} c[t',S] & \text{if } v \notin S \\ c[t',S \setminus \{v\}] + \boldsymbol{w}(v) & \text{otherwise} \end{cases}$$

- forget node t of vertex v with child t':

$$c[t,S] = \max\big\{c[t',S],\, c[t',S \cup \{v\}]\big\}$$

- join node t with children t_1 and t_2:

$$c[t,S] = c[t_1,S] + c[t_2,S] - \boldsymbol{w}(S)$$

Dyn. programming using tree-width (Weigh. Ind. Set)

Let $\langle \mathcal{T} = \langle T, F, r\rangle, \{B_t\}_{t\in T}\rangle$ be a nice tree decomposition of $\mathcal{G} = \langle V, E\rangle$ of width k. For every $t \in T$, and every independent $S \subseteq B_t$:

$$c[t,S] := \begin{cases} \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent } \wedge\ S \subseteq \hat{S} \subseteq V_t = \bigcup_{s\in T_t} B_s \ \wedge\ \hat{S} \cap B_i = S \end{cases}$$

Time Complexity: Based on the values of $c[t,S]$, the maximum possible weight of an independent set $S \subseteq V$ can be computed as:

$$= \max_{S\subseteq B_r} c[r,S] \quad = c[r,\varnothing]$$

Then for all $t \in T$:

- $|B_t| \le k+1$,
- $\Rightarrow$ number of values $c[t,S]$ at index t : $2^{|B_t|} = 2^{k+1}$,
- $\Rightarrow$ adjacency/independence check for $S \subseteq B_t$ possible in: $O(k^2)$ using a datastructure computable in time $O(k^{O(1)} \cdot n)$,
- time for comp. a value at t, using map of values at $t-1$: $O(k)$
- time for comp. all values at t, using values at $t-1$: $2^{k+1} \cdot O(k^2)$

$\Rightarrow$ the time for computing all values at the root r :
$(2^{k+1} \cdot O(k^2)) \cdot |T| + O(k^{O(1)} \cdot n) \in O(2^k \cdot k^{O(1)} \cdot n)$, since $|T| \in O(k \cdot n)$.

Dynamical programming with tree width (example)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and tree-width tw($\mathcal{G}$) $= k$,

p^*-WEIGHTED-INDEPENDENT-SET $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$.

S is a *minimal* vertex cover
$\iff V \setminus S$ is a *maximal* independent set.

Corollary

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and tree-width tw($\mathcal{G}$) $= k$,

p^*-VERTEX-COVER $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$.

Dyn. programming with tree-width: general strategy

We consider problem P for graphs $\mathcal{G} = \langle V, E\rangle$ of size n and nice tree decompositions $\langle \mathcal{T} = \langle T, F, r\rangle, \{B_t\}_{t\in T}\rangle$ of tree width k.

- Formulate a family of properties that can be restricted to subtrees of $\mathcal{T}$ such that
  - a solution of P can be obtained from the properties at the root of $\mathcal{T}$.
- Find recursion equations for bottom-up evaluation on $\mathcal{T}$.
- Prove correctness of these recursion equations by showing two inequalities for each type of node:
  - one relating an optimum solution for the node to some solutions for its children,
  - one relating optimum solutions for a node's children to a solution for the node.
- Obtain an estimate of the time needed to compute the properties in a node t depending on n and k.
- Sum up the time needed to compute the solution(s) at root r of $\mathcal{T}$.
- Add time needed to obtain the solution of P from properties at r.

Dynamical programming: similar results (I)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and tw($\mathcal{G}$) = k,

- p^*-VERTEX-COVER, INDEPENDENT-SET ∈ DTIME($2^k \cdot k^{O(1)} \cdot n$),
- p^*-DOMINATING-SET ∈ DTIME($4^k \cdot k^{O(1)} \cdot n$),
- p^*-ODD CYCLE TRAVERSAL ∈ DTIME($3^k \cdot k^{O(1)} \cdot n$),
- p^*-MAXCUT ∈ DTIME($2^k \cdot k^{O(1)} \cdot n$),
- p^*-q-COLORABILITY ∈ DTIME($q^k \cdot k^{O(1)} \cdot n$).

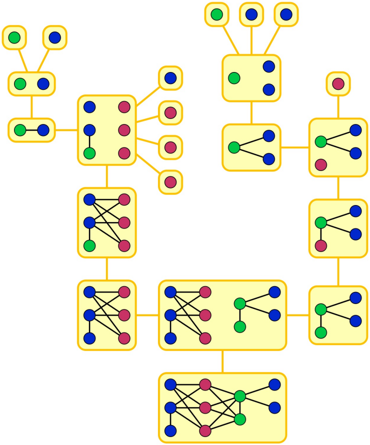
Dynamical programming: similar results (II)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and tw$(\mathcal{G}) = k$, the following problems are in DTIME$(k^{O(k)} \cdot n)$ *:*

- p^*-STEINER-TREE,
- p^*-FEEDBACK-VERTEX-SET,
- p^*-HAMILTONIAN-PATH *and* p^*-LONGEST-PATH,
- p^*-HAMILTONIAN-CYCLE *and* p^*-LONGEST-CYCLE,
- p^*-CHROMATIC-NUMBER,
- p^*-CYCLE-PACKING,
- p^*-CONNECTED-VERTEX-COVER,
- p^*-CONNECTED-FEEDBACK-VERTEX-SET.

Clique width (example)



Clique-Width

For $k \in \mathbb{N}$, the *k-expressions* are defined by:

$$\varphi, \varphi_1, \varphi_2 \ ::= \ \boldsymbol{i} \mid \text{edge}_{i-j}(\varphi) \mid \text{recolor}_{i \to j}(\varphi) \mid (\varphi_1 \oplus \varphi_2)$$

for $i, j \in [k]$ with $i \neq j$. k-expressions φ *generate* graphs $\mathcal{G}(\varphi)$:

- $\mathcal{G}(\boldsymbol{i})$ is the graph with a single vertex of color i.
- $\mathcal{G}(\text{edge}_{i-j}(\varphi))$ results from $\mathcal{G}(\varphi)$ by adding edges between every vertex of color i and every vertex of color j.
- $\mathcal{G}(\text{recolor}_{i \to j}(\varphi))$ results from $\mathcal{G}(\varphi)$ by recoloring every vertex of color i by color j.
- $\mathcal{G}(\varphi_1 \oplus \varphi_2)$ is the disjoint union of $\mathcal{G}(\varphi_1)$ and $\mathcal{G}(\varphi_2)$.

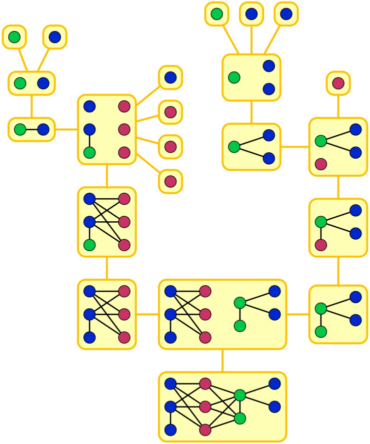
Definition (Courcelle, Engelfriet, Rozenberg, 1993, [2])

The *clique-width* $clw(\mathcal{G})$ of $\mathcal{G} = \langle V, E \rangle$ is defined by:

$clw(\mathcal{G})$:= the least $k \in \mathbb{N}$ such that, for some k-expression φ, $\mathcal{G} = \mathcal{G}(\varphi)$ (when removing colors)

Clique width (example)

Building a graph $\mathcal{G}$ of clique-width *clw*$(\mathcal{G}) = 3$:



Clique-Width (examples, properties, computability)

Example

- The class of cliques has clique-width 2.
- The class of stars has clique-width 2.
- The class of trees has clique-width 3.
- The class of $n \times n$ grids has clique-width $\Theta(n)$.

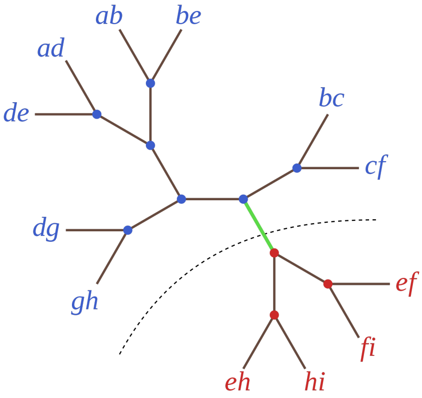
- subgraphs/induced subgraphs:
  - clique-width is preserved under taking induced subgraphs,
  - clique-width is not preserved under taking subgraphs (e.g. minors).
- *clw* $\prec$ *tw*:
  - *clw* $\preceq$ *tw*: $\mathit{clw}(\mathcal{G}) \leq 3 \cdot 2^{\mathit{tw}(\mathcal{G})-1}$
  - $\neg$(*tw* $\preceq$ *clw*): for example, $\mathit{clw}(K_n) = 2$, and $\mathit{tw}(K_n) = n - 1$.
- Deciding whether $\mathit{clw}(\mathcal{G}) \leq k$ is NP-hard. With parameter k it is in XP (slice-wise polynomial), but unknown to be in FPT.
- Every graph property expressible in MSO (monadic second-order logic) can be decided in linear time w.r.t. the graph's clique-width.

f-Width (of sets)

By a *cut function* or a *connectivity function* we mean a function $f : 2^U \to \mathbb{R}_0^+$ such that:

$$f \text{ is } \textit{symmetric} :\iff \forall X \subseteq U \left[f(X) = f(U \setminus X) \right];$$

$$f \text{ is } \textit{fair} :\iff f(\varnothing) = f(U) = 0\,.$$



Branch-Width

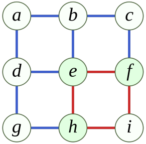
Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph. The *border (vertices)* of a set $X \subseteq E$ of edges is defined by:

$$\partial(X) := \Big\{ v \in V \mid \exists e_1 \in X \; \exists e_2 \in E \setminus X \; \big[v \text{ is incident to } e_1 \text{ and } e_2 \big] \Big\}$$

The *branch-width bw*$(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is defined as

$$\mathit{bw}(\mathcal{G}) := w_f(E) \quad \text{for} \quad f : 2^E \to \mathbb{R}_0^+, \; X \mapsto |\partial(X)|$$



Rank-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.
For $X \subseteq V$ we define the $GF(2)$-matrix:

$$B_{\mathcal{G}}(X) := (b_{x,y})_{x \in X, y \in V \setminus X}\ , \text{ where, for all } x \in X, y \in V \setminus X\text{:}$$

$$b_{x,y} = 1 \iff \{x, y\} \in E\,.$$

($B_{\mathcal{G}}(X)$ is the adjacency matrix of the bipartite graph induced by $\mathcal{G}$ between X and $V \setminus X$.)

The *rank-width rw*($\mathcal{G}$) of a graph $\mathcal{G} = \langle G, E \rangle$ is:

$$\textit{rw}(\mathcal{G}) := w_{\rho_{\mathcal{G}}}(E) \quad \text{for} \quad \rho_{\mathcal{G}} : 2^V \to \mathbb{N}_0,\ X \mapsto \text{rank of } B_{\mathcal{G}}(X)$$

Properties

- *rw*($\mathcal{G}$) ≤ *tw*($\mathcal{G}$).
- tree-width cannot be bounded functionally by rank-width: *rw*(K_n) = 1, but *tw*(K_n) = $n - 1$.

Carving-Width and Cut-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.
For $X \subseteq V$ the *edge-cut* of X is:

$$\textit{cut}_{\mathcal{G}}(X) := \{e = \{u, v\} \in E \mid u \in X, v \in V \setminus X\} .$$

The *carving-width* $\textit{carw}(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is:

$$\textit{carw}(\mathcal{G}) := w_{\textit{cut}}(E) \quad \text{for} \quad \textit{cut} : 2^V \to \mathbb{N}_0,\ X \mapsto |\textit{cut}_{\mathcal{G}}(X)| .$$

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph with $n = |V|$.
For a permutation $\pi : \{1, \ldots, n\} \to V$ on V we define:

$$\textit{width}(\pi) := \max_{1 \le i \le n} \textit{cut}_{\mathcal{G}}(\{\pi(j) \mid 1 \le j \le i\}) .$$

The *cut-width* $\textit{cutw}(\mathcal{G})$ of $\mathcal{G}$ is:

$$\textit{cutw}(\mathcal{G}) := \min_{\pi \text{ perm. of } V} \textit{width}(\pi) .$$

Coverage in Multi-Interface Networks

CMI(p) (for $p \in \mathbb{N}$)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, $W : V \to 2^{\{1,\ldots,a\}}$ available-interface allocation, $c : \{1, \ldots, a\} \to \mathbb{R}^{+}$ interface cost function.

Solution: An allocation $W_A : V \to 2^{\{1,\ldots,a\}}$ of active interfaces covering $\mathcal{G}$ such that $W_A(v) \subseteq W(v)$, and $|W_A(v)| \leq p$ for all $v \in V$, if possible; otherwise, a negative answer.

Problem: Obtain, if possible, a minimal solution with respect to the total cost of the interfaces that are activated, that is, $c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i)$.

Coverage in Multi-Interface Networks (parameterized)

Theorem

CMI$(2) \in$ NP-complete, *also for graphs with max. node degree* ≥ 4.

p^*-*CMI*(p) (for $p \in \mathbb{N}$)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, $W : V \to 2^{\{1,\ldots,a\}}$ available-interface allocation, $c : \{1, \ldots, a\} \to \mathbb{R}^+$ interface cost function.

Parameter: path-width / carving-width k

Problem: Obtain, if possible, a minimal solution with respect to the total cost of the interfaces that are activated, that is, $c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i)$.

Theorem (Aloisio, Navarra, 2020, [1])

- *For path-width pw*$(\mathcal{G}) = k$,
  p^*-*CMI*$(2) \in$ DTIME$\left(n \cdot \left(a + \binom{a}{2}\right)^{k+1}\right)$.
- *For carving-width carw*$(\mathcal{G}) = k$, p^*-*CMI*$(2) \in$ DTIME$(n \cdot a^{4k})$.

Coverage in Multi-Interface Networks (parameterized)

Theorem (Aloisio, Navarra, 2020, [1])

- *For path-width pw*$(\mathcal{G}) = k$,
  p^**-CMI*$(2) \in \mathrm{DTIME}(n \cdot (a + \binom{a}{2}))^{k+1})$.
- *For carving-width carw*$(\mathcal{G}) = k$, p^**-CMI*$(2) \in \mathrm{DTIME}(n \cdot a^{4k})$.

$(p^*)'$*-CMI*(p) (for $p \in \mathbb{N}$)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, $W : V \to 2^{\{1,\ldots,a\}}$ available-interface allocation, $c : \{1, \ldots, a\} \to \mathbb{R}^+$ interface cost function.

Parameter: a + (path-width / carving-width k)

Problem: Obtain, if possible, a minimal solution with respect to the total cost of the interfaces that are activated, that is, $c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i)$.

Corollary

$(p^*)'$*-CMI*$(p) \in$ FPT.

Comparing parameterizations

Definition (computably bounded)

Let $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ parameterizations.

- $\kappa_1 \geq \kappa_2 \;:\Longleftrightarrow\; \exists g : \mathbb{N} \to \mathbb{N}$ computable $\forall x \in \Sigma^* \big[\, g(\kappa_1(x)) \geq \kappa_2(x) \,\big]$.
- $\kappa_1 \approx \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \geq \kappa_2 \;\wedge\; \kappa_2 \geq \kappa_1$.
- $\kappa_1 > \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \geq \kappa_2 \;\wedge\; \neg\big(\kappa_2 \geq \kappa_1\big)$.

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ with $\kappa_1 \geq \kappa_2$:

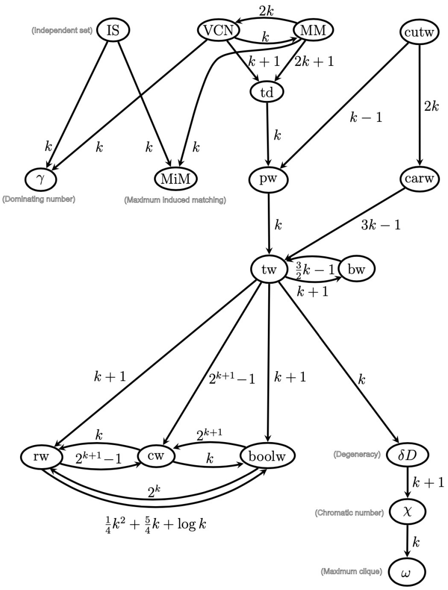
$$\begin{array}{ccc} \langle Q, \kappa_1 \rangle \in \text{FPT} & \Longleftarrow & \langle Q, \kappa_2 \rangle \in \text{FPT} \\ \langle Q, \kappa_1 \rangle \notin \text{FPT} & \Longrightarrow & \langle Q, \kappa_2 \rangle \notin \text{FPT} \end{array}$$

Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \succeq wd_2 \;:\Leftarrow\; wd_1 \xrightarrow{g(k)} wd_2$$

- FPT-results
  transfer upwards
  (and conversely to $\xrightarrow{g}$)
- ($\notin$ FPT)-results
  transfer downwards
  (and along $\xrightarrow{g}$)



Summary

- comparing parameterizations
- dynamical programming on trees, example:
  - WEIGHTED-INDEPENDENT-SET (and VERTEX-COVER)
- path-width
  - example: fpt-algorithm for bounded path-width
- tree-width
  - example: fpt-algorithm for bounded path-width
- fpt-results for other problems, obtained similarly
- other notions of width
  - clique-width
  - using f-width to define:
    - carving-width (and cut-width)
    - branch-width
    - rank-width
- example problem: coverage in multi-interface networks
- comparing width-notions

Course overview

Monday, June 16 10.30 – 12.30	Tuesday, June 17 10.30 – 12.30	Wednesday, June 18	Thursday, June 19 10.30 – 12.30	Friday, June 20
Algorithmic Techniques			*Formal-Method* & *Algorithmic Techniques*	
Introduction & basic FPT results	**Notions of bounded graph width**		**Algorithmic Meta-Theorems**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width		1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	(Fair Division)
				14.30 – 16.30
				FPT-Intractability Classes & Hierarchies
			(Fair Division)	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Thursday

- recalling notions from logic:
  - propositional, and first-order logic
  - monadic second-order logic (MSO)
- Courcelle's Theorem: obtaining FPT-results by
  - model-checking of MSO-properties
    on graphs and structures of bounded tree-/clique-width

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