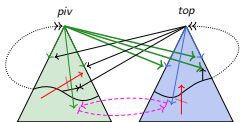


# Milner's Proof System for Regular Expressions Modulo Bisimilarity is Complete

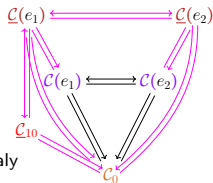
Crystallization: Near-Collapsing Process Graph Interpretations  
of Regular Expressions



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LICS 2022

Technion, Haifa, Israel

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# Process semantics of regular expressions $\llbracket \cdot \rrbracket_{\mathbf{P}}$ (Milner, 1984)

$0 \xrightarrow{\llbracket \cdot \rrbracket_{\mathbf{P}}} \text{deadlock } \delta, \text{ no termination}$

$1 \xrightarrow{\llbracket \cdot \rrbracket_{\mathbf{P}}} \text{empty-step process } \epsilon, \text{ then terminate}$

$a \xrightarrow{\llbracket \cdot \rrbracket_{\mathbf{P}}} \text{atomic action } a, \text{ then terminate}$

$e + f \xrightarrow{\llbracket \cdot \rrbracket_{\mathbf{P}}} (\text{choice}) \text{ execute } \llbracket e \rrbracket_{\mathbf{P}} \text{ or } \llbracket f \rrbracket_{\mathbf{P}}$

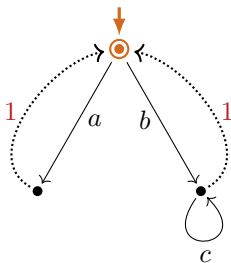
$e \cdot f \xrightarrow{\llbracket \cdot \rrbracket_{\mathbf{P}}} (\text{sequentialization}) \text{ execute } \llbracket e \rrbracket_{\mathbf{P}}, \text{ then } \llbracket f \rrbracket_{\mathbf{P}}$

$e^* \xrightarrow{\llbracket \cdot \rrbracket_{\mathbf{P}}} (\text{iteration}) \text{ repeat (terminate or execute } \llbracket e \rrbracket_{\mathbf{P}})$

$\llbracket e \rrbracket_{\mathbf{P}} := [\mathcal{C}(e)]_{\leftrightarrow} \quad (\text{bisimilarity equivalence class of chart } \mathcal{C}(e))$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

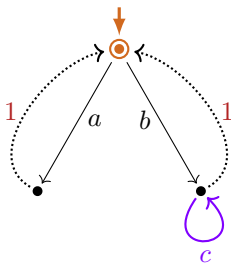
1-chart



$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

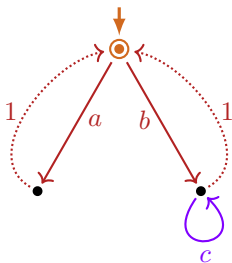
1-chart



$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

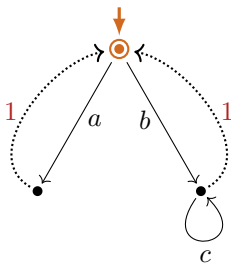
1-chart



$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

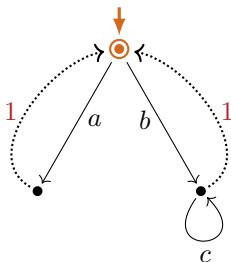
1-chart



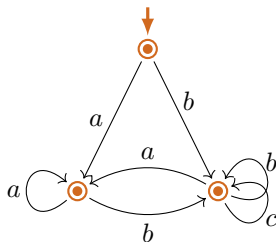
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart



chart



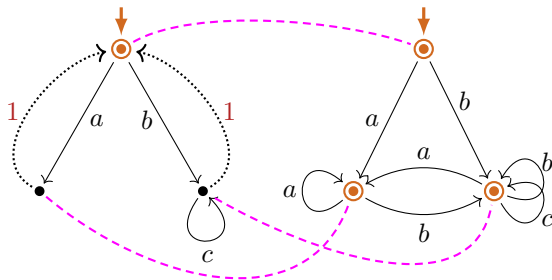
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart

chart



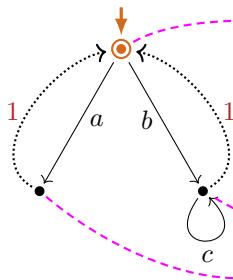
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$



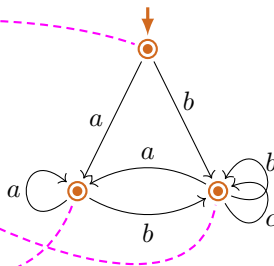
# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart



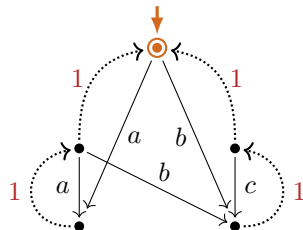
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

chart



$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

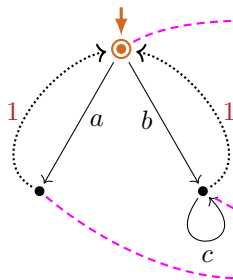
1-chart



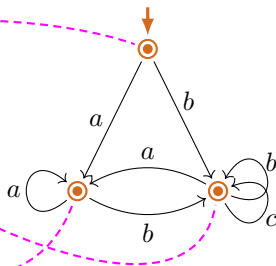
$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

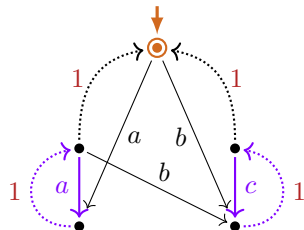
1-chart



chart



1-chart



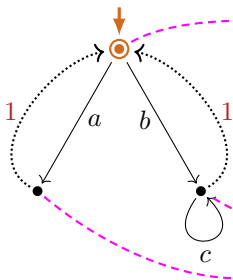
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

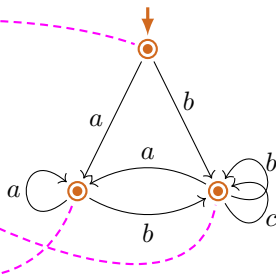
$$\begin{aligned} &\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \\ &\quad \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 \\ &\quad + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*) \end{aligned}$$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

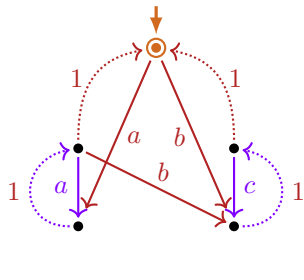
1-chart



chart



1-chart

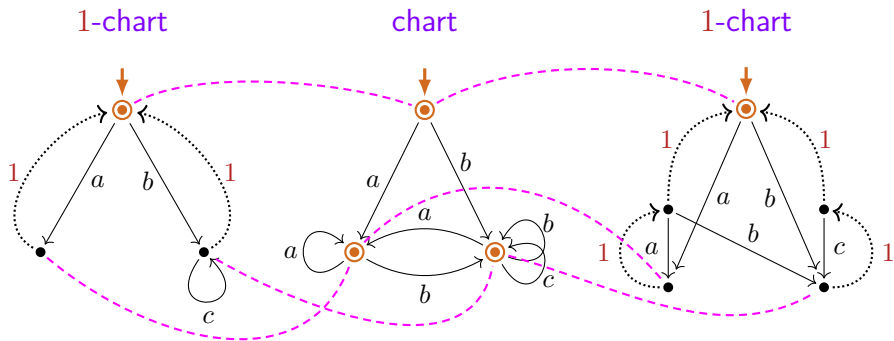


$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\begin{aligned} &\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \\ &\quad \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 \\ &\quad + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*) \end{aligned}$$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)



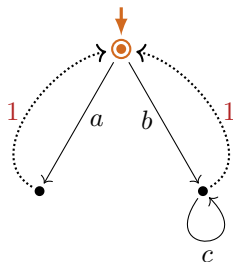
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$

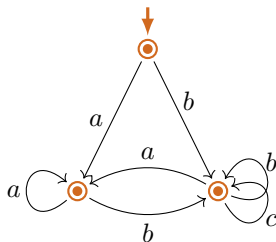
# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart



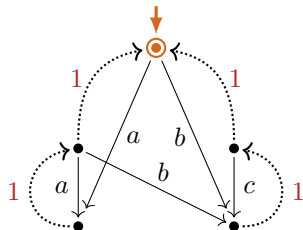
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

chart



$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

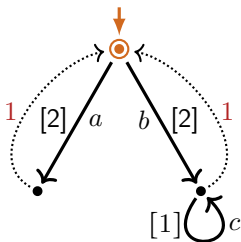
1-chart



$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

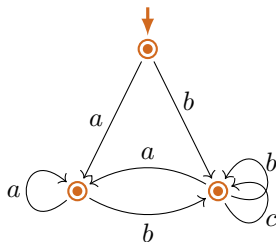
1-chart



LLEE

$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

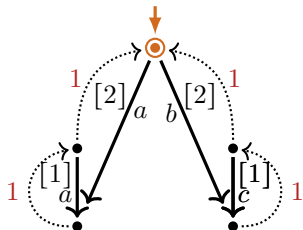
chart



LLEE

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

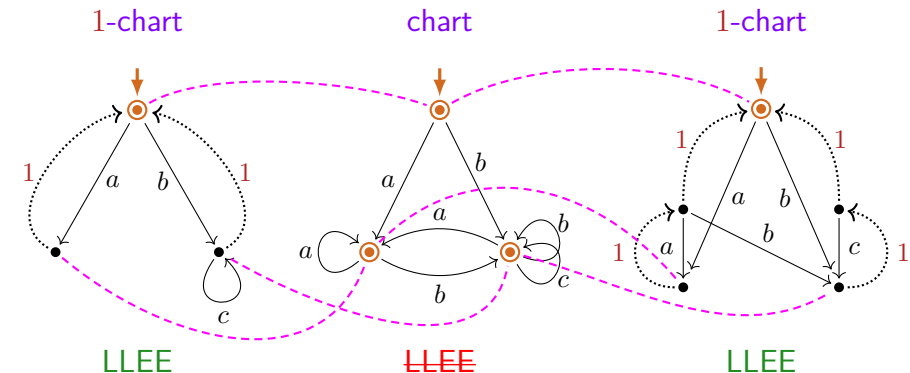
1-chart



LLEE

$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)



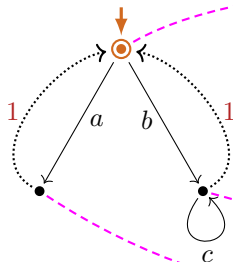
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart

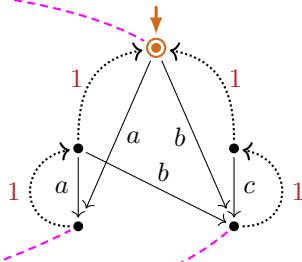


LLEE

$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$\longleftrightarrow$

1-chart

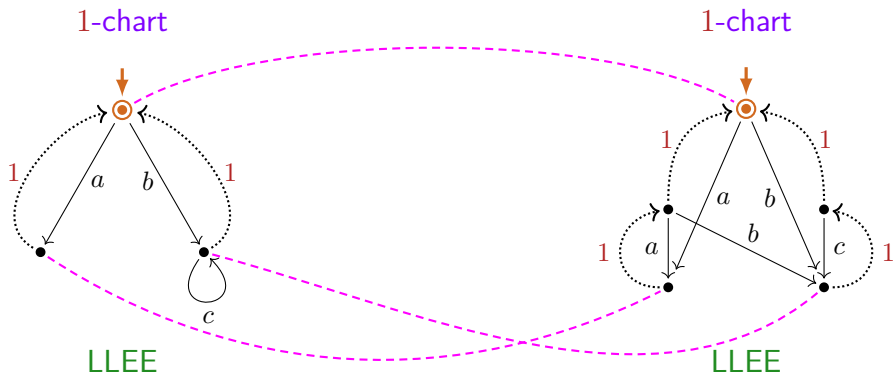


LLEE

$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$



# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

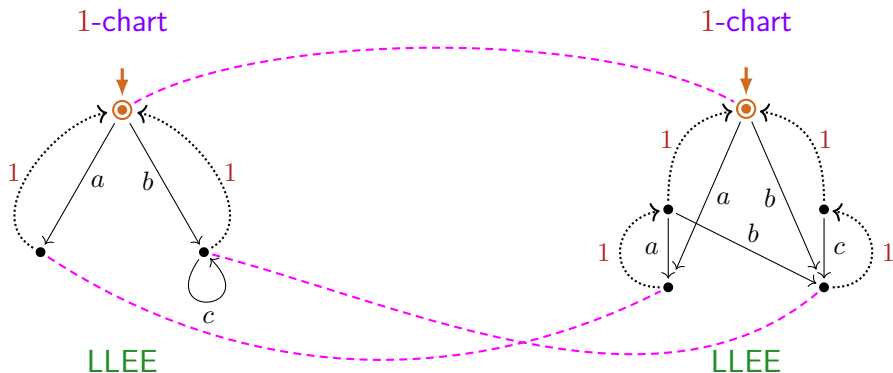


$$\llbracket (a \cdot 1 + b \cdot (c^* \cdot 1))^* \rrbracket_P$$

=

$$\begin{aligned} & \llbracket (a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \\ & \quad \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 \\ & \quad + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^* \rrbracket_P \end{aligned}$$

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)



$$(a \cdot 1 + b \cdot (c^* \cdot 1))^*$$

$=_P$

$$\begin{aligned} & (a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \\ & \quad \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 \\ & \quad + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^* \end{aligned}$$

# Milner's proof system Mil

*Axioms:*

$$(A1) \quad e + (f + g) = (e + f) + g$$

$$(A2) \quad e + 0 = e$$

$$(A3) \quad e + f = f + e$$

$$(A4) \quad e + e = e$$

$$(A5) \quad e \cdot (f \cdot g) = (e \cdot f) \cdot g$$

$$(A6) \quad (e + f) \cdot g = e \cdot g + f \cdot g$$

$$(A7) \quad e = 1 \cdot e$$

$$(A8) \quad e = e \cdot 1$$

$$(A9) \quad 0 = 0 \cdot e$$

$$(A10) \quad e^* = 1 + e \cdot e^*$$

$$(A11) \quad e^* = (1 + e)^*$$

$$\text{But: } e \cdot (f + g) \neq e \cdot f + e \cdot g$$

$$\text{But: } e \cdot 0 \neq 0$$

*Inference rules:* rules of equational logic plus

$$\frac{e = f \cdot e + g}{e = f^* \cdot g} \text{RSP}^* \text{ (if } f \text{ does not terminate immediately)}$$

Milner's question (1984)

Is Mil complete with respect to  $=_P$ ? (Does  $=_P \subseteq =_{\text{Mil}}$  hold?)

# Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

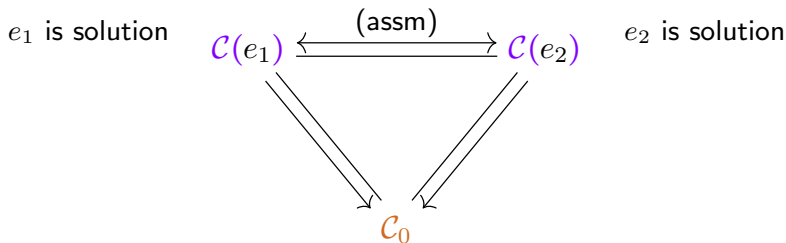
For 1-free regular expressions  $e_1$  and  $e_2$ :

$$e_1 \text{ is solution} \quad \mathcal{C}(e_1) \xrightleftharpoons{\text{(assm)}} \mathcal{C}(e_2) \quad e_2 \text{ is solution}$$

$$e_1 =_{\text{Mil}} e_2$$

# Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

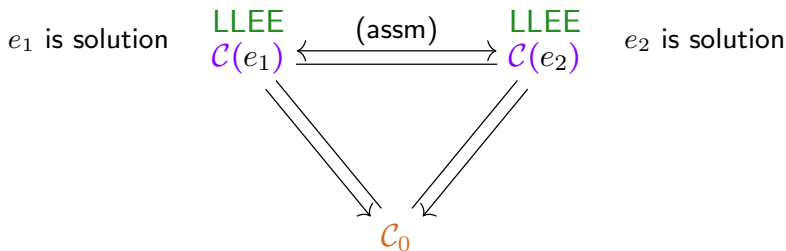
For **1-free** regular expressions  $e_1$  and  $e_2$ :



$$e_1 =_{\text{Mil}} e_2$$

# Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

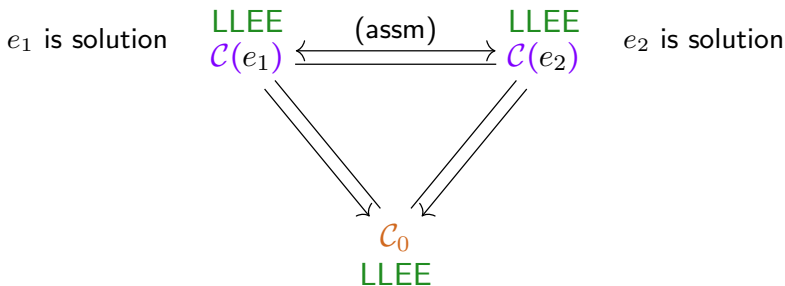
For **1-free** regular expressions  $e_1$  and  $e_2$ :



$$e_1 =_{\text{Mil}} e_2$$

# Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

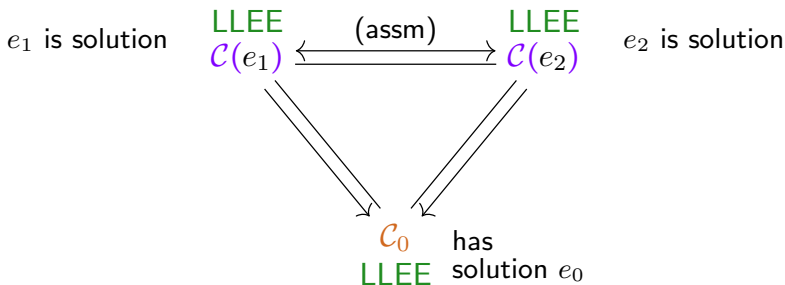
For **1-free** regular expressions  $e_1$  and  $e_2$ :



$$e_1 =_{\text{Mil}} e_2$$

# Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

For **1-free** regular expressions  $e_1$  and  $e_2$ :

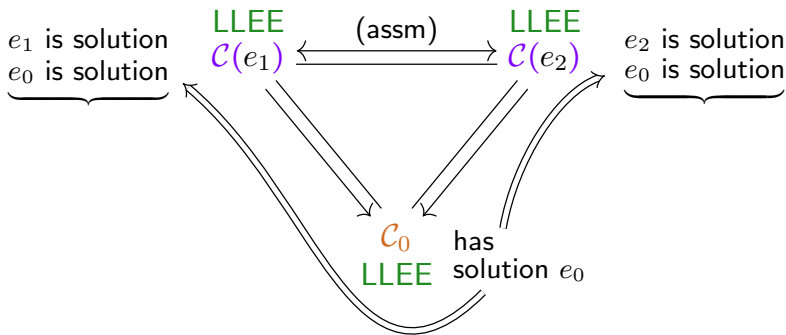


$$e_1 =_{\text{Mil}} e_2$$



# Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

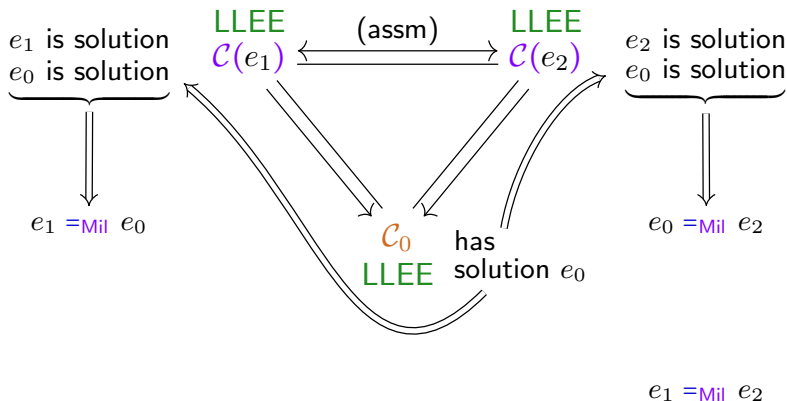
For **1-free** regular expressions  $e_1$  and  $e_2$ :



$$e_1 =_{\text{Mil}} e_2$$

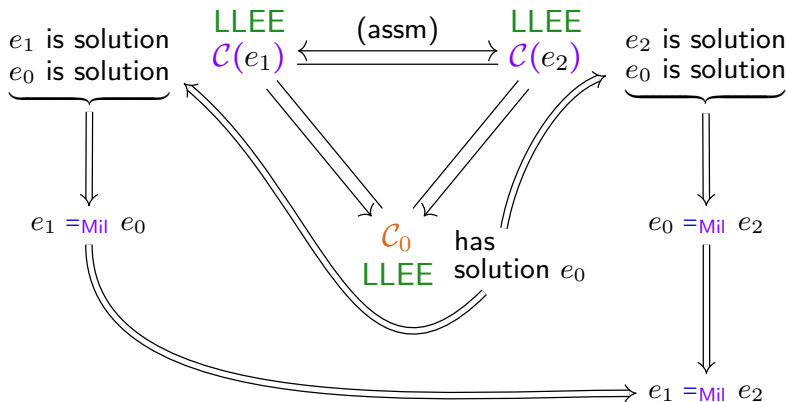
# Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

For **1-free** regular expressions  $e_1$  and  $e_2$ :



# Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

For **1-free** regular expressions  $e_1$  and  $e_2$ :



# Bisimulation collapse proof strategy (general case)

## Problem 1

$$\text{chart interpretations} \quad \text{LLEE} \quad \mathcal{C}(e_1) \xrightleftharpoons{(\text{assm})} \mathcal{C}(e_2) \quad \text{LLEE}$$

# Bisimulation collapse proof strategy (general case)

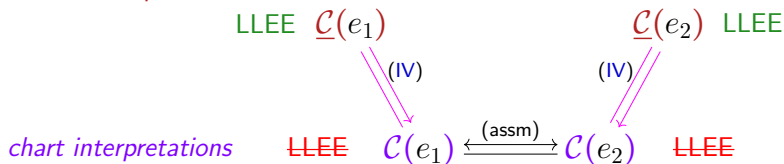
Remedy for Problem 1 (*G, TERMGRAPH 2020*)

chart interpretations  $\text{LLEE} \quad \mathcal{C}(e_1) \xrightleftharpoons{(\text{assm})} \mathcal{C}(e_2) \quad \text{LLEE}$

# Bisimulation collapse proof strategy (general case)

## Remedy for Problem 1 (*G, TERMGRAPH 2020*)

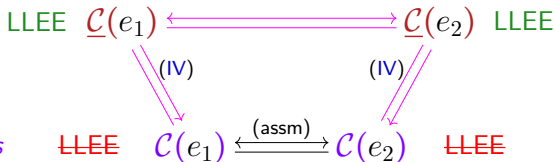
1-chart interpretations



# Bisimulation collapse proof strategy (general case)

## Remedy for Problem 1 (*G, TERMGRAPH 2020*)

1-chart interpretations



# Bisimulation collapse proof strategy (general case)

## Remedy for Problem 1 (*G, TERMGRAPH 2020*)

1-chart interpretations

LLEE  $\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$  LLEE

chart interpretations

$\text{LLEE } C(e_1) \xleftrightarrow{\text{(assm)}} C(e_2) \text{ LLEE}$

bisimulation collapse

$C_0$



# Bisimulation collapse proof strategy (general case)

## Remedy for Problem 1 (*G, TERMGRAPH 2020*)

*1-chart interpretations*

LLEE  $\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$  LLEE

*chart interpretations*

LLEE  $C(e_1) \xleftrightarrow{\text{(assm)}} C(e_2)$  LLEE

*bisimulation collapse*

$C_0$

# Bisimulation collapse proof strategy (general case)

## Remedy for Problem 1 (*G*, TERMGRAPH 2020)

1-chart interpretations

LLEE  $\underline{C}(e_1)$   $\longleftrightarrow$   $\underline{C}(e_2)$  LLEE

chart interpretations

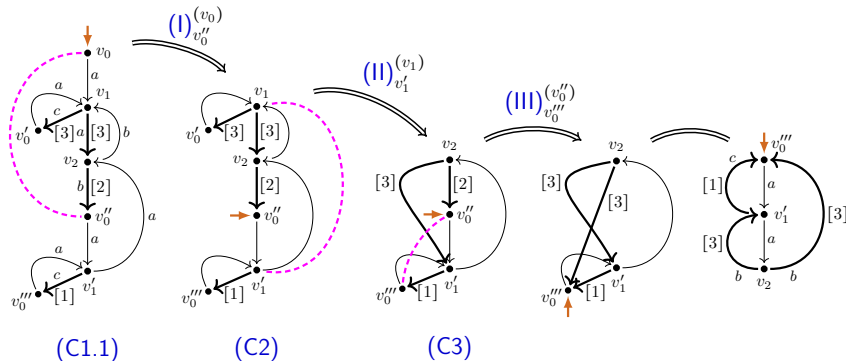
LLEE  $C(e_1) \xleftrightarrow{\text{(assm)}} C(e_2)$  LLEE

bisimulation collapse

$C_0$  LLEE?

# LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20)

(no 1-transitions!)



## Lemma

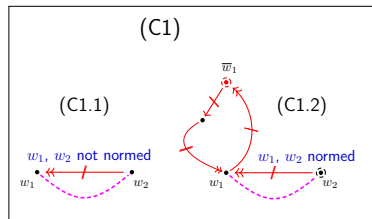
The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

# Reduced bisimilarity redundancies in LLEE-charts (no 1-trans.!) (G/Fokkink, LICS'20)

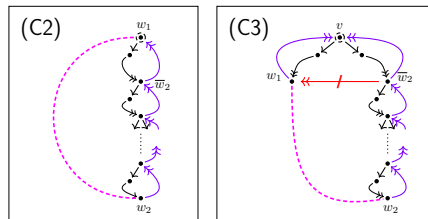
## Lemma

Every *not collapsed* LLEE-chart contains bisimilar vertices  $w_1 \neq w_2$  of kind (C1), (C2), or (C3) (a *reduced bisimilarity redundancy*  $\langle w_1, w_2 \rangle$ ):

$w_1, w_2$  in *different* scc's



$w_1, w_2$  in the *same* scc



## Lemma

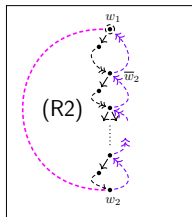
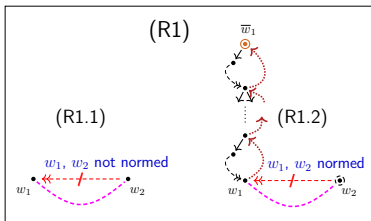
Every *reduced bisimilarity redundancy* in a LLEE-chart can be eliminated LLEE-preservingly.

# Reduced 1-bisimilarity redundancies in LLEE-1-charts

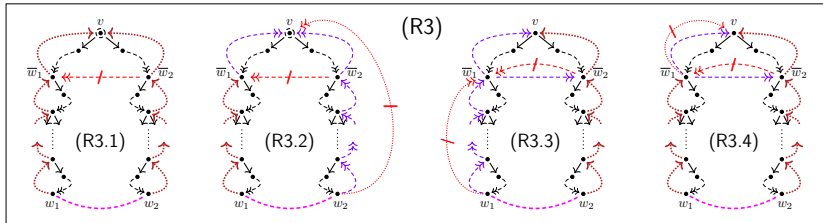
## Lemma

Every *not collapsed LLEE-1-chart* contains 1-bisimilar vertices  $w_1 \neq w_2$  (a *reduced 1-bisimilarity redundancy*  $\langle w_1, w_2 \rangle$ ) of kind (R1), (R2), (R3):

$w_1, w_2$  in  
different  
scc's



$w_1, w_2$  in  
the same  
scc

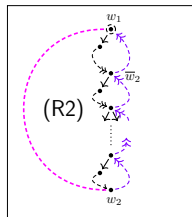
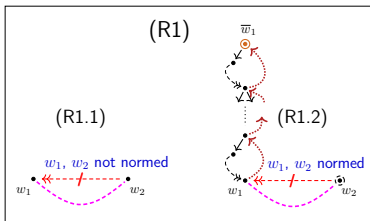


# Reduced 1-bisimilarity redundancies in LLEE-1-charts

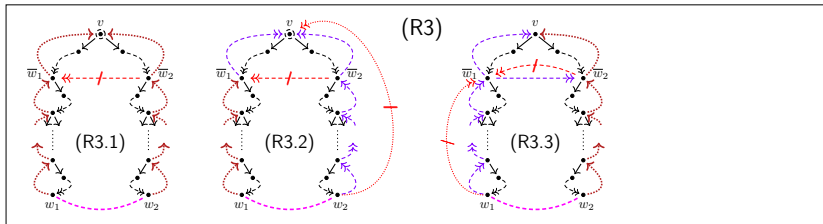
## Lemma

Every *simple* reduced 1-bisimilarity redundancies in a LLEE-1-chart can be eliminated LLEE-preservingly.

$w_1, w_2$  in  
different  
scc's



$w_1, w_2$  in  
the same  
scc



# Reduced 1-bisimilarity redundancies in LLEE-1-charts

## Stumbling Block

How to LLEE-preservingly eliminate  
reduced 1-bisimilarity redundancies of kind (R3.4)?

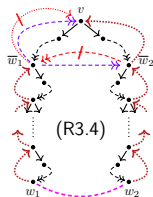
$w_1, w_2$  in  
different  
scc's

(R1)

(R2)

$w_1, w_2$  in  
the same  
scc

(R3)



# Reduced 1-bisimilarity redundancies in LLEE-1-charts

## Stumbling Block

How to LLEE-preservingly eliminate  
precrySTALLINE reduced 1-bisimilarity redundancies?

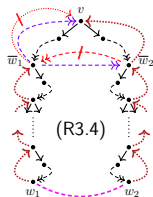
$w_1, w_2$  in  
different  
scc's

(R1)

(R2)

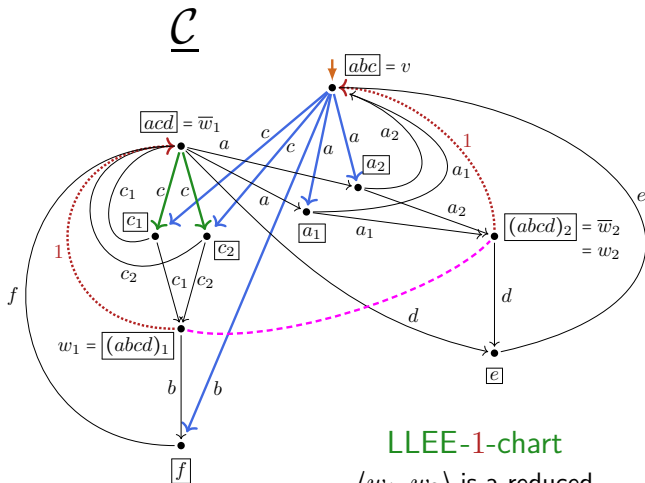
$w_1, w_2$  in  
the same  
scc

(R3)

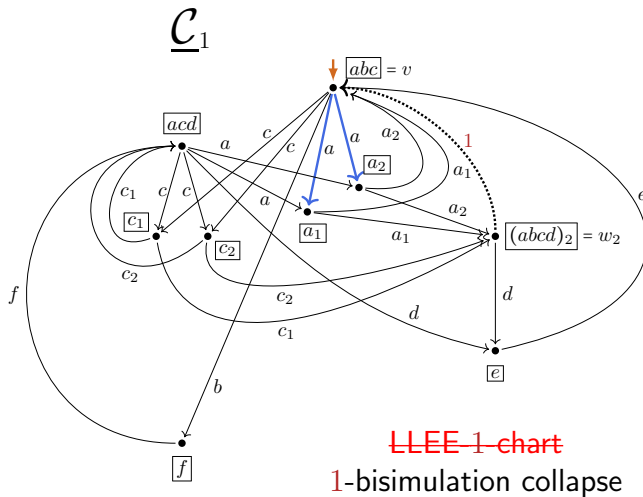




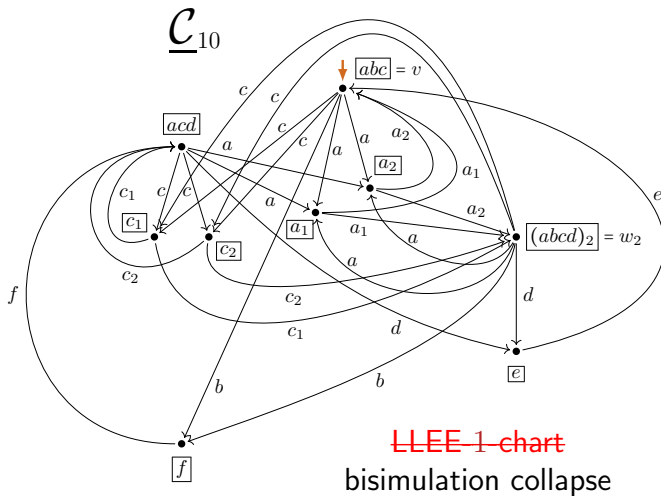
# Counterexample LLEE-preserving collapse



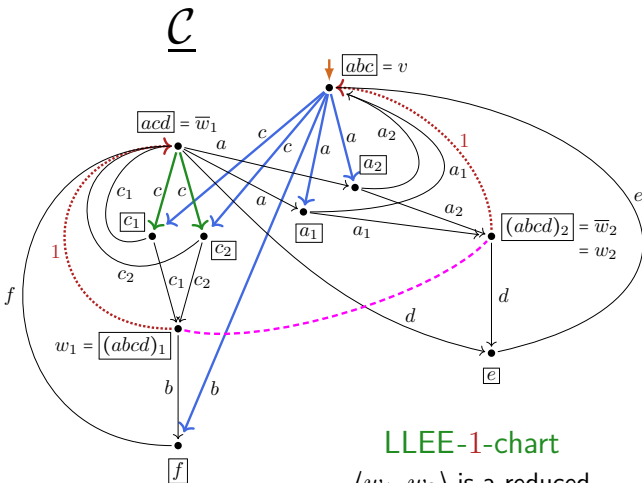
## Counterexample LLEE-preserving collapse



# Counterexample LLEE-preserving collapse



## Counterexample LLEE-preserving collapse



## LEE-1-chart

$\langle w_1, w_2 \rangle$  is a reduced

### 1-bisimilarity redundancy of kind (R3.4)

# Bisimulation collapse proof strategy (general case)

**Problem 2:** There are regular expressions  $e_1$  and  $e_2$  such that:

1-chart interpretations

LLEE  $\underline{C}(e_1)$   $\longleftrightarrow$   $\underline{C}(e_2)$  LLEE

chart interpretations

LLEE  $C(e_1)$   $\xleftrightarrow{\text{(assm)}} C(e_2)$  LLEE

bisimulation collapse

$C_0$   
LLEE

# Bisimulation collapse proof strategy (general case)

**Problem 2:** There are regular expressions  $e_1$  and  $e_2$  such that:

1-chart interpretations

LLEE  $\underline{\mathcal{C}}(e_1)$   $\longleftrightarrow$   $\underline{\mathcal{C}}(e_2)$  LLEE

chart interpretations

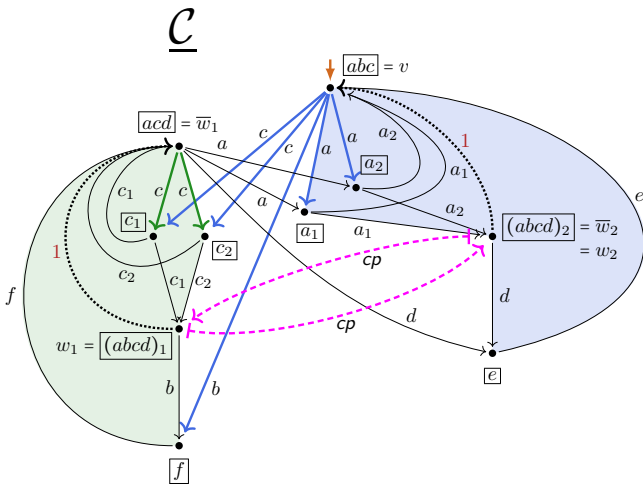
LLEE  $\mathcal{C}(e_1)$   $\xleftrightarrow{\text{(assm)}} \mathcal{C}(e_2)$  LLEE

for all  $\mathcal{C}_0$ :

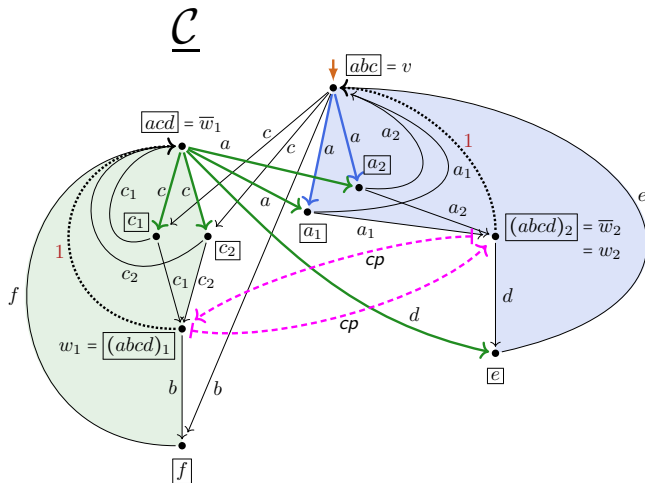
LLEE

$\mathcal{C}(e_1)$ ,  $\mathcal{C}(e_2)$ ,  $\underline{\mathcal{C}}(e_1)$  and  $\underline{\mathcal{C}}(e_2)$  are **not** LLEE-preservingly jointly minimizable under bisimilarity.

# Twin-Crystal

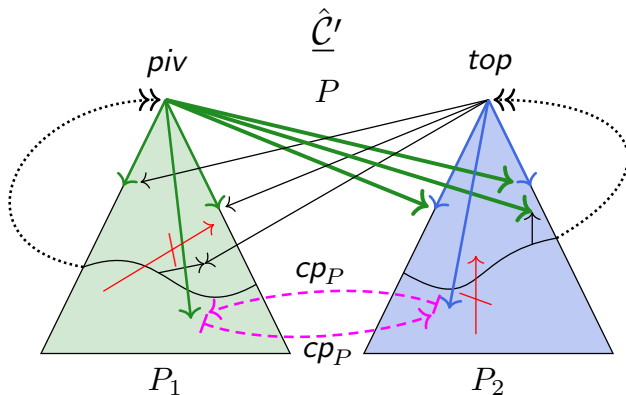


# Twin-Crystal

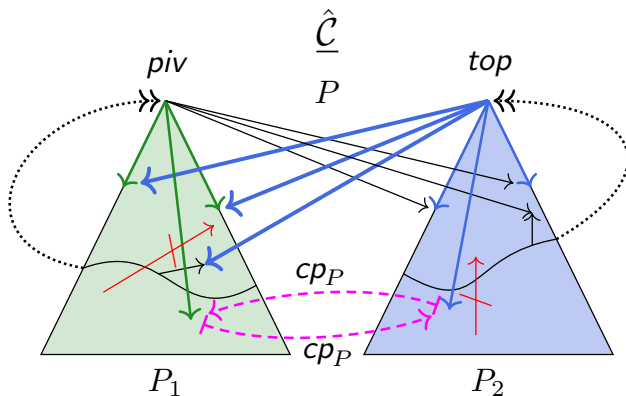




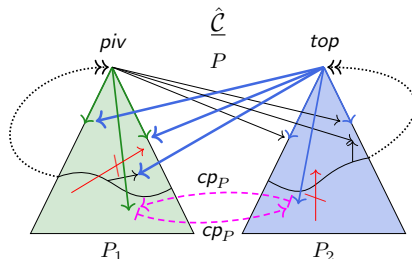
# Twin-Crystal



# Twin-Crystal



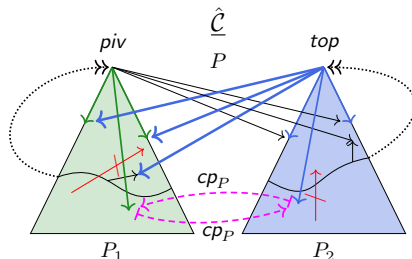
# Crystallization



twin-crystal

Crystallized 1-charts = LLEE-1-charts that are collapsed apart from strongly connected components of twin-crystal form.

# Crystallization

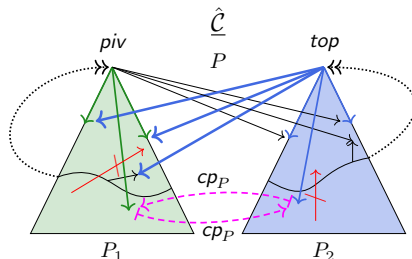


twin-crystal

Crystallized 1-charts = LLEE-1-charts that are collapsed apart from strongly connected components of twin-crystal form.

(CR) Crystallization: Every LLEE-1-chart can be reduced under bisimilarity to a 1-bisimilar crystallized 1-chart.

# Crystallization



twin-crystal

**Crystallized 1-charts** = **LLEE-1-charts** that are collapsed apart from strongly connected components of **twin-crystal** form.

**(CR) Crystallization:** Every **LLEE-1-chart** can be reduced under bisimilarity to a 1-bisimilar **crystallized 1-chart**.

**(CC)** Every **Mil-provable** solution of a **crystallized 1-chart** give rise to **Mil-provable** solution on the **bisimulation collapse**.

# Completeness proof of Mil (structure)

*chart interpretations*

$$\mathcal{C}(e_1) \xLeftrightarrow{(\text{assm})} \mathcal{C}(e_2)$$

$\xRightarrow{?}$

$$e_1 =_{\text{Mil}} e_2$$

# Completeness proof of Mil (structure)

*chart interpretations*

$$\mathcal{C}(e_1) \xLeftrightarrow{(\text{assm})} \mathcal{C}(e_2)$$

# Completeness proof of Mil (structure)

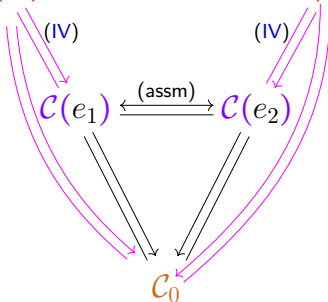
## 1-chart interpretations

guarded, LLEE  
 $e_1$  is solution

$$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$$

LLEE, guarded  
 $e_2$  is solution

## chart interpretations



## bisimulation collapse



# Completeness proof of Mil (structure)

## 1-chart interpretations

guarded, LLEE  
 $e_1$  is solution

$\underline{C}(e_1)$

$\underline{C}(e_2)$

LLEE, guarded  
 $e_2$  is solution

## chart interpretations

(CR)

$C(e_1)$

(assm)

$C(e_2)$

## crystallized 1-chart

guarded, LLEE

$\underline{C}_{10}$

## bisimulation collapse

$C_0$

# Completeness proof of Mil (structure)

## 1-chart interpretations

guarded, LLEE  
 $e_1$  is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE, guarded  
 $e_2$  is solution

## chart interpretations

(IV)  $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$   $C(e_1) \xrightarrow{(IV)} \underline{C}(e_2)$   
(CR)  $\underline{C}(e_1) \xrightarrow{(CR)} C(e_1)$   $C(e_2) \xrightarrow{(CR)} \underline{C}(e_2)$   
(assm)  $C(e_1) \xleftrightarrow{(assm)} C(e_2)$

## crystallized 1-chart

guarded, LLEE

$\underline{C}_{10}$

## bisimulation collapse

$C_0$

# Completeness proof of Mil (structure)

## 1-chart interpretations

guarded, LLEE  
 $e_1$  is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE, guarded  
 $e_2$  is solution

## chart interpretations

(IV)  $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$   
 (IV)  $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$   
 (assm)  $C(e_1) \longleftrightarrow C(e_2)$   
 (CR)  $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

## crystallized 1-chart

guarded, LLEE  
 $e_{10}$  is complete solution

$\underline{C}_{10}$

## bisimulation collapse

$C_0$

# Completeness proof of Mil (structure)

## 1-chart interpretations

guarded, LLEE  
 $e_1$  is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE, guarded  
 $e_2$  is solution

## chart interpretations

(IV)  $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$   
 (IV)  $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$   
 (assm)  $C(e_1) \longleftrightarrow C(e_2)$   
 (CR)  $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

## crystallized 1-chart

guarded, LLEE  
 $e_{10}$  is complete solution

$\underline{C}_{10}$

## bisimulation collapse

$C_0$

$e_{10}$  is solution (by (CC))

# Completeness proof of Mil (structure)

## 1-chart interpretations

guarded, LLEE  
 $e_1$  is solution  
 (by (T))  $e_{10}$  is solution

$\underline{C}(e_1)$

$\underline{C}(e_2)$

LLEE, guarded  
 $e_2$  is solution  
 (by (T))  $e_{10}$  is solution

## chart interpretations

(IV)

(IV)

(CR)

$C(e_1)$

$C(e_2)$

(assm)

## crystallized 1-chart

guarded, LLEE  
 $e_{10}$  is complete solution

$\underline{C}_{10}$

## bisimulation collapse

$C_0$

$e_{10}$  is solution (by (CC))

# Completeness proof of Mil (structure)

## 1-chart interpretations

guarded, LLEE  $\underline{C}(e_1)$   
 $e_1$  is solution  
 (by (T))  $e_{10}$  is solution

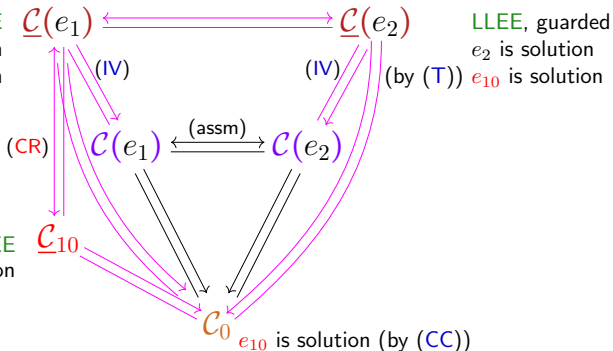
## chart interpretations

## crystallized 1-chart

guarded, LLEE  $\underline{C}_{10}$   
 $e_{10}$  is complete solution

## bisimulation collapse

$C(e_1)$  guarded, LLEE  
 $e_1$  is solution  
 $e_{10}$  is solution



$C(e_2)$  guarded, LLEE  
 $e_2$  is solution  
 $e_{10}$  is solution

# Completeness proof of Mil (structure)

## 1-chart interpretations

guarded, LLEE  $\underline{C}(e_1)$   
 $e_1$  is solution  
 (by (T))  $e_{10}$  is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE, guarded  $\underline{C}(e_2)$   
 $e_2$  is solution  
 (by (T))  $e_{10}$  is solution

## chart interpretations

(IV)  $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$   
 (IV)  $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$   
 (assm)  $C(e_1) \longleftrightarrow C(e_2)$   
 (CR)  $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

## crystallized 1-chart

guarded, LLEE  $C_{10}$   
 $e_{10}$  is complete solution

## bisimulation collapse

$C_0$   $e_{10}$  is solution (by (CC))

$C(e_1)$  guarded, LLEE  
 $e_1$  is solution  
 $e_{10}$  is solution  
 $\left. \vphantom{\begin{matrix} C(e_1) \text{ guarded, LLEE} \\ e_1 \text{ is solution} \\ e_{10} \text{ is solution} \end{matrix}} \right\} \xRightarrow{(SE)} e_1 =_{\text{Mil}} e_{10}$

$C(e_2)$  guarded, LLEE  
 $e_2$  is solution  
 $e_{10}$  is solution

# Completeness proof of Mil (structure)

## 1-chart interpretations

guarded, LLEE  $\underline{C}(e_1)$   
 $e_1$  is solution  
 (by (T))  $e_{10}$  is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE, guarded  $\underline{C}(e_2)$   
 $e_2$  is solution  
 (by (T))  $e_{10}$  is solution

## chart interpretations

(IV)  $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$   
 (IV)  $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$   
 (assm)  $C(e_1) \longleftrightarrow C(e_2)$   
 (CR)  $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

## crystallized 1-chart

guarded, LLEE  $\underline{C}_{10}$   
 $e_{10}$  is complete solution

## bisimulation collapse

$C_{10} \xrightarrow{(CC)} C_0$   
 $C(e_1) \xrightarrow{(CC)} C_0$   
 $C(e_2) \xrightarrow{(CC)} C_0$   
 $C_0$   $e_{10}$  is solution (by (CC))

$$\left\{ \begin{array}{l} C(e_1) \text{ guarded, LLEE} \\ e_1 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right\} \xRightarrow{(SE)} e_1 =_{\text{Mil}} e_{10} \quad e_{10} =_{\text{Mil}} e_2 \xleftarrow{(SE)} \left\{ \begin{array}{l} C(e_2) \text{ guarded, LLEE} \\ e_2 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right.$$



# Completeness proof of Mil (structure)

## 1-chart interpretations

guarded, LLEE  $\underline{C}(e_1)$   
 $e_1$  is solution  
 (by (T))  $e_{10}$  is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE, guarded  $\underline{C}(e_2)$   
 $e_2$  is solution  
 (by (T))  $e_{10}$  is solution

## chart interpretations

(IV)  $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$   
 (IV)  $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$   
 (assm)  $C(e_1) \longleftrightarrow C(e_2)$   
 (CR)  $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

## crystallized 1-chart

guarded, LLEE  $\underline{C}_{10}$   
 $e_{10}$  is complete solution

## bisimulation collapse

$C_{10} \xrightarrow{} C_0$   
 $C(e_1) \xrightarrow{} C_0$   
 $C(e_2) \xrightarrow{} C_0$   
 $C_0$   $e_{10}$  is solution (by (CC))

$\left. \begin{array}{l} C(e_1) \text{ guarded, LLEE} \\ e_1 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right\} \xRightarrow{(SE)} e_1 =_{\text{Mil}} e_{10} \quad e_{10} =_{\text{Mil}} e_2 \xleftarrow{(SE)} \left\{ \begin{array}{l} C(e_2) \text{ guarded, LLEE} \\ e_2 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right.$   
 $\xRightarrow{} e_1 =_{\text{Mil}} e_2$

# Completeness proof of Mil (structure)

## 1-chart interpretations

guarded, LLEE  
 $e_1$  is solution  
 (by (T))  $e_{10}$  is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE, guarded  
 $e_2$  is solution  
 (by (T))  $e_{10}$  is solution

## chart interpretations

(IV)  $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$   
 (IV)  $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$   
 (assm)  $C(e_1) \longleftrightarrow C(e_2)$   
 (CR)  $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

## crystallized 1-chart

guarded, LLEE  
 $e_{10}$  is complete solution

$\underline{C}_{10}$

## bisimulation collapse

$C_0$   $e_{10}$  is solution (by (CC))

## Theorem

Milner's proof system Mil is complete  
 for process semantics equivalence  $=_P$  of regular expressions.

Since:  $e_1 =_P e_2 \implies \llbracket e_1 \rrbracket_P = \llbracket e_2 \rrbracket_P \implies C(e_1) \leftrightarrow C(e_2) \implies e_1 =_{\text{Mil}} e_2$ .

# Outlook

## poster presentation

- ▶ tomorrow, 10–10.30

## next steps and projects

- ▶ monograph project: proof in fine-grained detail
- ▶ computation/animation tool for crystallization
- ▶ use crystallization for [recognition problem](#)

## resources on Github:

- ▶ <https://github.com/clegra/crystallization/blob/main>
  - ▶ article (after rebuttal): [/cryst-article.pdf](#)
  - ▶ poster: [/poster-lics2022.pdf](#)
  - ▶ presentation: [/presentation-lics2022.pdf](#)

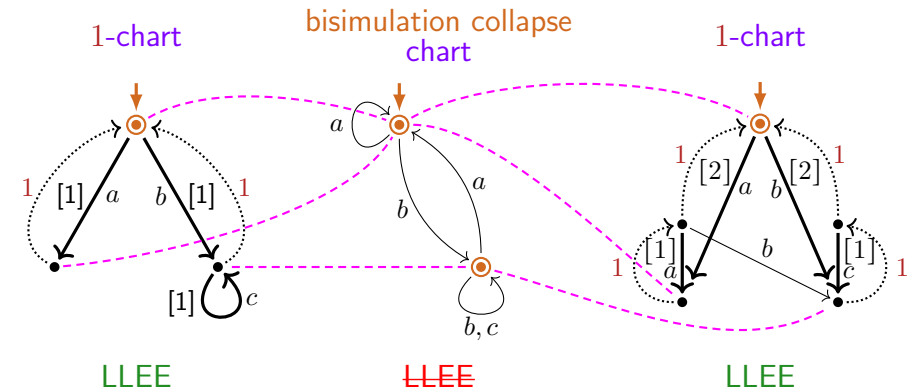
## acknowledgment & thanks to:

- ▶ [Wan Fokkink](#) (for long collaboration)

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# Thank you for your attention!

# Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)



$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

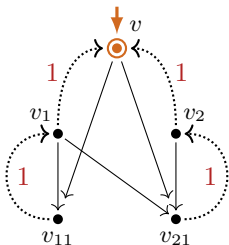
$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$

$$\underline{\mathcal{C}}((a + b \cdot c^*)^*)$$

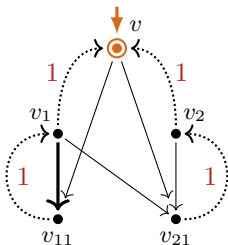
$$\mathcal{C}((a + b \cdot c^*)^*)$$

$$\underline{\mathcal{C}}((a^* \cdot (1 + b \cdot c^*))^*)$$

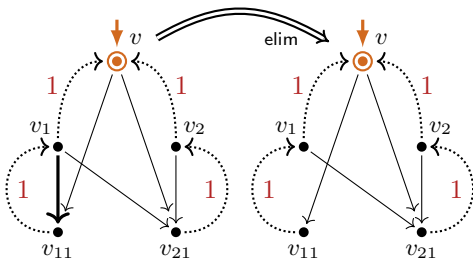
# Layered loop existence/elimination and LLEE-witnesses



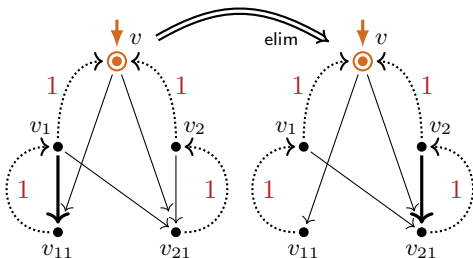
# Layered loop existence/elimination and LLEE-witnesses



# Layered loop existence/elimination and LLEE-witnesses

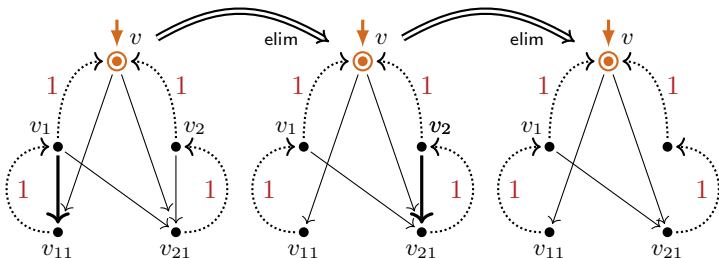


# Layered loop existence/elimination and LLEE-witnesses

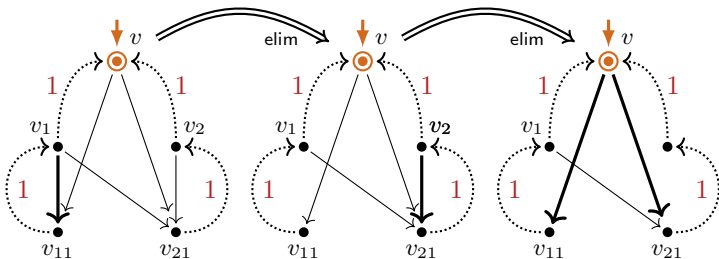




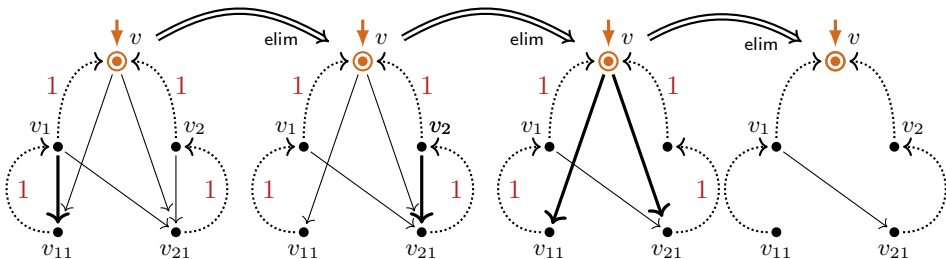
# Layered loop existence/elimination and LLEE-witnesses



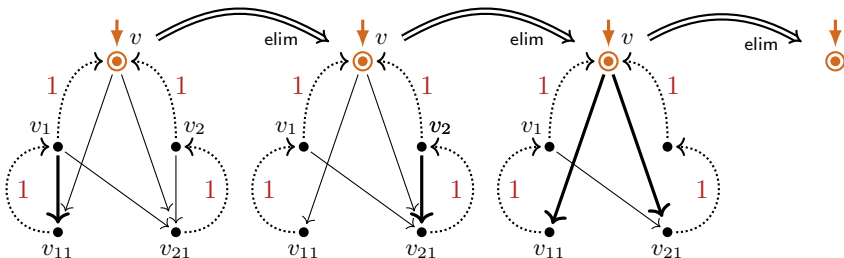
# Layered loop existence/elimination and LLEE-witnesses



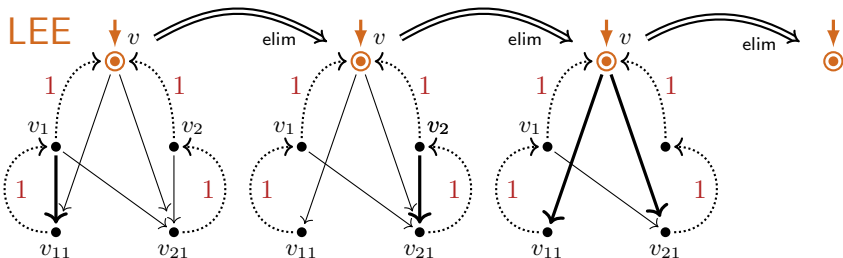
# Layered loop existence/elimination and LLEE-witnesses



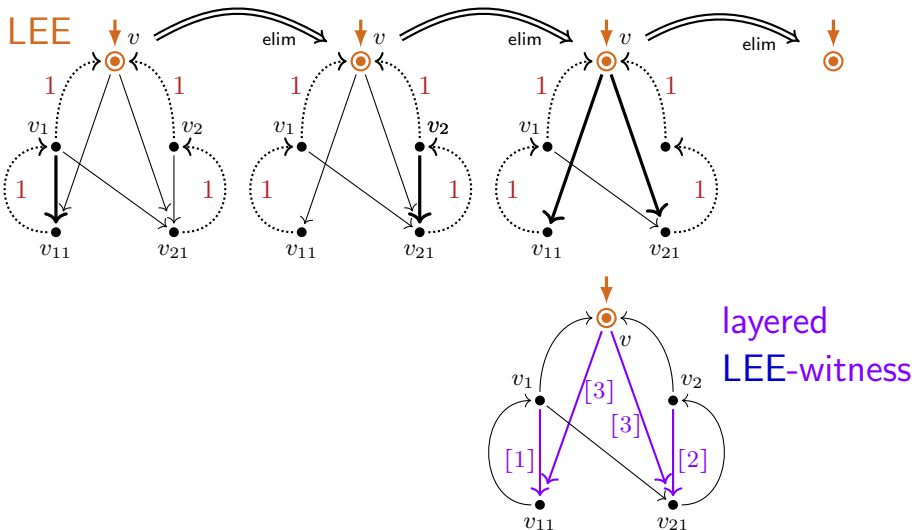
# Layered loop existence/elimination and LLEE-witnesses



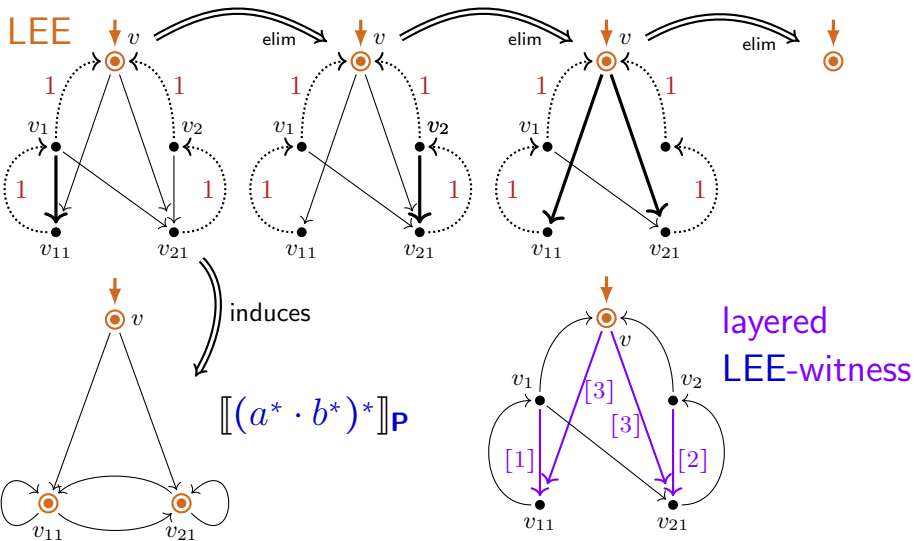
# Layered loop existence/elimination and LLEE-witnesses



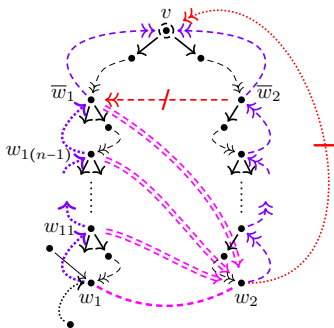
# Layered loop existence/elimination and LLEE-witnesses



# Layered loop existence/elimination and LLEE-witnesses



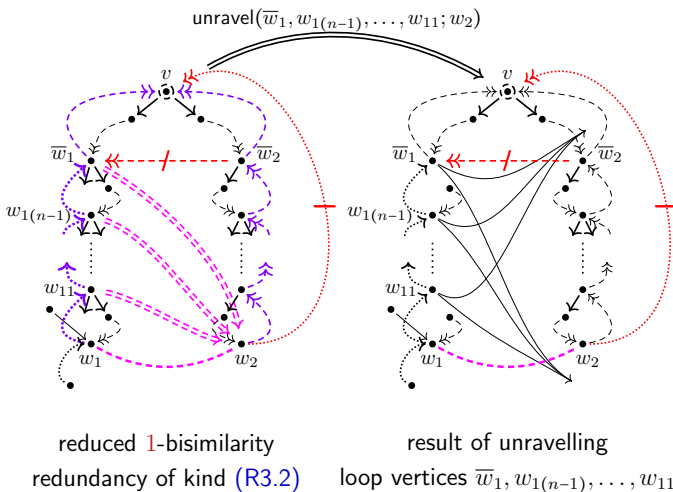
# Eliminating reduced 1-bisimilarity redundancies (example)



reduced 1-bisimilarity  
redundancy of kind (R3.2)



# Eliminating reduced 1-bisimilarity redundancies (example)



# Eliminating reduced 1-bisimilarity redundancies (example)

