From Compactifying Lambda-Letrec Terms to Recognizing Regular-Expression Processes

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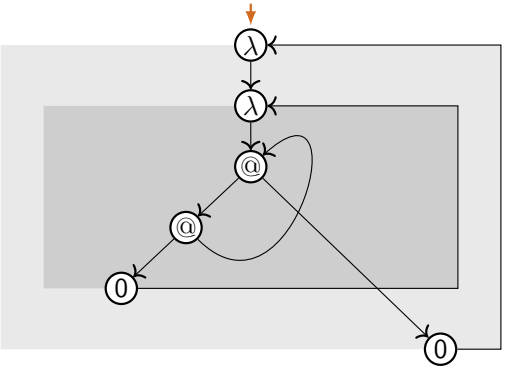

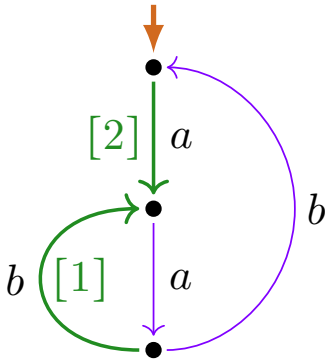

DCM'23
Sapienza Università di Roma
July 2, 2023

Overview

1. Compactifying λ-terms with `letrec` (maximal sharing of functional programs)
   - higher-order λ-term graphs

2. Recognizing regular-expression processes
   - LEE-witnesses: graph labelings based on a loop–condition LEE

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   - from terms in the λ-calculus with letrec to:
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     - first-order λ-term graphs
     - λ-NFAs, and λ-DFAs
   - minimization / readback / efficiency / Haskell implementation
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   - Milner's questions, known results
   - structure-constrained process graphs:
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     - preservation under bisimulation collapse
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- Comparison desiderata

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- Comparison results

Comparison original desiderata

λ-calculus with letrec under the unfolding semantics

Well-known: graph representations implemented by compilers

- but were not intended for use under $\underline{\leftrightarrow}$

Not well-known: term graph interpretation that is studied under $\underline{\leftrightarrow}$

Regular expressions under process semantics (bisimilarity $\underline{\leftrightarrow}$)

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- not closed under $\underline{\rightarrow}$, and $\underline{\leftrightarrow}$, modulo $\underline{\leftrightarrow}$ incomplete

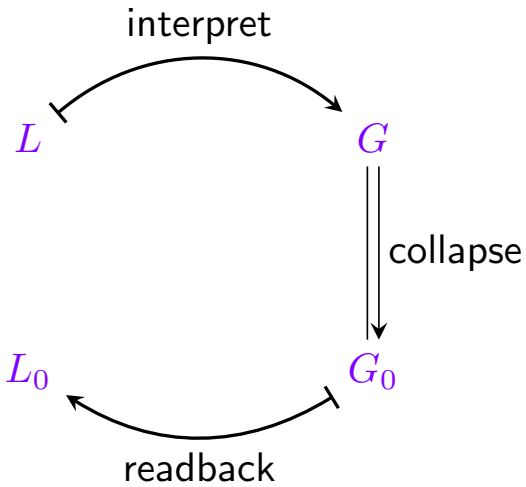
Desired: reason with graphs that are $P(\cdot)$-expressible modulo $\underline{\leftrightarrow}$
(at least with 'sufficiently many')

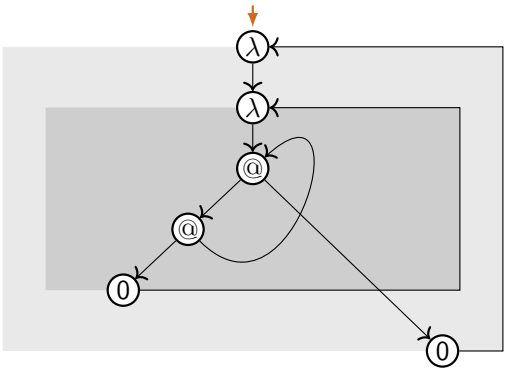
understand incompleteness by a structural graph property

structure constraints (L'Aquila)

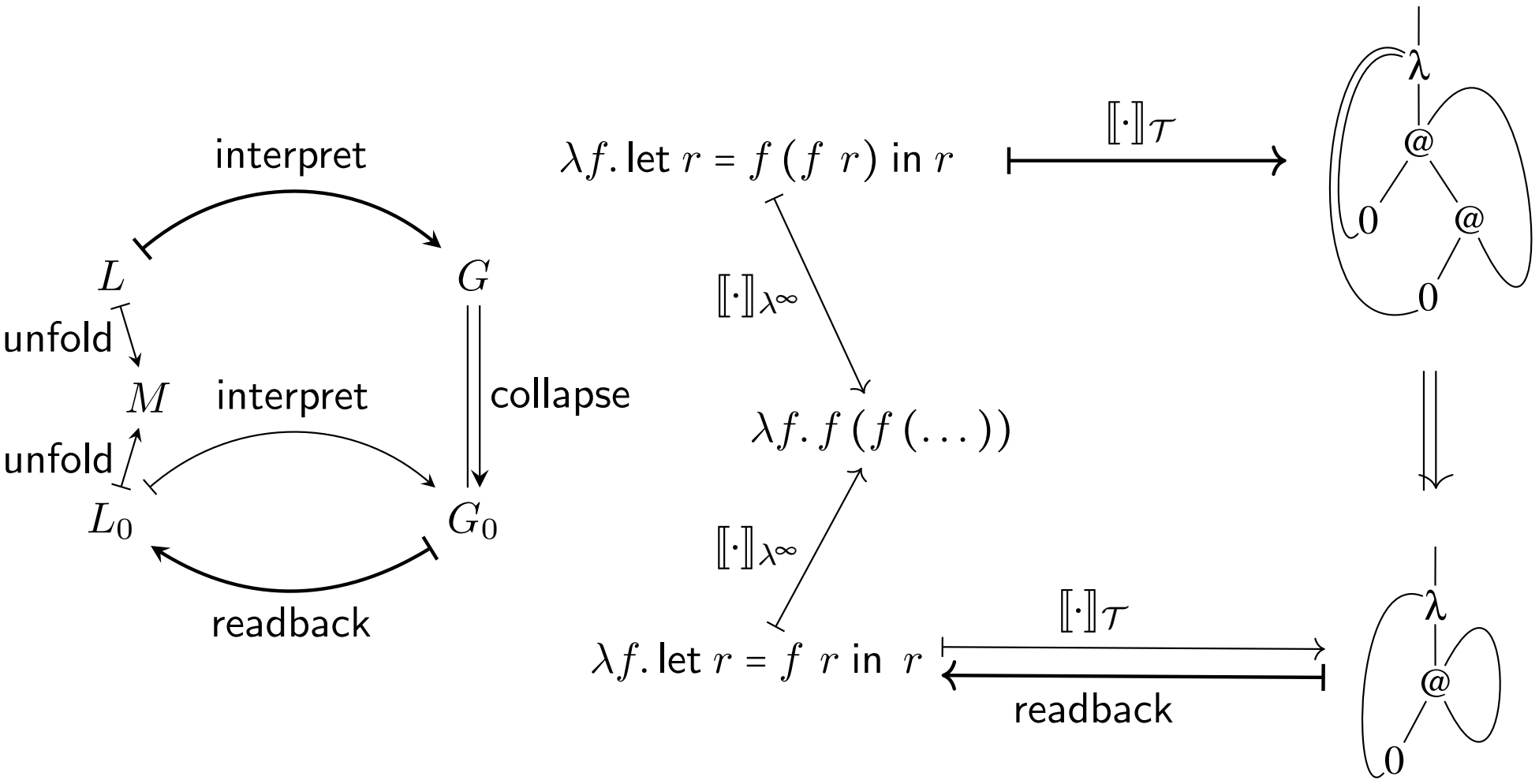
Maximal sharing of functional programs

(joint work with Jan Rochel)





Maximal sharing: example (fix)



Maximal sharing: the method

$$L \xmapsto{[\![\cdot]\!]_{\mathcal{H}}} \mathcal{G}$$

1. term graph interpretation $[\![\cdot]\!]$.
   of $\boldsymbol{\lambda}_{\mathsf{letrec}}$-term L as:
   a. higher-order term graph
      $\mathcal{G} = [\![L]\!]_{\mathcal{H}}$

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Maximal sharing: the method

$$\begin{tikzcd} L \arrow[r, mapsto, bend left, "[\![\cdot]\!]_{\mathcal{H}}"] \arrow[rr, mapsto, bend left, "[\![\cdot]\!]_{\mathcal{T}}"] & \mathcal{G} \arrow[r, mapsto] & G \end{tikzcd}$$

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   of f-o term graph G into G_0

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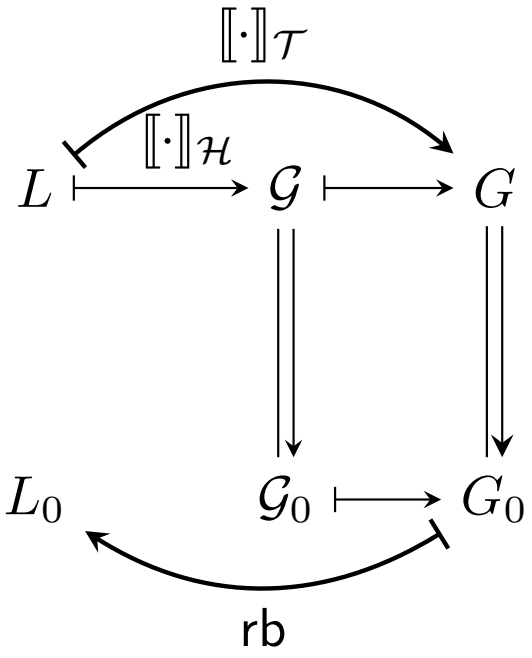

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   yielding program $L_0 = \text{rb}(G_0)$.

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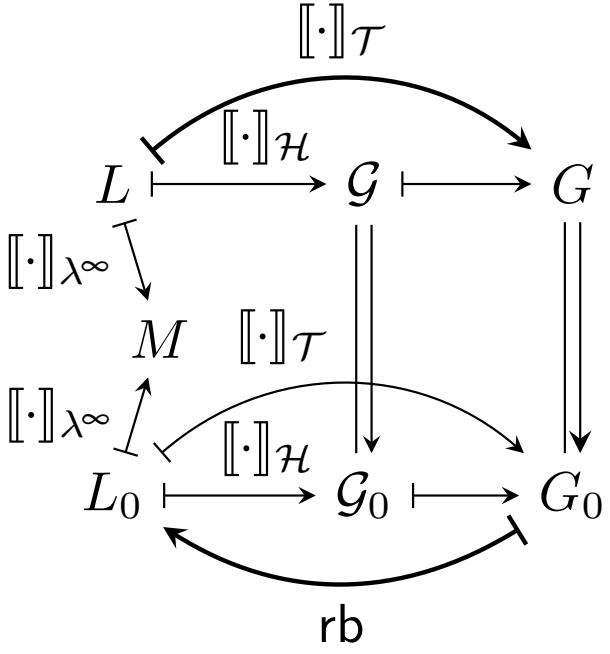

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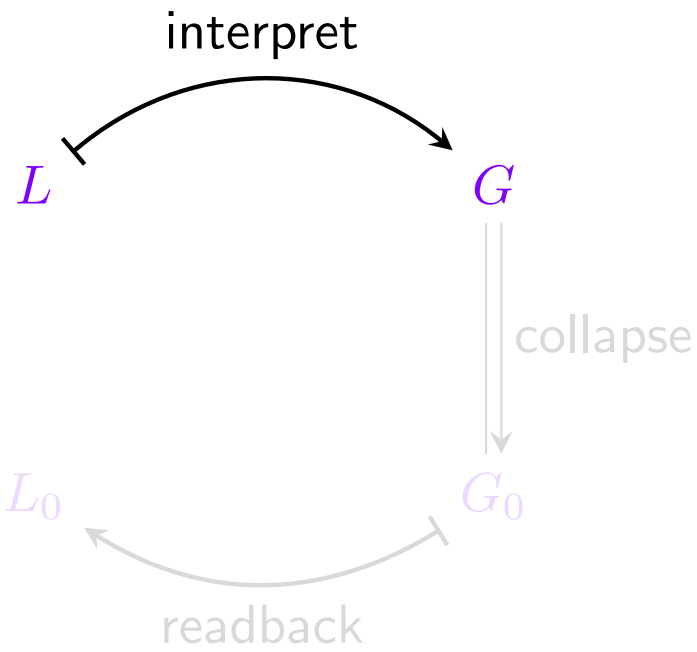
Interpretation



Running example

instead of:

$\lambda f.\,\mathsf{let}\ r = f\ (f\ r)\ \mathsf{in}\ r \quad \longmapsto_{\text{max-sharing}} \quad \lambda f.\,\mathsf{let}\ r = f\ r\ \mathsf{in}\ r$

we use:

$\lambda x.\,\lambda f.\,\mathsf{let}\ r = f\ (f\ r\ x)\ x\ \mathsf{in}\ r \quad \longmapsto_{\text{max-sharing}} \quad \lambda x.\,\lambda f.\,\mathsf{let}\ r = f\ r\ x\ \mathsf{in}\ r$

$L \quad \longmapsto_{\text{max-sharing}} \quad L_0$

Graph interpretation (example 1)

$L_0 = \lambda x.\, \lambda f.\, \mathsf{let}\ r = f\ r\ x\ \mathsf{in}\ r$

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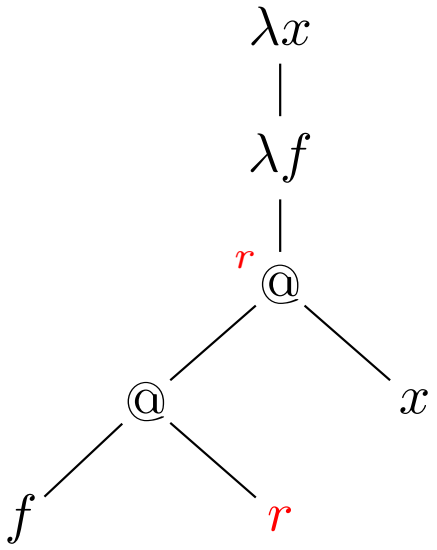

syntax tree

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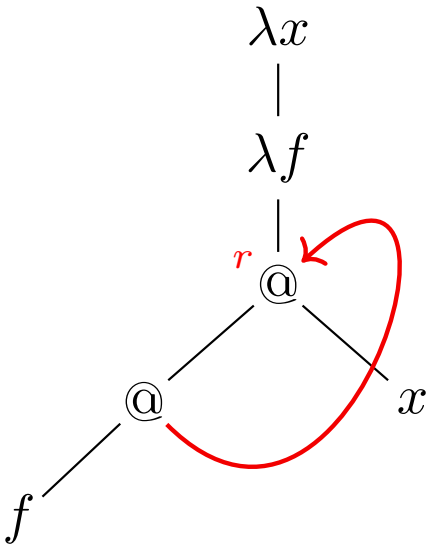

syntax tree (+ recursive backlink)

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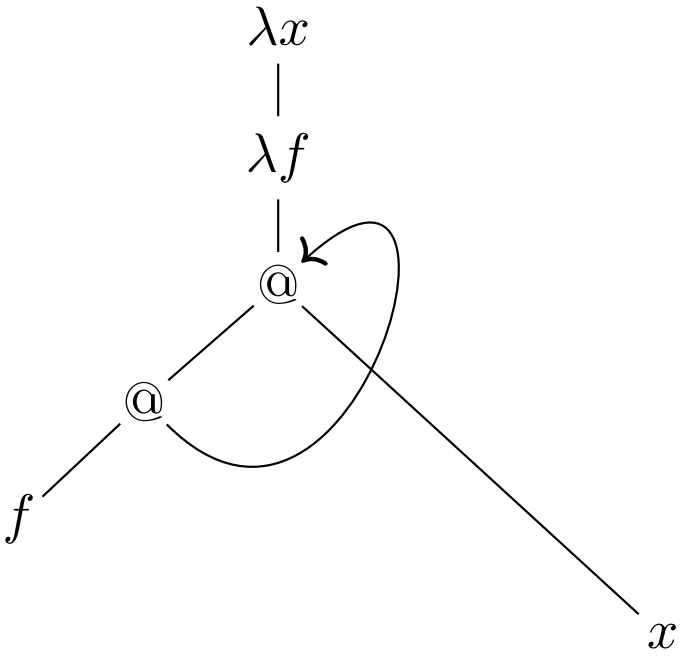

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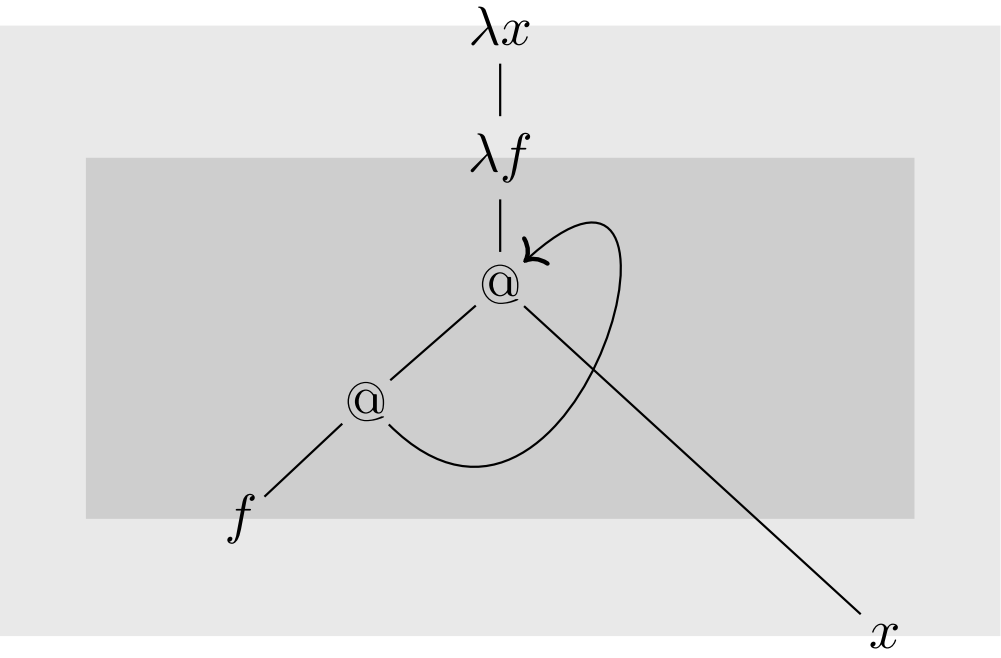

syntax tree (+ recursive backlink, + scopes)

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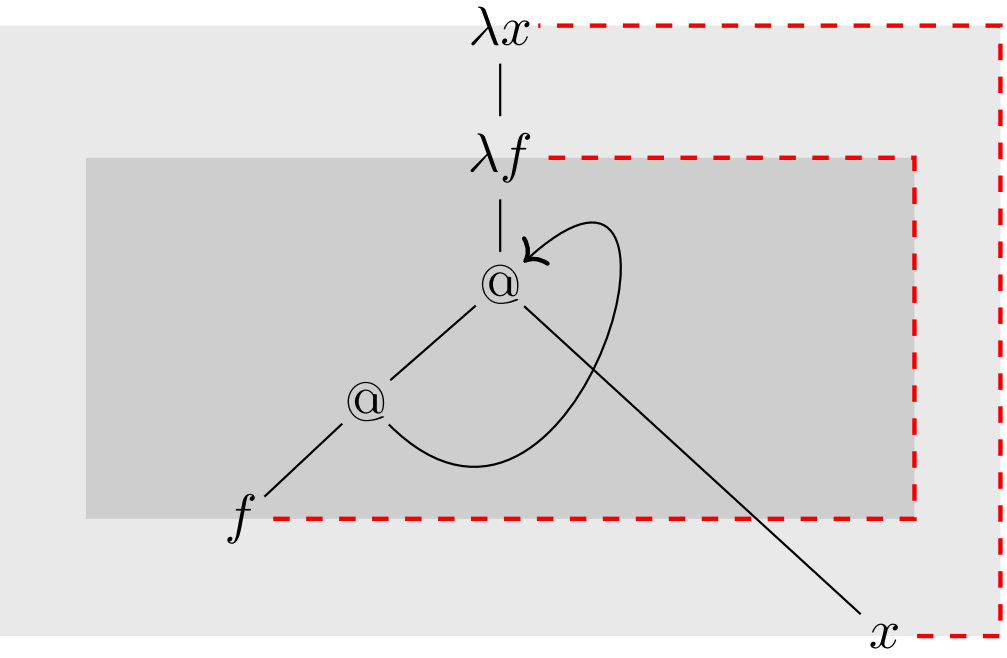

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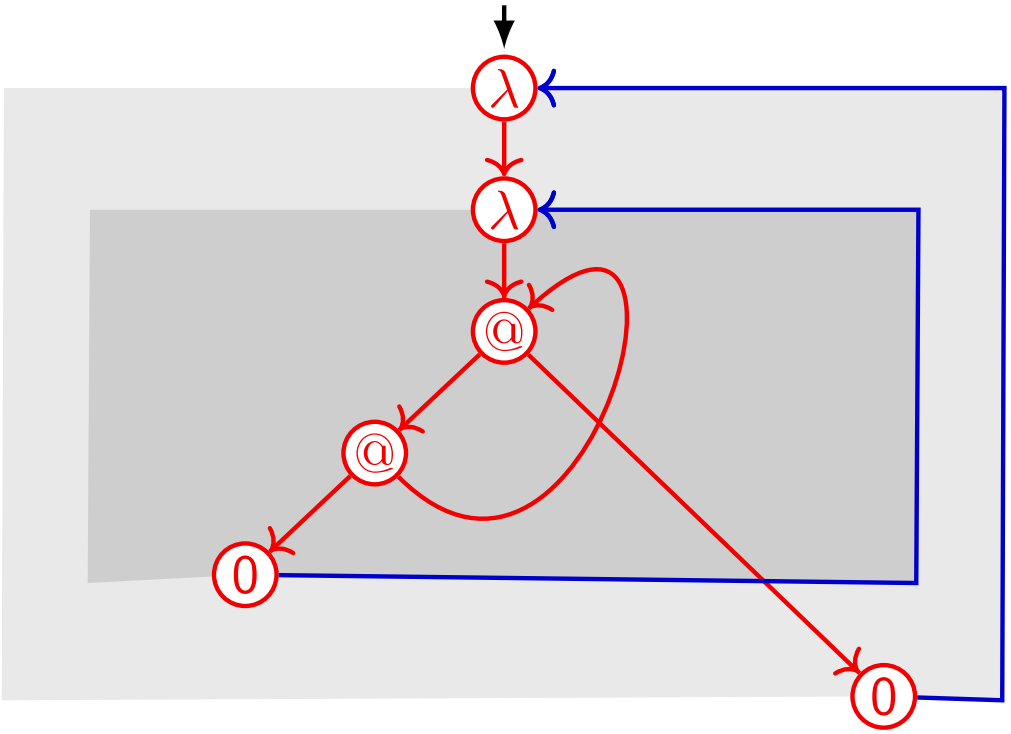

first-order term graph with binding backlinks (+ scope sets)

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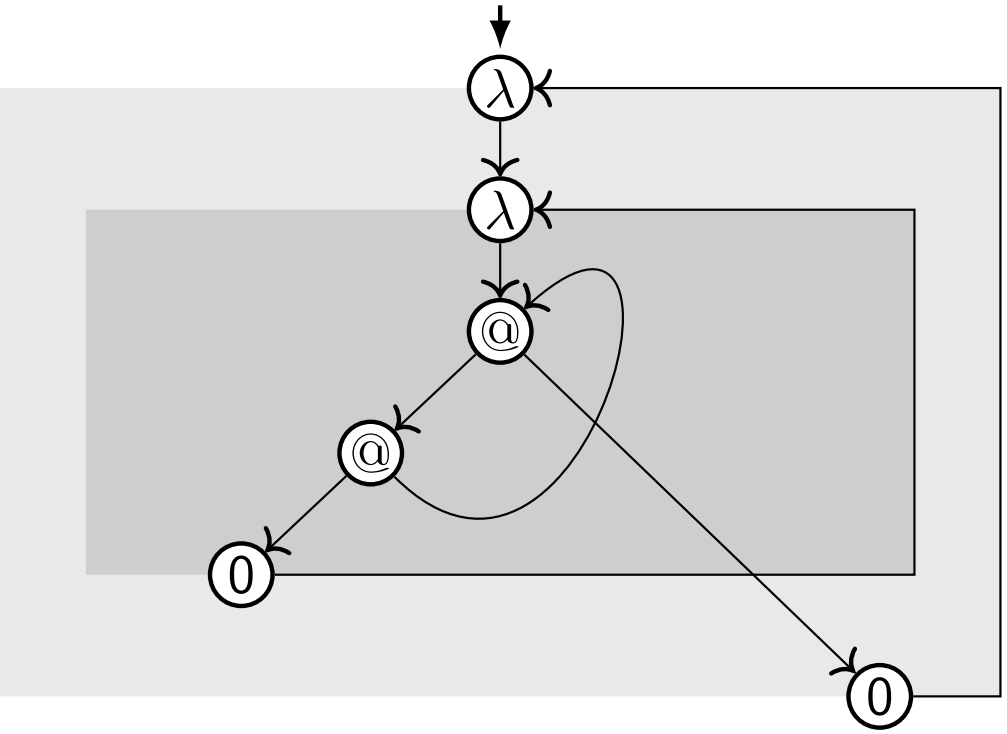

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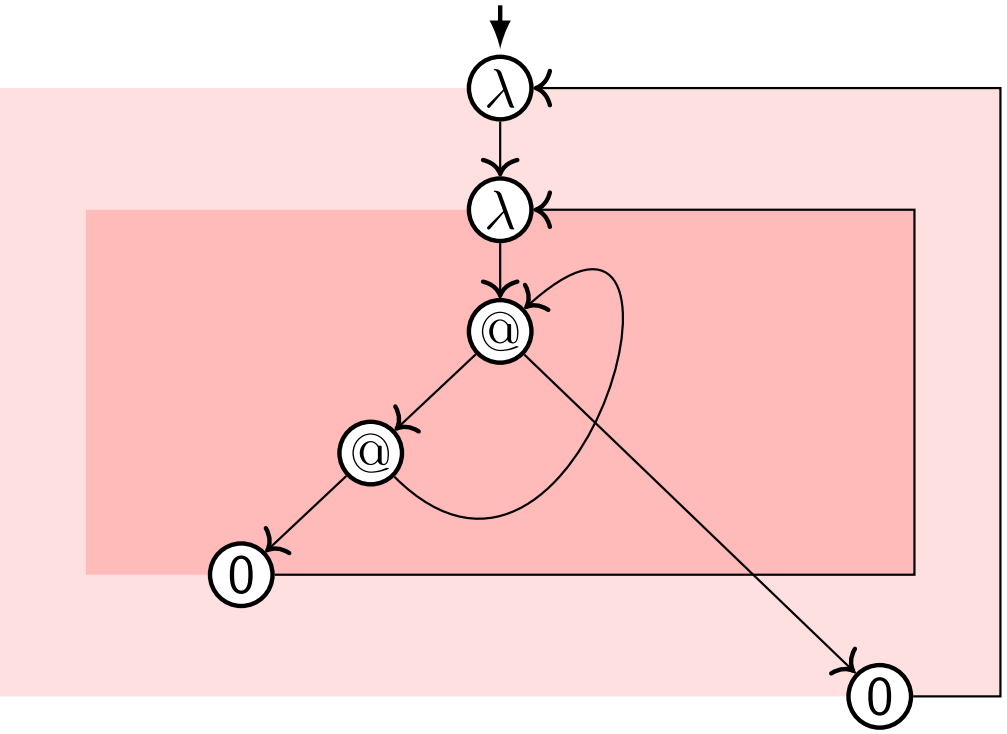

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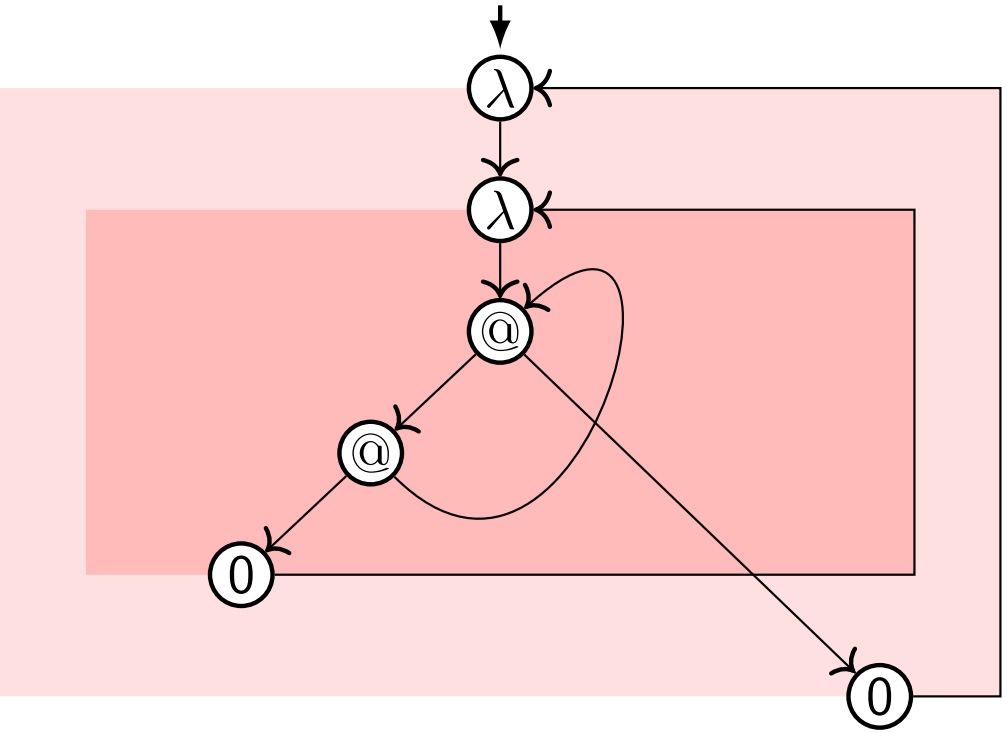

higher-order term graph (with scope sets, Blom [2003])

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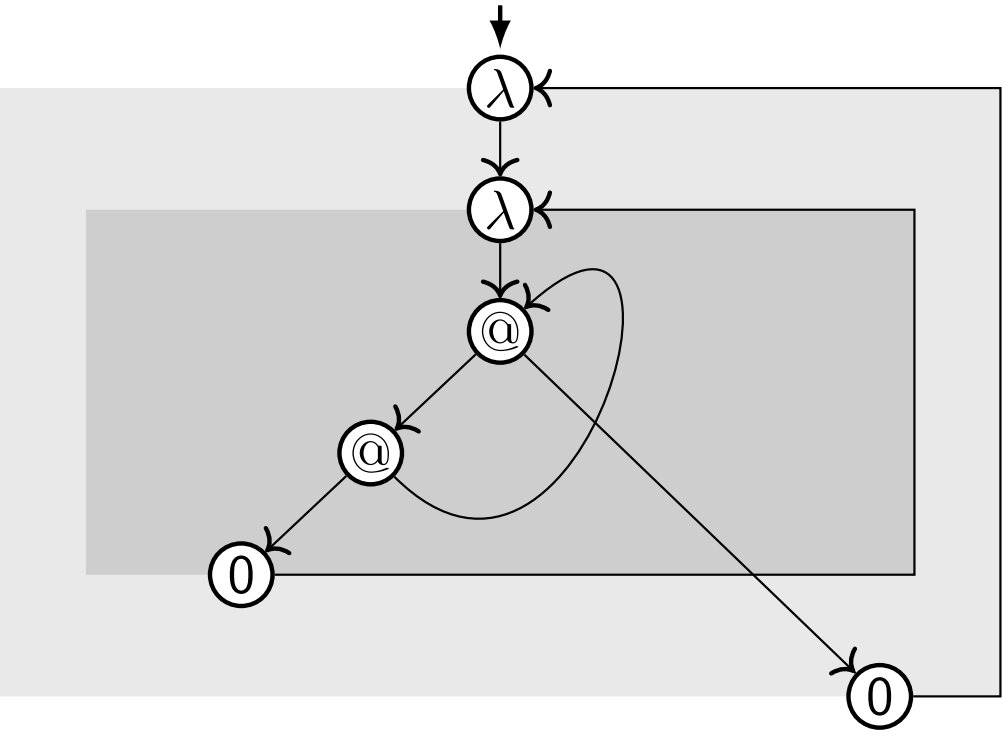

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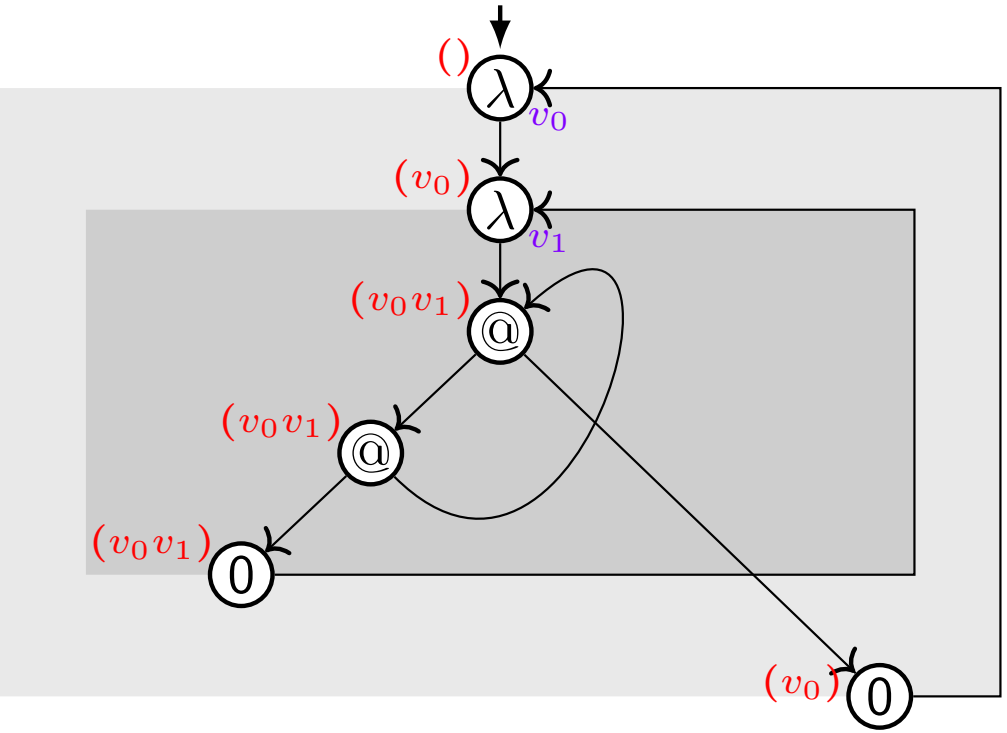

higher-order term graph (with scope sets, + abstraction-prefix function)

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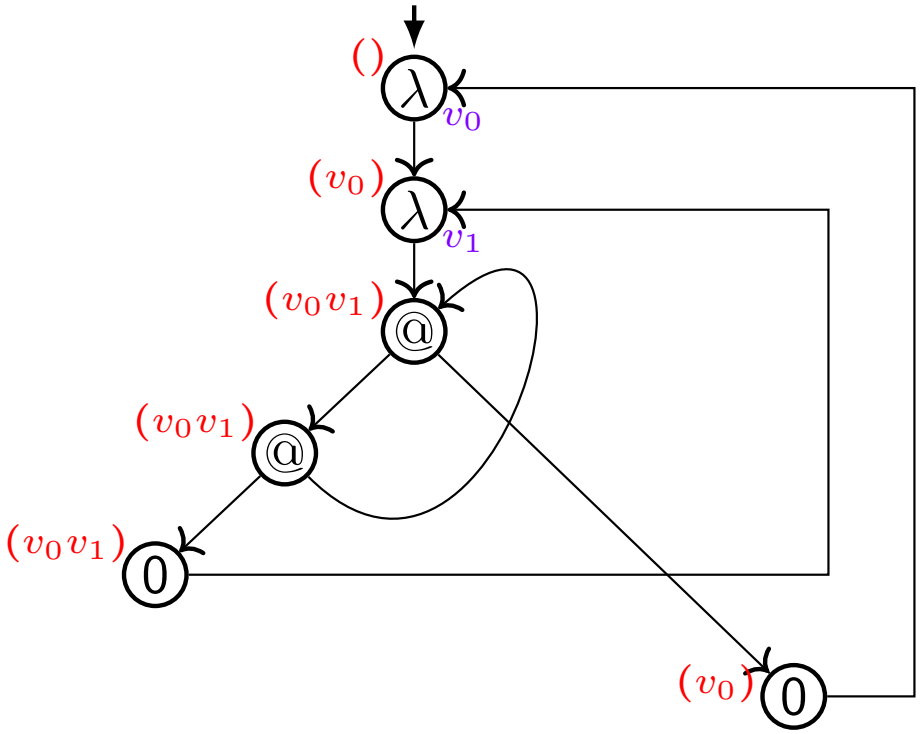

higher-order term graph (with abstraction-prefix function)

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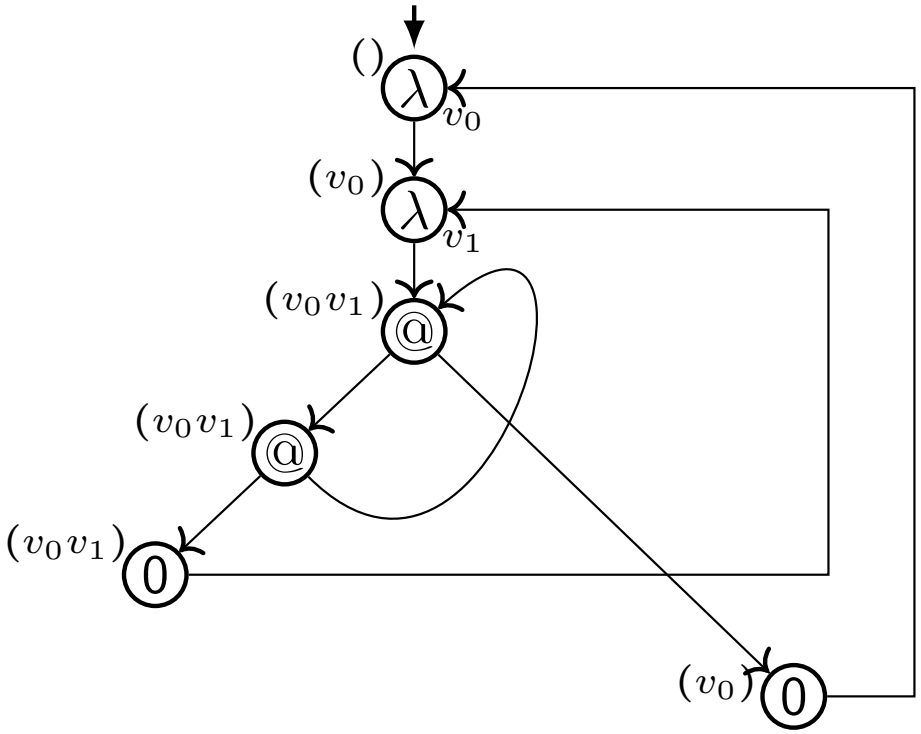

λ-higher-order-term-graph $[\![L_0]\!]_{\mathcal{H}}$

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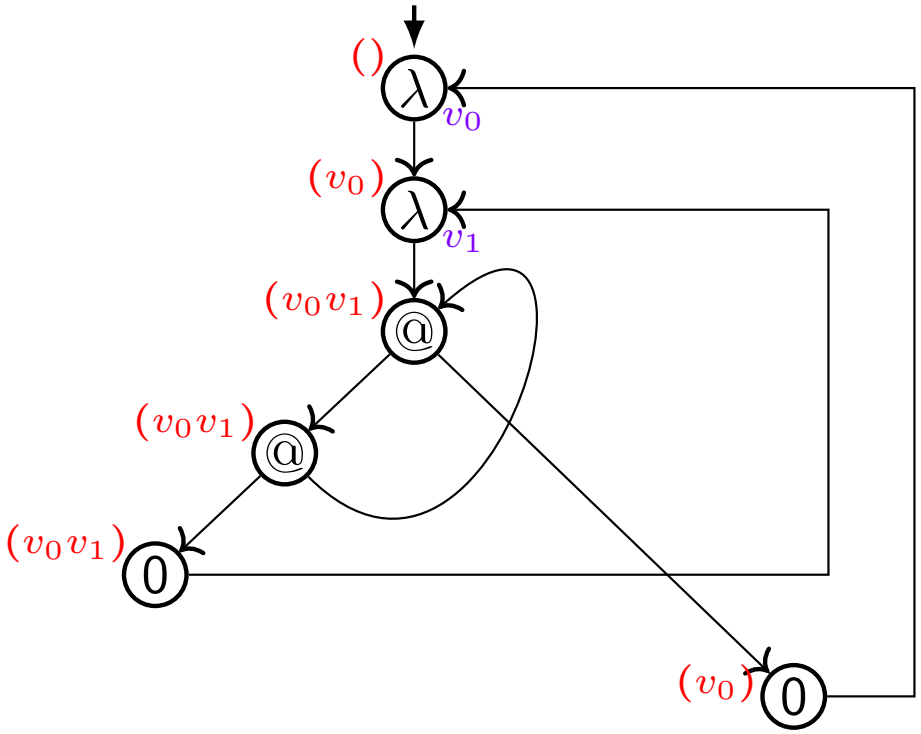

first-order term graph (+ abstraction-prefix function)

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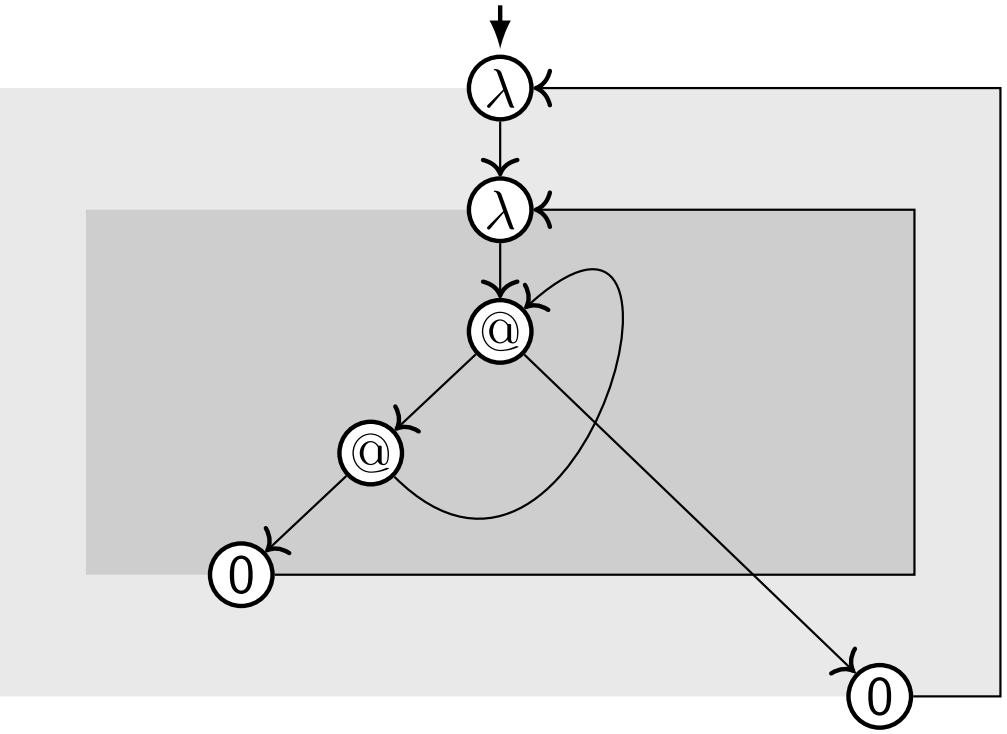

first-order term graph with binding backlinks (+ scope sets)

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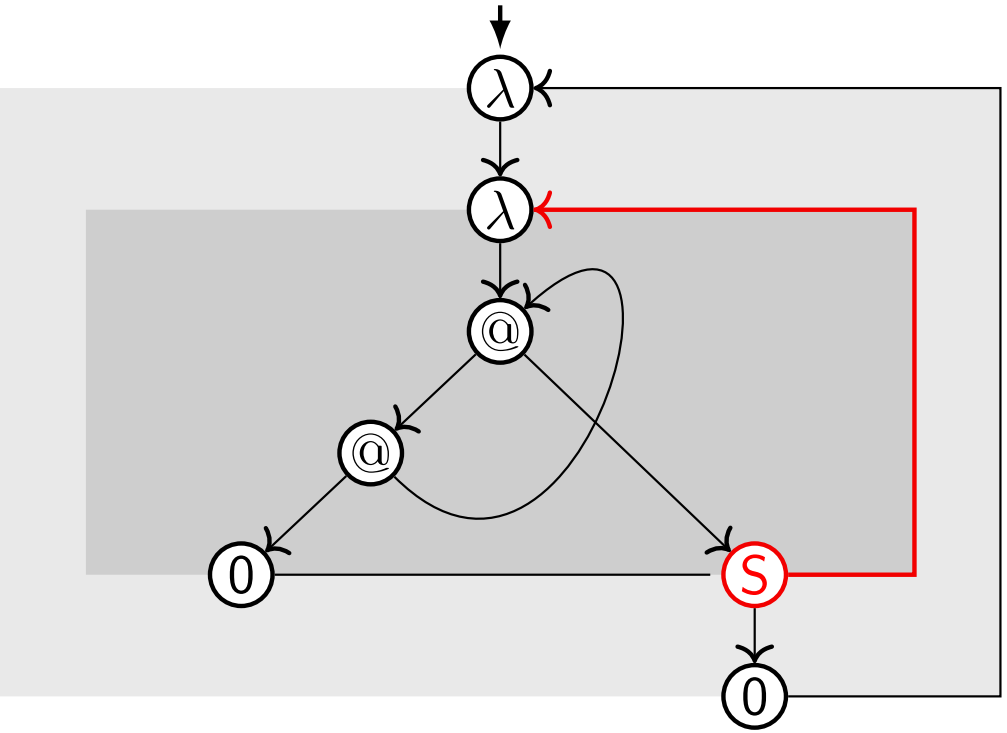

first-order term graph with scope vertices with backlinks (+ scope sets)

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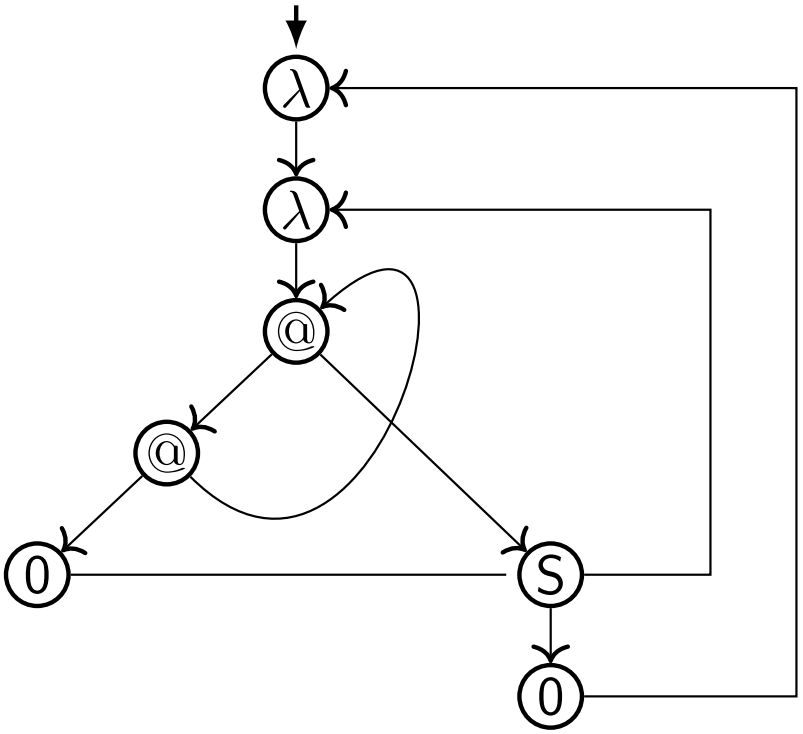

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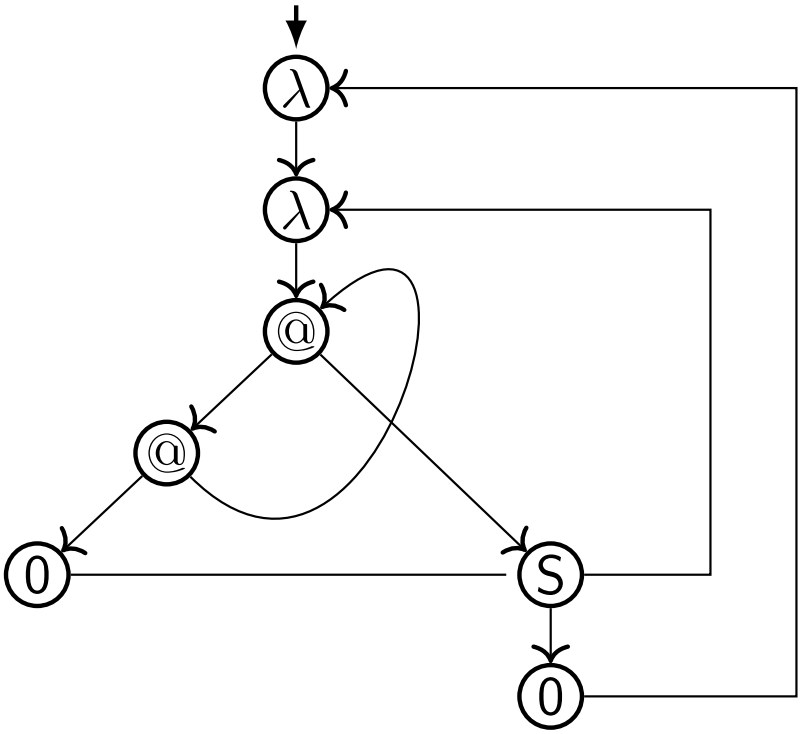

λ-term-graph $[\![L_0]\!]_{\mathcal{T}}$

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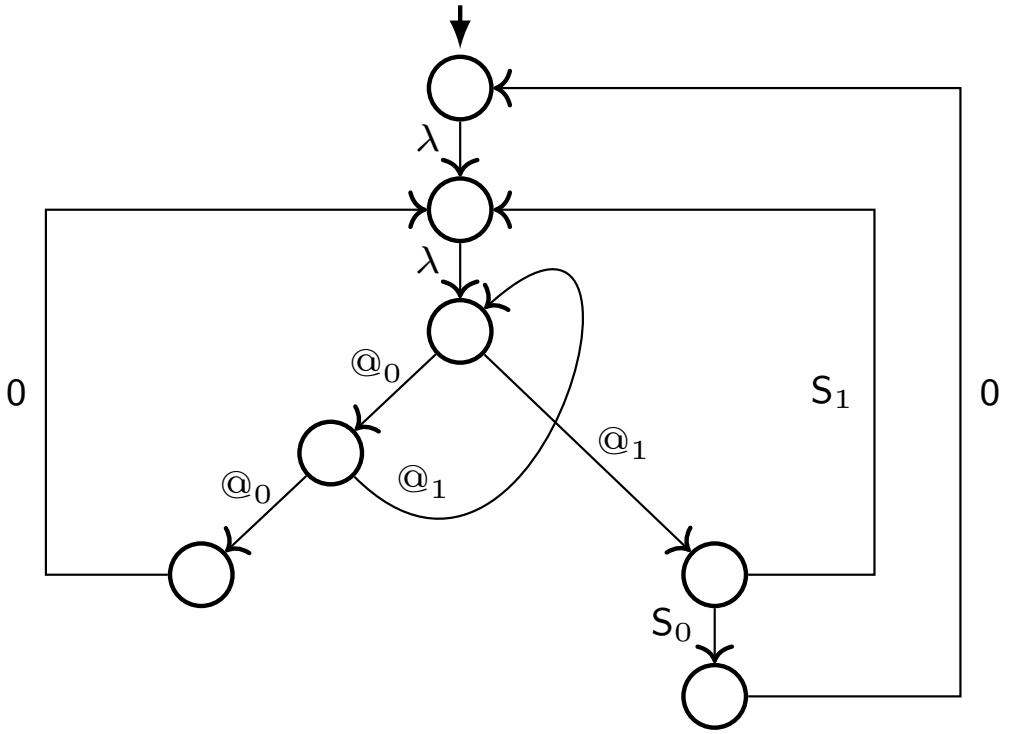

incomplete DFA

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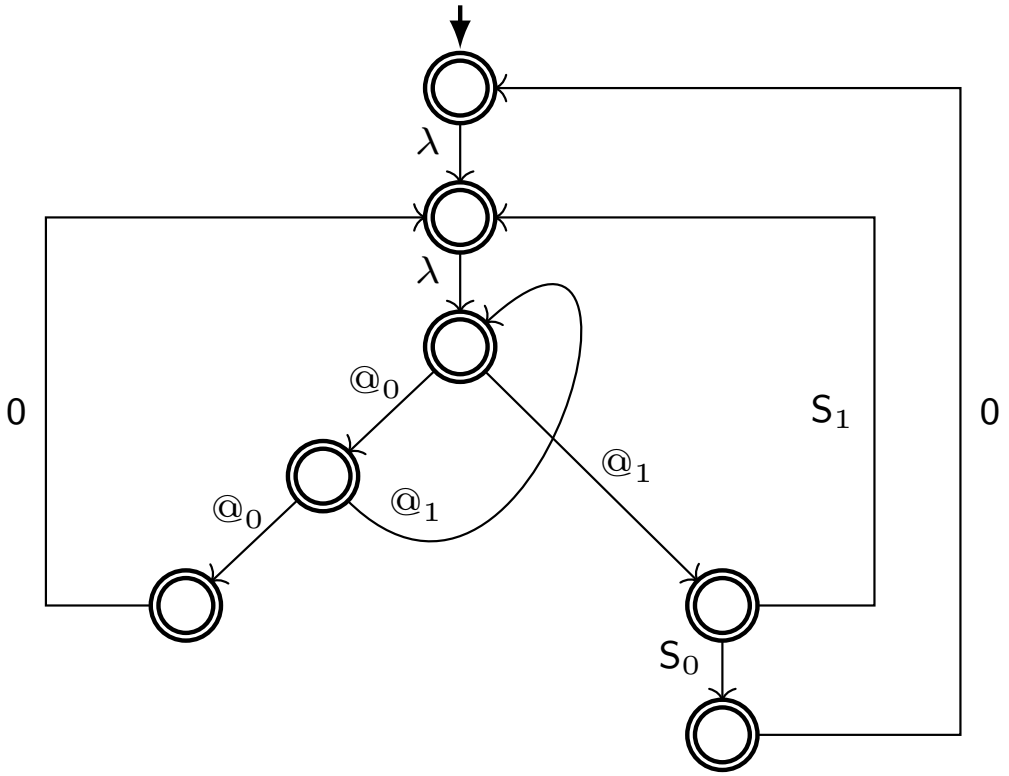

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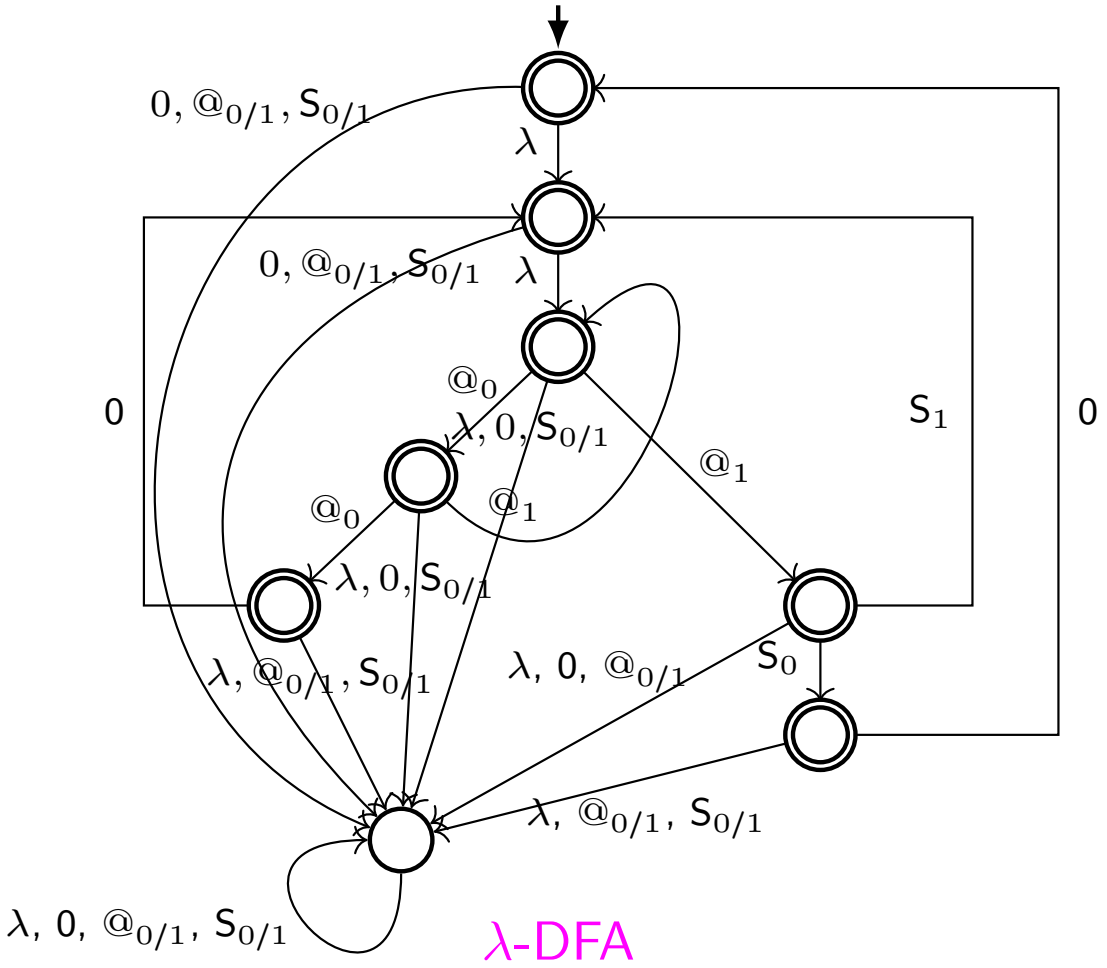

Graph interpretation (example 2)

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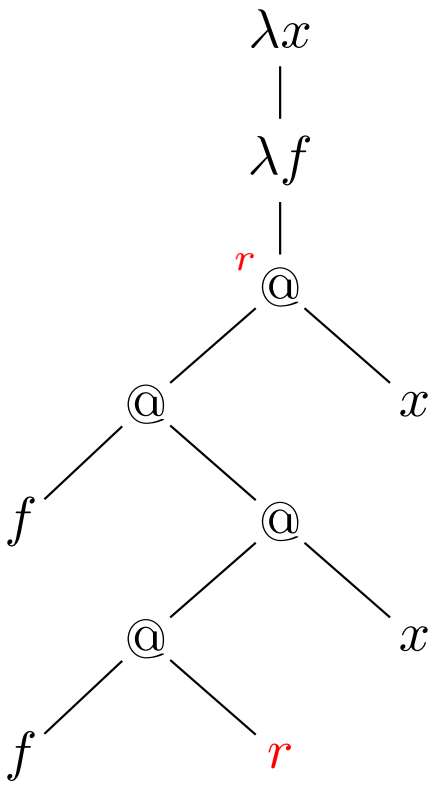

syntax tree

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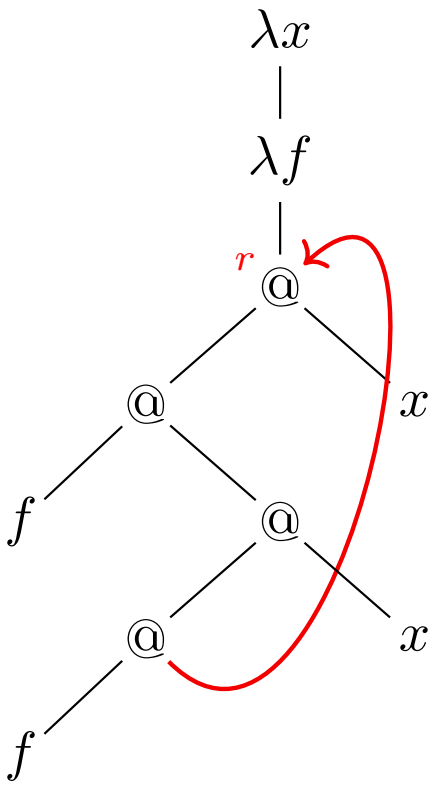

syntax tree (+ recursive backlink)

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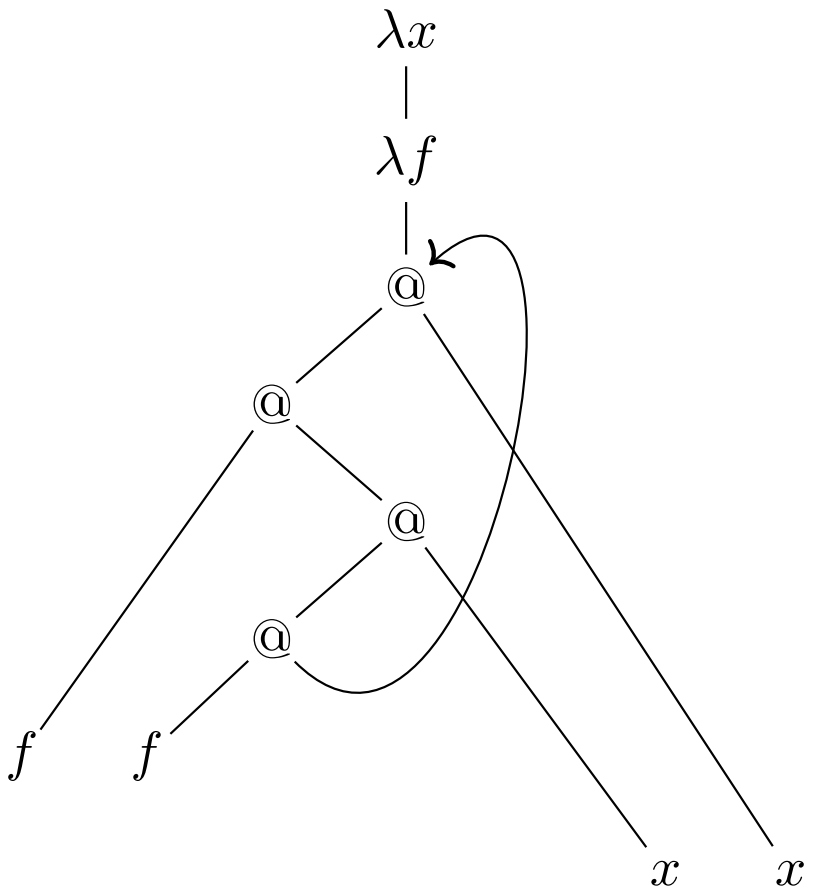

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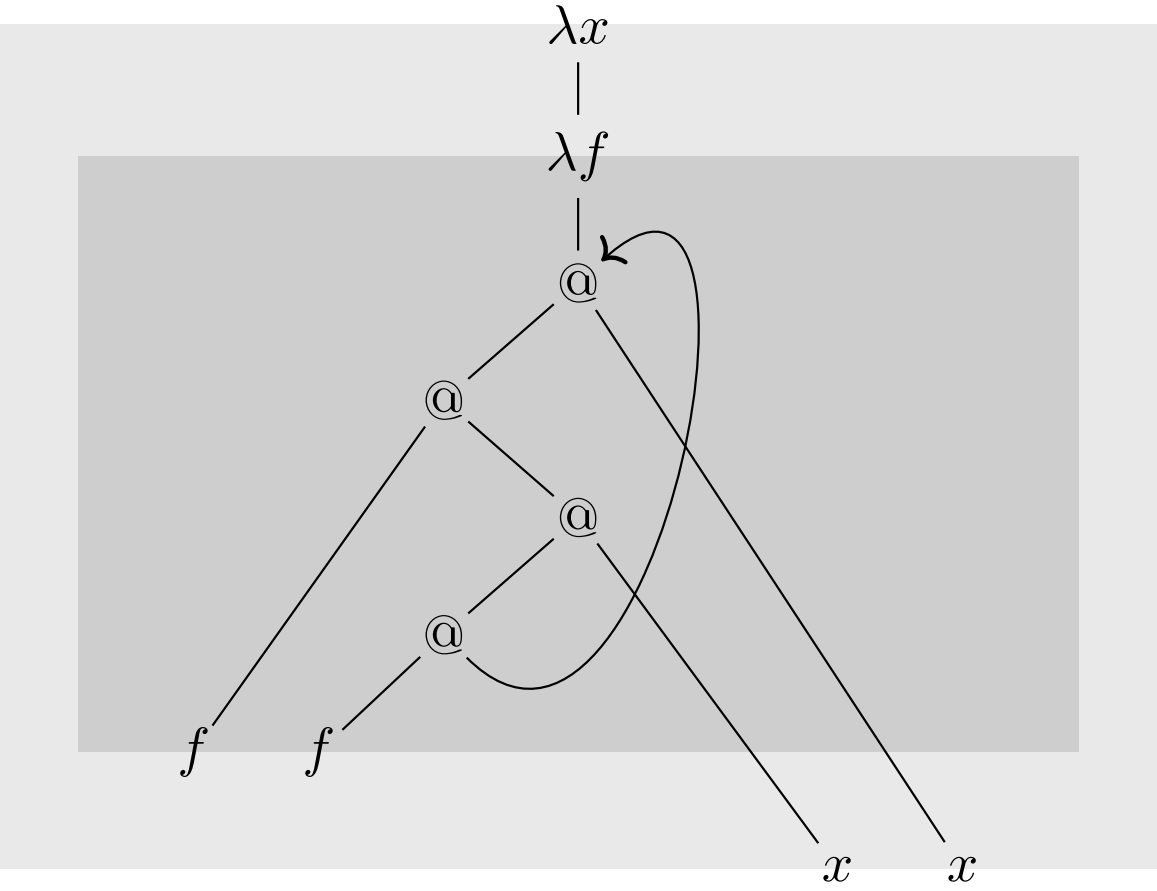

syntax tree (+ recursive backlink, + scopes)

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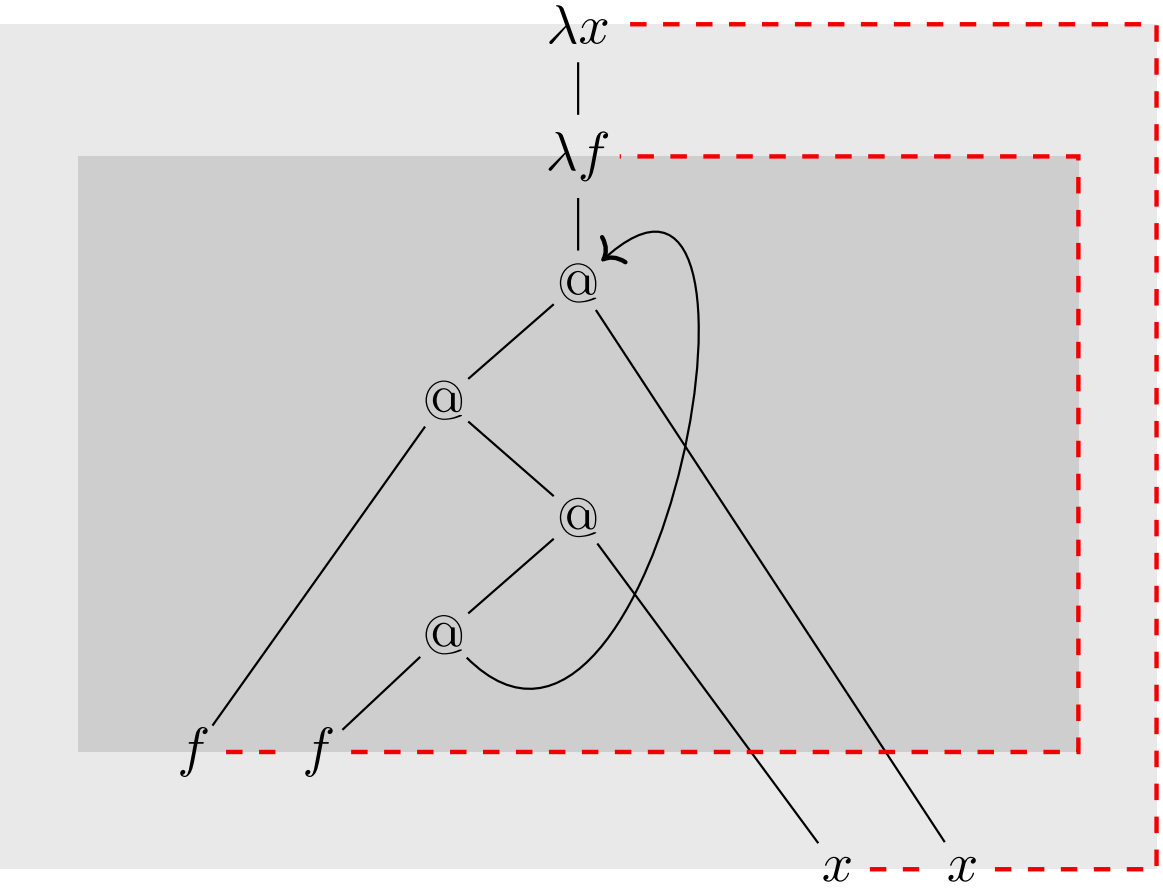

syntax tree (+ recursive backlink, + scopes, + binding links)

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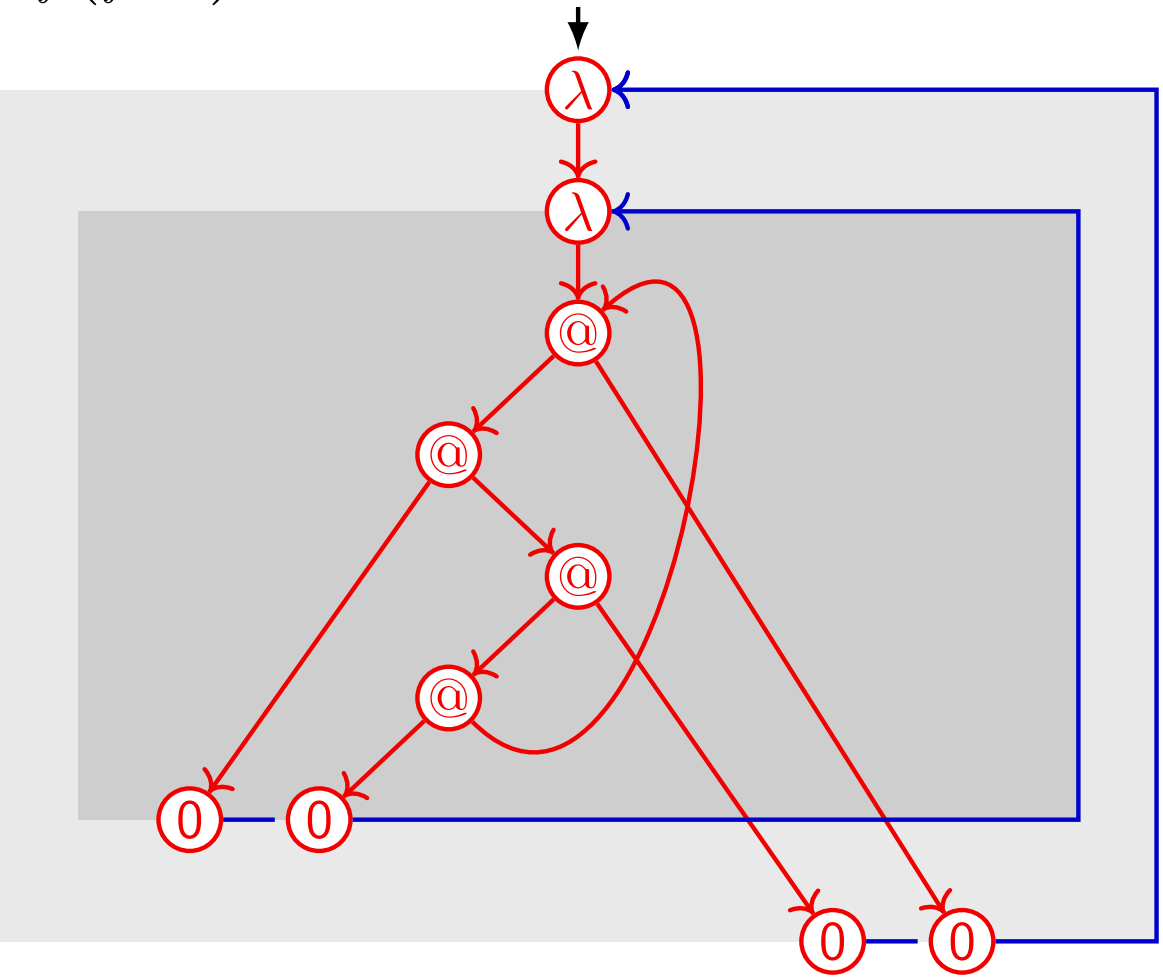

first-order term graph with binding backlinks (+ scope sets)

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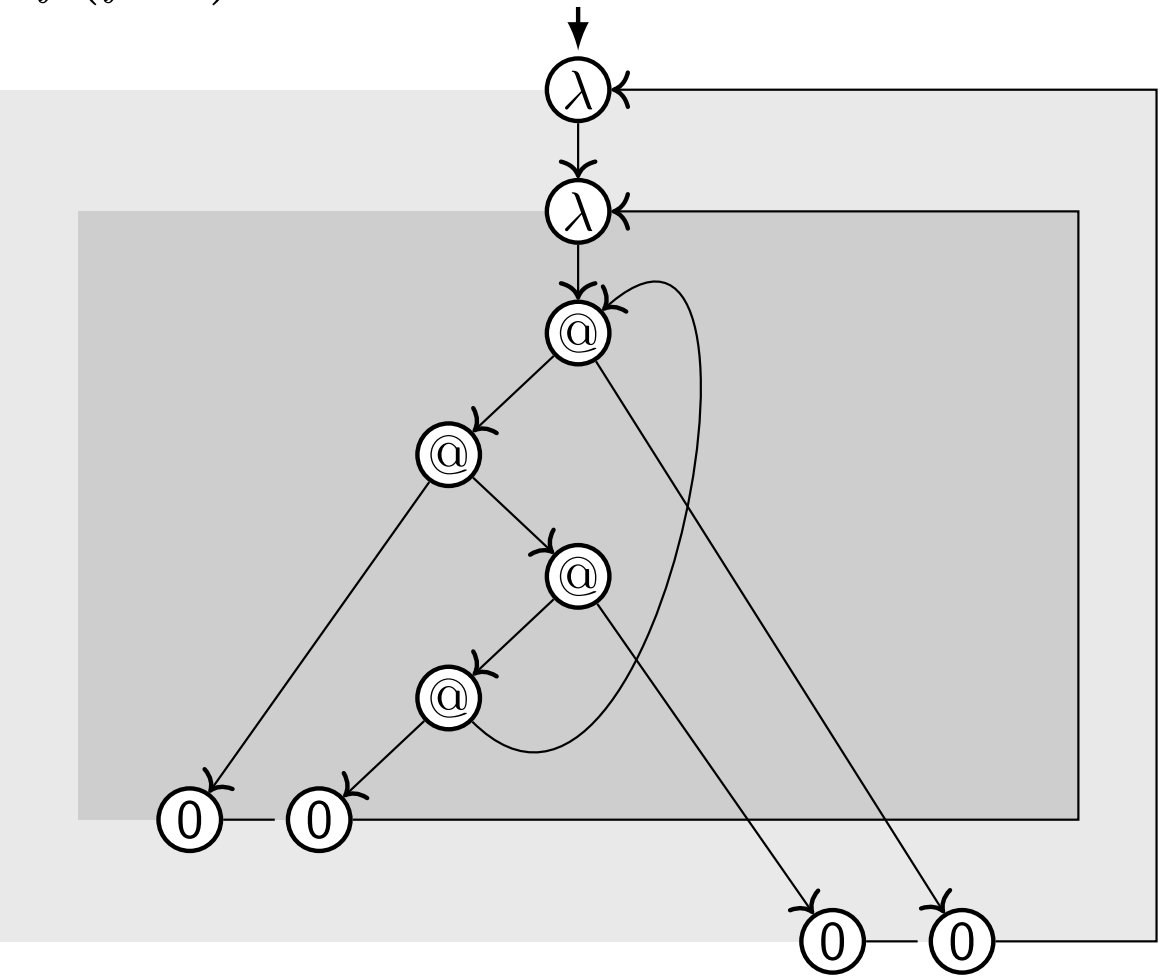

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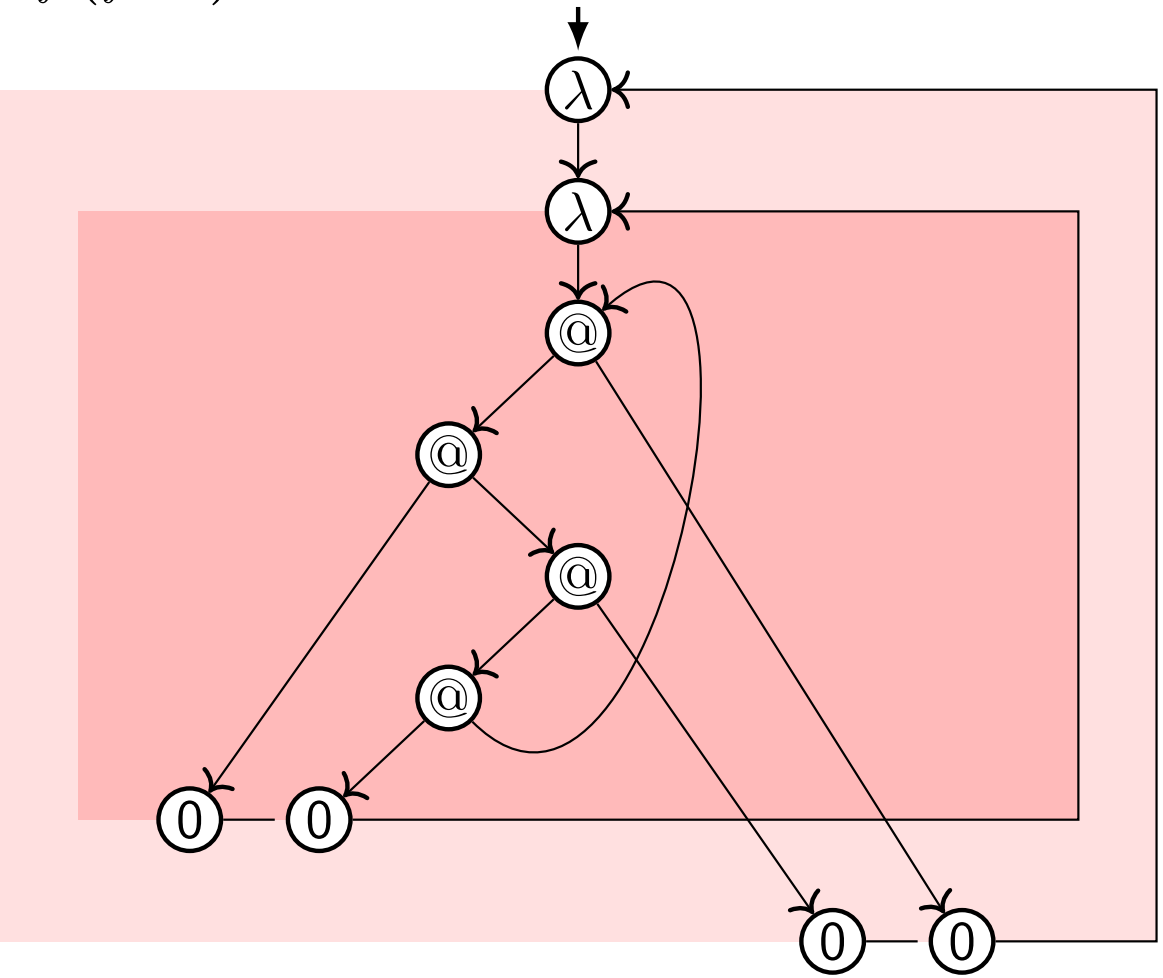

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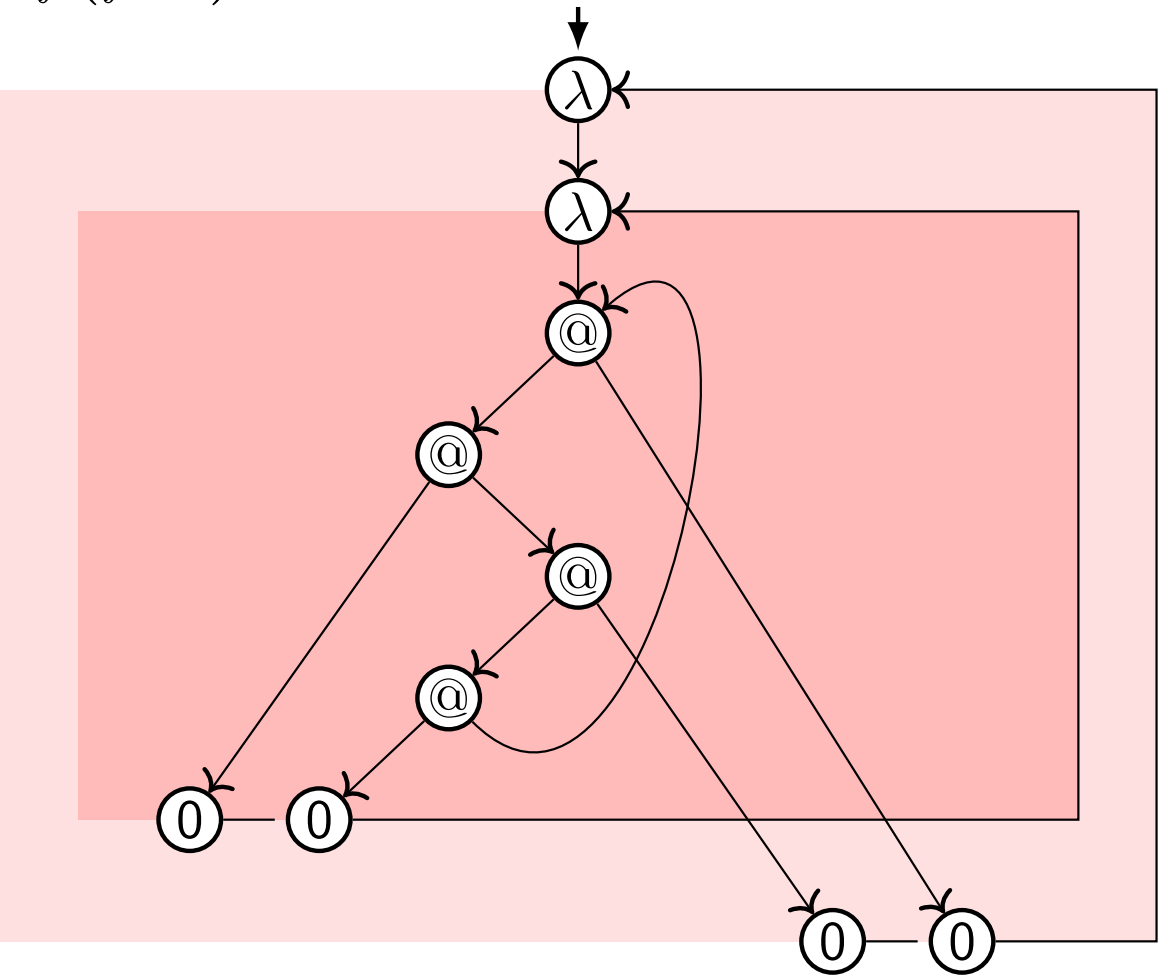

higher-order term graph (with scope sets, Blom [2003])

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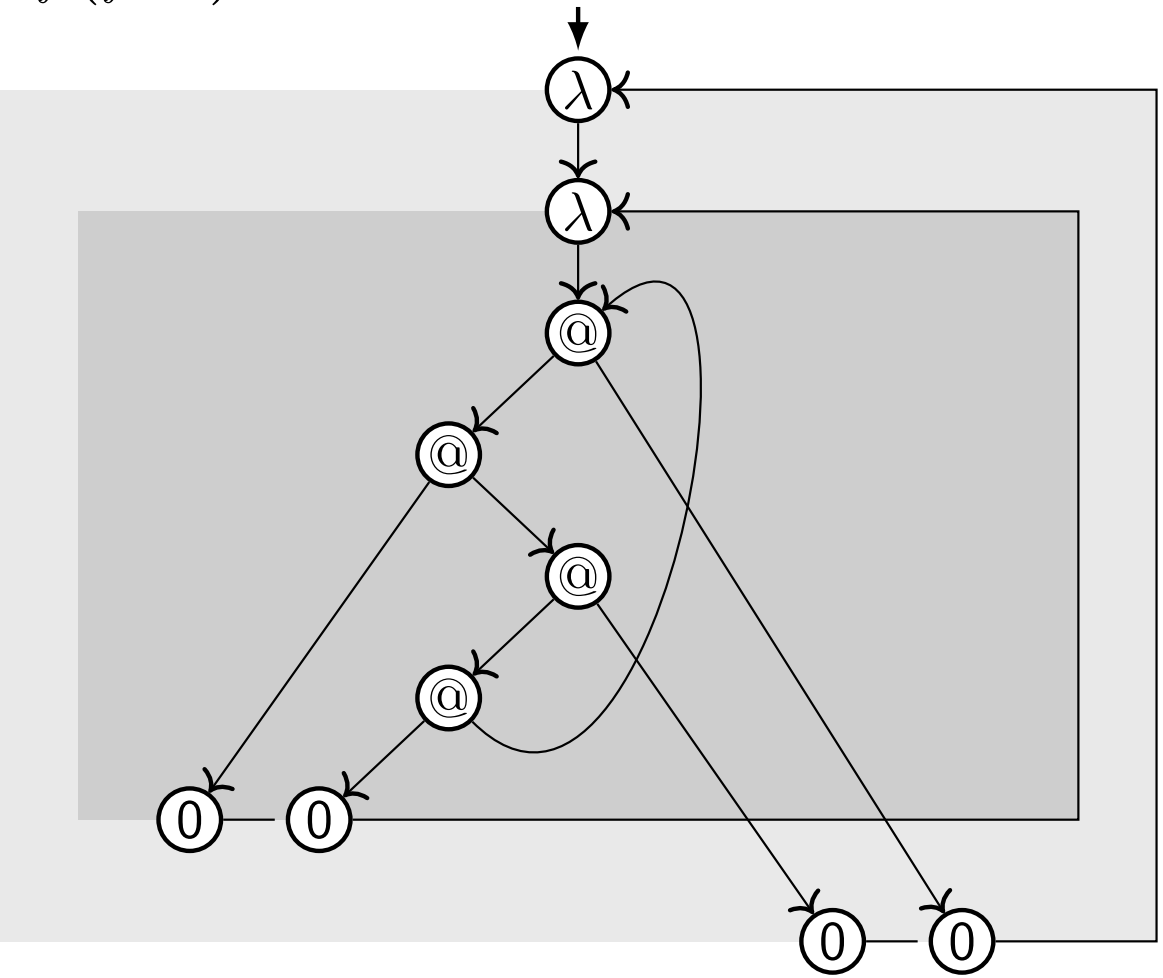

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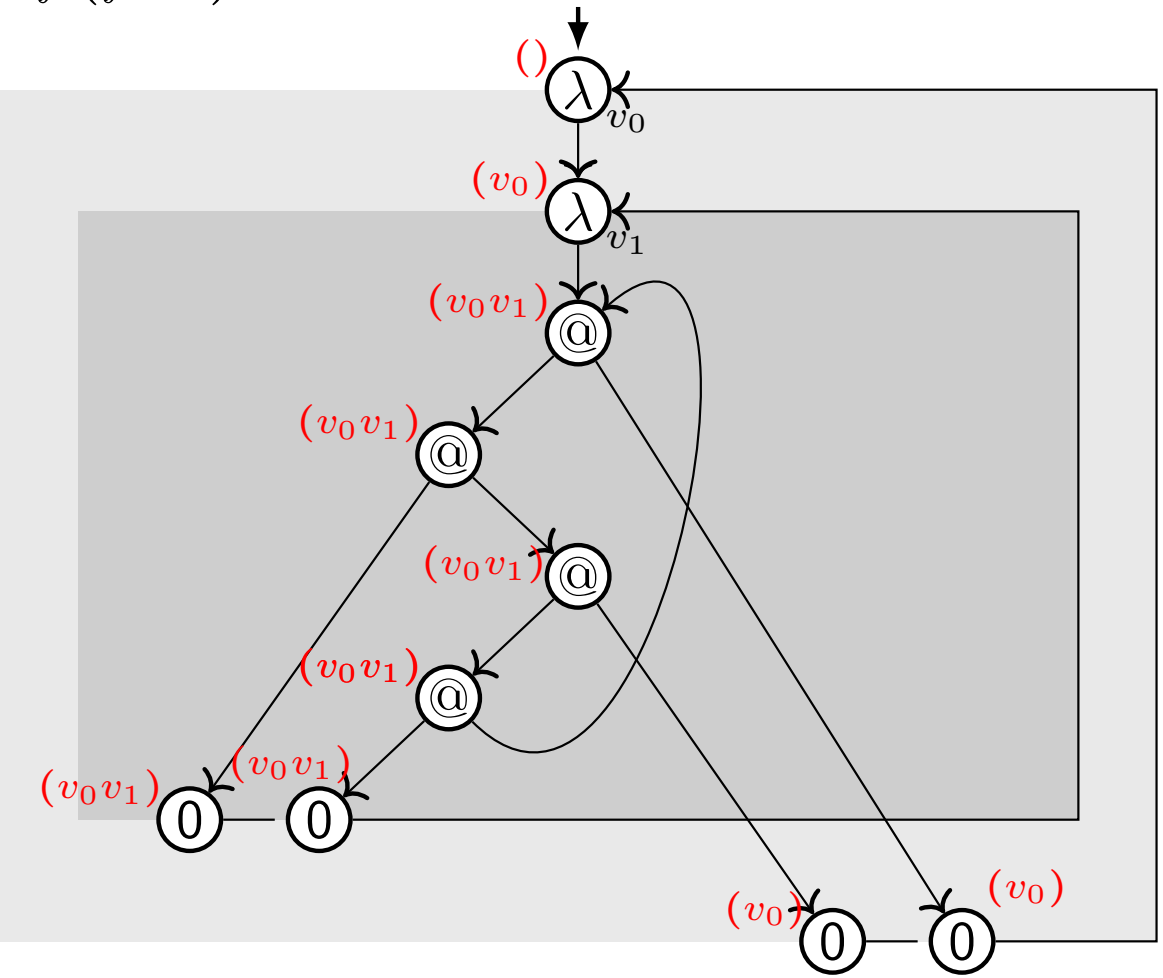

higher-order term graph (with scope sets, + abstraction-prefix function)

Graph interpretation (example 2)

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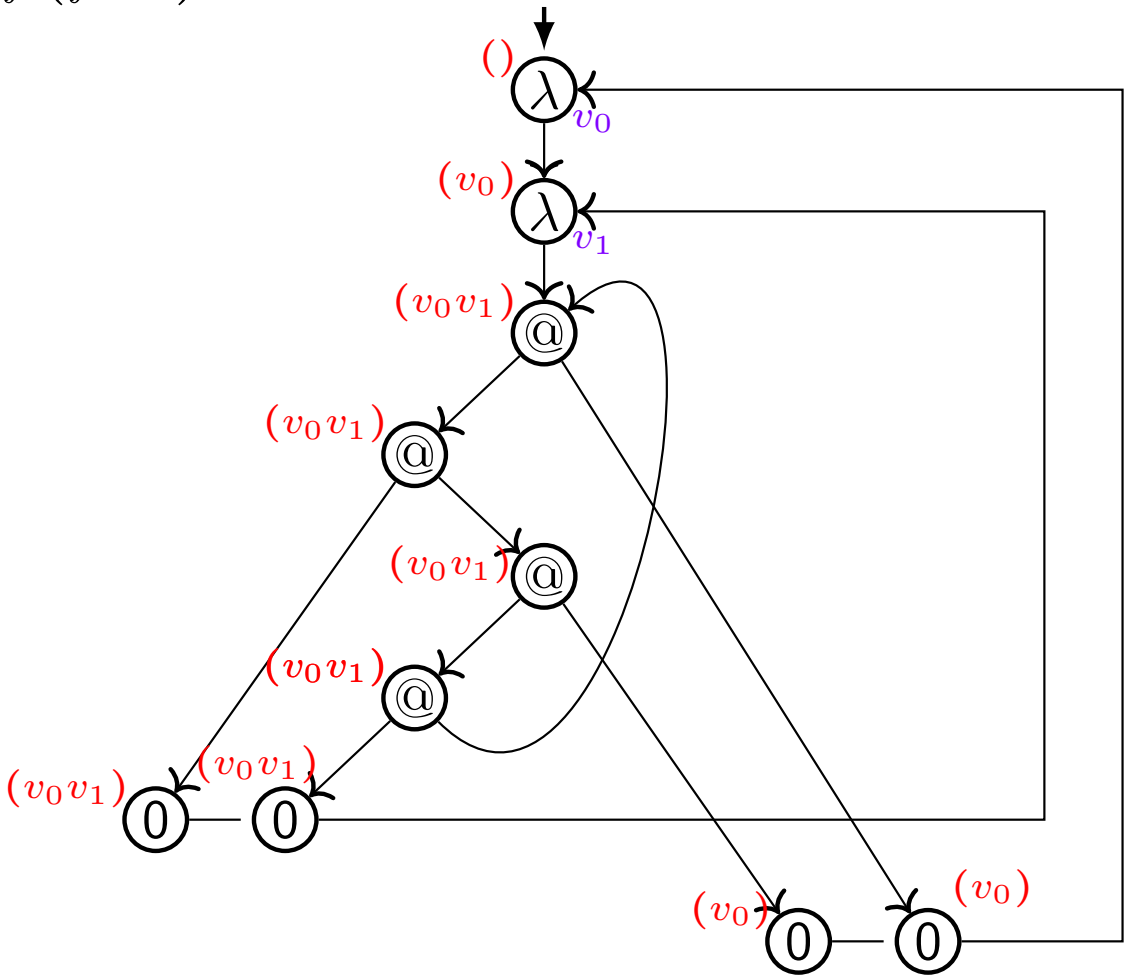

higher-order term graph (with abstraction-prefix function)

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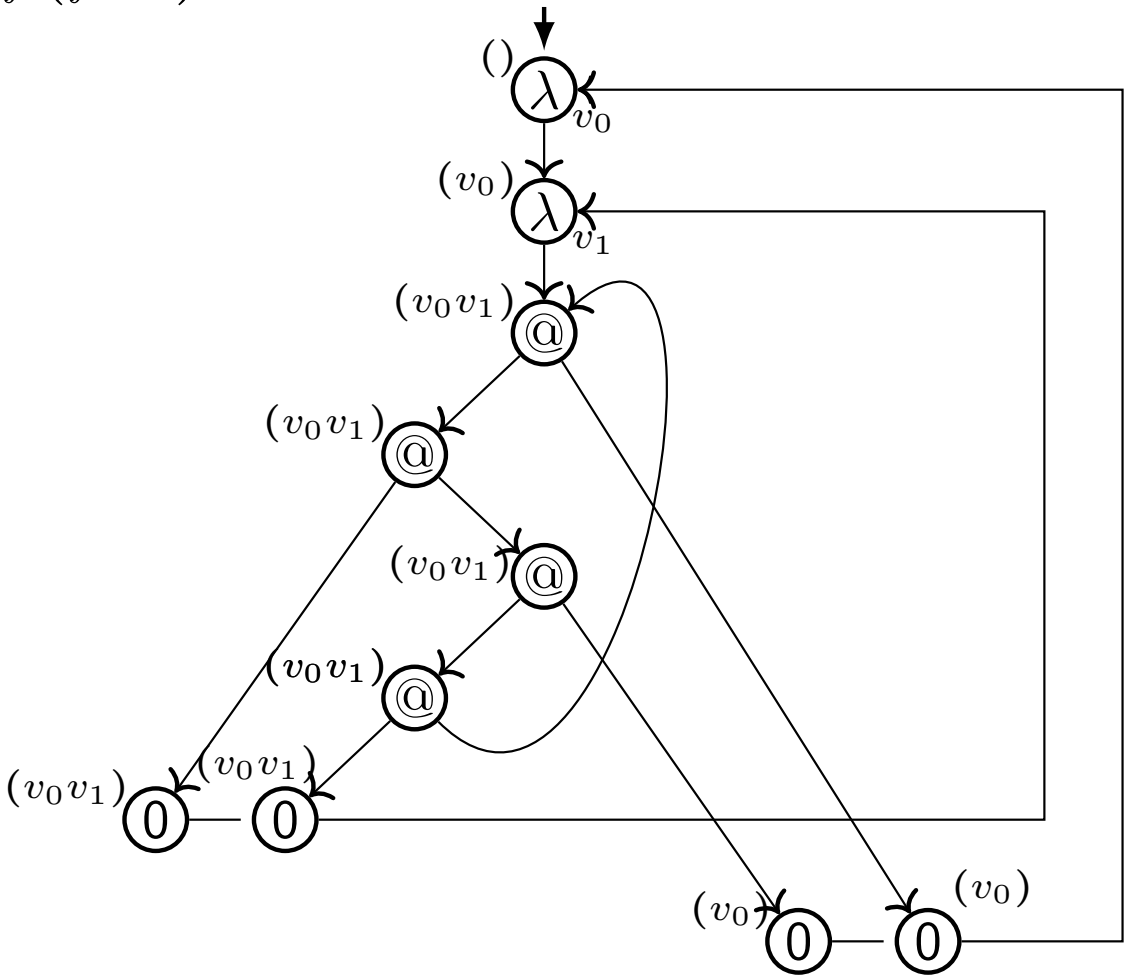

λ-higher-order-term-graph $[\![L]\!]_{\mathcal{H}}$

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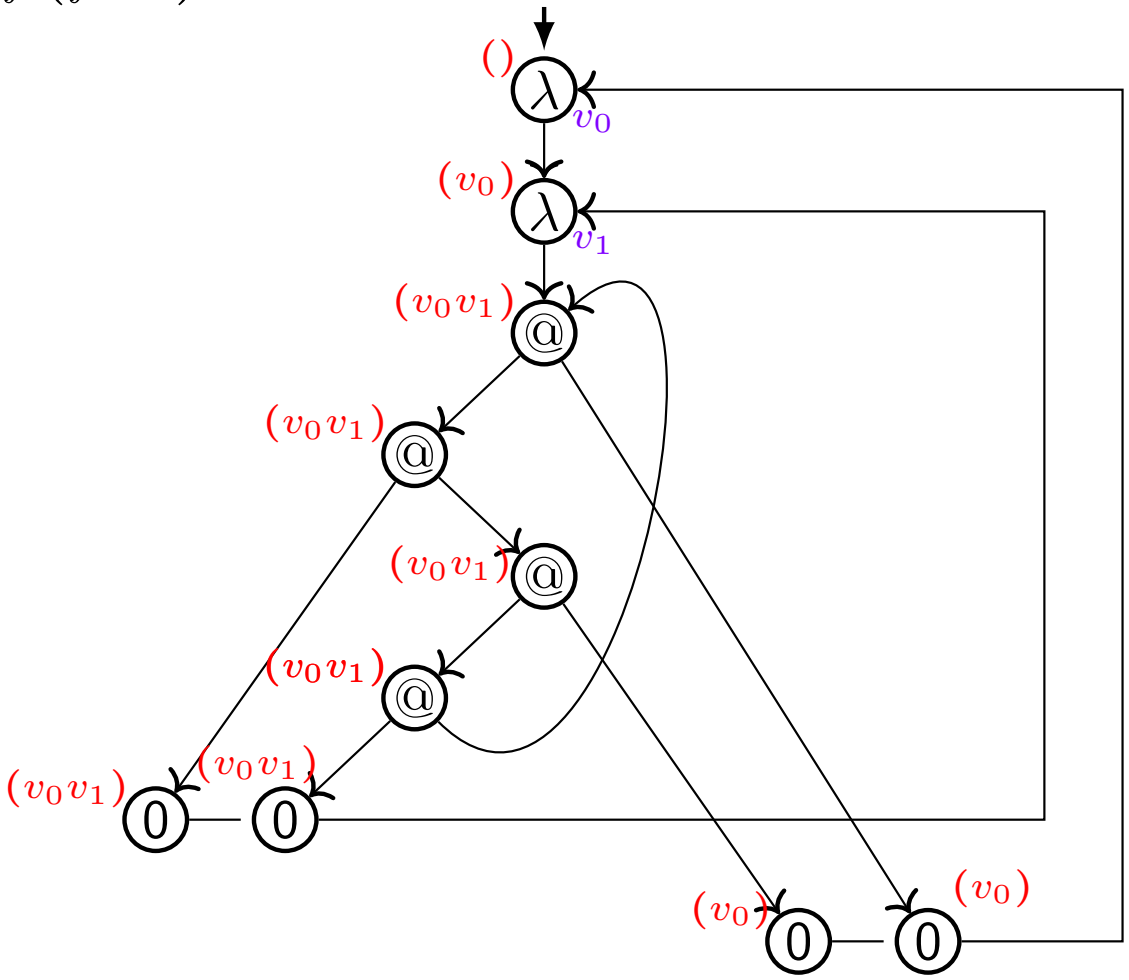

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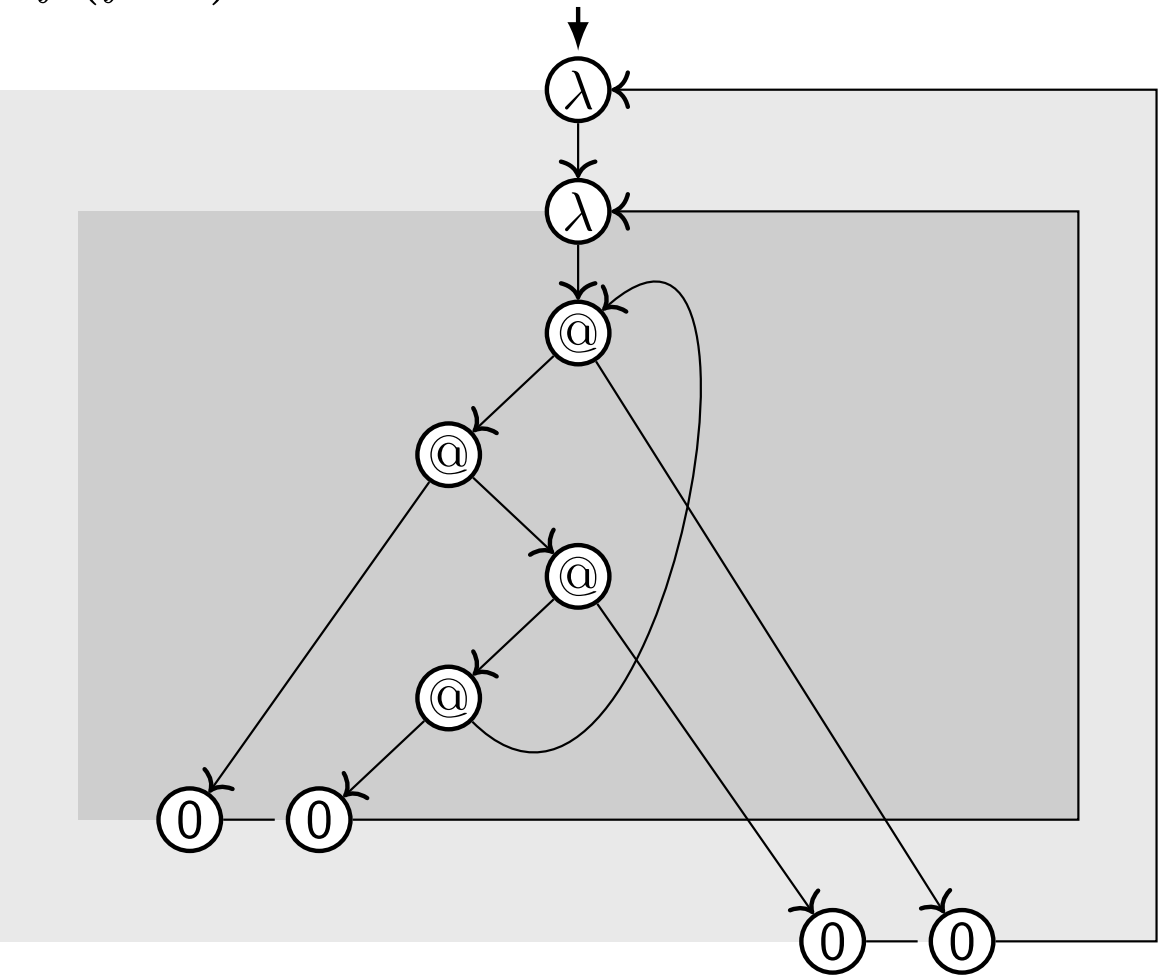

first-order term graph with binding backlinks (+ scope sets)

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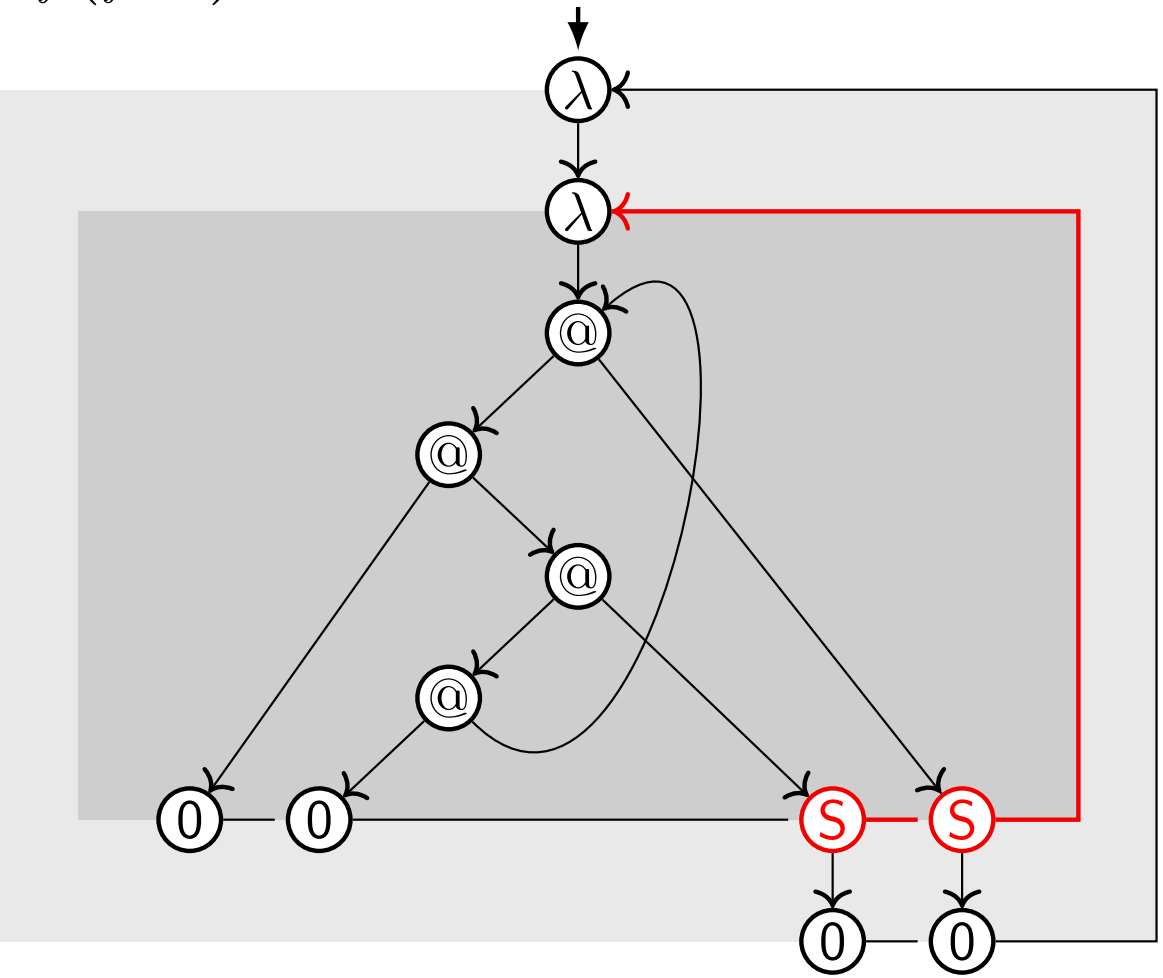

first-order term graph with scope vertices with backlinks (+ scope sets)

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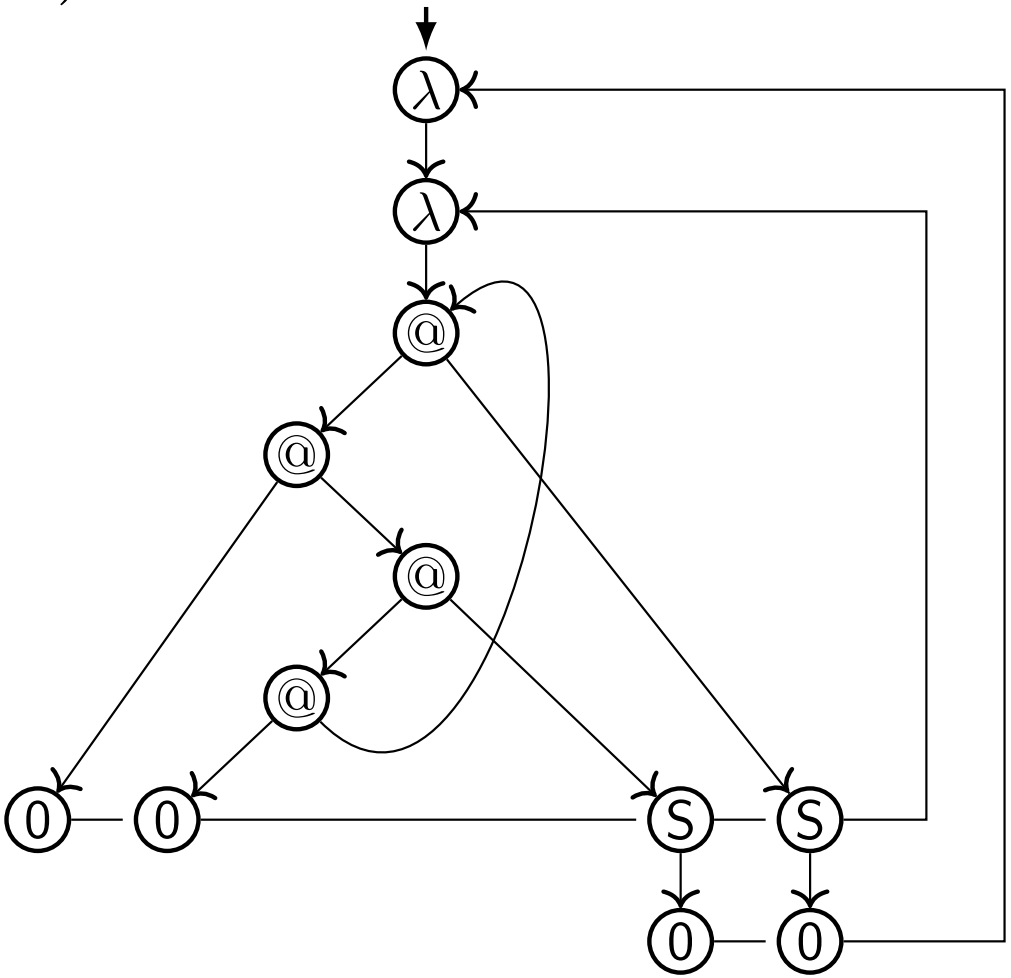

first-order term graph with scope vertices with backlinks

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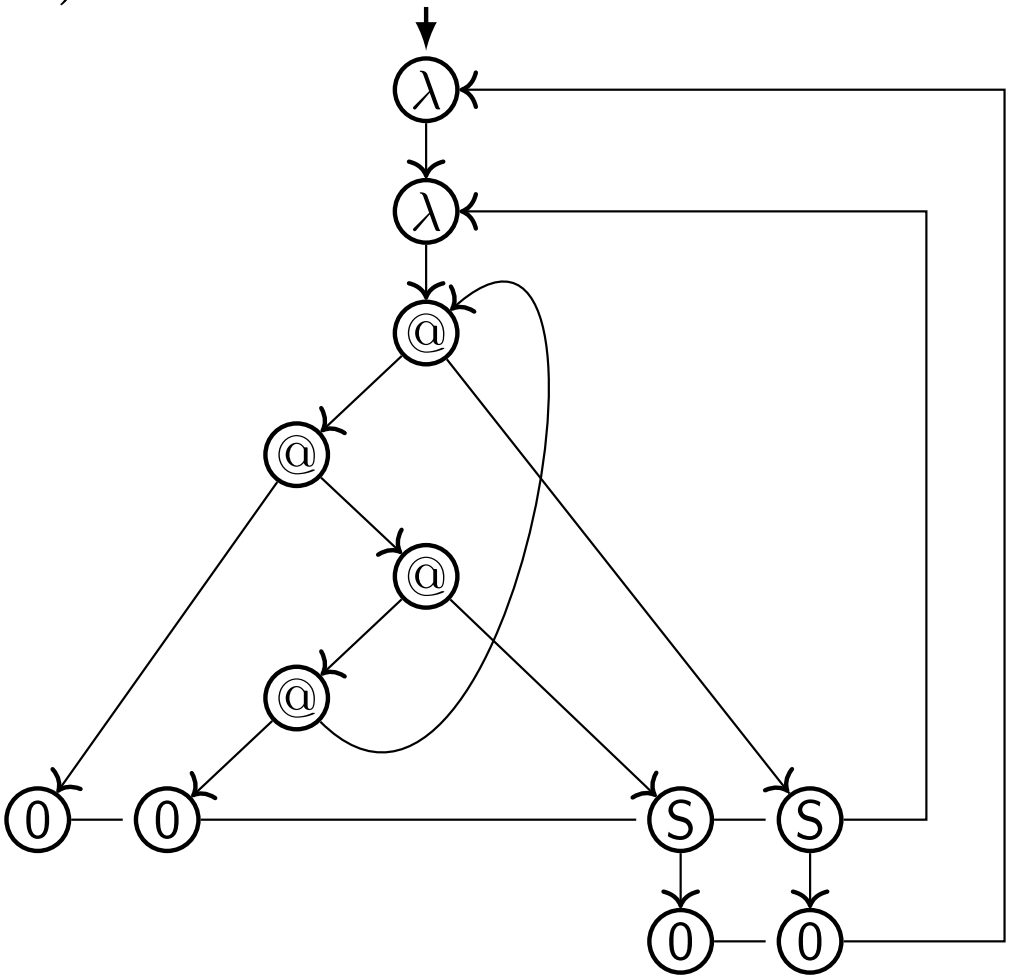

λ-term-graph $[\![L]\!]_{\mathcal{T}}$

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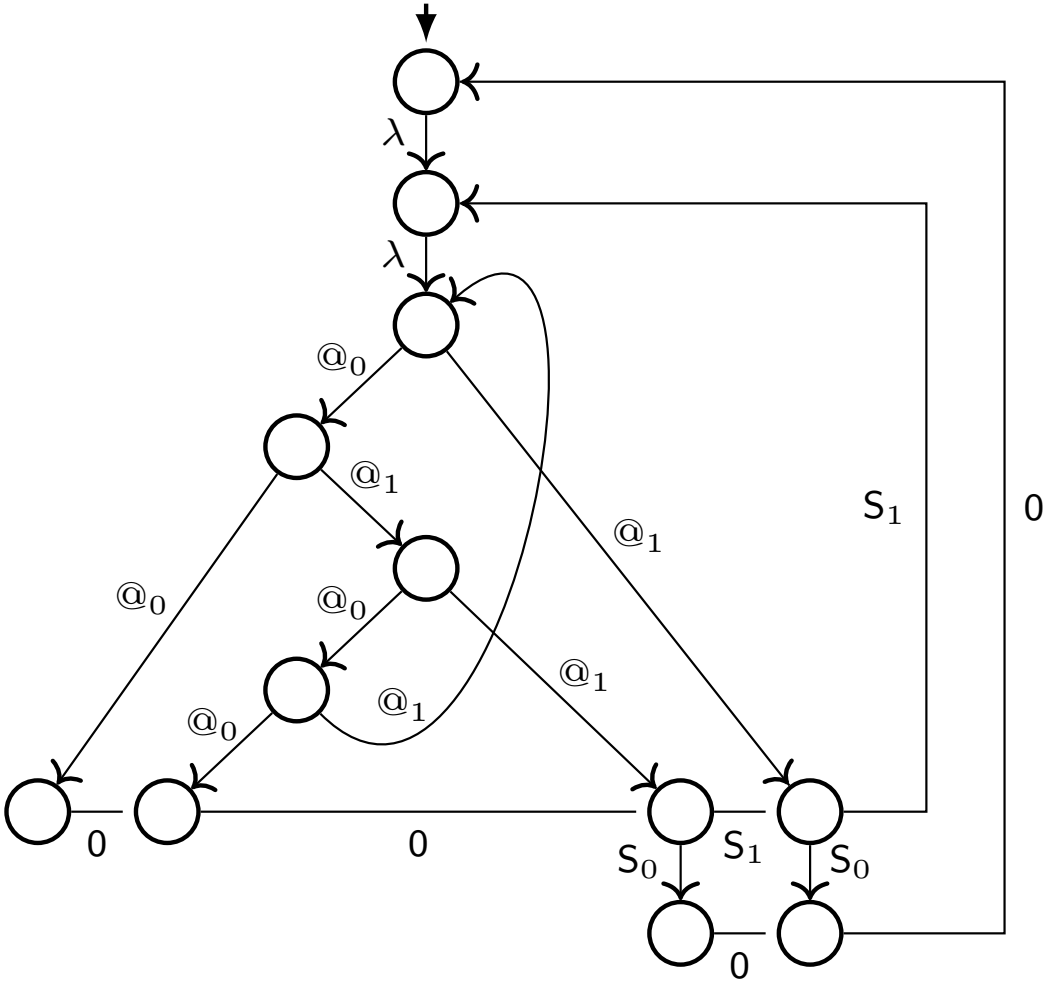

incomplete DFA

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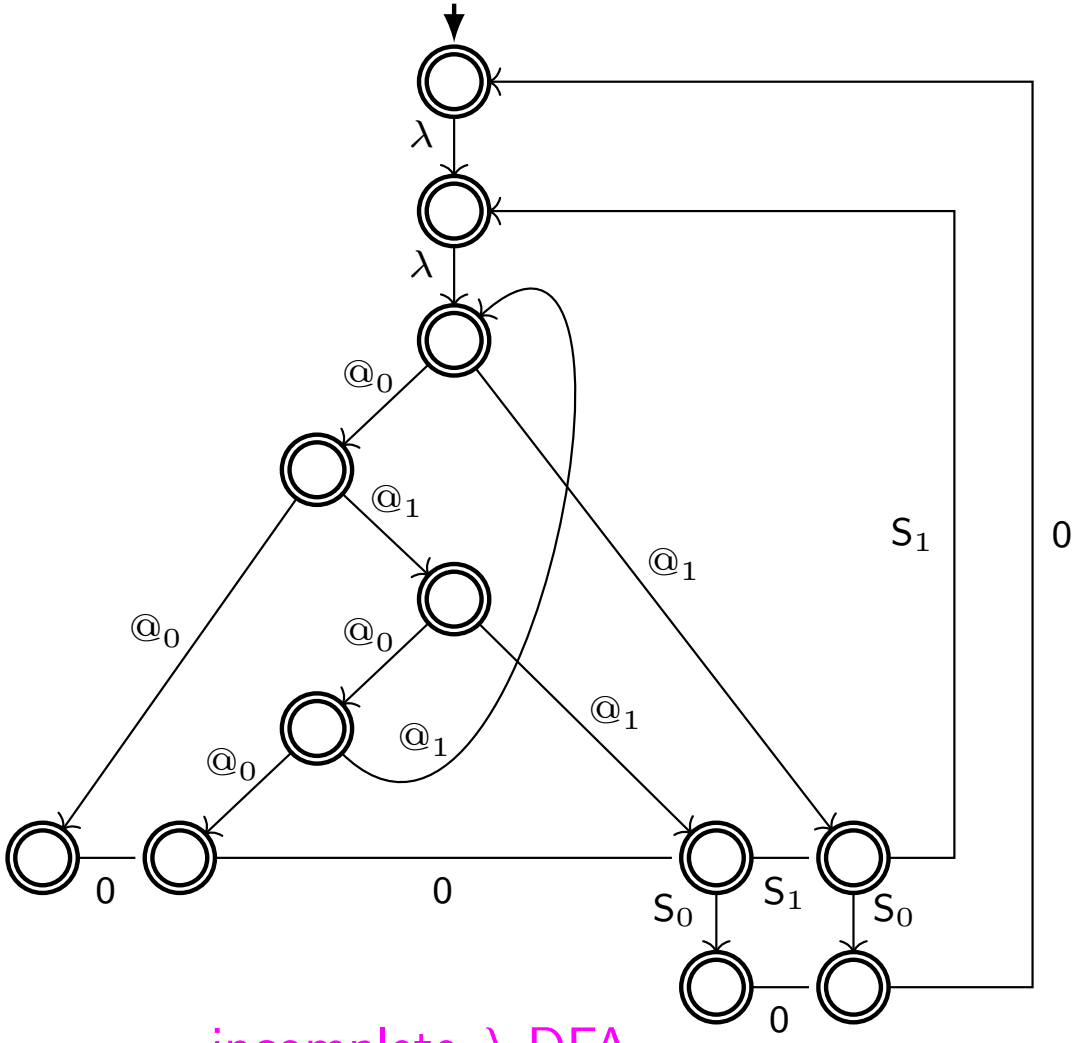

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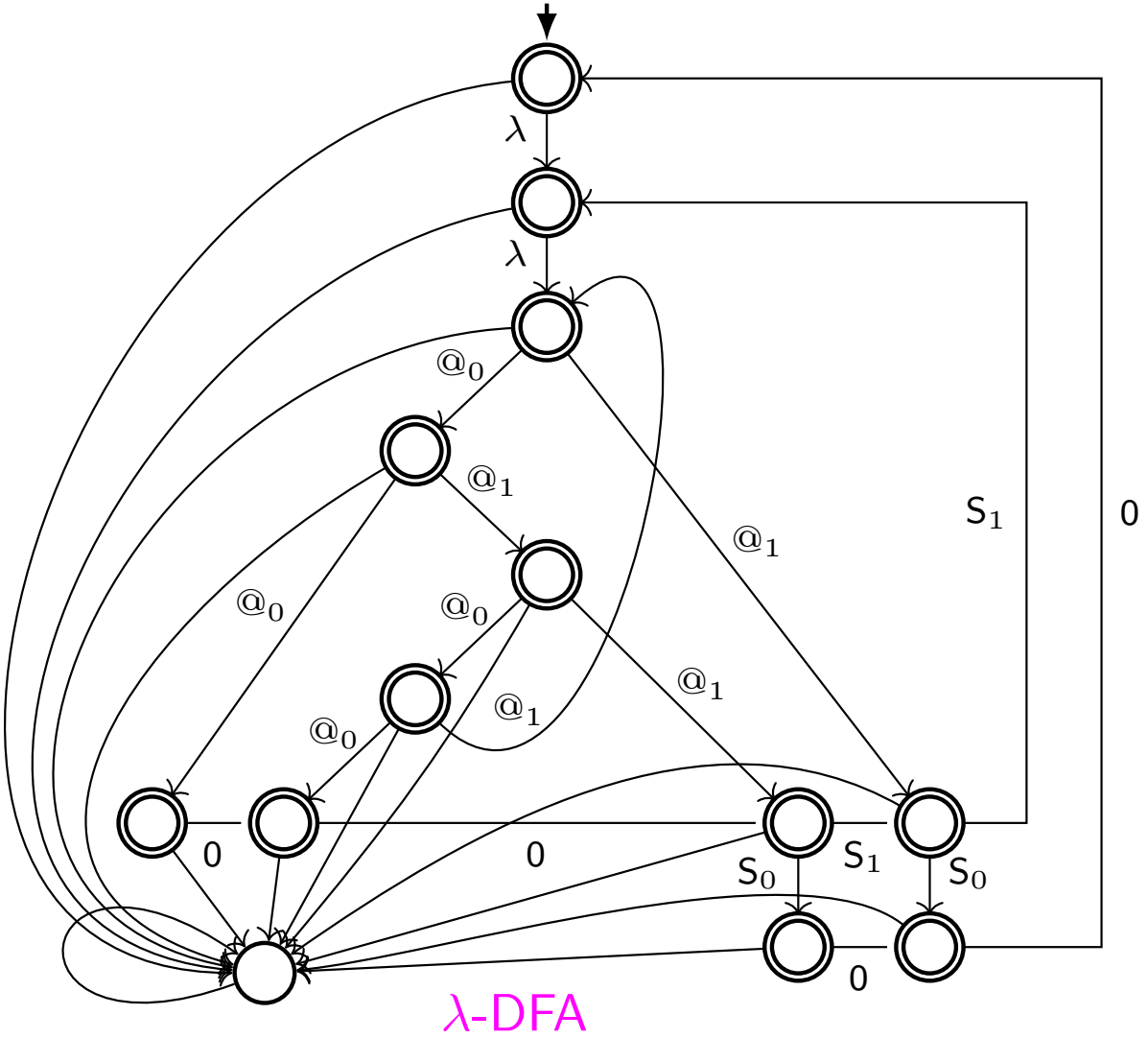
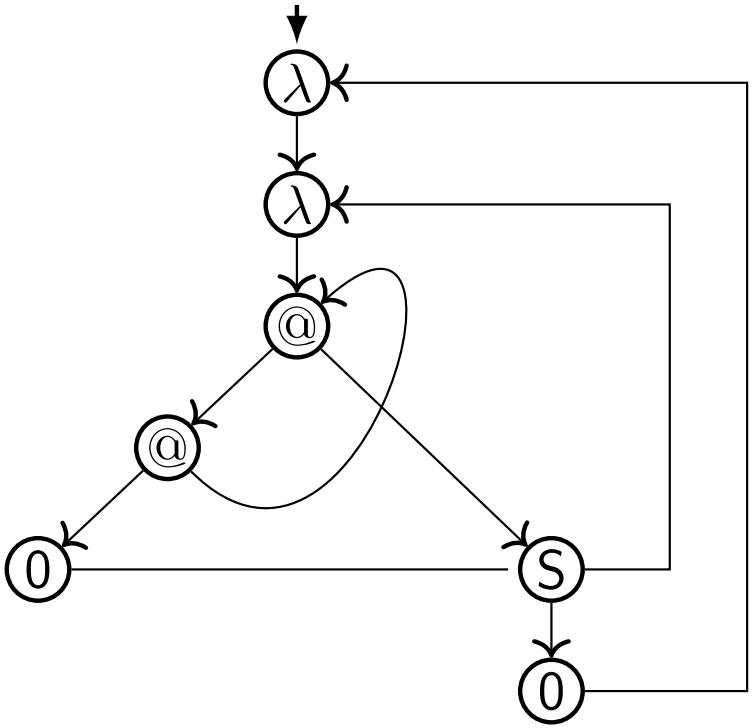
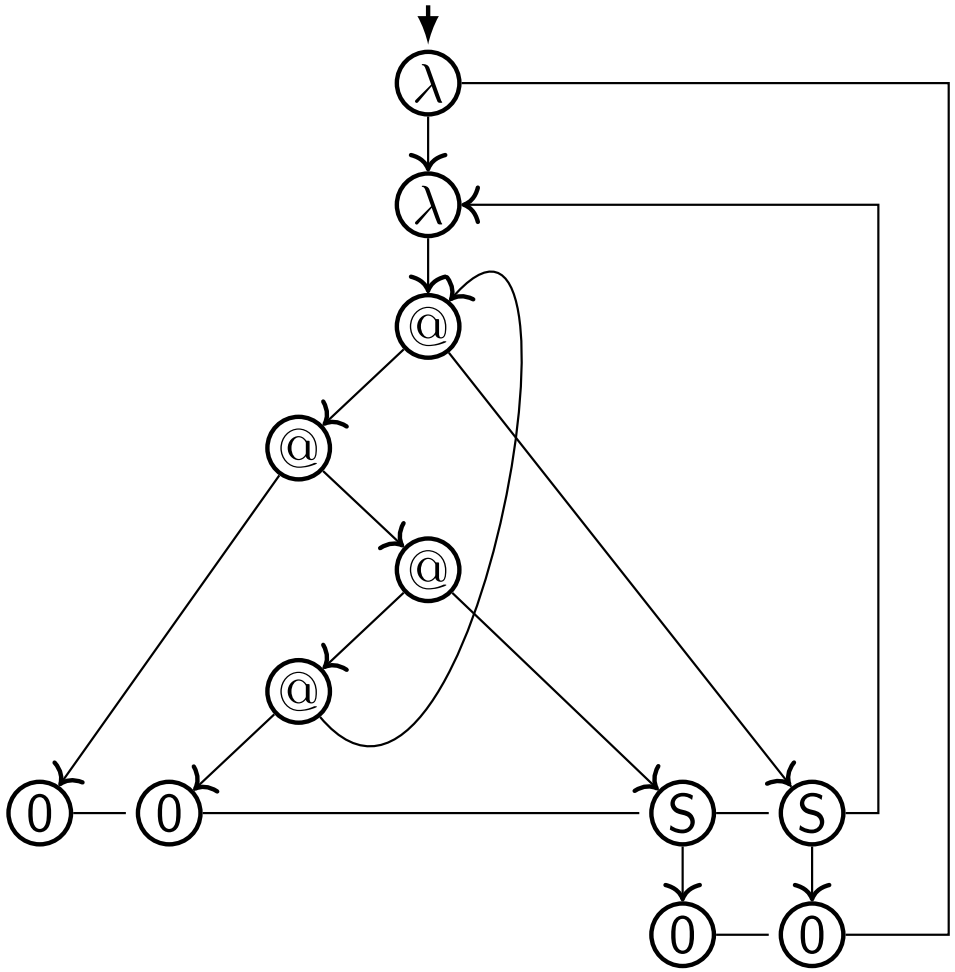


Graph interpretation (examples 1 and 2)



$[\![L_0]\!]_{\mathcal{T}}$



$[\![L]\!]_{\mathcal{T}}$

Interpretation $[\![\cdot]\!]_{\mathcal{T}}$: properties (cont.)

interpretation $\boldsymbol{\lambda}_{\text{letrec}}$-term $L \longmapsto$ λ-term-graph $[\![L]\!]_{\mathcal{T}}$

- defined by induction on structure of L
- similar analysis as fully-lazy lambda-lifting
- yields eager-scope λ-term-graphs: $\sim$ minimal scopes

Theorem

For $\boldsymbol{\lambda}_{\text{letrec}}$-terms L_1 and L_2 it holds: Equality of infinite unfolding coincides with bisimilarity of λ-term-graph interpretations:

$$[\![L_1]\!]_{\lambda^\infty} = [\![L_2]\!]_{\lambda^\infty} \iff [\![L_1]\!]_{\mathcal{T}} \leftrightarrow [\![L_2]\!]_{\mathcal{T}}$$

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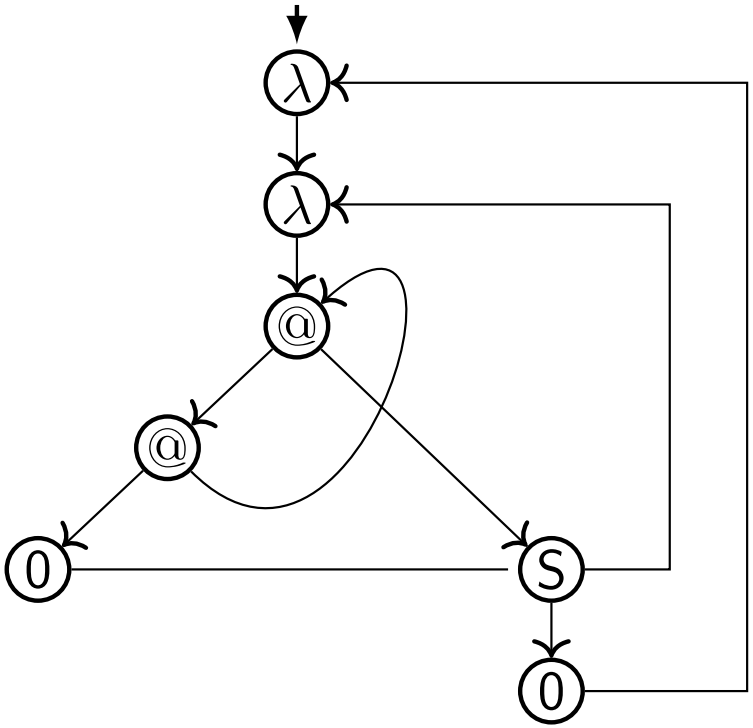
Collapse

$$
\begin{tikzcd}
L \arrow[r, mapsto, bend left, "\text{interpret}"] & G \arrow[d, Rightarrow, "\text{collapse}"] \\
L_0 & G_0 \arrow[l, mapsto, bend left, "\text{readback}"]
\end{tikzcd}
$$

Bisimulation check between λ-term-graphs



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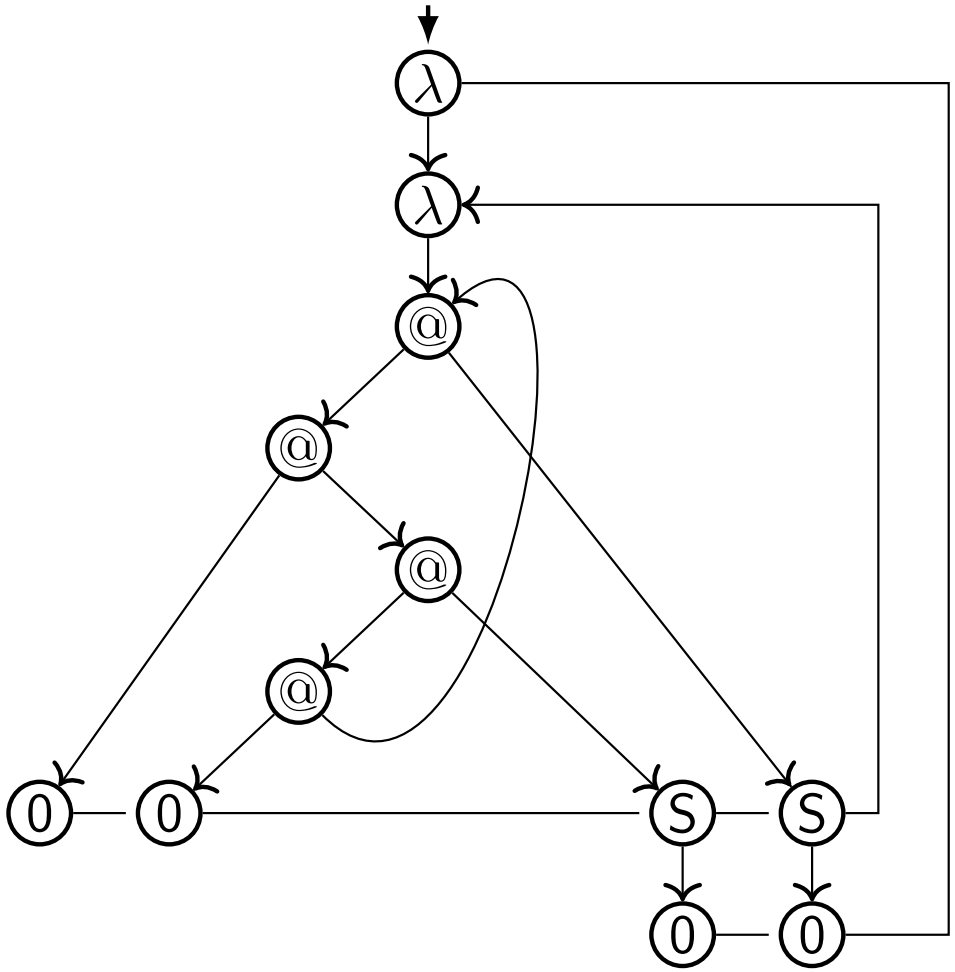

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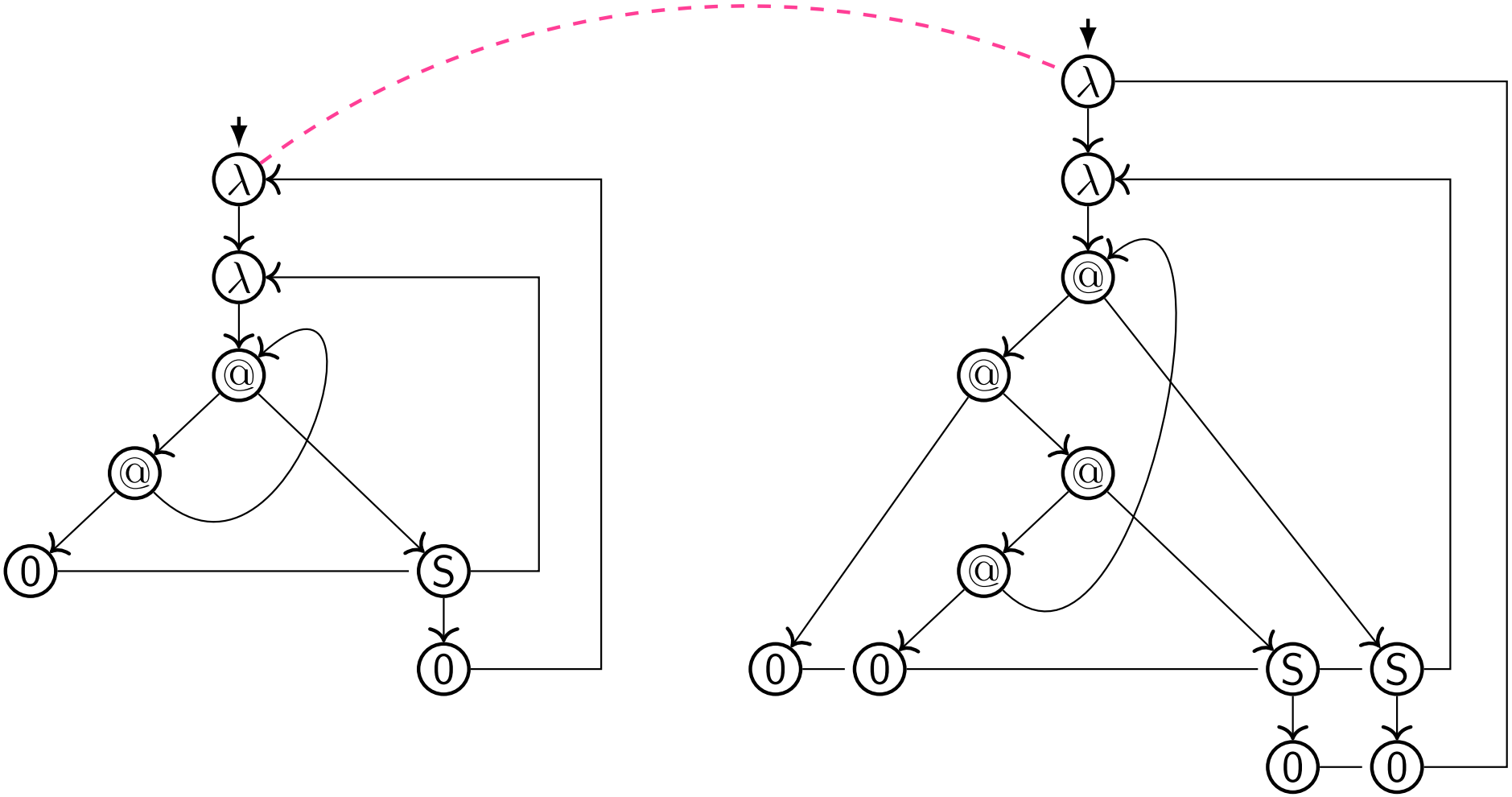

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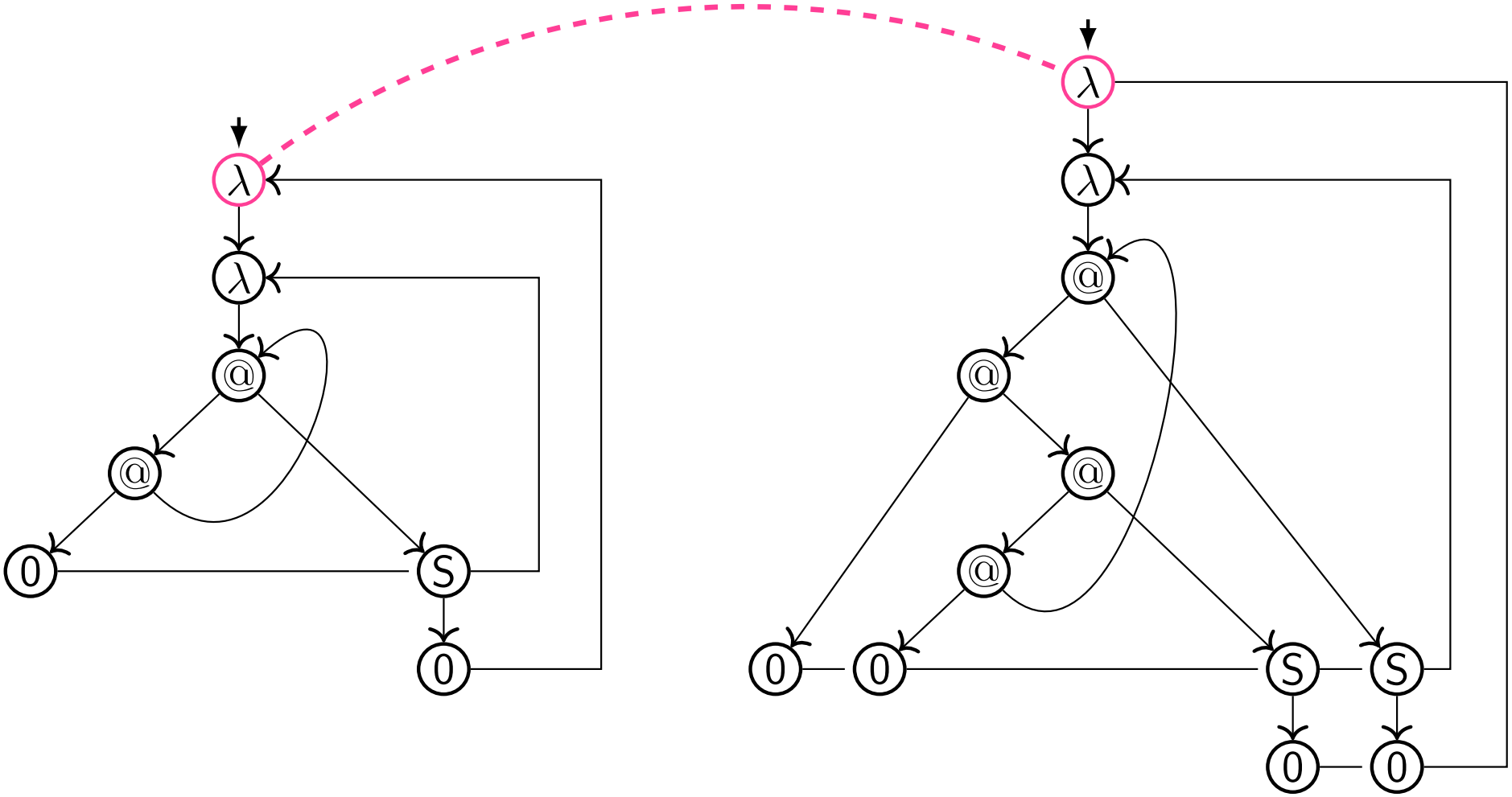

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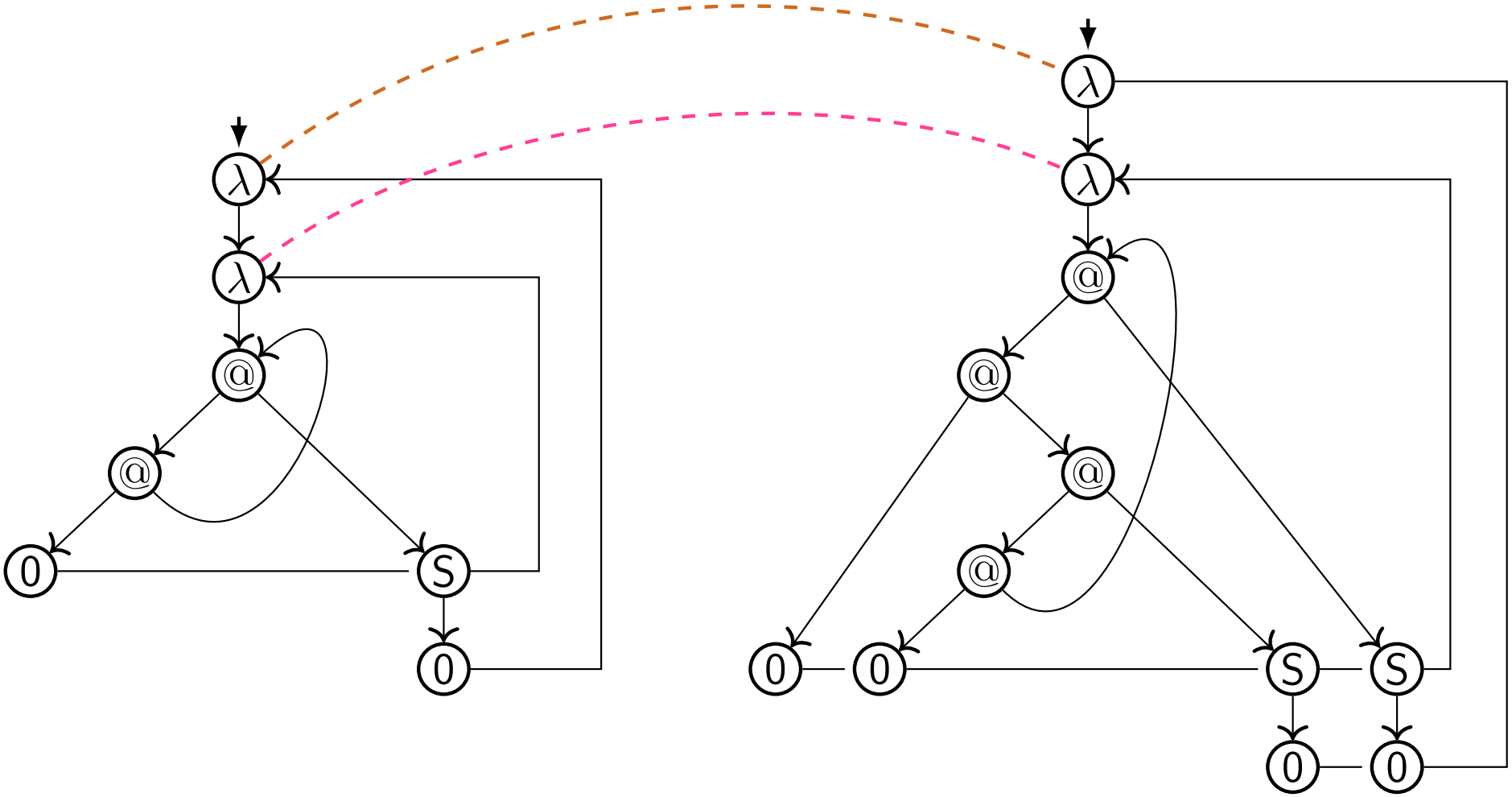

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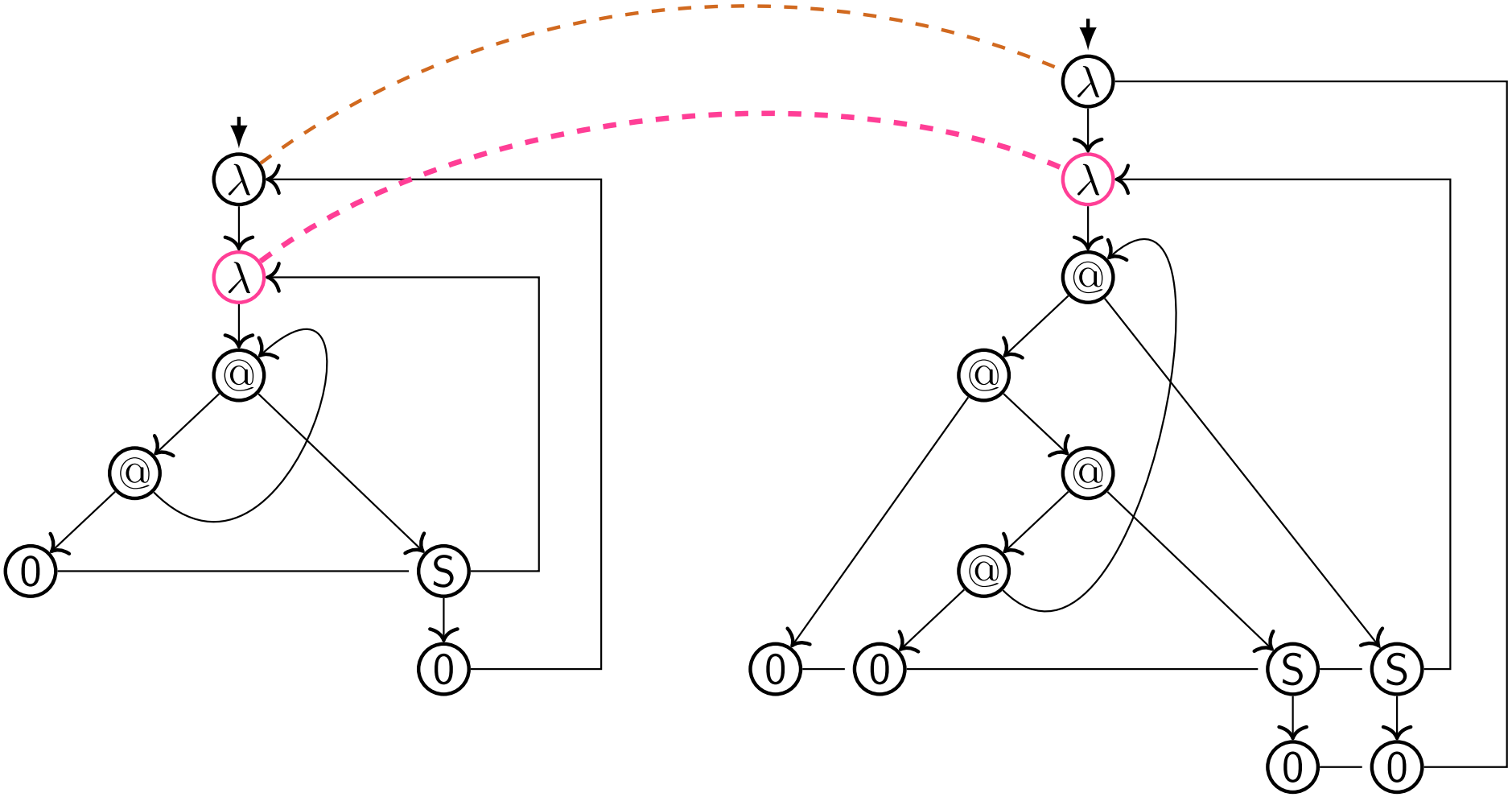

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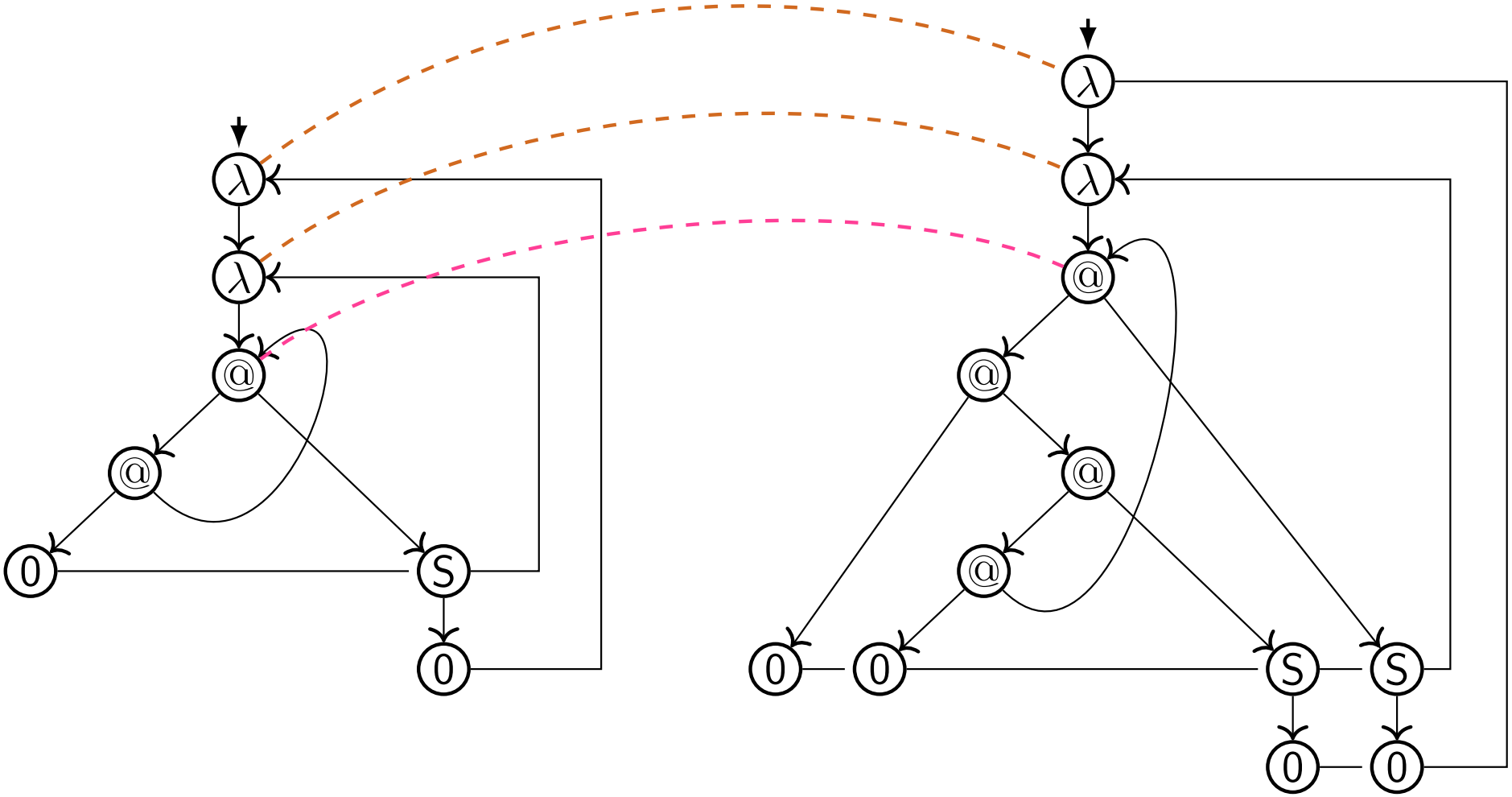

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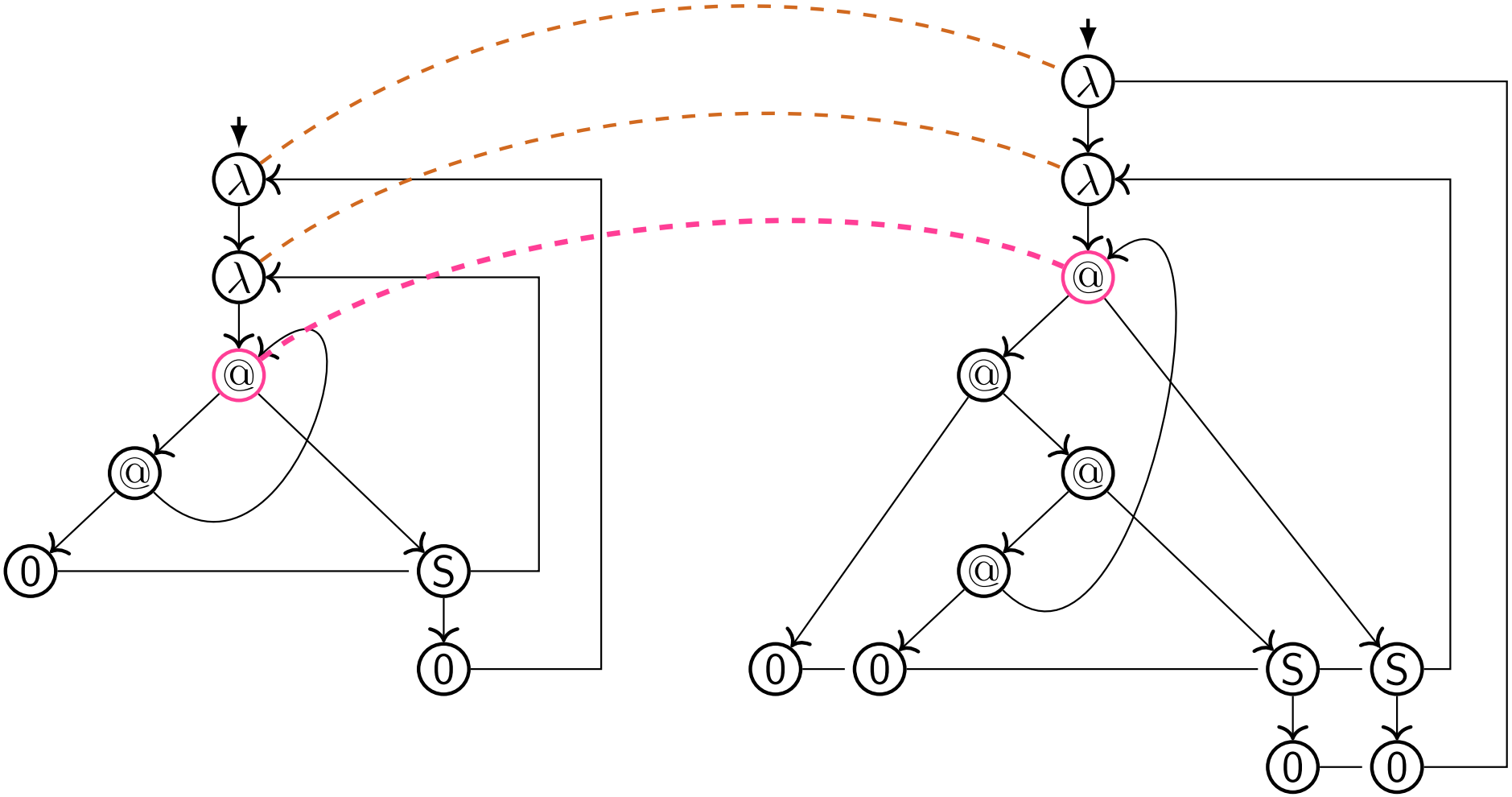

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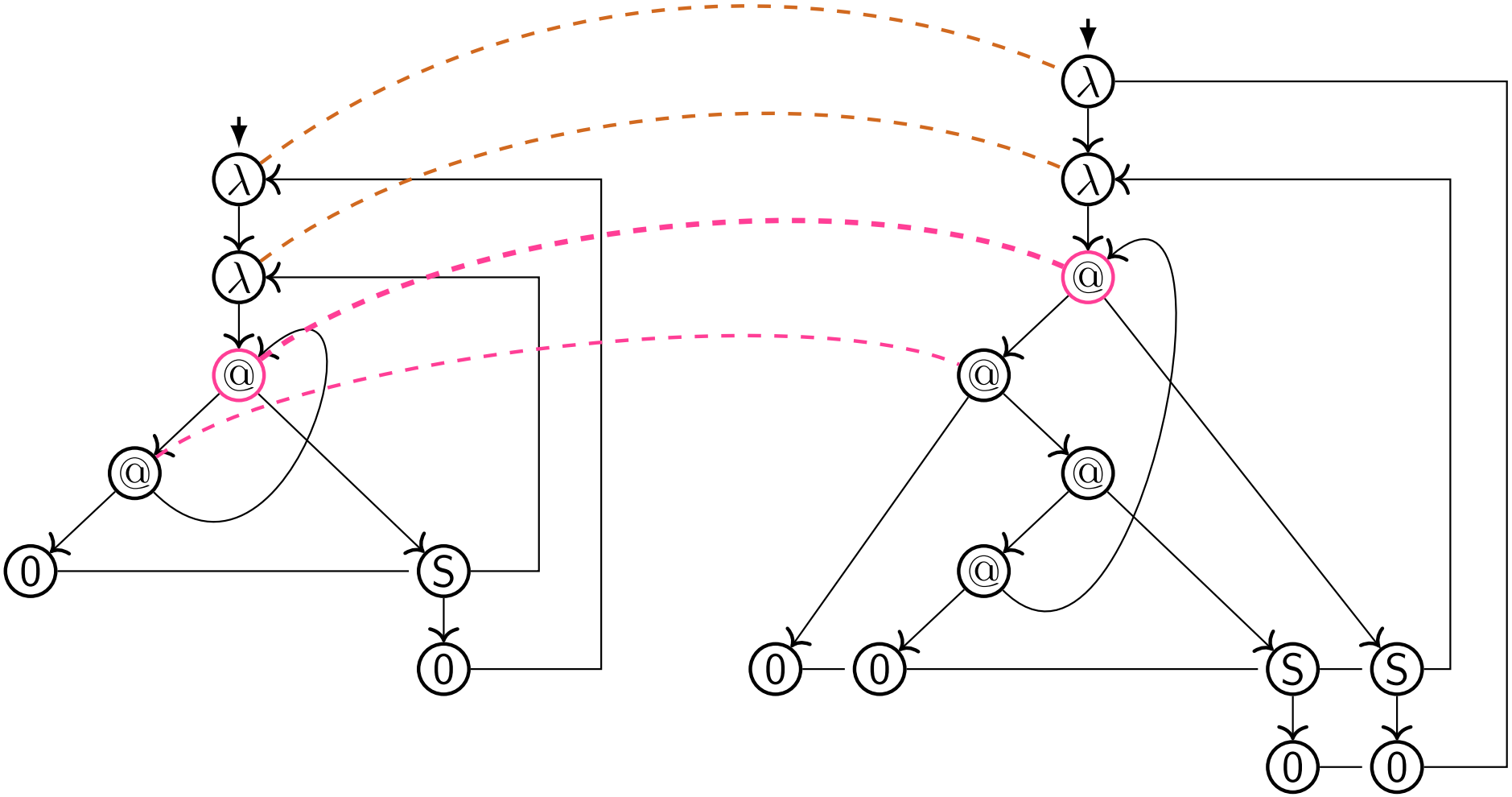

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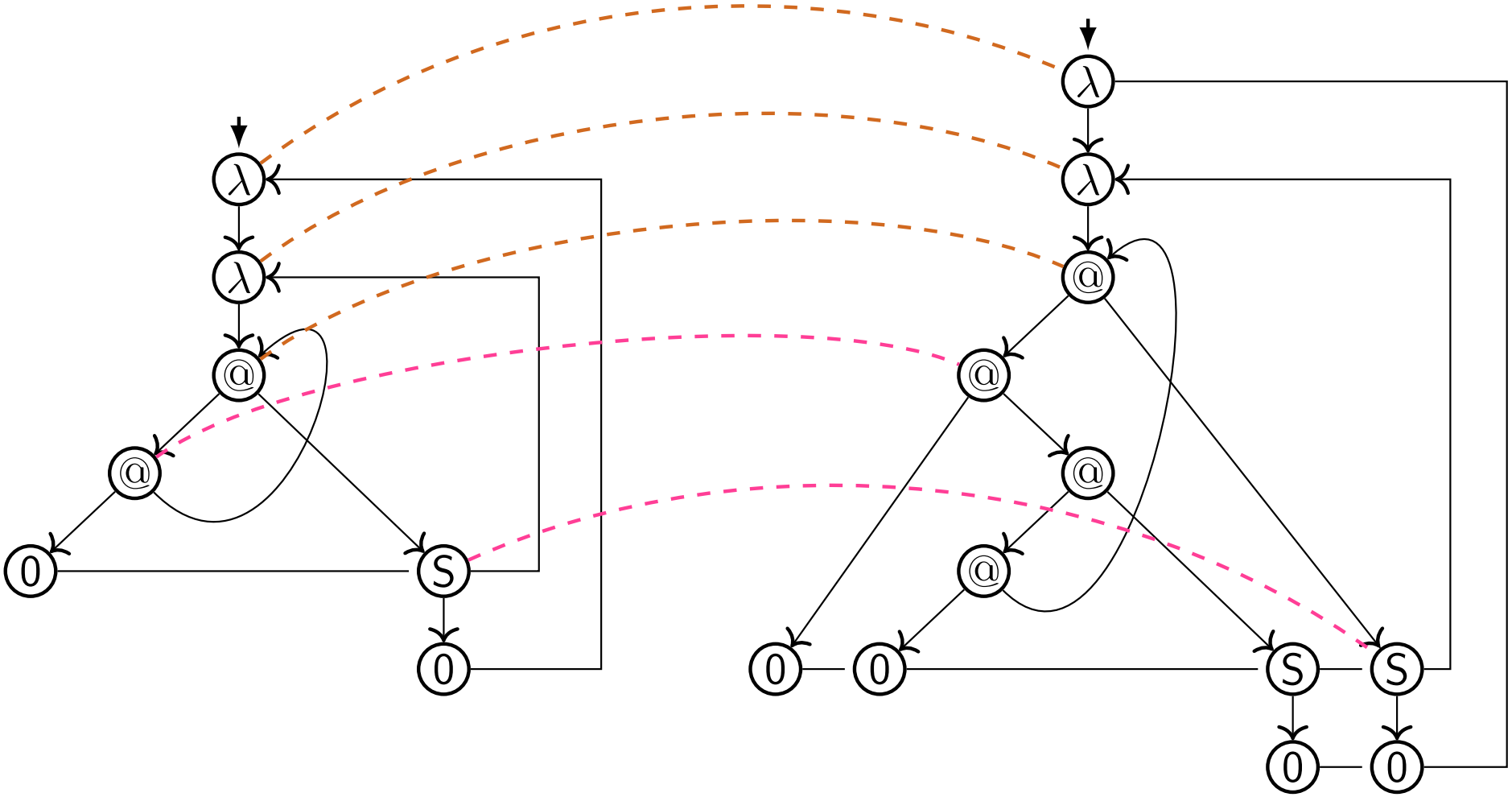

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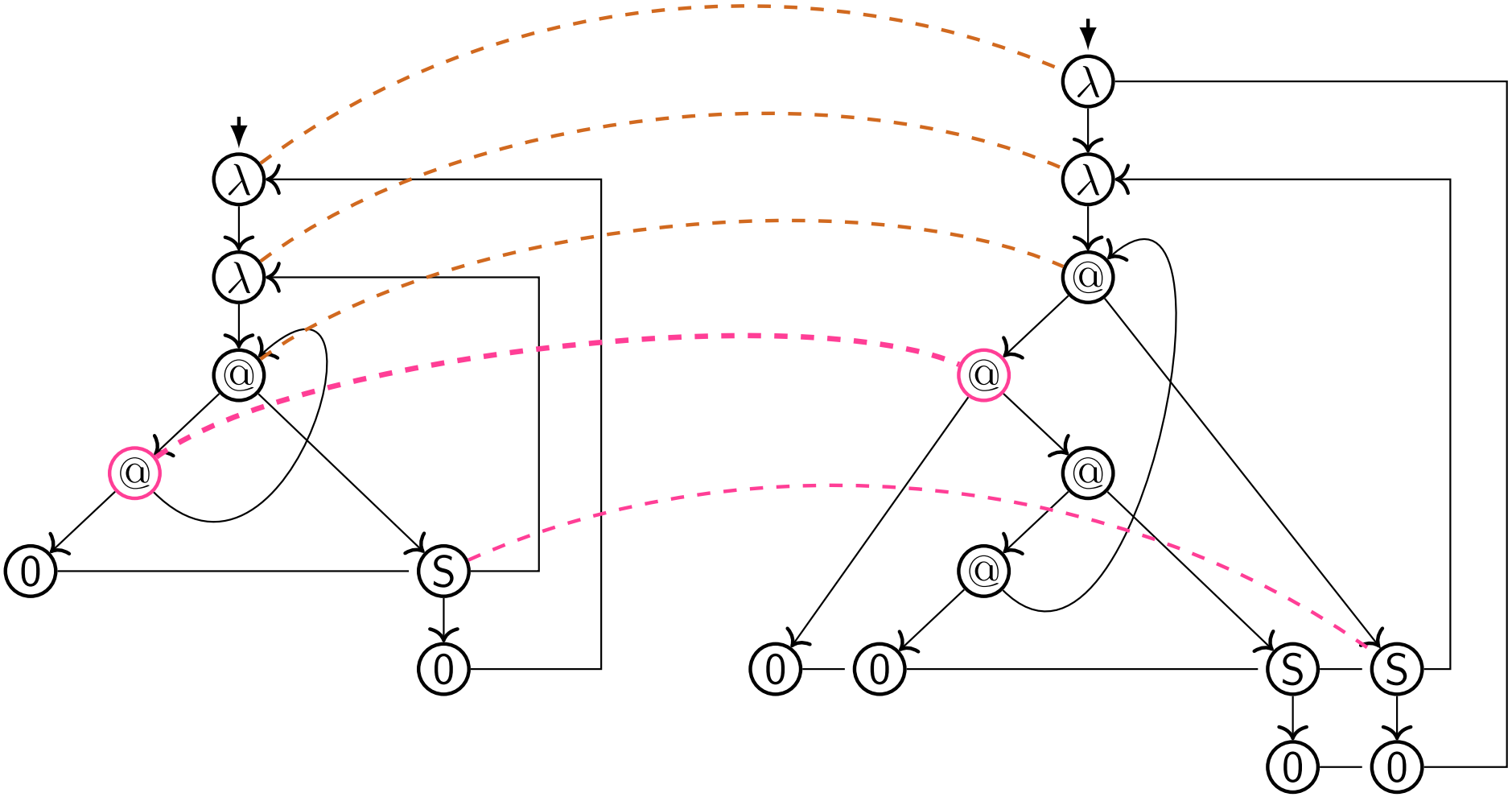

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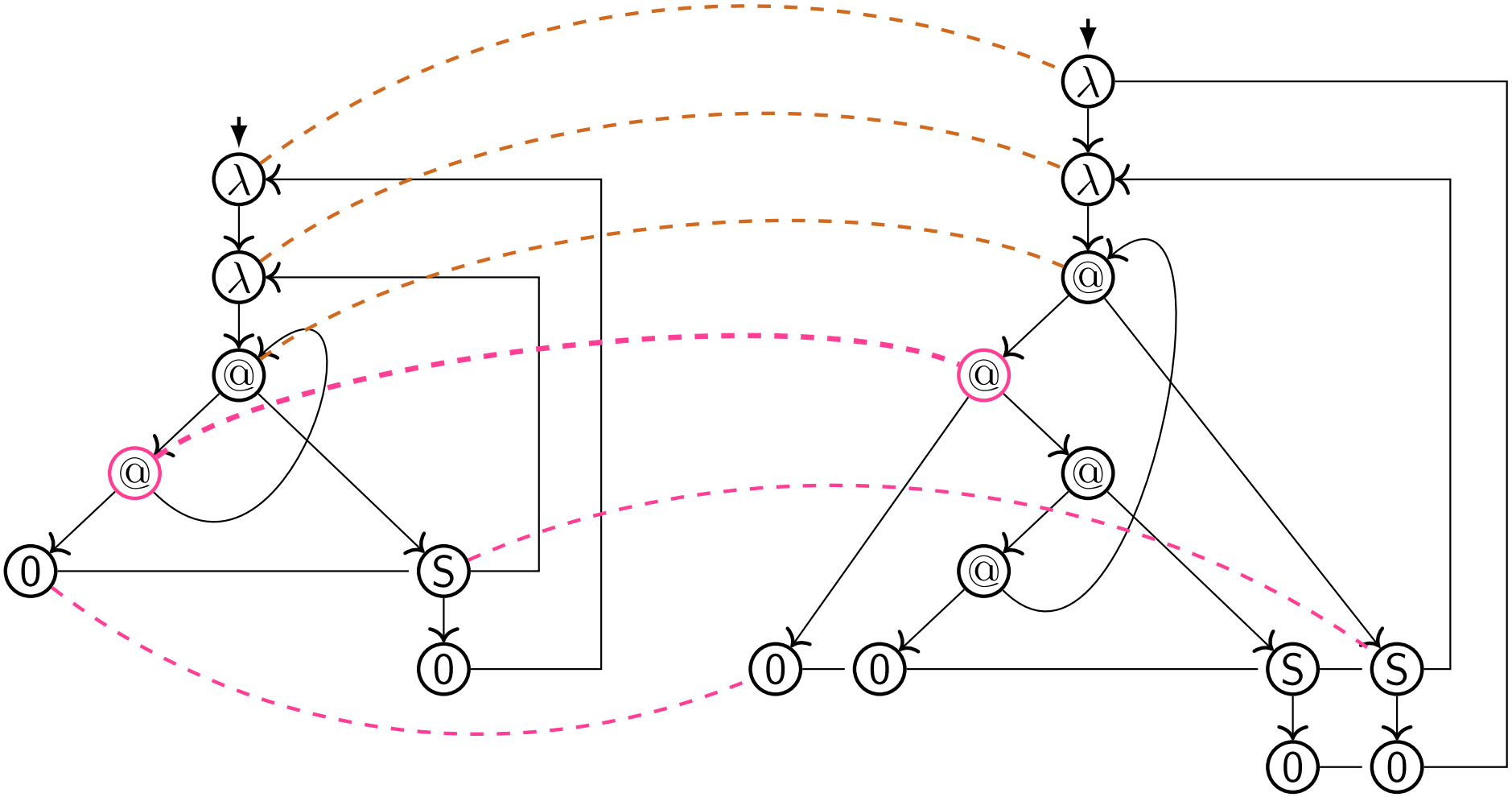

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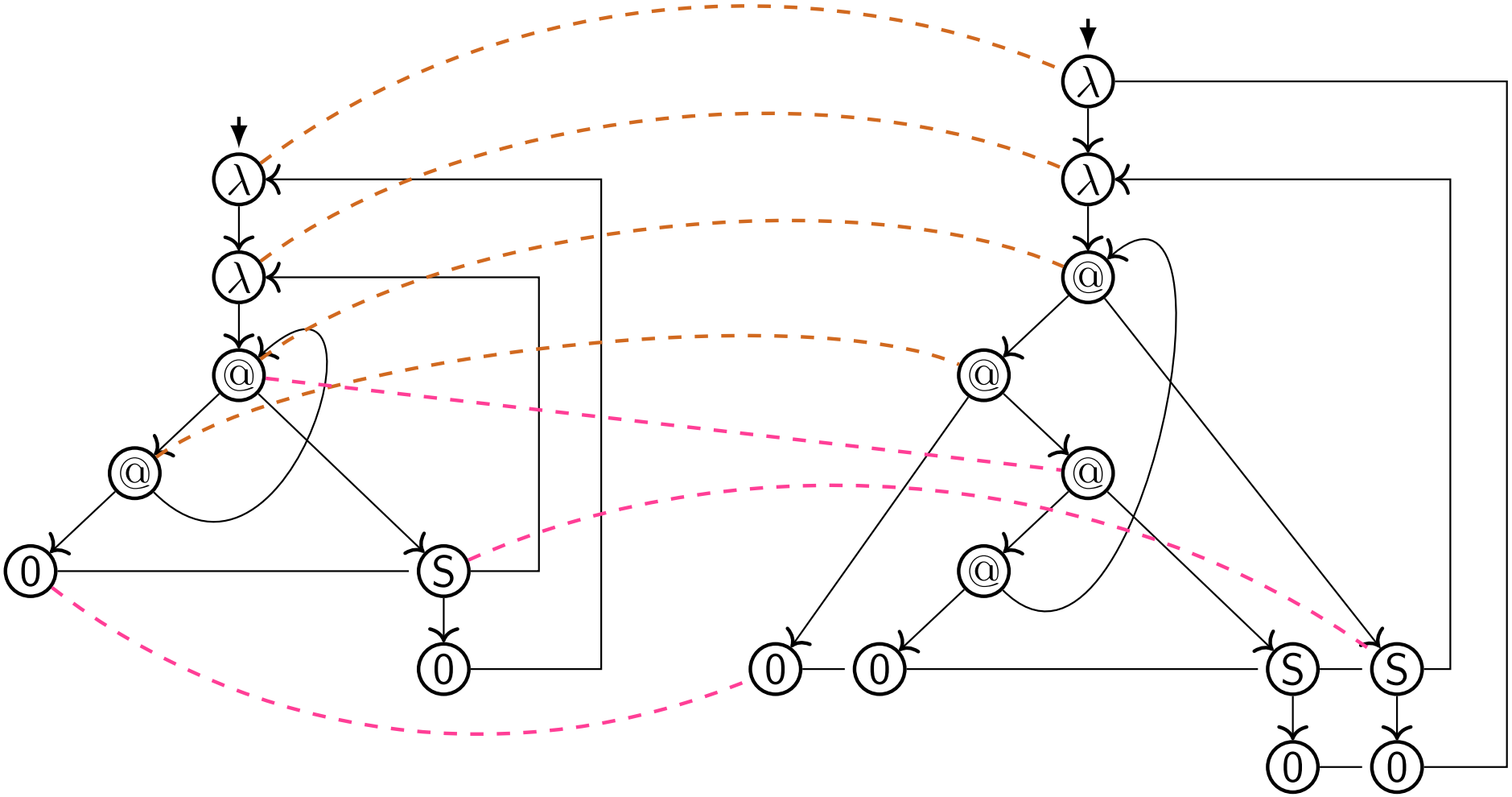

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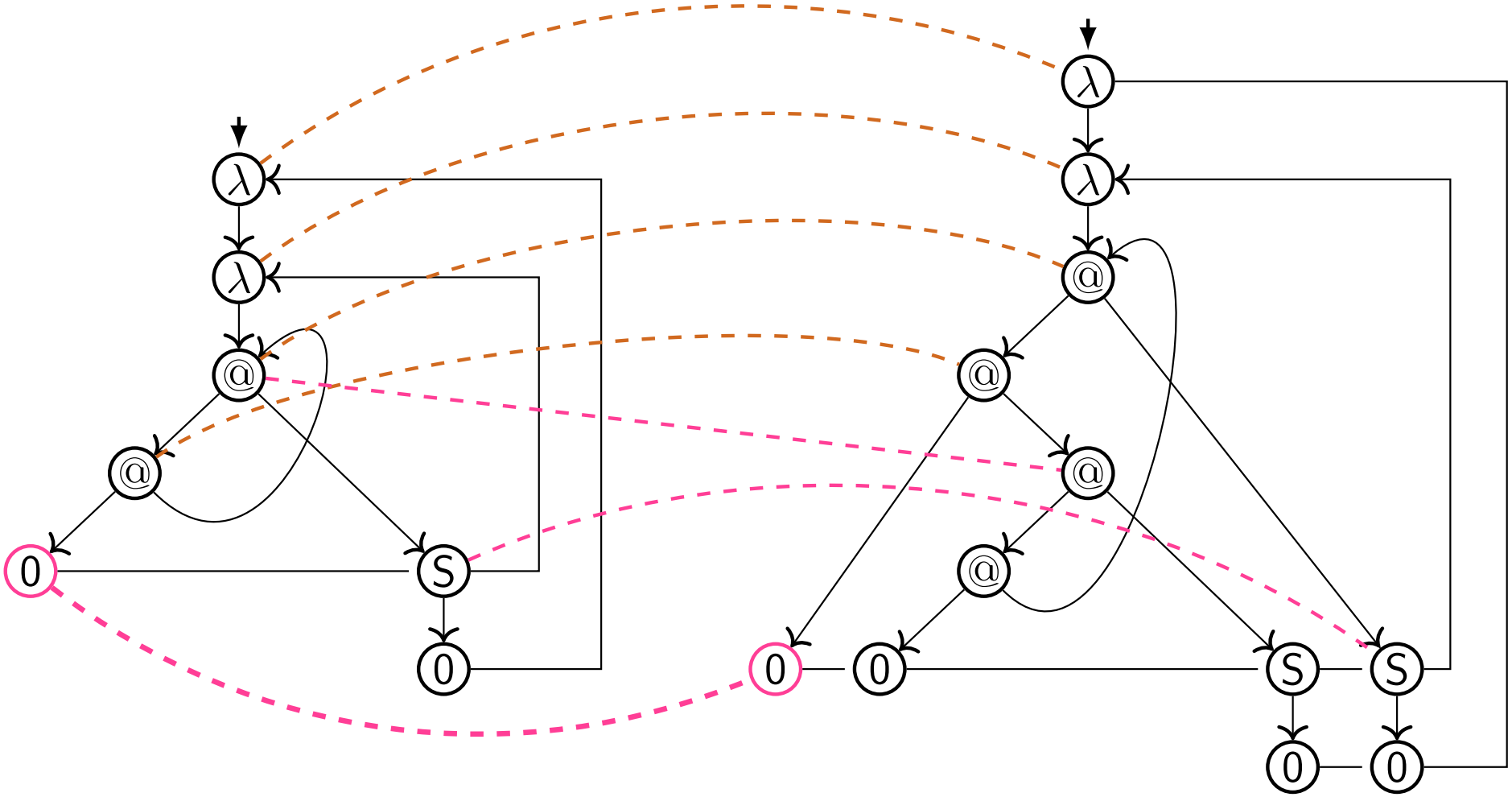

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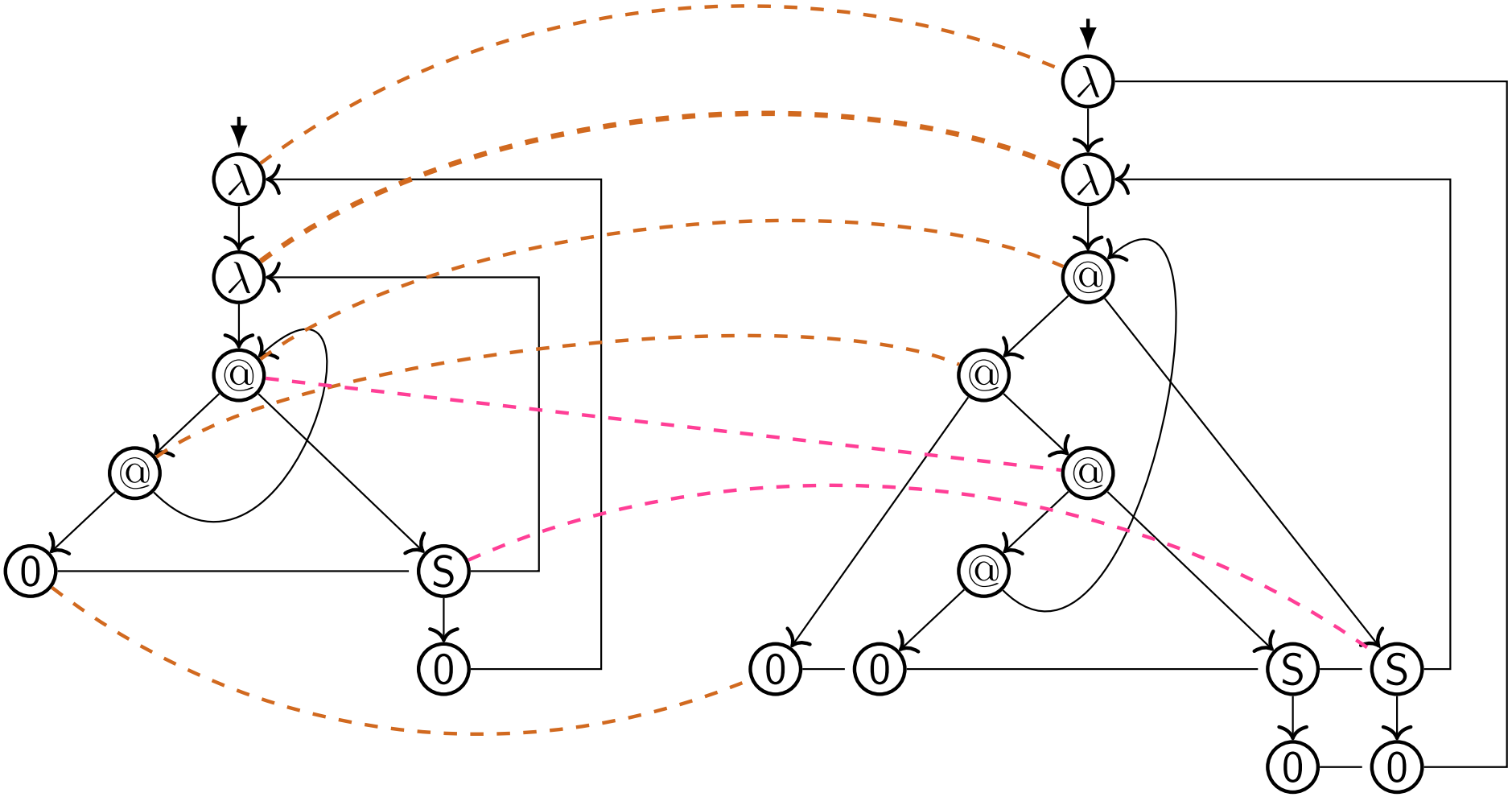

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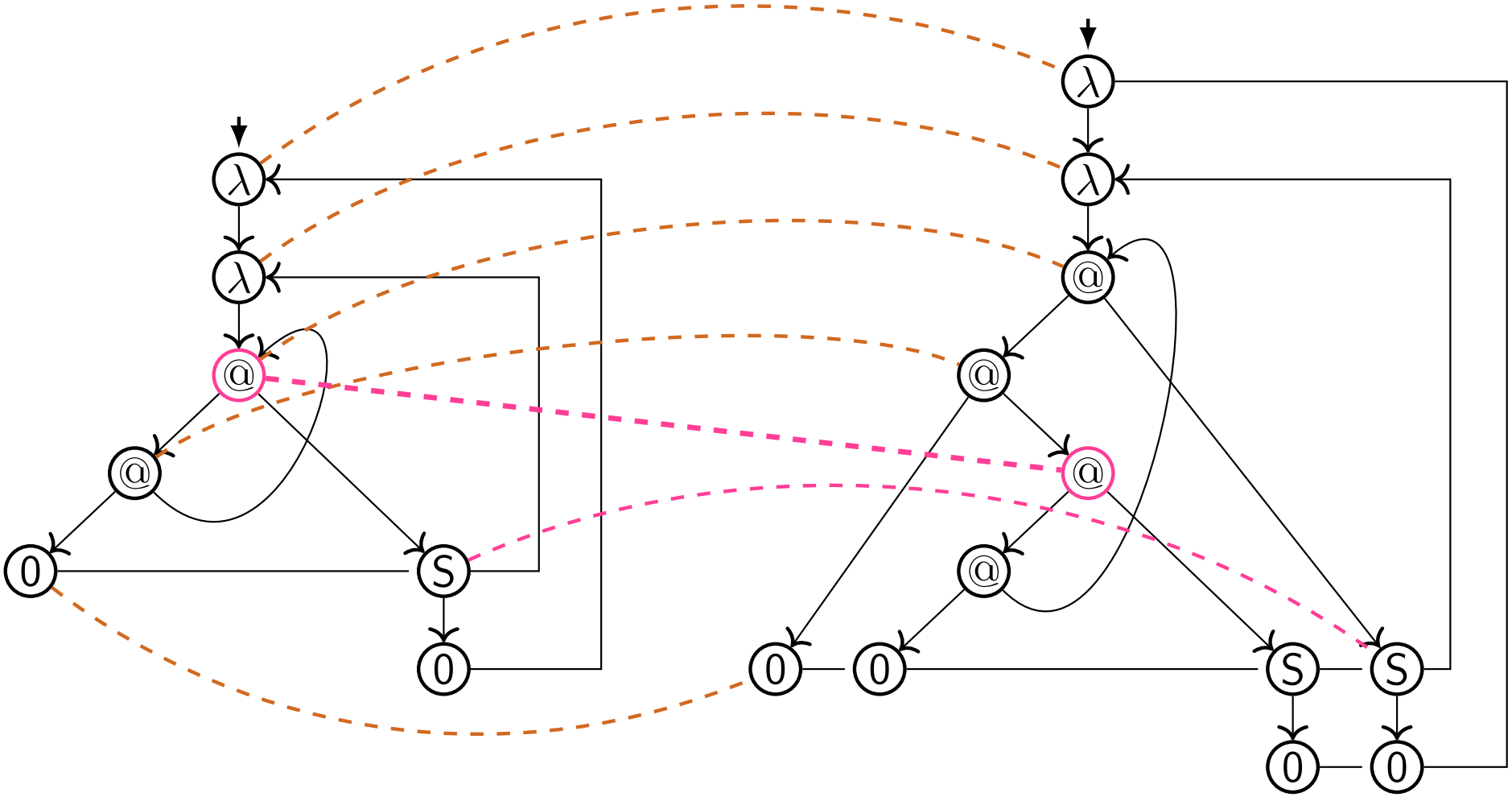

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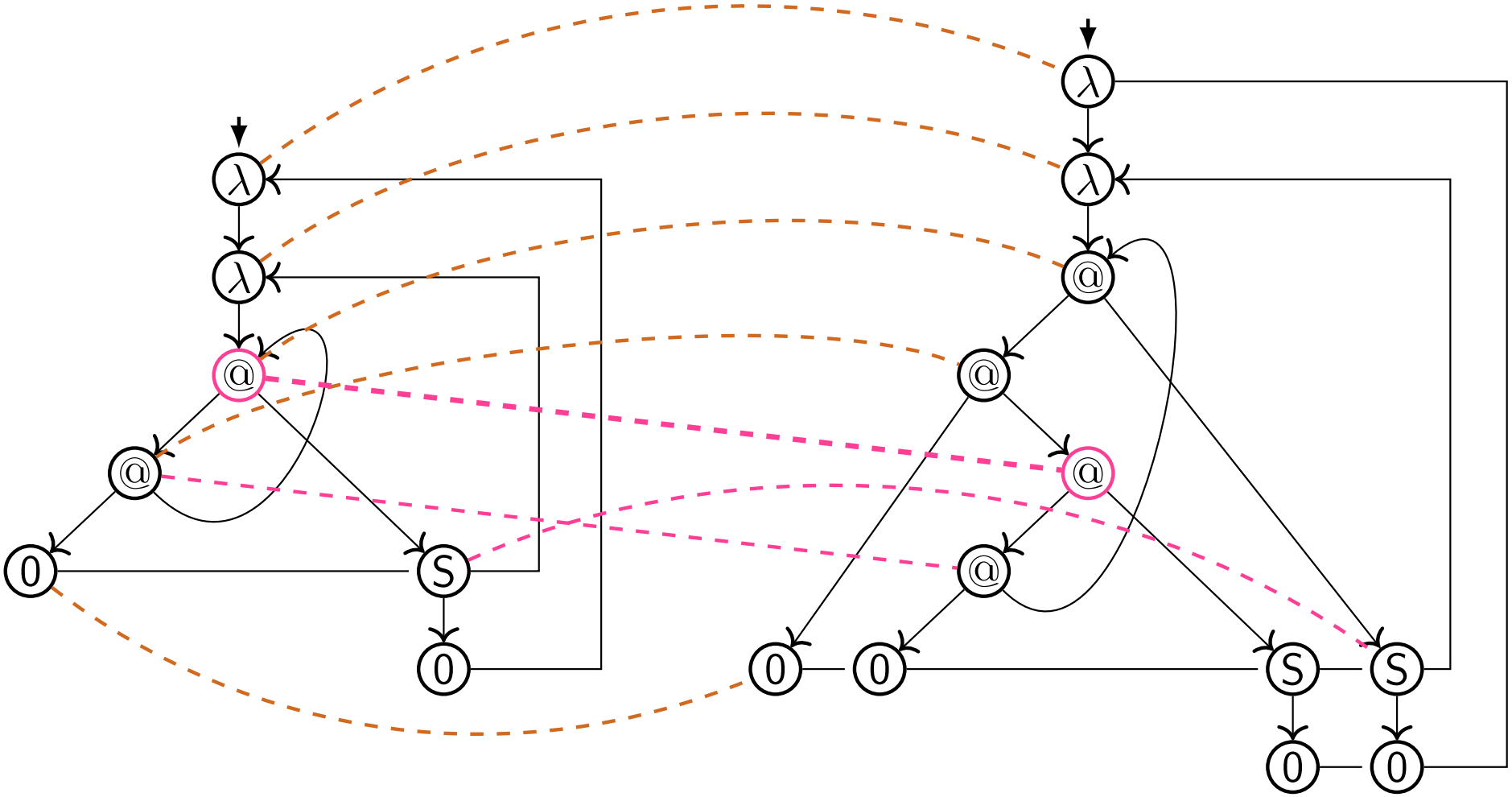
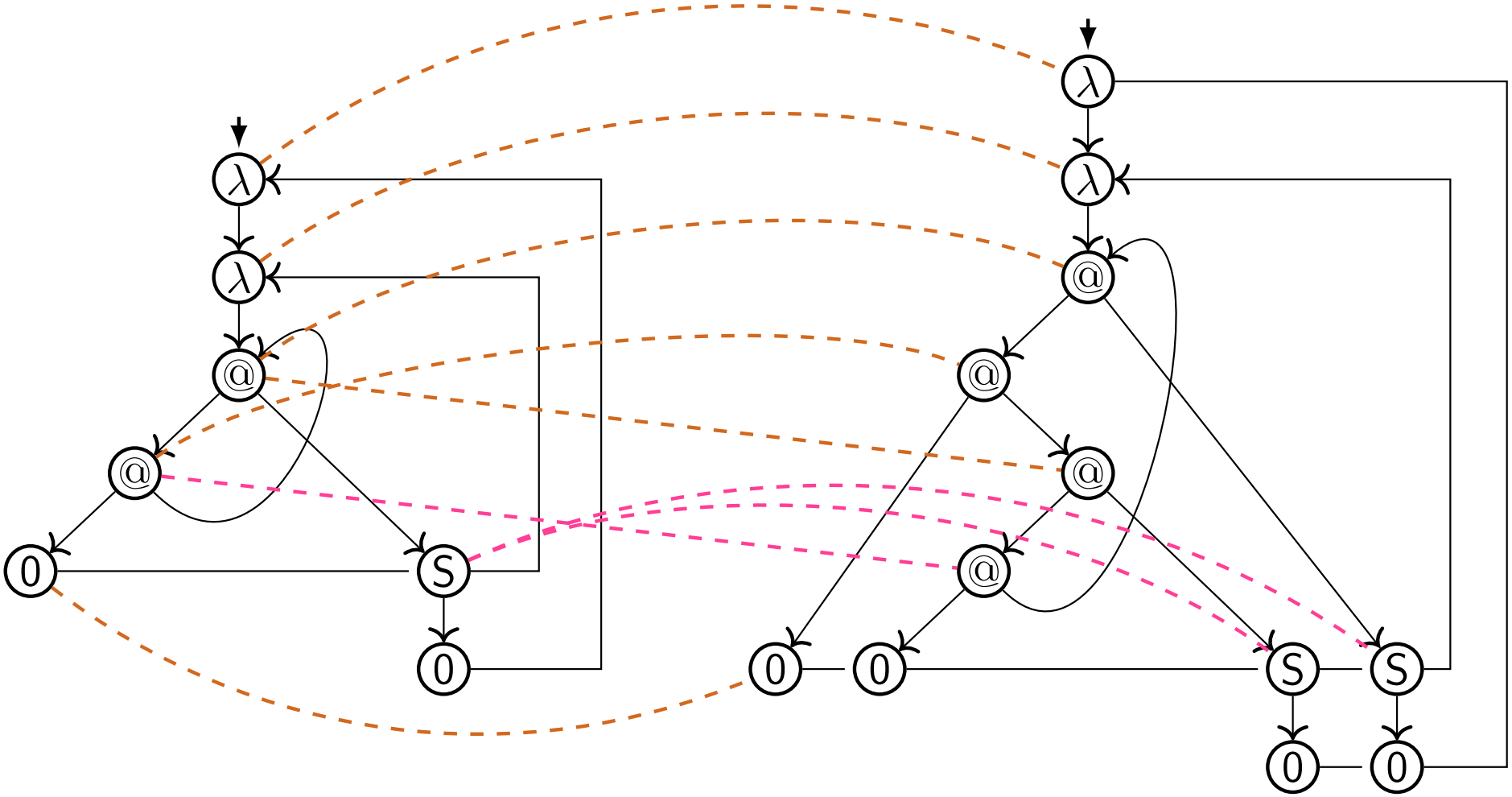


$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

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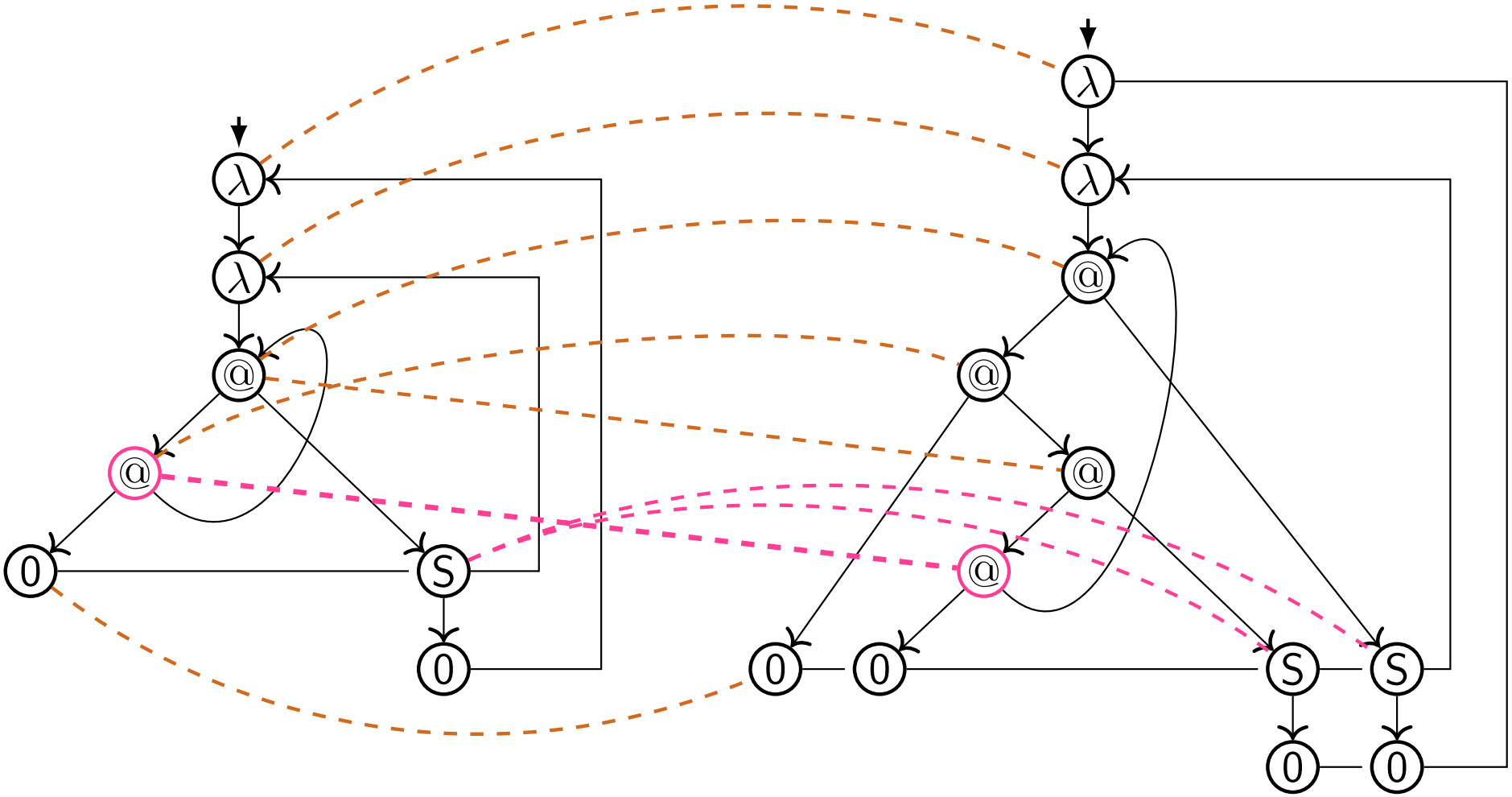
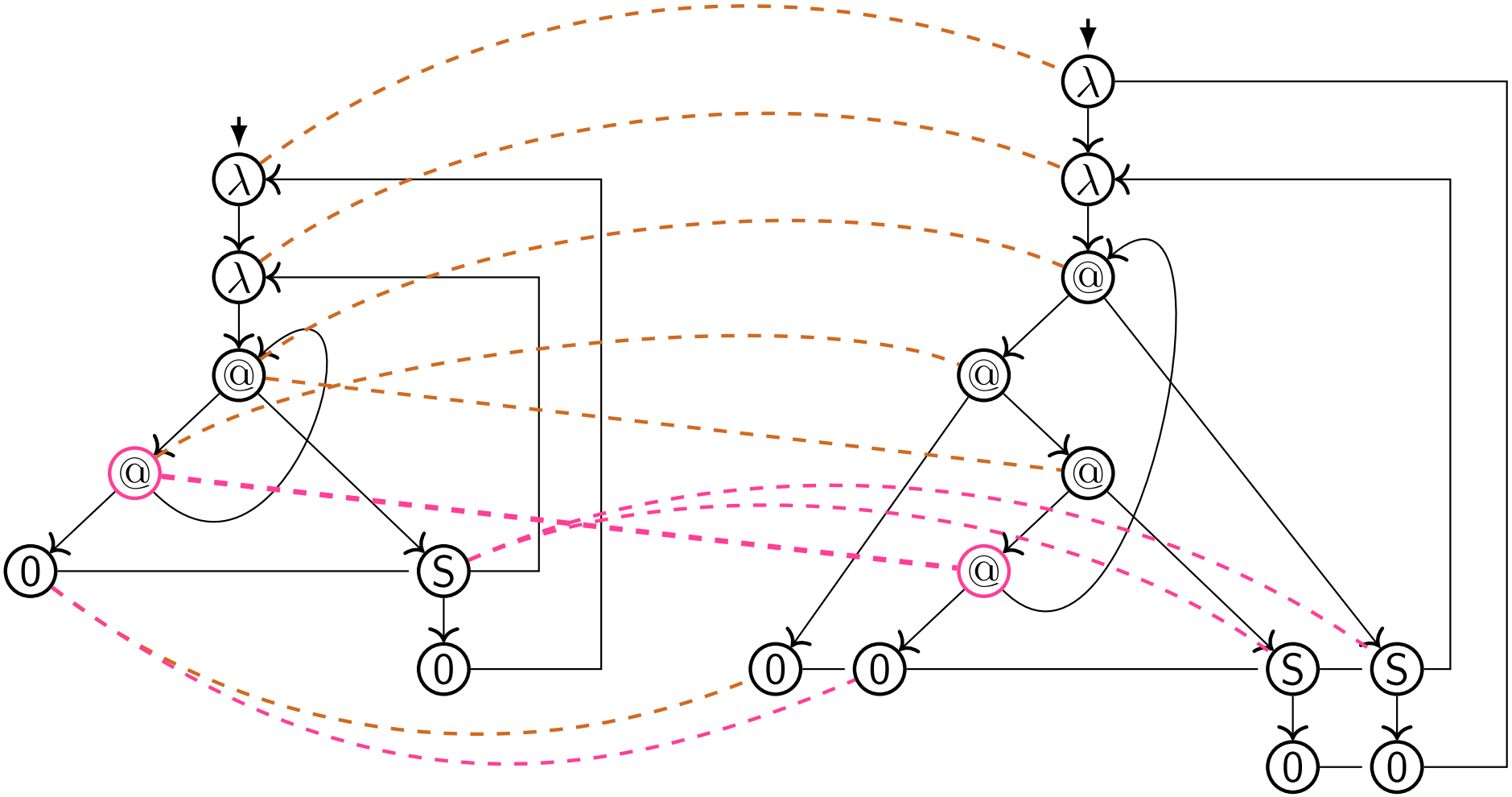


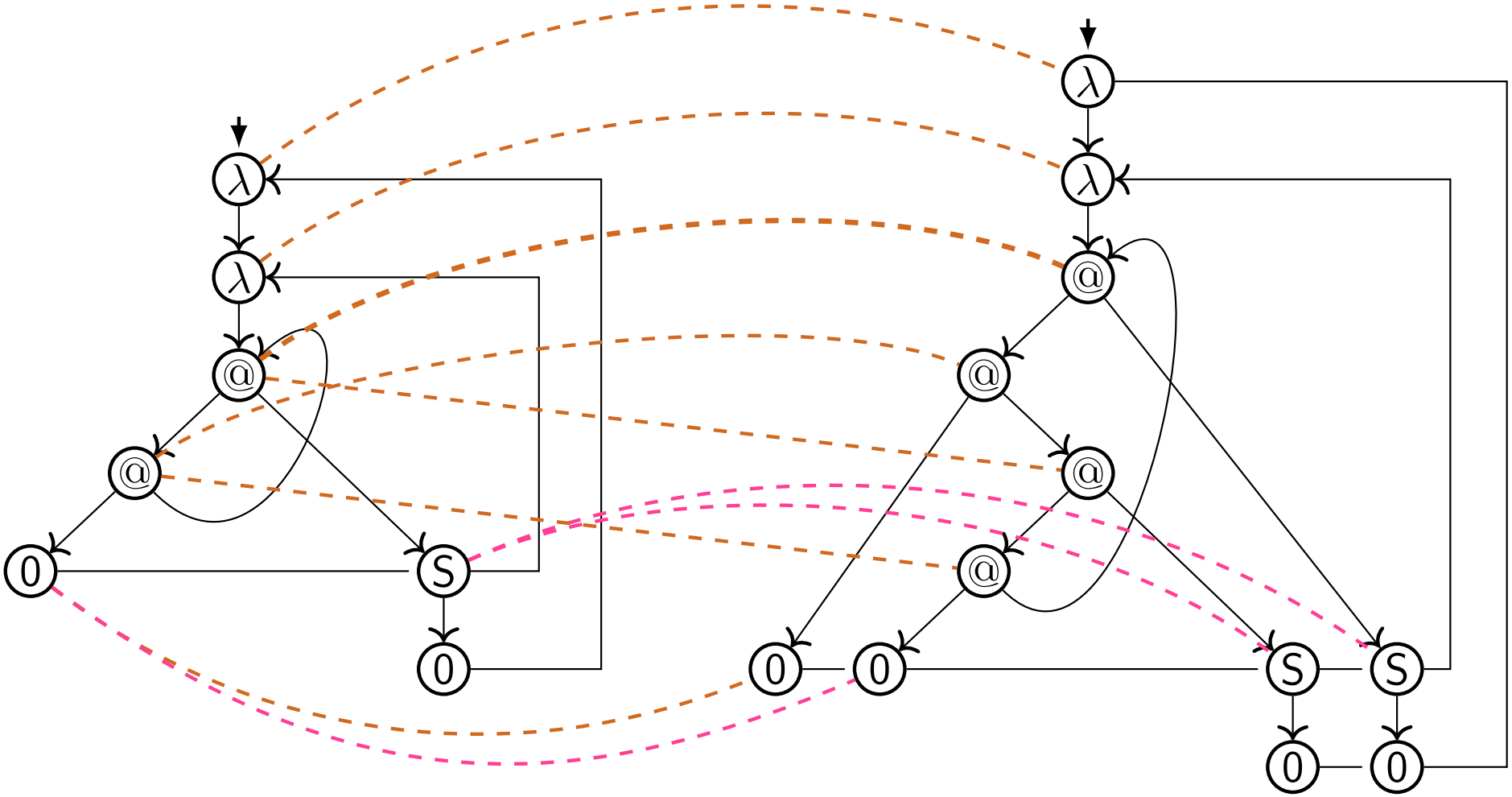
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



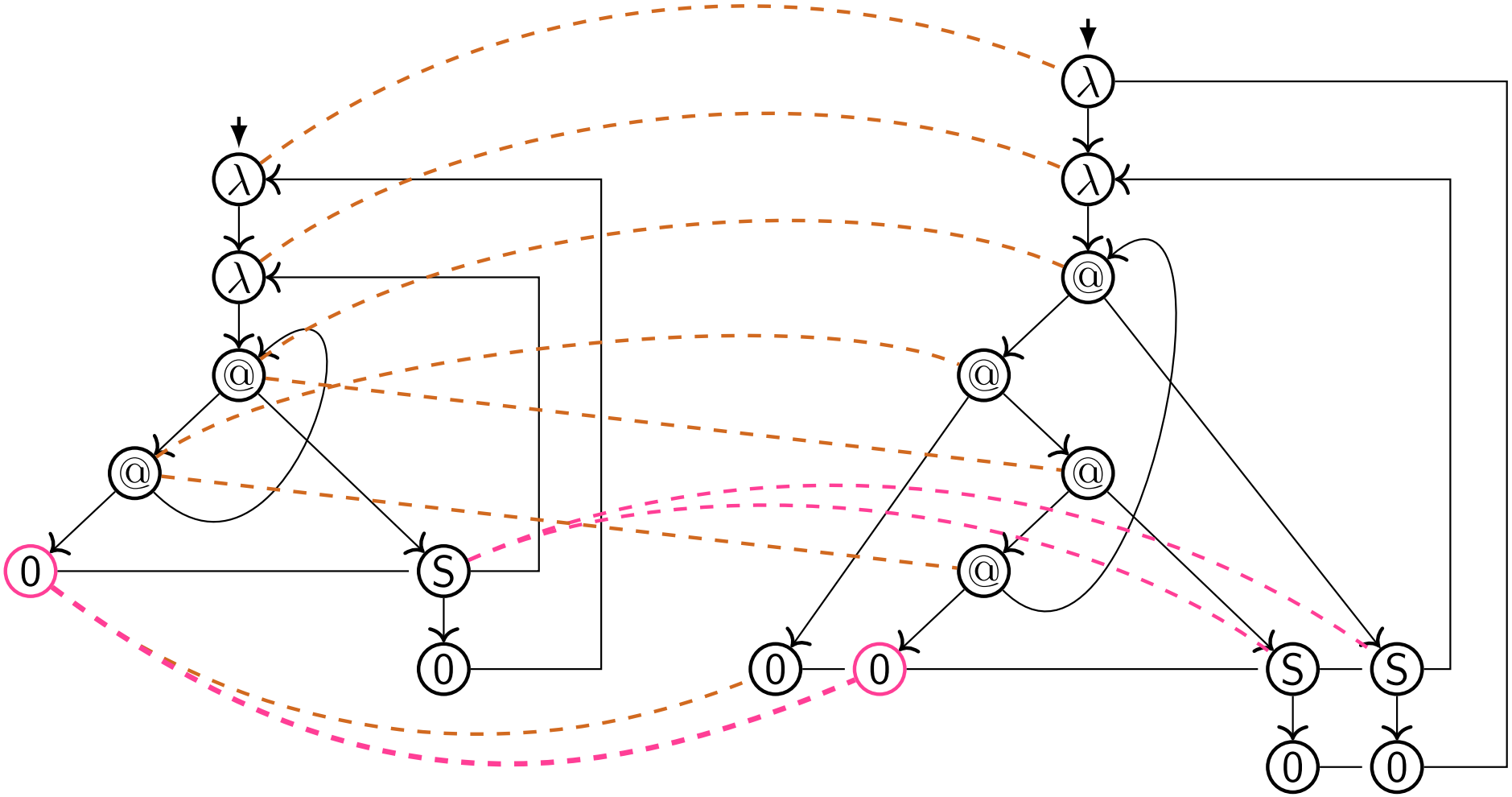
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



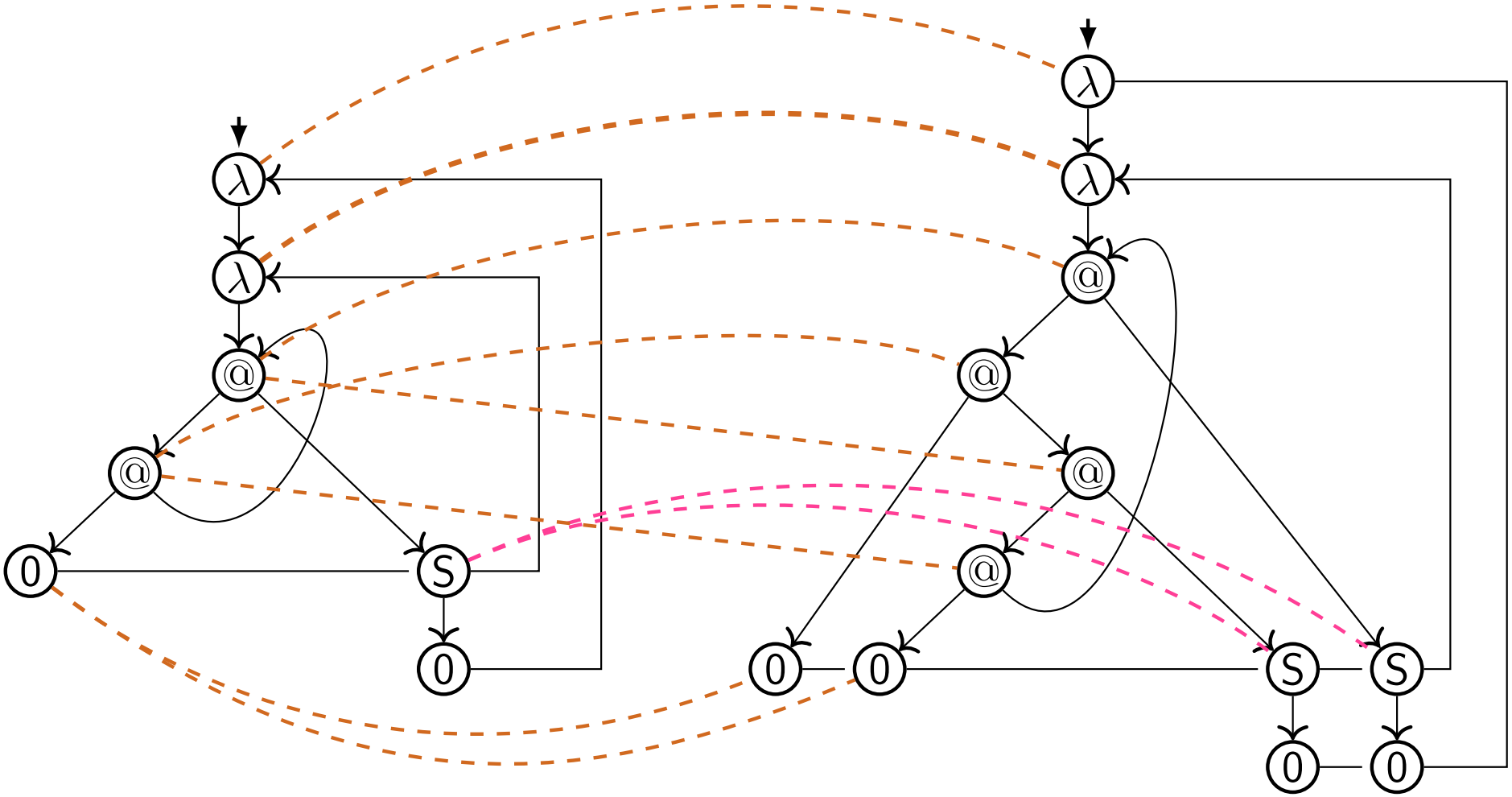
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



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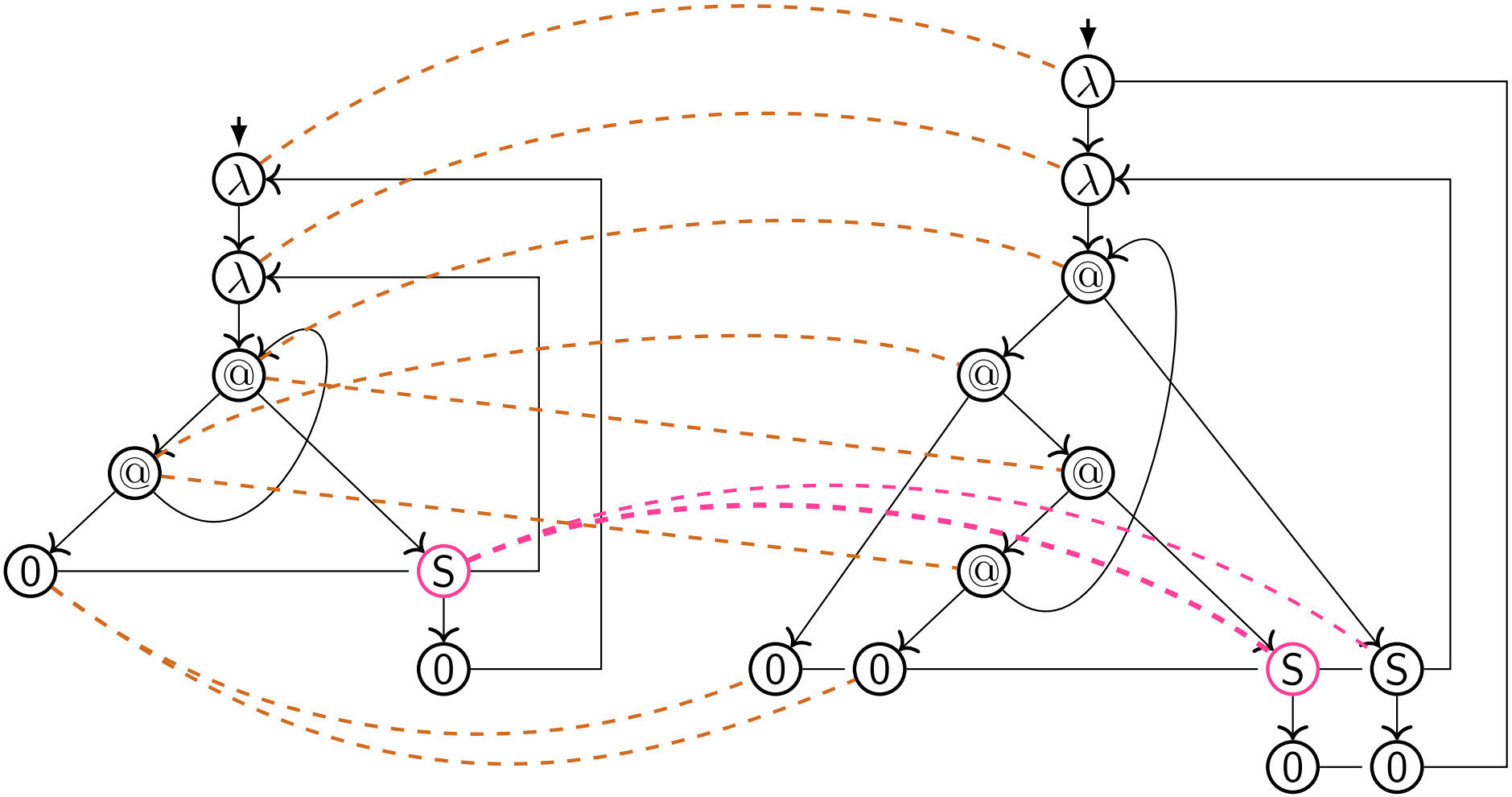

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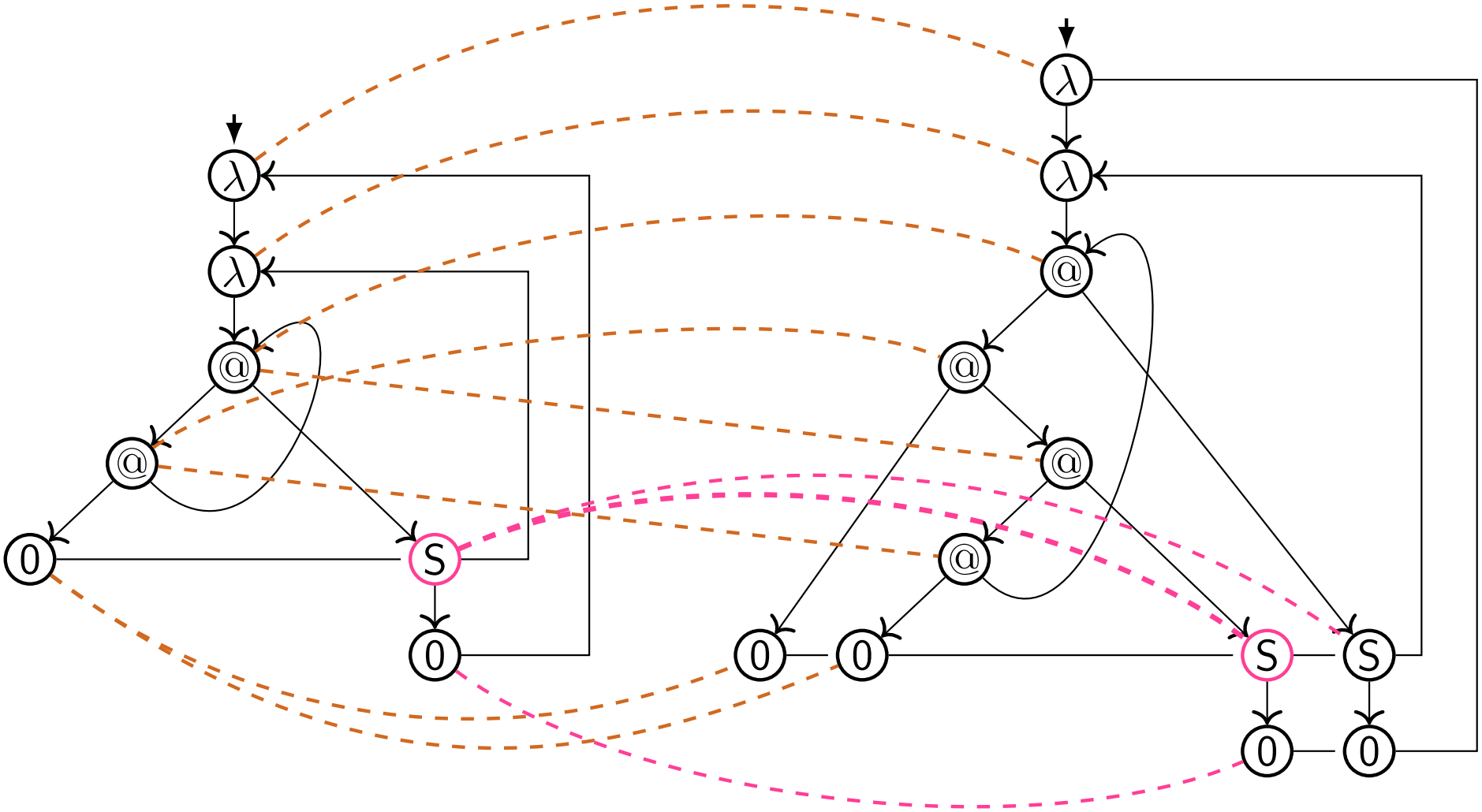

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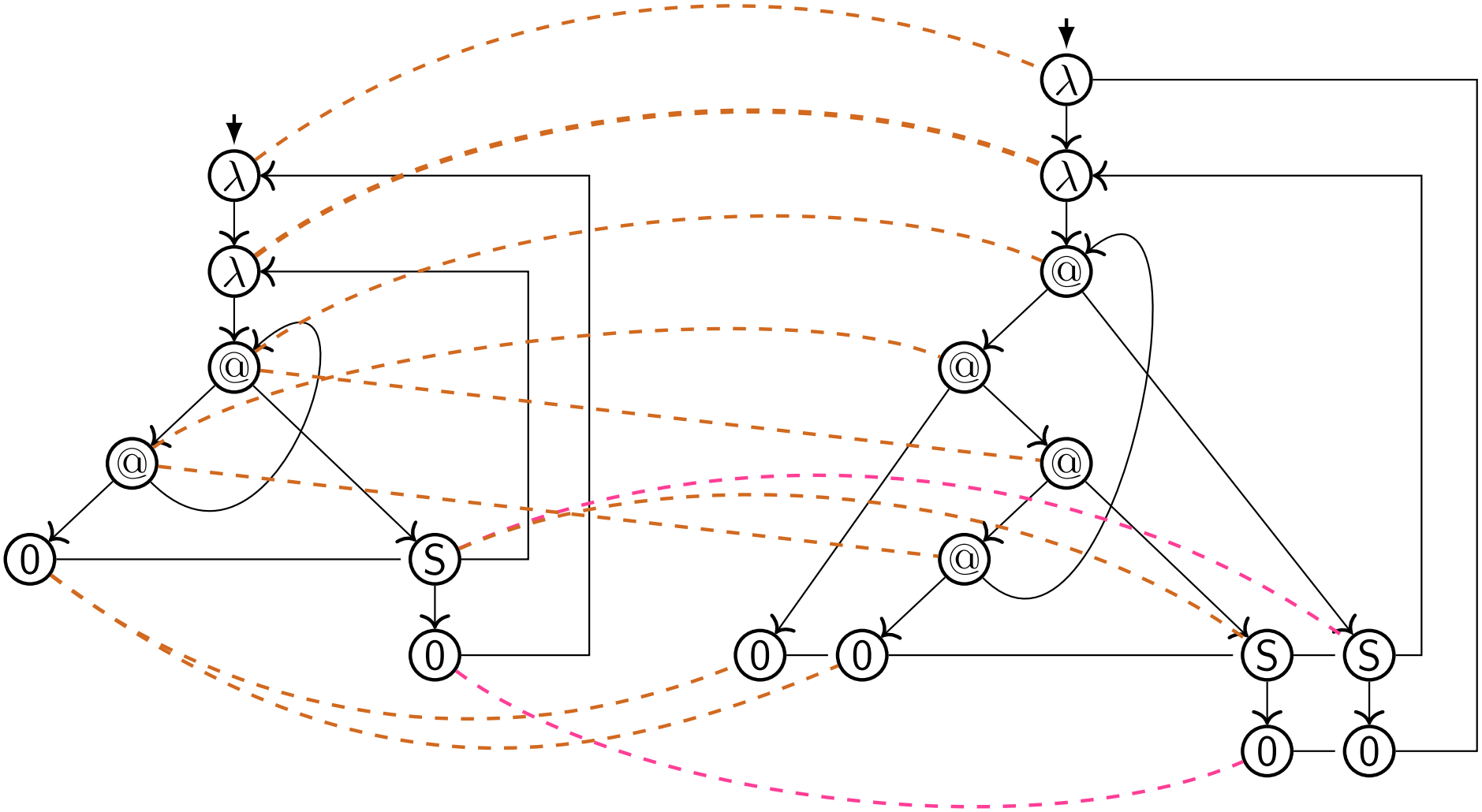
Bisimulation check between λ-term-graphs



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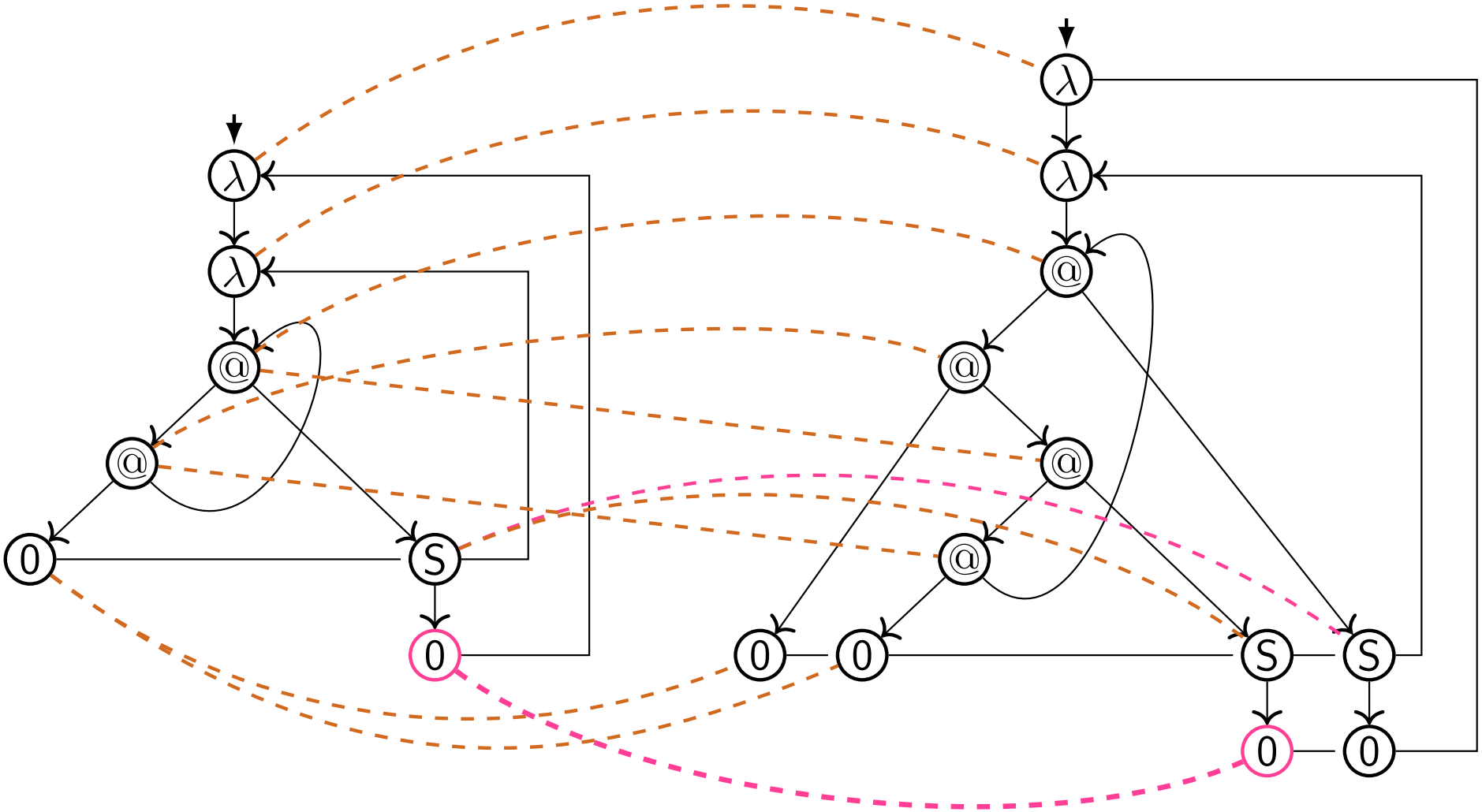

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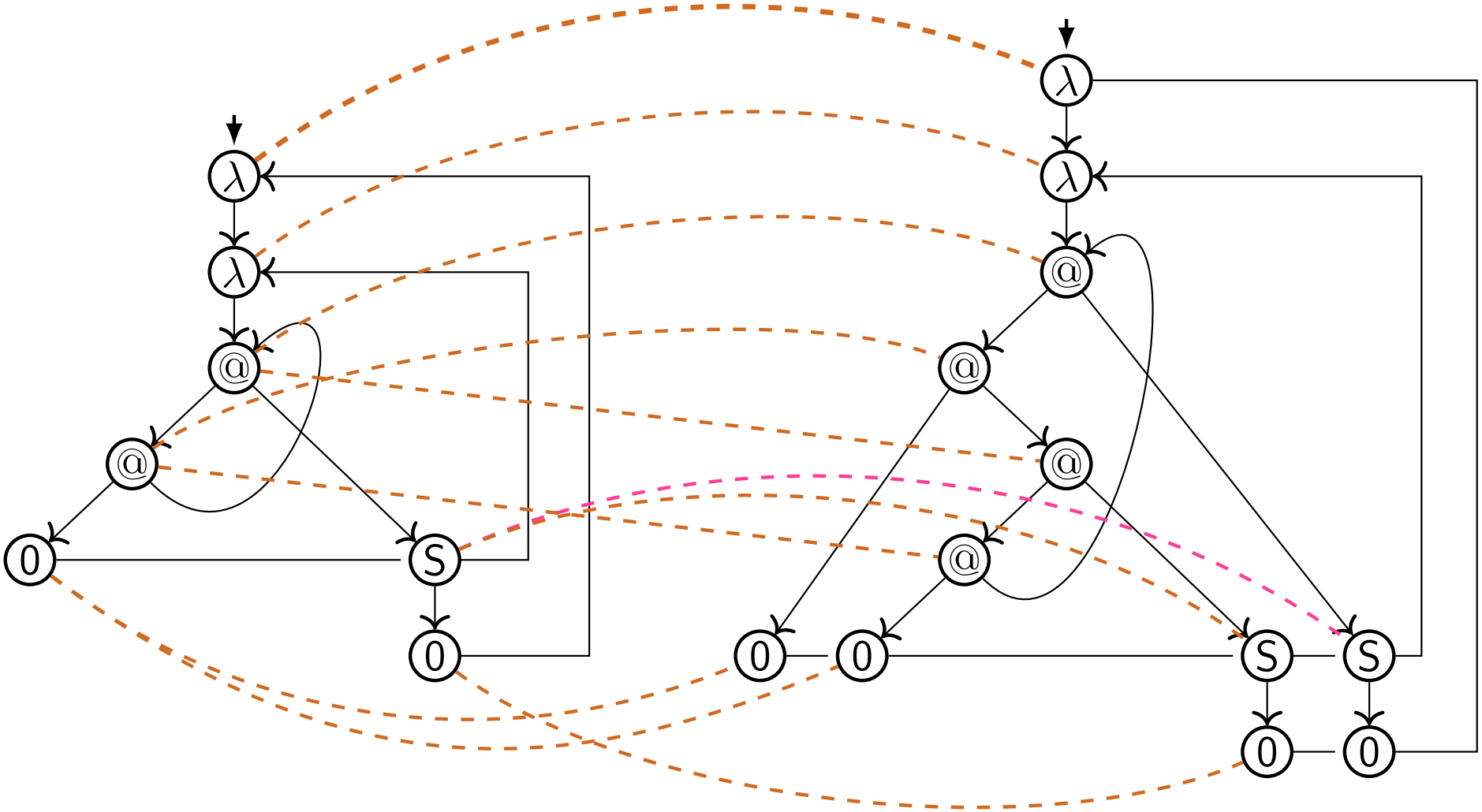

$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

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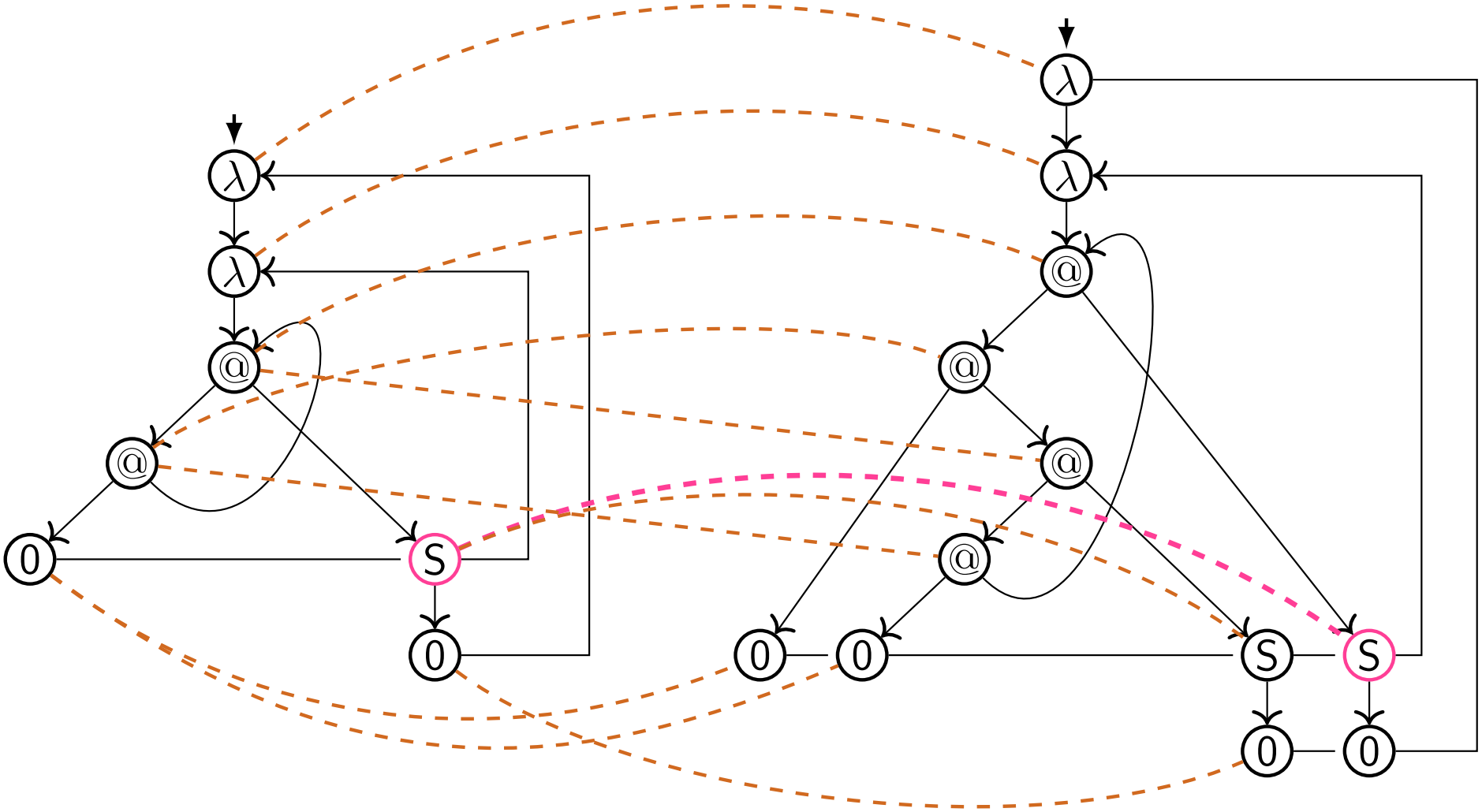
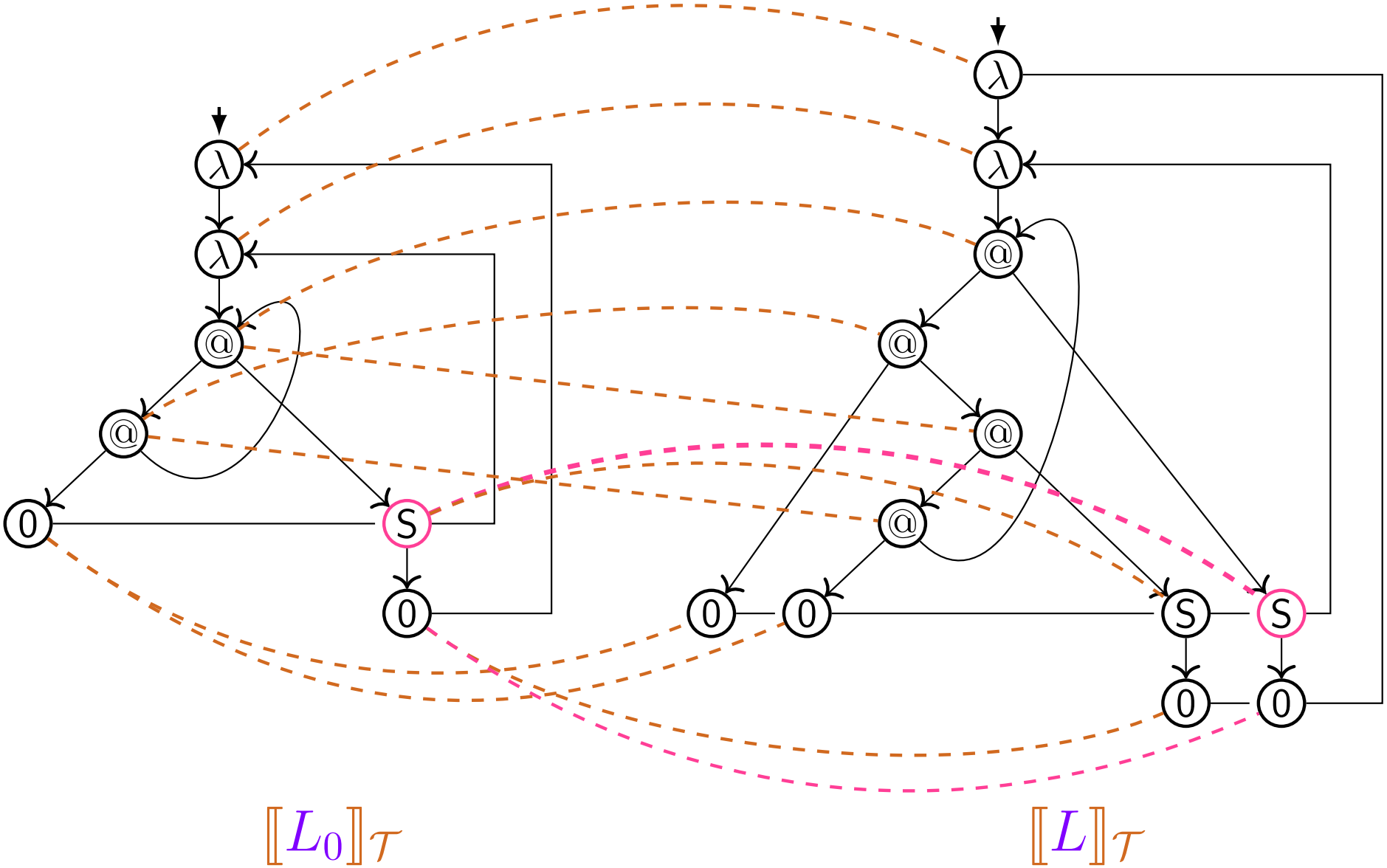
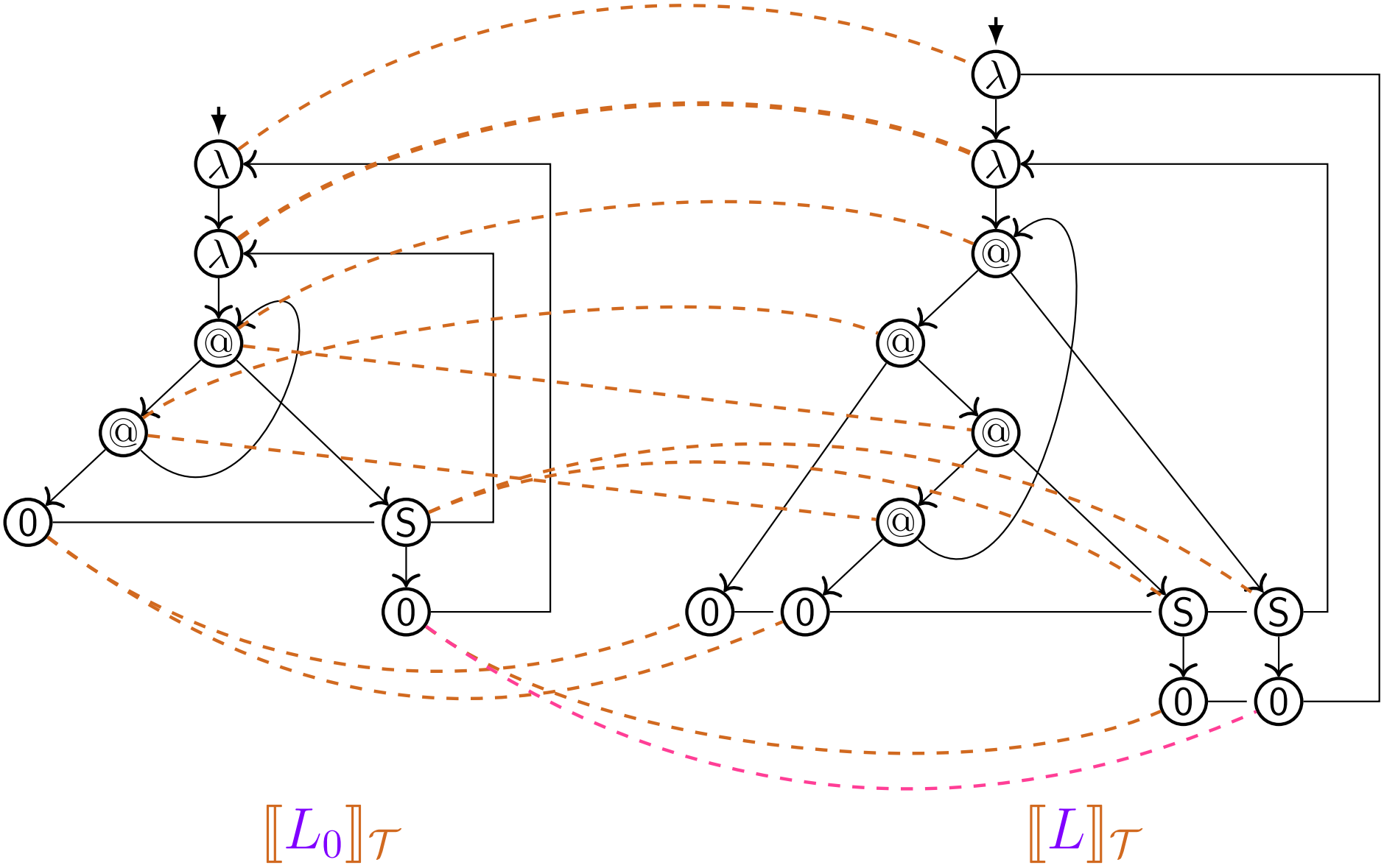


$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

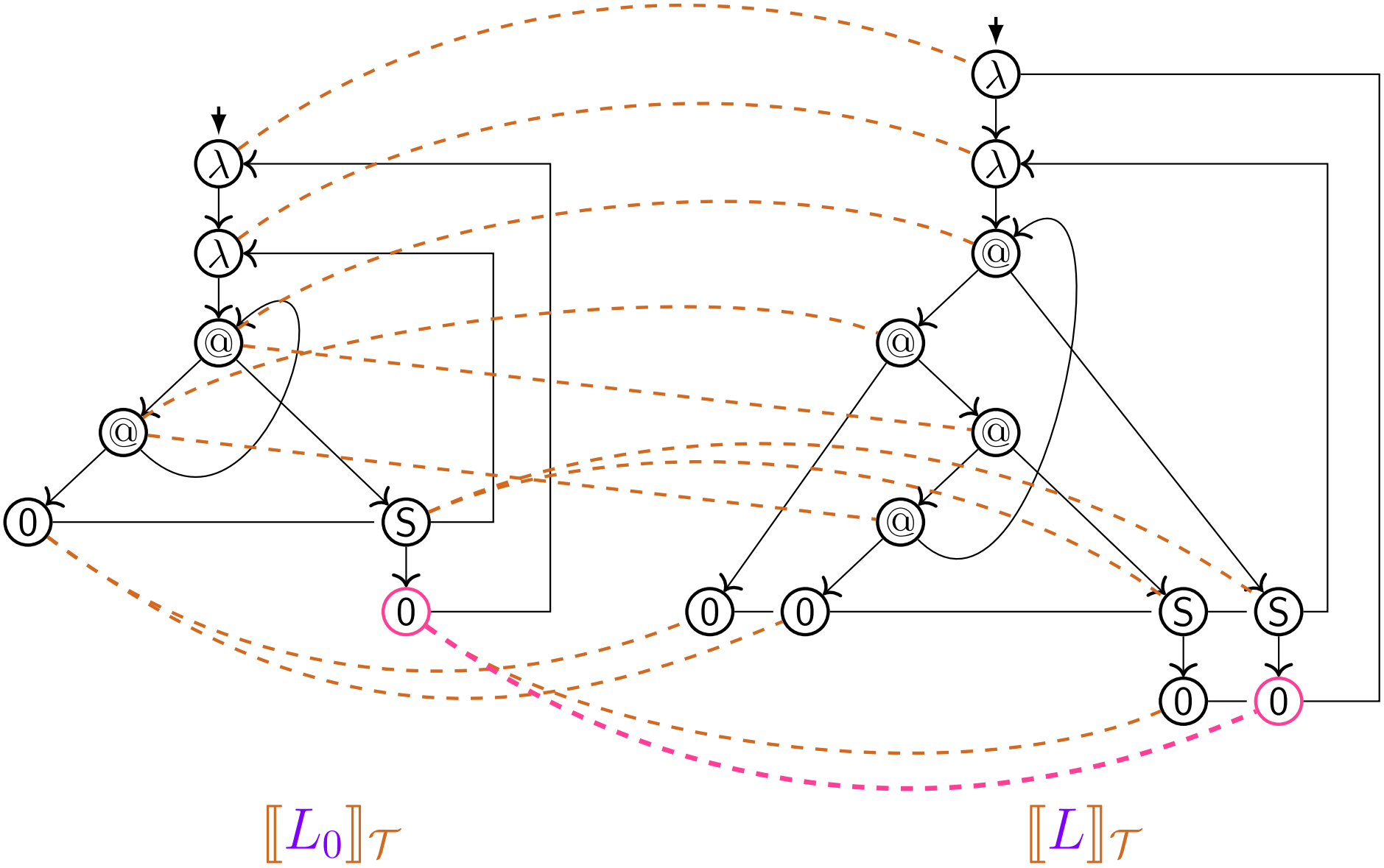
Bisimulation check between λ-term-graphs



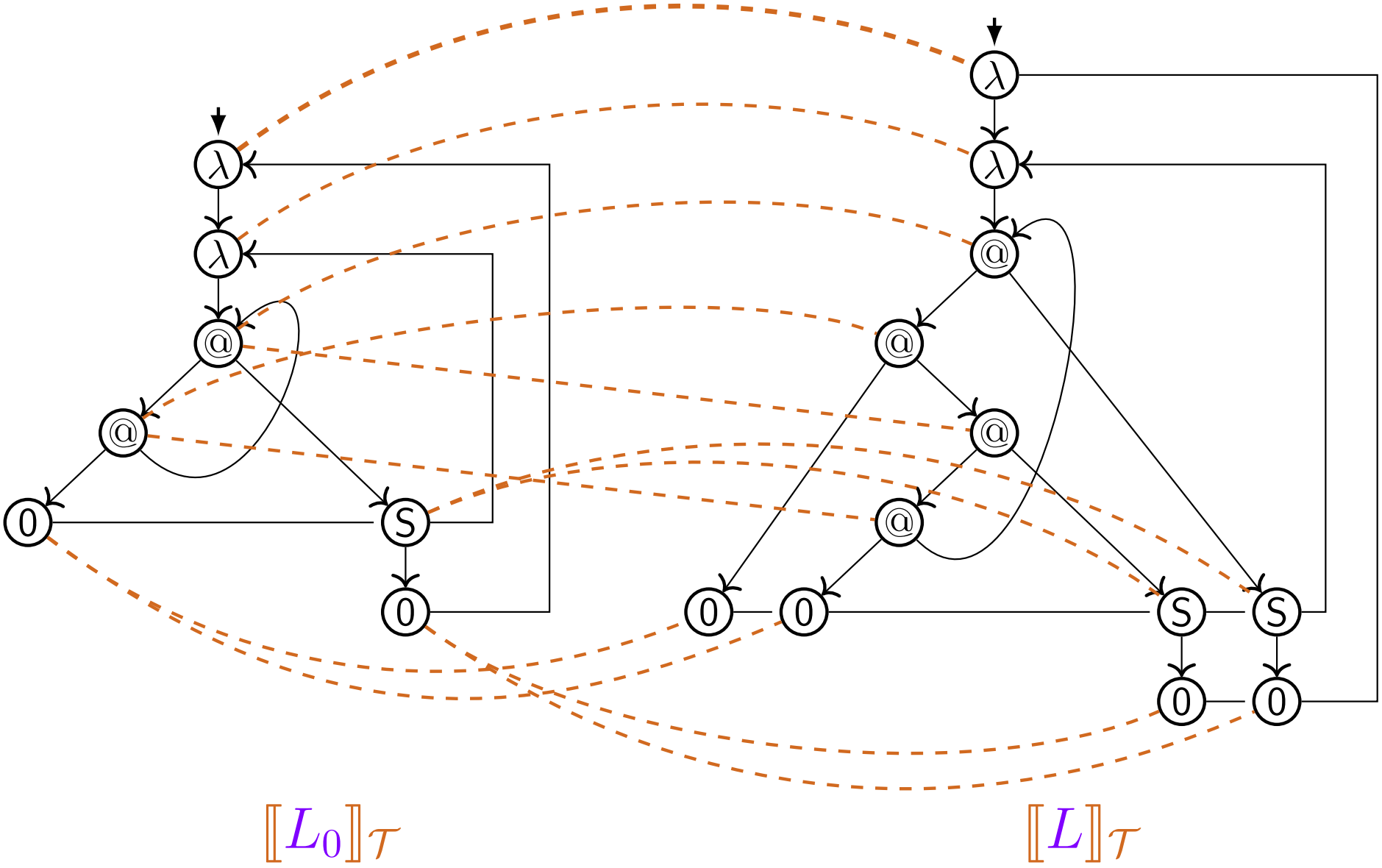
Bisimulation check between λ-term-graphs



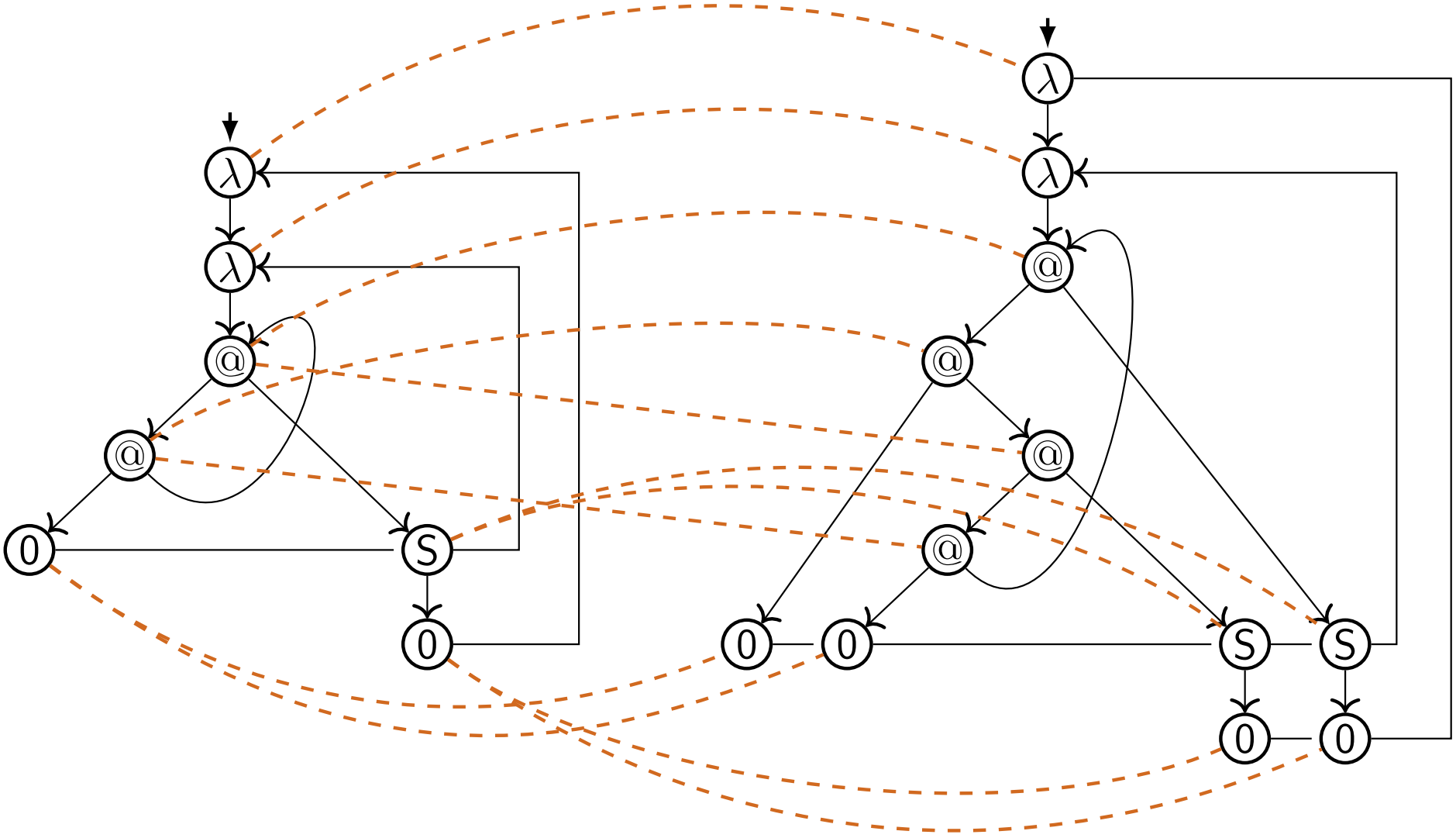
Bisimulation check between λ-term-graphs



Bisimulation check between λ-term-graphs

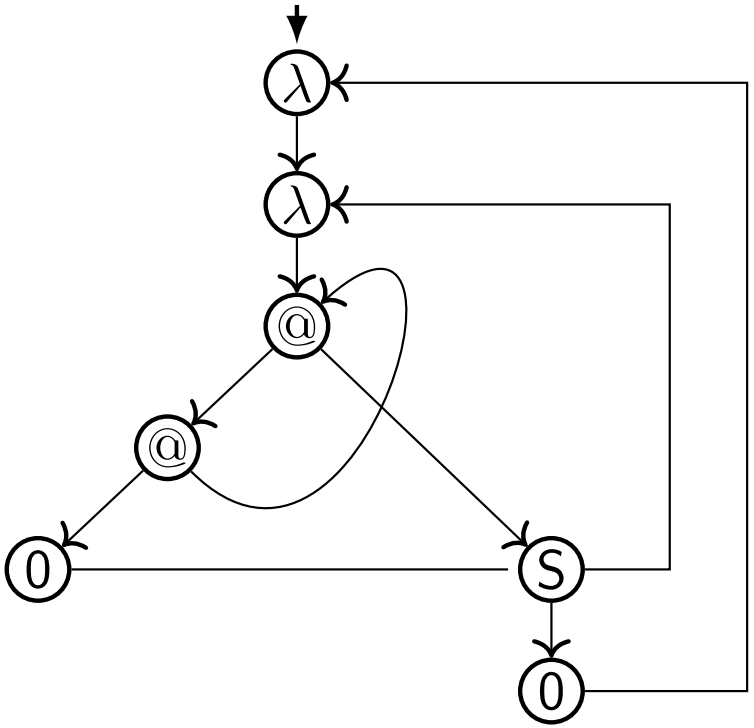


bisimulation between λ-term-graphs

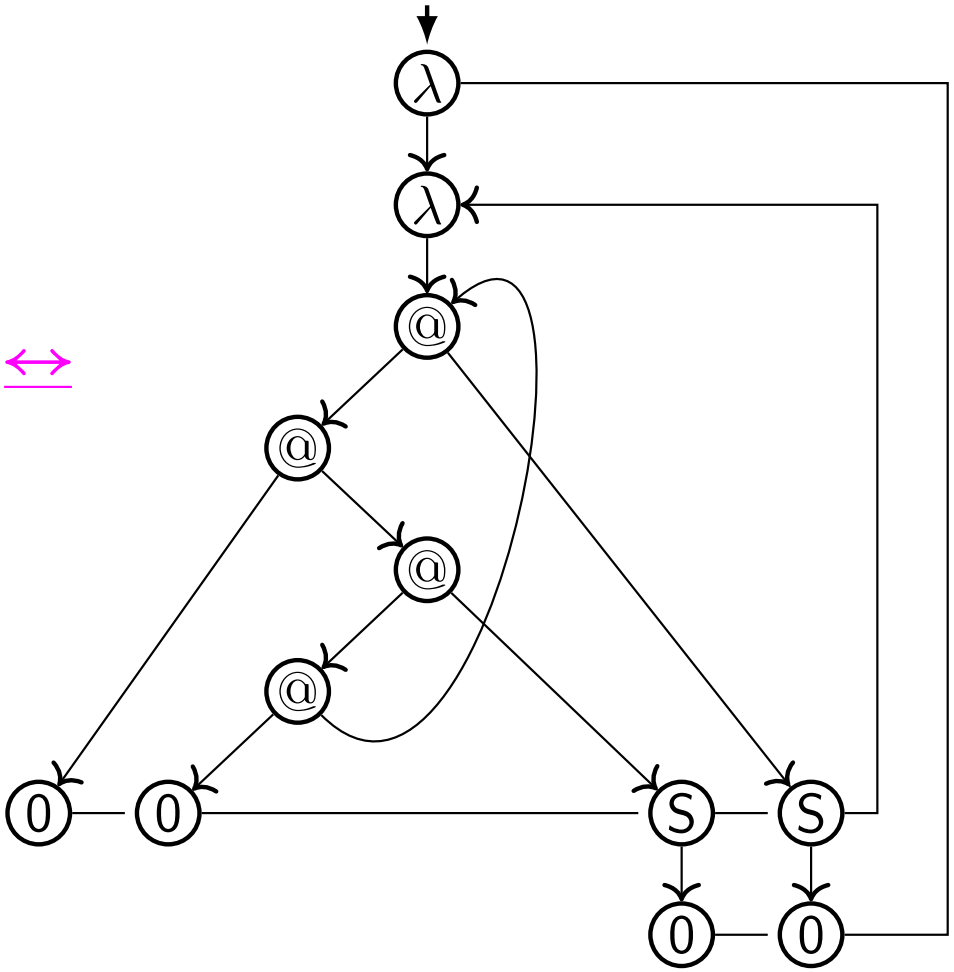


$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

bisimilarity between λ-term-graphs

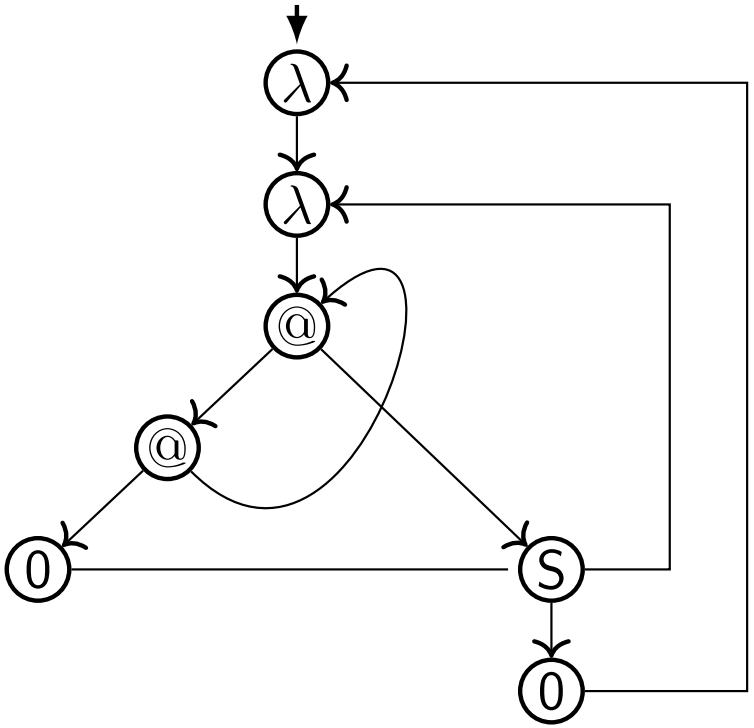


$\leftrightarrow$

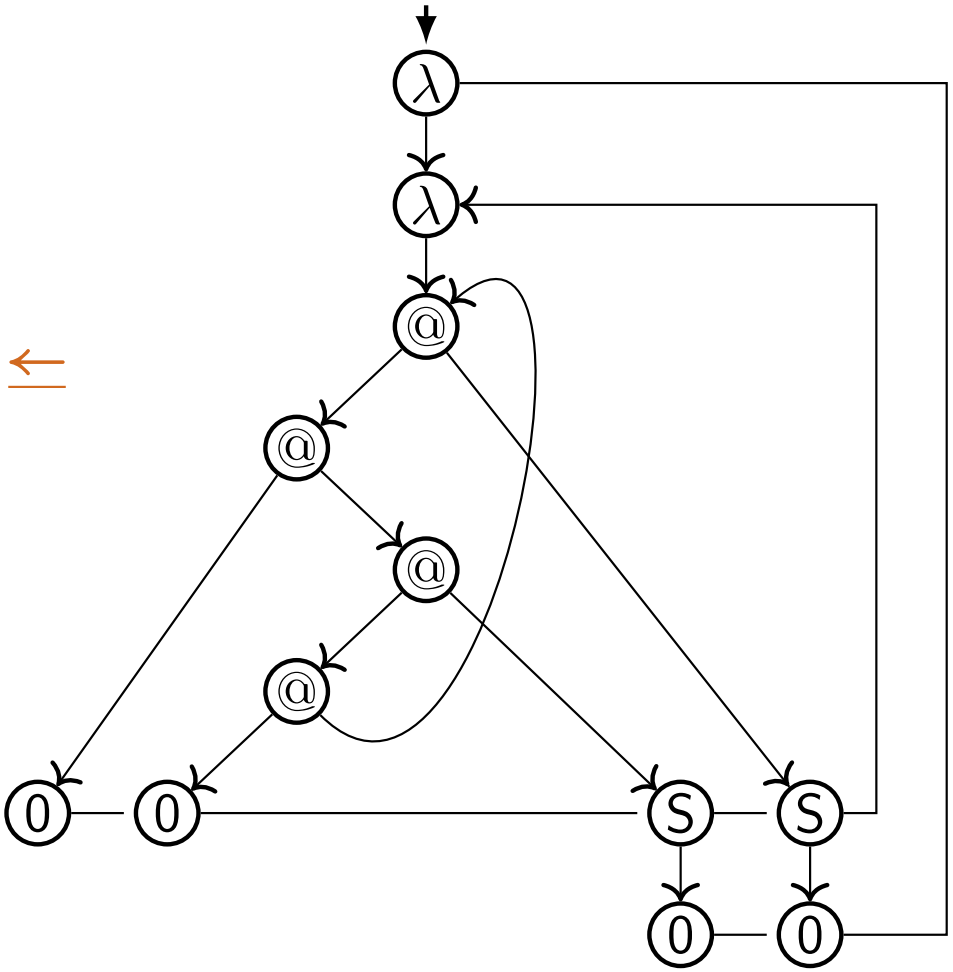


$[\![L_0]\!]_{\mathcal{T}}$ $\leftrightarrow$ $[\![L]\!]_{\mathcal{T}}$

functional bisimilarity and bisimulation collapse



$\leftarrow$



$[\![L_0]\!]_{\mathcal{T}}$ $\leftarrow$ $[\![L]\!]_{\mathcal{T}}$

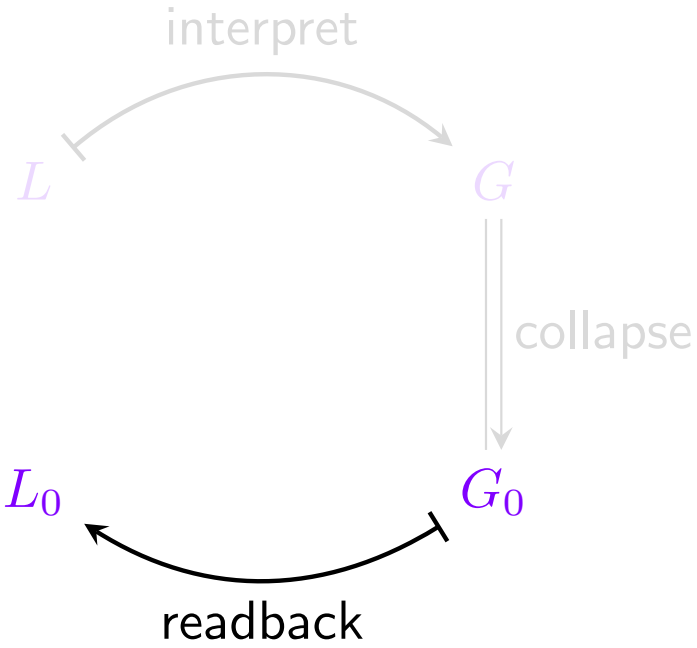
Bisimulation collapse: property

Theorem

The class of eager-scope λ-term-graphs
is closed under functional bisimilarity $\rightrightarrows$.

$\Longrightarrow$ For a $\boldsymbol{\lambda}_{\mathsf{letrec}}$-term L
the bisimulation collapse of $[\![L]\!]_{\mathcal{T}}$ is again an eager-scope λ-term-graph.

Readback



Readback

defined with property:

$$L \xleftarrow{\text{rb}} G \quad \text{eager-scope}$$

Readback

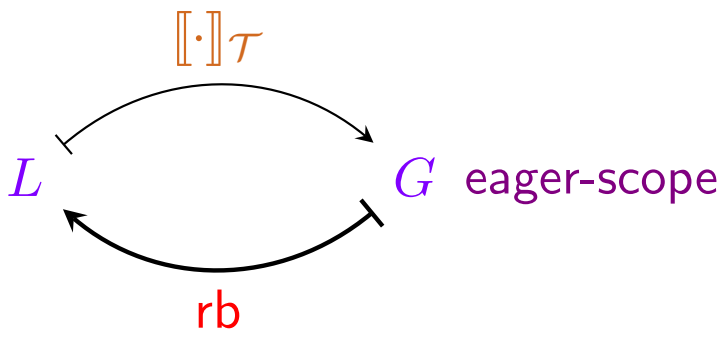
defined with property:

$$\begin{tikzcd} L \arrow[r, bend left, "{[\![\cdot]\!]\mathcal{T}}", mapsto] & G \arrow[l, bend left, "\text{rb}", mapsto] \end{tikzcd}$$

eager-scope

Readback

defined with property:



Theorem

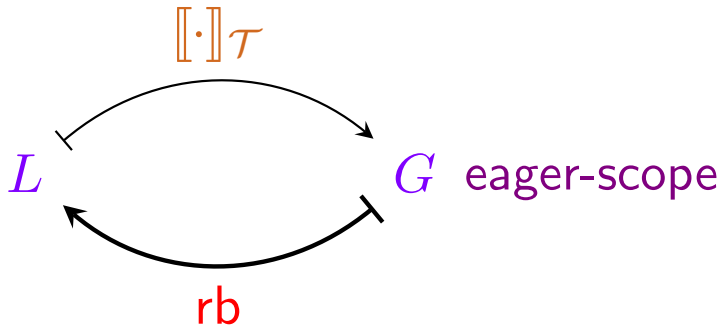
For all eager-scope λ-term-graphs G:

$$([\![\cdot]\!]_{\mathcal{T}} \circ \mathsf{rb})(G) \simeq G$$

The readback rb *is a right-inverse of* $[\![\cdot]\!]_{\mathcal{T}}$ *modulo isomorphism* $\simeq$.

Readback

defined with property:



Theorem

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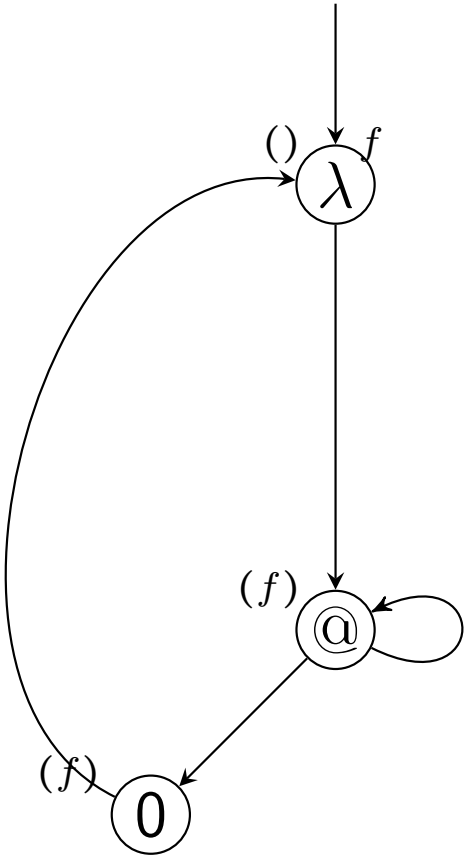
$$([\![\cdot]\!]_{\mathcal{T}} \circ \mathsf{rb})(G) \simeq G$$

The readback rb is a right-inverse of $[\![\cdot]\!]_{\mathcal{T}}$ modulo isomorphism $\simeq$.

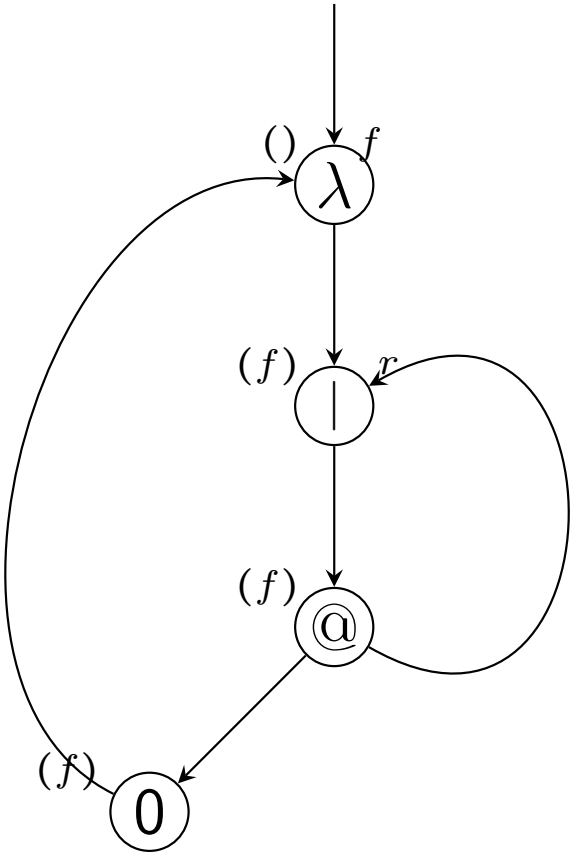
idea:

1. construct a spanning tree T of G
2. using local rules, in a bottom-up traversal of T synthesize $L = \mathsf{rb}(G)$

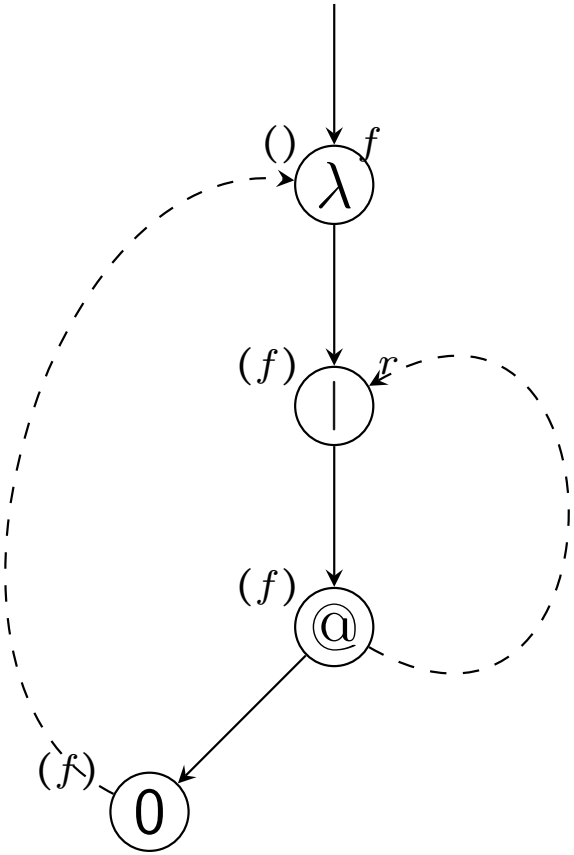
Readback: example (fix)



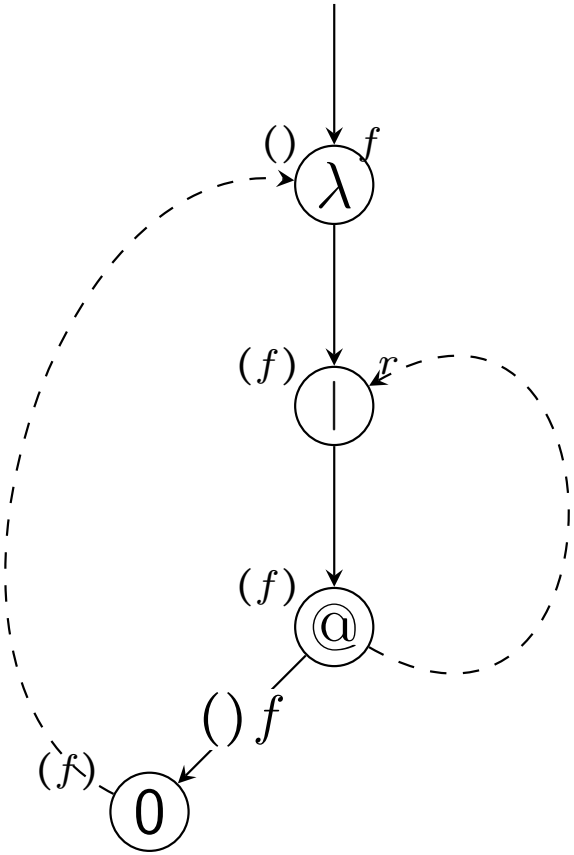
Readback: example (fix)



Readback: example (fix)



Readback: example (fix)



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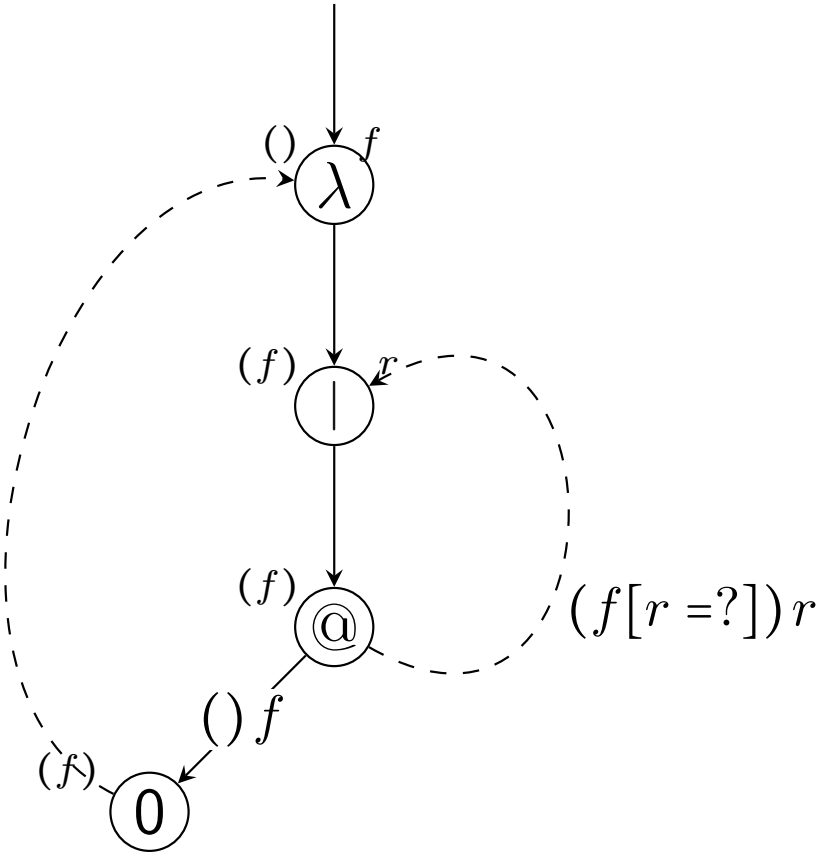

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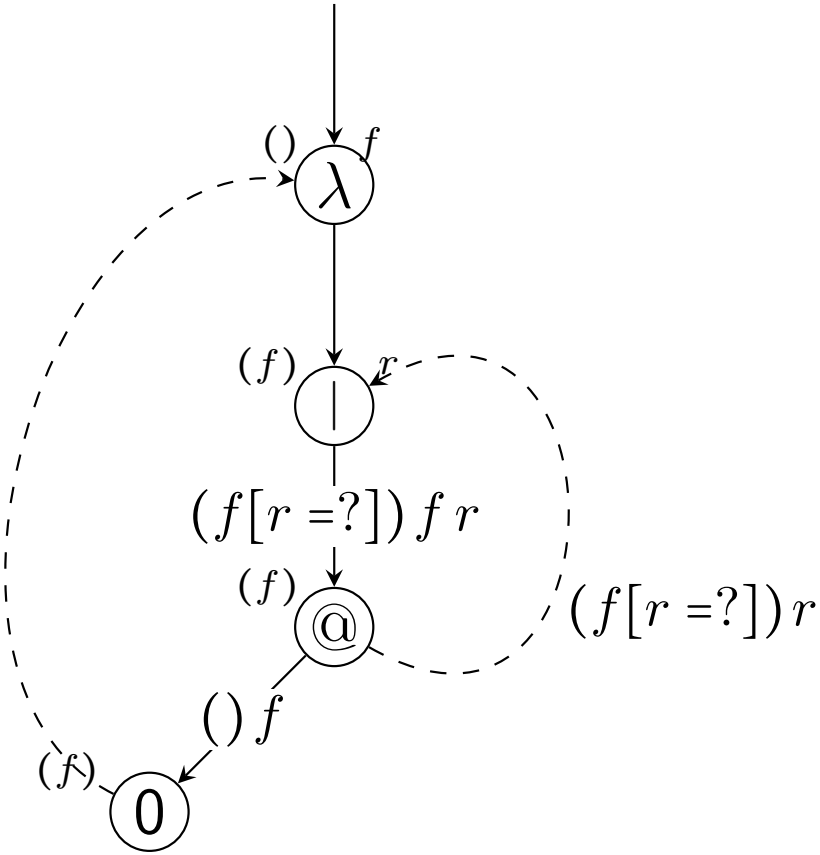

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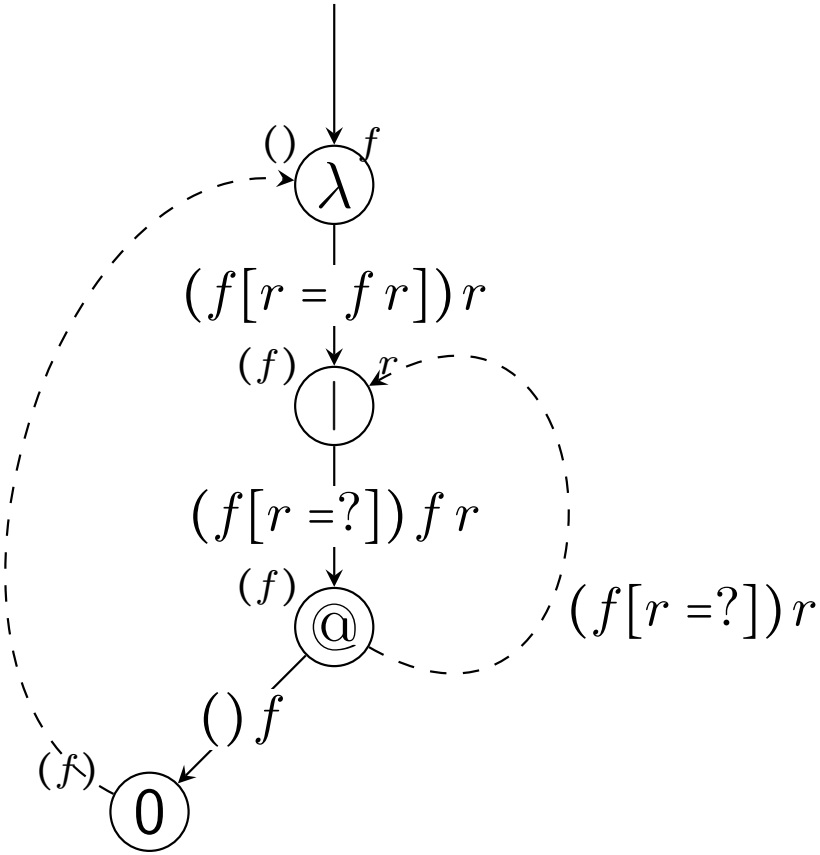

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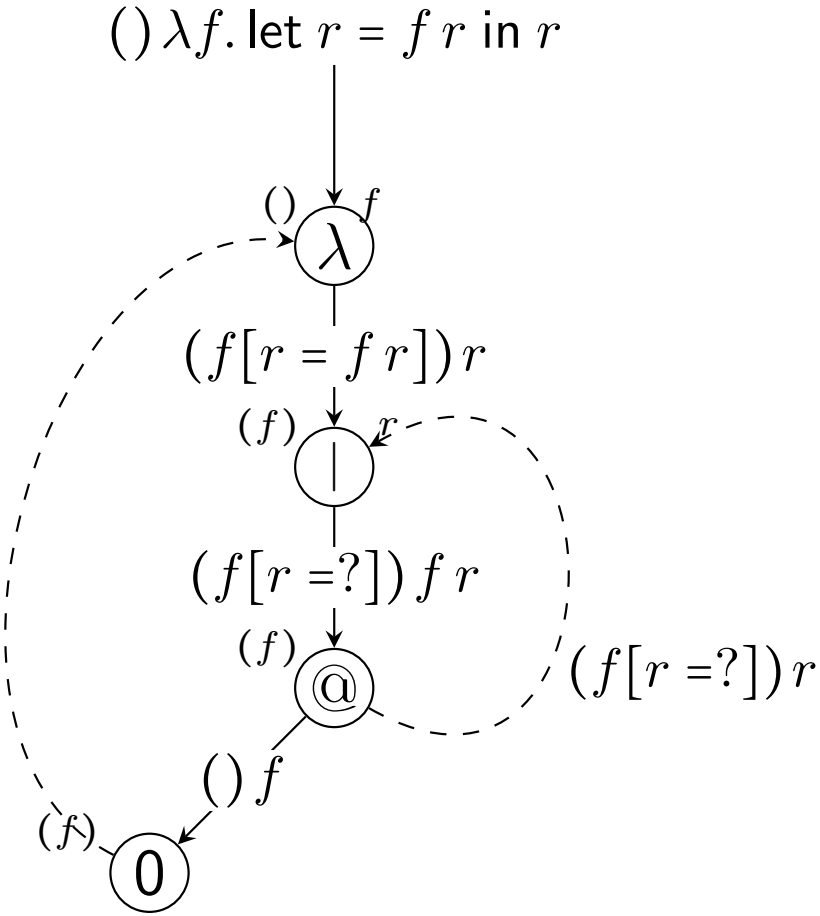

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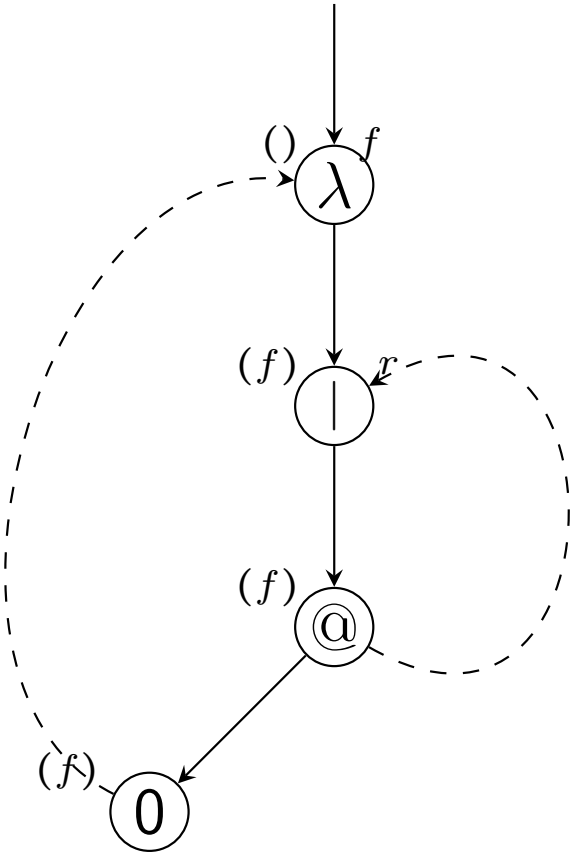

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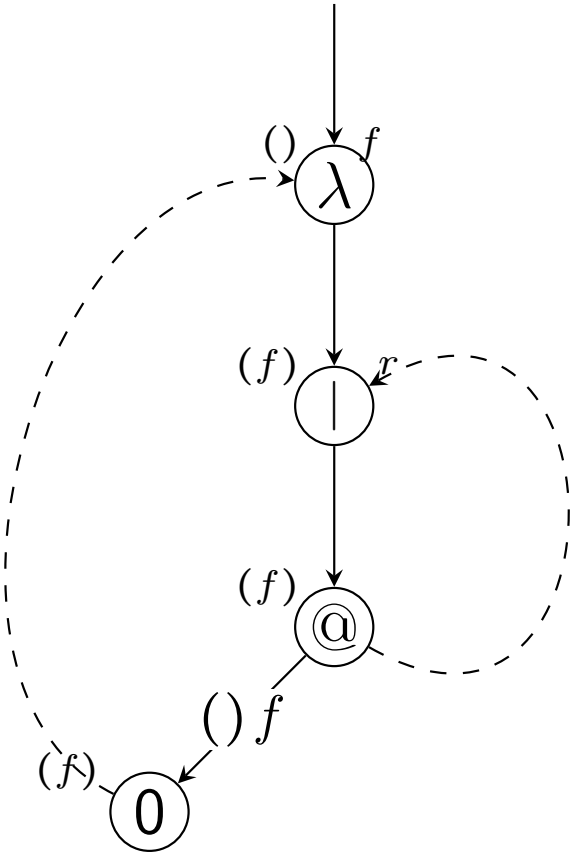

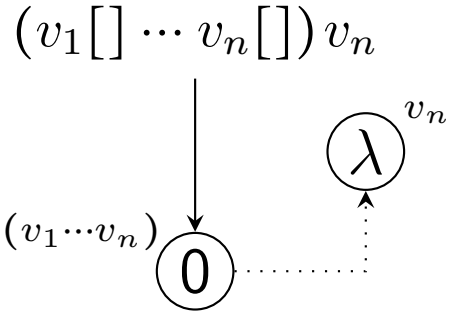

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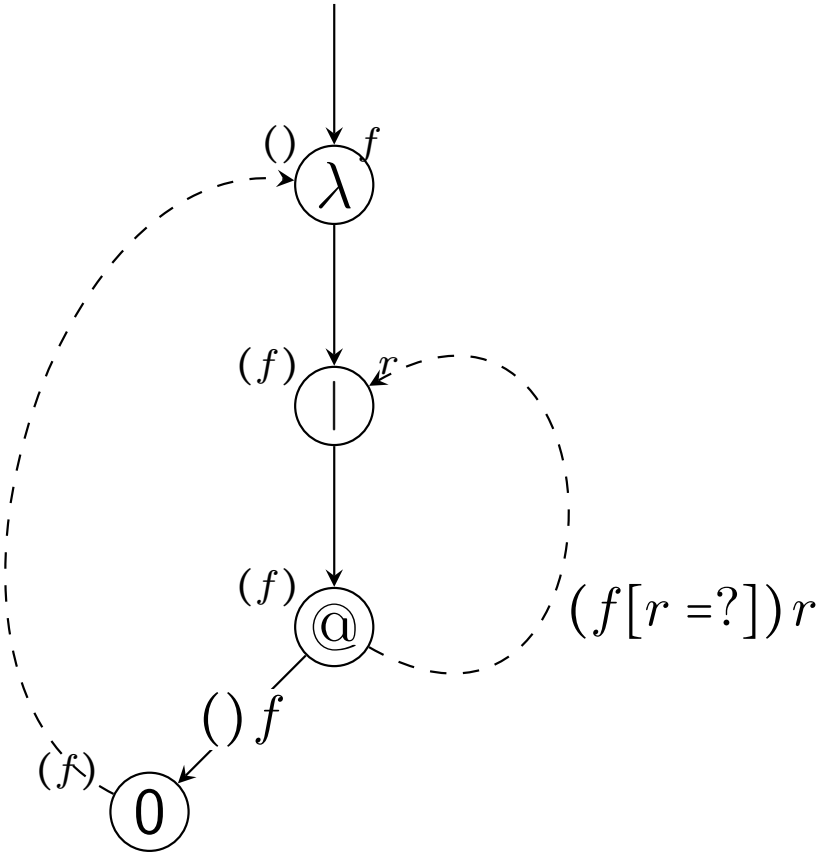

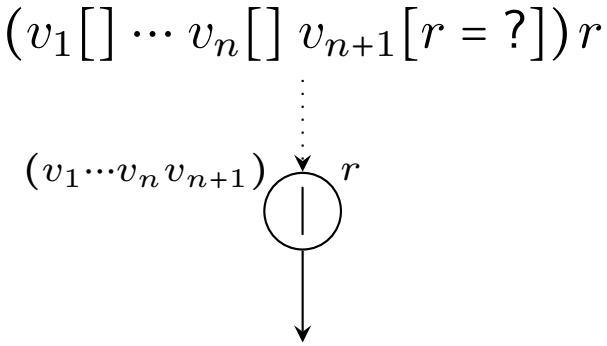

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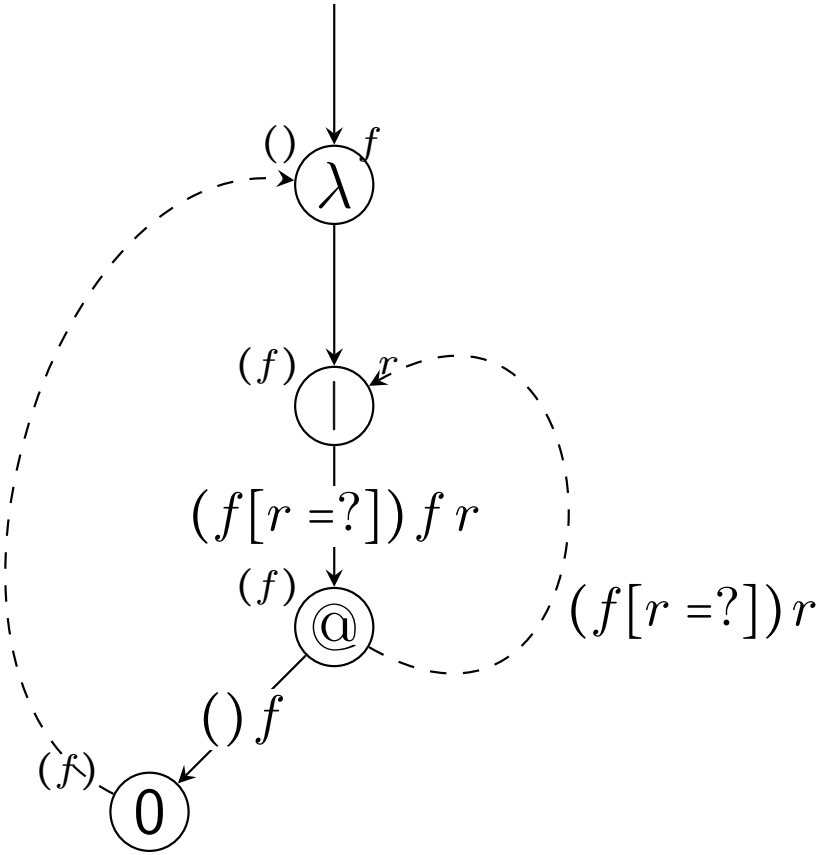

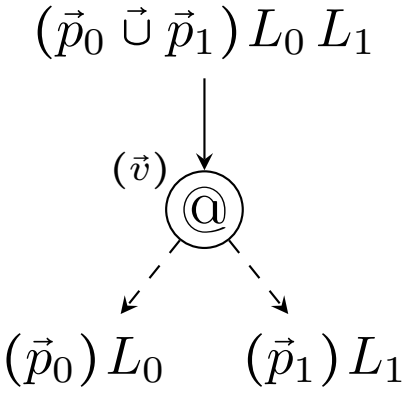

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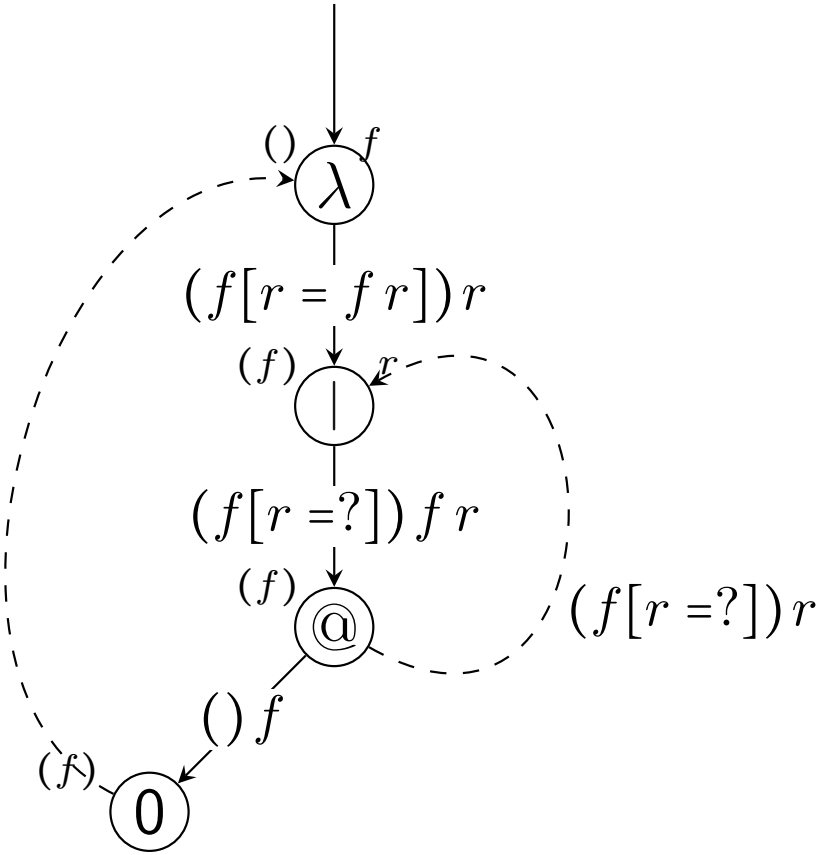

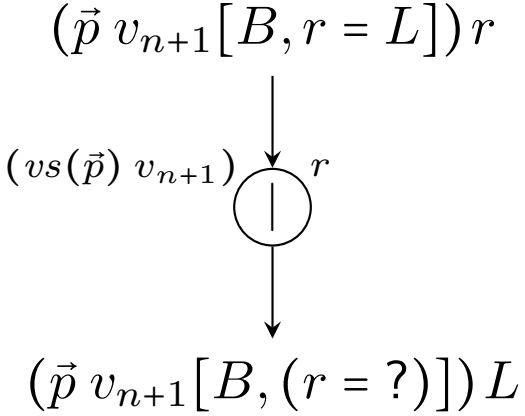

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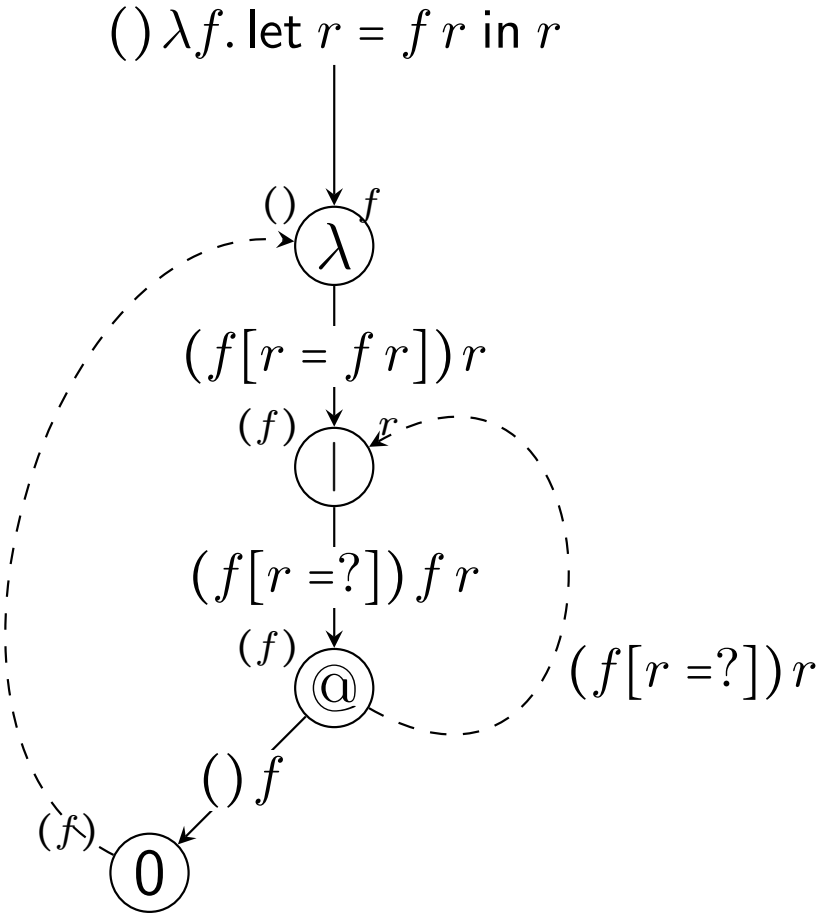
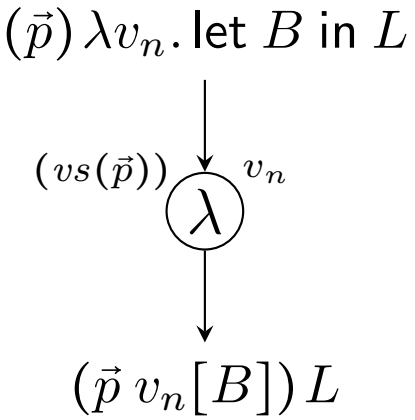
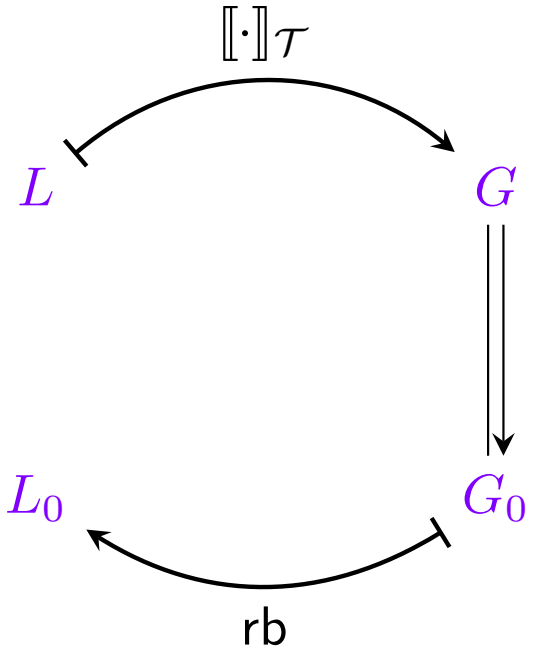




Maximal sharing: complexity



1. interpretation

   of $\boldsymbol{\lambda}_{\text{letrec}}$-term L with $|L| = n$

   as λ-term-graph $G = [\![L]\!]_{\mathcal{T}}$

   - in time $O(n^2)$, size $|G| \in O(n^2)$.

2. bisimulation collapse $|\downarrow$

   of f-o term graph G into G_0

   - in time $O(|G| \log |G|) = O(n^2 \log n)$

3. readback rb

   of f-o term graph G_0

   yielding $\boldsymbol{\lambda}_{\text{letrec}}$-term $L_0 = \text{rb}(G_0)$.

   - in time $O(|G| \log |G|) = O(n^2 \log n)$

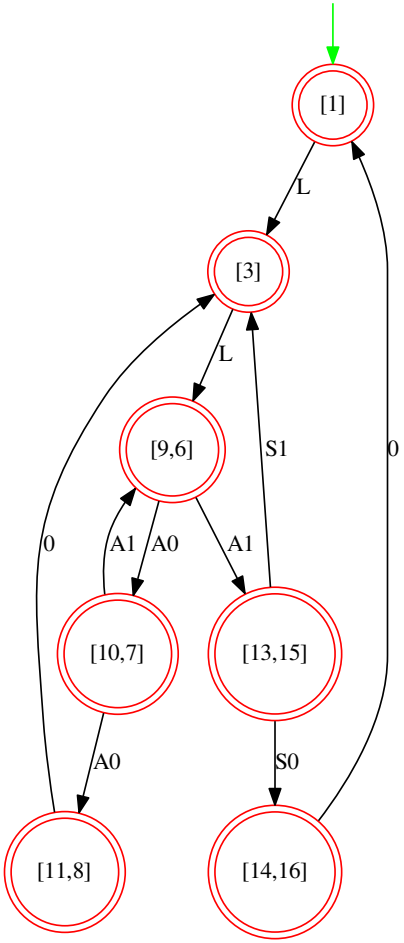
Theorem

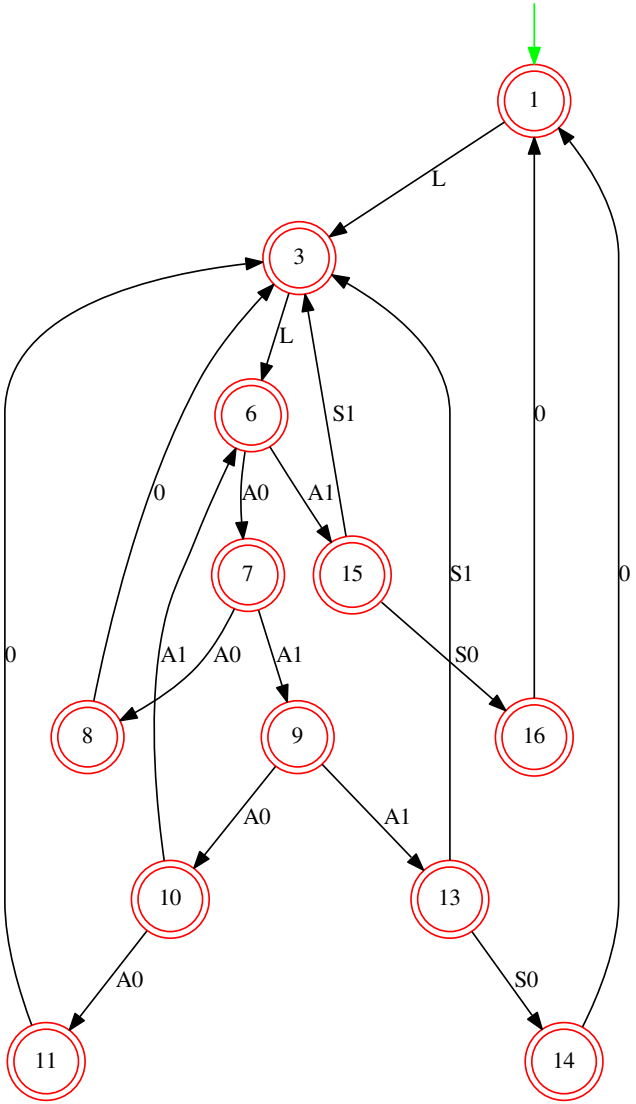
Computing a maximally compact form $L_0 = (\text{rb} \circ |\downarrow \circ [\![\cdot]\!]_{\mathcal{T}})(L)$ of L for a $\boldsymbol{\lambda}_{\text{letrec}}$-term L requires time $O(n^2 \log n)$, where $|L| = n$.

Demo: console output

```
jan:~/papers/maxsharing-ICFP/talks/ICFP-2014> maxsharing running.l
λ-letrec-term:
λx. λf. let r = f (f r x) x in r

derivation:
                ---------- 0                 ------- 0
                (x f[r]) f    (x f[r]) r    (x) x
                ------------------------ @  ---------- S
                (x f[r]) f r                (x f[r]) x
------------ 0  ---------------------------------------- @    ------- 0
(x f[r]) f      (x f[r]) f r x                               (x) x
------------------------------------------------------------ @  ---------- S
(x f[r]) f (f r x)                                             (x f[r]) x
-------------------------------------------------------------------------- @
(x f[r]) f (f r x) x                                                          (x f[r]) r
------------------------------------------------------------------------------------------------------ let
(x f) let r = f (f r x) x in r
------------------------------------------------------------------------------------------------------- λ
(x) λf. let r = f (f r x) x in r
-------------------------------------------------------------------------------------------------------- λ
() λx. λf. let r = f (f r x) x in r

writing DFA to file: running-dfa.pdf

readback of DFA:
λx. λy. let F = y (y F x) x in F

writing minimised DFA to file: running-mindfa.pdf

readback of minimised DFA:
λx. λy. let F = y F x in F
jan:~/papers/maxsharing-ICFP/talks/ICFP-2014>
```

Demo: generated λ-NFAs



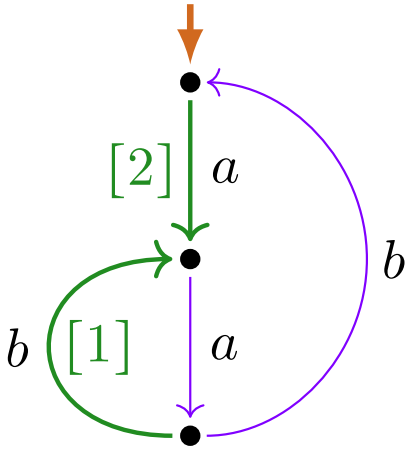


Resources (maximal sharing)

- tool maxsharing on `hackage.haskell.org`
- articles and reports
  - Maximal Sharing in the Lambda Calculus with Letrec
    - ICFP 2014 paper
    - accompanying report arXiv:1401.1460
  - Term Graph Representations for Cyclic Lambda Terms
    - TERMGRAPH 2013 proceedings
    - extended report arXiv:1308.1034
  - Vincent van Oostrom, CG: Nested Term Graphs
    - TERMGRAPH 2014 post-proceedings in EPTCS 183
- thesis Jan Rochel
  - Unfolding Semantics of the Untyped λ-Calculus with letrec
    - Ph.D. Thesis, Utrecht University, 2016

Process interpretation of regular expressions

(based on joint work with Wan Fokkink)



Regular expressions *(S.C. Kleene, 1951)*

Definition

The set Reg(A) of regular expressions over alphabet A is defined by the grammar:

$$e, f \ ::= \ 0 \mid 1 \mid a \mid (e + f) \mid (e \cdot f) \mid (e^\star) \qquad \text{(for } a \in A\text{)}.$$

Regular expressions *(S.C. Kleene, 1951)*

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Note, here:

- symbol 0 instead of $\varnothing$
- symbol 1 used (often dropped, definable as $0^\star$)
- no complementation operation $\overline{e}$
  - which is not expressible under language interpretation

Language semantics $[\![\cdot]\!]_L$ of reg. expr's *(Copi–Elgot–Wright, 1958)*

$$\mathbf{0} \xmapsto{L} \text{empty language } \varnothing$$

$$\mathbf{1} \xmapsto{L} \{\epsilon\} \qquad (\epsilon \text{ the empty word})$$

$$a \xmapsto{L} \{a\}$$

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$$e + f \xmapsto{L} \text{union of } L(e) \text{ and } L(f)$$

$$e \cdot f \xmapsto{L} \text{element-wise concatenation of } L(e) \text{ and } L(f)$$

$$e^* \xmapsto{L} \text{set of words formed by concatenating words in } L(e), \text{ and adding the empty word } \epsilon$$

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$$[\![e]\!]_L := L(e) \quad (\text{language defined by } e)$$

Process semantics of regular expressions $[\![\cdot]\!]_P$ *(Milner, 1984)*

$\mathbf{0} \overset{P}{\longmapsto}$ deadlock $\boldsymbol{\delta}$, no termination

$\mathbf{1} \overset{P}{\longmapsto}$ empty-step process $\boldsymbol{\epsilon}$, then terminate

$a \overset{P}{\longmapsto}$ atomic action a, then terminate

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$e + f \xmapsto{P}$ *(choice)* execute $[\![e]\!]_P$ or $[\![f]\!]_P$

$e \cdot f \xmapsto{P}$ *(sequentialization)* execute $[\![e]\!]_P$, then $[\![f]\!]_P$

$e^* \xmapsto{P}$ *(iteration)* repeat (terminate or execute $[\![e]\!]_P$)

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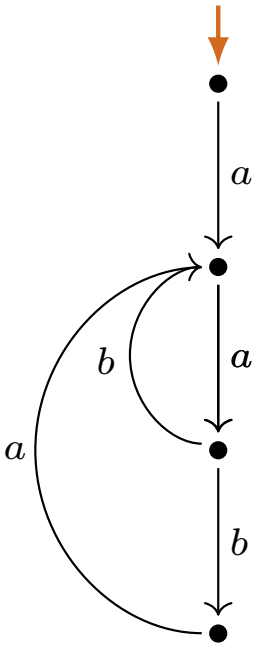
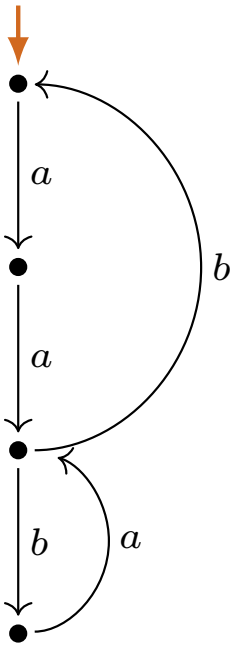
$$[\![e]\!]_P := [P(e)]_{\leftrightarrow} \quad \text{(bisimilarity equivalence class of process } P(e)\text{)}$$

Process interpretation of regular expressions



$P(a(a(b+ba))^*0)$



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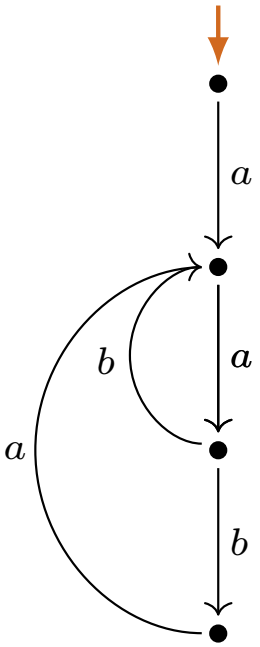

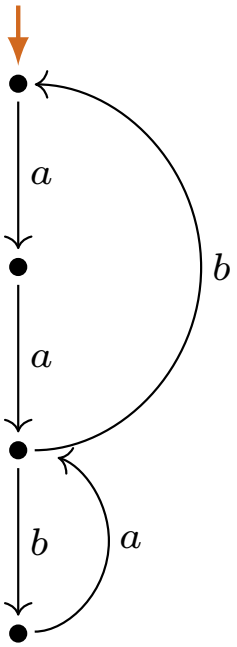

$P(a \cdot (a \cdot (b + b \cdot a))^* \cdot 0)$

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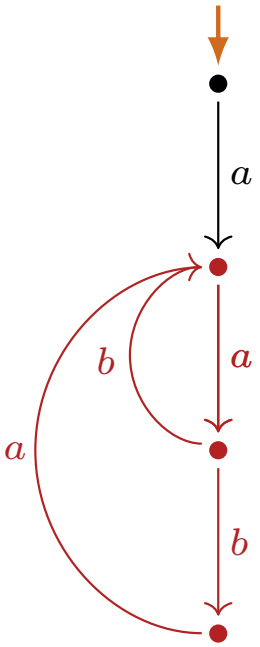
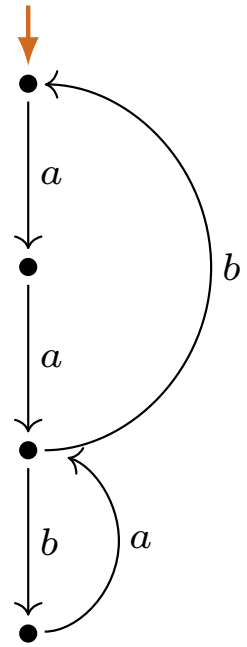
Process interpretation of regular expressions





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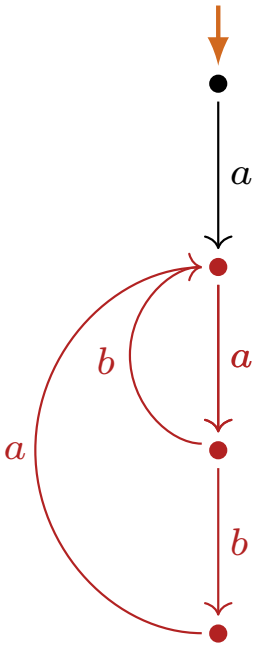

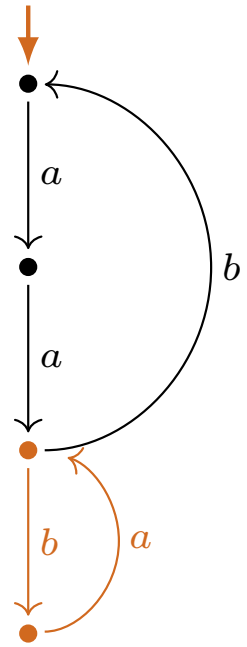

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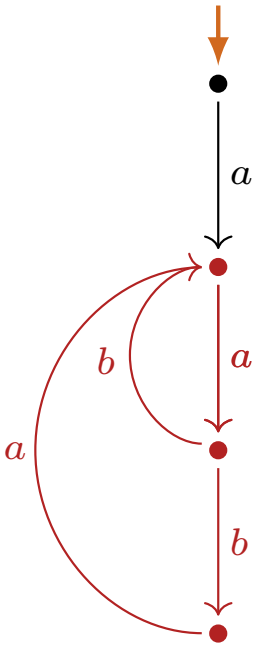

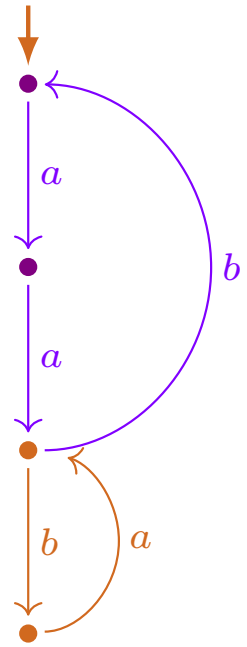

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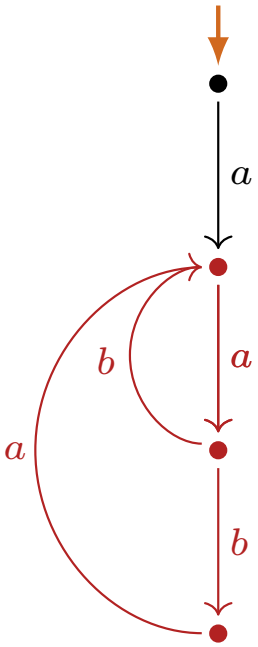

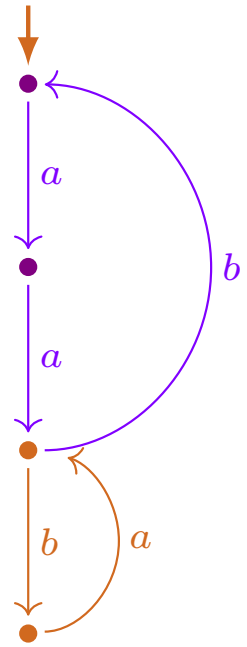

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Process interpretation of regular expressions

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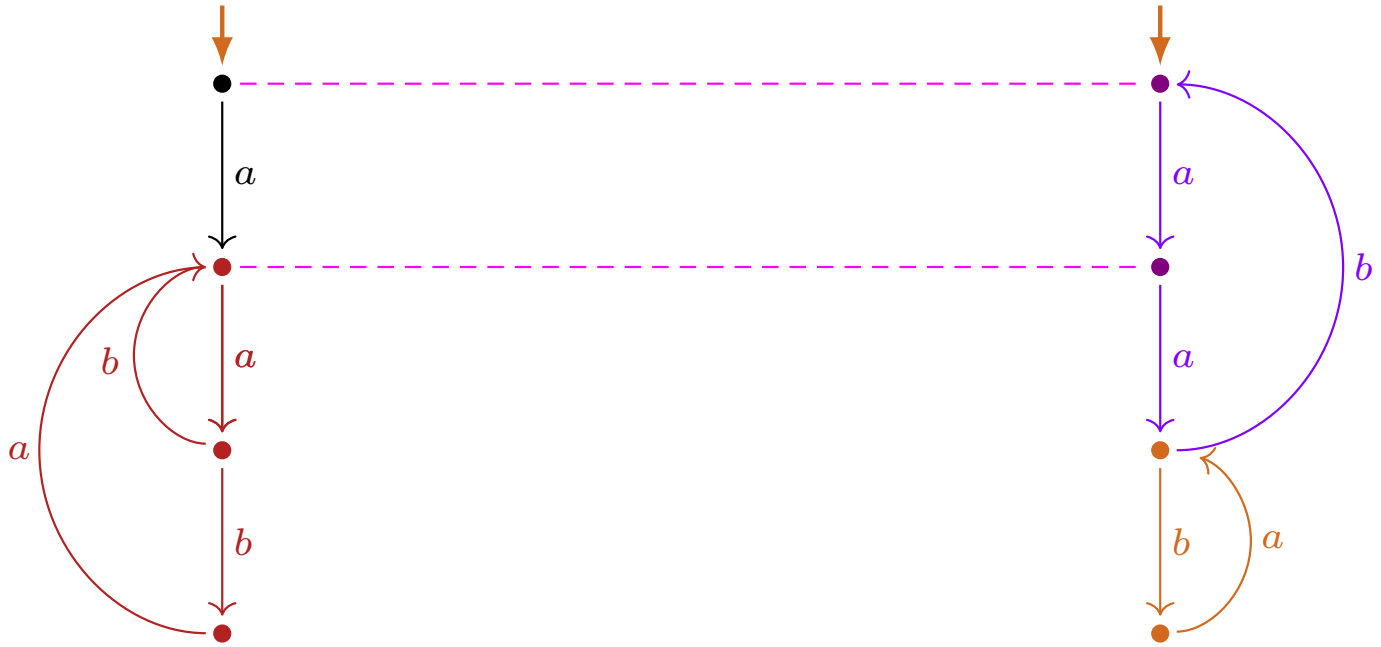

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Process interpretation of regular expressions

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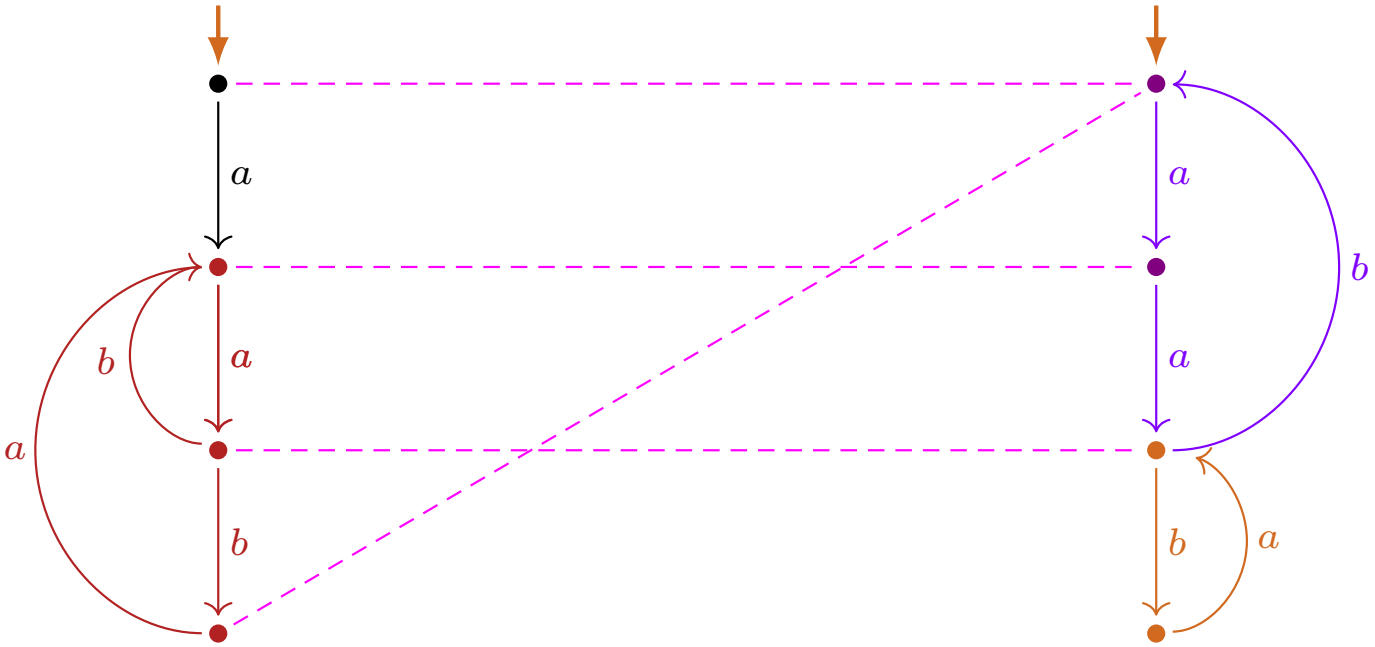

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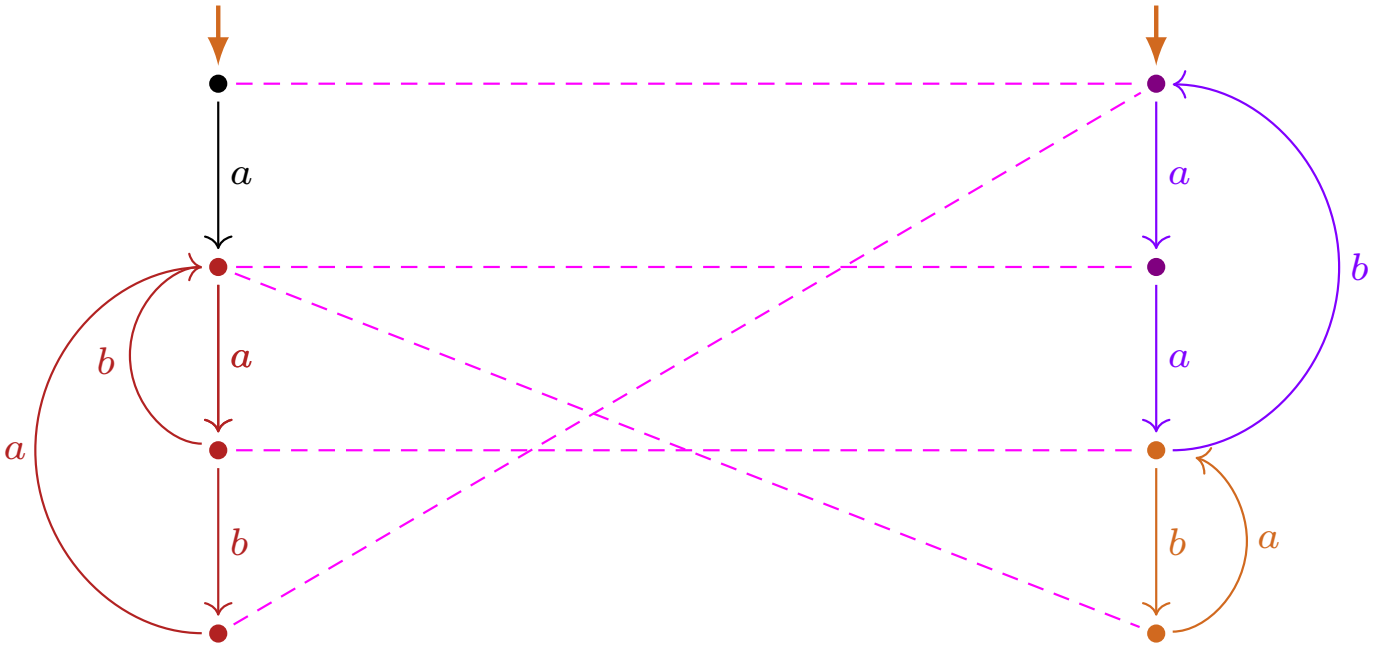

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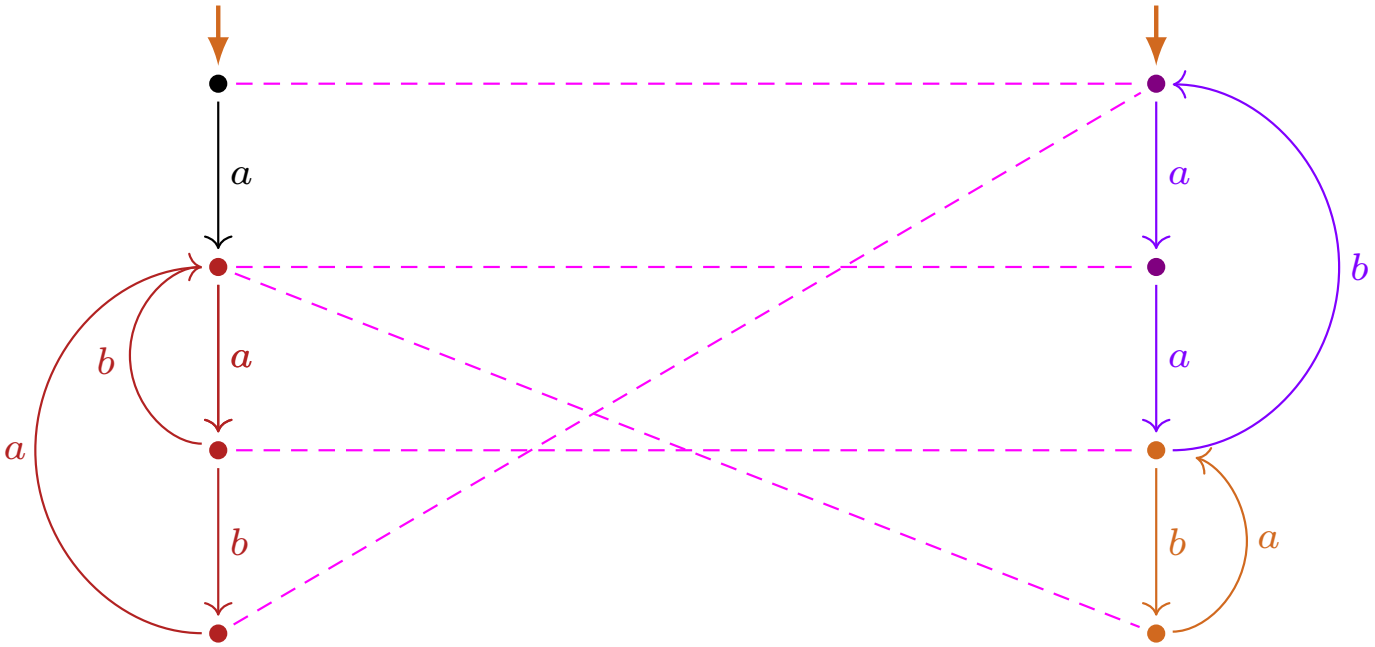
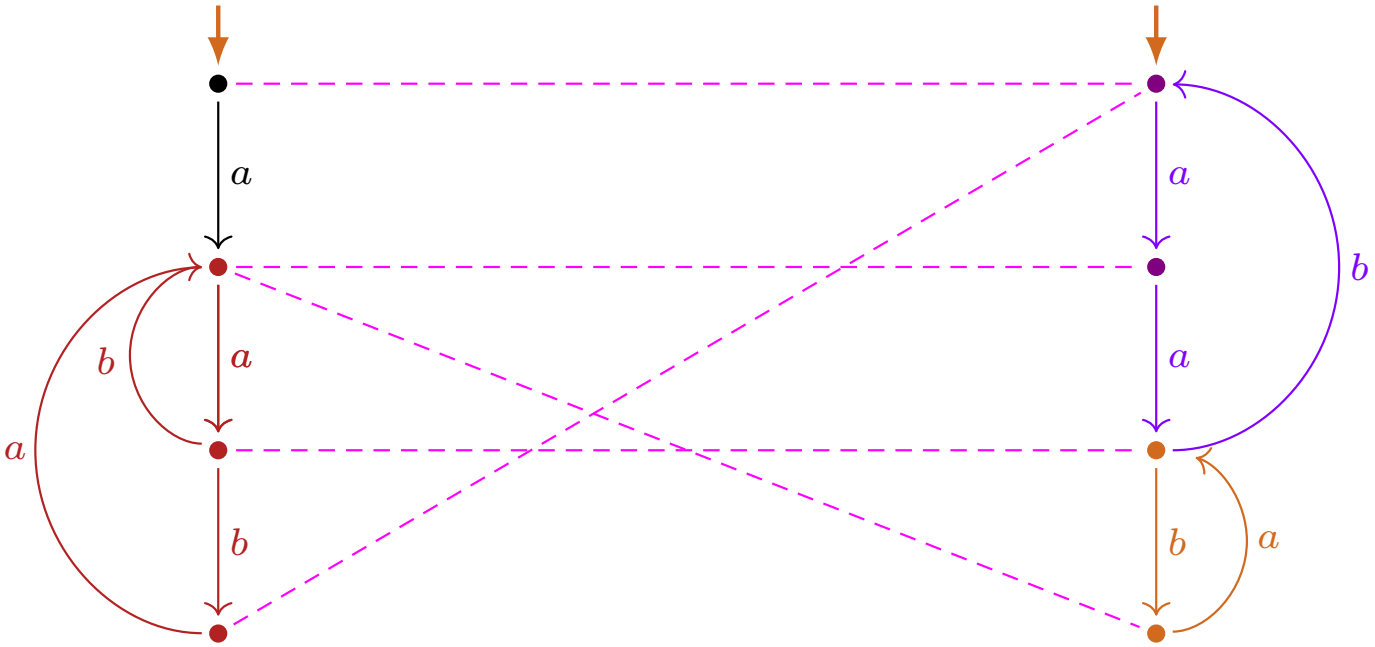


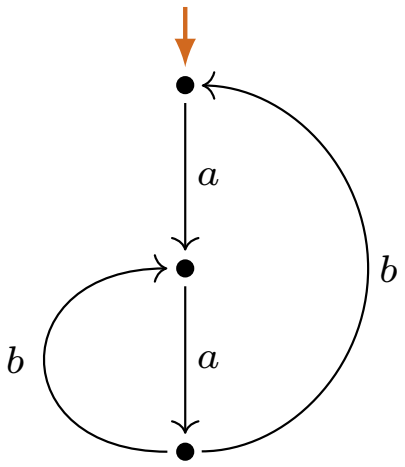
$$P(a(a(b+ba))^*0) \quad \underline{\leftrightarrow} \quad P((aa(ba)^*b)^*0)$$

Process interpretation of regular expressions



$$a(a(b+ba))^*0 \quad \underline{\leftrightarrow}_P \quad (aa(ba)^*b)^*0$$

Expressible process graphs (under bisimulation $\underline{\leftrightarrow}$)



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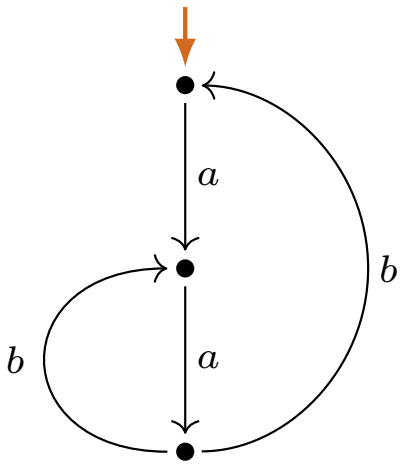
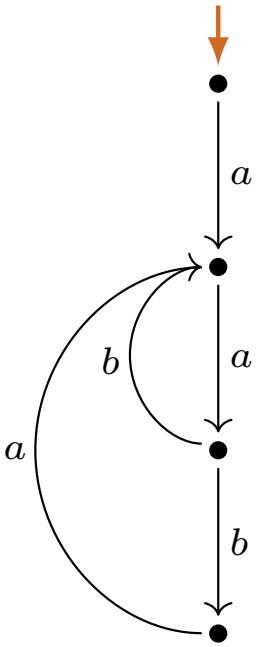
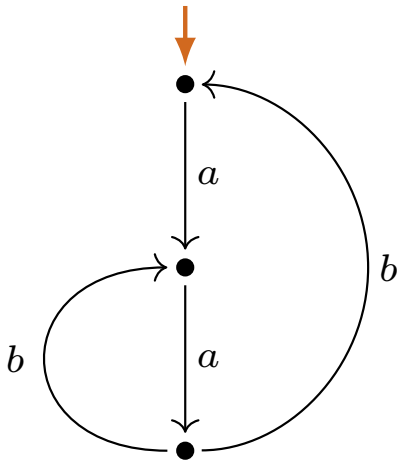


$? \in im(P(\cdot))\ ?$

Expressible process graphs (under bisimulation $\underline{\leftrightarrow}$)



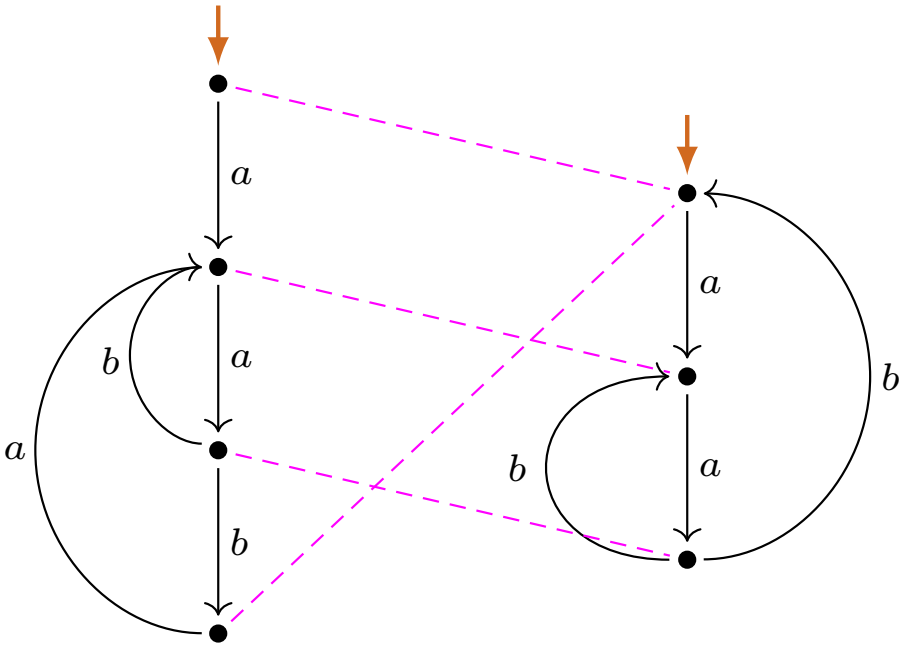


$\in im(P(\cdot))$

$P(\cdot)$-expressible

$? \in im(P(\cdot))\ ?$

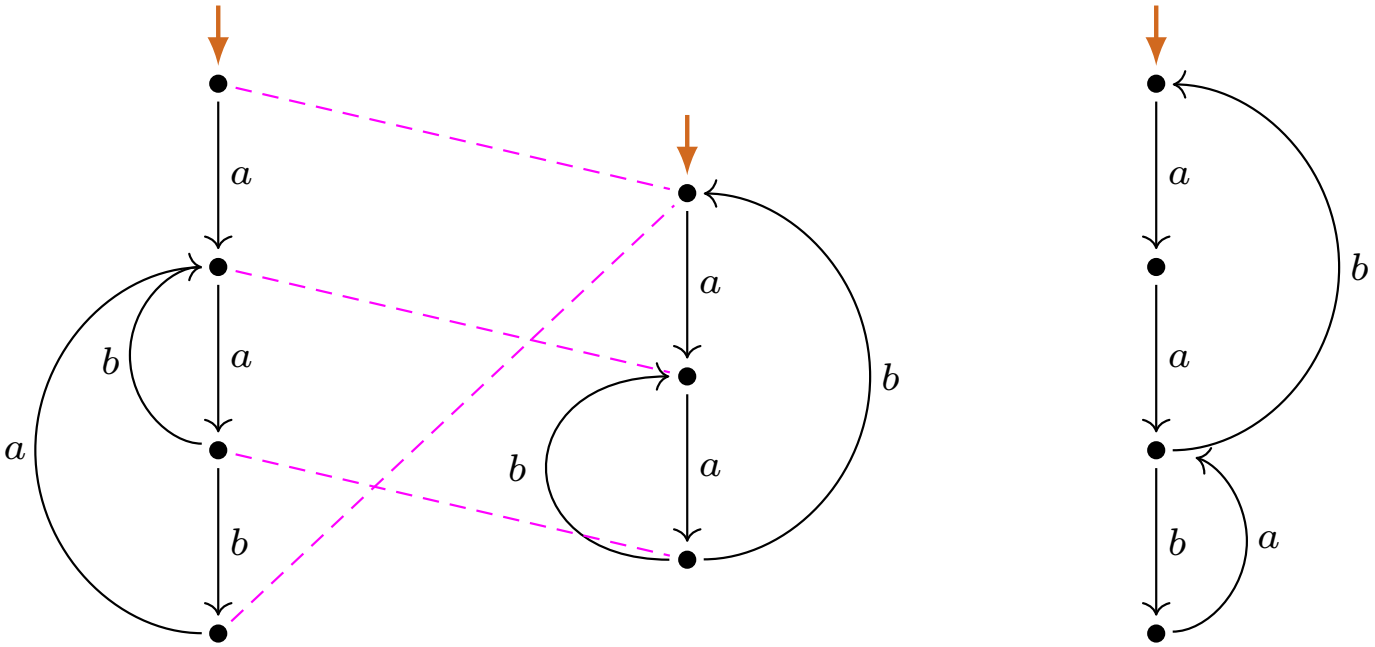
Expressible process graphs (under bisimulation $\underline{\leftrightarrow}$)



$\in im(P(\cdot))$ $\qquad$ $? \in im(P(\cdot))\ ?$

$P(\cdot)$-expressible

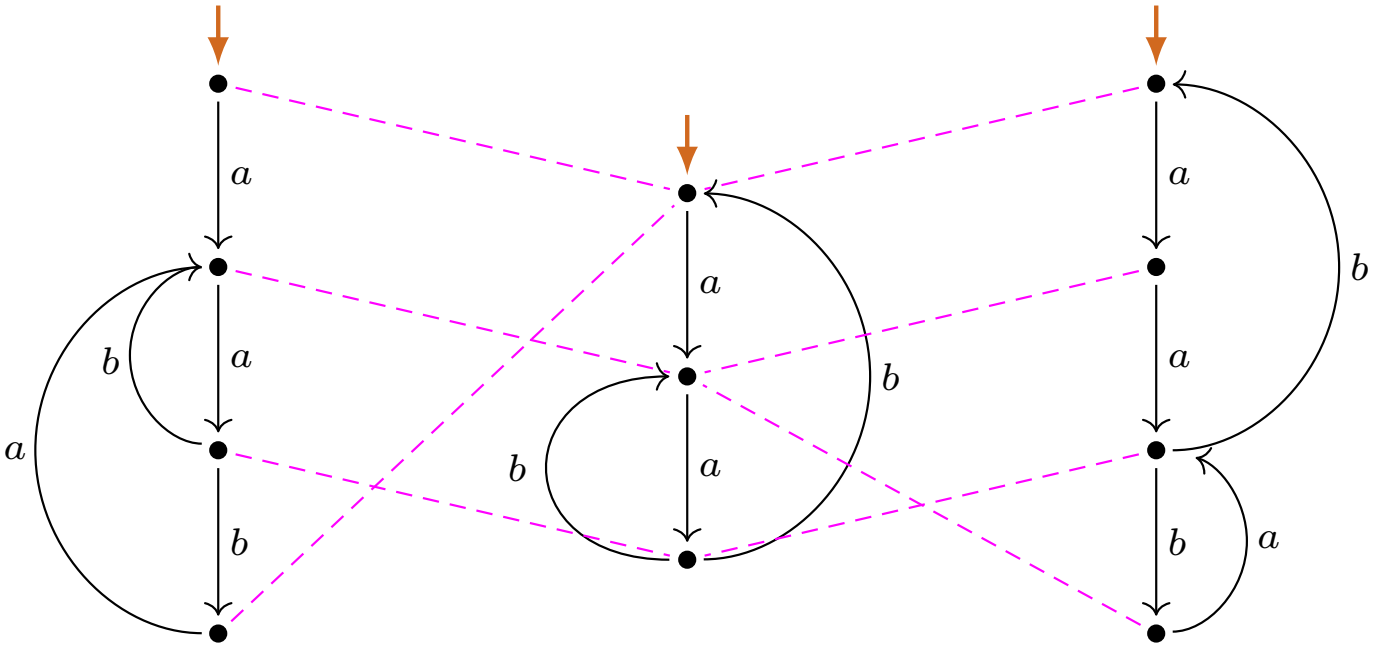
Expressible process graphs (under bisimulation $\leftrightarrow$)



$\in im(P(\cdot))$ $\quad$? $\in im(P(\cdot))$? $\quad$ $\in im(P(\cdot))$

$P(\cdot)$-expressible $\quad$ $P(\cdot)$-expressible

Expressible process graphs (under bisimulation $\underline{\leftrightarrow}$)



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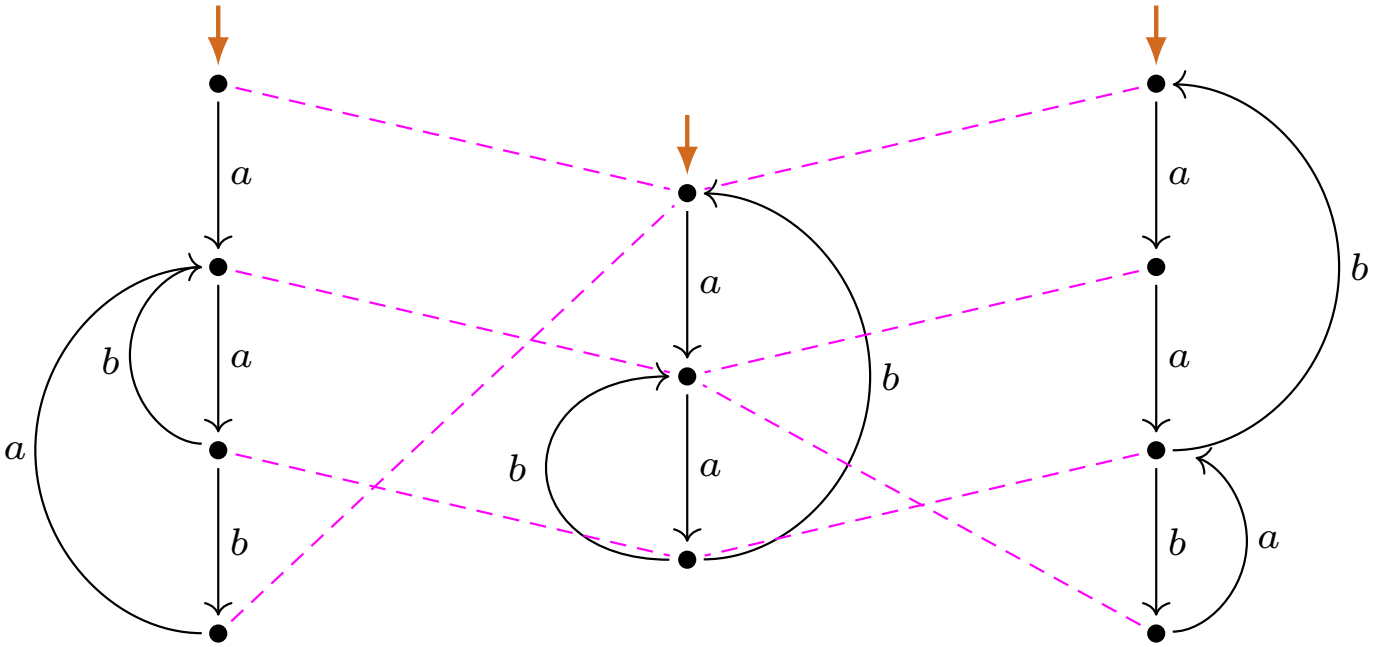
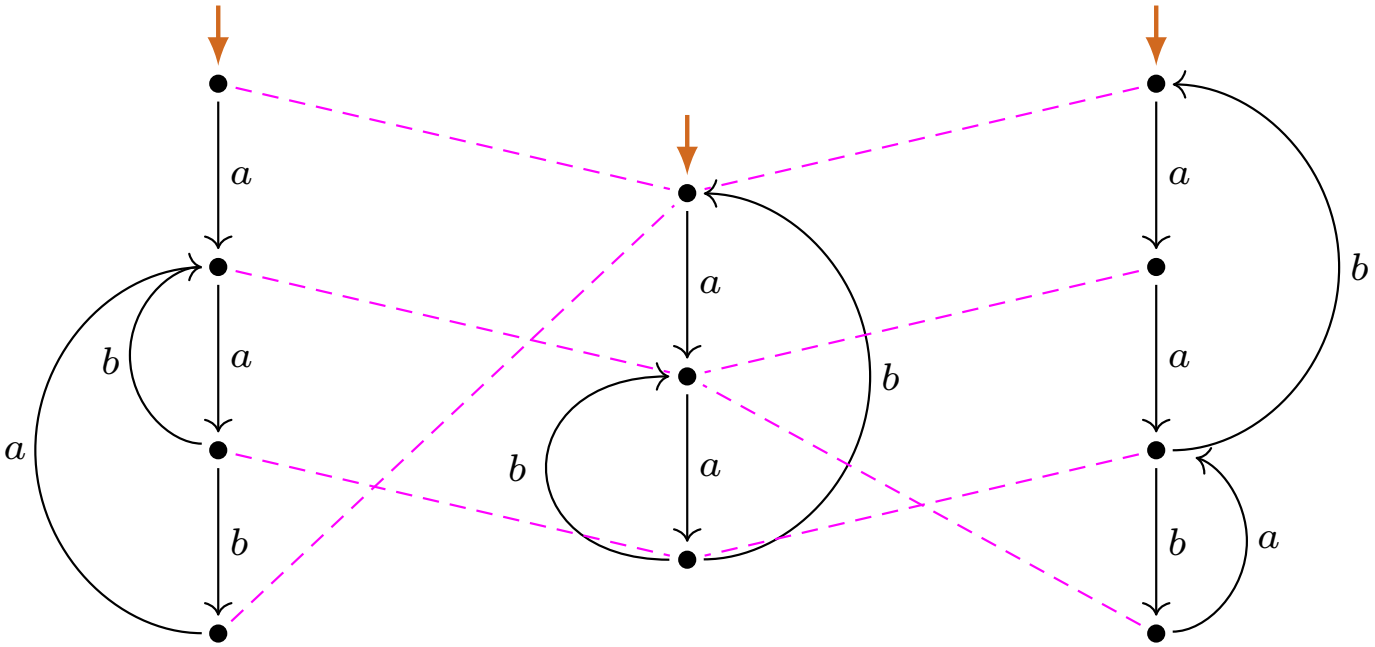
$P(\cdot)$-expressible $\quad$ $P(\cdot)$-expressible

Expressible process graphs (under bisimulation $\underline{\leftrightarrow}$)



$\in im(P(\cdot))$	$? \in im(P(\cdot))\ ?$	$\in im(P(\cdot))$
$P(\cdot)$-expressible	$P(\cdot)$-expressible modulo $\underline{\leftrightarrow}$	$P(\cdot)$-expressible

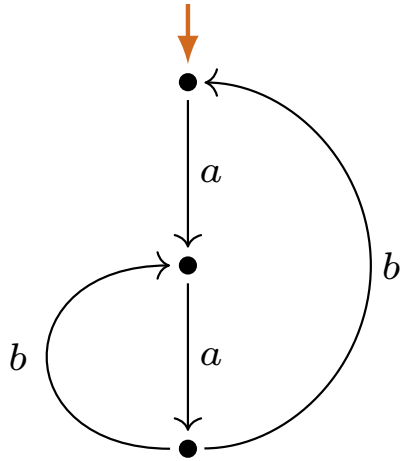
Expressible process graphs (under bisimulation $\underline{\leftrightarrow}$)



$\in im(P(\cdot))$	$? \in im(P(\cdot))\ ?$	$\in im(P(\cdot))$
$P(\cdot)$-expressible	$P(\cdot)$-expressible modulo $\underline{\leftrightarrow}$	$P(\cdot)$-expressible
$[\![\cdot]\!]_P$-expressible	$[\![\cdot]\!]_P$-expressible	$[\![\cdot]\!]_P$-expressible

Properties of P and $[\![\cdot]\!]_P$

- Not every finite-state process is $P(\cdot)$-expressible.



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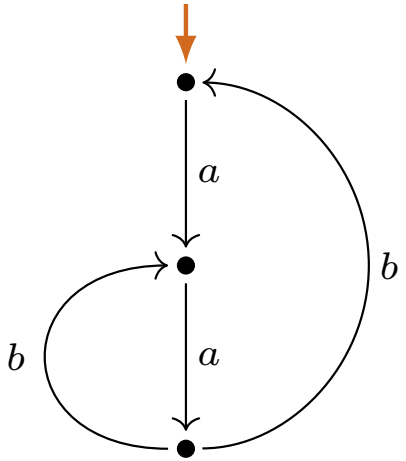

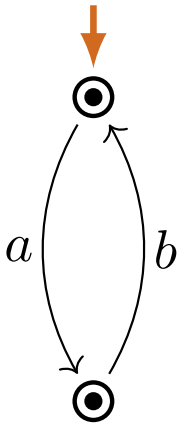

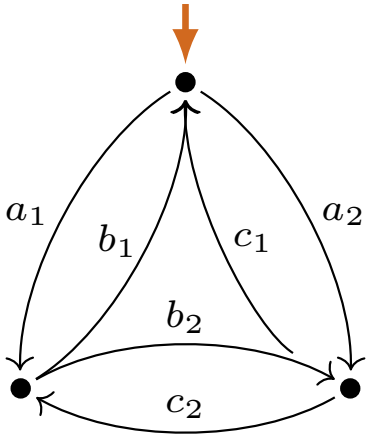

? $P(\cdot)$-expressible ?

$[\![\cdot]\!]_P$-expressible

not $P(\cdot)$-expressible

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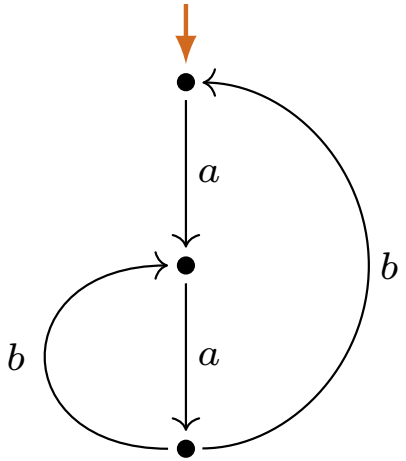

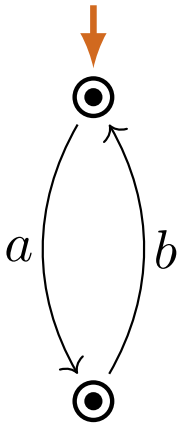

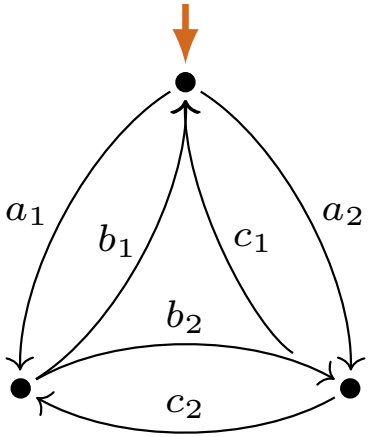

? $P(\cdot)$-expressible ?

$[\![\cdot]\!]_P$-expressible

not $P(\cdot)$-expressible

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Properties of P and $[\![\cdot]\!]_P$

- Not every finite-state process is $P(\cdot)$-expressible.
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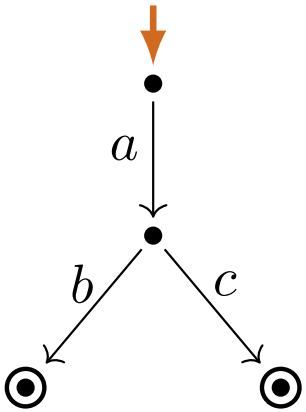

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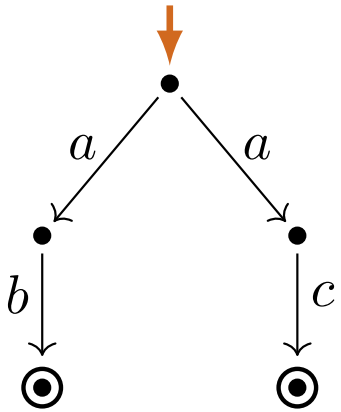

$a \cdot (b + c)$ $=_L$ $a \cdot b + a \cdot c$

Properties of P and $[\![\cdot]\!]_P$

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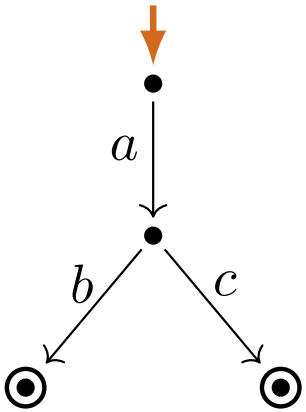

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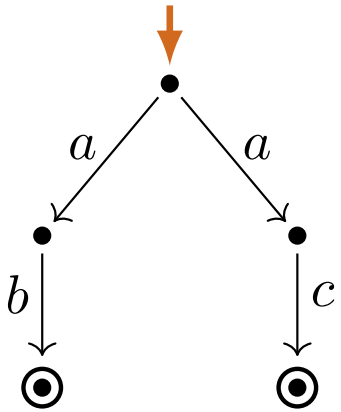

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Complete axiomatization of $=_L$ *(Aanderaa/Salomaa, 1965/66)*

Axioms:

(B1)	$e + (f + g) = (e + f) + g$	(B7)	$e \cdot 1 = e$
(B2)	$(e \cdot f) \cdot g = e \cdot (f \cdot g)$	(B8)	$e \cdot 0 = 0$
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Inference rules: equational logic *plus*

$$\frac{e = f \cdot e + g}{e = f^* \cdot g}\ \text{FIX} \quad (\text{if } \underbrace{\{\epsilon\} \notin L(f)}_{\textit{non-empty-word property}})$$

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Adaptation for $\underline{\leftrightarrow}_P$ *(Milner, 1984)* (Mil = Mil$^-$ + RSP*)

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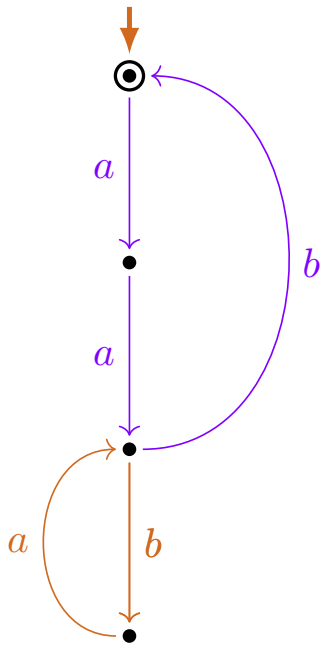
- Mil is complete *(G, 2022, proof overview)*

Well-behaved form, looping palm trees



$P((aa(ba)^*b)^*)$

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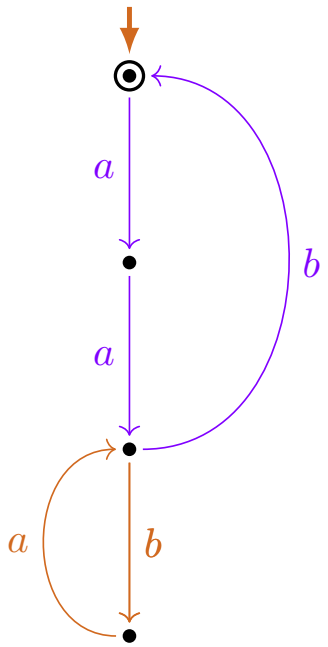

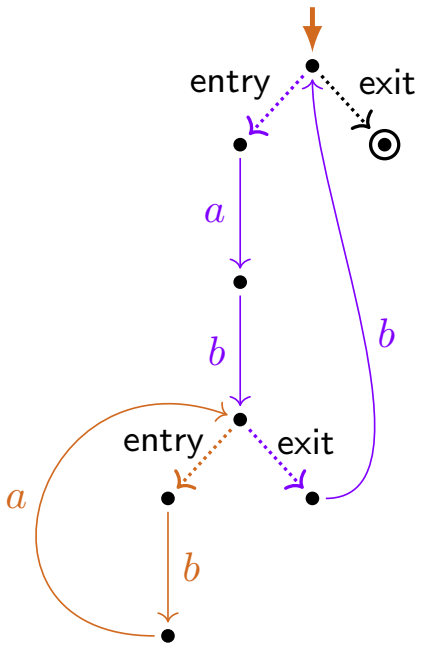

well-behaved form

(Corradini, Baeten)

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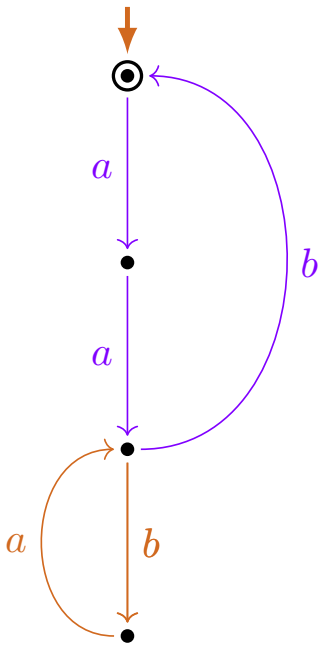

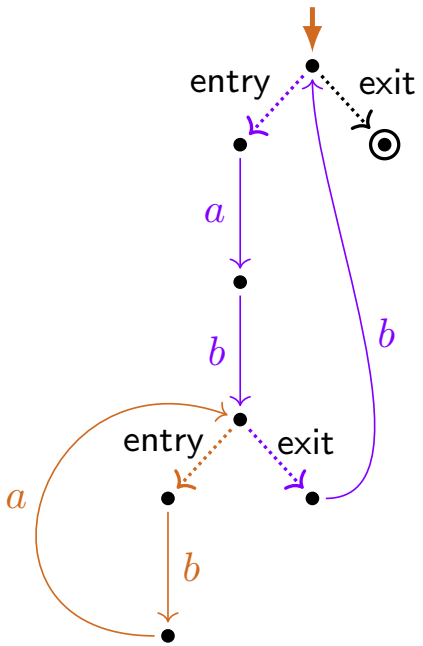

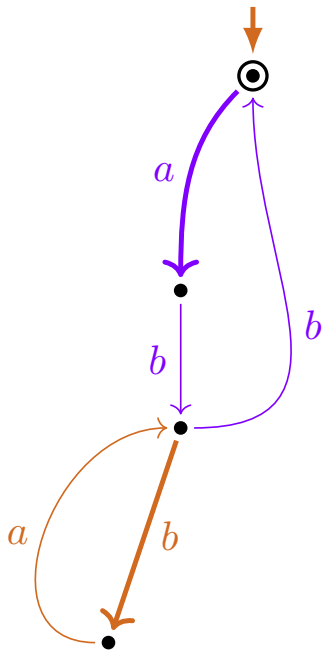

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Loop charts (interpretations of innermost iterations)

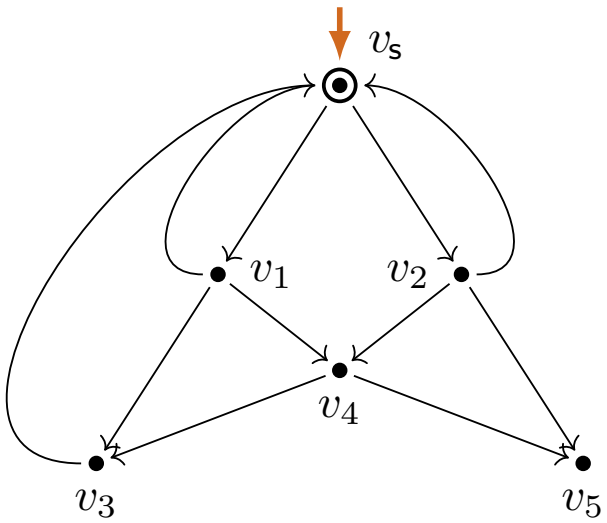
Definition

A process graph is a loop chart if:

L-1.

L-2.

L-3.



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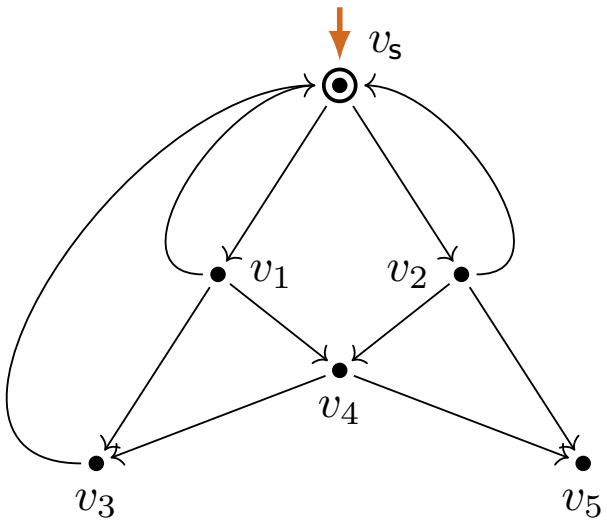

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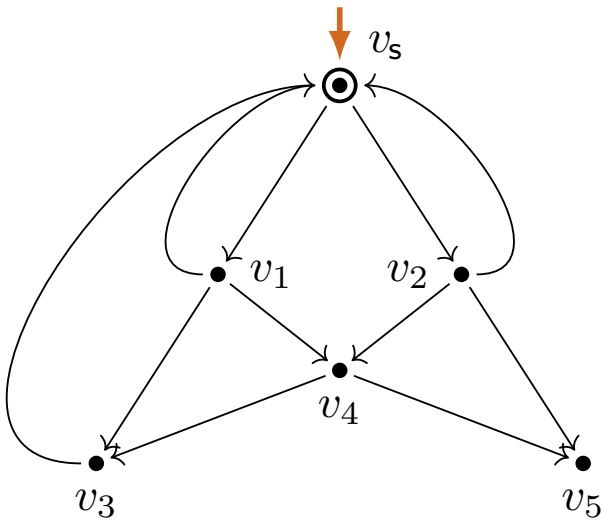

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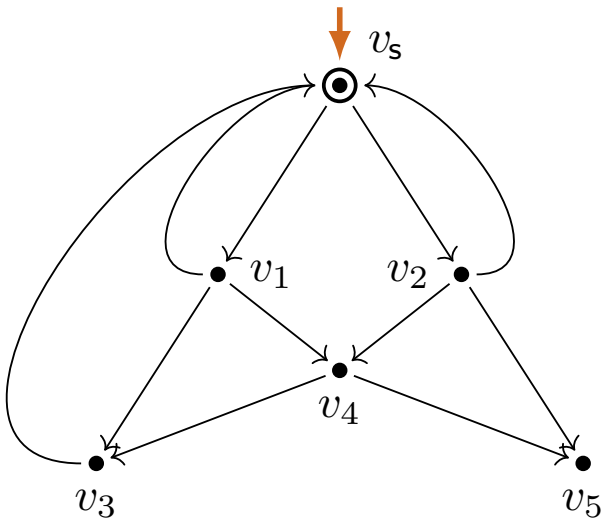

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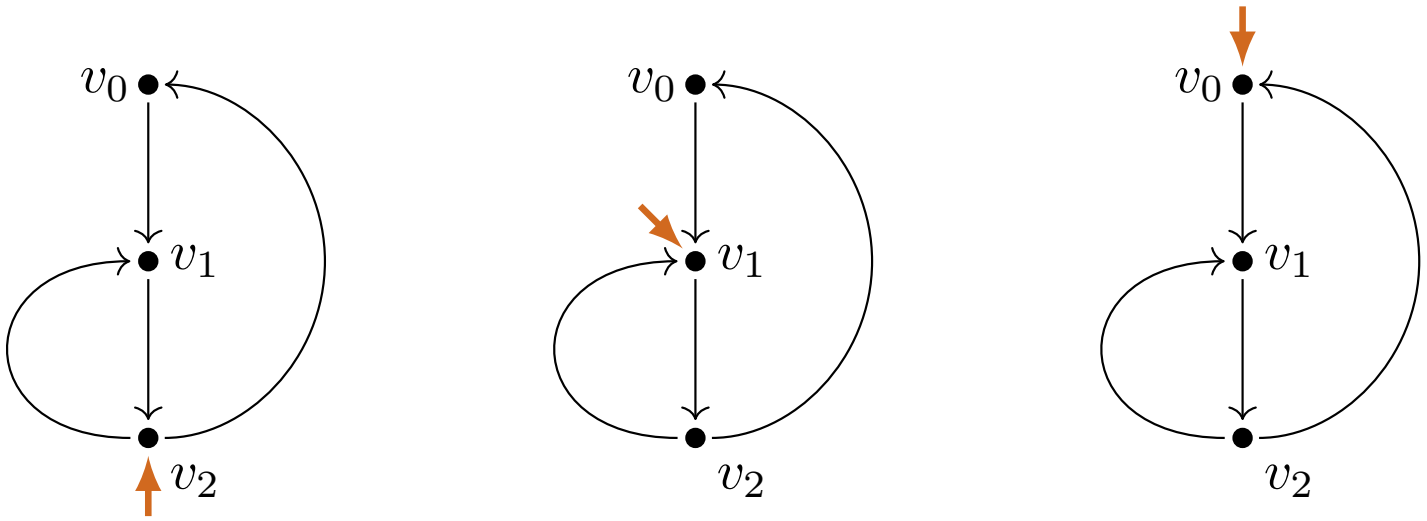

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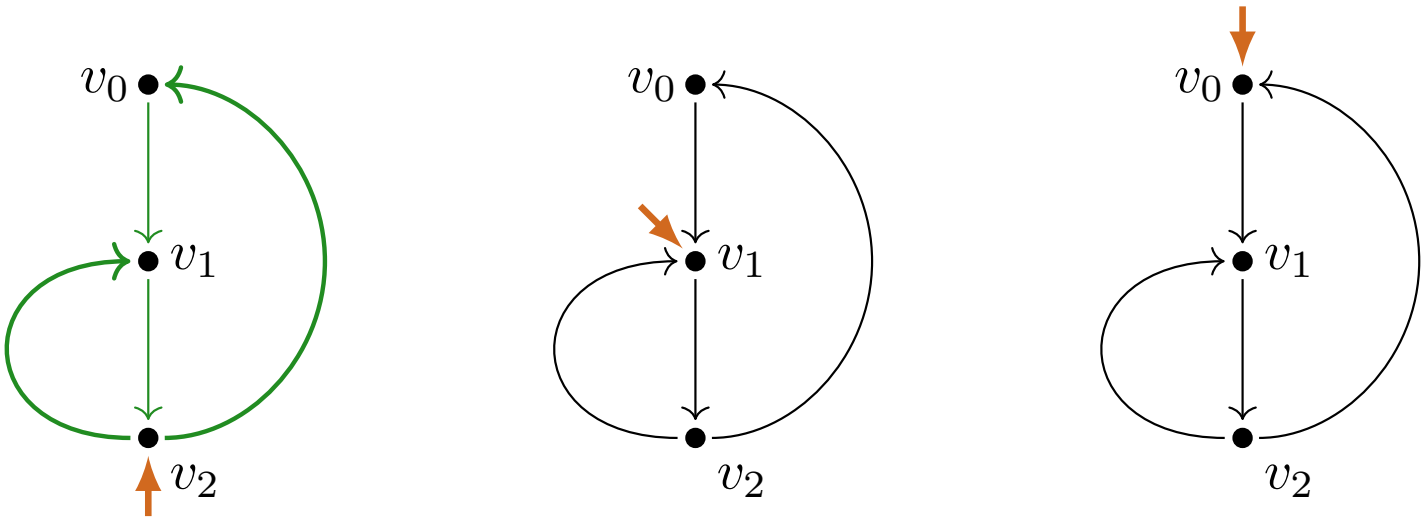

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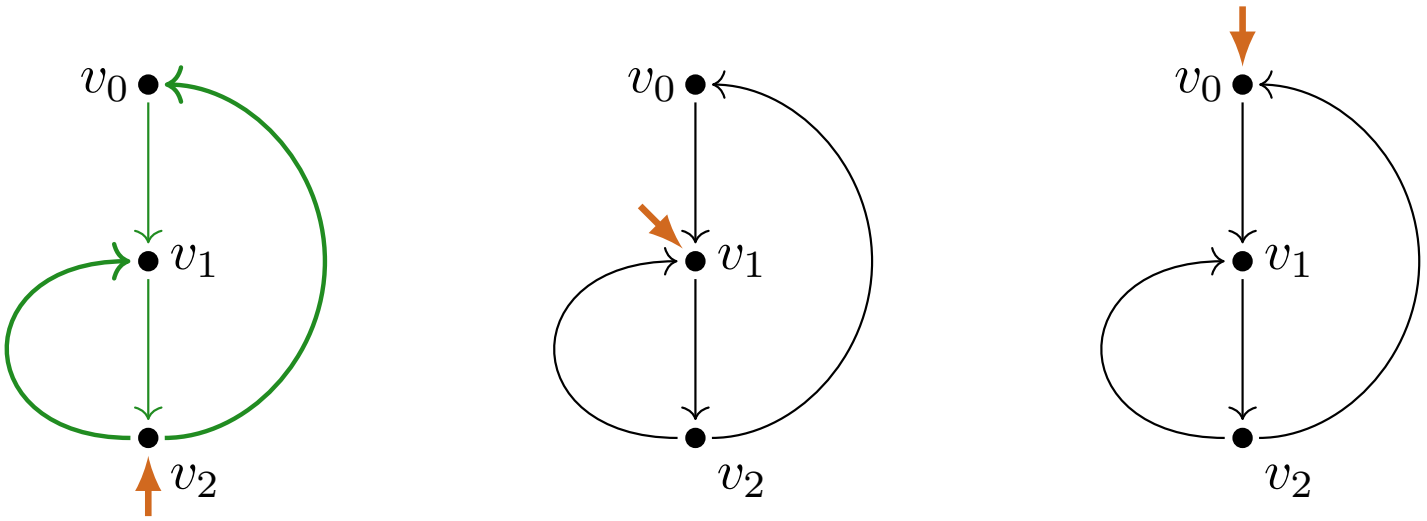

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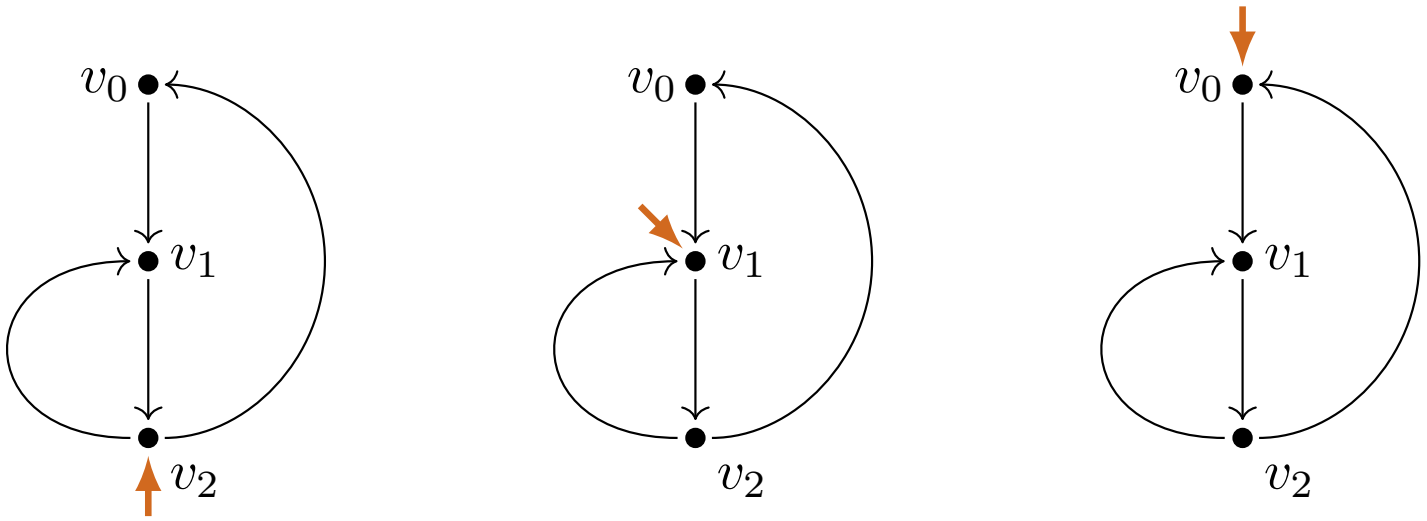

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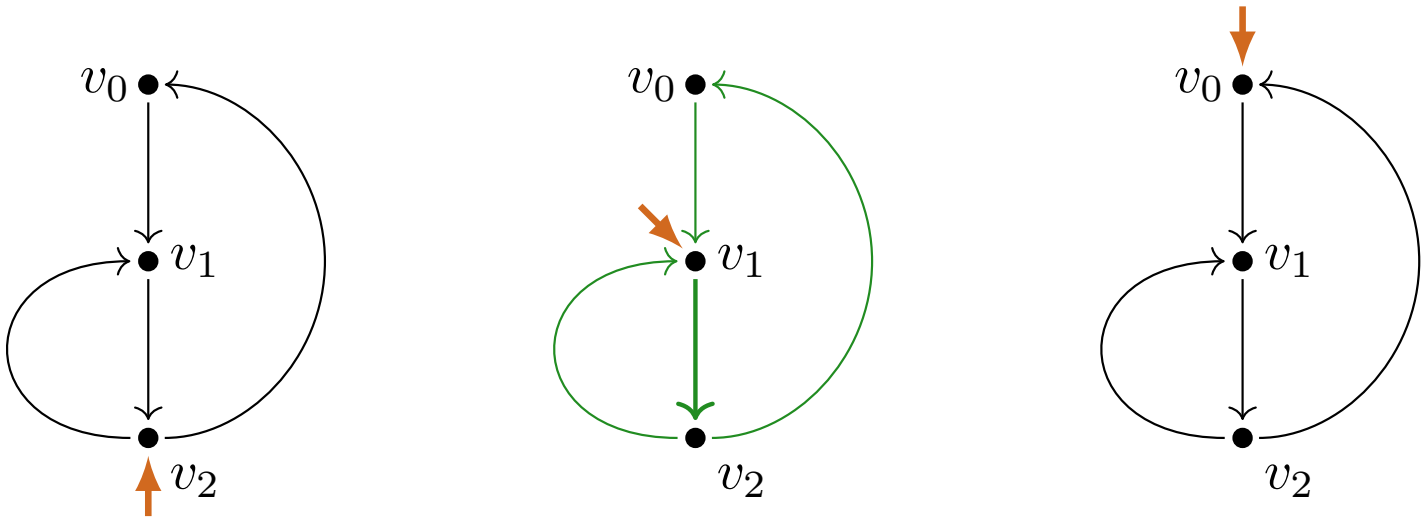

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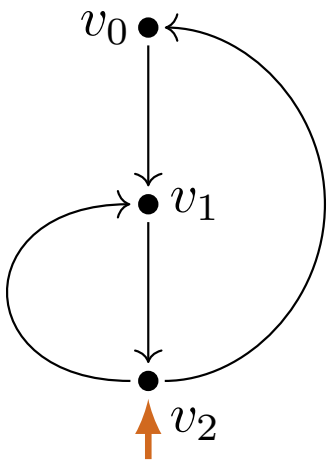

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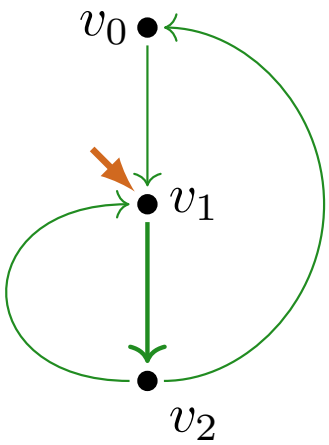

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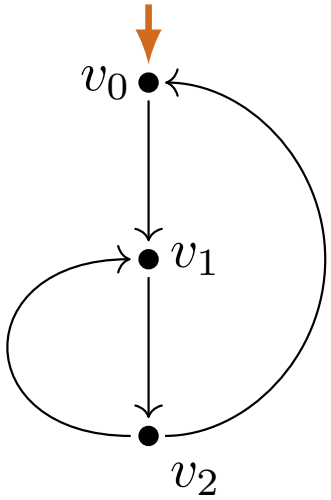

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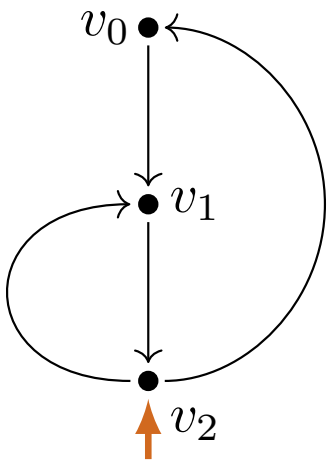

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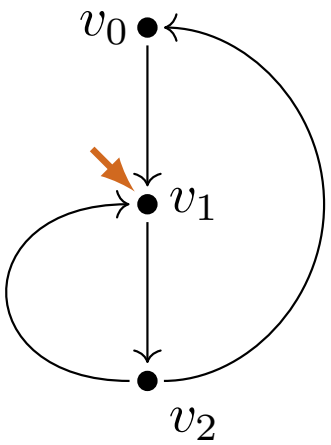

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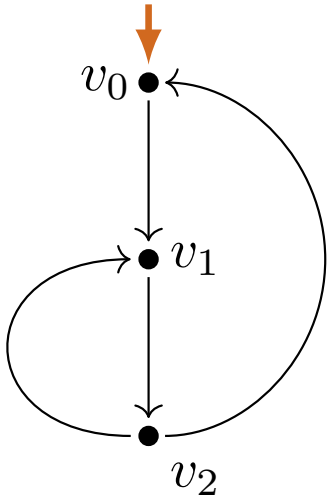

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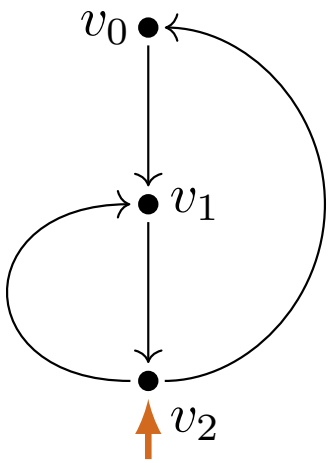

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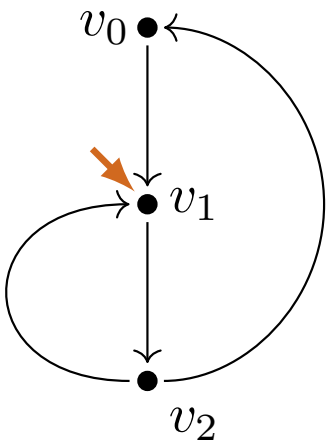

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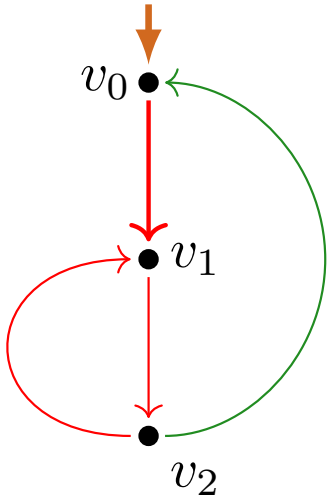

Loop charts (interpretations of innermost iterations)

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A process graph is a loop chart if:

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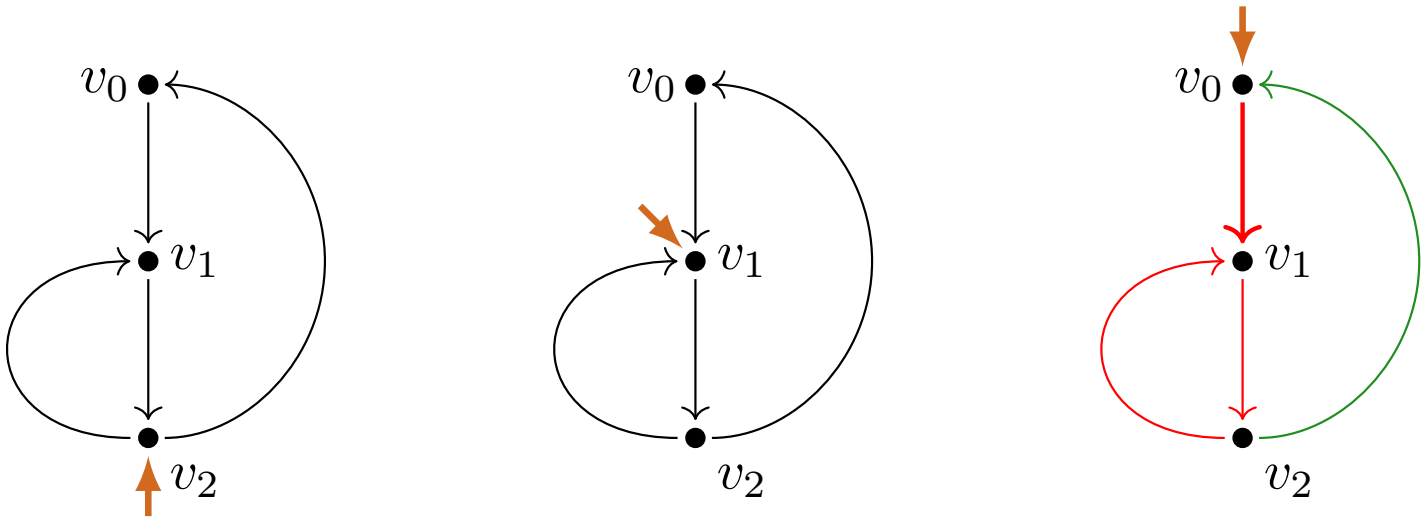

loop chart loop chart no loop chart

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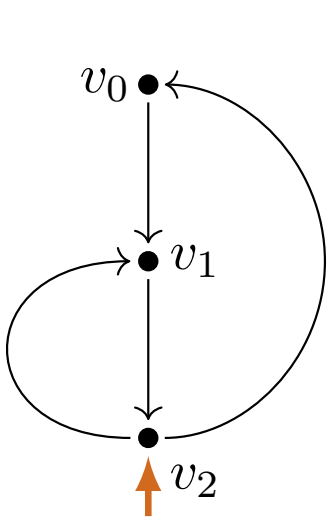

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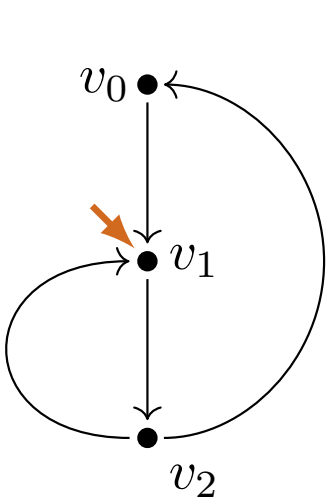

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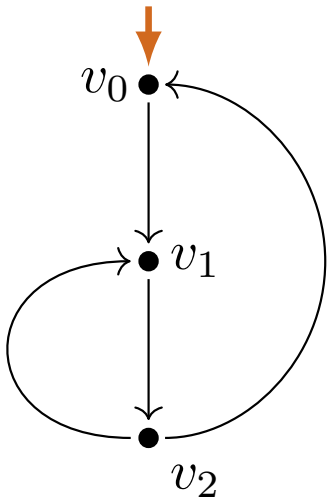
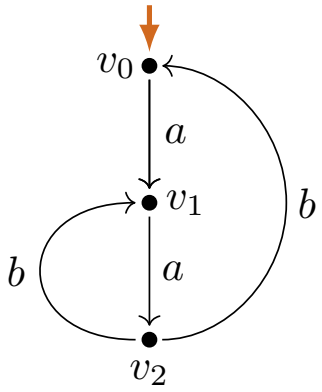


no loop chart

Loop elimination



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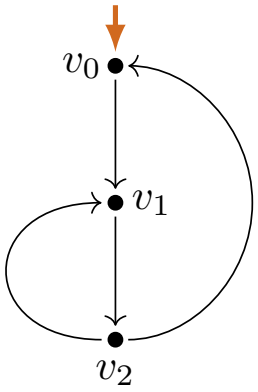

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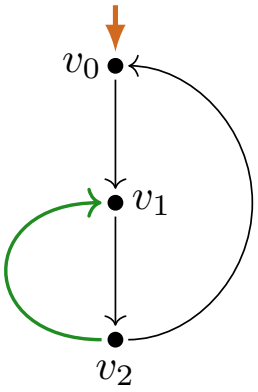

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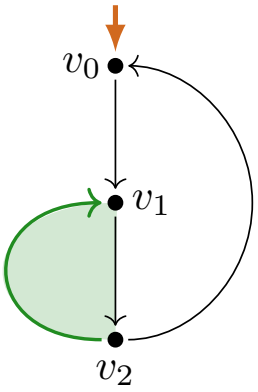

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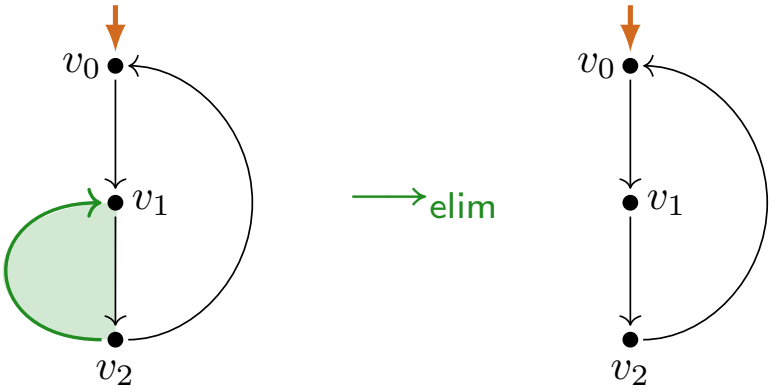

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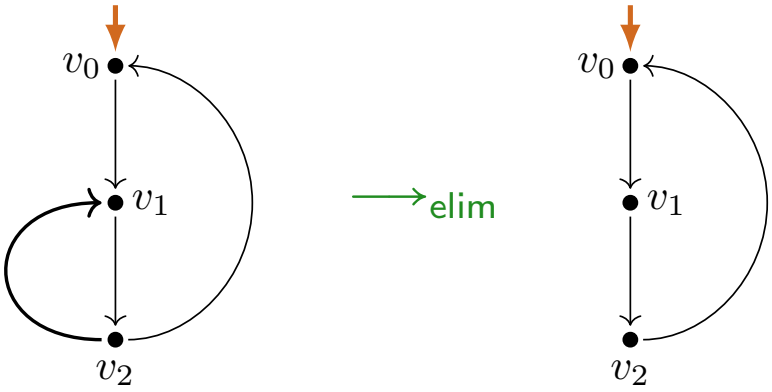

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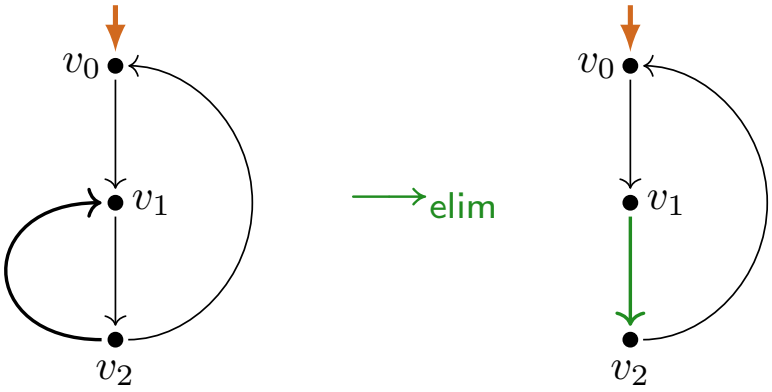

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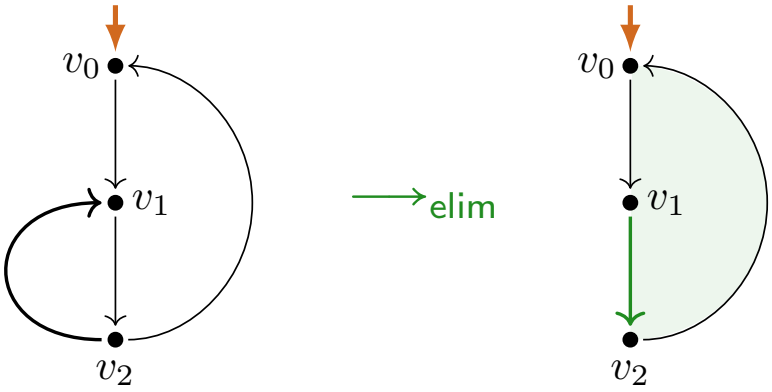

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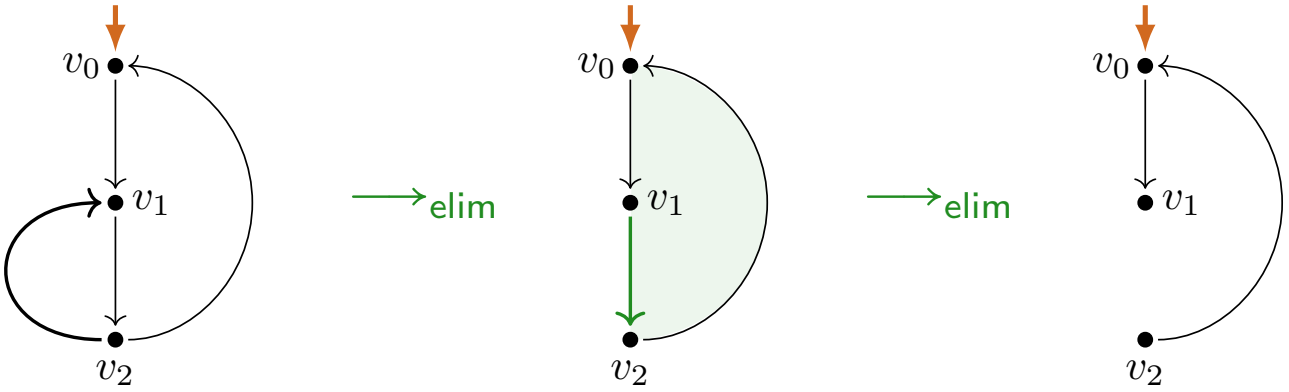

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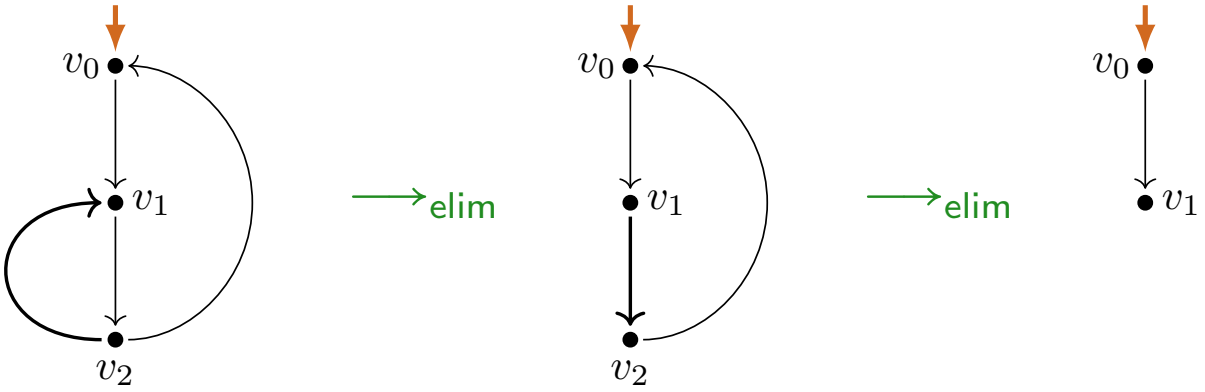

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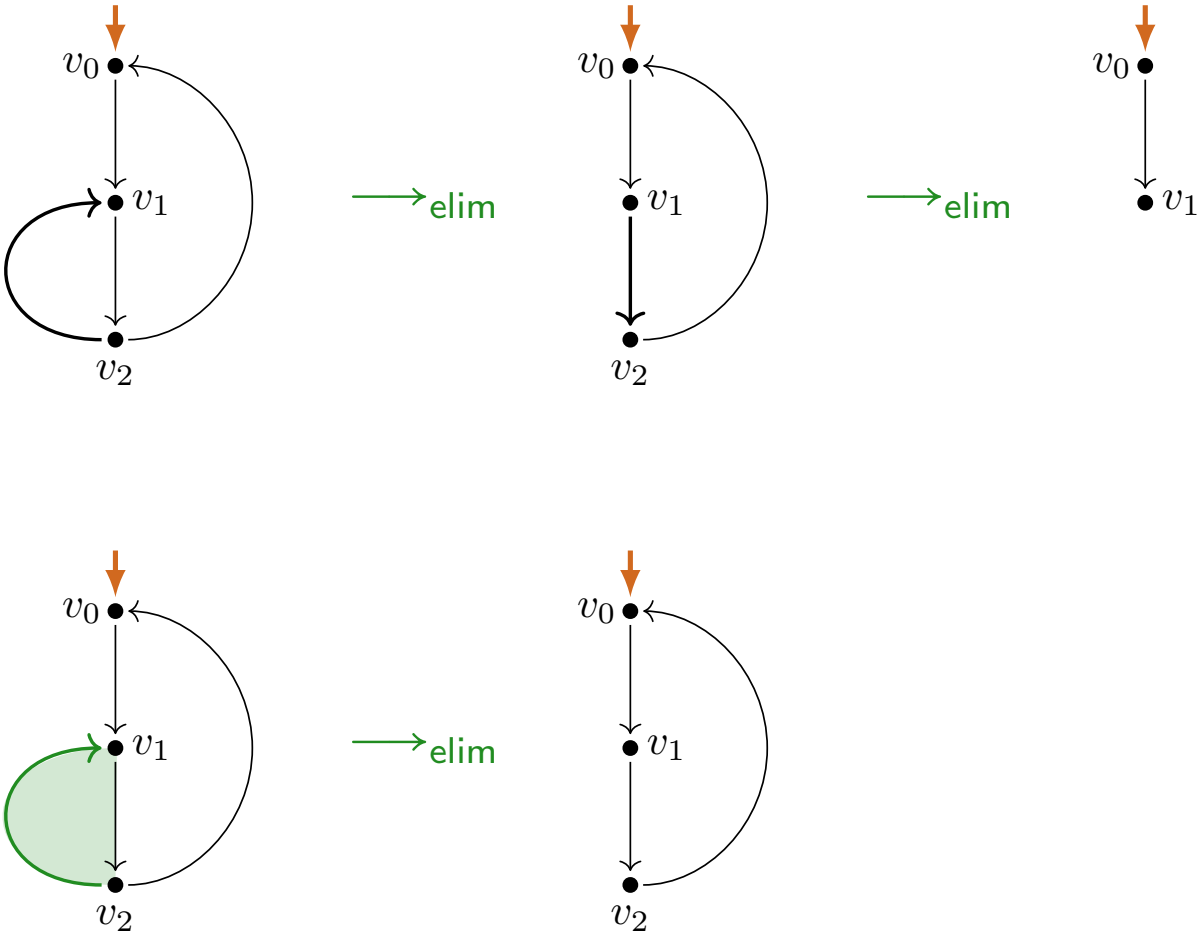

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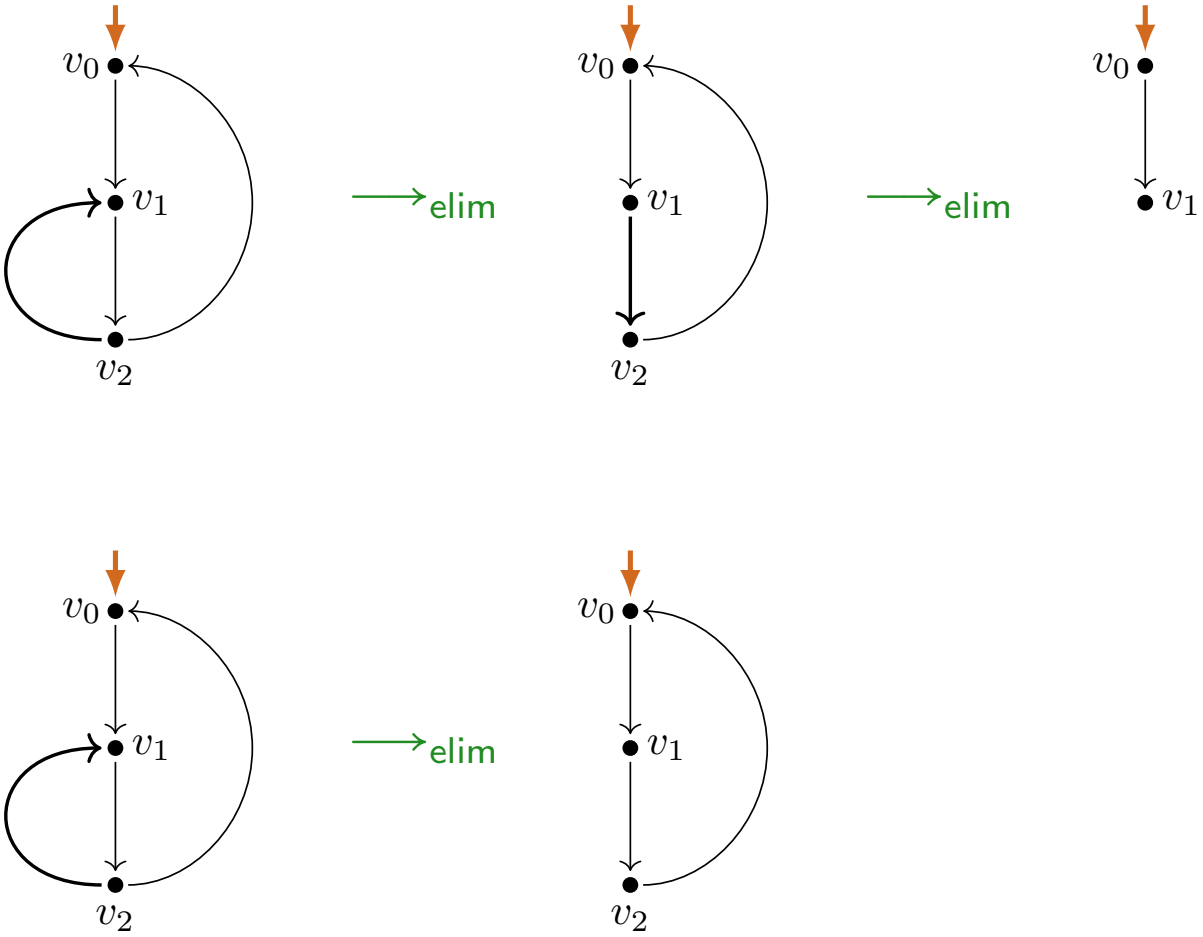

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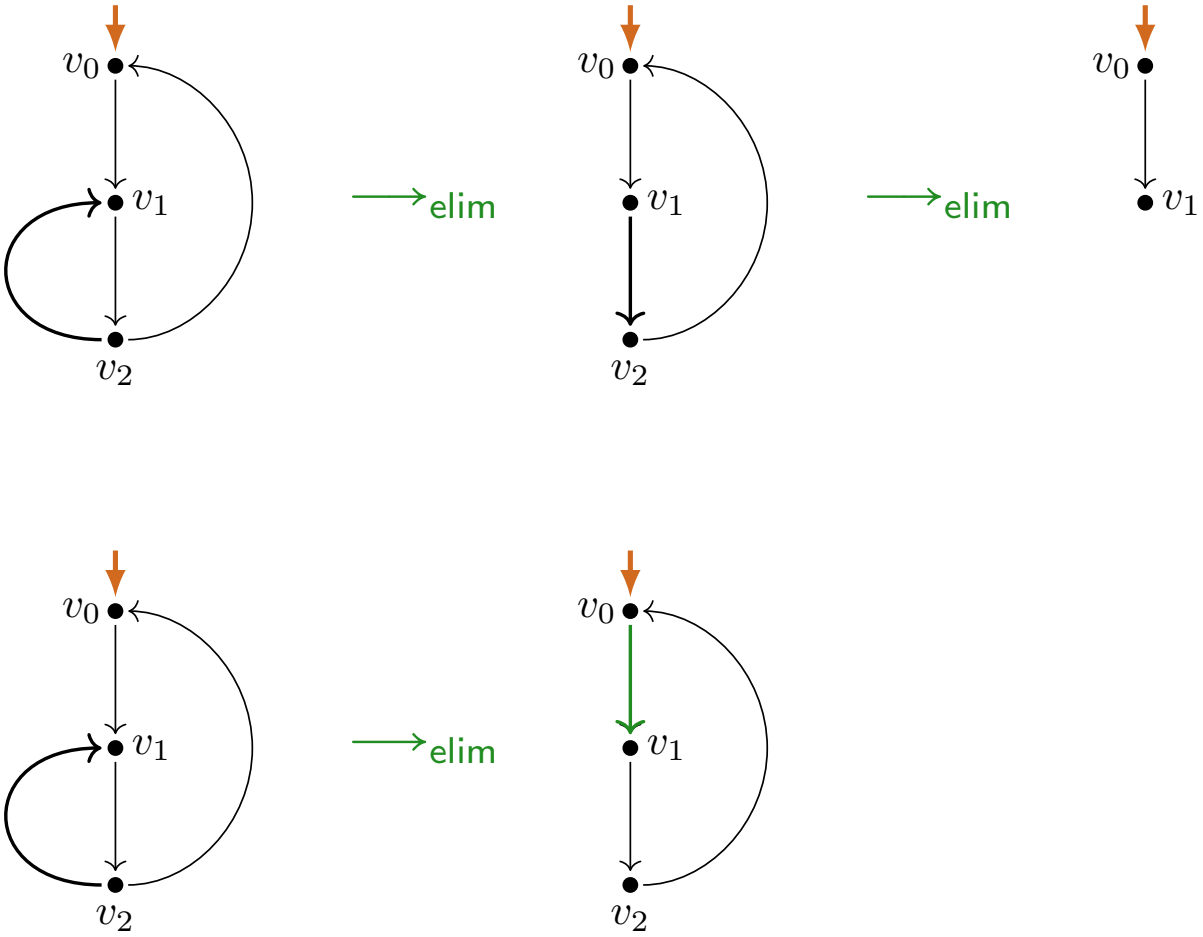

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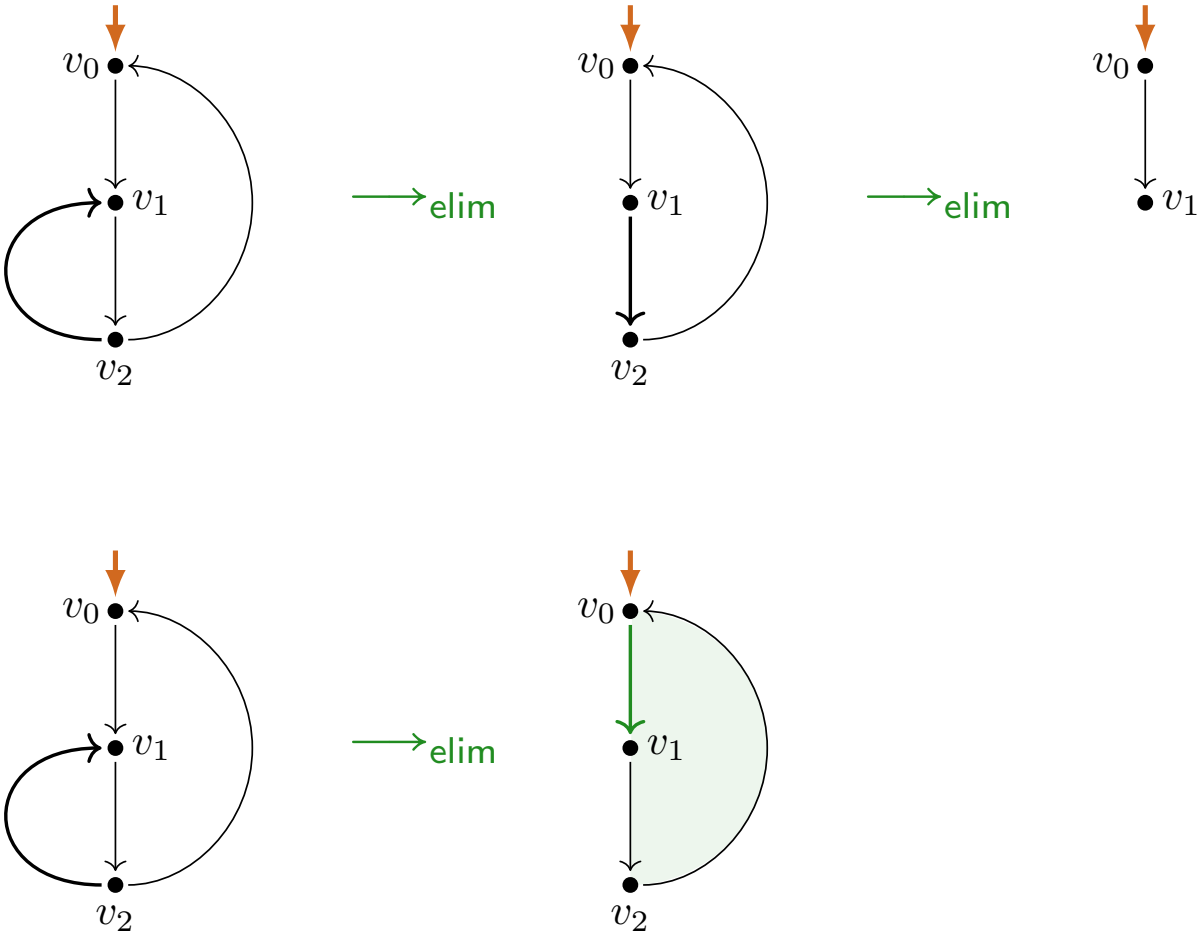

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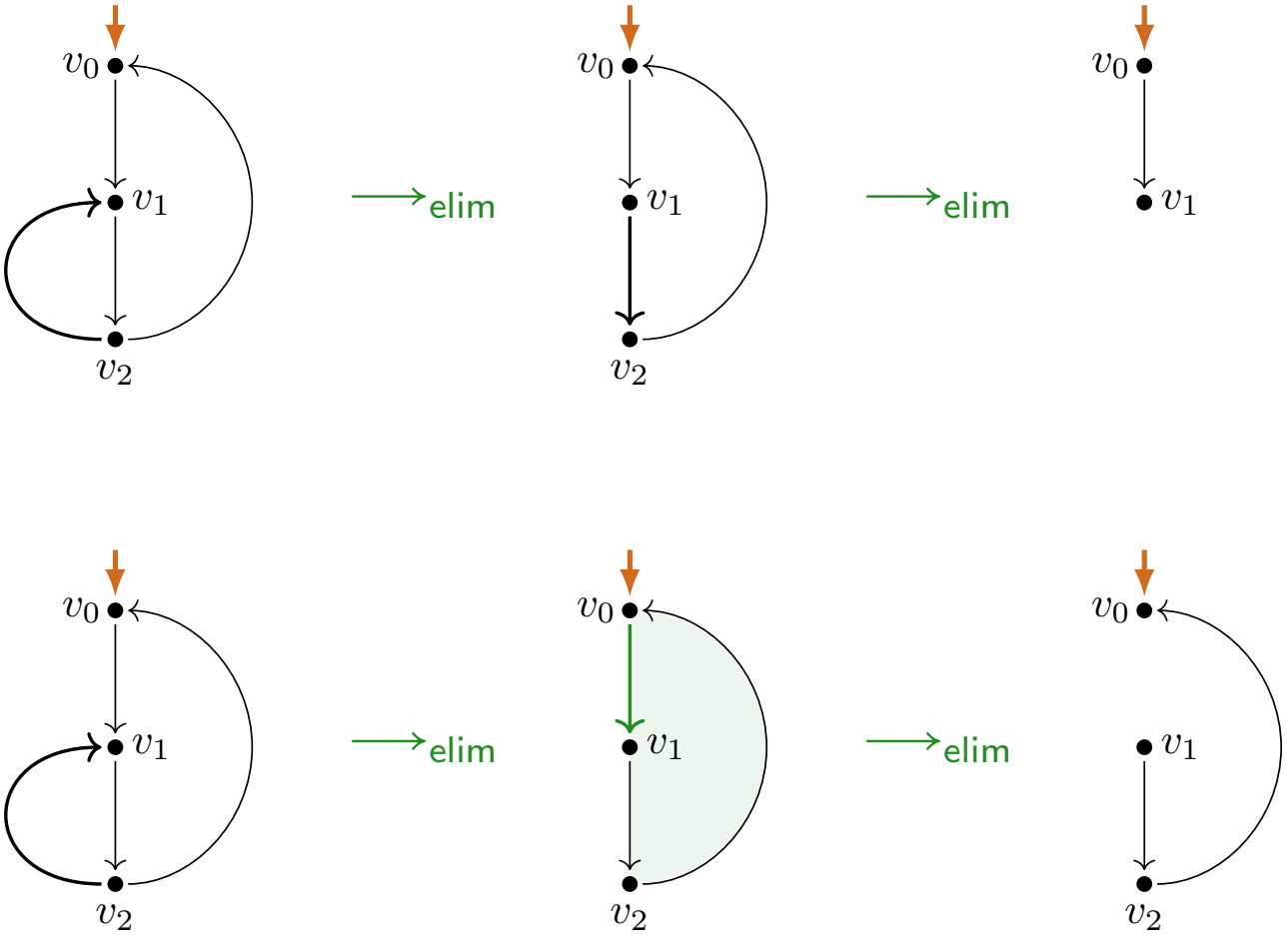

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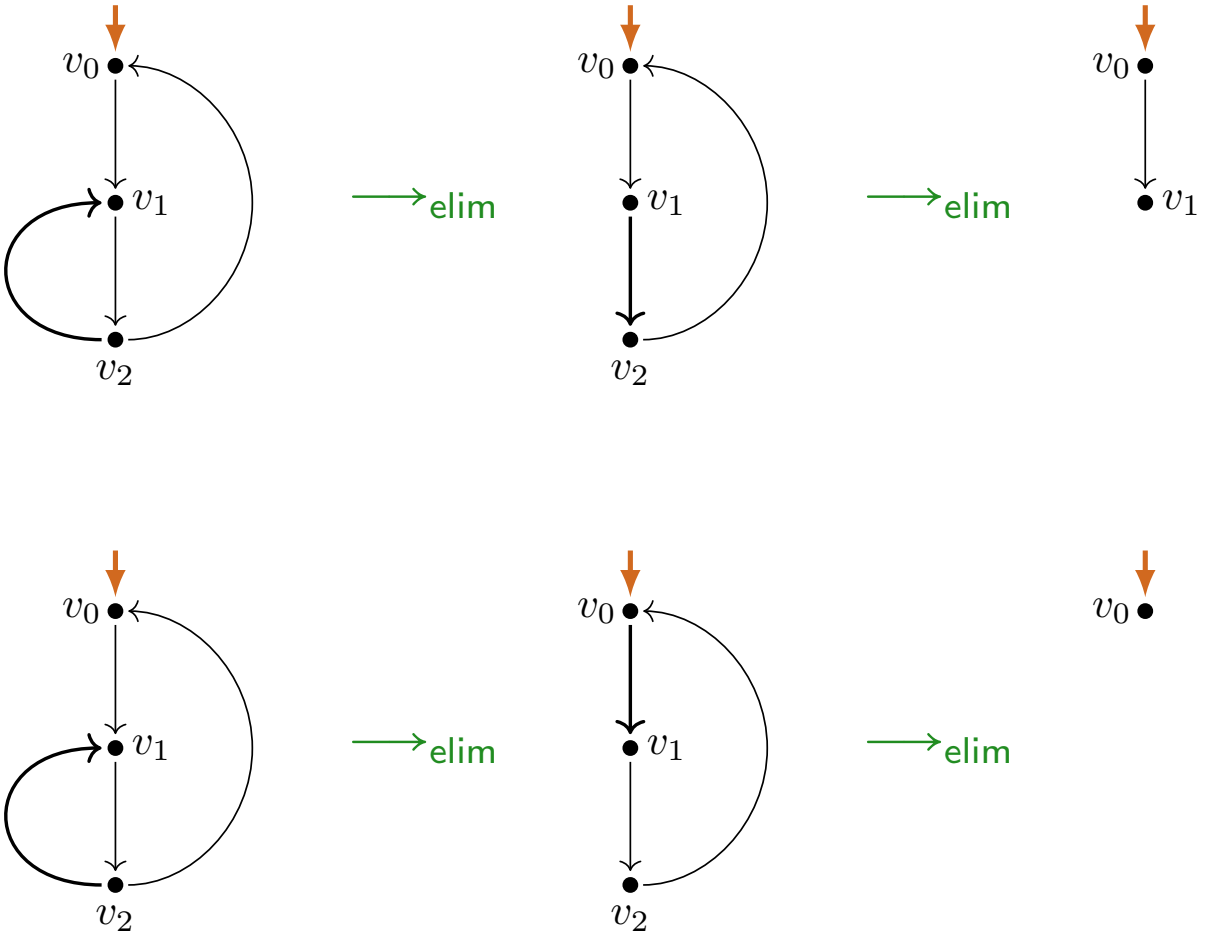

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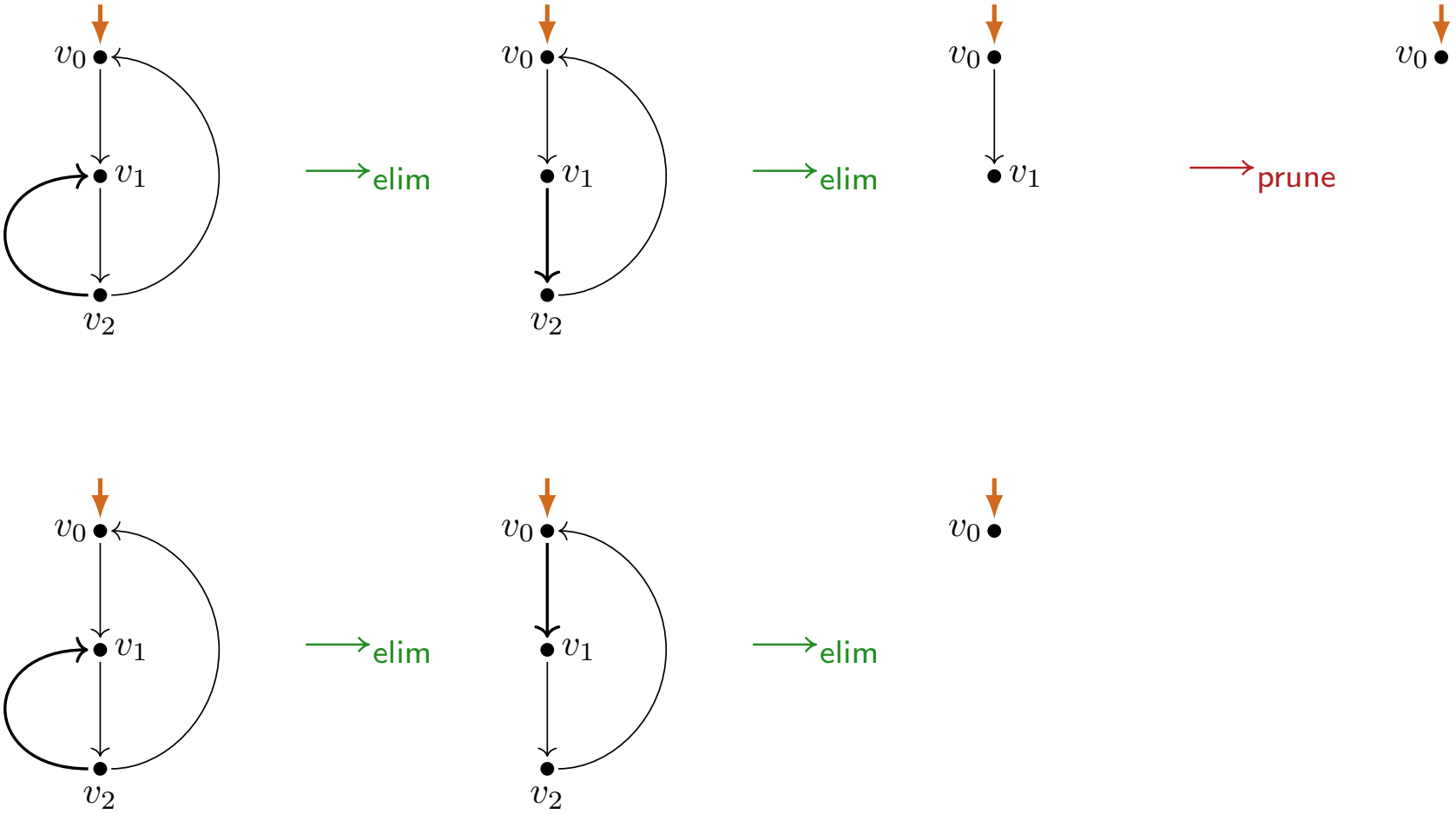

Loop elimination, and properties

$\longrightarrow_{\text{elim}}$: eliminate a transition-induced loop by:

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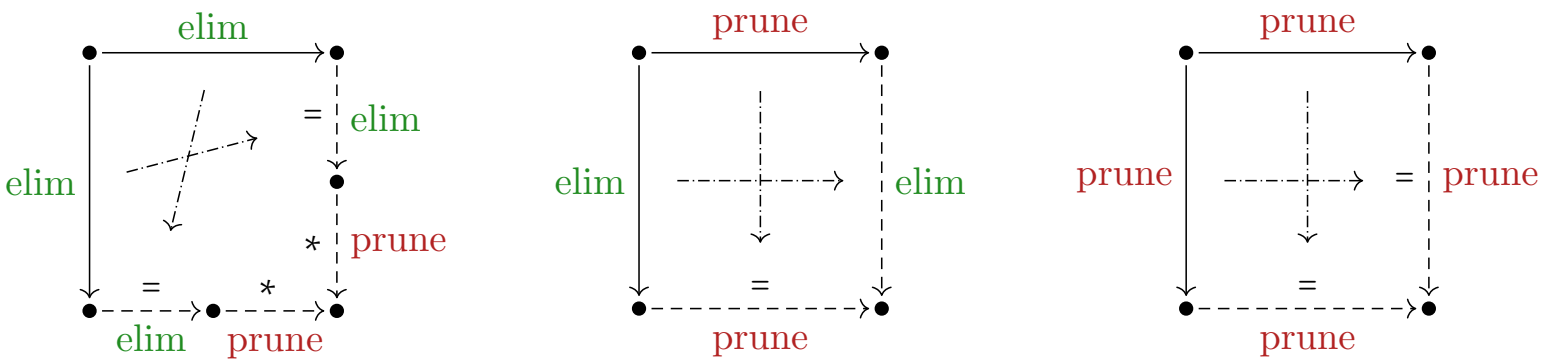

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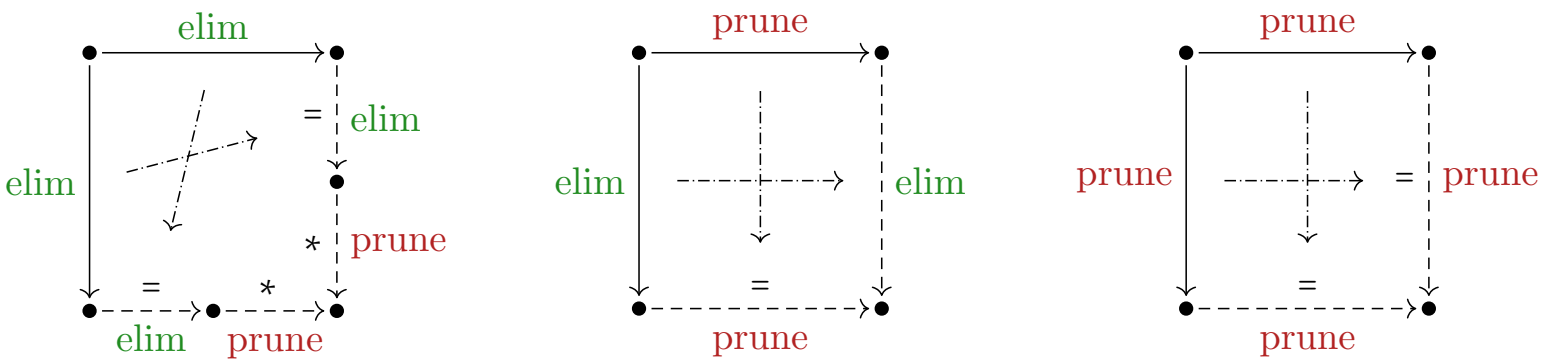

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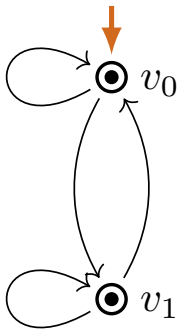

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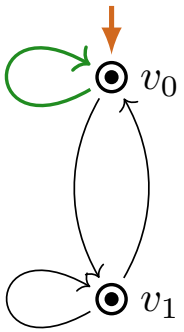

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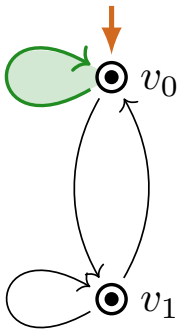

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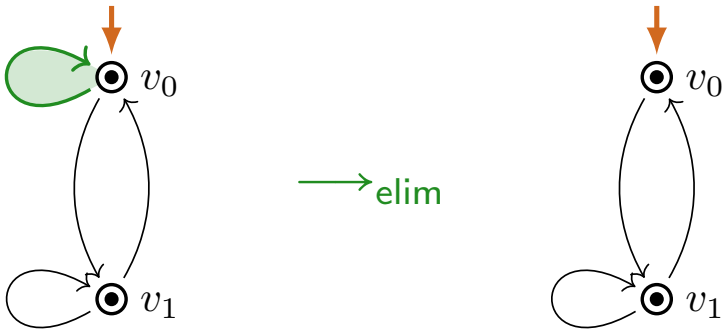

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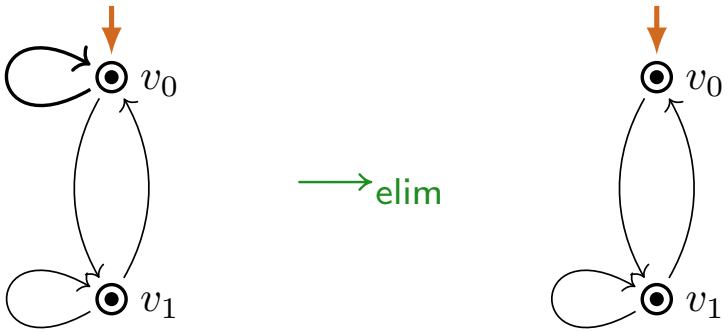

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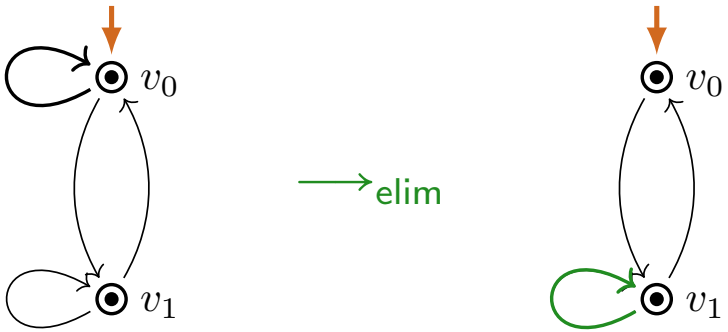

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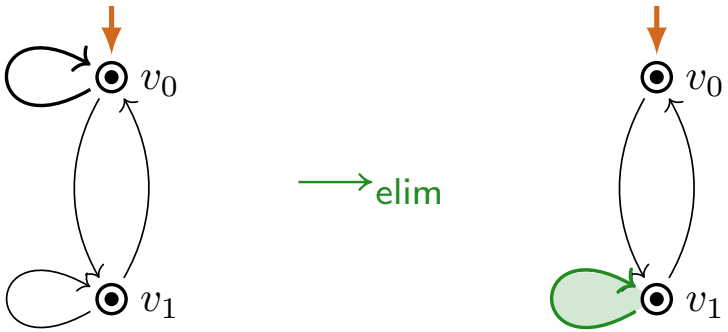

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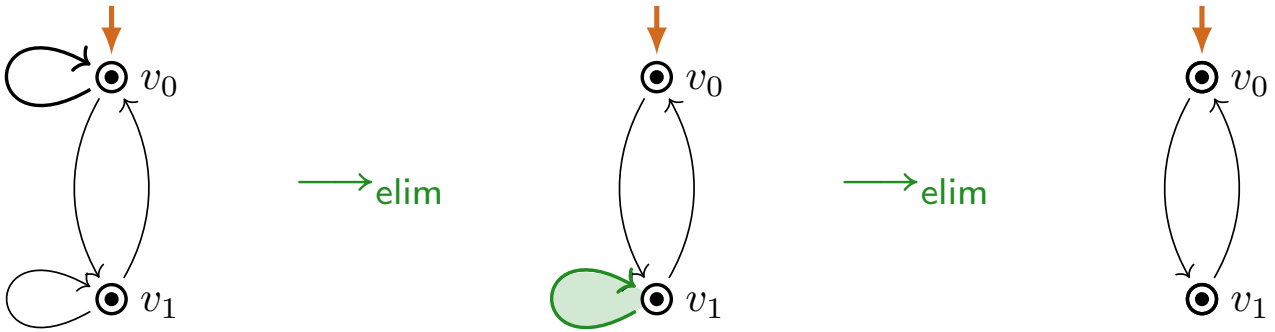

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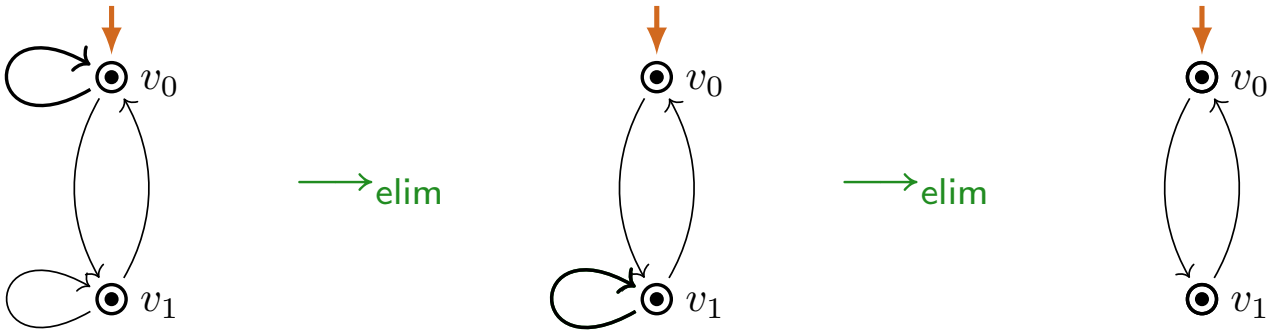

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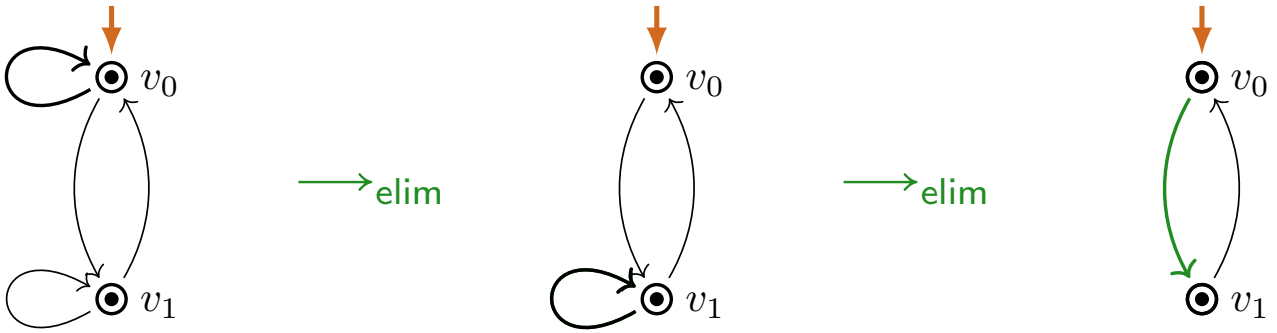

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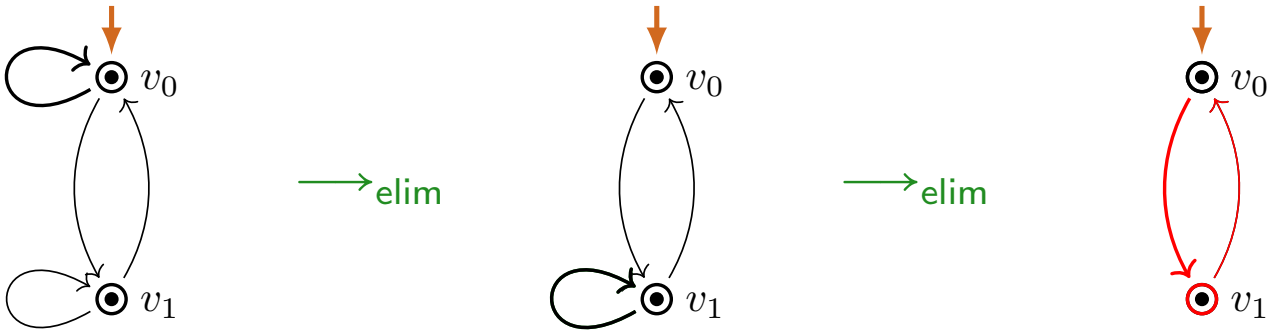

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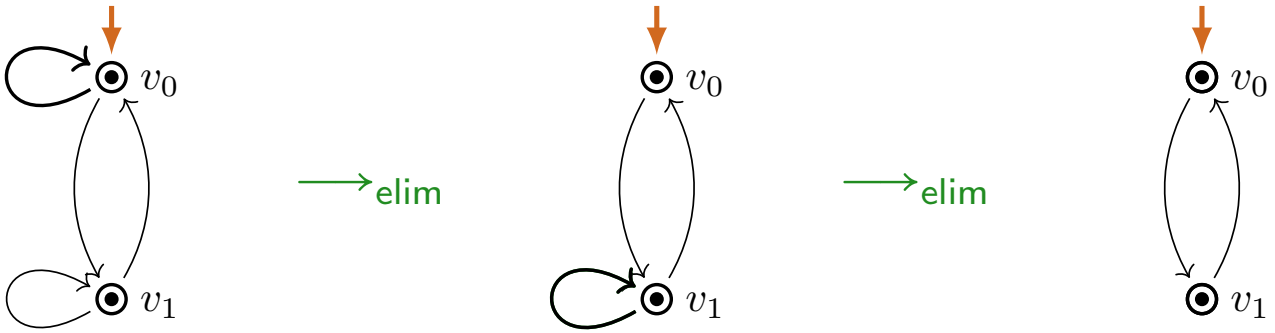

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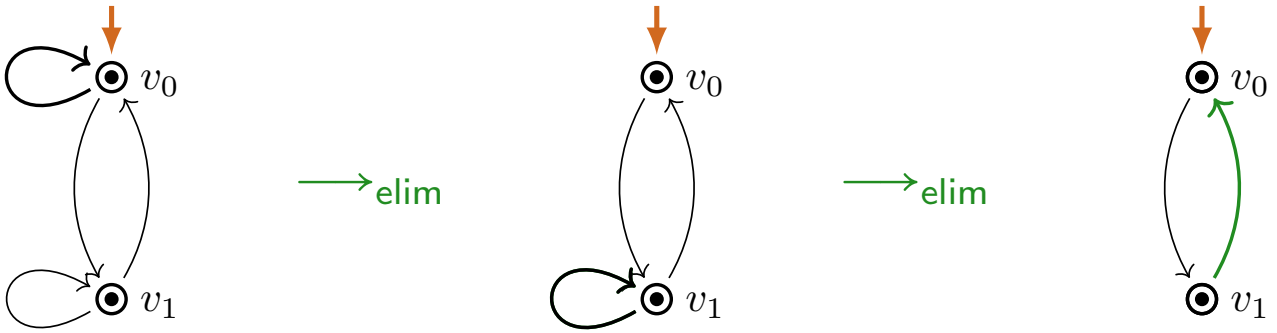

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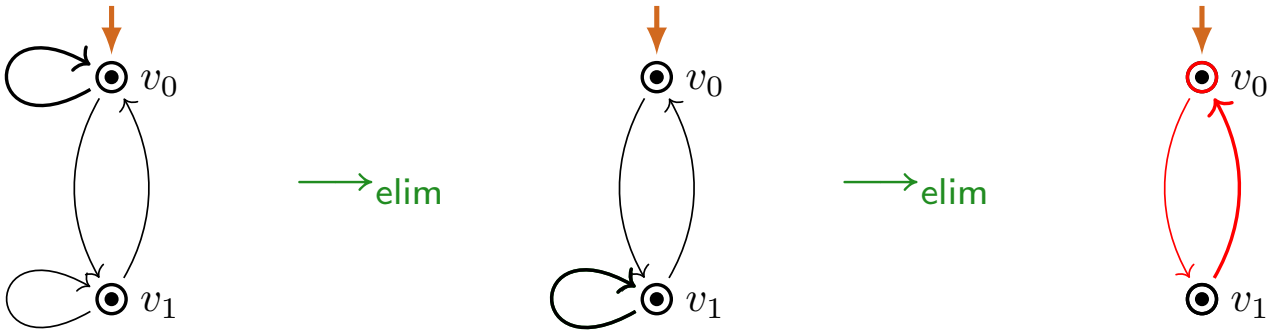

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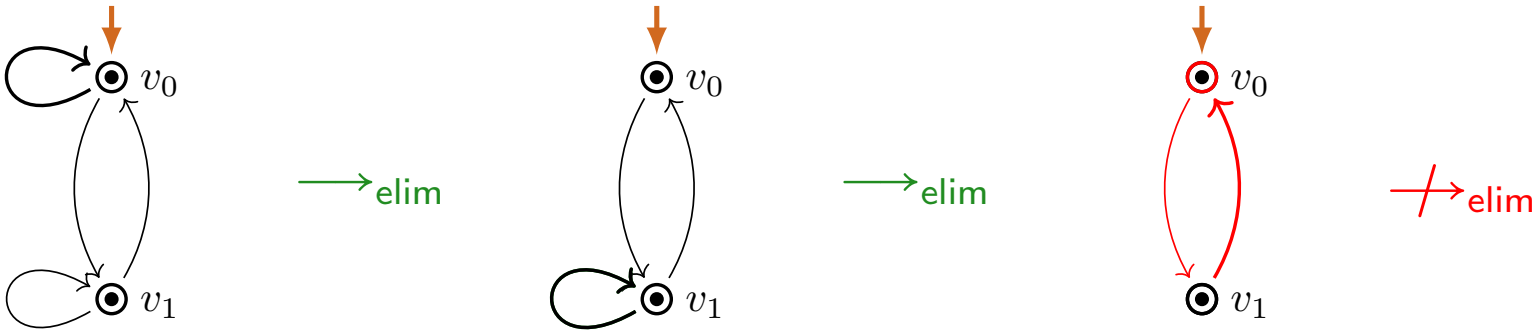

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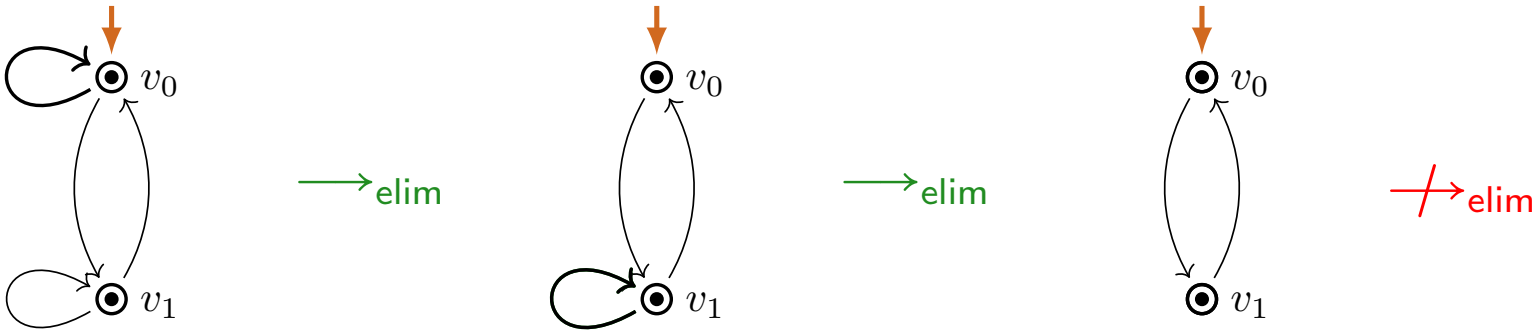

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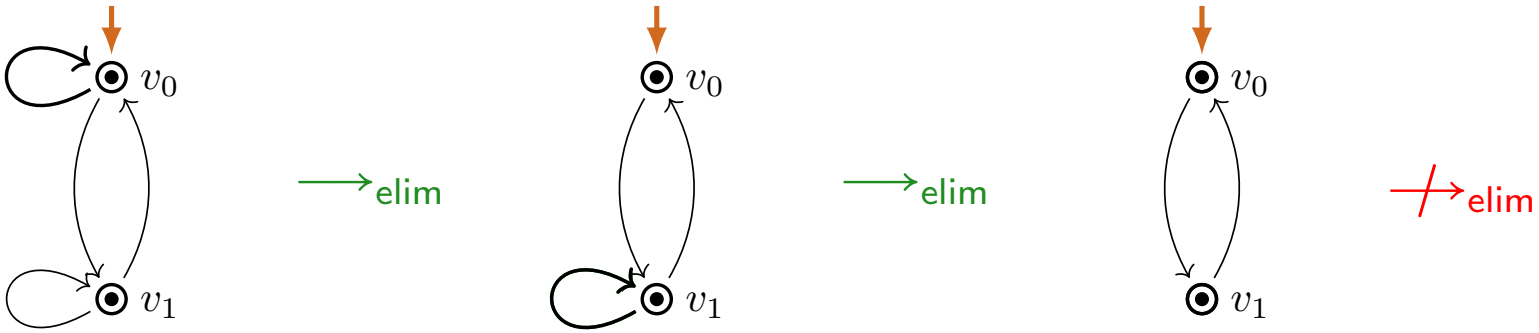

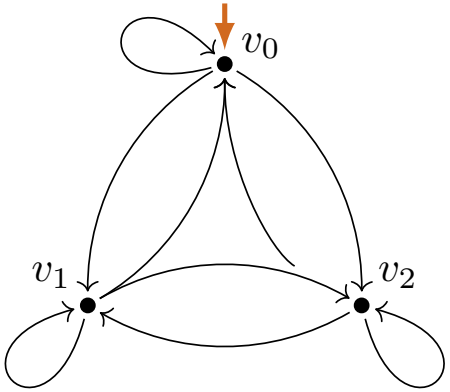

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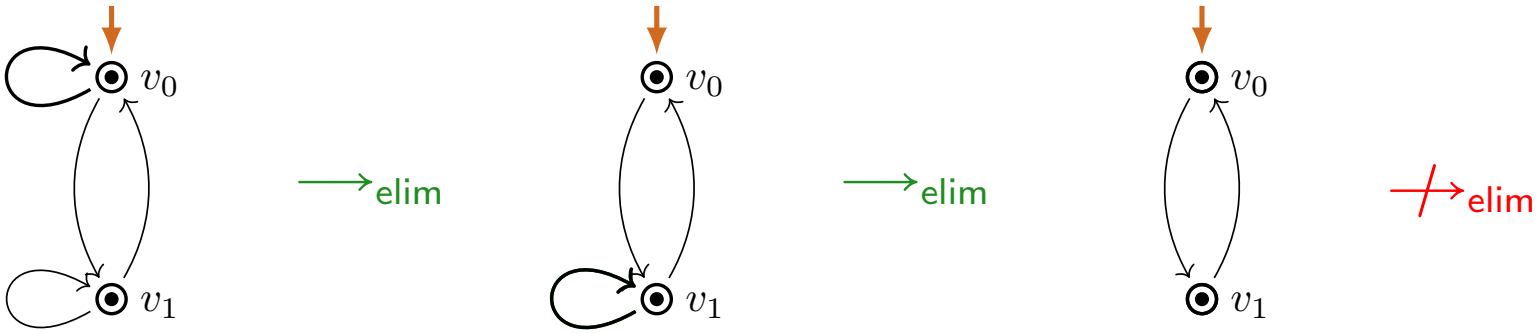

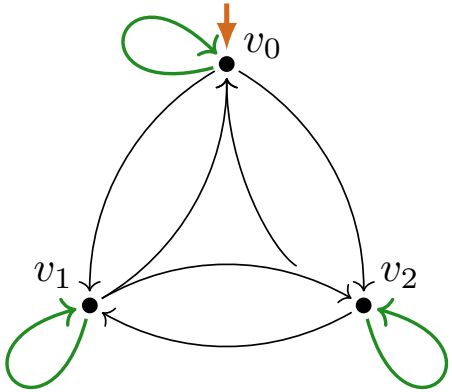

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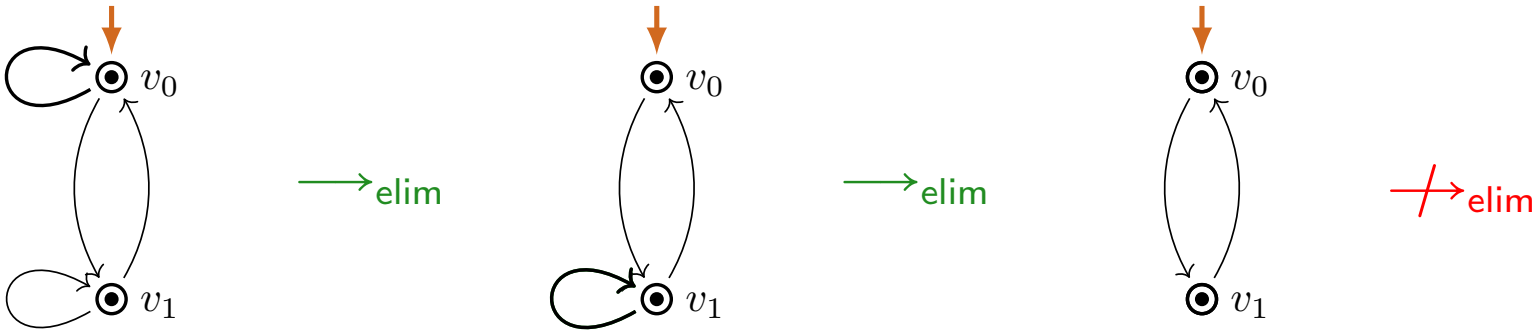

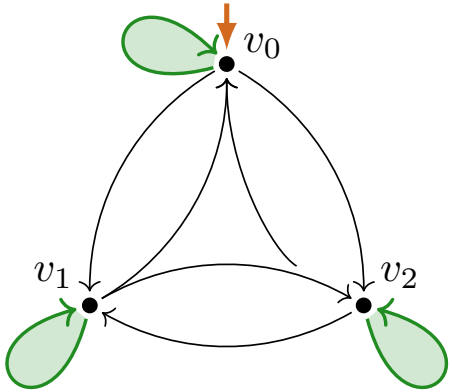

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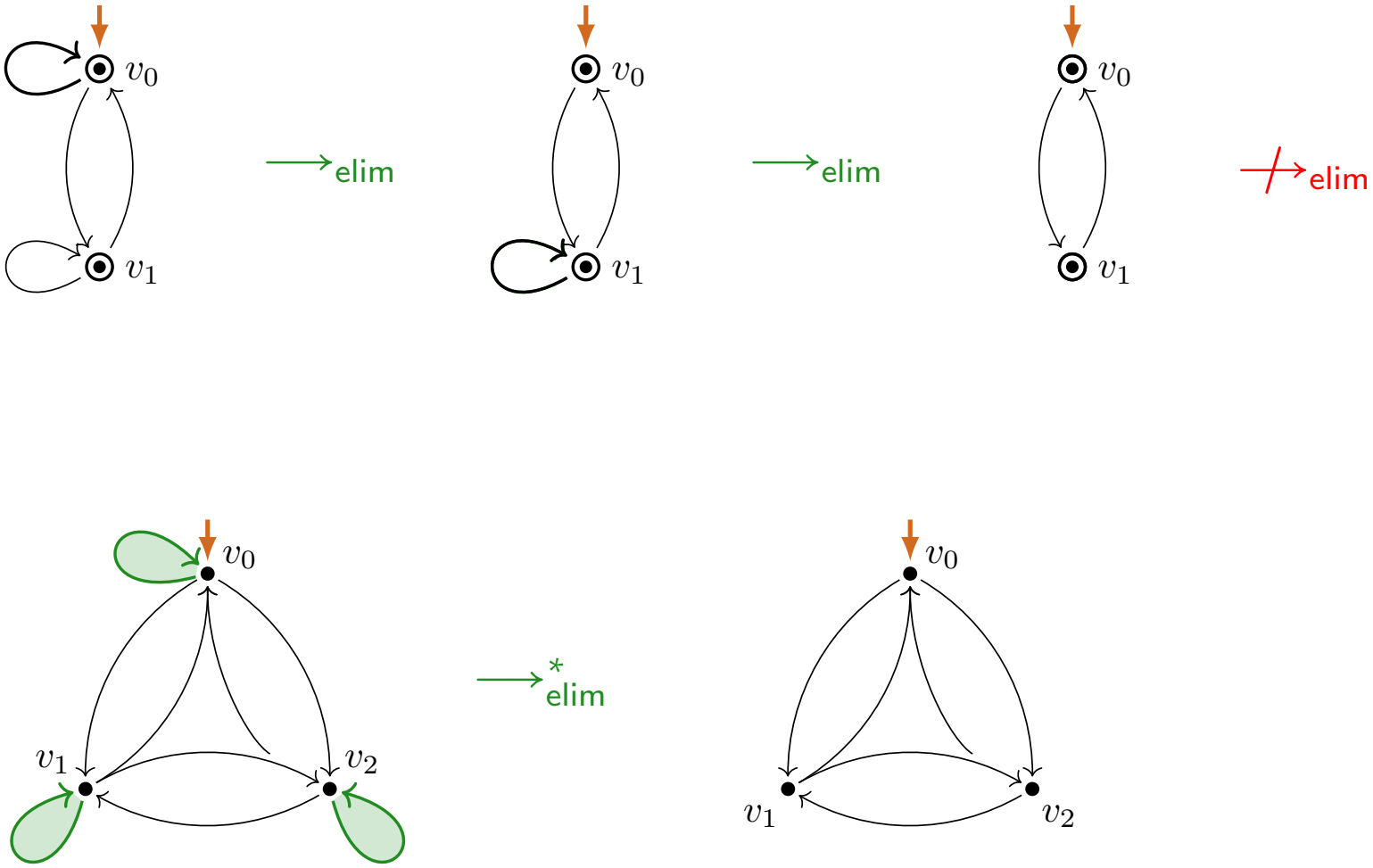

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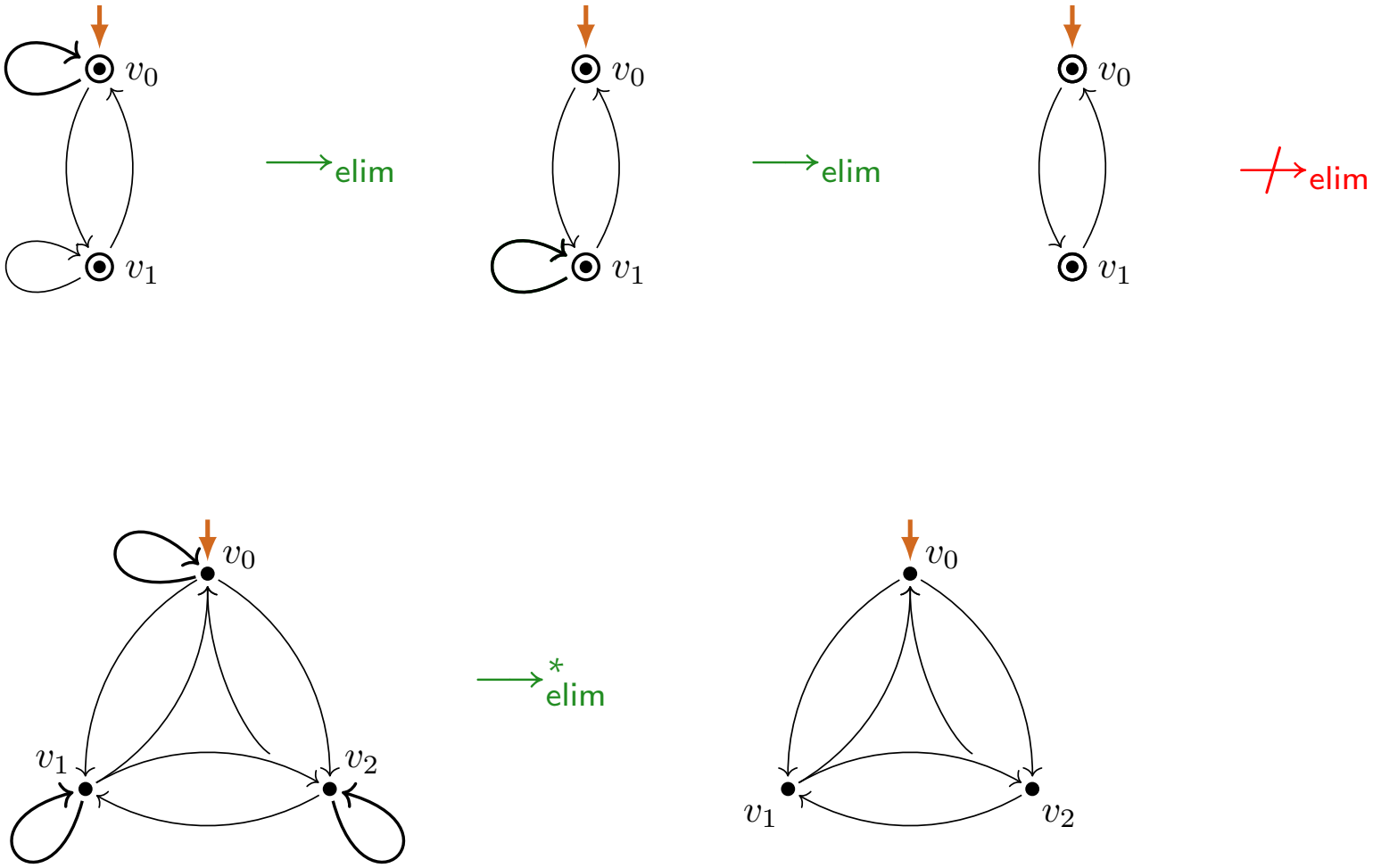

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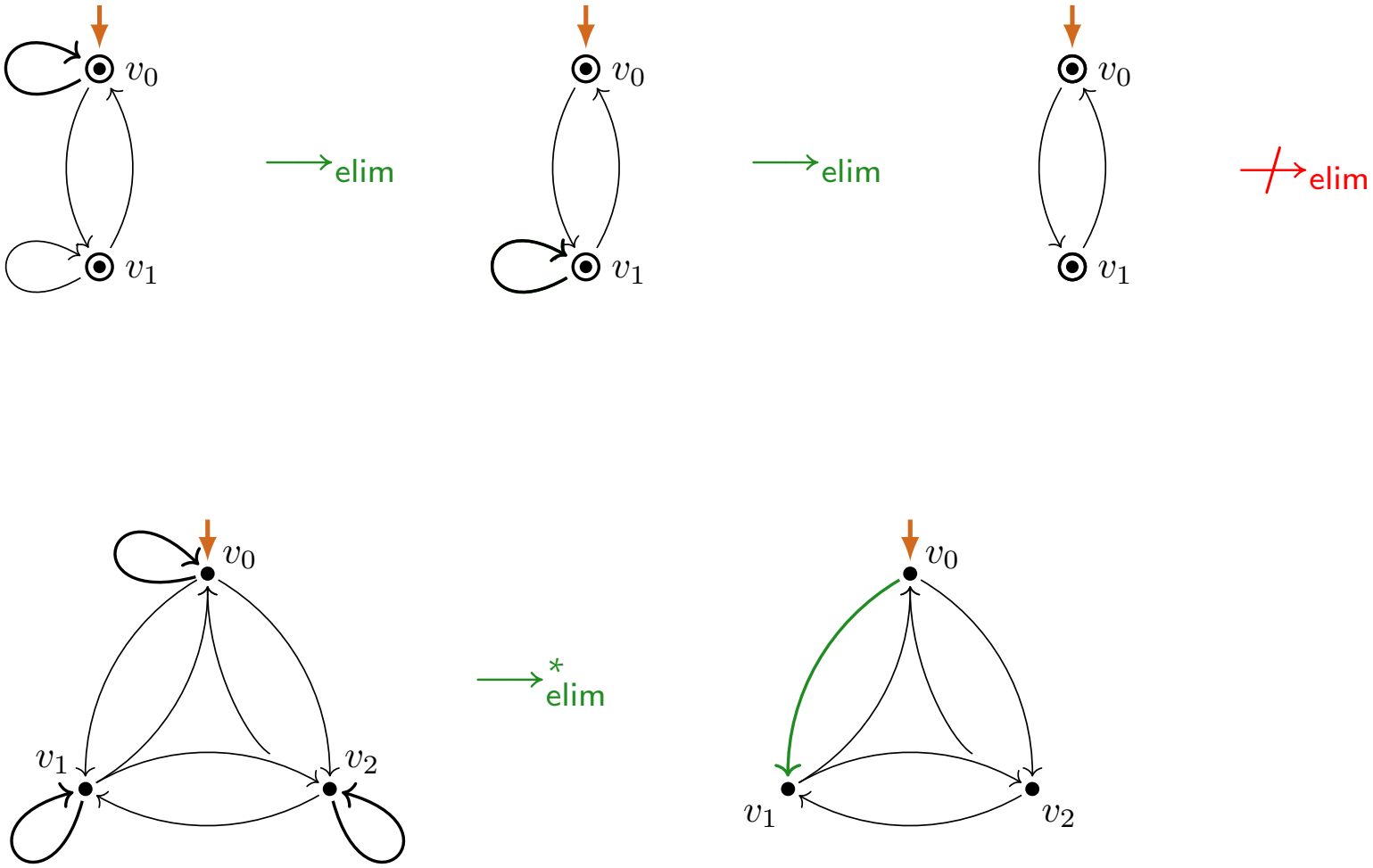

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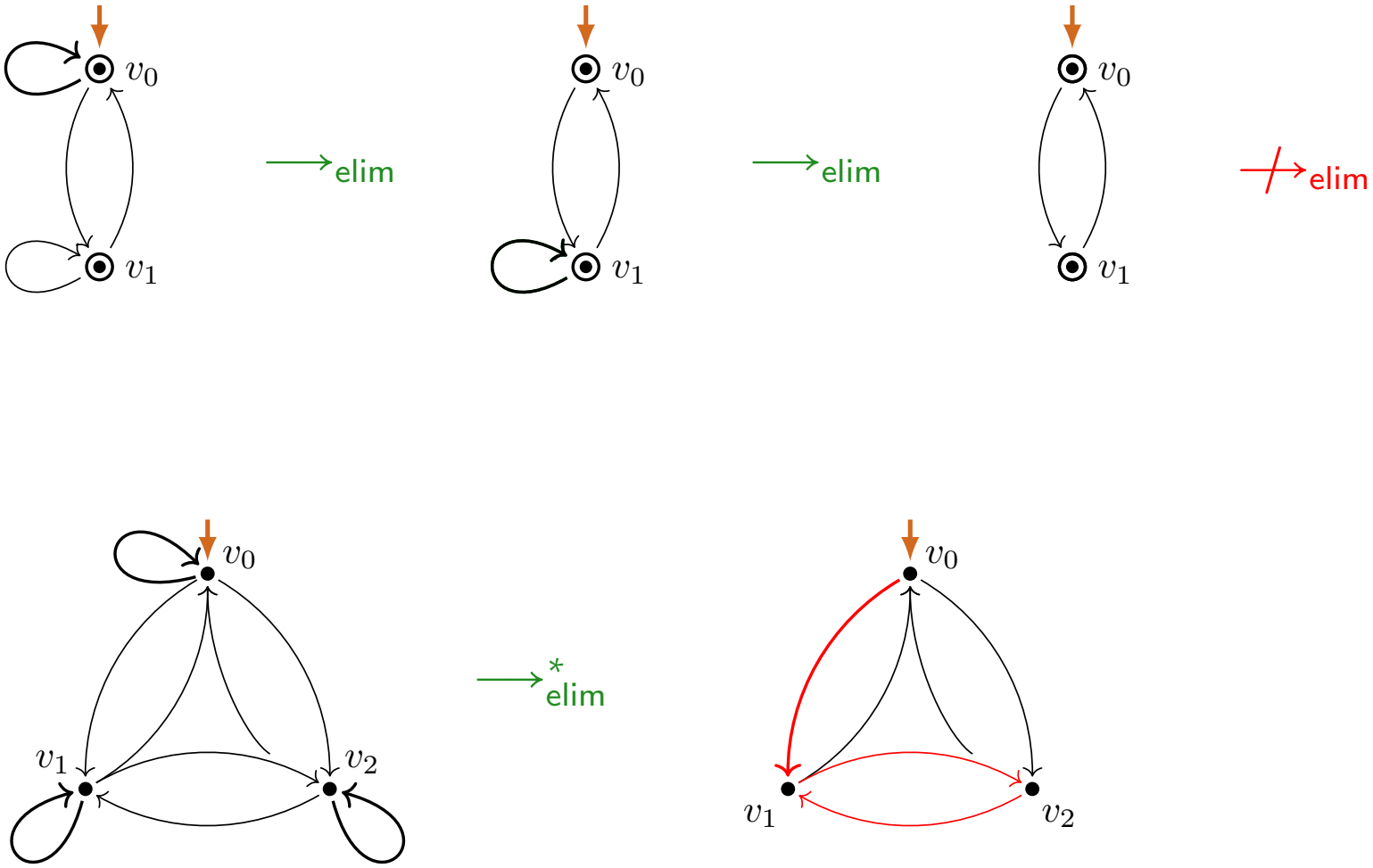

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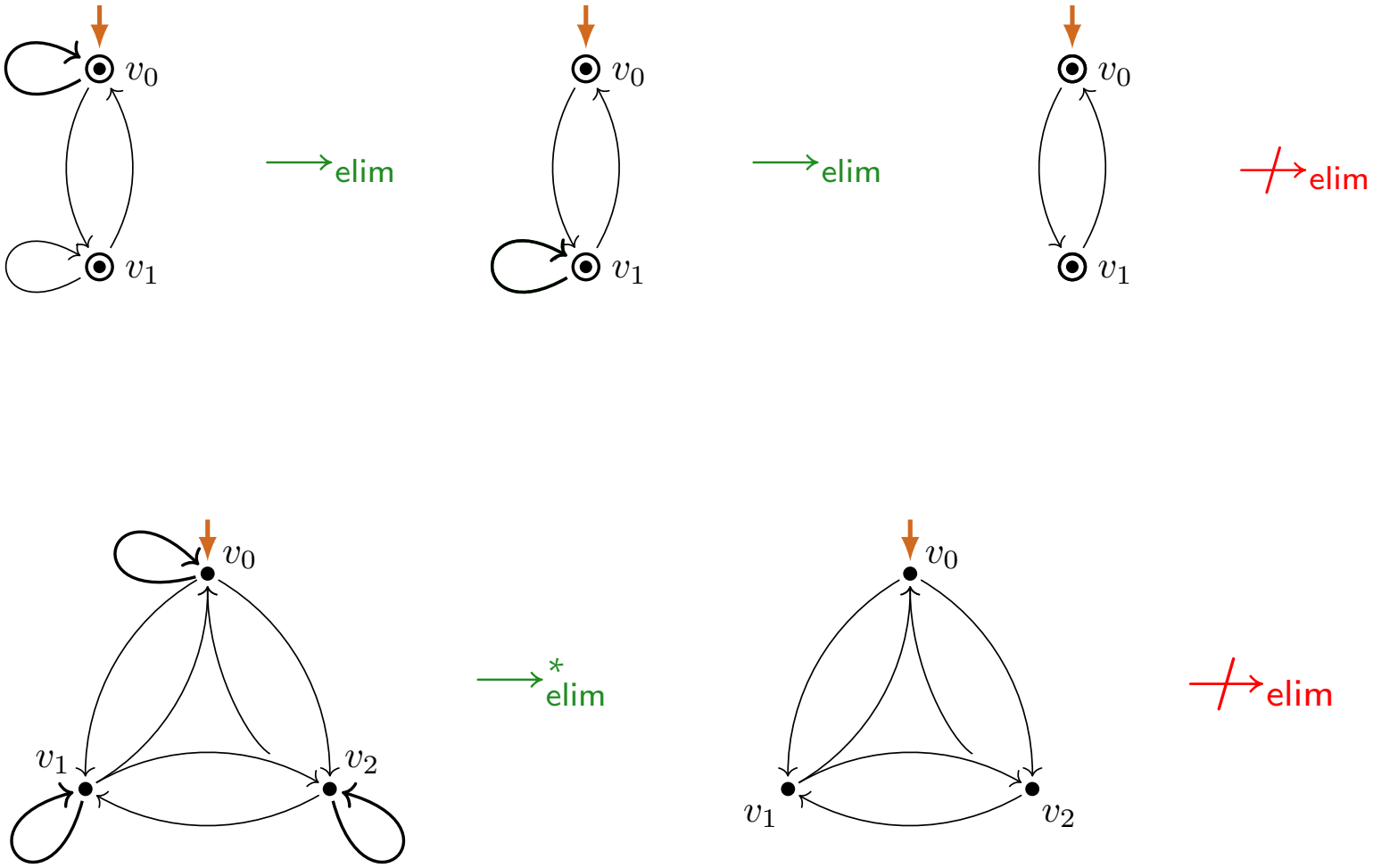

Structure property LEE

Definition

A process graph G satisfies LEE (*loop existence and elimination*) if:

$$\exists G_0 \left(G \longrightarrow^*_{\text{elim}} G_0 \not\longrightarrow_{\text{elim}} \right.$$
$$\left. \wedge\ G_0 \text{ has no infinite trace} \right) .$$

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Lemma (by using termination and confluence)

For every process graph G the following are equivalent:

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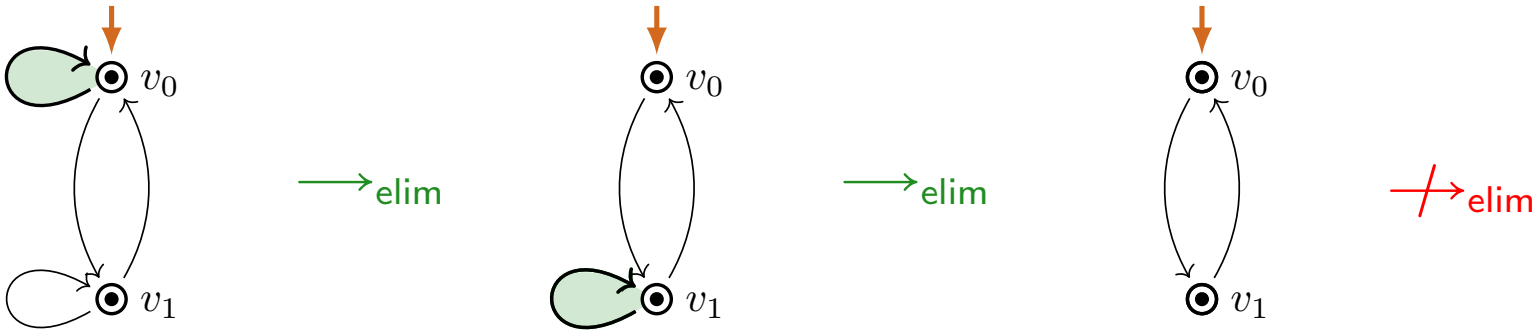
Theorem (efficient decidability)

The problem of deciding LEE(G) for process graphs G is in PTIME.

LEE fails



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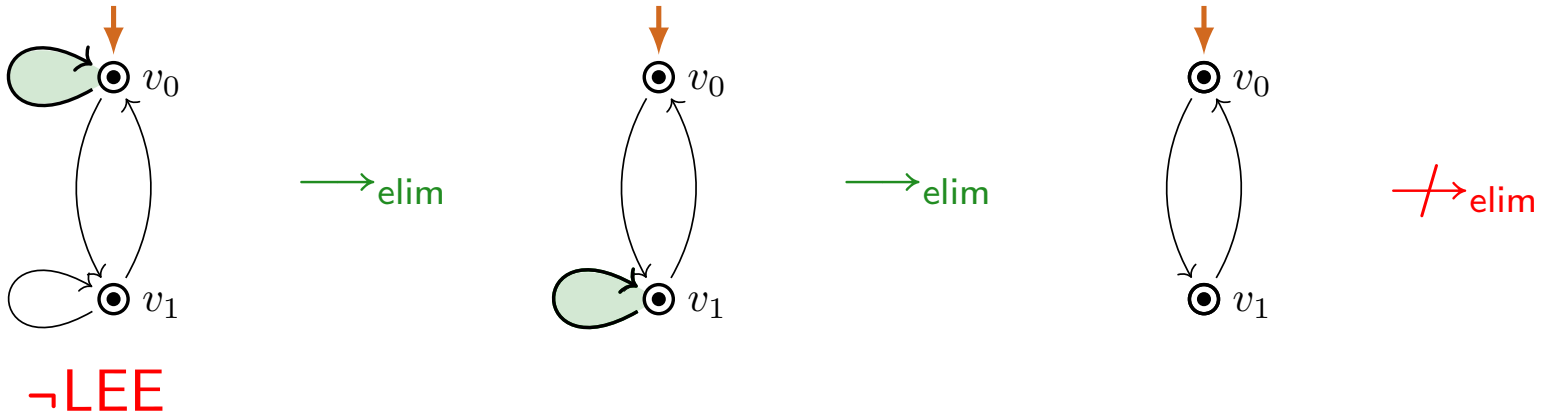

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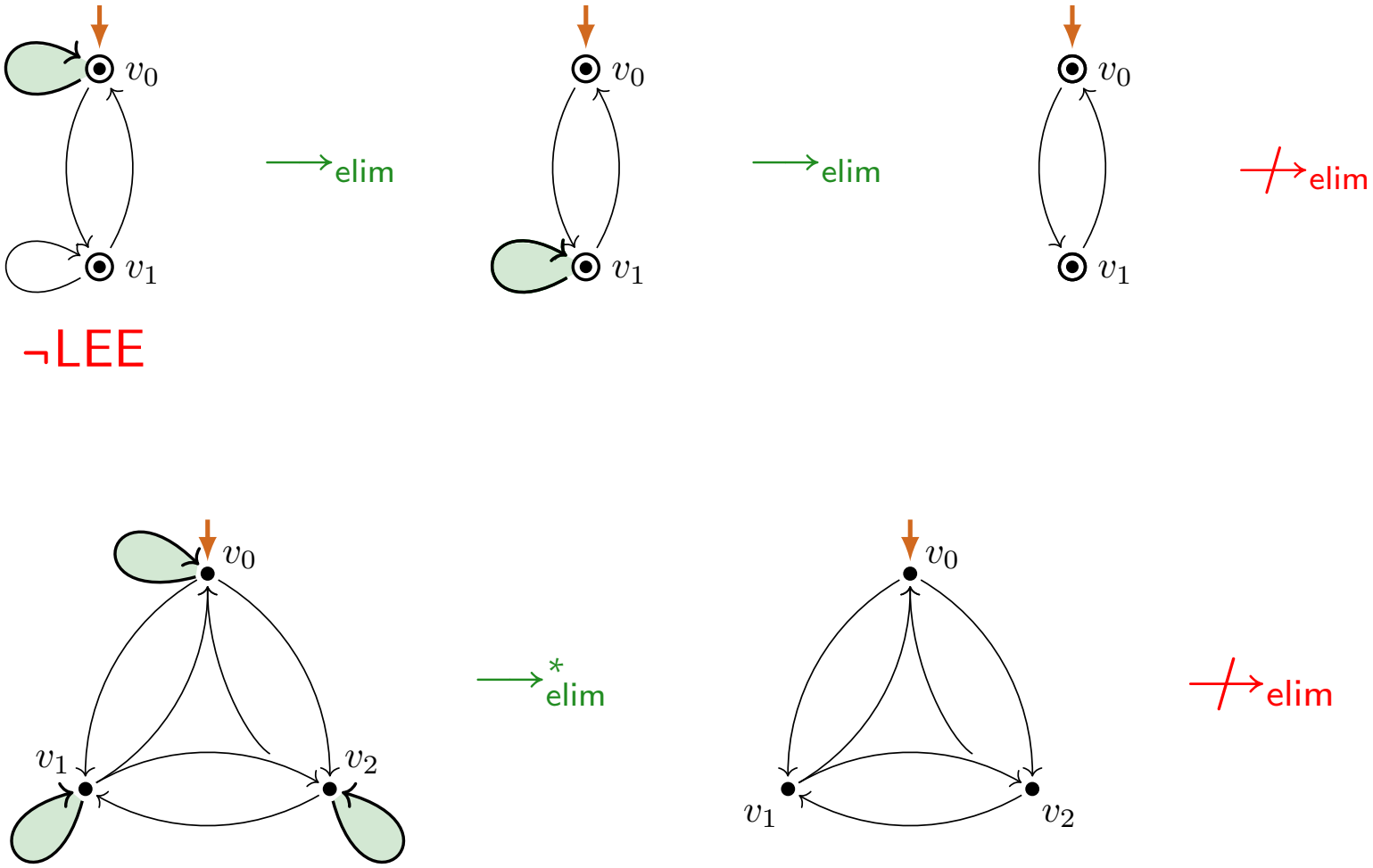

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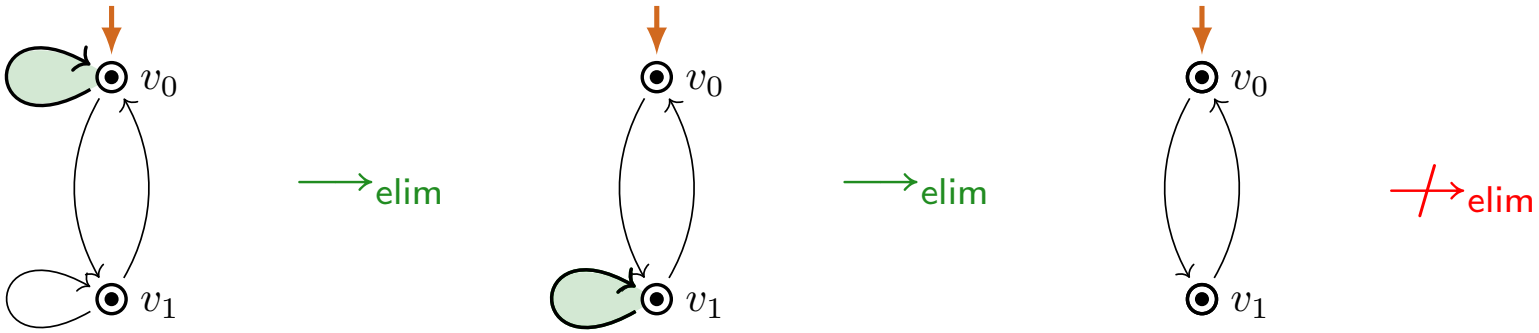

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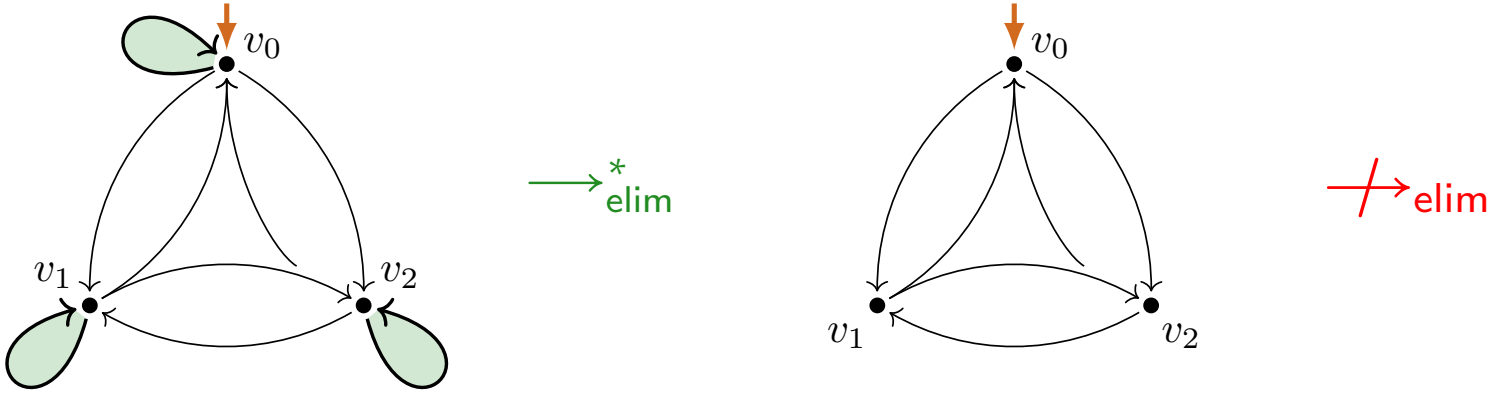

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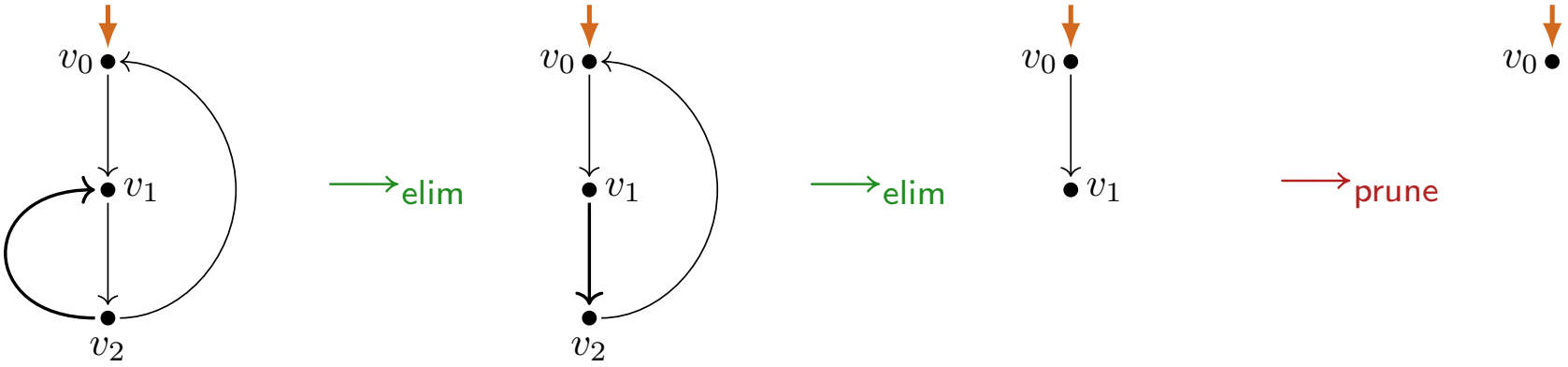

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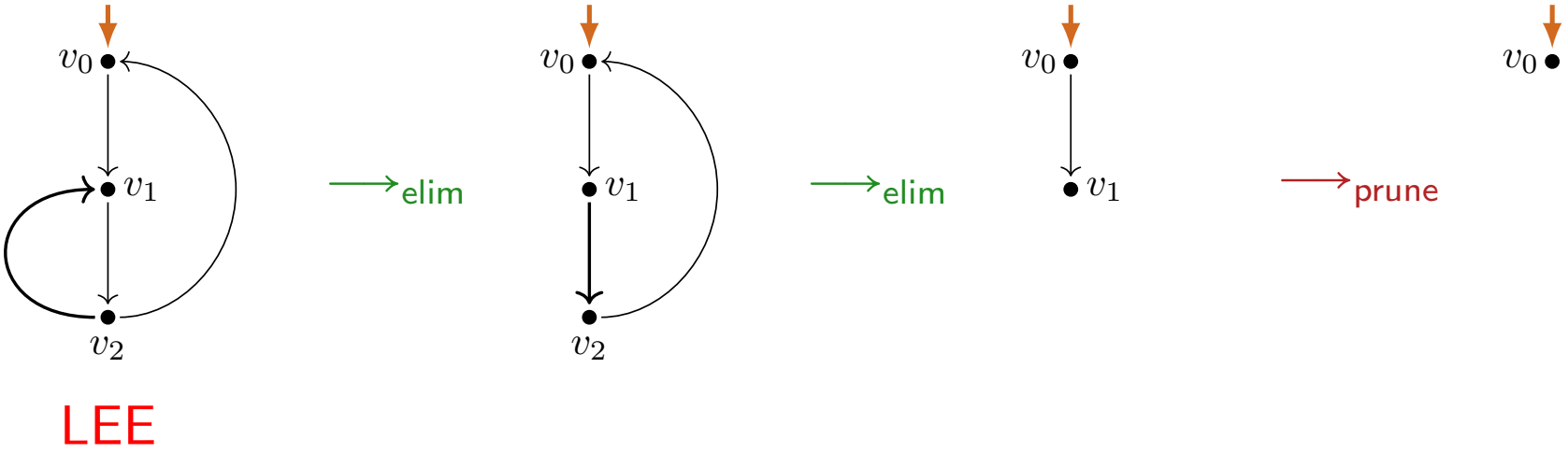
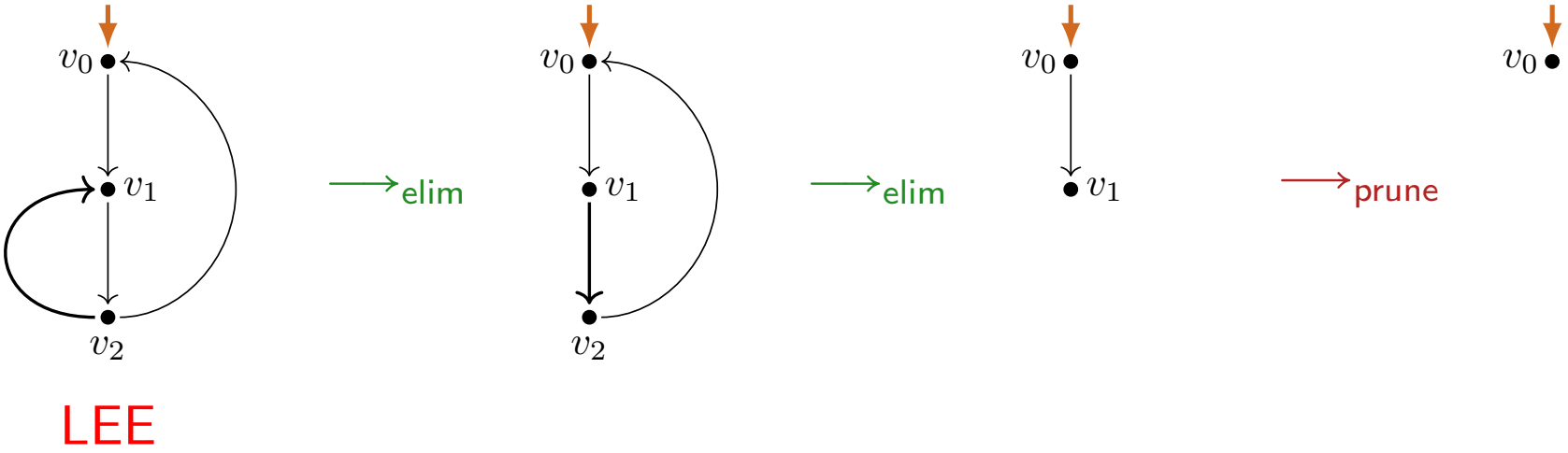


LEE holds / Recording loop elimination



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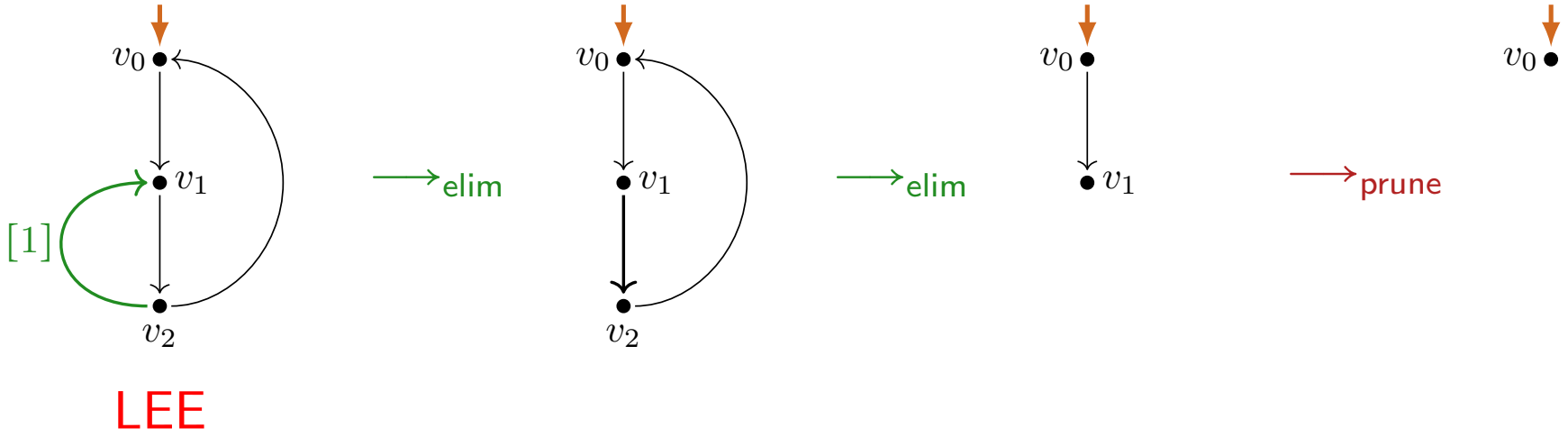

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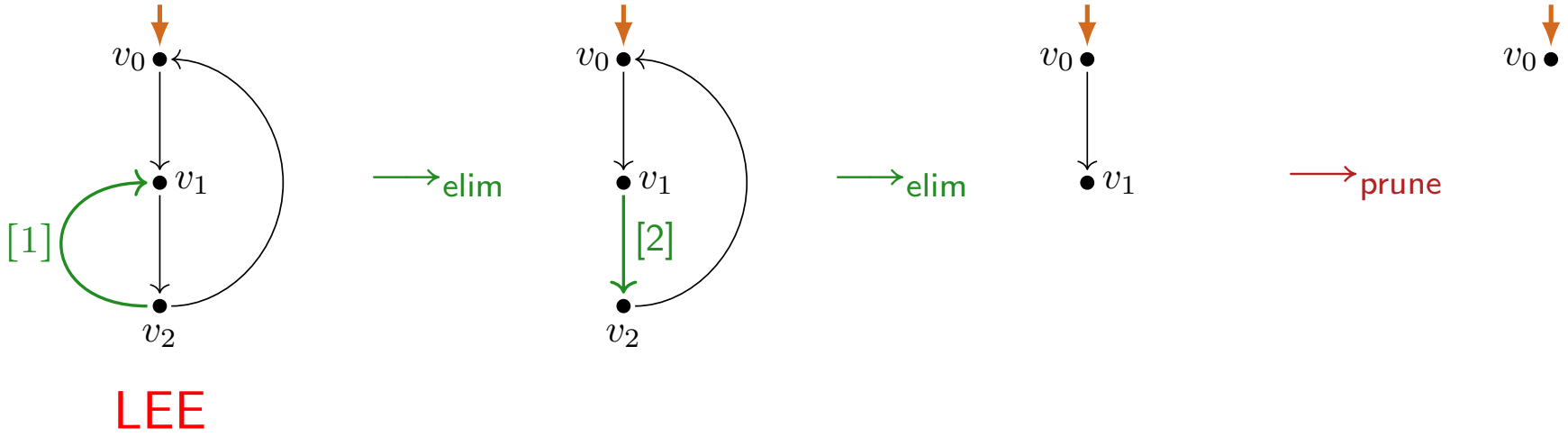

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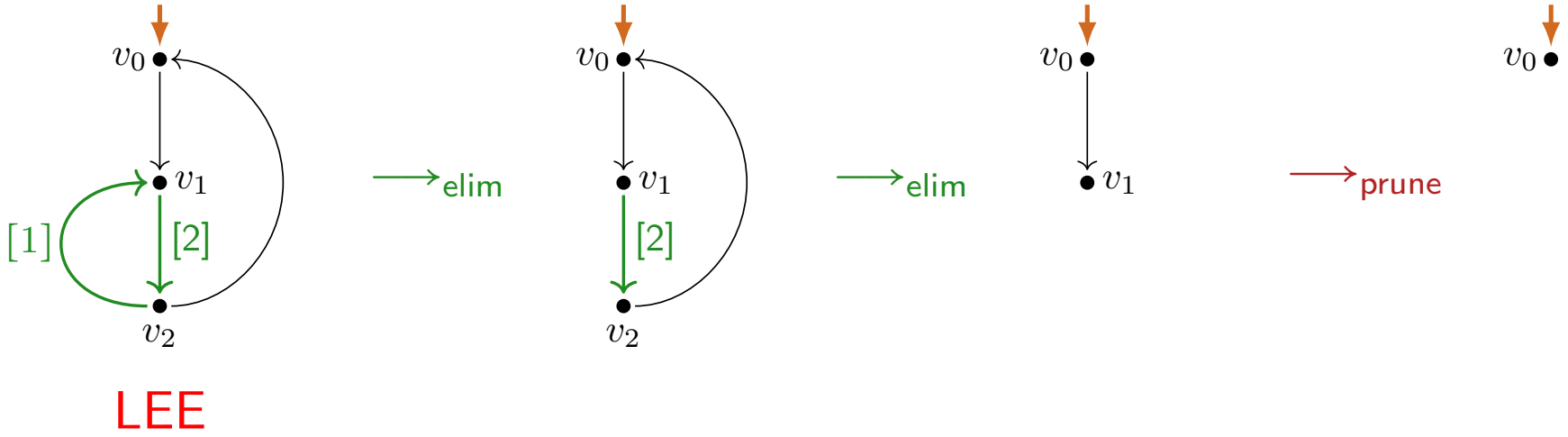

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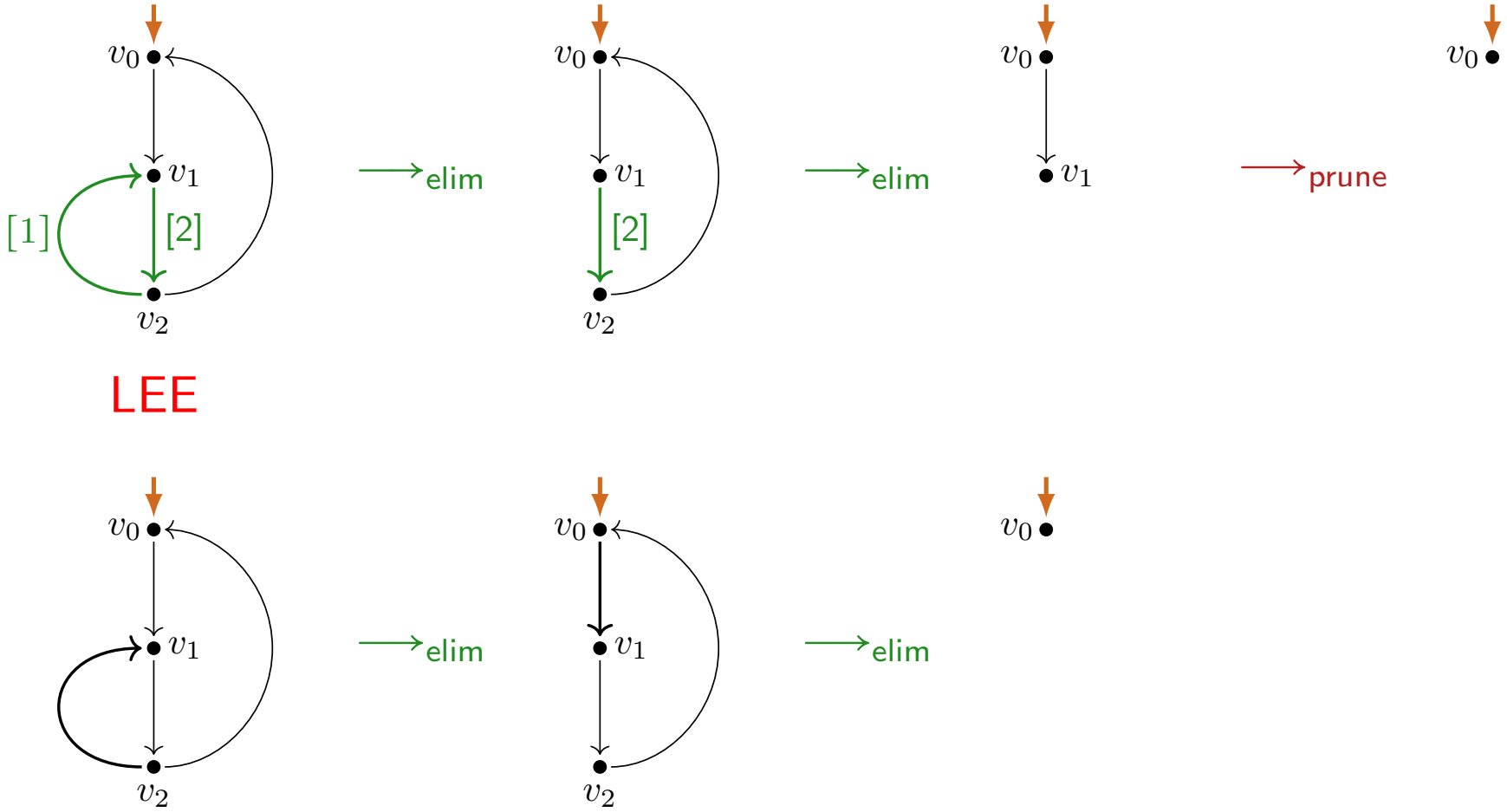

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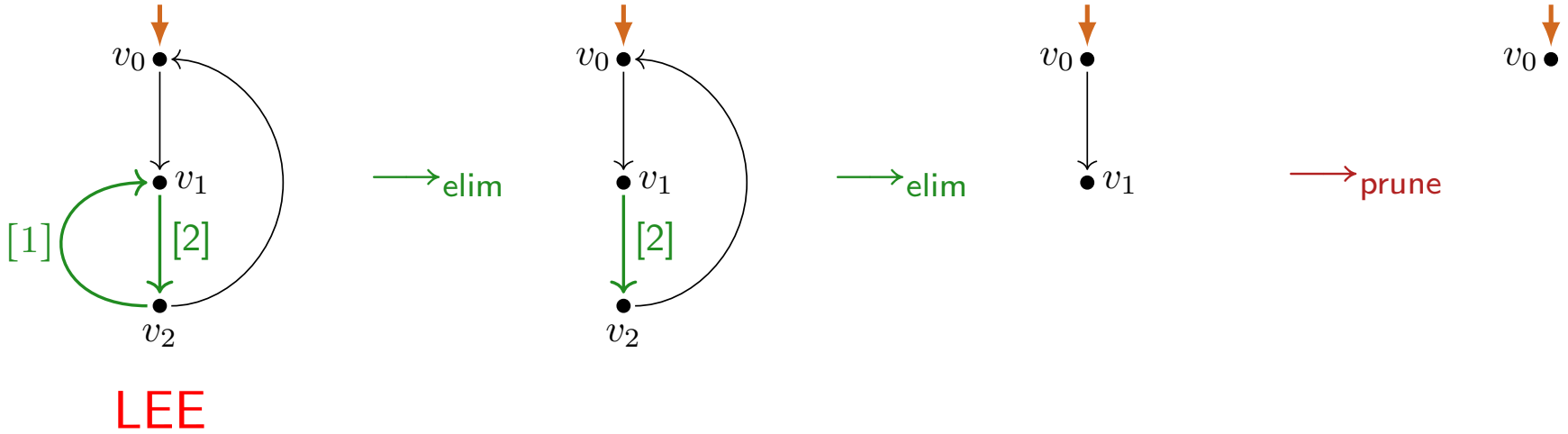

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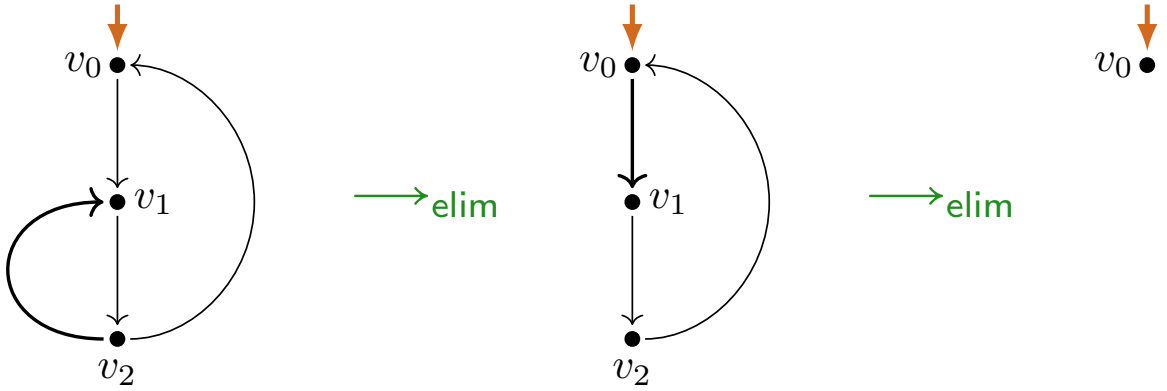

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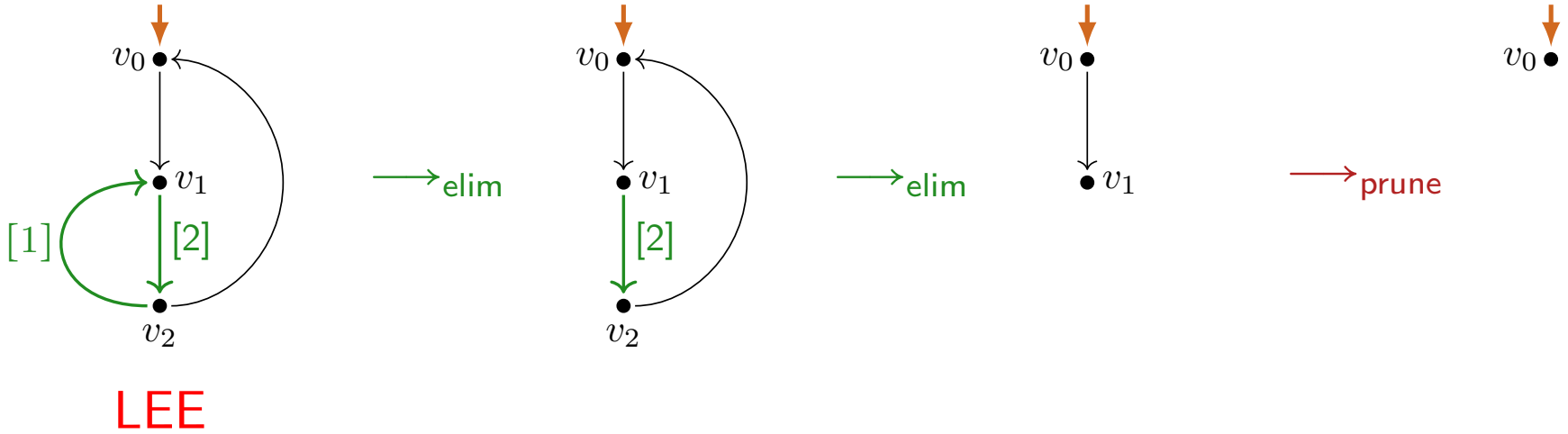

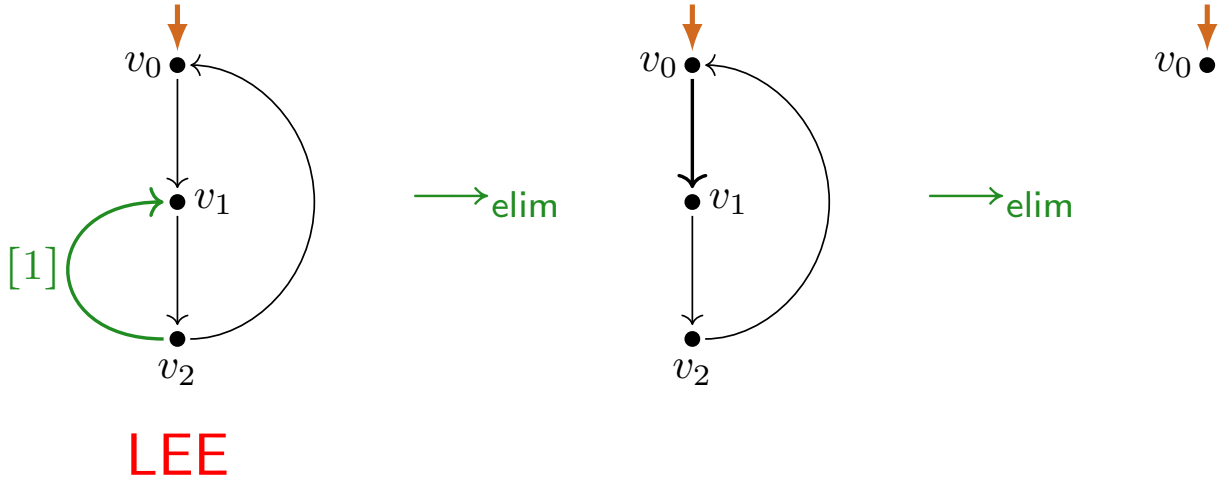

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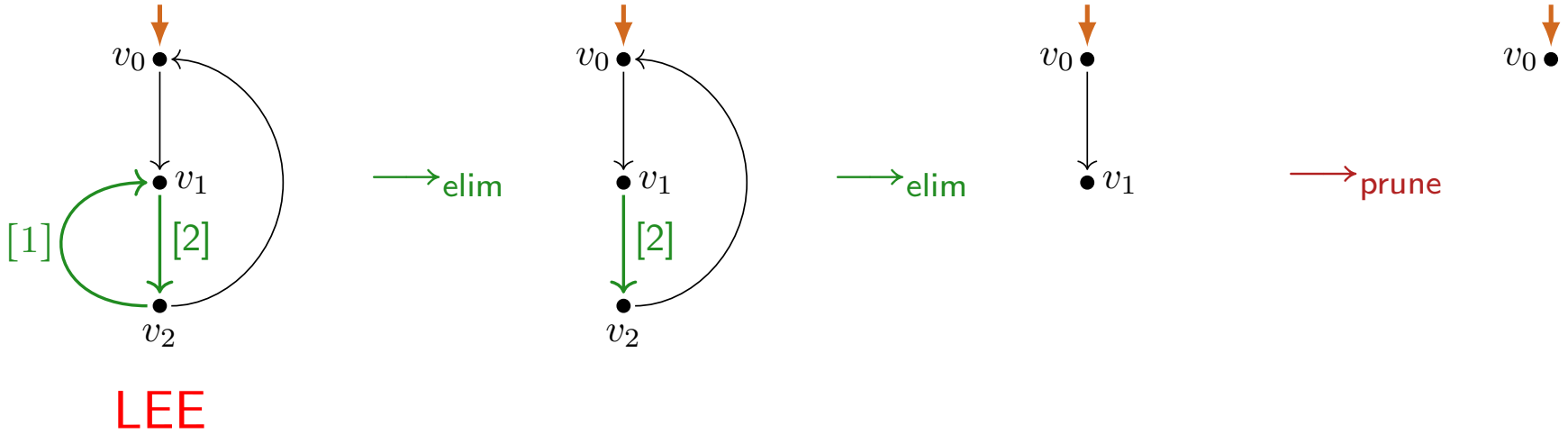

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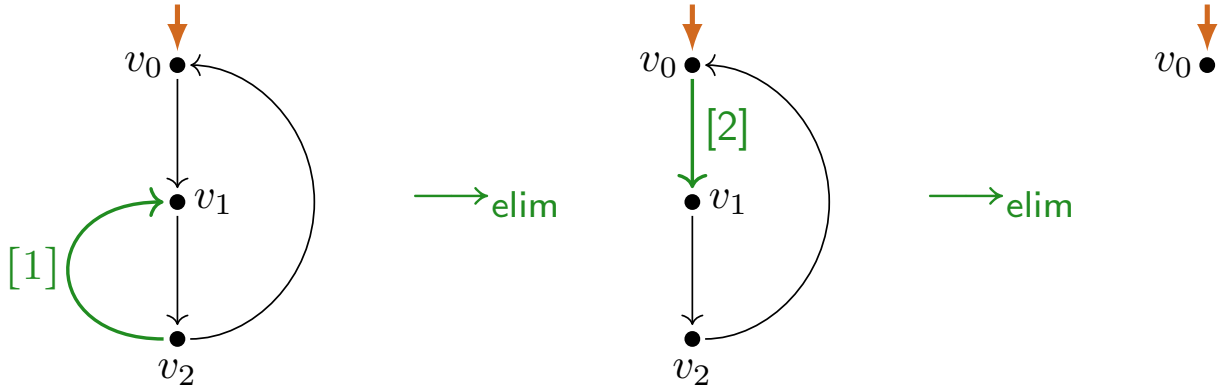

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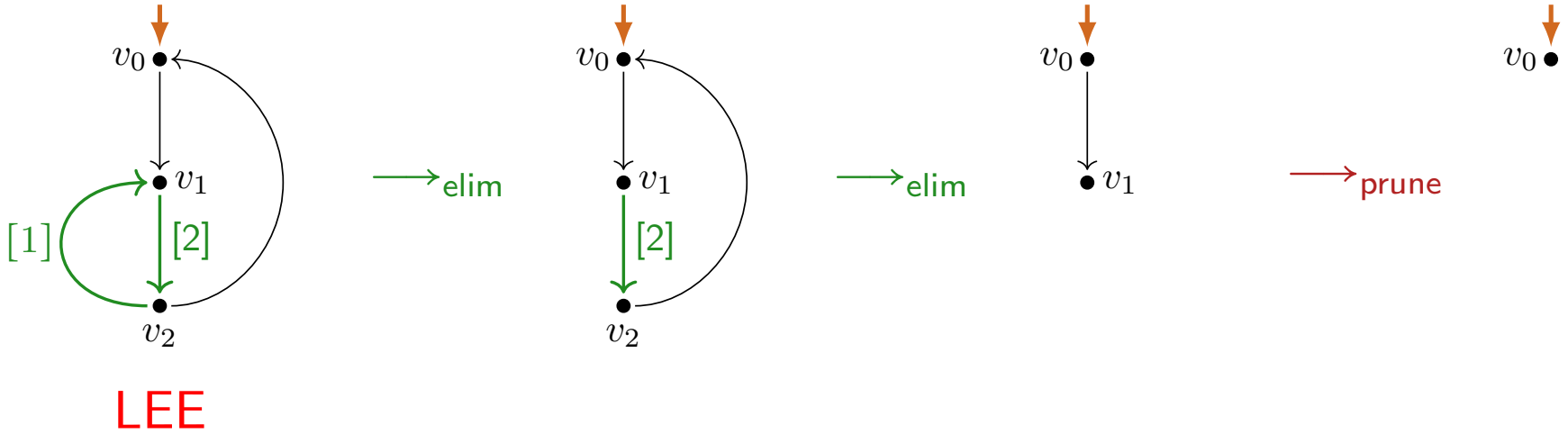

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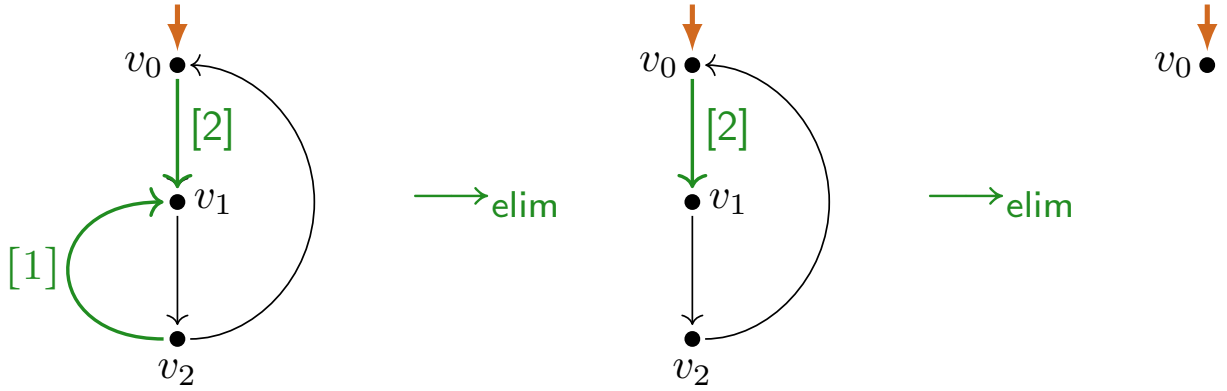

LEE

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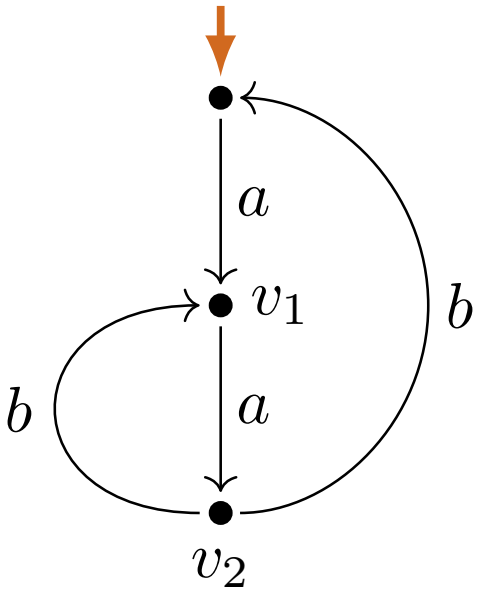

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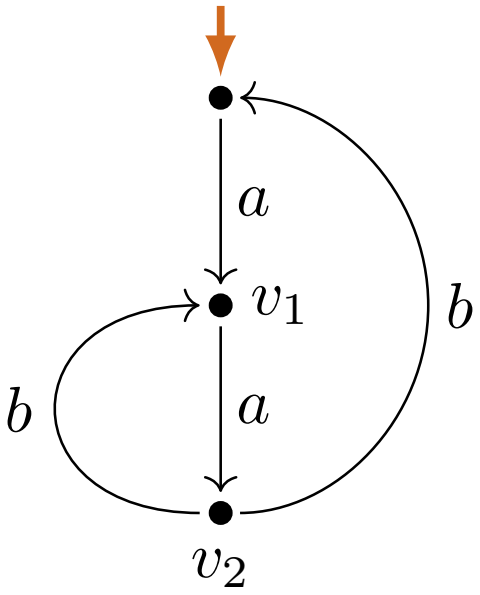
loop–branch labeling: marking transitions $\xrightarrow{a}$ as:



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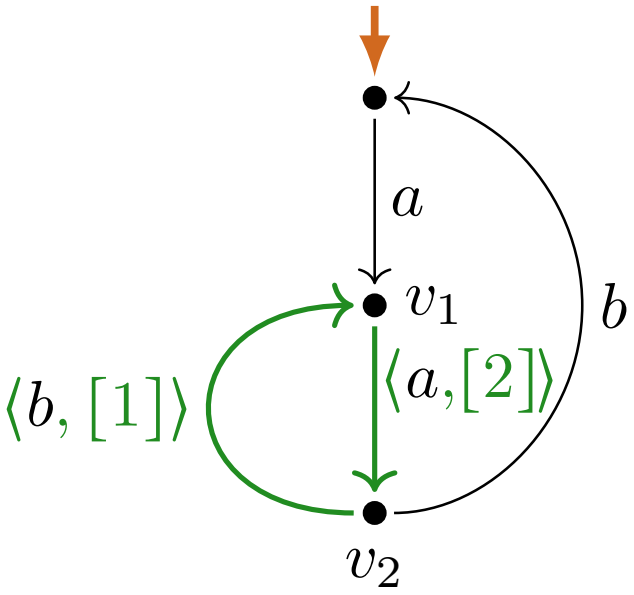

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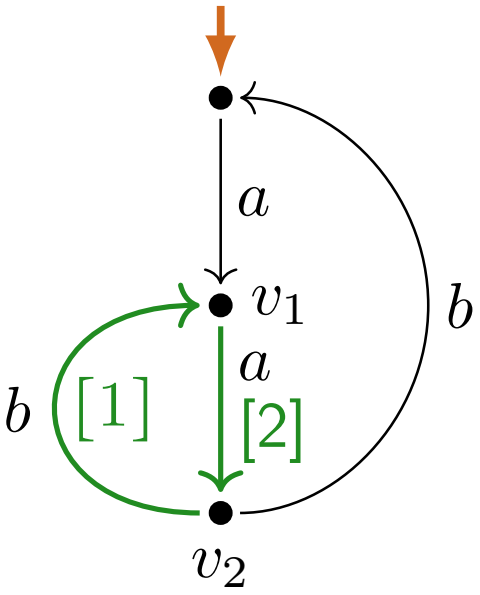

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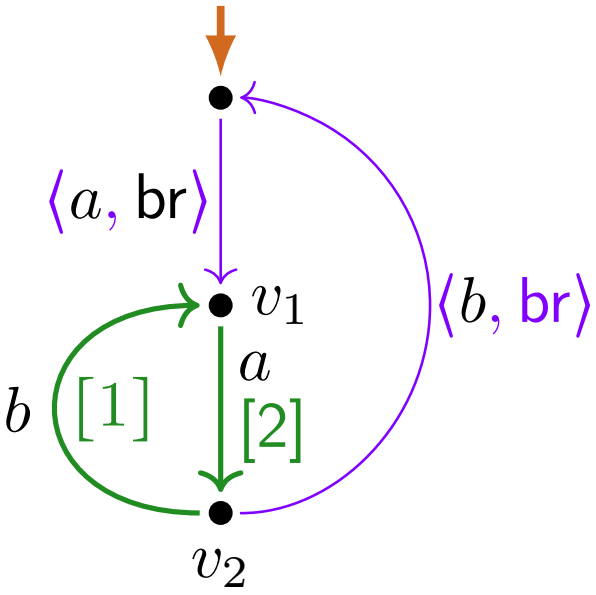

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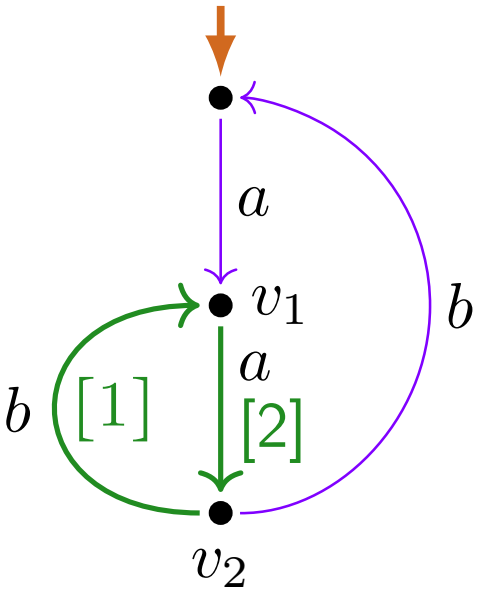

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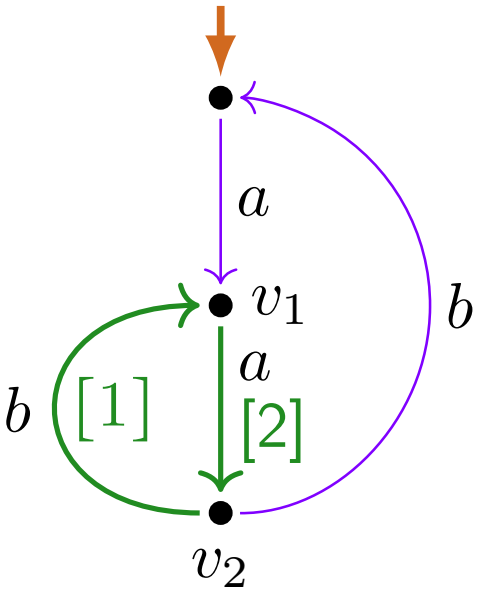

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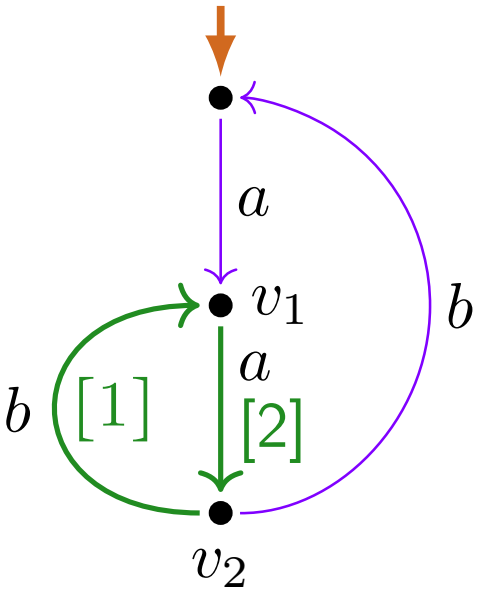

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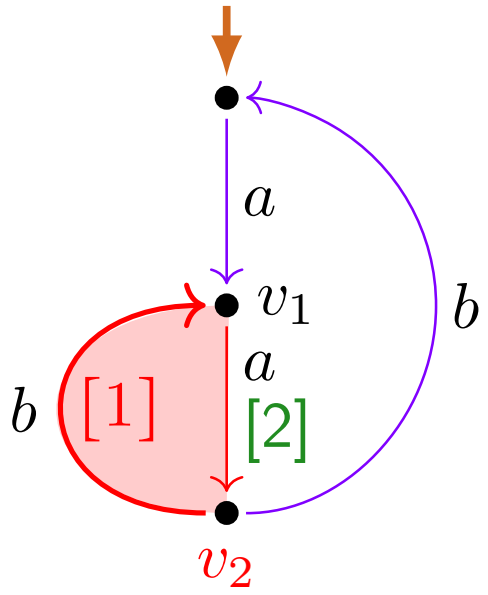

$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$

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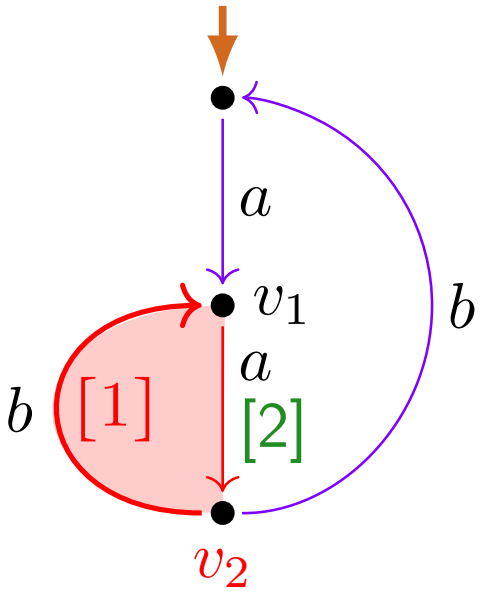

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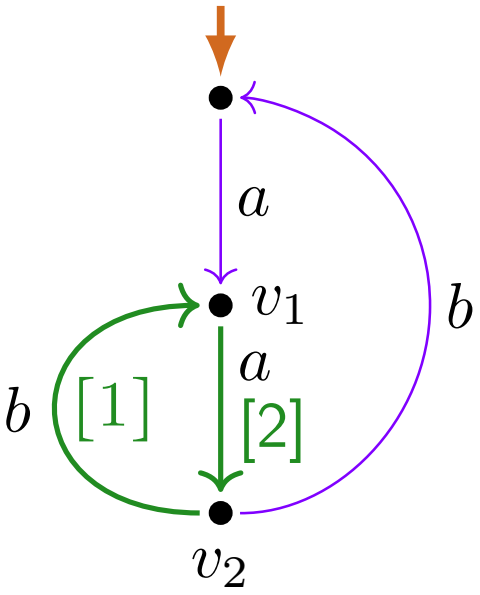

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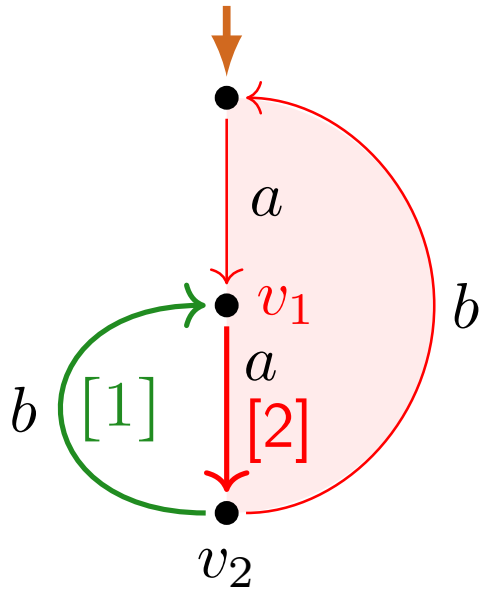

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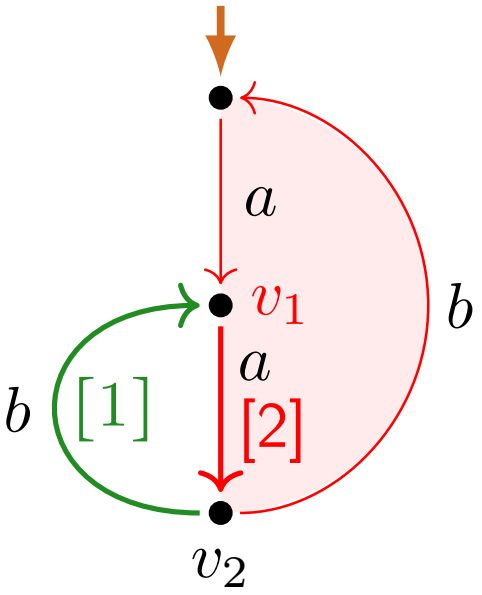

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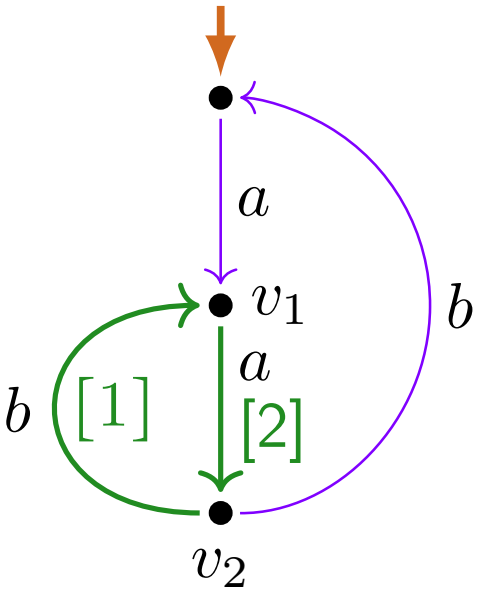

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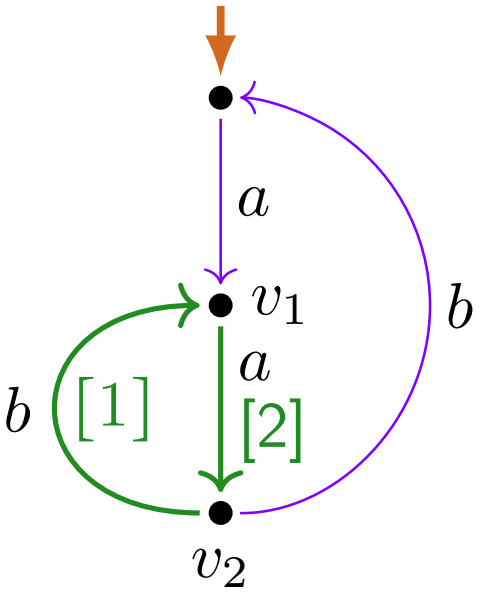

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L2. No infinite $\to_{\mathsf{br}}$ path from start vertex.

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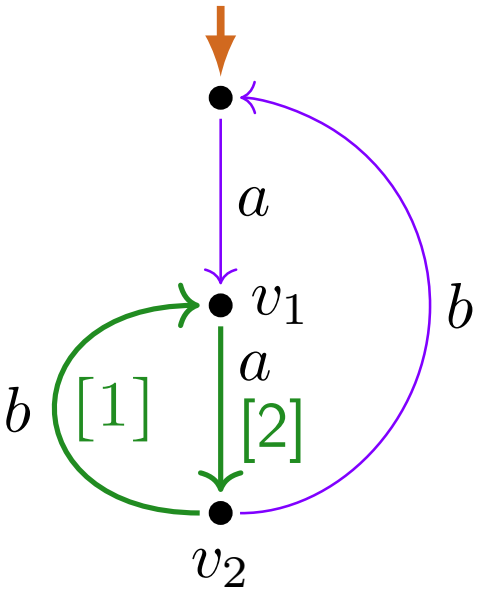
LEE-witness



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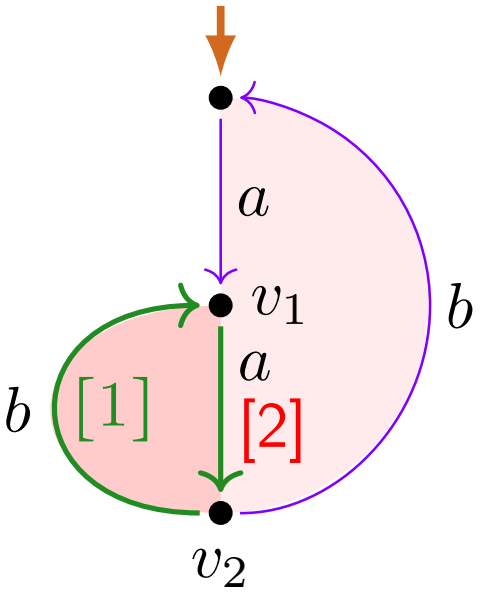

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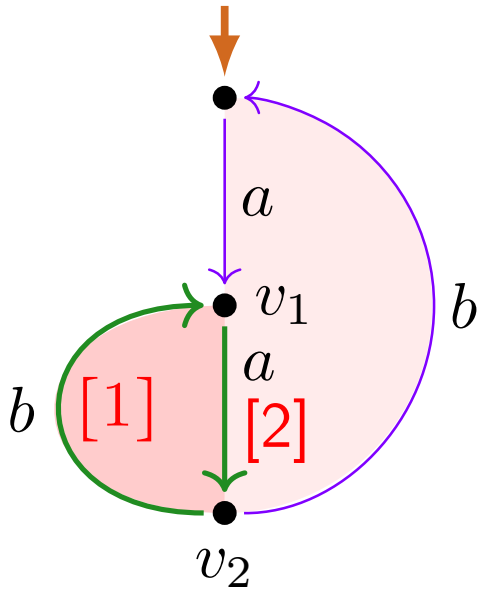

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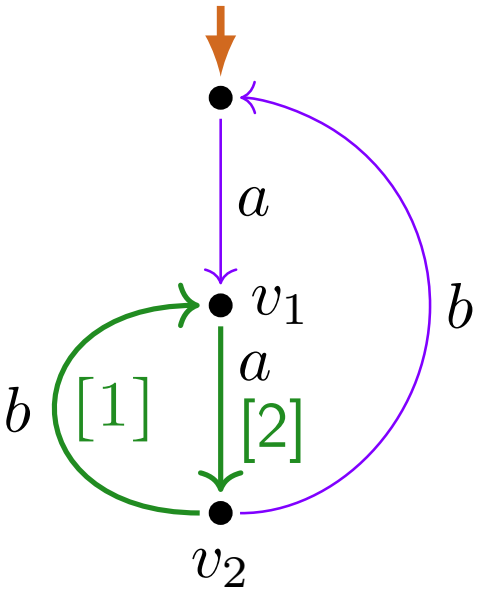

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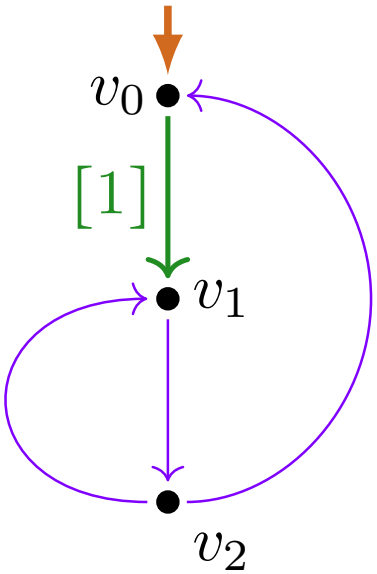

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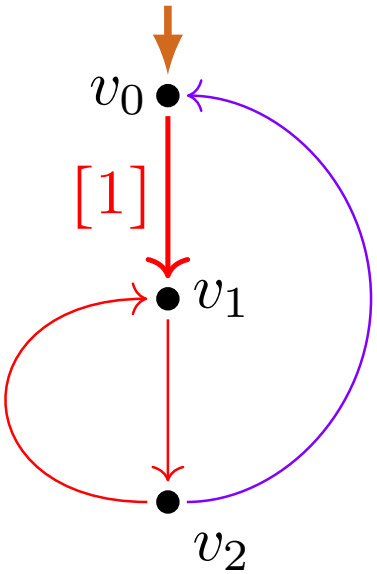

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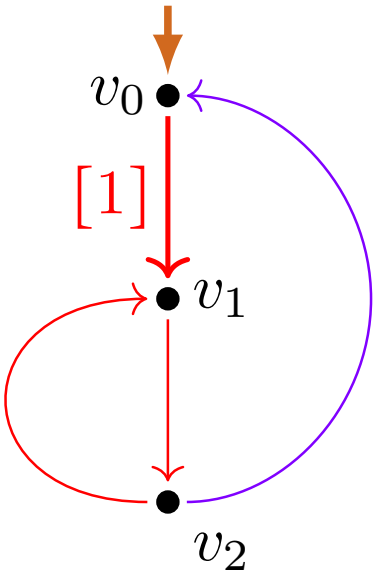

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$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$

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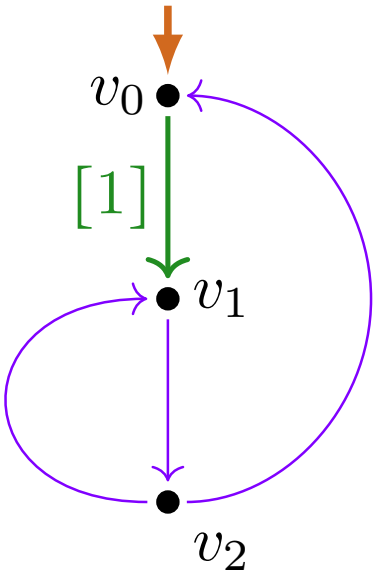

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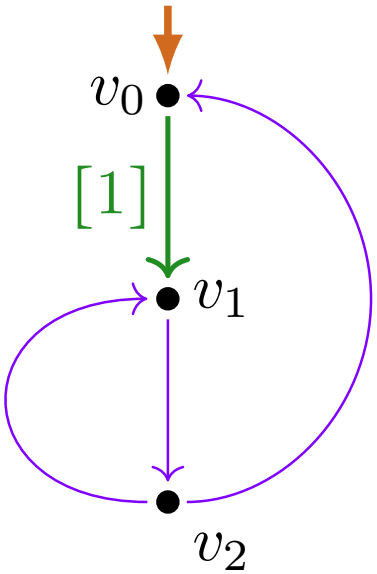

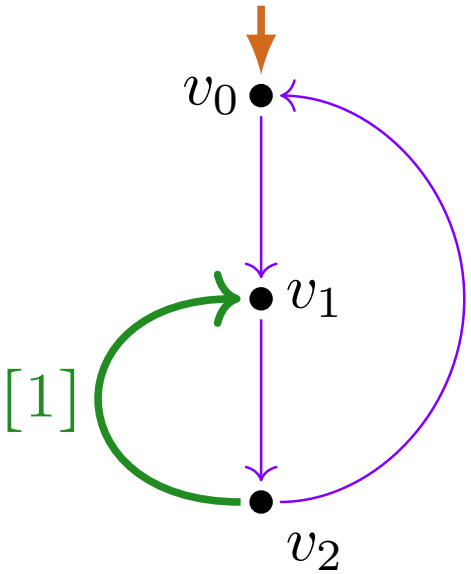

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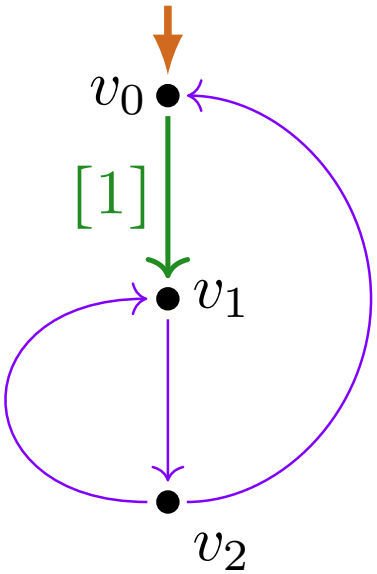

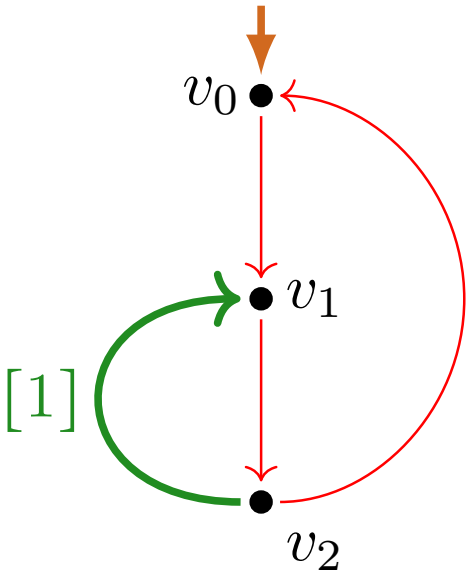

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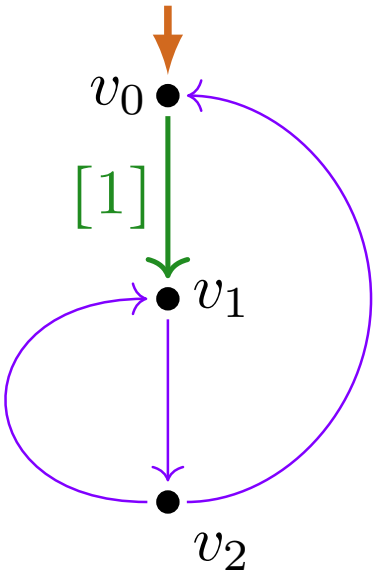

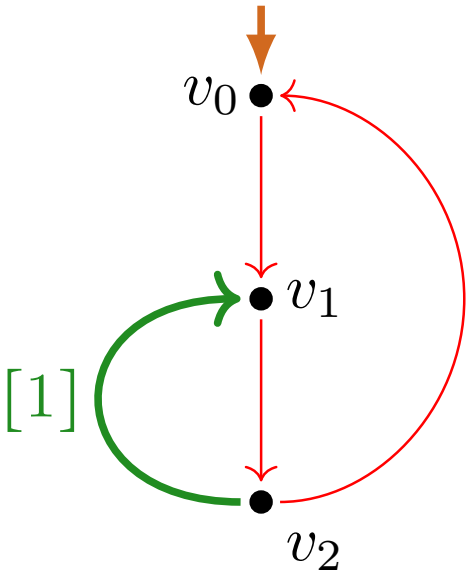

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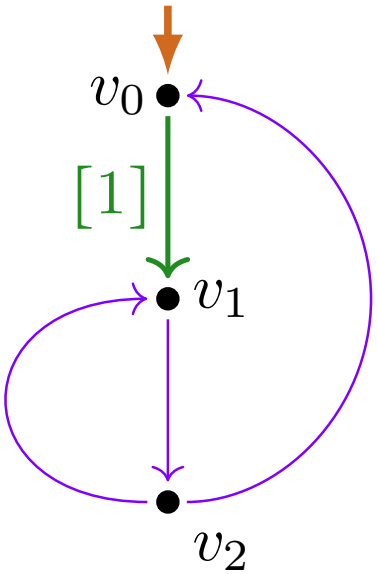
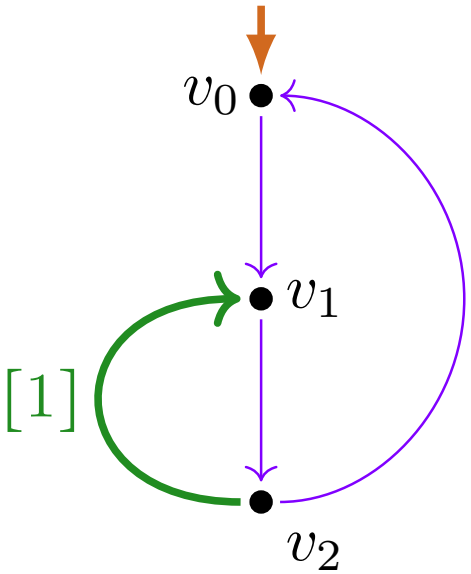
infinite $\rightarrow_{\mathsf{br}}$ path

from start vertex

LEE-witness ?





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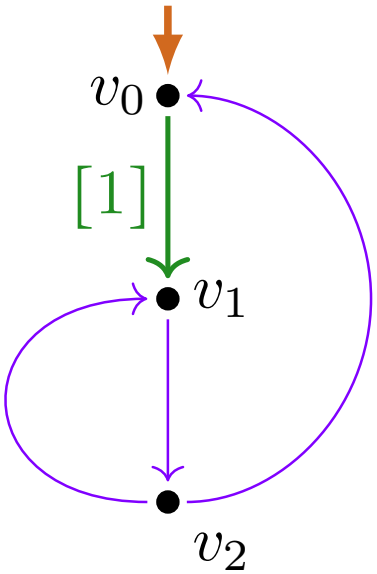

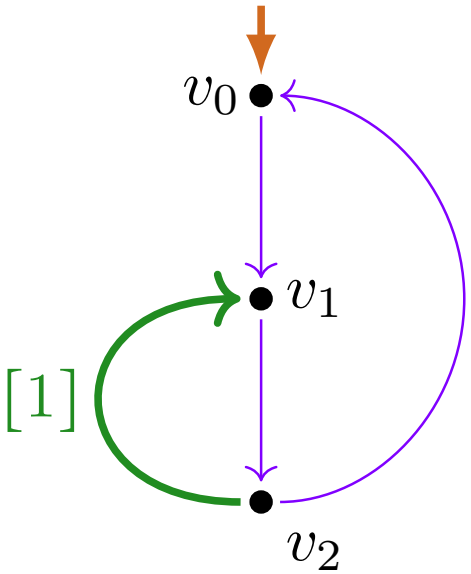

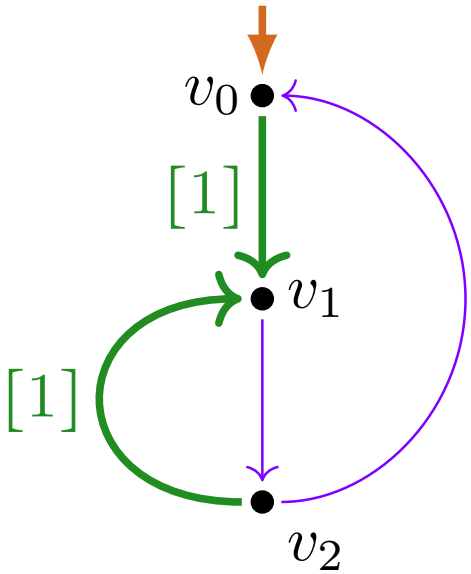

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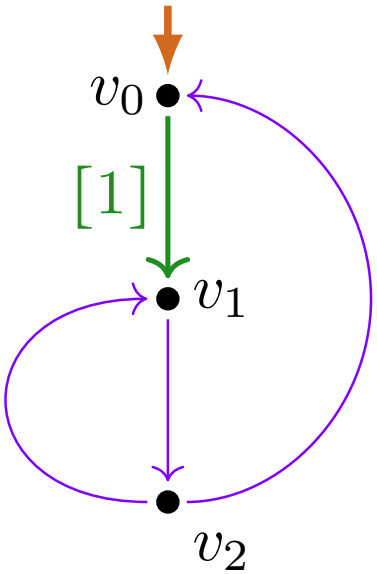

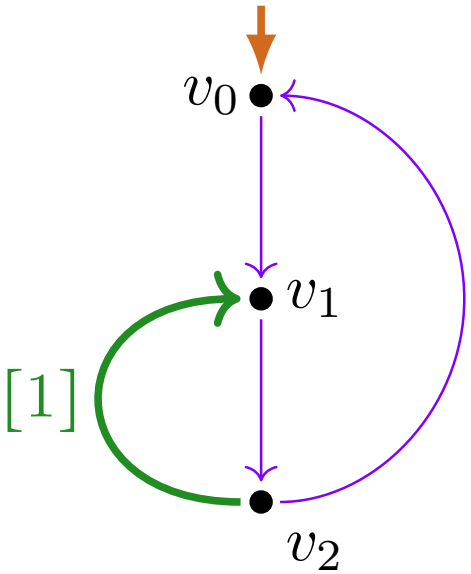

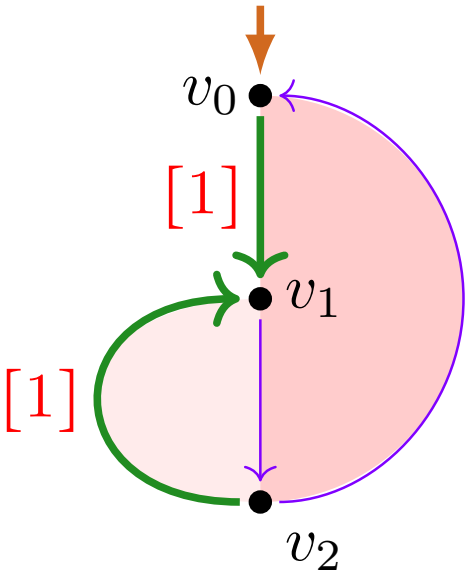

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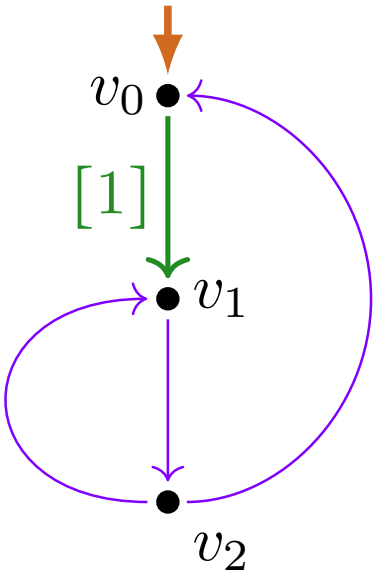

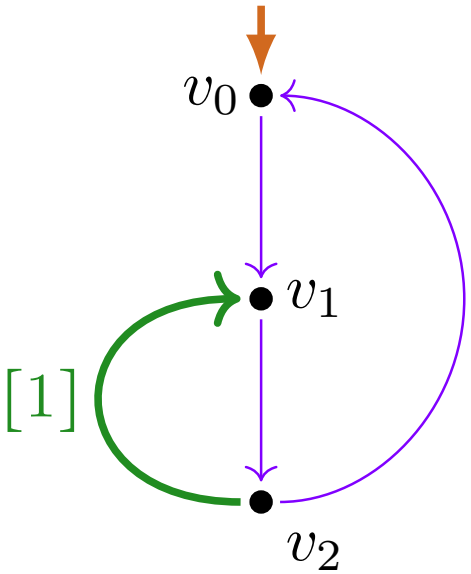

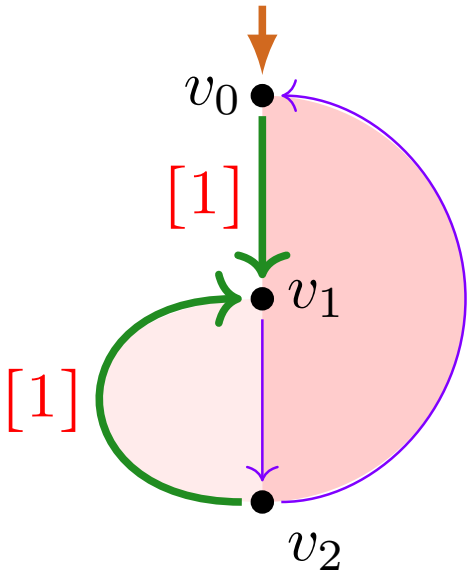

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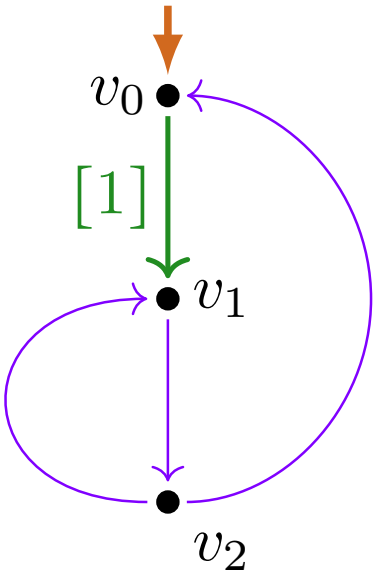
no!

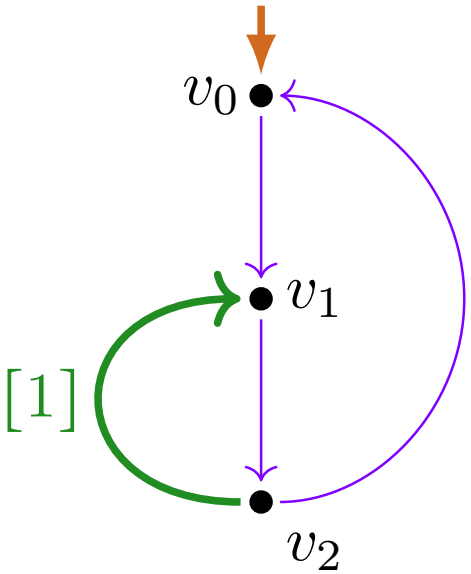
(L3.) violated:

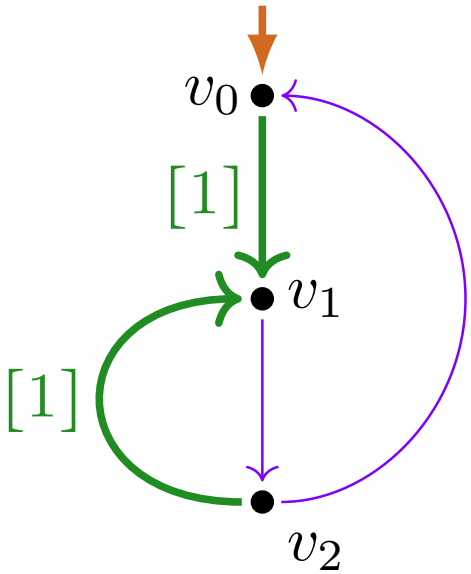
overlapping loop charts

have same level

LEE-witness ?







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LEE versus LEE-witness

Theorem

For every process graph G :

$$\mathsf{LEE}(G) \iff G \text{ has a LEE-witness.}$$

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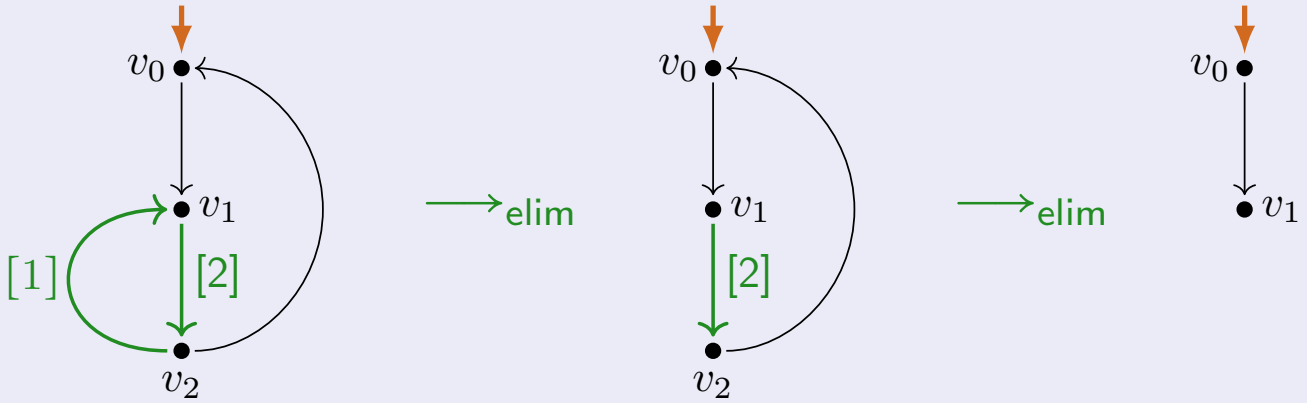
$\Leftarrow$: carry out loop-elimination as indicated in the LEE-witness, in *inside–out* direction, e.g.:



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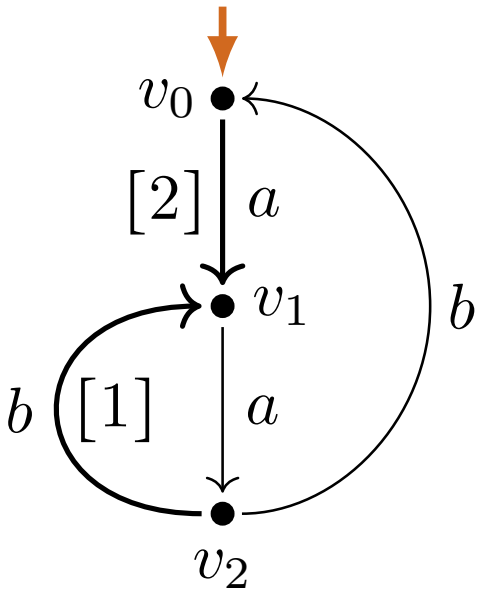

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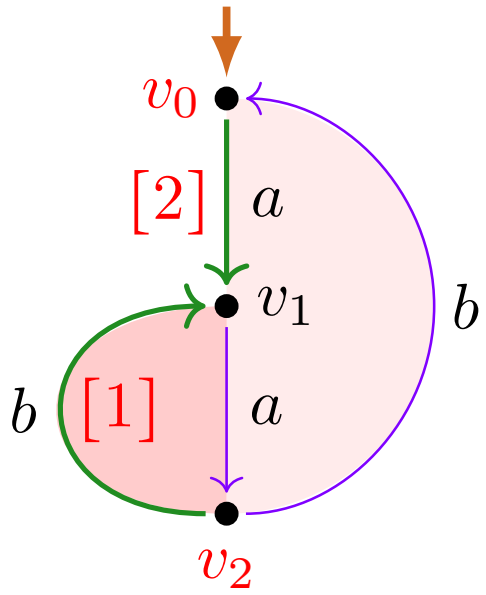

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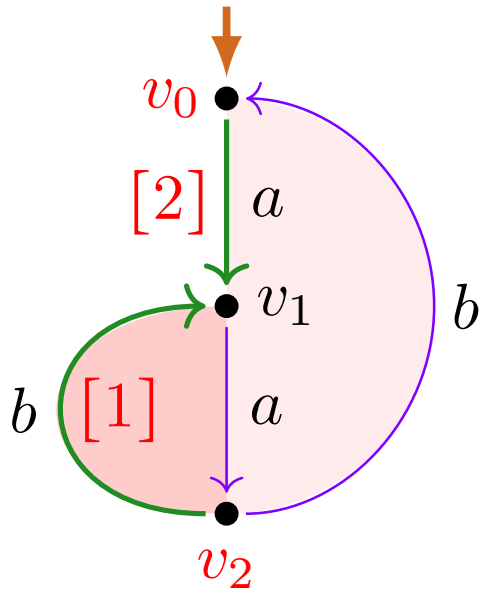
Layered LEE-witness



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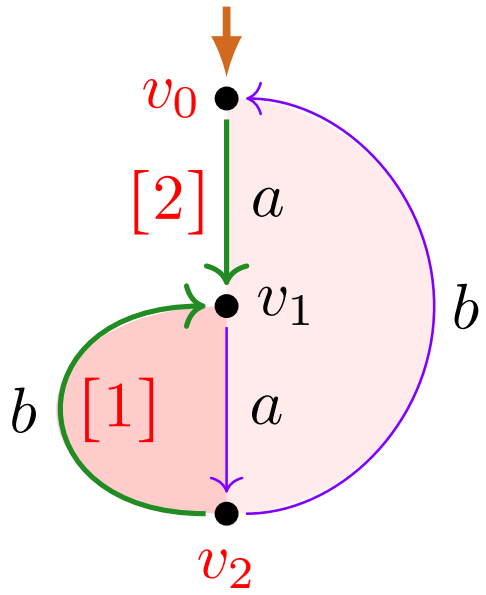

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I-L1. $\forall n \in \mathbb{N} \forall v \in V \Big(v \to_{[n]} \Rightarrow \mathcal{L}(v, \to_{[n]}, \to_{\mathsf{br}})$ is a loop subchart $\Big)$.

I-L2. No infinite $\to_{\mathsf{br}}$ path from start vertex.

I-L3. $\mathcal{L}(w_i, \to_{[n_i]}, \to_{\mathsf{br}})$ for $i \in \{1,2\}$ loop charts $\wedge\ w_1 \neq w_2 \ \wedge\ w_1 \in \mathcal{L}(w_2, \ldots, \ldots) \implies n_1 < n_2$.

$\mathcal{L}(v, \to_{[n]}, \to_{\mathsf{br}})$:= subchart induced
by entry steps $\to_{[n]}$ from v
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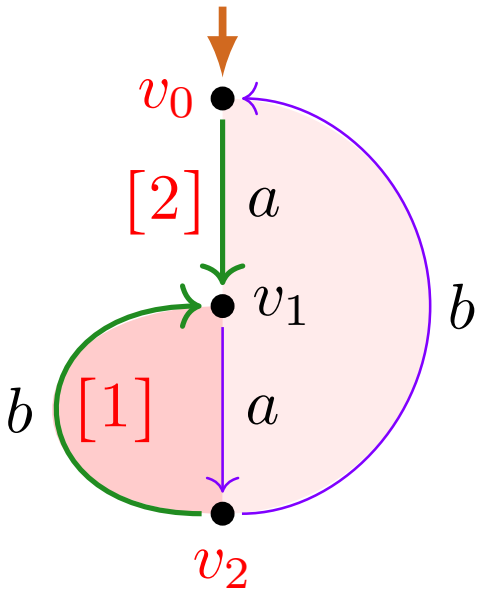
or entry steps $\to_{[m]}$ with $m > n$,
until v is reached again

Layered LEE-witness



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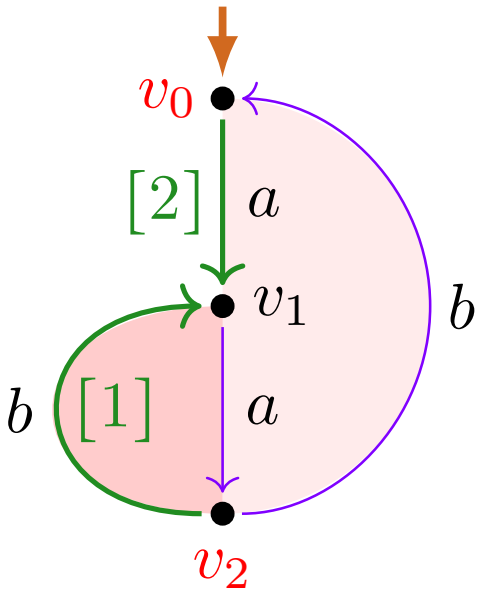

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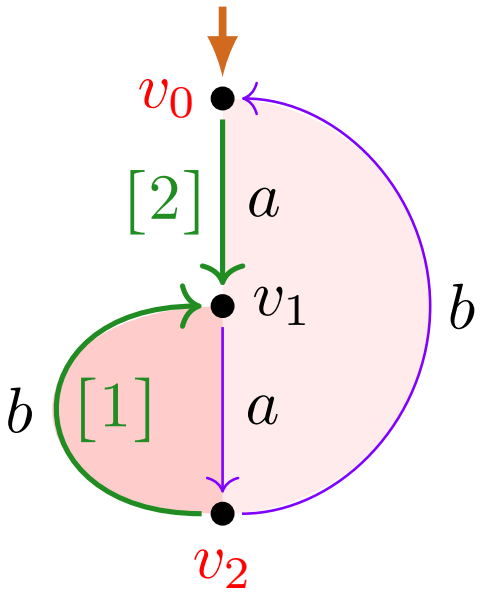

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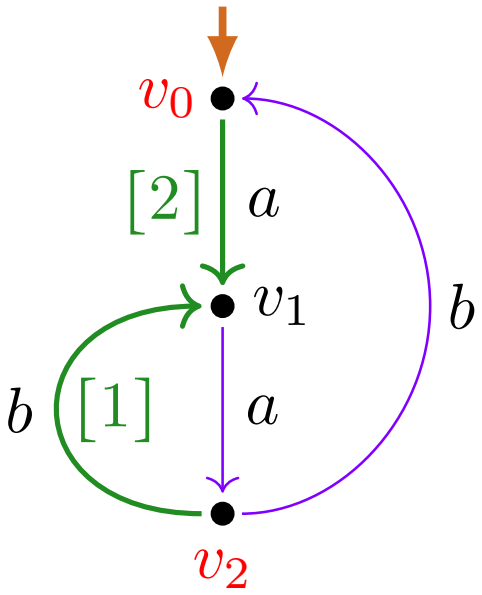

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LEE and (layered) LEE-witness

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Every layered LEE-witness is a LEE-witness.

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Every LEE-witness $\widehat{G}$ of a process graph G
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Lemma

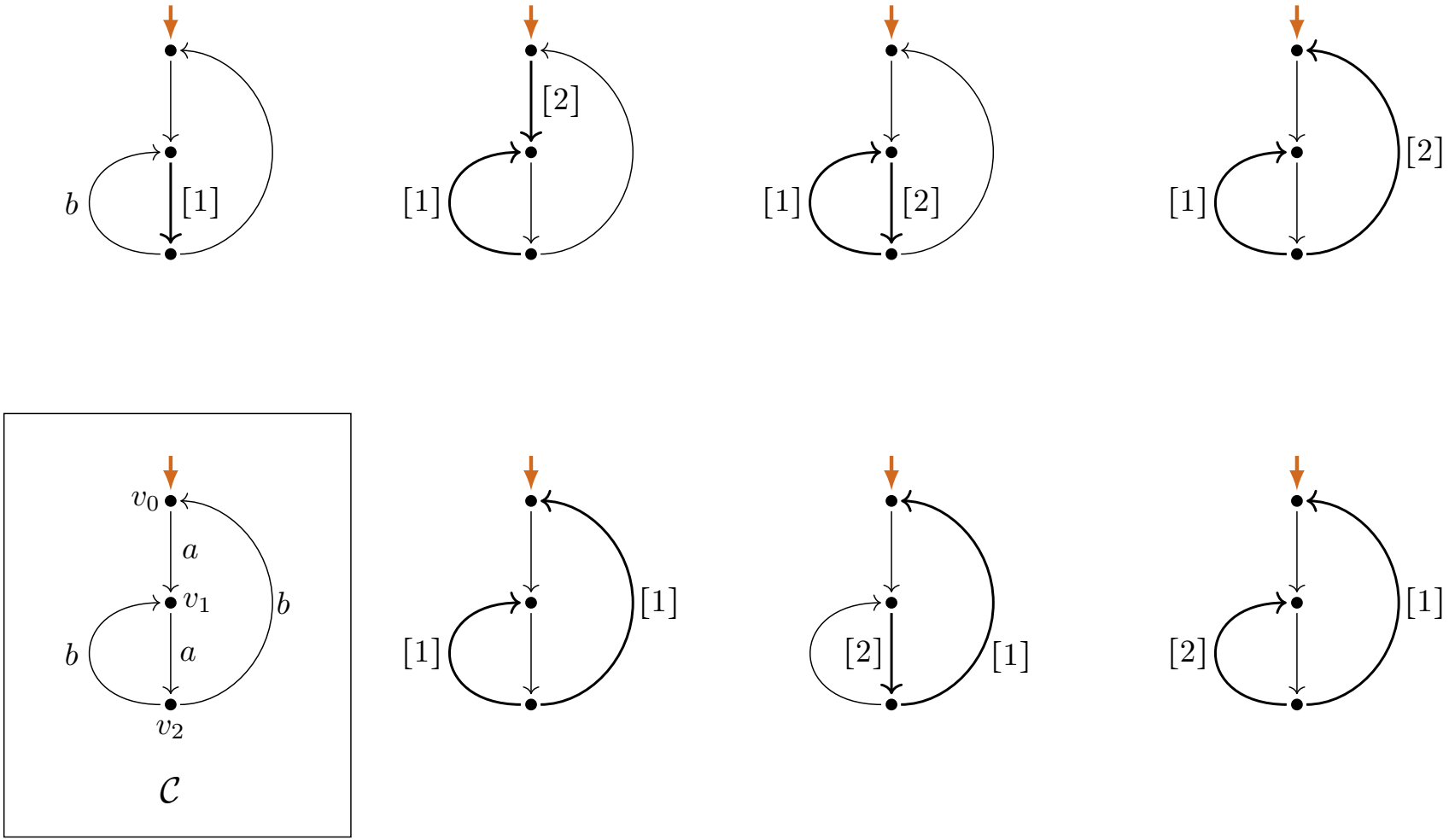
For every process graph G the following are equivalent:

(i) LEE(G).

(ii) G *has a LEE-witness.*

(iii) G *has a layered LEE-witness.*

7 LEE-witnesses



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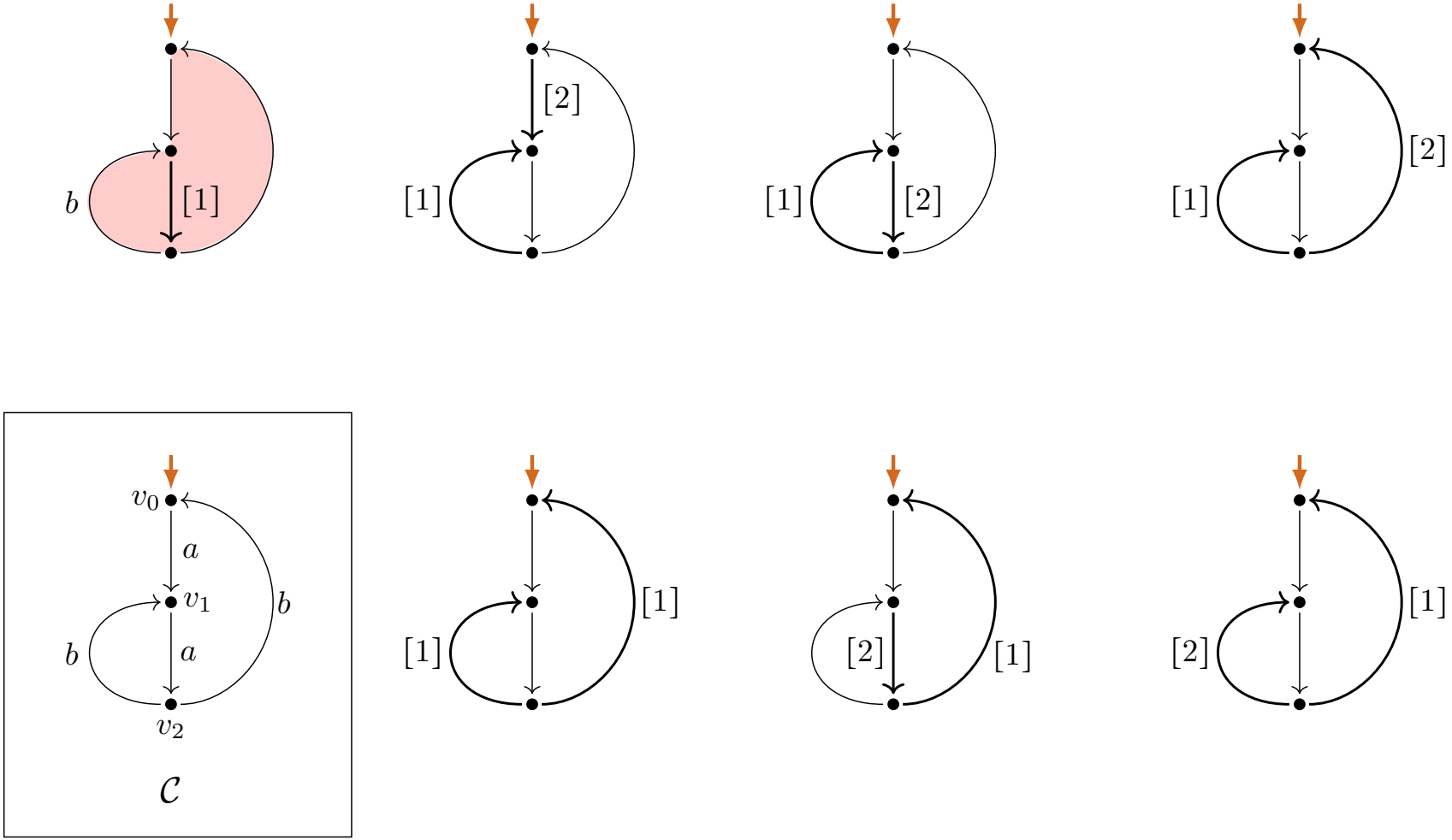

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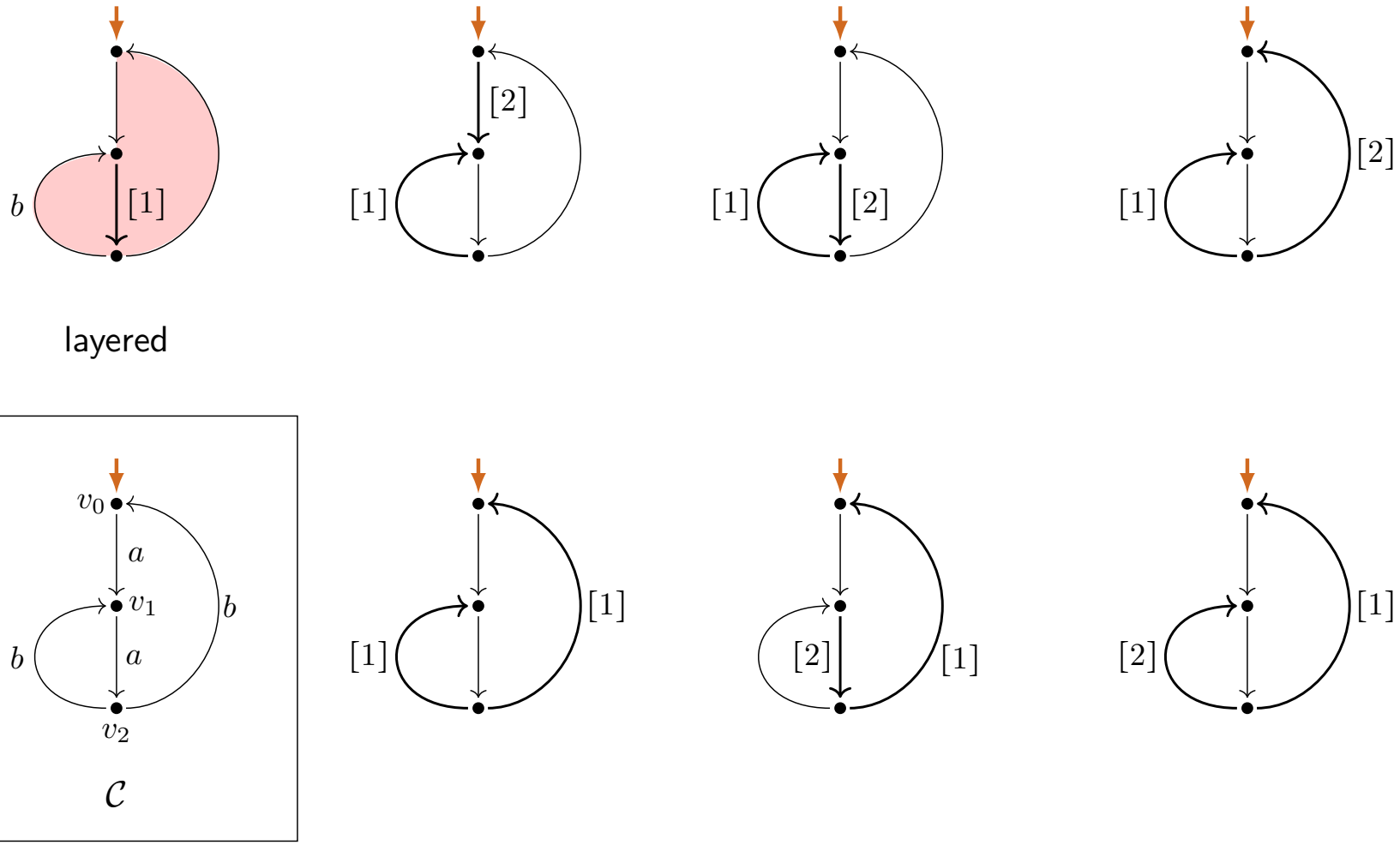

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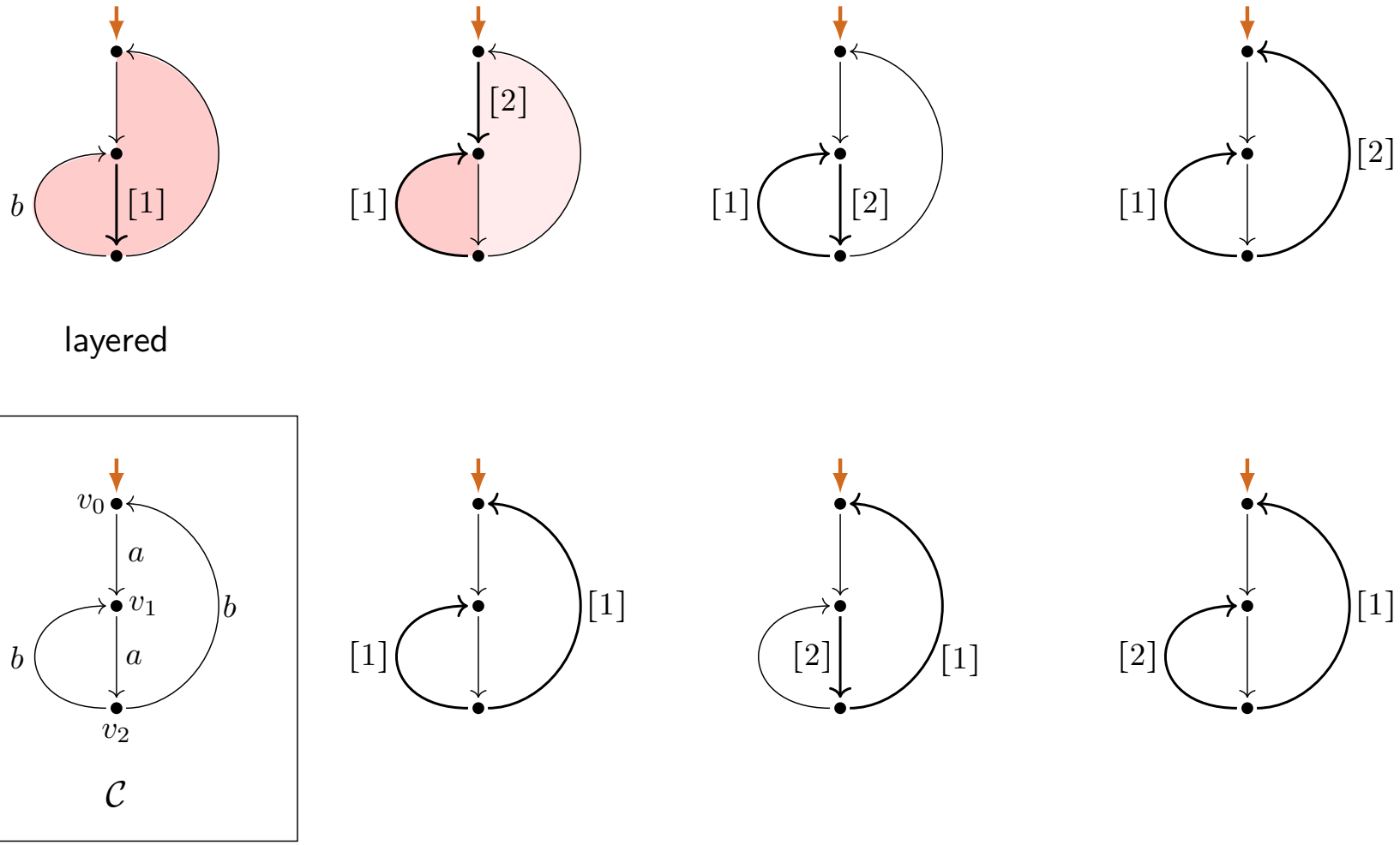

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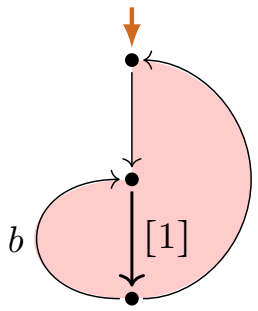

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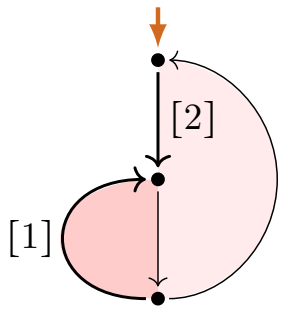

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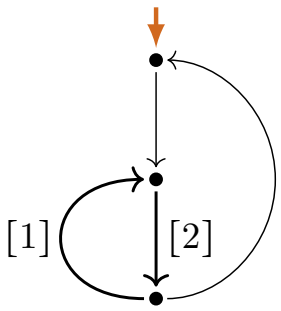

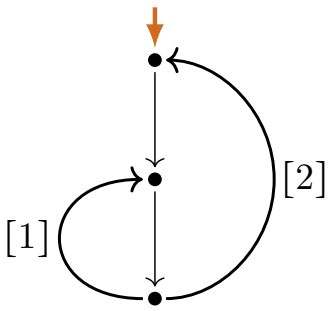

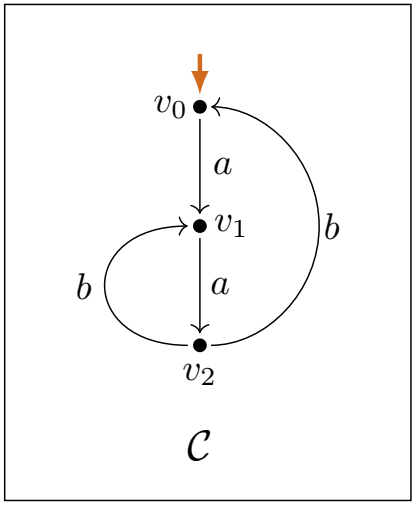

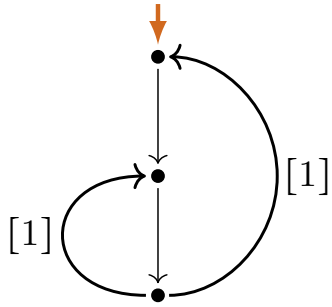

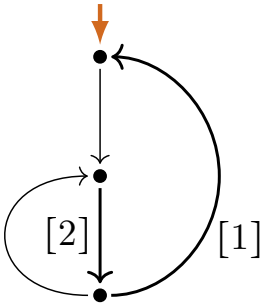

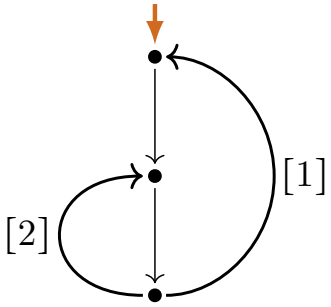

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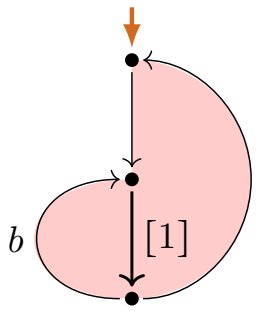

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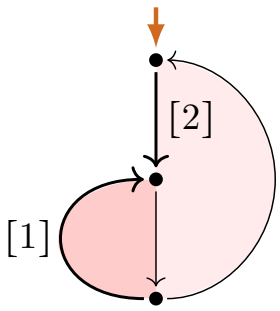

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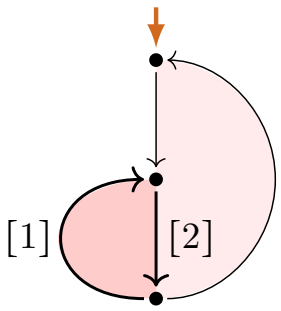

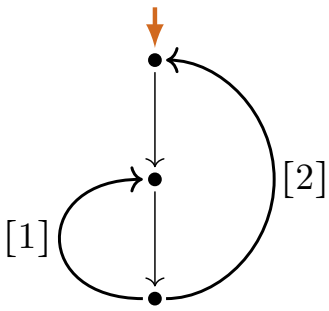

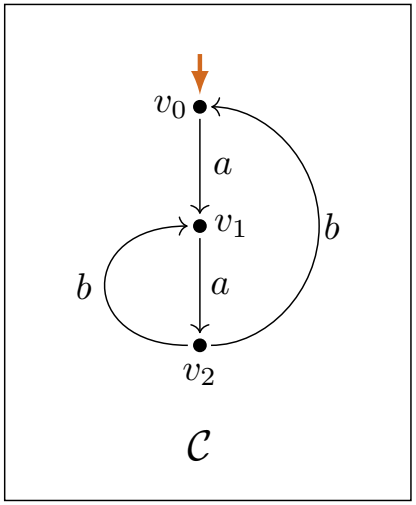

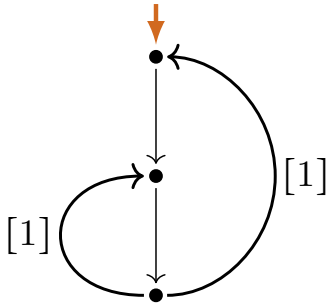

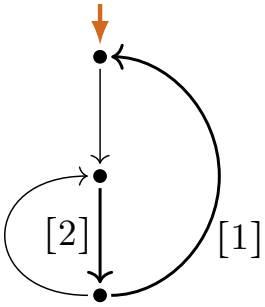

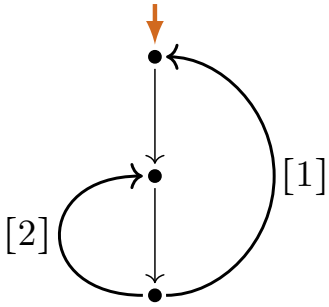

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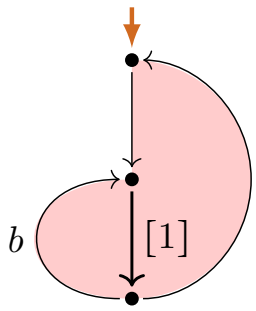

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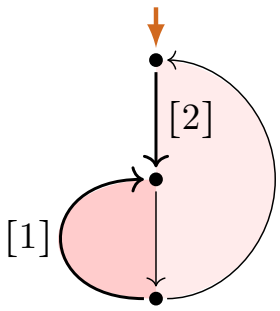

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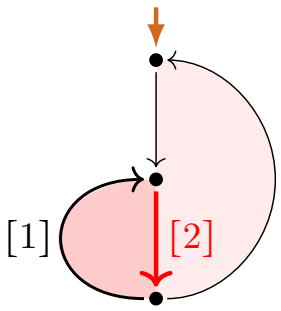

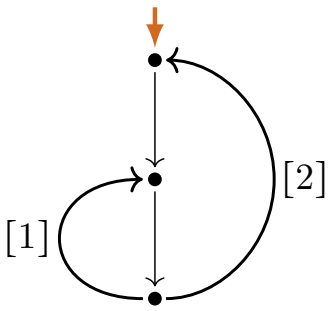

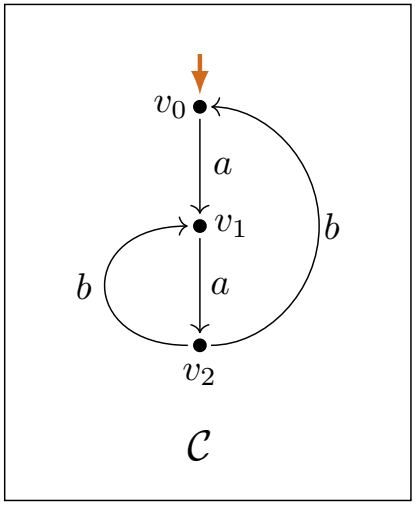

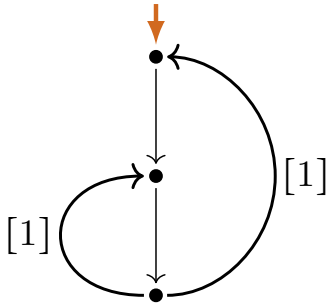

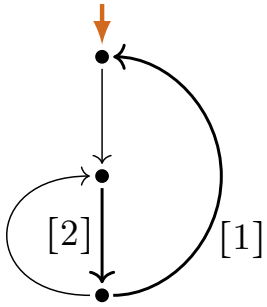

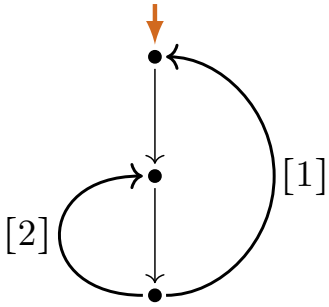

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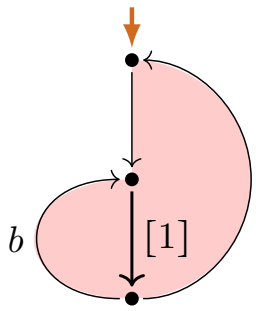

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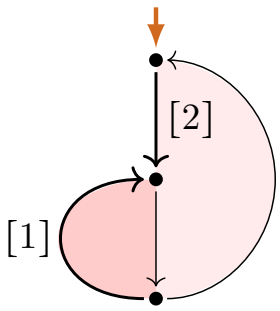

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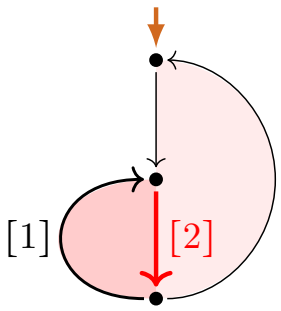

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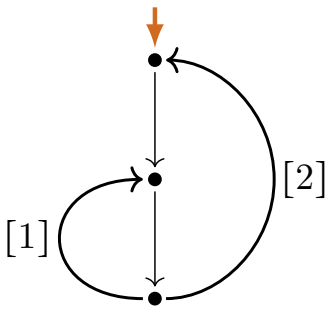

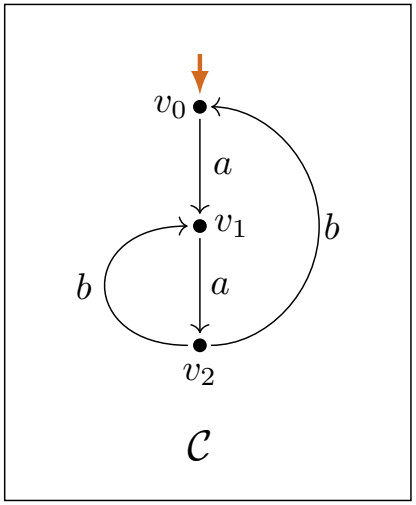

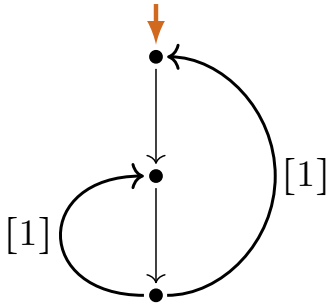

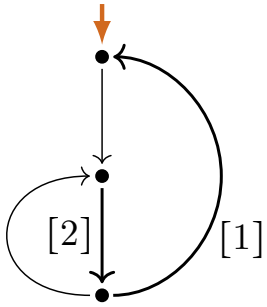

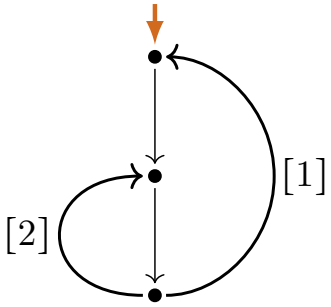

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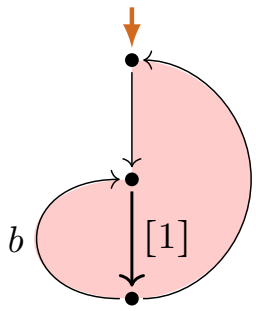

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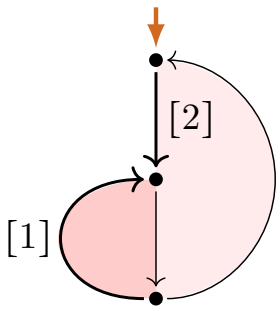

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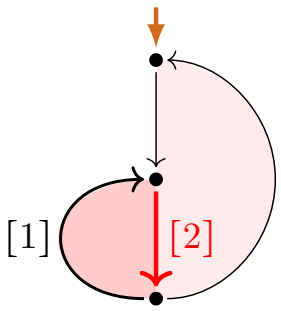

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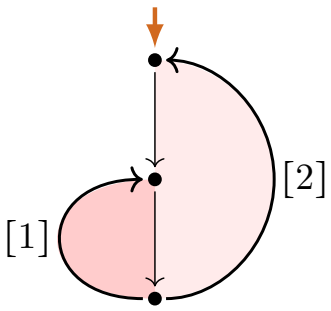
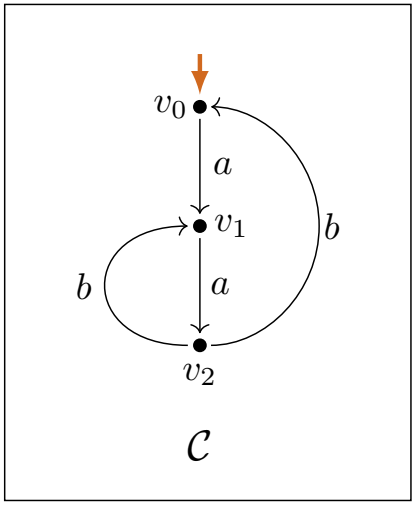
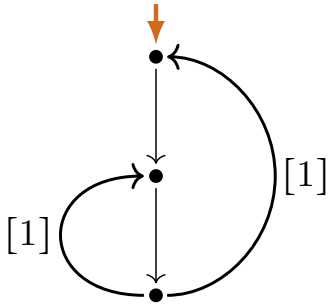
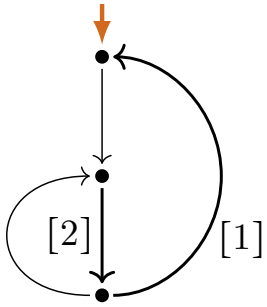
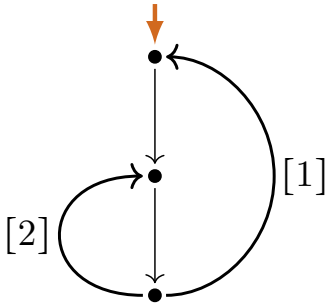


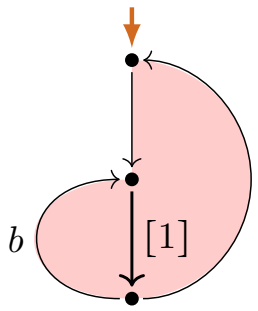








7 LEE-witnesses



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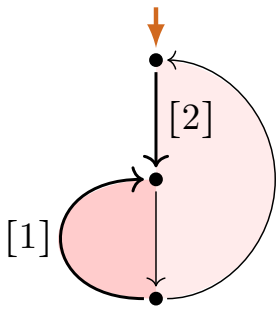

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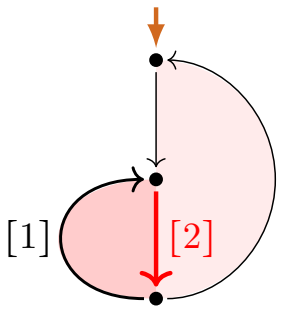

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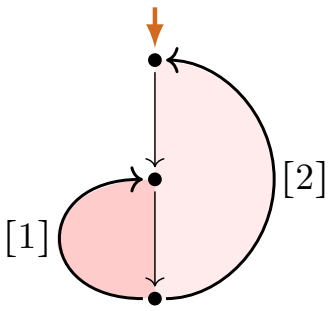

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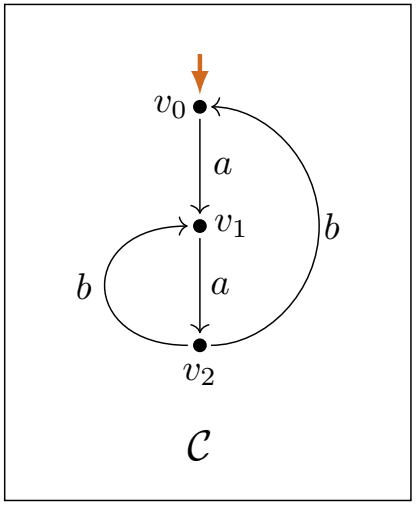

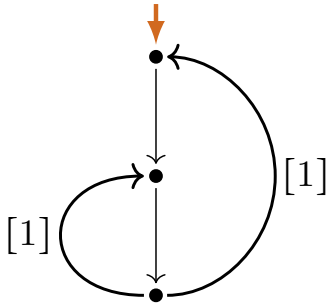

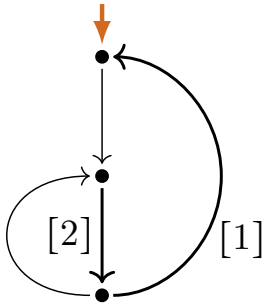

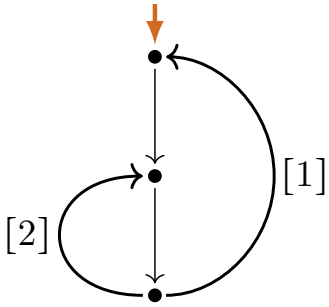

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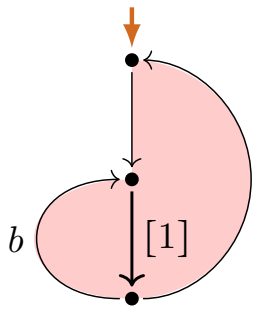

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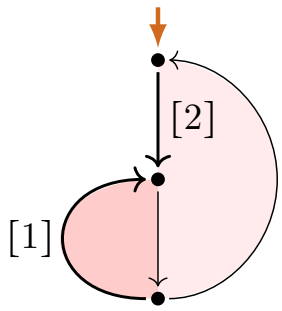

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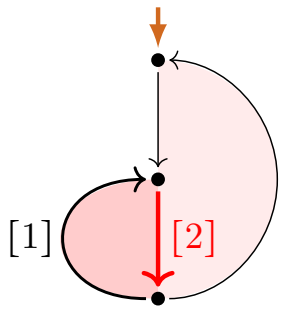

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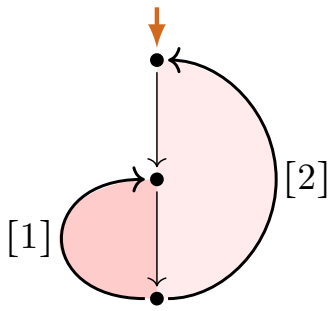

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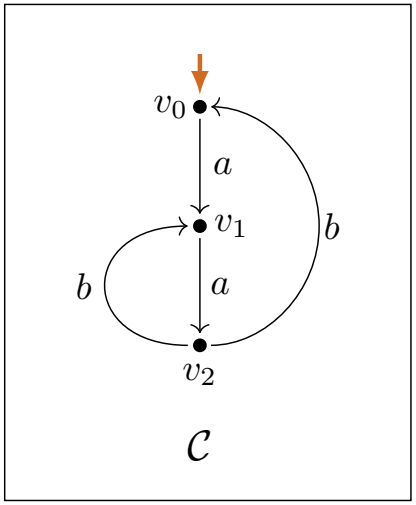

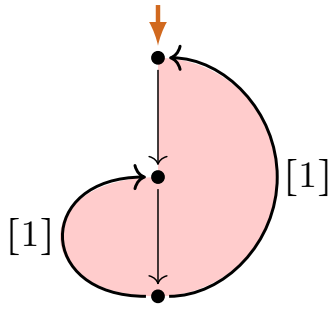

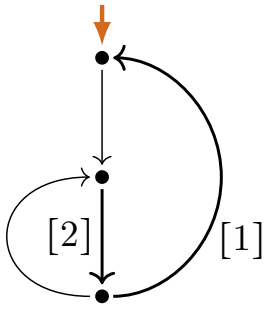

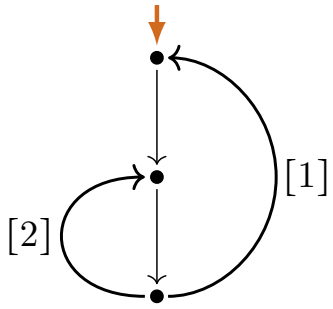

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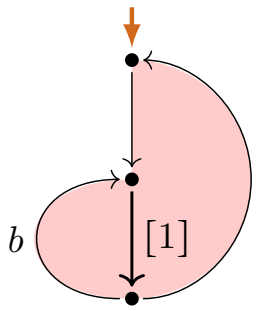

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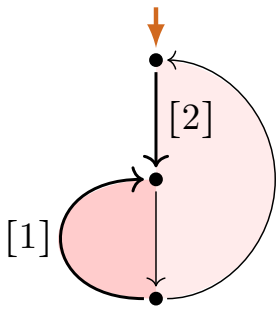

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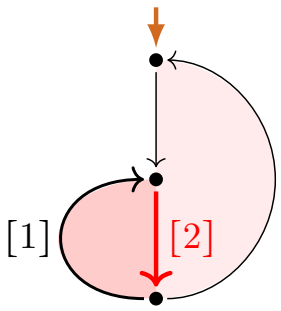

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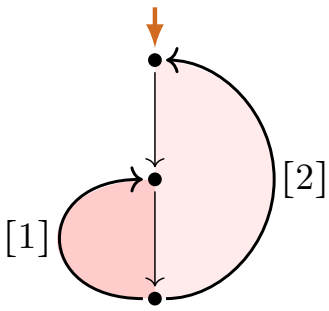

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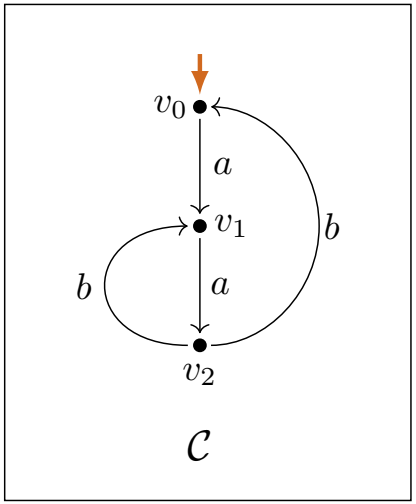

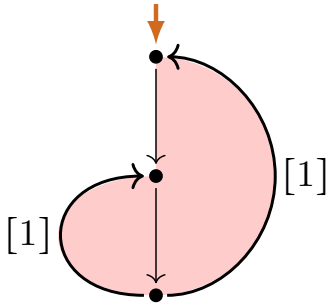

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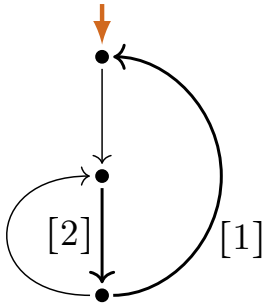

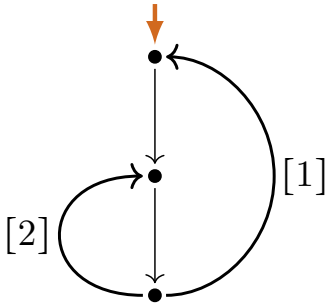

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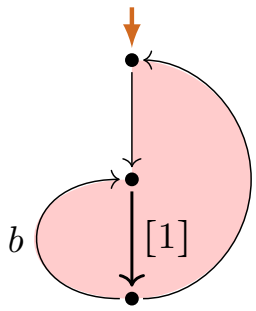

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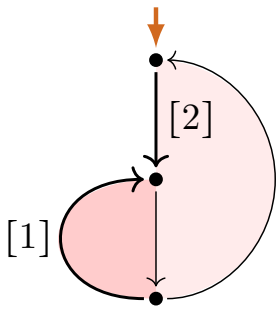

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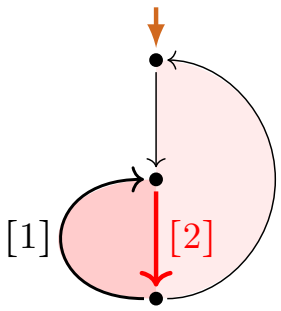

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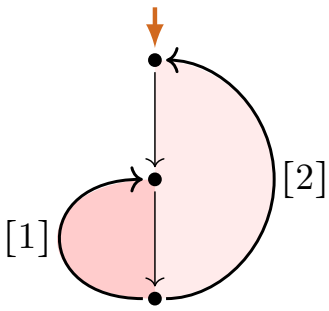

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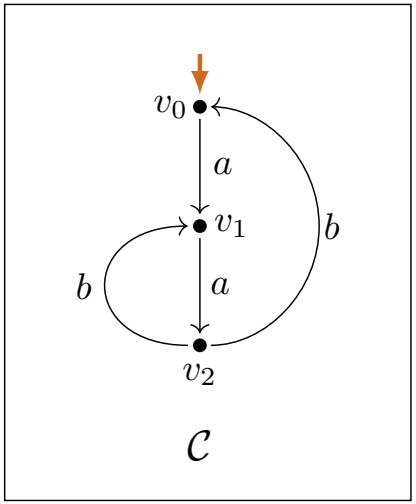

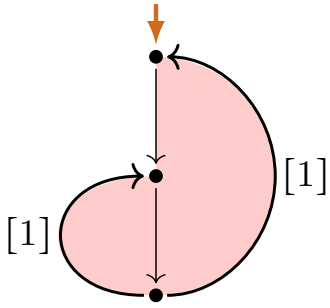

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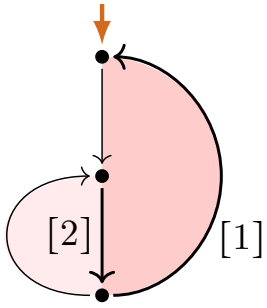

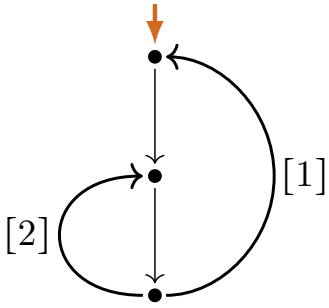

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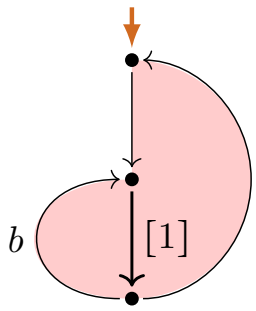

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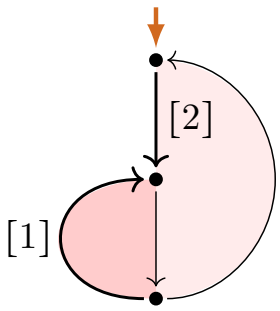

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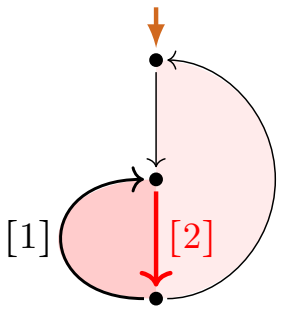

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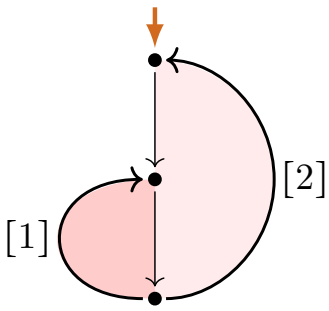

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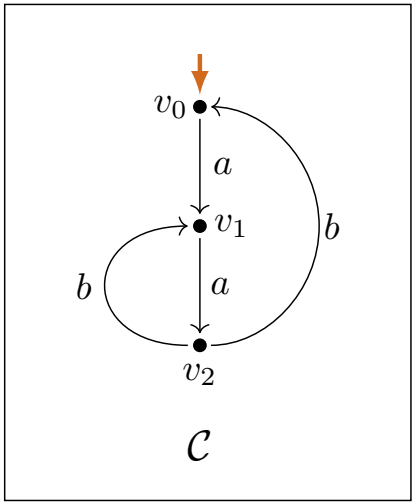

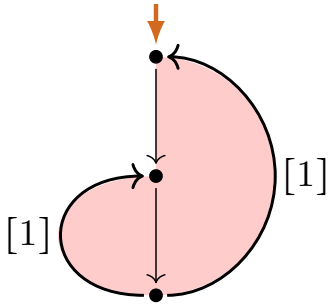

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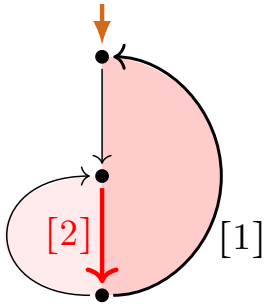

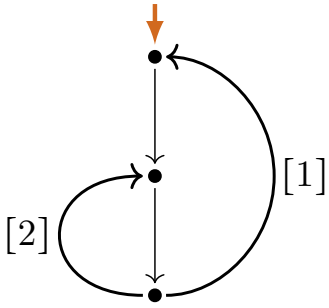

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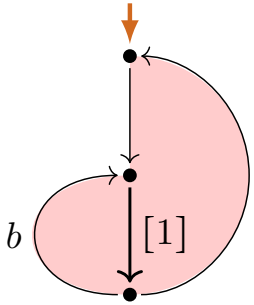

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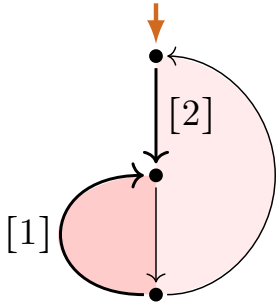

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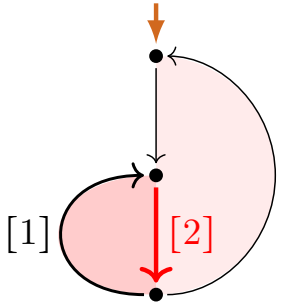

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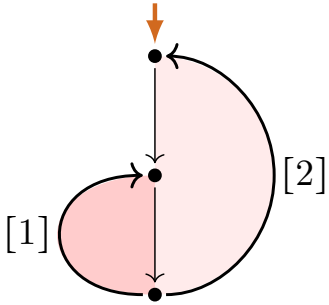

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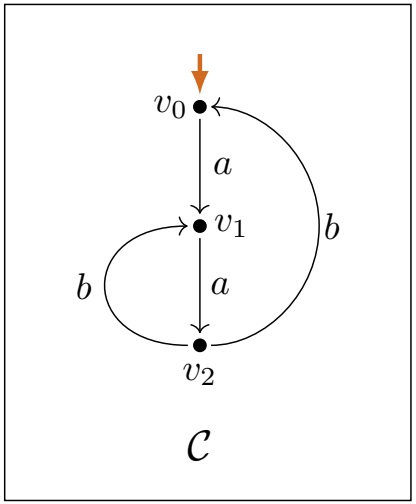

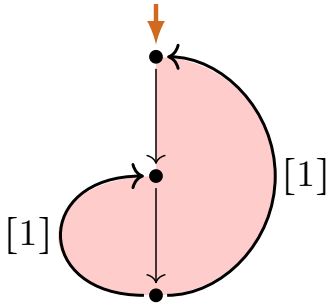

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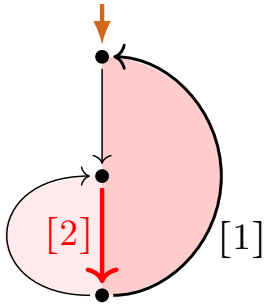

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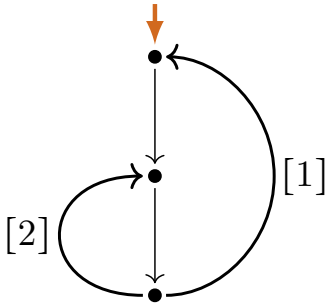

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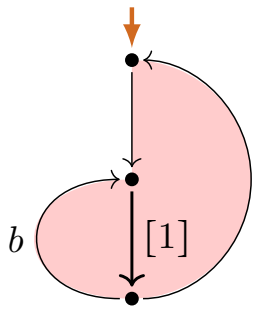

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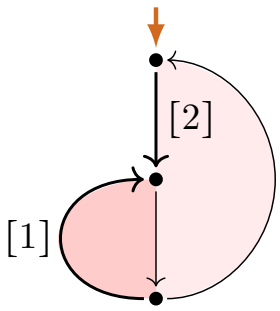

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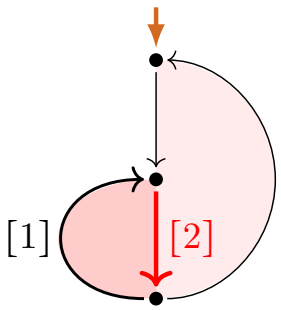

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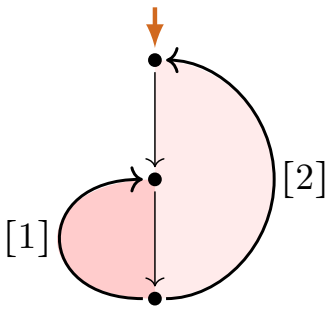

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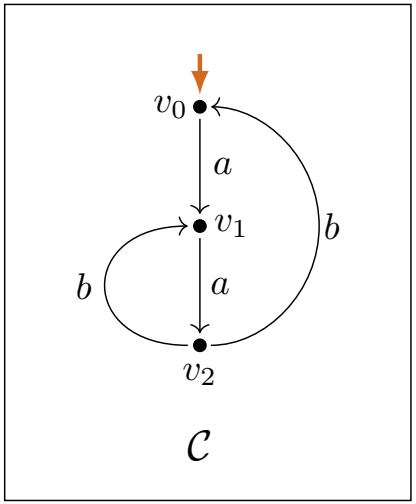

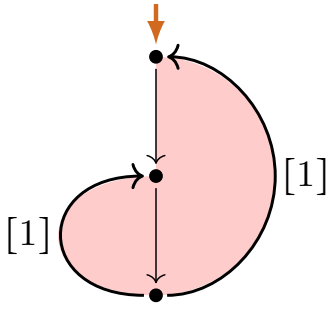

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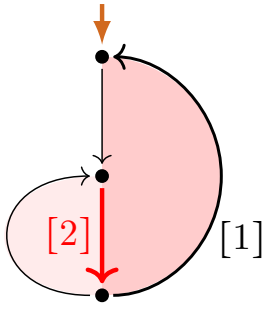

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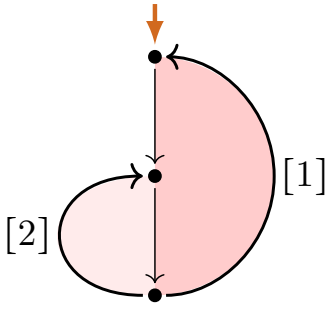
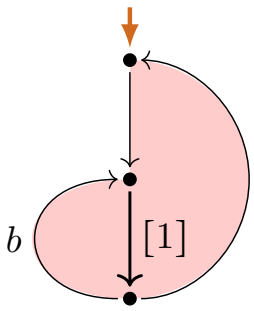


7 LEE-witnesses



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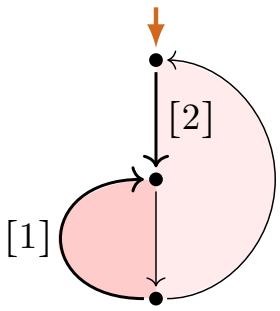

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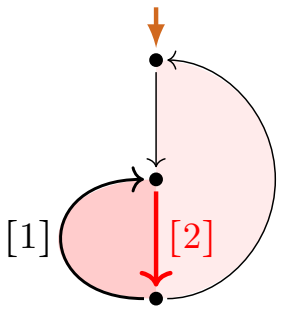

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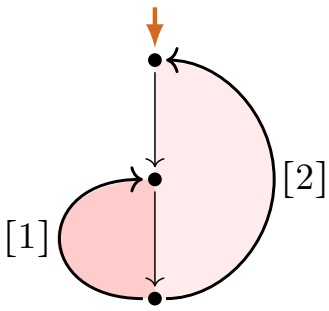

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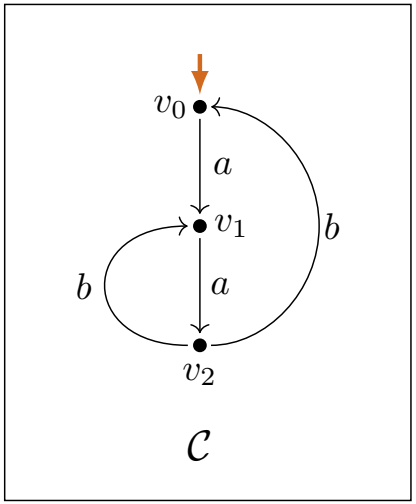

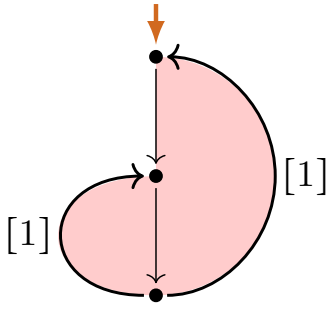

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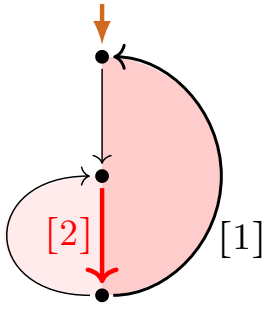

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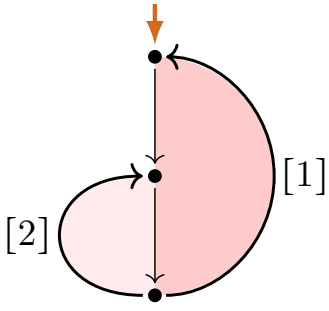

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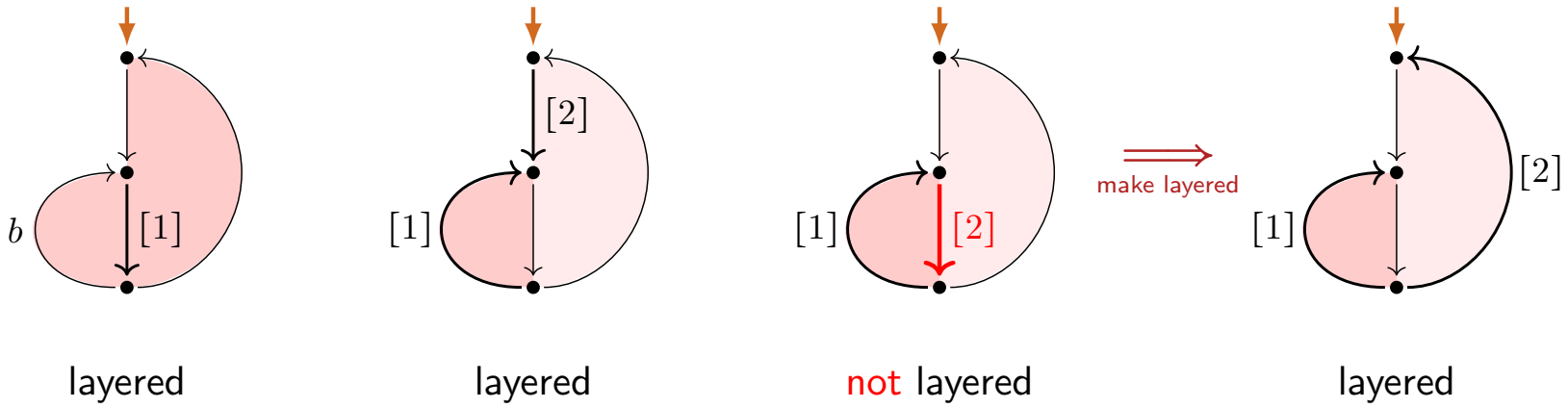
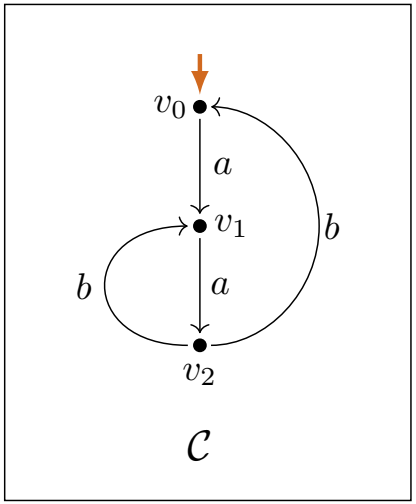
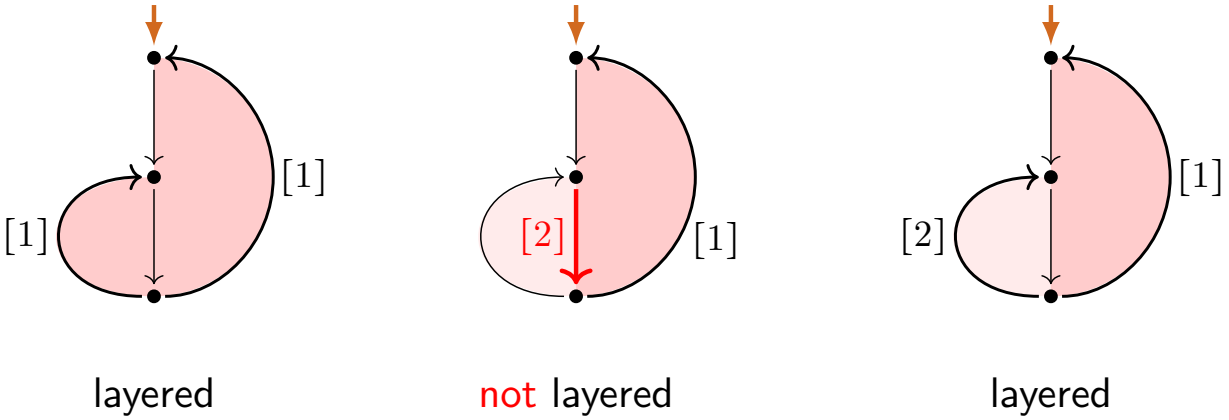
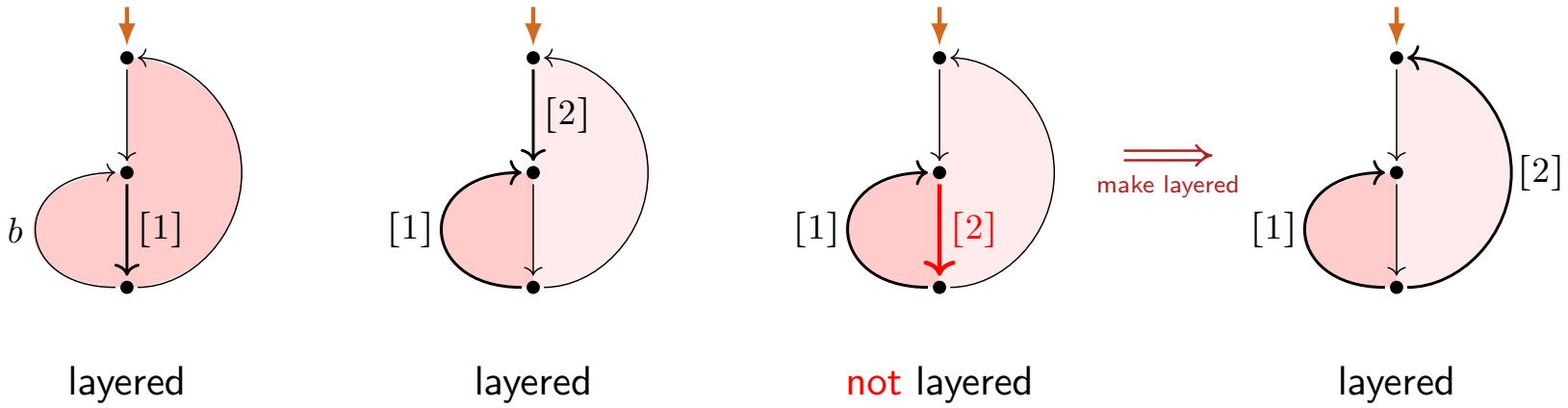
7 LEE-witnesses

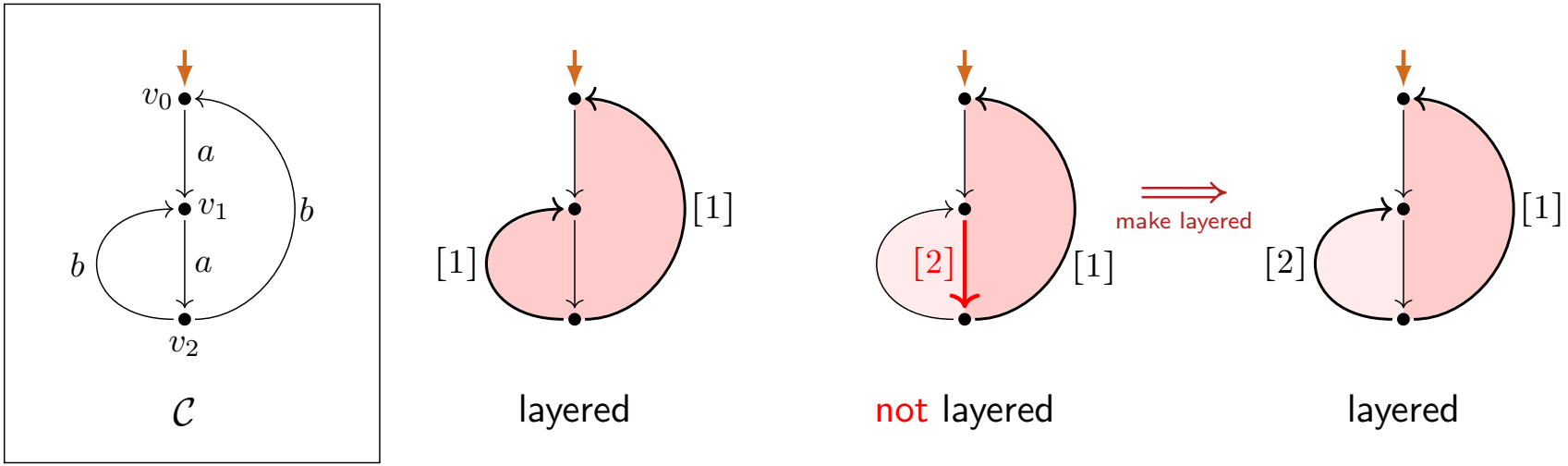




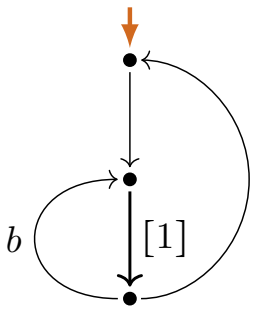


7 LEE-witnesses





7 LEE-witnesses



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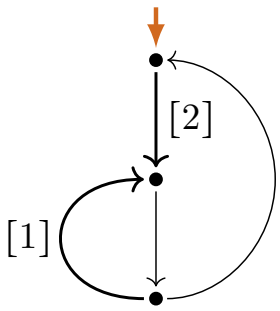

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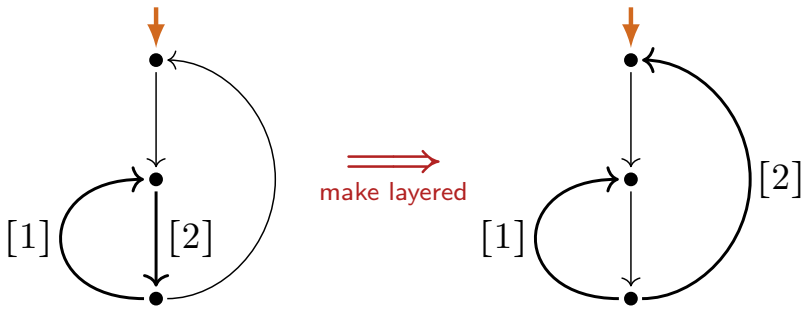

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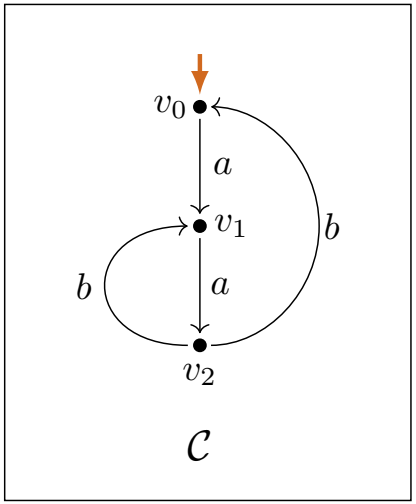

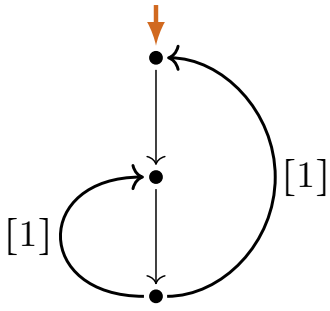

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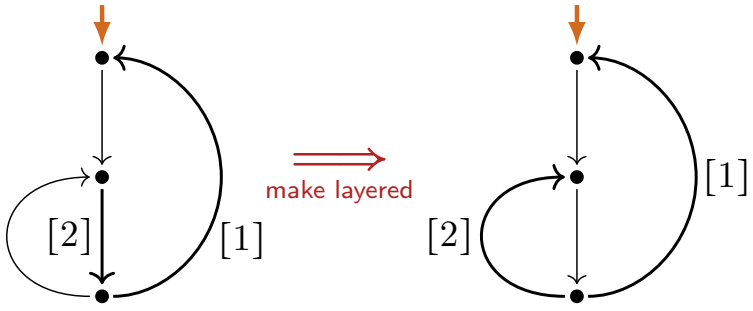

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LEE under bisimulation?

LEE under bisimulation

Observation

- LEE is not invariant under bisimulation.

LEE under bisimulation

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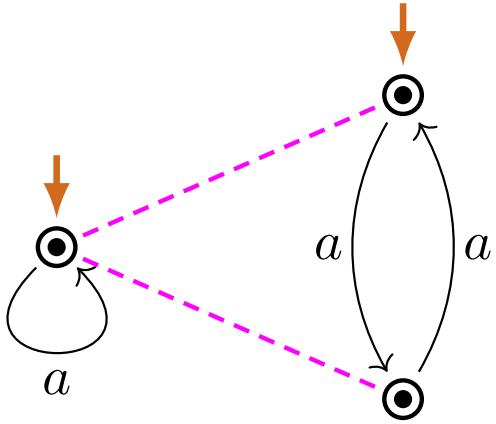
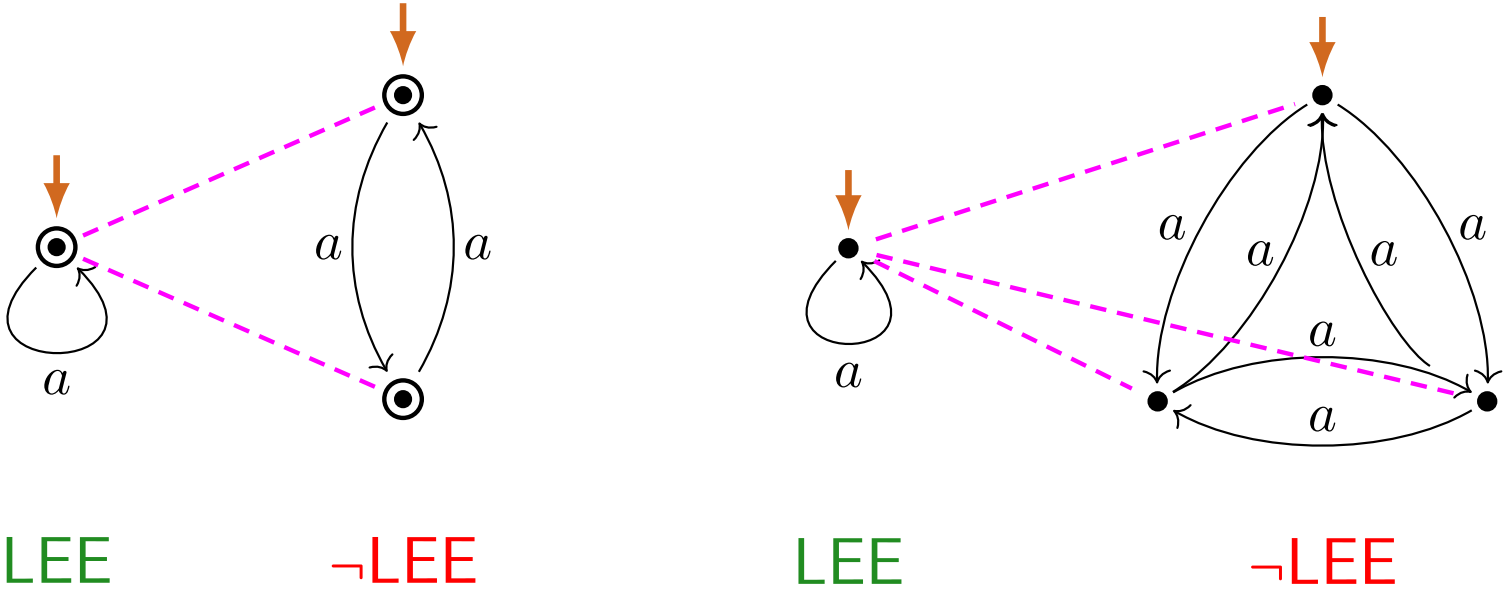


LEE ¬LEE

LEE under bisimulation

Observation

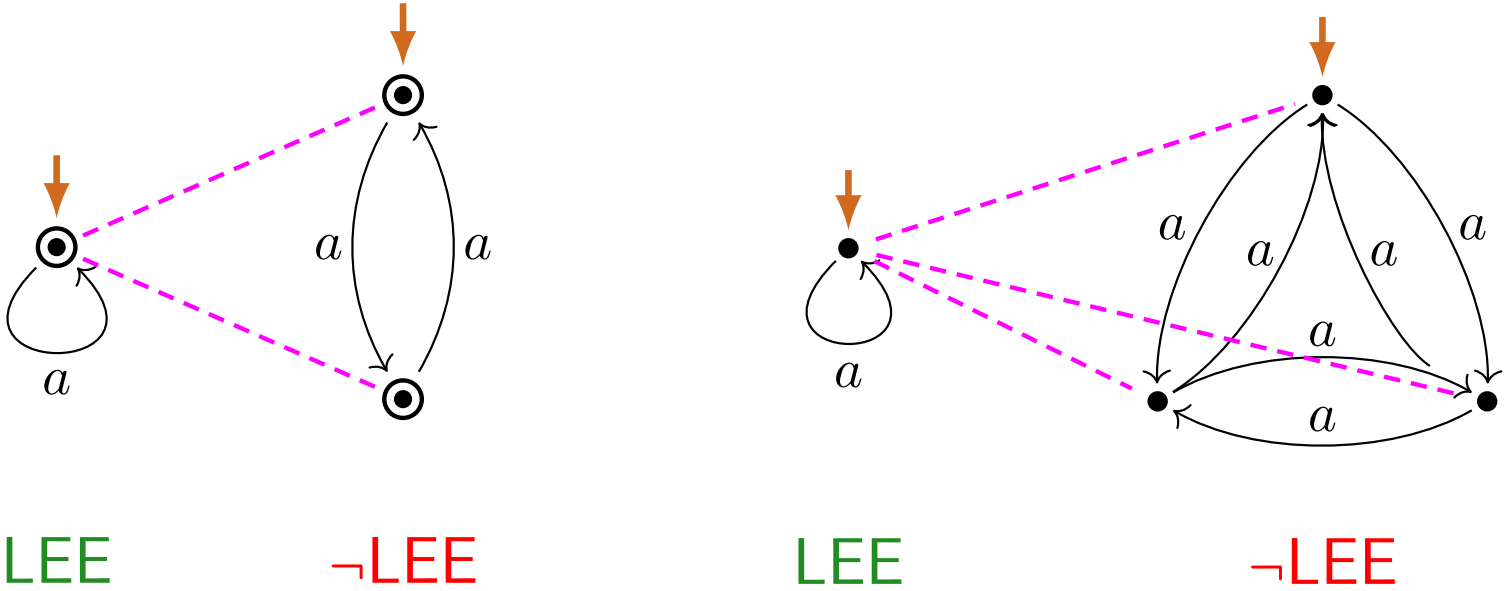
- LEE is not invariant under bisimulation.



LEE under bisimulation

Observation

- LEE is not invariant under bisimulation.
- LEE is not preserved by converse functional bisimulation.



LEE under functional bisimulation

Lemma

(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \mathrel{\underline{\to}} G_2 \implies \mathsf{LEE}(G_2)\ .$$

LEE under functional bisimulation

Lemma

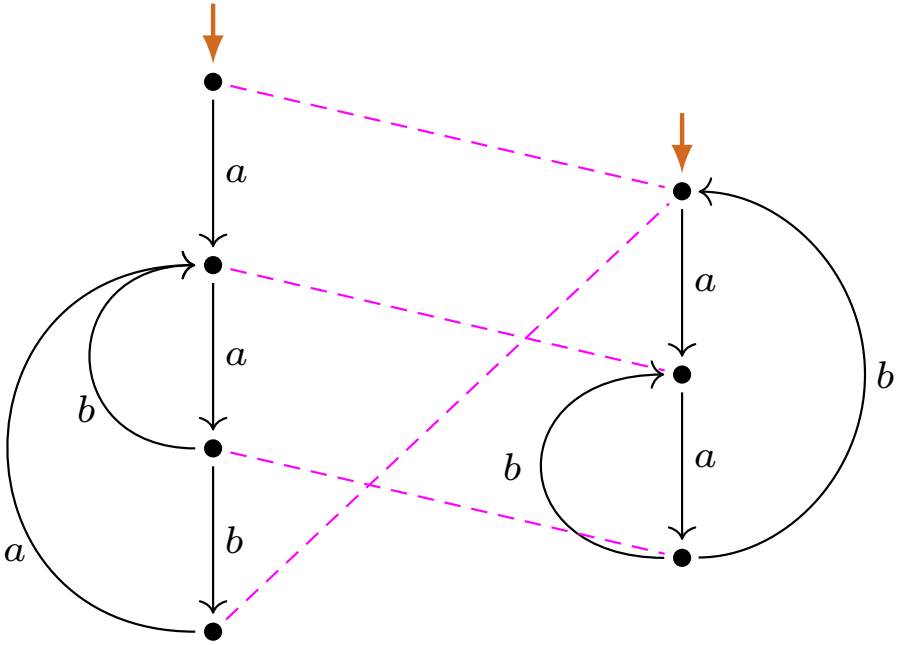
(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \mathrel{\underline{\to}} G_2 \implies \mathsf{LEE}(G_2)\ .$$

Proof (Idea).

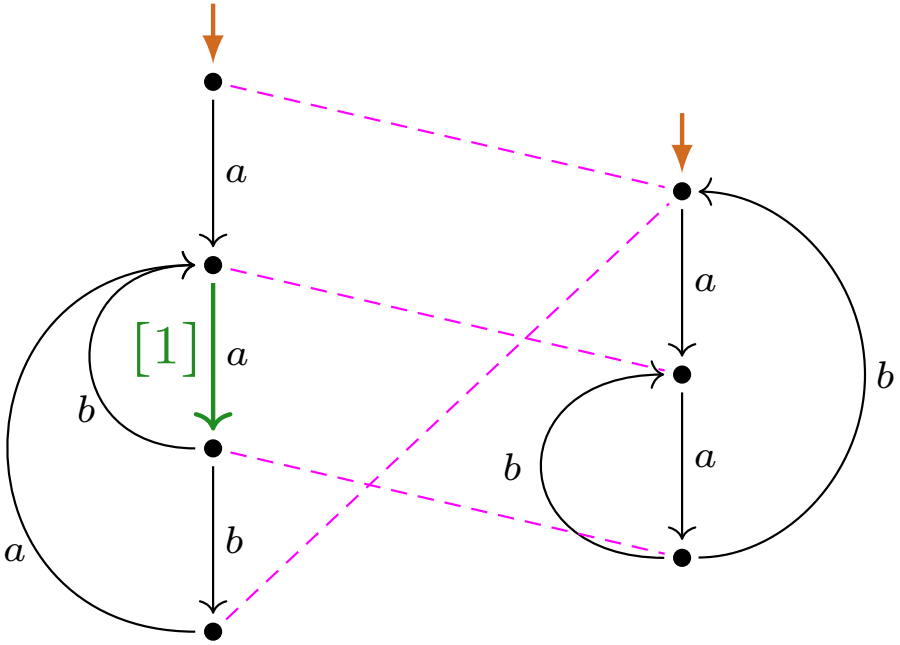
Use loop elimination in G_1 to carry out loop elimination in G_2.

Collapsing LEE-witnesses



$P(a(a(b+ba))^*0)$

Collapsing LEE-witnesses



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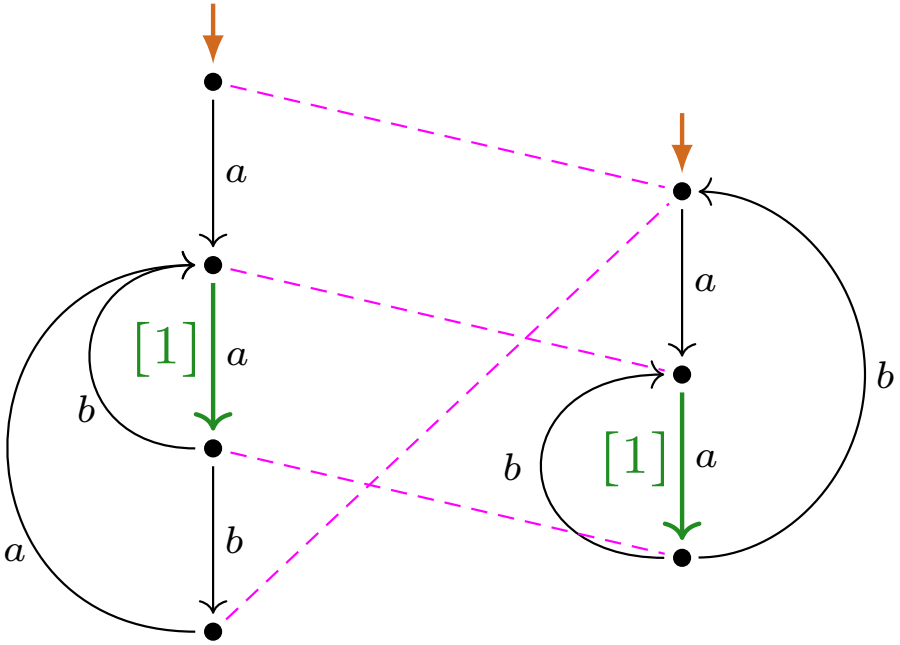
Collapsing LEE-witnesses



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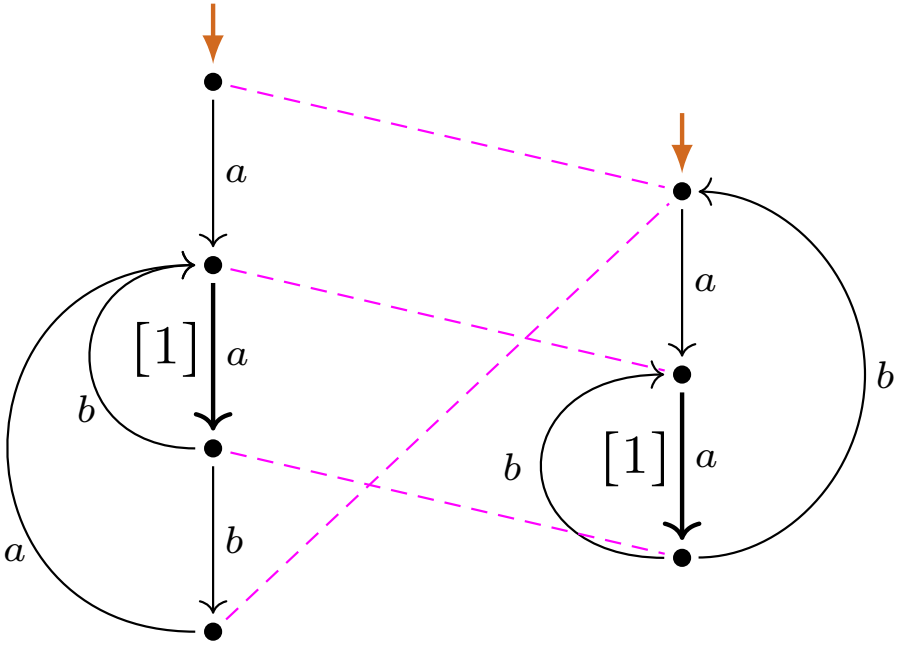
Collapsing LEE-witnesses



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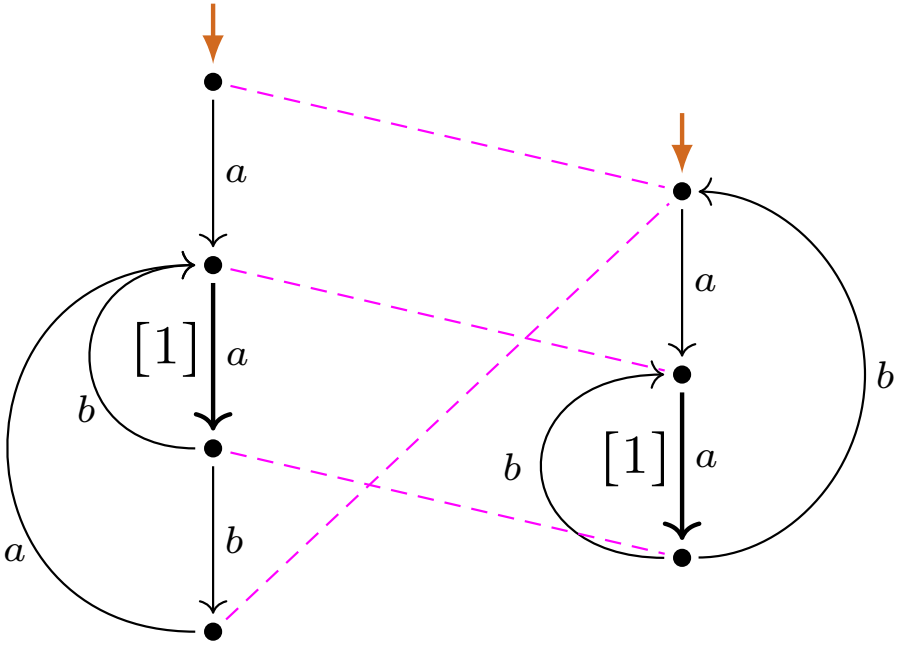
Collapsing LEE-witnesses



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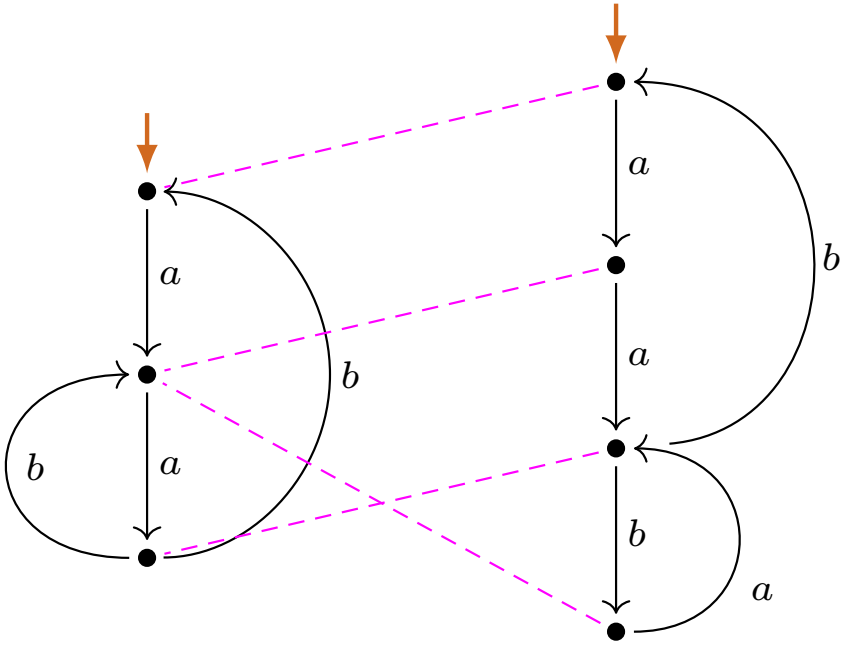
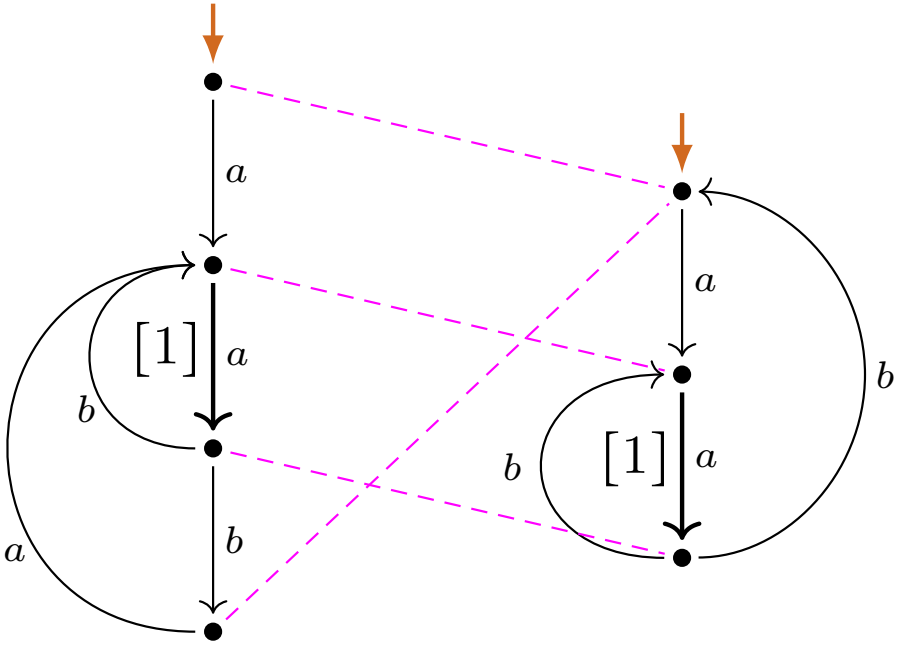


$P((aa(ba)^*b)^*0)$

Collapsing LEE-witnesses



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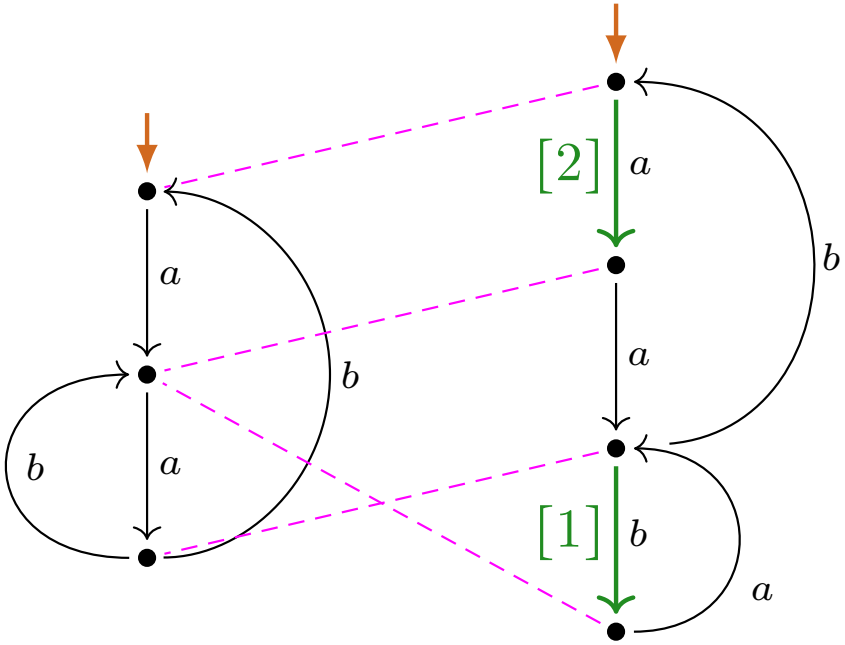

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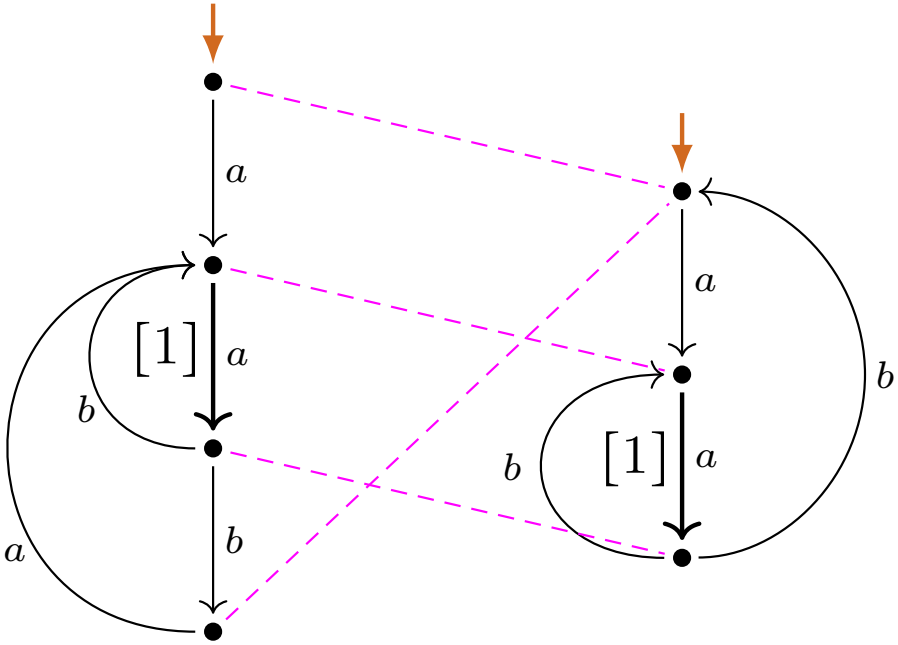
Collapsing LEE-witnesses



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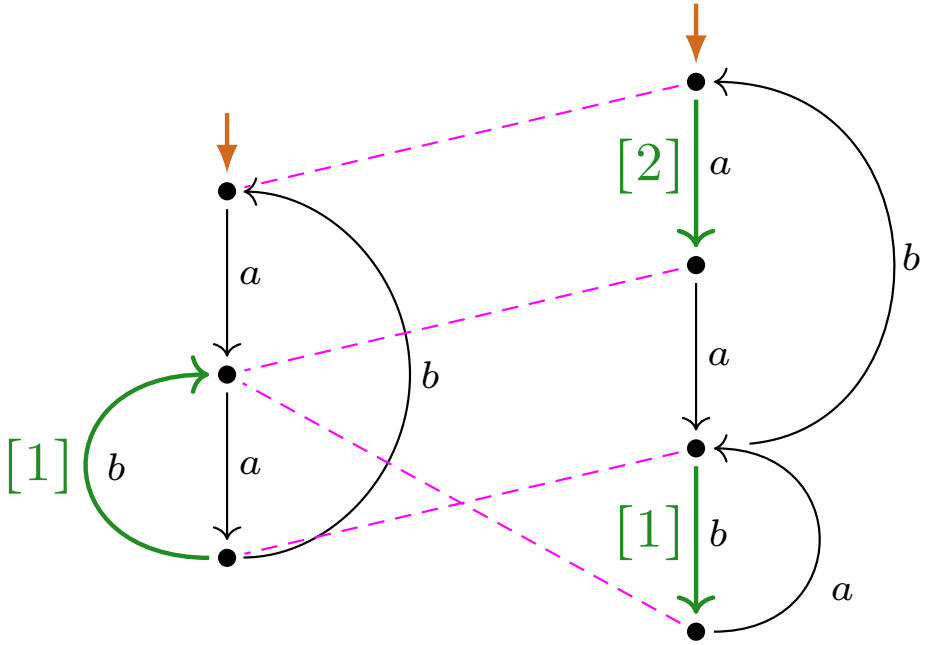

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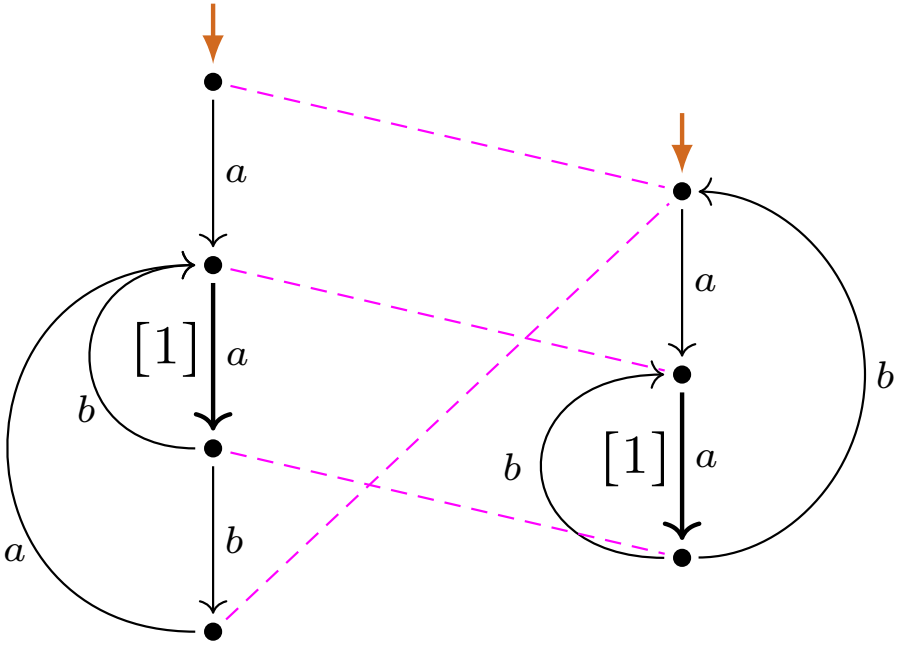
Collapsing LEE-witnesses



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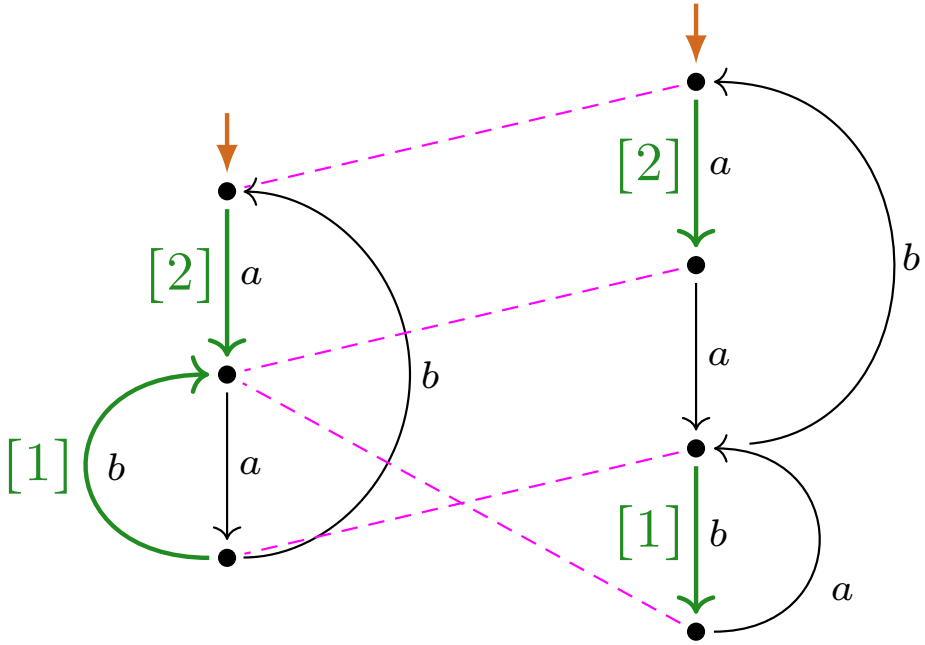

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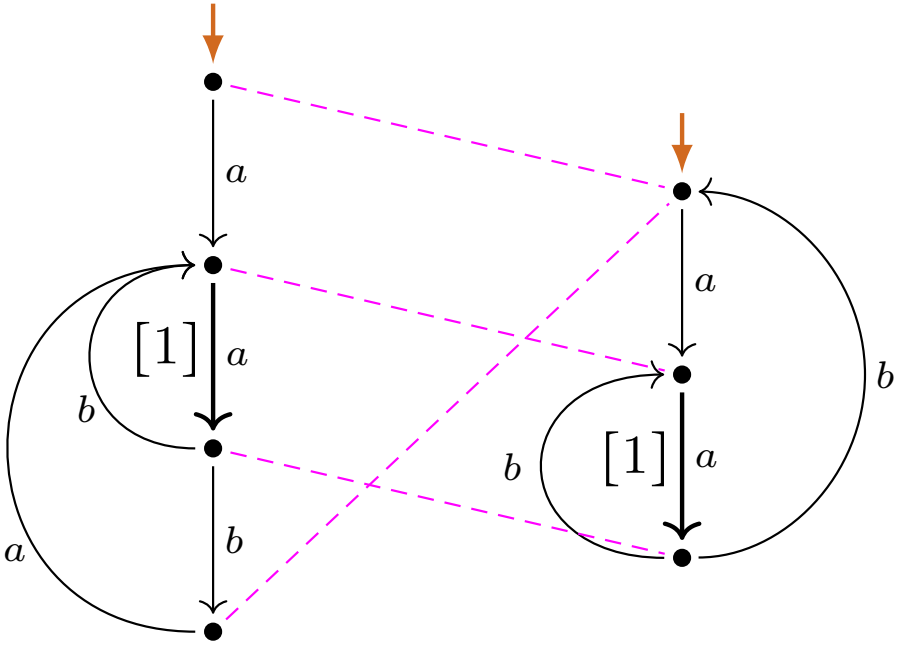
Collapsing LEE-witnesses



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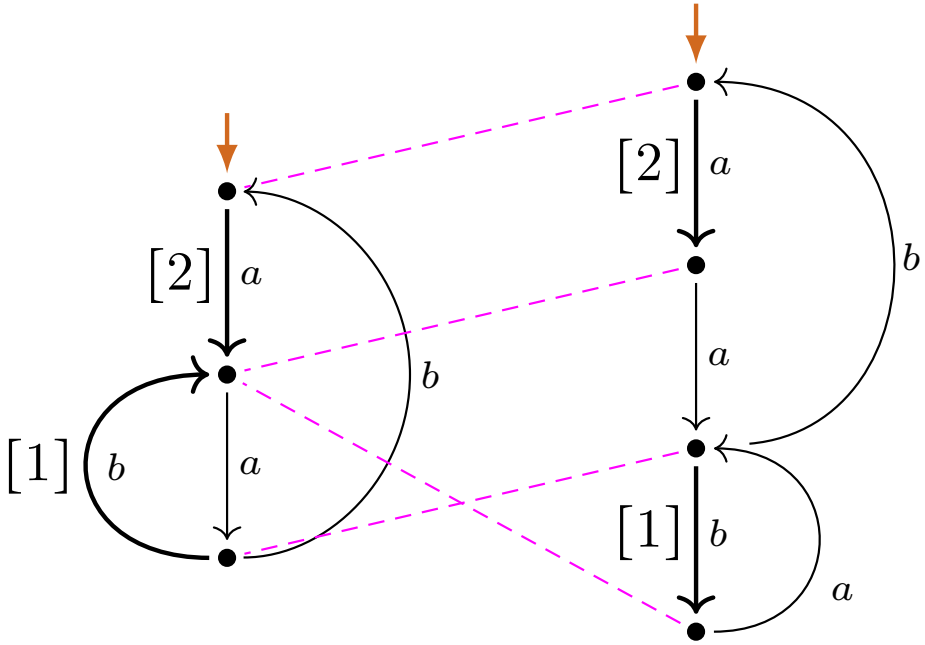
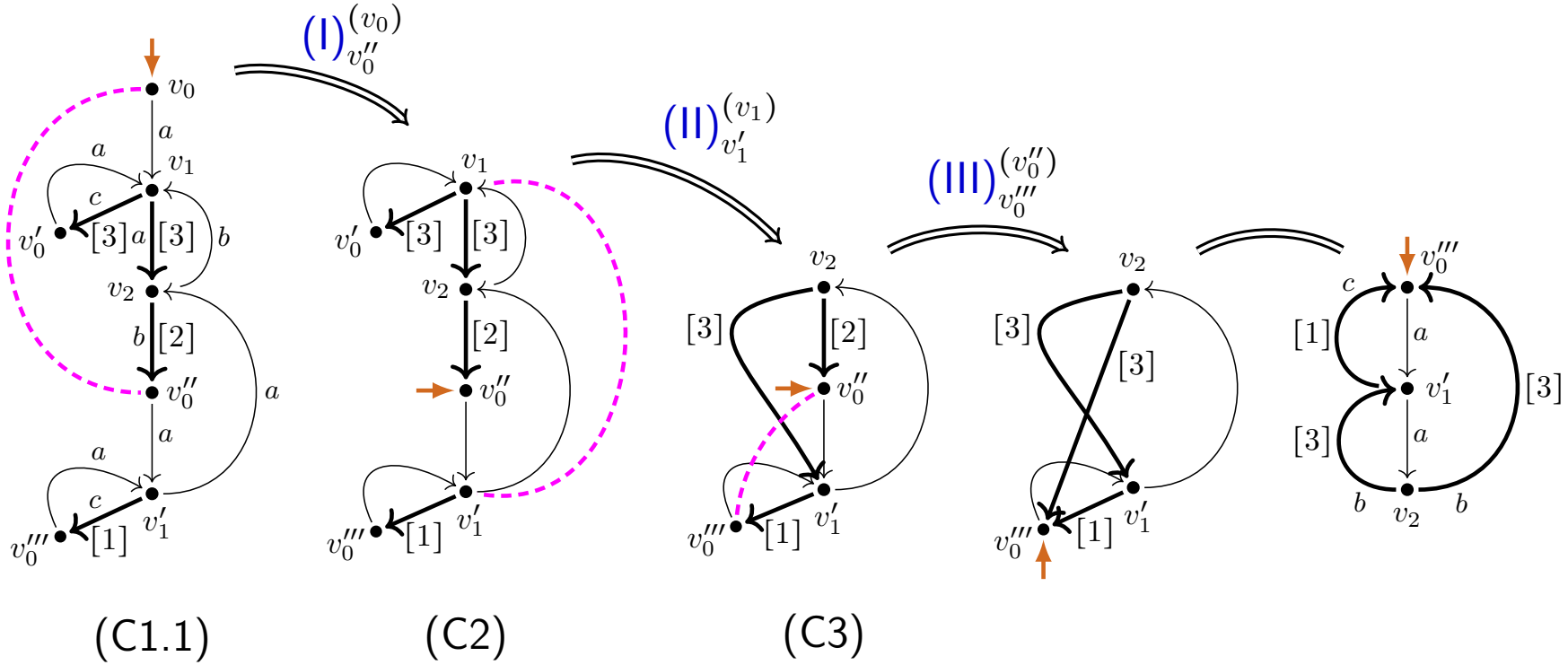


$P((aa(ba)^*b)^*0)$

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20)

(no 1-transitions!)



Lemma

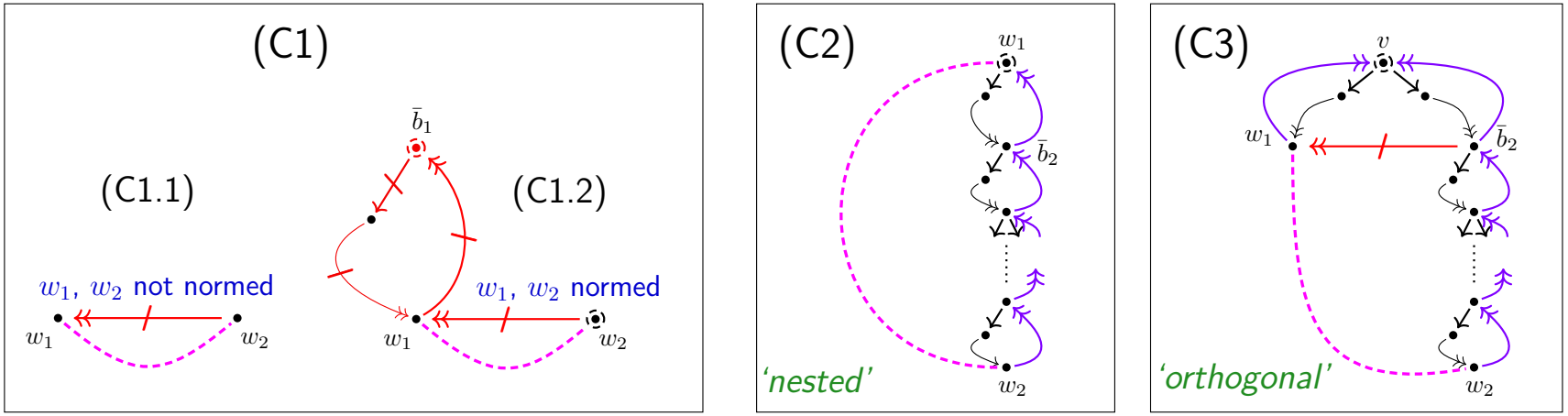
The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

Reduced bisimilarity redundancies in LLEE-charts (no 1-trans.!)

(G/Fokkink, LICS'20)

w_1, w_2 in different scc's

w_1, w_2 in the same scc



Lemma

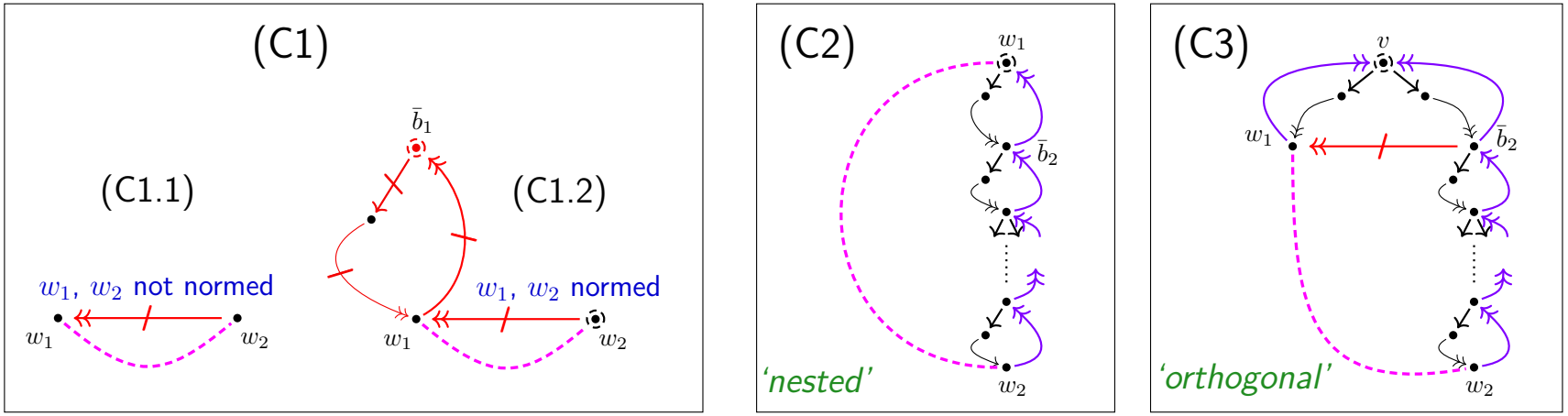
Every not collapsed LLEE-chart contains bisimilar vertices $w_1 \neq w_2$ of kind (C1), (C2), *or* (C3) *(a reduced bisimilarity redundancy $\langle w_1, w_2 \rangle$):*

Reduced bisimilarity redundancies in LLEE-charts (no 1-trans.!)

(G/Fokkink, LICS'20)

w_1, w_2 in different scc's

w_1, w_2 in the same scc



Lemma

Every not collapsed LLEE-chart contains bisimilar vertices $w_1 \neq w_2$ of kind (C1), (C2), or (C3) (a reduced bisimilarity redundancy $\langle w_1, w_2 \rangle$):

Lemma

Every reduced bisimilarity redundancy in a LLEE-chart can be eliminated LLEE-preservingly.

LEE under functional bisimulation

Lemma

(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \mathrel{\underline{\to}} G_2 \implies \mathsf{LEE}(G_2)\ .$$

Idea of Proof for (i)

Use loop elimination in G_1 to carry out loop elimination in G_2.

LEE under functional bisimulation / bisimulation collapse

Lemma

(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \rightrightarrows G_2 \implies \mathsf{LEE}(G_2)\ .$$

(ii) LEE *is preserved from a process graph to its bisimulation collapse:*

$$\mathsf{LEE}(G) \wedge C \text{ is bisimulation collapse of } G \implies \mathsf{LEE}(C)\ .$$

Idea of Proof for (i)

Use loop elimination in G_1 to carry out loop elimination in G_2.

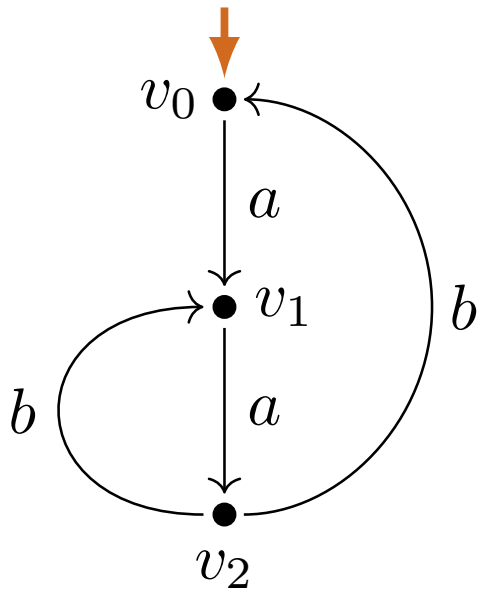
Readback

Lemma

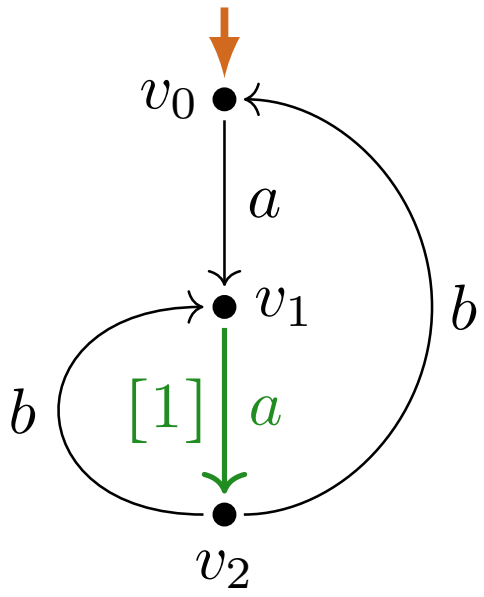
Process graphs with LEE are $P(\cdot)$-expressible:

$$\mathsf{LEE}(G) \implies \exists\, e \in \mathsf{Reg}(A) \big(G \leftrightarrow P(e) \big) .$$

Readback from layered LEE-witness (example)

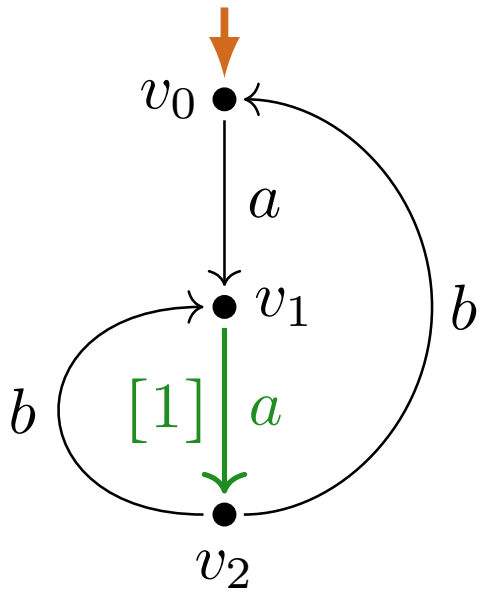


Readback from layered LEE-witness (example)



layered
LEE-witness

Readback from layered LEE-witness (example)

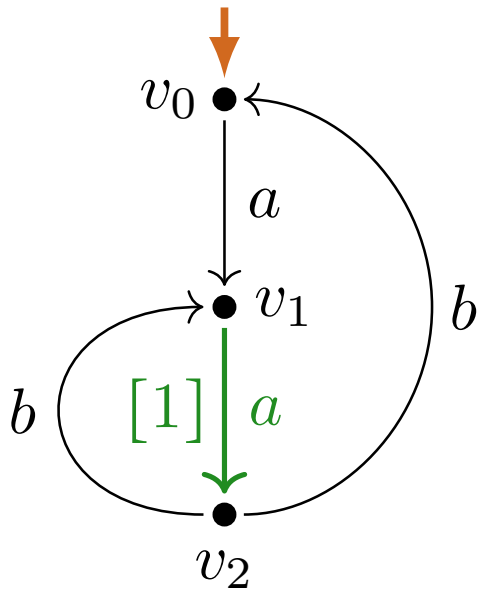


layered
LEE-witness

$$
\begin{aligned}
s(v_0) &= 0^* \cdot a \cdot s(v_1) \\
&=_{\mathsf{Mil}^-} a \cdot s(v_1) \\
&=_{\mathsf{Mil}^-} a \cdot \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
s(v_1) &= \big(a \cdot s(v_2, v_1)\big)^* \cdot 0 \\
&=_{\mathsf{Mil}^-} \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
s(v_2, v_1) &= 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big) \\
&=_{\mathsf{Mil}^-} 0^* \cdot \big(b \cdot 1 + b \cdot a\big) \\
&=_{\mathsf{Mil}^-} b + b \cdot a \\
s(v_1, v_1) &= 1 \\
s(v_0, v_1) &= 0^* \cdot a \cdot s(v_1, v_1) \\
&= 0^* \cdot a \cdot 1 \\
&=_{\mathsf{Mil}^-} a
\end{aligned}
$$

Readback from layered LEE-witness (example)

$$s(v_0) = \quad 0^* \cdot a \cdot s(v_1)$$



layered
LEE-witness

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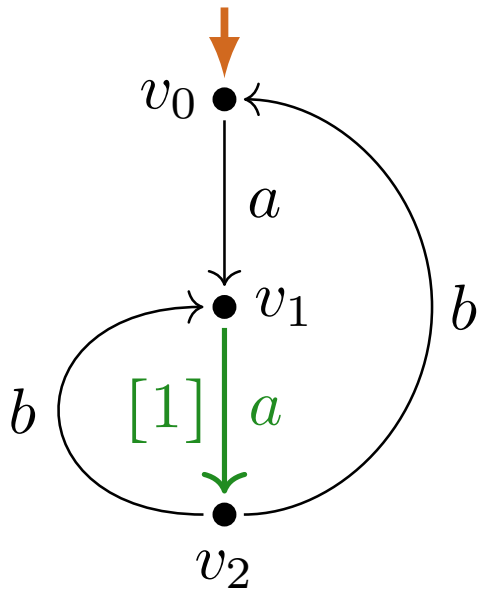
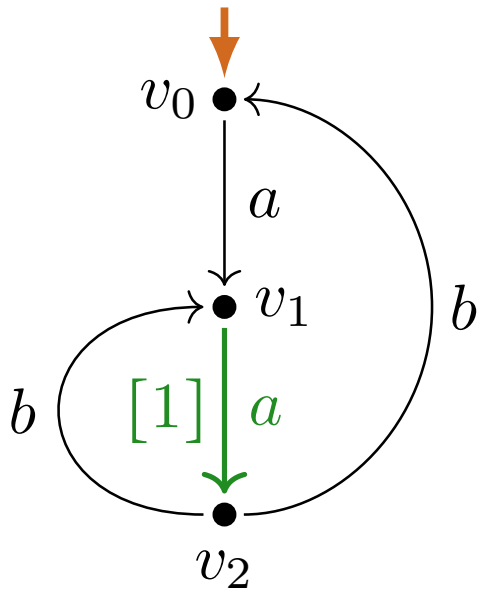


layered
LEE-witness

$$s(v_0) = \quad 0^* \cdot a \cdot s(v_1)$$

$$s(v_1) = \quad \big(a \cdot s(v_2, v_1)\big)^* \cdot 0$$

Readback from layered LEE-witness (example)



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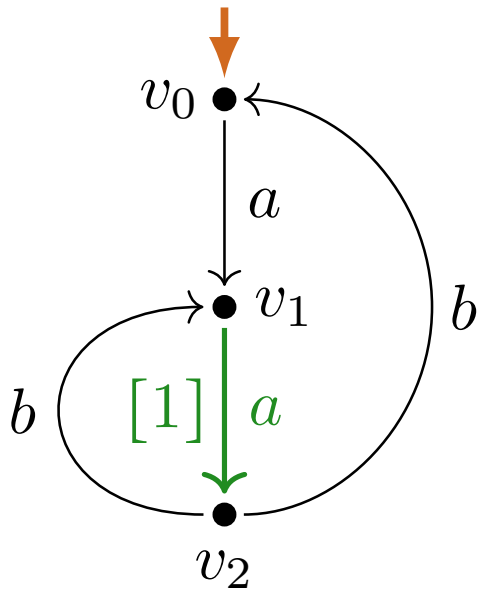

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$$s(v_1, v_1) = \quad 1$$

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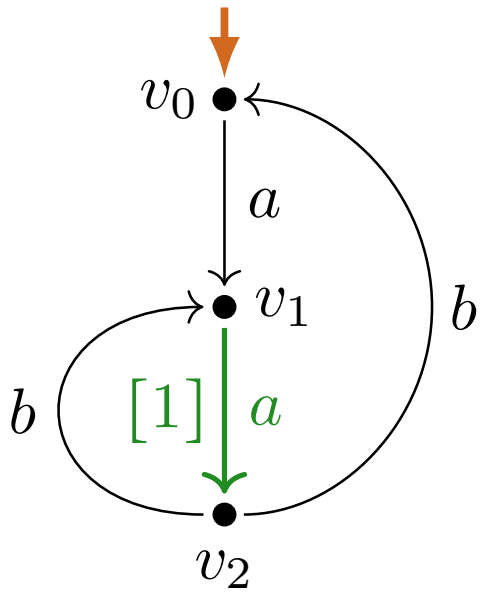

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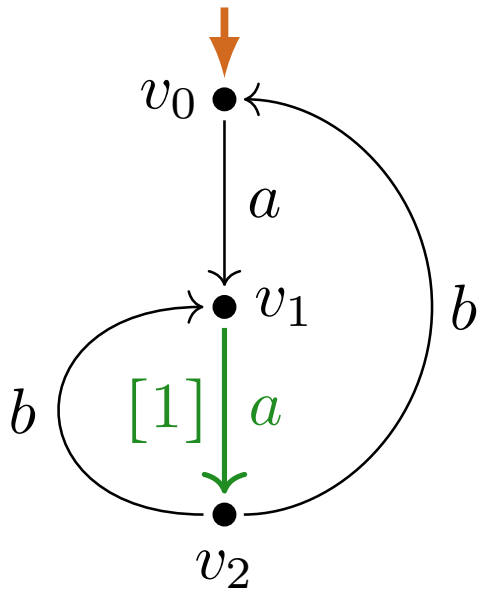

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$$\begin{aligned} s(v_1, v_1) &= \quad 1 \\ s(v_0, v_1) &= \quad 0^* \cdot a \cdot s(v_1, v_1) \\ &= \quad 0^* \cdot a \cdot 1 \end{aligned}$$

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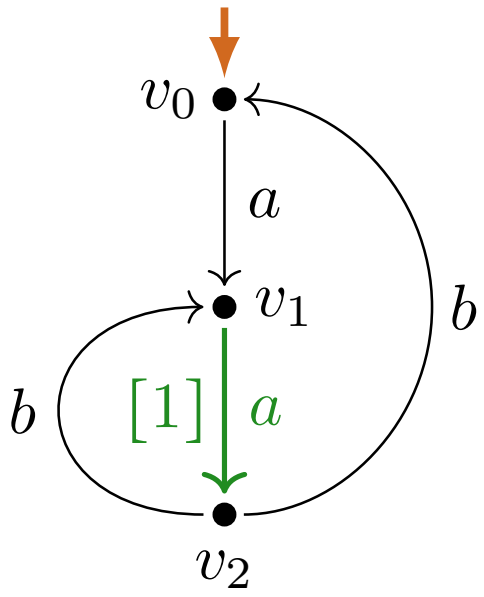

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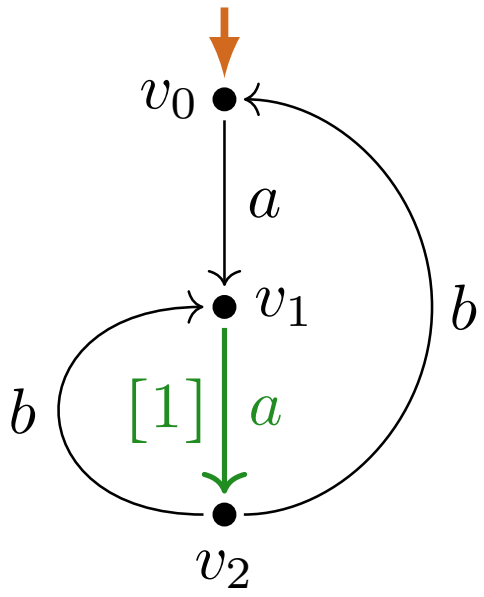

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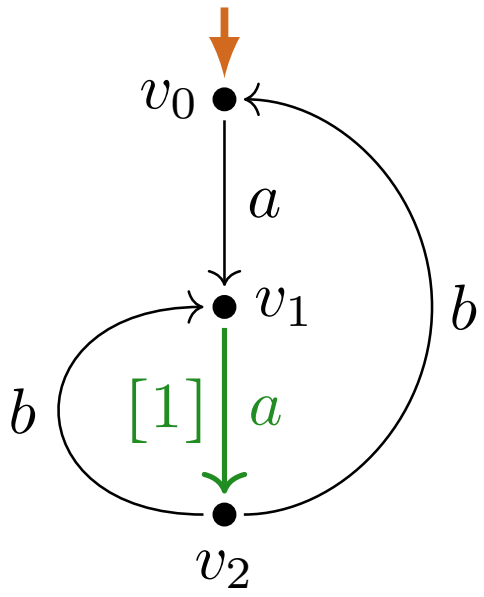

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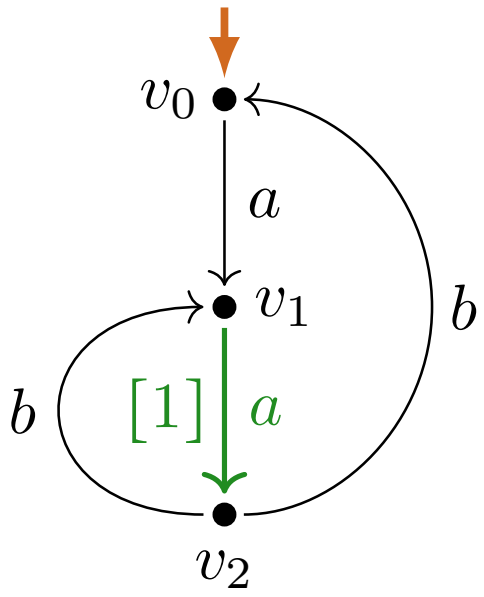

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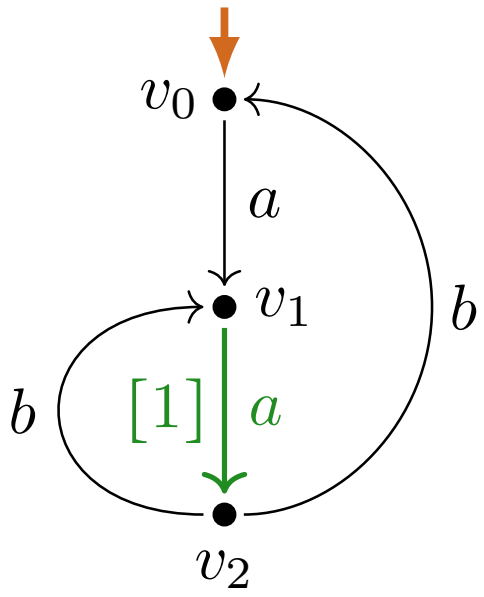

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$$

$$
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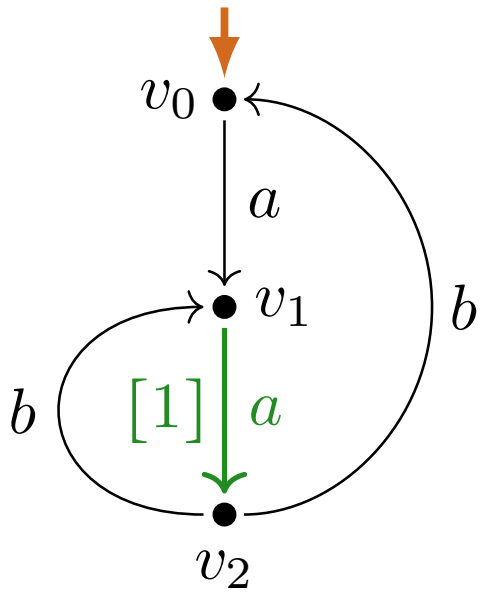

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s(v_1) &= \big(a \cdot s(v_2, v_1)\big)^* \cdot 0 \\
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1-return-less regular expressions

Lemma

Process graphs with LEE are $P(\cdot)$-expressible:

$$\mathsf{LEE}(G) \implies \exists\, e \in \mathsf{Reg}(A) \big(G \leftrightarrow P(e) \big) .$$

1-return-less regular expressions

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Process graphs with LEE are $[\![\cdot]\!]_P^{\cancel{1r}\backslash\star}$-expressible:

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Definition (Corradini, De Nicola, Labella (here intuitive version))

A regular expression e is 1-return-less(-under-$\star$) ($e \in \mathsf{Reg}^{\cancel{1r}\backslash\star}(A)$) if:

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  - p has the option to immediately terminate, and
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Non-/Examples of 1-return-less regular expressions

- $(a \cdot (1 + b))^*$

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- $(a \cdot (0^* + b))^*$ ✗
- $a \cdot \big(a \cdot (b + b \cdot a)\big)^* \cdot 0$ ✓
- $(a^*(b^* + c \cdot 0)^*)^*$ ✗
- $(a^*(b^* + c \cdot 0))^*$ ✗
- $(a^*(b + c \cdot 0))^*$

1-return-less regular expressions

Lemma

Process graphs with LEE are $[\![\cdot]\!]_P^{\cancel{1r}\backslash\star}$-expressible:

$$\mathsf{LEE}(G) \implies \exists\, e \in \mathsf{Reg}^{\cancel{1r}\backslash\star}(A)\,\big(\, G \leftrightarrow P(e)\,\big)\,.$$

Definition (Corradini, De Nicola, Labella (here intuitive version))

A regular expression e is 1-return-less(-under-$\star$) ($e \in \mathsf{Reg}^{\cancel{1r}\backslash\star}(A)$) if:

- for no iteration subexpression f^* of e does $P(f)$ proceed to a process p such that:
  - p has the option to immediately terminate, and
  - p has the option to do a proper step, and terminate later.

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Characterization of expressibility$^{\cancel{1r}\backslash\star}$

Theorem

For every process graph G with bisimulation collapse C the following are equivalent:

(i) G *is* $[\![\cdot]\!]_P^{\cancel{1r}\backslash\star}$*-expressible.*

(ii) LEE(C).

(iii) C *has a* LEE*-witness.*

(iv) C *has a layered* LEE*-witness.*

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Milners characterization question:

Q1. Which structural property of finite process graphs characterizes $P(\cdot)$-expressibility?

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Milners characterization question restricted:

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Milners characterization question restricted, and adapted:

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Answering Milners characterization question restricted, and adapted:

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- The loop-existence and elimination property LEE.

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Also yields: efficient decision method of $[\![\cdot]\!]_P^{\cancel{1r}\backslash\star}$-expressibility?

Structure constrained finite process graphs

graphs with LEE / a (layered) LEE-witness

Benefits of the class of process graphs with LEE:

- is closed under $\rightarrow$
- forth-/back-correspondence with 1-return-less regular expressions

Structure constrained finite process graphs

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\begin{aligned}
& \text{graphs with LEE / a (layered) LEE-witness} \\
\subsetneq\ & \text{graphs whose collapse satisfies LEE} \\
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Application to Milner's questions yields partial results:

Q1: characterization/efficient decision of $[\![\cdot]\!]_P^{\cancel{1r}\backslash\star}$-expressibility

Q2: alternative compl. proof of Mil on 1-return-less expressions (C/DN/L)

Milner's Proof System for Regular Expressions Modulo Bisimilarity is Complete

Crystallization: Near-Collapsing Process Graph Interpretations of Regular Expressions

Clemens Grabmayer (Department of Computer Science, Gran Sasso Science Institute, Viale F. Crispi, 7, 67100 L'Aquila AQ, Italy)

Abstract

We report on a lengthy completeness proof for Robin Milner's proof system Mil (1984) for bisimilarity of regular expressions in the process semantics. Central for our proof are the recognitions:

1. Process graphs with 1-transitions (1-charts) and the loop existence/elimination property LLEE are **not** closed under bisimilation collapse,
2. Such process graphs can be '**crystallized**' to 'near-collapsed' 1-charts with some strongly connected components of 'twin-crystal' form.

The Process Semantics of Regular Expressions

Milner (1984) introduced a process semantics for regular expressions: the interpretation of 0 is *deadlock*, of 1 is an *empty step to termination*, letters a are *atomic actions*, the operators $+$ and $\cdot$ stand for *choice* and *concatenation* of processes, and unary Kleene star $(\cdot)^*$ represents *(unbounded) iteration*. Formally, Milner defined chart (finite process graph) interpretations $\mathcal{C}(e)$ of regular expressions e.

Milner's Proof System

As axiomatization of the relation $e_1 =_{\boldsymbol{P}} e_2$ on regular expressions e_1 and e_2 defined by $\mathcal{C}(e_1) \leftrightarrows \mathcal{C}(e_2)$ (as bisimilarity $\leftrightarrows$ of chart interpretations), Milner asked whether the following system Mil is complete:

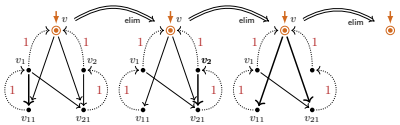
(A1)	$e + (f + g) = (e + f) + g$	(A7)	$e = 1 \cdot e$
(A2)	$e + 0 = e$	(A8)	$e = e \cdot 1$
(A3)	$e + f = f + e$	(A9)	$0 = 0 \cdot e$
(A4)	$e + e = e$	(A10)	$e^* = 1 + e \cdot e^*$
(A5)	$e \cdot (f \cdot g) = (e \cdot f) \cdot g$	(A11)	$e^* = (1 + e)^*$
(A6)	$(e + f) \cdot g = e \cdot g + f \cdot g$		

$$\frac{e = f \cdot e + g}{e = f^* \cdot g}\,\mathrm{RSP}^* \text{ (if } f \text{ does not terminate immediately)}$$

This system is a variation of Salomaa's complete axiom system (1966) for language equality of regular expressions, missing left-distributivity $e \cdot (f + g) = e \cdot f + e \cdot g$ and $e \cdot 0 = 0$, which are unsound here.

Loop Existence and Elimination

The process semantics is incomplete: not every finite process graph is *expressible by* (=bisimilar to the interpretation of) a regular expression. A sufficient condition for expressibility is the *(layered) loop existence and elimination property* LLEE. It is defined via elimination of 'loops' (loop subcharts):



LLEE holds if a graph without infinite behavior can be obtained. Important features of LLEE:

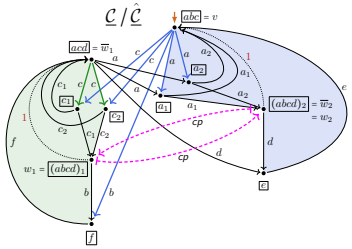
(US) Every guarded LLEE-1-chart (chart, maybe 1-transitions, with LLEE) is uniquely Mil-provably solvable modulo provability in Mil (CALCO 2021).

(IV) The chart interpretation $\mathcal{C}(e)$ of a regular expression e can always be expanded under bisimilarity to a LLEE-1-chart $\underline{\mathcal{C}}(e)$ (TERMGRAPH 2020).

($C_{\pm}$) LLEE-charts (without 1-transitions) are preserved by bisimulation collapse (G/Fokkink, LICS'20).

LLEE-preserving Collapse Fails

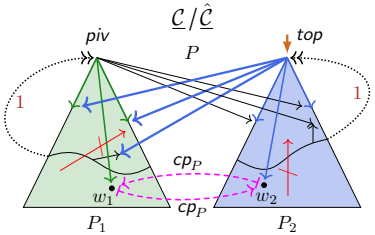
LLEE-1-charts with 1-transitions, however, are **not** preserved under bisimulation collapse. A counterexample is provided by the following LLEE-1-chart $\underline{\mathcal{C}}$:



Identifying the bisimilar vertices w_1 and w_2 yields a chart for which LLEE fails. Also, the subcharts of $\underline{\mathcal{C}}$ that are rooted at w_1 and w_2 are **not** *LLEE-preservingly jointly minimizable under bisimilarity.*

Twin-Crystals

The counterexample to LLEE-preserving collapse is symmetric, and its structure can be abstracted as:



It is a LLEE-1-chart with a single scc (strongly connected component) P that consists of a *pivot part* P_1 below *pivot vertex* ***piv***, and a *top part* P_2 below *top vertex* ***top***. P_1 and P_2 are connected only via transitions from ***piv*** and from ***top***. While both P_1 and P_2 are collapsed, P contains *bisimilarity redundancies* (= distinct bisimilar vertices) such as $\{w_1, w_2\}$ that are linked by a self-inverse counterpart function cp_P. We call such an scc a *twin-crystal*. We have:

(CC) Every Mil-provable solution of a twin-crystal gives rise to a Mil-provable solution of its bisimulation collapse (which often is not a LLEE-1-chart).

Crystallization of LLEE-1-charts

By *crystallization* of a LLEE-1-chart $\underline{\mathcal{C}}$ we mean:

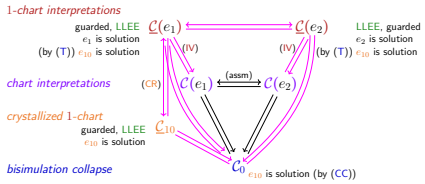
- a process of minimization of $\underline{\mathcal{C}}$ under bisimilarity by steps that **eliminate most** (all but crystalline) bisimilarity redundancies $\{w_1, w_2\}$, roughly by redirecting transitions that target w_1 over to w_2;
- hereby only '**reduced**' bisimilarity redundancies can be eliminated LLEE-preservingly, which exist whenever a LLEE-1-chart is **not** collapsed;
- the result is a *crystallized* LLEE-1-chart that is bisimilar to $\underline{\mathcal{C}}$, and collapsed **apart from** within some its scc's that are twin-crystals.

The crystallization process facilitates to show:

(CR) From every LLEE-1-chart a bisimilar crystallized LLEE-1-chart can be obtained.

Completeness Proof

Let $\mathcal{C}(e_1) \leftrightarrows \mathcal{C}(e_2)$ be bisimilar chart interpretations of regular expressions e_1 and e_2. To secure LLEE, $\mathcal{C}(e_1)$ and $\mathcal{C}(e_2)$ are expanded to their 1-chart interpretations $\underline{\mathcal{C}}(e_1)$ and $\underline{\mathcal{C}}(e_2)$. One of them, say $\underline{\mathcal{C}}(e_1)$, is crystallized to $\underline{\mathcal{C}}_{10}$. All (1-)charts are linked by (1-)bisimulations to their bisimulation collapse $\mathcal{C}_0$.



$$\left.\begin{array}{l}\mathcal{C}(e_1) \text{ guarded, LLEE}\\ e_1 \text{ is solution}\\ e_{10} \text{ is solution}\end{array}\right\} \xRightarrow{(US)} e_1 =_{\mathrm{Mil}} e_{10} \quad e_{10} =_{\mathrm{Mil}} e_2 \xLeftarrow{(US)} \left\{\begin{array}{l}\mathcal{C}(e_2) \text{ guarded, LLEE}\\ e_2 \text{ is solution}\\ e_{10} \text{ is solution}\end{array}\right.$$
$$\Longrightarrow e_1 =_{\mathrm{Mil}} e_2$$

From $\underline{\mathcal{C}}_{10}$ a provable solution e_{10} can be extracted due to LLEE, transferred (T) to the collapse $\mathcal{C}_0$, and then to $\mathcal{C}(e_1)$ and $\mathcal{C}(e_2)$. On the LLEE-1-charts $\underline{\mathcal{C}}(e_1)$ and $\underline{\mathcal{C}}(e_2)$, e_{10} can be proved equal to the solutions e_1 and e_2 there, respectively. By transitivity, $e_1 =_{\mathrm{Mil}} e_2$ (provability of $e_1 = e_2$ in Mil) follows.

Theorem. *Milner's system* Mil *is complete:* $e_1 =_{\boldsymbol{P}} e_2$ *implies* $e_1 =_{\mathrm{Mil}} e_2$, *for reg. expr's* e_1, e_2.

Next Steps and Projects

- Monograph project: proof in fine-grained details.
- Build an animation tool for crystallization.
- Apply crystallization to find an efficient algorithm for expressibility of finite process graphs by a regular expression modulo bisimilarity.

Contact

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Scuola Universitaria Superiore

Resources (process interpretation)

- CG: Modeling Terms by Graphs with Structure Constraints
  - TERMGRAPH 2018 Post-Proceedings, EPTCS 288, arXiv:1902.02010.
- CG: Structure-Constrained Process Graphs for the Process Semantics of Regular Expressions
  - TERMGRAPH 2020 Post-Proceedings, EPTCS 334, arXiv:2012.10869.
- CG, Wan Fokkink: A Complete Proof System for 1-Free Regular Expressions Modulo Bisimilarity
  - LICS 2020, arXiv:2004.12740, video on youtube.
- CG: Milner's Proof System for Regular Expressions Modulo Bisimilarity is Complete
  - LICS 2022, arXiv:2209.12188, poster.
- CG: A Coinductive Version/Reformulation of Milner's Proof System for Regular Expressions Modulo Bisimilarity
  - CALCO 2021, arXiv:2108.13104.
  - LMCS 2023, arXiv:2303.14219.

Outlook

correspondences found

- process graphs with LEE
  ~ $P(\cdot)$-interpretations of 1-return-less regular expressions
- process graphs with 1-transitions and with LEE
  ~ $P(\cdot)$-interpretations of regular expressions
- facilitate/may facilitate:
  efficient manipulation/recognition of $P(\cdot)/[\![\cdot]\!]_P$-expressible graphs

slides and resources: `clegra.github.io`

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current projects

- PTIME-decidability of LEE (LLEE) and $[\![\cdot]\!]_P^{\cancel{1r}\backslash\star}$-expressibility
- refinability into LEE-graphs by adding 1-transitions (in PTIME?)
- $[\![\cdot]\!]_P$-expressibility: $\iff$ expansion and refinability into a crystallized LLEE-1-process-graph (in FPT?)
- full completeness proof of Mil via crystallization
  (two parts: motivation / procedure)

slides and resources: `clegra.github.io`

Comparison results: structure-constrained graphs

λ-calculus with letrec under $=_{\lambda^\infty}$

Not available: graph interpretation that is studied modulo $\underline{\leftrightarrow}$

Regular expressions under $\underline{\leftrightarrow}_P$

Given: graph interpretation $P(\cdot)$, studied modulo bisimulation $\underline{\leftrightarrow}$

- not closed under $\underline{\rightarrow}$, and $\underline{\leftrightarrow}$, incomplete under $\underline{\leftrightarrow}$

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Defined: int's $[\![\cdot]\!]_{\mathcal{H}}/[\![\cdot]\!]_{\mathcal{T}}$ as higher-order/first-order λ-term graphs

- closed under $\underline{\rightarrow}$ (hence under collapse)
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  - efficient translation and readback
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Given: graph interpretation $P(\cdot)$, studied modulo bisimulation $\underline{\leftrightarrow}$

- not closed under $\underline{\rightarrow}$, and $\underline{\leftrightarrow}$, incomplete under $\underline{\leftrightarrow}$

Defined: class of process graphs with LEE / (layered) LEE-witness

- closed under $\underline{\rightarrow}$ (hence under collapse)
- back-/forth correspondence with 1-return-less expr's
- contains the collapse of a process graph G
  $\iff$ G is $[\![\cdot]\!]_P^{\cancel{1r}\backslash\star}$-expressible