Milner's Proof System for Regular Expressions Modulo Bisimilarity is Complete

Crystallization: Near-Collapsing Process Graph Interpretations of Regular Expressions

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Abstract

Milner (1984) defined a process semantics for regular expressions. He formulated a sound proof system for bisimilarity of process interpretations of regular expressions, and asked whether this system is complete.

We report conceptually on a proof that shows that Milner's system is complete, by motivating and describing all of its main steps. We substantially refine the completeness proof by Grabmayer and Fokkink (2020) for the restriction of Milner's system to '1-free' regular expressions. As a crucial complication we recognize that process graphs with empty-step transitions that satisfy the layered loop-existence/elimination property LLEE are not closed under bisimulation collapse (unlike process graphs with LLEE that only have proper-step transitions). We circumnavigate this obstacle by defining a LLEE-preserving 'crystallization procedure' for such process graphs. By that we obtain 'near-collapsed' process graphs with LLEE whose strongly connected components are either collapsed or of 'twin-crystal' shape. Such near-collapsed process graphs guarantee provable solutions for bisimulation collapses of process interpretations of regular expressions.

CCS Concepts: • **Theory of computation → Process calculi**; Regular languages.

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Kleene [14] (1951) introduced regular expressions, which are widely studied in formal language theory. In a typical formulation, they are constructed from constants 0, 1, letters a from some alphabet (interpreted as the formal languages $\varnothing$, $\{\epsilon\}$, and $\{a\}$, where ϵ is the empty word) and binary operators $+$ and $\cdot$, and the unary Kleene star * (which are interpreted as language union, concatentation, and iteration).

Milner [15] (1984) introduced a process semantics for regular expressions. He defined an interpretation $C(e)$ of regular expressions e as charts (finite process graphs): the interpretation of 0 is deadlock, of 1 is successful termination, letters a are atomic actions, the operators $+$ and $\cdot$ stand for choice and concatenation of processes, and (unary) Kleene star $(\cdot)^*$ represents iteration with the option to terminate successfully before each execution of the iteration body. He then defined the process semantics of 'star expressions' (regular expressions in this context) e as 'star behaviors' $[\![e]\!]_P := [C(e)]_{\underline{\leftrightarrow}}$, that is, as equivalence classes of chart interpretations with respect to bisimilarity $\underline{\leftrightarrow}$. Milner was interested in an axiomatization of equality of 'star behaviors'. For this purpose he adapted Salomaa's complete proof system [16] for language equivalence on regular expressions to a system Mil (see Def. 2.6) that is sound for equality of denoted star behaviors. Recognizing that Salomaa's proof strategy cannot be followed directly, he left completeness as an open question.

Over the past 38 years, completeness results have been obtained for restrictions of Milner's system to the following subclasses of star expressions: (a) without 0 and 1, but with binary star iteration $e_1{}^{\circledast}e_2$ instead of unary star [6], (b) with 0, with iterations restricted to exit-less ones $(\cdot)^* \cdot 0$, without 1 [5] and with 1 [4], (c) without 0, and with only restricted occurrences of 1 [3], and (d) '1-free' expressions formed with 0, without 1, but with binary instead of unary iteration [13]. By refining concepts developed in [13] for the proof of (d) we can finally establish completeness of Mil.

The aim of this article. We provide an outline of the completeness proof for Mil. Hereby our focus is on the main new concepts and results. For scrutiny of most of the crucial arguments of the proof we refer to the appendix, which is the kernel of a monograph on the proof that we are writing. While details are sometimes only hinted at in this article, we think that the crystallization technique we present opens up a wide space for other applications (we suggest one in Sect. 9). We want to communicate it in summarized form to the community in order to stimulate its further development.

1 Motivation for the chosen proof strategy

We explain the main obstacle we encountered for developing our proof strategy through explaining shortcomings of existing approaches. Finally we describe crucial new concepts that we use for adapting the collapse strategy from [12, 13].

Obstacle for the 'bisimulation chart' proof strategy. Milner [15] recognized that completeness of the proof system Mil cannot be established along the lines of Salomaa's completeness proof for his proof system $\mathbf{F_1}$ of language equivalence of regular expressions [16]. The reason is as follows. Adopting Salomaa's proof strategy would mean (i) to link given bisimilar chart interpretations $C(e_1)$ and $C(e_2)$ of star

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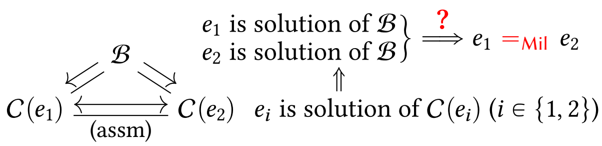
expressions e_1 and e_2 via a chart $\mathcal{B}$ that represents a bisimulation between $C(e_1)$ and $C(e_2)$, (ii) to use this link via functional bisimulations from $\mathcal{B}$ to $C(e_1)$ and $C(e_2)$ to prove equal in Mil the provable solutions e_1 of $C(e_1)$, and e_2 of $C(e_2)$, (iii) to extract from $\mathcal{B}$ a star expression e that provably solves $\mathcal{B}$, and then is provably equal to e_1 and e_2. Here a 'provable solution' of a chart C is a function s from its set of vertices to star expressions such that the value $s(v)$ at a vertex v can be reconstructed, provably in Mil, from the transitions to, and the expressions at, immediate successor vertices of v in C, and (non-)termination at v. By the 'principal value' of a provable solution we mean its value at the start vertex. In pictures we write 'e is solution' for 'e is the principal value of a solution'.



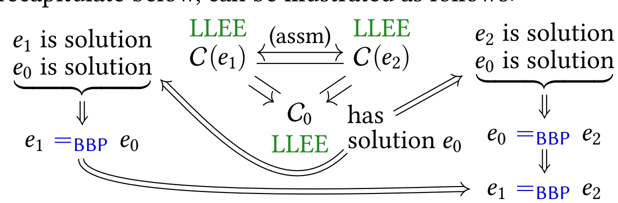
First by (i) star expressions e_1 and e_2 can be shown to be the principal values of provable solutions of their chart interpretations $C(e_1)$ and $C(e_2)$, respectively. These solutions can be transferred backwards by (ii) over the functional bisimulations from the bisimulation chart $\mathcal{B}$ to $C(e_1)$ and $C(e_2)$, respectively. It follows that e_1 and e_2 are the principal values of two provable solutions of $\mathcal{B}$. However, now the obstacle appears, because the extraction procedure in (iii) of a proof of $e_1 = e_2$ in Mil cannot work, like Salomaa's, for all charts $\mathcal{B}$ irrespective of the actions of its transitions. An example that demonstrates that is the chart C_{12} in Ex. 4.1 in [12, 13]. This is because some charts are unsolvable ("[In] contrast with the case for languages—an arbitrary system of guarded equations in [star]-behaviours cannot in general be solved in star expressions" [15]), but turn into a solvable one if all actions in it are replaced by a single one. The reason for the failure of Salomaa's extraction procedure is then that the absence in Mil of the *left*-distributivity law $x \cdot (y + z) = x \cdot y + x \cdot z$ (it is not sound under bisimilarity) frequently prevents applications of the fixed-point rule RSP* in Mil unlike for the system $\mathbf{F_1}$ that Salomaa proved complete. We conclude that such a bisimulation-chart proof strategy, inspired by Salomaa [16], is not expedient for showing completeness of Mil.

However, if the fixed-point rule RSP* in Milner's system is replaced in Mil by a general unique-solvability rule scheme USP for guarded systems of equations, then a proof system arises to which the bisimulation-chart proof strategy is applicable. That system can therefore be shown to be complete comparatively easily (as noted in [9]).

Loop existence and elimination. A sufficient structural condition for solvability of a chart, and correspondingly of a linear system of recursion equations, by a regular expression modulo bisimilarity was given by Grabmayer and Fokkink in [13]: the 'loop existence and elimination' condition LEE, and its 'layered' specialization LLEE, which is independent of the specific actions in a chart. These properties are refinements for graphs of 'well-behaved specifications' due to Baeten and Corradini in [2] that single out a class of 'palm trees' (trees with back-links) that specify star expressions under the process interpretation. For showing that the tailored restriction BBP of Milner's system Mil to '1-free' star expressions (without 1, but with binary instead of unary star iteration) is complete, the following properties were established in [13]: **($\mathbf{I}_{\nmid}$)** Chart interpretations of 1-free star expressions are LLEE-charts. **($\mathbf{S}_{\nmid}$)** Every 1-free star expression e is the principal value of a provable solution of its chart interpretation $C(e)$. **($\mathbf{E}_{\nmid}$)** From every LLEE-chart C a provable solution of C (by 1-free star expressions) can be extracted. **($\mathbf{SE}_{\nmid}$)** All provable solutions of a LLEE-chart are provably equal. **($\mathbf{P}_{\nmid}$)** Every provable solution can be pulled back from the target to the source chart of a functional bisimulation to obtain a provable solution of the source chart. **($\mathbf{C}_{\nmid}$)** The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

As a consequence of these properties, a finite chart C is expressible by a 1-free star expression modulo bisimilarity if and only if the bisimulation collapse of C satisfies LLEE.

The 'bisimulation collapse' proof strategy for BBP ([13]). For the completeness proof of the tailored restriction BBP of Milner's system Mil to 1-free star expressions, Grabmayer and Fokkink in [13] linked bisimilar chart interpretations $C(e_1)$ and $C(e_2)$ of 1-free star expressions e_1 and e_2 via the joint bisimulation collapse C_0. That argument, which we recapitulate below, can be illustrated as follows:



By ($\mathbf{S}_{\nmid}$), the star expressions e_1 and e_2 are the principal values of provable solutions s_1 and s_2 of their chart interpretations $C(e_1)$ and $C(e_2)$, respectively. Furthermore, by ($\mathbf{I}_{\nmid}$) the chart interpretations of the 1-free star expressions $C(e_1)$ and $C(e_2)$ have the property LLEE. Since LLEE is preserved under the operation of bisimulation collapse due to ($\mathbf{C}_{\nmid}$), the joint bisimulation collapse C_0 of $C(e_1)$ and $C(e_2)$ is again a LLEE-chart. Therefore a provable solution s_0 can be extracted from C_0 due to ($\mathbf{E}_{\nmid}$). Let e_0 be its principal value. The solution s_0 can be pulled back from C_0 conversely over the functional bisimulations from $C(e_1)$ and $C(e_2)$ to C_0 due to ($\mathbf{P}_{\nmid}$), and thereby defines provable solutions $\tilde{s}_1$ of $C(e_1)$ and $\tilde{s}_2$ of $C(e_2)$, both with e_0 as the principal value. Now having provable solutions s_1 and $\tilde{s}_1$ of the LLEE-chart $C(e_1)$, these solutions are BBP-provably equal by ($\mathbf{SE}_{\nmid}$), and hence also their principal values e_1 and e_0, that is $e_1 =_{\mathsf{BBP}} e_0$. Analogously $e_1 =_{\mathsf{BBP}} e_0$ can be established. Then $e_1 =_{\mathsf{BBP}} e_2$ follows by applying symmetry and transitivity proof rules of equational logic.

Obstacles for a 'bisimulation collapse' strategy for Mil. A generalization of this argument for arbitrary star expressions runs into two problems that can be illustrated as:

$$\cancel{(\mathrm{I})}:\quad C(e)^{\cancel{\text{LLEE}}} \qquad\qquad \cancel{(\mathrm{C})}:\quad \begin{array}{c} \underline{C}^{\,\text{LLEE}} \\ \downarrow\!| \\ \text{any 1-bisimulation collapse }\ \underline{C}_0^{\,\cancel{\text{LLEE}}}\ \text{ of } \underline{C} \end{array}$$

First, see ~~(I)~~, there are star expressions e whose chart interpretation $C(e)$ satisfies neither LEE nor LLEE, as was noted in [11]. In order to still be able to utilize LLEE, in [11] a variant chart interpretation $\underline{C}(e)$ was defined for star expressions e such that $\underline{C}(e)$ is 'LLEE-1-chart', that is, a chart with '1-transitions' (explicit empty-step transitions) that satisfies LLEE, and $\underline{C}(e)$ is '1-bisimilar' to the chart interpretation $C(e)$. Hereby '1-bisimulations' and '1-bisimilarity' are adaptations of bisimulations and bisimilarity to 1-charts.

However, use of the variant chart interpretation encounters the second obstacle ~~(C)~~ as illustrated above. A part of it that was also observed in [11] is that LLEE-1-charts are not closed under bisimulation collapse, unlike LLEE-charts. While this renders the bisimulation collapse proof strategy unusable, we here show that an adaptation to a '1-bisimulation collapse' strategy is not possible, either, if it is based on '1-bisimulation collapsed' 1-charts in which none of its vertices are 1-bisimilar. In particular we show ~~(C)~~ as the first of our key observations and concepts as listed below:

1. *LLEE-1-charts are not in general collapsible to (collapsed) LLEE-1-charts.* Nor do 1-bisimilar LLEE-1-charts always have a joint (1-bisimilarity) minimization. We demonstrate this by a example (see Fig. 4).

The second part of 1 also prevents the generalization of a variation of the proof in [13] sketched above that uses that any two bisimilar LLEE-charts (thus without 1-transitions) are jointly minimizable under bisimilarity (which was shown by Schmid, Rot, and Silva in [17]).

How we recover the collapse proof strategy for Mil. We define 'crystallized' approximations with LLEE of collapsed LLEE-1-charts in order to show that bisimulation collapses of LLEE-1-charts have provable solutions. For this purpose we combine the following concepts and their properties:

2. *Twin-Crystals*: These are 1-charts with a single strongly connected comTonent (scc) that exhibit a self-inverse symmetry function that links 1-bisimilar vertices. Twin-Crystals abstract our example that demonstrates 1.
3. *Near-Collapsed* 1-charts: These are 1-charts in which 1-bisimilar vertices appear as pairs that are linked by a self-inverse function that induces a 'grounded 1-bisimulation slice'. Twin-Crystals are near-collapsed LLEE-1-charts.
4. *Crystallization:* By this we understand a process of stepwise minimization of LLEE-1-charts under 1-bisimilarity that produces 1-bisimilar 'crystallized' LLEE-1-charts in which all strongly connected comTonents are collapsed or of twin-crystal shape. This process uses the connect-through operation from [13] for 1-bisimilar vertices. We show that crystallized 1-charts are near-collapsed.
5. *Complete* Mil*-provable solution* of a 1-chart $\underline{C}$: This is a Mil-provable solution of $\underline{C}$ with the property that its values for 1-bisimilar vertices of $\underline{C}$ are Mil-provably equal. Any complete Mil-provable solution of a 1-chart $\underline{C}$ yields a Mil-provable solution of the bisimulation collapse of $\underline{C}$.
6. *Elevation* of vertex sets above 1-charts: This is a concept of partially unfolding 1-charts that facilitates us to show that near-collapsed weakly guarded LLEE-1-charts have complete Mil-provable solutions.

With these conceptual tools we will be able to recover the collapse proof strategy for LLEE-1-charts. The idea is to establish, for given 1-bisimilar LLEE-1-charts $\underline{C}_1$ and $\underline{C}_2$, a link via which solutions of $\underline{C}_1$ and $\underline{C}_2$ with the same principal value can be obtained. We create such a link via the crystallized LLEE-1-chart $\underline{C}_{10}$ of (one of them, say) $\underline{C}_1$ and the joint bisimulation collapse C_0 of $\underline{C}_1$ and $\underline{C}_2$. Then a solution of $\underline{C}_{10}$ can obtained, transferred first to C_0, and then to $\underline{C}_1$ and to $\underline{C}_2$. This argument will be illustrated in Fig. 3 in Section 4, where we give the completeness proof for Mil based on lemmas.

2 Preliminaries

Let A be a set whose members we call *actions*. The set $StExp(A)$ of *star expressions over actions in* A is defined by the following grammar, where $a \in A$:

$$e, e_1, e_2 \ ::= \ 0 \ \mid \ 1 \ \mid \ a \ \mid \ e_1 + e_2 \ \mid \ e_1 \cdot e_2 \ \mid \ e^*$$

Definition 2.1 (1-charts and 1-LTSs)**.** A 1-*chart* is a 6-tuple $\langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ with V a finite set of *vertices*, A a finite set of *(proper) action labels*, $1 \notin A$ the specified *empty step label*, $v_s \in V$ the *start vertex* (hence $V \neq \emptyset$), $\rightarrow \subseteq V \times \underline{A} \times V$ the *labeled transition relation*, where $\underline{A} := A \cup \{1\}$ is the set of action labels including 1, and $\downarrow \subseteq V$ a set of *vertices with immediate termination* (or *terminating* vertices). In such a 1-chart, we call a transition in $\rightarrow \cap\, (V \times A \times V)$ (labeled by a *proper action* in A) a *proper transition*, and a transition in $\rightarrow \cap\, (V \times \{1\} \times V)$ (labeled by 1) a 1-*transition*. Reserving non-underlined action labels like $a, b, \ldots$ for proper actions, we use highlighted underlined action label symbols like $\underline{a}$ for actions labels in the set $\underline{A}$ that includes the label 1.

A 1-*LTS* (a *labeled transition system with* 1*-transitions and immediate termination*) is a 5-tuple $\langle V, A, 1, \rightarrow, \downarrow\rangle$ with concepts as explained above but without a start vertex. For every 1-chart $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ we denote by $\underline{\mathcal{L}}(\underline{C}) = \langle V, A, 1, \rightarrow, \downarrow\rangle$ the 1-LTS *underlying* $\underline{C}$ (or *that underlies* $\underline{C}$).

We say that a 1-chart $\underline{C}$ (a 1-LTS $\underline{\mathcal{L}}$) is *weakly guarded* (*w.g.*) if $\underline{C}$ (resp. $\underline{\mathcal{L}}$) does not have an infinite path of 1-transitions.

By an *induced a-transition* $v \xrightarrow{(a)} w$, for a proper action $a \in A$, in a 1-chart $\underline{C}$ (in a 1-LTS $\underline{\mathcal{L}}$) we mean a path of the form $v \xrightarrow{1} \cdots \xrightarrow{1} \cdot \xrightarrow{a} w$ in $\underline{C}$ (in $\underline{\mathcal{L}}$) that consists of a finite number of 1-transitions that ends with a proper a-transition. By *induced termination* $v{\downarrow}^{(1)}$, for $v \in V$ we mean that there is a path $v \xrightarrow{1} \cdots \xrightarrow{1} \tilde{v}$ with $\tilde{v}{\downarrow}$ in $\underline{C}$ (in $\underline{\mathcal{L}}$).

By a *chart* (a *LTS*) we mean a 1-transition free 1-chart (a 1-transition free LTS). Let $\underline{C} = \langle V, A, 1, v_s, \to, \downarrow\rangle$ be a 1-chart. We define by $\underline{C}_{(\cdot)} := \langle V, A, 1, v_s, \xrightarrow{(\cdot)}, {\downarrow}^{(1)}\rangle$ *the induced chart of* $\underline{C}$ whose transitions are the induced transitions of $\underline{C}$, and whose terminating vertices are the vertices of $\underline{C}$ with induced termination. Note that $\underline{C}_{(\cdot)}$ is 1-transition free. Also, for every vertex $w \in V$ we denote by $\underline{C}{\downarrow}^{w}_{*} := \langle V, A, 1, w, \to, \downarrow\rangle$ the *generated sub(-1)-chart of* $\underline{C}$ *at* w.

Definition 2.2 (1-bisimulating slices, 1-bisimulations for 1-LTSs)**.** Let $\underline{\mathcal{L}}_i = \langle V_i, A, 1, \to_i, {\downarrow}_i\rangle$ for $i \in \{1,2\}$ be 1-LTSs.

A *1-bisimulating slice between* $\underline{\mathcal{L}}_1$ *and* $\underline{\mathcal{L}}_2$ is a binary relation $B \subseteq V_1 \times V_2$, with active domain $W_1 := \mathrm{dom}_{\mathrm{act}}(B) = \pi_1(B)$, and active codomain $W_2 := \mathrm{cod}_{\mathrm{act}}(B) = \pi_2(B)$, where $\pi_i : V_1 \times V_2 \to V_i, \pi_i(\langle v_1, v_2\rangle) = v_i$, for $i \in \{1,2\}$, such that $B \neq \emptyset$, and for all $\langle v_1, v_2\rangle \in B$ the three conditions hold:

$$\text{(forth)}_{\mathrm{s}}\ \forall a \in A\, \forall v_1' \in V_1 \big(v_1 \xrightarrow{(a)}_1 v_1' \ \wedge\ \underline{v_1' \in W_1} \implies \exists v_2' \in V_2 \big(v_2 \xrightarrow{(a)}_2 v_2' \wedge \langle v_1', v_2'\rangle \in B \big)\big),$$

$$\text{(back)}_{\mathrm{s}}\ \forall a \in A\, \forall v_2' \in V_2 \big(v_2 \xrightarrow{(a)}_2 v_2' \ \wedge\ \underline{v_2' \in W_2} \implies \exists v_1' \in V_1 \big(v_1 \xrightarrow{(a)}_1 v_1' \wedge \langle v_2', v_1'\rangle \in B \big)\big),$$

$$\text{(termination)}\ v_1{\downarrow}_1^{(1)} \iff v_2{\downarrow}_2^{(1)}.$$

Here $\text{(forth)}_{\mathrm{s}}$ entails $v_2' \in W_2$, and $\text{(back)}_{\mathrm{s}}$ entails $v_1' \in W_1$.

A *1-bisimulation between* $\underline{\mathcal{L}}_1$ *and* $\underline{\mathcal{L}}_2$ is a 1-bisimulation slice B between $\underline{\mathcal{L}}_1$ and $\underline{\mathcal{L}}_2$ such that the active domain of B, and the active codomain of B, are transition-closed (that is, closed under $\to_1$ and $\to_2$, respectively), or equivalently, a non-empty relation $B \subseteq V_1 \times V_2$ such that for every $\langle w_1, w_2\rangle \in B$ the conditions (forth), (back), (termination) hold, where (forth), and (back) result from $\text{(forth)}_{\mathrm{s}}$ and $\text{(back)}_{\mathrm{s}}$ by dropping the underlined conjuncts.

By a *1-bisimulation slice between* $\underline{\mathcal{L}}_1$ *and* $\underline{\mathcal{L}}_2$ we mean a 1-bisimulating slice between $\underline{\mathcal{L}}_1$ and $\underline{\mathcal{L}}_2$ that is contained in a 1-bisimulation between $\underline{\mathcal{L}}_1$ and $\underline{\mathcal{L}}_2$.

A 1-bisimulating slice (a 1-bisimulation slice, a 1-bisimulation) *on* a 1-LTS $\underline{\mathcal{L}}$ is a 1-bisimulating slice (and resp., a 1-bisimulation slice, a 1-bisimulation) between $\underline{\mathcal{L}}$ and $\underline{\mathcal{L}}$.

Definition 2.3 ((funct.) 1-bisimulation between 1-charts)**.** We consider 1-charts $\underline{C}_i = \langle V_i, A, 1, v_{s,i}, \to_i, {\downarrow}_i\rangle$ for $i \in \{1,2\}$.

A *1-bisimulation* between 1-charts $\underline{C}_1$ and $\underline{C}_2$ is a 1-bisimulation $B \subseteq V_1 \times V_2$ between the 1-LTSs $\underline{\mathcal{L}}(\underline{C}_1)$ and $\underline{\mathcal{L}}(\underline{C}_2)$ underlying $\underline{C}_1$ and $\underline{C}_2$, respectively, such that additionally:

(start) $\langle v_{s,1}, v_{s,2}\rangle \in B$ (B relates start vertices of $\underline{C}_1$ and $\underline{C}_2$)

holds; thus B must satisfy (start), and, for all $\langle w_1, w_2\rangle \in B$, the conditions (forth), (back), (termination) from Def. 2.2.

By a *functional* 1-bisimulation *from* $\underline{C}_1$ *to* $\underline{C}_2$ we mean a 1-bisimulation between $\underline{C}_1$ and $\underline{C}_2$ that is the graph of a partial function from V_1 to V_2. By $\underline{C}_1 \leftrightarrow \underline{C}_2$ (by $\underline{C}_1 \rightarrow \underline{C}_2$) we denote that there is a 1-bisimulation between $\underline{C}_1$ and $\underline{C}_2$ (respectively, a functional 1-bisimulation from $\underline{C}_1$ to $\underline{C}_2$).

Definition 2.4. Let $\underline{C} = \langle V, A, 1, v_s, \to, \downarrow\rangle$ be a 1-chart.

By $\leftrightarrow_{\underline{C}}$ we denote *1-bisimilarity on* $\underline{C}$, the largest 1-bisimulation (which is the union of all 1-bisimulations) between $\underline{C}$ and $\underline{C}$ itself. If $w_1 \leftrightarrow_{\underline{C}} w_2$ holds for vertices $w_1, w_2 \in V$, then we say that w_1 *and* w_2 *are 1-bisimilar in* $\underline{C}$.

We call $\underline{C}$ *1-collapsed*, and a *1-bisimulation collapse*, if $\leftrightarrow_{\underline{C}} = \mathrm{id}_V$ holds, that is, if 1-bisimilar vertices of $\underline{C}$ are identical. If, additionally, $\underline{C}$ does not contain any 1-transitions, then we call $\underline{C}$ *collapsed*, and a *bisimulation collapse*.

Let $w_1, w_2 \in V$. We say that w_1 *is a substate of* w_2, denoted by $w_1 \sqsubseteq_{\underline{C}} w_2$, if the pair $\langle w_1, w_2\rangle$ forth-progresses to 1-bisimilarity on $\underline{C}$ in the sense of the following conditions:

$$\text{(prog-forth)}\quad \forall w_1' \in V_1 \forall a \in A \big(w_1 \xrightarrow{(a)} w_1' \implies \exists w_2' \in V_2 \big(w_2 \xrightarrow{(a)} w_2' \wedge w_1' \leftrightarrow_{\underline{C}} w_2' \big)\big),$$

$$\text{(prog-termination)}\quad w_1{\downarrow}^{(1)} \implies w_2{\downarrow}^{(1)}.$$

Definition 2.5. The *chart interpretation of* a star expression $e \in \mathit{StExp}(A)$ is the (1-tr. free) chart $C(e) = \langle V(e), A, 1, e, \to \cap (V(e) \times V(e)), {\downarrow} \cap V(e)\rangle$ where $V(e)$ consists of all star expressions that are reachable from e via transitions of the labeled transition relation $\to \subseteq \mathit{StExp}(A) \times A \times \mathit{StExp}(A)$, which is defined, together with the imm.-termination relation ${\downarrow} \subseteq \mathit{StExp}(A)$, by derivability in the transition system specification (TSS) $\mathcal{T}(A)$, where $a \in A$, $e, e_1, e_2, e' \in \mathit{StExp}(A)$:

$$\frac{}{1{\downarrow}} \qquad \frac{e_i{\downarrow}}{(e_1 + e_2){\downarrow}} \qquad \frac{e_1{\downarrow} \quad e_2{\downarrow}}{(e_1 \cdot e_2){\downarrow}} \qquad \frac{}{(e^*){\downarrow}}$$

$$\frac{}{a \xrightarrow{a} 1} \qquad \frac{e_i \xrightarrow{a} e_i'}{e_1 + e_2 \xrightarrow{a} e_i'} \qquad \frac{e \xrightarrow{a} e'}{e^* \xrightarrow{a} e' \cdot e^*}$$

$$\frac{e_1 \xrightarrow{a} e_1'}{e_1 \cdot e_2 \xrightarrow{a} e_1' \cdot e_2} \qquad \frac{e_1{\downarrow} \quad e_2 \xrightarrow{a} e_2'}{e_1 \cdot e_2 \xrightarrow{a} e_2'}$$

Definition 2.6. Milner's proof system Mil on star expressions has the following axioms (here numbered differently):

(A1)	$e + (f + g) = (e + f) + g$	(A7)	$e = 1 \cdot e$
(A2)	$e + 0 = e$	(A8)	$e = e \cdot 1$
(A3)	$e + f = f + e$	(A9)	$0 = 0 \cdot e$
(A4)	$e + e = e$	(A10)	$e^* = 1 + e \cdot e^*$
(A5)	$e \cdot (f \cdot g) = (e \cdot f) \cdot g$	(A11)	$e^* = (1 + e)^*$
(A6)	$(e + f) \cdot g = e \cdot g + f \cdot g$		

The rules of Mil are the basic inference rules of equational logic (reflexivity, symmetry, transitivity of $=$, compatibility of $=$ with $+, \cdot, (\cdot)^*$) as well as the fixed-point rule RSP*:

$$\frac{e = f \cdot e + g}{e = f^* \cdot g}\ \text{RSP}^*\ (\text{if } f{\not\downarrow})$$

By $e_1 =_{\mathsf{Mil}} e_2$ we denote that $e_1 = e_2$ is derivable in Mil.

By Mil^- we denote the purely equational part of Mil that results by dropping the rule scheme RSP* from Mil.

Definition 2.7. While we formulate the stipulations below for 1-LTSs, we will use them also for 1-charts. So we let $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow \rangle$ be a 1-LTS, and we let $\mathcal{S} \in \{\mathsf{Mil}, \mathsf{Mil}^{-}\}$.

By a *star expression function on* $\underline{\mathcal{L}}$ we mean a function $s : V \rightarrow \mathit{StExp}(A)$ on the vertices of $\underline{\mathcal{L}}$. Now we let $v \in V$. We say that such a star expression function s on $\underline{\mathcal{L}}$ is an $\mathcal{S}$*-provable solution of* $\underline{\mathcal{L}}$ *at* v if it holds that:

$$s(v) =_{\mathcal{S}} \tau_{\underline{\mathcal{L}}}(v) + \sum_{i=1}^{n} \underline{a_i} \cdot s(v_i)$$

given that $\mathit{Tr}_{\underline{\mathcal{L}}}(v) = \left\{ v \xrightarrow{\underline{a_i}} v_i \;\middle|\; i \in \{1, \ldots, n\} \right\}$ is a (possibly redundant) list representation of transitions from v in $\underline{\mathcal{L}}$, and where $\tau_{\underline{\mathcal{L}}}(v)$ is the *termination constant* $\tau_{\underline{\mathcal{L}}}(v)$ *of* $\underline{\mathcal{L}}$ *at* v defined as 0 if $v\not\downarrow$, and as 1 if $v\downarrow$. This definition does not depend on the specifically chosen list representation of $\mathit{Tr}_{\underline{\mathcal{L}}}(v)$, because $\mathcal{S}$ contains the associativity, commutativity, and idempotency axioms for $+$.

By an $\mathcal{S}$*-provable solution of* $\underline{\mathcal{L}}$ (with *principal value* $s(v_s)$ at the start vertex v_s) we mean a star expression function s on $\underline{\mathcal{L}}$ that is an $\mathcal{S}$-provable solution of $\underline{\mathcal{L}}$ at every vertex of $\underline{\mathcal{L}}$.

We say that an $\mathcal{S}$-provable solution s of $\underline{\mathcal{L}}$ is $\mathcal{S}$*-complete* if:

$$w_1 \;\underline{\leftrightarrow}_{\underline{\mathcal{L}}}\; w_2 \implies s(w_1) =_{\mathcal{S}} s(w_2)\,, \qquad \text{(concept 5)}$$

holds for all $w_1, w_2 \in V$, that is, if values of the solution s at 1-bisimilar vertices of $\underline{\mathcal{L}}$ are $\mathcal{S}$-provably equal.

The following lemma gathers preservation statements of (complete) provable solutions under (functional) 1-bisimilarity that are crucial for the completeness proof.

Lemma 2.8. *On weakly guarded* 1*-charts, the following statements hold for all star expressions* $e \in \mathit{StExp}$*:*

- (i) Mil*-Provable solvability with principal value* e *is preserved under converse functional* 1*-bisimilarity.*
- (ii) Mil*-Complete* Mil*-provable solvability with principal value* e *of a w.g.* 1*-chart* $\underline{C}$ *implies* Mil*-provable solvability with principal value* e *of the bisimulation collapse of* $\underline{C}$*. (See 5.)*
- (iii) Mil*-Complete* Mil*-provable solvability with principal value* e *is preserved under* 1*-bisimilarity.*

3 LLEE-1-Charts

We use the adaptation of the 'loop existence and elimination property' LEE from [13] to 1-charts as described in [7, 9]. Here we only briefly explain the concept by examples, and refer to [7, 9] and to the appendix for the definitions. Crucially, we gather statements from [7, 9] that we need for the proof.

LEE is defined by a stepwise elimination process of 'loop sub-1-charts' from a given 1-chart $\underline{C}$. A run of this process is illustrated in Fig. 1. Hereby a 1-chart $\underline{\mathcal{L}C} = \langle V, A, 1, v_s, \rightarrow, \downarrow \rangle$ is called a *loop* 1*-chart* if it satisfies three conditions:

- (L1) There is an infinite path from the start vertex v_s.
- (L2) Every infinite path from v_s returns to v_s after a positive number of transitions.

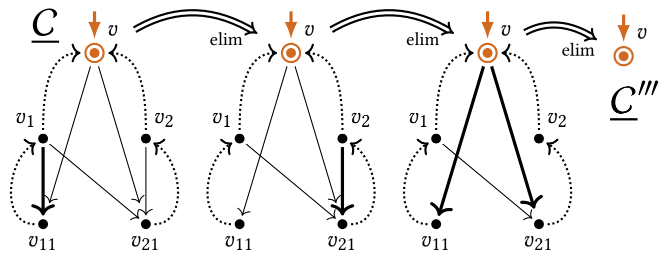


Figure 1. A successful run of the loop elimination procedure. The start vertex is indicated by ➔, immediate termination by a boldface ring. Loop-entry transitions of loop sub-1-charts eliminated in the next step are marked in bold. Action labels are neglected, however dotted arrows indicate 1-transitions.

- (L3) Immediate termination is only permitted at the start vertex, that is, $\downarrow \subseteq \{v_s\}$.

A *loop sub-*1*-chart of* a 1-chart $\underline{C}$ is a loop 1-chart $\underline{\mathcal{L}C}$ that is a sub-1-chart of $\underline{C}$ with some vertex $v \in V$ of $\underline{C}$ as start vertex, such that $\underline{\mathcal{L}C}$ is constructed, for a nonempty set U of transitions of $\underline{C}$ from v, by all paths that start with a transition in U and continue onward until v is reached again (so the transitions in U are the loop-entry transitions of $\underline{\mathcal{L}C}$). *Eliminating a loop sub-*1*-chart* $\underline{\mathcal{L}C}$ *from a* 1*-chart* $\underline{C}$ consists of removing all loop-entry transitions of $\underline{\mathcal{L}C}$ from $\underline{C}$, and then also removing all vertices and transitions that become unreachable. Fig. 1 shows a successful three-step run of the loop elimination procedure. A 1-chart $\underline{C}$ has the *loop existence and elimination property* (LEE) if the procedure, started on $\underline{C}$, of repeated eliminations of loop sub-1-charts results in a 1-chart without an infinite path. If, in a successful elimination process from a 1-chart $\underline{C}$, loop-entry transitions are never removed from the body of a previously eliminated loop sub-1-chart, then we say that $\underline{C}$ satisfies *layered* LEE (LLEE), and is a LLEE-1*-chart*. LLEE-1-LTSs are 1-LTSs that are defined analogously. While the property LLEE leads to a formally easier concept of 'witness', it is equivalent to LEE. Since the resulting 1-chart $\underline{C'''}$ in Fig. 1 does not have an infinite path, and no loop-entry transitions have been removed from a previously eliminated loop sub-1-chart, we conclude that the initial 1-chart $\underline{C}$ satisfies LLEE as well as LEE.

A LLEE-*witness* $\hat{\underline{C}}$ *of* a 1-chart $\underline{C}$ is the recording of a successful run of the loop elimination procedure by attaching to a transition τ of $\underline{C}$ the marking label n for $n \in \mathbb{N}^{+}$ (in pictures indicated as $[n]$, in steps as $\rightarrow_{[n]}$) forming a *loop-entry transition* if τ is eliminated in the n-th step, and by attaching marking label 0 to all other transitions of $\underline{C}$ (in pictures neglected, in steps indicated as $\rightarrow_{\mathrm{bo}}$) forming a *body transition*. Formally, LLEE-witnesses arise as *entry/body-labelings* from 1-charts, and are charts in which the transition labels are pairs of action labels over A, and marking labels in $\mathbb{N}$.

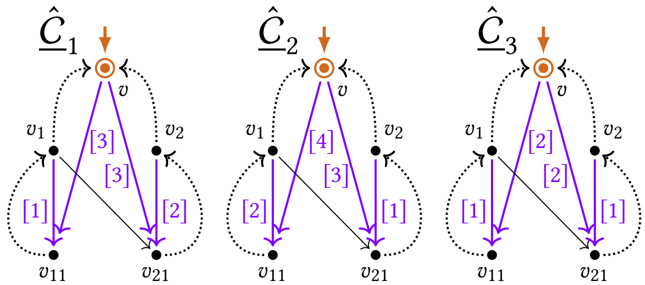


Figure 2. Three LLEE-witnesses of the 1-chart $\underline{C}$ in Fig. 1. $\hat{\underline{C}}_1$ is a recording of the successful procedure run in Fig. 1 of the order in which loop-entry transitions have been removed.

The LLEE-witness $\hat{\underline{C}}_1$ in Fig. 2 arises from the run of the loop elimination procedure in Fig. 1. The LLEE-witnesses $\hat{\underline{C}}_2$ and $\hat{\underline{C}}_3$ of $\underline{C}$ in Fig. 2 record two other successful runs of the loop elimination procedure of length 4 and 2, respectively, where for $\hat{\underline{C}}_3$ we have permitted to eliminate two loop subcharts at different vertices together in the first step.

Definition 3.1 (1-charts with LLEE−1-lim)**.** Let $\underline{C}$ be a 1-chart.

Let $\hat{\underline{C}}$ be a LLEE-witness of $\underline{C}$. We say that $\hat{\underline{C}}$ *is 1-transition limited* if every 1-transition of $\underline{C}$ lifts to a *backlink* (a transition from the body of a loop sub-1-chart back to its start) in $\hat{\underline{C}}$. Fixing also a weaker property, we say that $\hat{\underline{C}}$ is *guarded* if all of its loop-entry transitions are proper transitions.

We say that $\underline{C}$ *satisfies* LLEE−1-lim, and *is 1-transition limited*, if $\underline{C}$ has a 1-transition limited LLEE-witness. We say that $\underline{C}$ *is guarded* if $\underline{C}$ has a guarded LLEE-witness.

We note that 1-transition limited LLEE-1-charts are guarded (since loop-entry transition are not backlinks), and guarded LLEE-1-charts are weakly guarded (since in a guarded LLEE-witness every 1-transition path is a body transition path, which as in every LLEE-witness is guaranteed to be finite).

Lemma 3.2. *Every weakly guarded* LLEE-1*-chart is* 1*-bisimilar to a (guarded)* 1*-chart with* LLEE−1-lim.

Two crucial properties of LLEE-1-charts that motivate their use, like for LLEE-charts earlier in [12, 13], are their provable solvability and unique solvability modulo provability in Mil. The two lemmas below that express these properties are generalizations to LLEE-1-charts of Prop. 5.5 and Prop. 5.8 in [12, 13]), and have been proved in [8, 9].

Lemma 3.3. *From every guarded* LLEE*-witness* $\hat{\underline{C}}$ *of a (weakly guarded)* LLEE-1*-chart* $\underline{C}$ *a* Mil^-*-provable solution* $s_{\hat{\underline{C}}}$ *of* $\underline{C}$ *can be extracted effectively.*

Lemma 3.4. *For every guarded* LLEE-1*-chart* $\underline{C}$ *it holds that any two* Mil*-provable solutions of* $\underline{C}$ *are* Mil*-provably equal.*

In Sect. 8 we will need a consequence of Lem. 3.4, namely provable invariance of provable solutions under 'transfer functions', which define functional 1-bisimulations. While this statement holds for also LLEE-1-charts, we have to formulate it for LLEE-1-LTSs for use later in Sect. 8.

Definition 3.5. A *transfer (partial) function* between 1-LTSs $\underline{\mathcal{L}}_1$ and $\underline{\mathcal{L}}_2$, for $\underline{\mathcal{L}}_i = \langle V_i, A, 1, \rightarrow_i, \downarrow_i\rangle$ where $i \in \{1,2\}$, is a partial function $\phi : V_1 \rightharpoonup V_2$ whose graph $\{\langle v, \phi(v)\rangle \mid v \in V_1\}$ is a 1-bisimulation between $\underline{\mathcal{L}}_1$ and $\underline{\mathcal{L}}_2$.

Lemma 3.6. *Let* $\phi : V_1 \rightharpoonup V_2$ *be a transfer function between* LLEE-1*-LTS* $\underline{\mathcal{L}}_i = \langle V_i, A, 1, \rightarrow_i, \downarrow_i\rangle$*, for* $i \in \{1,2\}$*. Then for all* Mil*-provable solutions* s_1 *of* $\underline{\mathcal{L}}_1$*, and* s_2 *of* $\underline{\mathcal{L}}_2$ *it holds that* s_1 *coincides* Mil*-provably with the precomposition* $s_2 \circ \phi$ *of* s_2 *and* ϕ*:*

$$s_1(w) =_{\mathsf{Mil}} s_2(\phi(w)), \quad \textit{for all } w \in \mathrm{dom}(\phi)\,.$$

A substantial obstacle for the use of LLEE-1-charts was already recognized in [13]: the chart interpretation of star expressions does not in general define LLEE-charts. This obstacle can, however, be navigated successfully by using the result from [7, 11] that a variant chart interpretation can be defined that produces 1-bisimilar LLEE-1-charts instead.

Lemma 3.7. *For every star expression* e*, there is a* 1-chart interpretation $\underline{C}(e)$ of e *that has the following properties:*

(i) $\underline{C}(e)$ *is a* 1*-transition limited (guarded)* LLEE-1*-chart,*
(ii) $\underline{C}(e) \rightrightarrows C(e)$*, and hence also* $\underline{C}(e) \leftrightarrows C(e)$*.*
(iii) e *is the principal value of a* Mil*-provable solution of* $\underline{C}(e)$*.*

4 Completeness proof based on lemmas

We anticipate the completeness proof for Milner's system by basing it on the following lemmas, which are faithful abbreviations of statements as formulated in other sections. The chosen acronyms for these lemmas stem from the letters that are typeset in boldface italics in their statements:

(IV) For every star expression e, there is a 1*-chart **i**nterpretation (**v**ariant)* $\underline{C}(e)$ *of* e with the properties (i), (ii), and (iii) in Lem. 3.7 (see above).

(T) Provable solutions can be ***t***ransferred backwards over a transfer function between weakly guarded 1-charts. (See Lem. 2.8, (i)).

(E) From every guarded LLEE-1-chart $\underline{C}$ a provable solution of $\underline{C}$ can be ***e***xtracted. (See Lem. 3.3.)

(CR) Every guarded LLEE-1-chart can be transformed into a 1-bisimilar (guarded) ***cr***ystallized (LLEE-)1-chart. (See Thm. 7.9, based on Def. 7.5.)

(CN) Every ***c***rystallized 1-chart is ***n***ear-collapsed. (Lem. 7.8.)

(NC) Solutions extracted from ***n***ear-collapsed guarded LLEE-1-charts are ***c***omplete provable solutions. (See Lem. 8.2.)

(CC) If a weakly guarded 1-chart $\underline{C}$ has a ***c***omplete provable solution with principal value e, then also the (1-transition free) bisimulation ***c***ollapse of $\underline{C}$ has a provable solution with principal value e. (See Lem. 2.8, (ii).)

(SE) All provable ***s***olutions of a guarded LLEE-1-chart are provably ***e***qual. (See Lem. 3.4.)

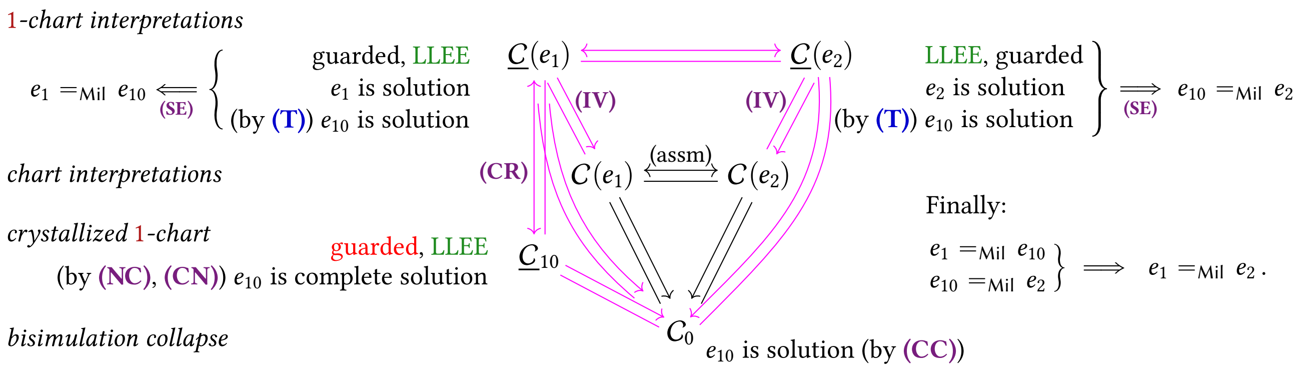


Figure 3. Structure of the completeness proof (see proof of Thm. 4.1): The argument starts from the assumption $C(e_1) \mathrel{\underline{\leftrightarrow}} C(e_2)$, that the chart interpretations of e_1 and e_2 are bisimilar. It uses 1-bisimilarity with the 1-chart interpretations $\underline{C}(e_1)$ and $\underline{C}(e_2)$ of e_1 and e_2 (which expand the chart interpretations), the crystallization $\underline{C}_{10}$ of $\underline{C}(e_1)$ (which arises by LLEE-preservingly minimizing $\underline{C}(e_1)$ and crystallization operations), and the joint bisimulation collapse C_0 of all of these (1-)charts. The conclusion is $e_1 =_{\mathsf{Mil}} e_2$, that e_1 and e_2 are provably equal in Milner's system Mil. By "f is (complete) solution of $\underline{C}$" we here mean that "f is the principal value of a (complete) provable solution of $\underline{C}$". The indicated lemmas are explained in Sect. 4.

Theorem 4.1. *Milner's proof system* Mil *is complete with respect to the process semantics equality of regular expressions.*

Proof. (*See Fig. 3 for an illustration.*) Let $e_1, e_2 \in \mathit{StExp}(A)$ be star expressions such that $[\![e_1]\!]_P = [\![e_2]\!]_P$ holds, that is, their process interpretations coincide. This means that the behaviors $[C(e_1)]_{\leftrightarrow}$ and $[C(e_2)]_{\leftrightarrow}$ of the chart interpretations $C(e_1)$ of e_1 and $C(e_2)$ of e_2 coincide. Therefore $C(e_1) \mathrel{\underline{\leftrightarrow}} C(e_2)$ holds, that is, $C(e_1)$ and $C(e_2)$ are bisimilar. We have to show $e_1 =_{\mathsf{Mil}} e_2$, that is, that $e_1 = e_2$ can be proved in Mil.

Due to **(IV)**, the 1-chart interpretations $\underline{C}(e_1)$ of e_1 and $\underline{C}(e_2)$ of e_2 are guarded LLEE-1-charts that are 1-bisimilar to $C(e_1)$ and to $C(e_2)$, respectively. As a consequence we get $\underline{C}(e_1) \mathrel{\underline{\leftrightarrow}} \underline{C}(e_2)$, that is, $\underline{C}(e_1)$ and $\underline{C}(e_2)$ are 1-bisimilar. By part (iii) of **(IV)**, e_1 and e_2 are the principal values of provable solutions s_1 and s_2 of $\underline{C}(e_1)$ and $\underline{C}(e_2)$, respectively.

We now focus on $\underline{C}(e_1)$, leaving aside $\underline{C}(e_2)$ for the moment. (Equally we could start from $\underline{C}(e_2)$, and build up a symmetrical argument). Due to **(CR)** the guarded LLEE-1-chart $\underline{C}(e_1)$ can be transformed into a 1-bisimilar crystallized 1-chart $\underline{C}_{10}$, which is a guarded LLEE-1-chart. Then $\underline{C}_{10} \mathrel{\underline{\leftrightarrow}} \underline{C}(e_1)$ holds. Due to **(CN)**, $\underline{C}_{10}$ is also near-collapsed. Now we can apply **(NC)** to $\underline{C}_{10}$ in order to conclude that the provable solution s_{10} that is extracted from $\underline{C}_{10}$ by **(E)** is a complete provable solution of the near-collapsed 1-chart $\underline{C}_{10}$. Let e_{10} be the principal value of s_{10}.

Now let C_0 be the (1-transition free) bisimulation collapse of $\underline{C}_{10}$. By applying **(CC)** to the complete provable solution s_{10} of $\underline{C}_{10}$, which is weakly guarded since it is guarded, we obtain a provable solution s_0 of C_0 that also has the principal value e_{10}. Due to $\underline{C}(e_1) \mathrel{\underline{\leftrightarrow}} \underline{C}(e_2)$, C_0 is the joint bisimulation collapse of $\underline{C}(e_1)$ and $\underline{C}(e_2)$. It follows that there are functional 1-bisimulations from $\underline{C}(e_1)$ and from $\underline{C}(e_2)$ to C_0, that is, $\underline{C}(e_1) \mathrel{\underline{\Rightarrow}} C_0 \mathrel{\underline{\Leftarrow}} \underline{C}(e_2)$. Now we can use **(T)** to transfer the provable solution s_0 from C_0 to $\underline{C}(e_1)$ and to $\underline{C}(e_2)$. We obtain provable solutions $\tilde{s}_1$ of $\underline{C}(e_1)$, and $\tilde{s}_2$ of $\underline{C}(e_2)$, both of which have e_{10} as their principal value.

Since both s_1 and $\tilde{s}_1$ are provable solutions of the guarded LLEE-1-chart $\underline{C}(e_1)$, we can apply **(SE)** to find that s_1 and $\tilde{s}_1$ are provably equal. In particular, the principal values e_1 of s_1 and e_{10} of $\tilde{s}_1$ are provably equal. That is, $e_1 =_{\mathsf{Mil}} e_{10}$ holds. Analogously, as both s_2 and $\tilde{s}_2$ are provable solutions of the guarded LLEE-1-chart $\underline{C}(e_2)$, **(SE)** also entails that s_2 and $\tilde{s}_2$ are provably equal. Therefore also the principal values e_2 of s_2 and e_{10} of $\tilde{s}_2$ are provably equal. That is, $e_2 =_{\mathsf{Mil}} e_{10}$ holds.

From $e_1 =_{\mathsf{Mil}} e_{10}$ and $e_2 =_{\mathsf{Mil}} e_{10}$ we obtain $e_1 =_{\mathsf{Mil}} e_2$, and hence that $e_1 = e_2$ is provable in Milner's system Mil. ◻

5 Failure of LLEE-preserving 1-collapse

Here we expand on observation (1), due to which we have realized in Sect. 1 that the bisimulation collapse strategy in [13] cannot be extended directly to a 1-bisimulation collapse strategy for showing completeness of Mil. We define the properties 'collapsible', '1-collapsible', and 'jointly minimizable' formally, formulate their failure for LLEE-1-charts, and suggestively explain the reason by means of an example.

Let $\underline{C}$ be a LLEE-1-chart. We say that $\underline{C}$ is LLEE-*preservingly 1-collapsible* (LLEE-*preservingly collapsible*) if $\underline{C}$ has a 1-bisimulation collapse that is a LLEE-1-chart (and respectively, the bisimulation collapse of $\underline{C}$ is a LLEE-1-chart).

We say that two LLEE-1-charts $\underline{C}_1$ and $\underline{C}_2$ that are 1-bisimilar (that is, with $\underline{C}_1 \mathrel{\underline{\leftrightarrow}} \underline{C}_2$) are LLEE-*preservingly jointly minimizable (under functional 1-bisimilarity $\underline{\Rightarrow}$)* if there is a LLEE-1-chart $\underline{C}_0$ such that $\underline{C}_1 \mathrel{\underline{\Rightarrow}} \underline{C}_0 \mathrel{\underline{\Leftarrow}} \underline{C}_2$.

Proposition 5.1 ((1)). *The following two statements hold:*

(i) W.g. LLEE*-1-charts are not in general* LLEE*-preservingly 1-collapsible, hence not in general* LLEE*-pres. collapsible.*

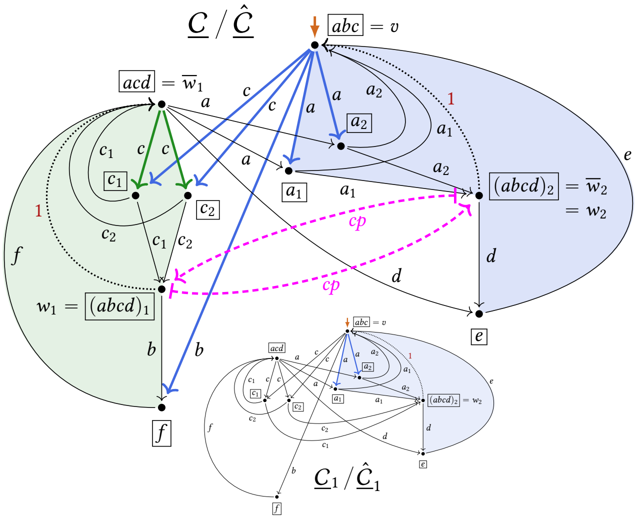


Figure 4. A 1-chart $\underline{C}$ with LLEE-witness $\hat{\underline{C}}$ (colored loop-entry transitions of level 1, green, of level 2, blue) that is not 1-bisimulation collapsed: the correspondences ⊢-→ via the 'counterpart function *cp*' indicate a (grounded) functional 1-bisimulation slice on $\underline{C}$. $\underline{C}$ is not LLEE-preservingly 1-collapsible. The result $\underline{C}_1$ (small) of connecting through w_1 to w_2 in $\underline{C}$ is a 1-bisimulation collapse of $\underline{C}$, but not a LLEE-1-chart. The colored regions explain $\underline{C}$ as a twin-crystal, see Sect. 6.

(ii) *Two 1-bisimilar w.g.* LLEE-1-*charts are not in general* LLEE-*preservingly jointly minimizable under* $\underline{\rightarrow}$.

Statement (i) is witnessed by the LLEE-1-*chart* $\underline{C}$ *in Fig. 4. Statement (ii) is witnessed by the 1-bisimilar generated sub-*LLEE-1-*charts* $\underline{C}{\downarrow}^{w_1}_{*}$ *and* $\underline{C}{\downarrow}^{w_2}_{*}$ *of the* LLEE-1-*chart* $\underline{C}$ *in Fig. 4.*

That the LLEE-1-chart $\underline{C}$ in Fig. 4 is not 1-collapsible is suggested there by exhibiting a natural 1-bisimulation collapse that is not a LLEE-1-chart: the 1-bisimilar 1-chart $\underline{C}_1$ that results by 'connecting through' all incoming transitions at w_1 in $\underline{C}$ over to w_2. From $\underline{C}_1$ it is easy to intuit that the bisimulation collapse of $\underline{C}$ cannot be a LLEE-1-chart, either, and hence that $\underline{C}$ is not LLEE-preservingly collapsible. While that is only part of the (remaining) proof of Prop. 5.1, (i), it is also easy to check that the connect-w_2-through-to-w_1 1-chart $\underline{C}_1$ of $\underline{C}$ cannot be a LLEE-1-chart, either. For the proof that $\underline{C}{\downarrow}^{w_1}_{*}$ and $\underline{C}{\downarrow}^{w_2}_{*}$ are not LLEE-preservingly jointly minimizable it is crucial to realize that a function that maps w_1 to w_2, and w_2 to w_1 cannot be extended into a transfer function on $\underline{C}$. (That function, however, defines a 'grounded' 1-bisimulation slice on $\underline{C}$, see Def. 6.1 later, and Fig. 5).

For motivating concepts in the next sections we will use the simplified version $\underline{C}_s$ in Fig. 5 of the LLEE-1-chart $\underline{C}$ in Fig. 4. We emphasize, however, that although $\underline{C}_s$ is not collapsible, it is 1-collapsible. Hence $\underline{C}_s$ does not witness statement (i), nor can it be used for showing (ii) in Prop. 5.1.

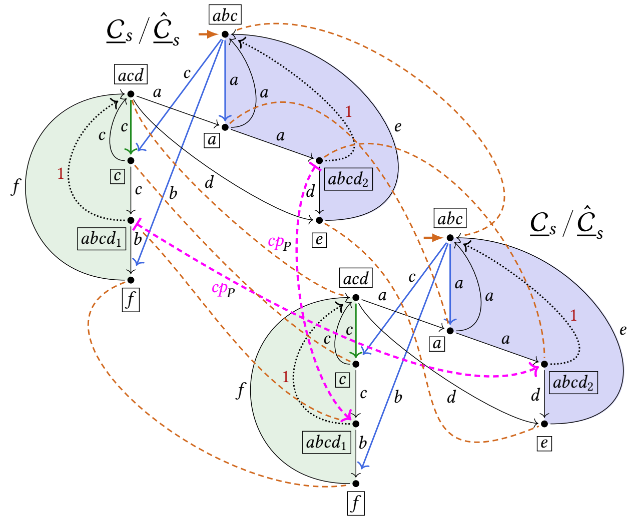


Figure 5. Two copies of a simplified version $\underline{C}_s$ (with LLEE-witness $\hat{\underline{C}}_s$) of the 1-chart $\underline{C}$ in Fig. 4, linked by a local transfer function cp_P (↦ links) whose graph (a grounded 1-bisimulation slice) is extended to a 1-bisimulation (added links). Like $\underline{C}$ in Fig. 4, also $\underline{C}_s$ is a twin-crystal shaped LLEE-1-chart.

6 Twin-Crystals

The 1-chart $\underline{C}$ in Fig. 4 and Prop. 5.1 will turn out to be prototypical for strongly connected comTonents in LLEE-1-charts that are 'nearly collapsed' but not LLEE-preservingly collapsible any further. By isolating a number of its properties from $\underline{C}$, and from its simplified version $\underline{C}_s$ in Fig. 5, we define the central concept of 'twin-crystal'. Each of the LLEE-1-charts $\underline{C}$ and $\underline{C}_s$ consists of a single scc that is of 'twin-crystal shape', which exhibits a certain kind of symmetry with respect to 1-bisimilarity. Our proof utilizes this symmetry for proving uniqueness of provable solutions for LLEE-1-charts of which all scc's are 1-collapsed or of twin-crystal shape. Before describing twin-crystals, we first define, also motivated by the two examples, 'grounded 1-bisimulation slices', 'local transfer functions', and 'near-collapsed' 1-charts.

On both of the LLEE-1-charts $\underline{C}$ in Fig. 4, and $\underline{C}_s$ in Fig. 5 there are non-trivial functional 1-bisimulation slices, which are suggested by the magenta links. These 1-bisimulation slices cannot be extended to 1-bisimulations that are defined by transfer functions. However, they have an expedient additional property that will permit us to work with 'local transfer functions' instead. Namely, that induced transitions from a vertex w_1 of pair $\langle w_1, w_2 \rangle$ of a slice B to a vertex w_1' outside of the domain of B can be joined by an induced transition with the same label from w_2 to w_1, and vice versa. We call 1-bisimulation slices with this property 'grounded', and functions that induce them 'local transfer functions'.

Definition 6.1. Let $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow\rangle$ be a 1-LTS.

By a *grounded 1-bisimulation slice on* $\underline{\mathcal{L}}$ we mean a 1-bisimulating slice $B \subseteq V \times V$ on $\underline{\mathcal{L}}$ such that for all $\langle w_1, w_2\rangle \in B$ the following additional forth/back conditions hold:

$$(\text{forth})_{\text{g}}\ \forall a \in A\, \forall w_1' \in V_1 \big(w_1 \xrightarrow{(a)} w_1' \wedge w_1' \notin W_1 \Longrightarrow w_2 \xrightarrow{(a)} w_1' \wedge w_1' \notin W_2 \big),$$

$$(\text{back})_{\text{g}}\ \forall a \in A\, \forall w_2' \in V_1 \big(w_1 \xrightarrow{(a)} w_2' \wedge w_2' \notin W_1 \Longleftarrow w_2 \xrightarrow{(a)} w_2' \wedge w_2' \notin W_2 \big).$$

where $W_1 := \mathrm{dom}_{\mathrm{act}}(B)$, and $W_2 := \mathrm{cod}_{\mathrm{act}}(B)$ are the active domain, and the active codomain of B, respectively.

Lemma 6.2. *For every grounded bisimulation slice $B \subseteq V \times V$ on a 1-LTS $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow\rangle$, the relation $B^{=} := B \cup =$ is a 1-bisimulation on $\underline{\mathcal{L}}$.*

Definition 6.3. A *local-transfer function* on a 1-LTS $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow\rangle$ is a partial function $\phi : V \rightharpoonup V$ whose graph $\{\langle v, \phi(v)\rangle \mid v \in V\}$ is a grounded 1-bisimulation slice on $\underline{\mathcal{L}}$.

Example 6.4. Both of the functions cp_P on the 1-charts $\underline{\mathcal{C}}$ in Fig. 4, and cp_P on $\underline{\mathcal{C}}_s$ in Fig. 5 are local transfer functions: in particular in Fig. 5 it can be checked easily that cp_P defines a grounded 1-bisimulation slice. Neither of these local transfer functions can be extended into a transfer function. For $\underline{\mathcal{C}}_s$ this can be checked in Fig. 4: all pairs of the identity function have to be added in order to extend $graph(cp_P)$ into a 1-bisimulation, thereby violating functionality of the relation.

We will say that a 1-chart $\underline{\mathcal{C}}$ (and respectively, a 1-LTS $\underline{\mathcal{L}}$) is 'near-collapsed' if 1-bisimilarity on $\underline{\mathcal{C}}$ (on $\underline{\mathcal{L}}$) is the reflexive–symmetrical closure of the union of the graphs of finitely many local transfer functions on $\underline{\mathcal{L}}$.

Definition 6.5 (③)**.** Let $\underline{\mathcal{L}}$ be a 1-LTS with state set V, and let $P \subseteq V$ be a subset of the vertices of $\underline{\mathcal{L}}$. We say that $\underline{\mathcal{L}}$ *is locally near-collapsed for P* if there are local transfer functions $\phi_1, \ldots, \phi_n : V \rightharpoonup V$ on $\underline{\mathcal{L}}$ such that:

$$\underline{\leftrightarrow}_{\underline{\mathcal{L}}} \cap (P \times P) \subseteq \leftrightarrow_R^{=}, \quad \text{for } R = \textstyle\bigcup_{i=1}^{n} graph(\phi_i),$$

where $\leftrightarrow_R^{=}$ means the reflexive-symmetric closure of R. We say that $\underline{\mathcal{L}}$ *is near-collapsed* if $\underline{\mathcal{L}}$ is near-collapsed on V.

A 1-chart $\underline{\mathcal{C}}$ is *(locally for P) near-collapsed* if the 1-LTS $\underline{\mathcal{L}}(\underline{\mathcal{C}})$ underlying $\underline{\mathcal{C}}$ is (locally for P) near-collapsed.

Example 6.1. Each of the 1-charts $\underline{\mathcal{C}}$ in Fig. 4, and $\underline{\mathcal{C}}_s$ in Fig. 5 contains only one pair of distinct non-1-bisimilar vertices. Since these vertices are related, respectively, by the appertaining transfer function $cp_{\underline{\mathcal{C}}}$, it follows that $\underline{\mathcal{C}}$ and $\underline{\mathcal{C}}_s$ are locally near-collapsed for the sets P of all their vertices, respectively. Therefore $\underline{\mathcal{C}}$ and $\underline{\mathcal{C}}_s$ are near-collapsed.

For vertices w and v in a LLEE-witness $\hat{\underline{\mathcal{C}}}$ we write $w \curvearrowright v$, *w loops-back-to v*, if $v \rightarrow_{[n]} \cdot \rightarrow_{\mathrm{bo}}^{*} w \rightarrow_{\mathrm{bo}}^{+} v$ holds with $n \in \mathbb{N}^{+}$ and such that v is only encountered again at the end. We fix $(\curvearrowright^{*} v) := \{w \mid w \curvearrowright^{*} v\}$, the *loops-back-to part of v*.

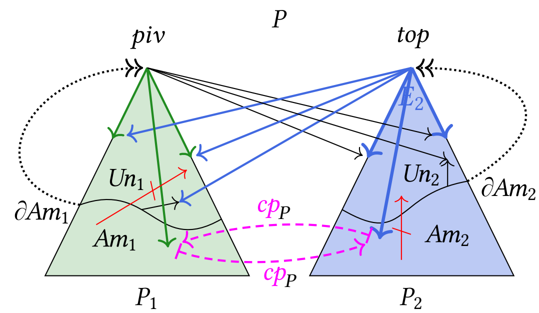


Figure 6. Structure schema of a twin-crystal with carrier P, with part P_1 of pivot vertex *piv*, and part P_2 of top vertex *top*, where P_2 is generated by the top entry transitions in E_2.

The structure of the LLEE-1-charts $\underline{\mathcal{C}}$ and $\underline{\mathcal{C}}_s$ above can be described by the illustration in Fig. 6, together with the following eight properties that define when a vertex set P in a LLEE-1-chart $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ is the *carrier of a twin-crystal* (and that the sub-LTS induced by P in $\underline{\mathcal{C}}$ is a *twin-crystal*) with respect to vertices $top, piv \in P \subseteq V$, sets $P_2, P_1 \subseteq P$, a LLEE-witness $\hat{\underline{\mathcal{C}}}$ of $\underline{\mathcal{C}}$ with binary loops-back-to relation $\curvearrowright$, and a non-empty set E_2 of transitions from *top*:

- (tc-1) $P = (\curvearrowright^{*} top)$ is a maximal loops-back-to part. Then *top* is $\curvearrowright$-maximal, and P is an scc.
- (tc-2) $P = P_1 \uplus P_2$ for $P_1 := (\curvearrowright^{*} piv)$ and $P_2 := ({}^{E_2}\!\curvearrowright^{*} top)$, the loops-back-to part generated by transitions in E_2. We call *piv pivot vertex* and *top top vertex*. Then $piv \in P_1 \subseteq P$, $top \in P_2 \subseteq P$, and $\{P_i\}_{i\in\{1,2\}}$ is a partition of P.
- (tc-3) $\underline{\mathcal{C}}$ is not 1-collapsed for P. Hence P contains 1-bisimilarity redundancies.
- (tc-4) (Using terminology from Def. 7.1 later:) All 'reduced' 1-bisimilarity redundancies in P are of 'precrystalline' form (R3.4), with one vertex in P_1, and the other in P_2.
- (tc-5) Proper transitions from *piv favor* P_1: whenever a proper transition from *piv* is 1-bisimilar to a vertex in P_1, then its target is in P_1.
- (tc-6) Proper transitions from *top favor* P_2 (confer (tc-5)).
- (tc-7) P is *squeezed in* $\underline{\mathcal{C}}$: no vertex in P is 1-bisimilar to a vertex outside of P.
- (tc-8) P is *grounded in* $\underline{\mathcal{C}}$: any two transitions from P with 1-bisimilar targets outside of P have the same target.

For carrier sets P of a twin-crystal in $\underline{\mathcal{C}}$ it can be shown that 1-bisimilarity redundancies in P occur with one vertex in P_1, and the other in P_2. Hence a vertex in P_1 may have a 1-bisimilar counterpart, which can only be in P_2; and vice versa. Then the following derived concepts can be introduced (see Fig. 6). The pivot part P_1 partitions into the sets Un_1 and Am_1 of unambiguous, and ambiguous vertices, in P that are unique, and respectively, are not unique up to $\underline{\leftrightarrow}_{\underline{\mathcal{C}}}$. The top part P_2 partitions into the sets Un_2 and Am_2 of unambiguous, and ambiguous vertices, respectively. The boundary vertices

∂Am_1 of P_1 have 1-transition paths to *piv*, and the boundary vertices ∂Am_2 of P_2 have 1-transition paths to *top*. There are no transitions directly from $Am_1 \setminus \partial Am_1$ to Un_1, and no transitions directly from $Am_2 \setminus \partial Am_2$ to Un_2.

1-Bisimilar vertices in Am_1 and Am_2 are related by the *counterpart (partial) function* $cp_P : V \rightharpoonup V$ *on* $\underline{C}$ with domain and range contained in P that is defined, for all $w \in V$, by:

$$cp_P(w) := \begin{cases} \overline{w} & \text{if } w \in Am, \text{ and } \overline{w} \text{ the 1-bisimilar} \\ & \quad \text{counterpart of } w \text{ in } P, \\ \text{undefined} & \text{if } w \notin Am. \end{cases}$$

Example 6.6. Each of the 1-transition limited LLEE-1-charts $\underline{C}$ in Fig. 4 and $\underline{C}_2$ in Fig. 5 consists of a single scc that is carrier of a twin-crystal. In the appertaining figures, the parts P_1 and P_2 are colored blue and green, respectively.

The picture in Fig. 6 suggests the symmetric nature of twin-crystals: if in the underlying LLEE-witness the loop-entry transitions from P_2 to P_1 (maximum level m) are relabeled into body transitions, and the body transitions from P_1 to P_2 into loop-entry transitions (level m), then a twin-crystal with permuted roles of *piv* and *top* arises. But for the completeness proof only the properties of twin-crystal shaped scc's that are formulated by the two lemmas below are crucial.

Lemma 6.7. *The counterpart function on the carrier P of a twin-crystal in a* LLEE-1*-chart* $\underline{C}$ *is a local transfer function.*

Lemma 6.8. *If P is the carrier of a twin-crystal in a* LLEE-1*-chart* $\underline{C}$*, then* $\underline{C}$ *is locally near-collapsed for P.*

7 Crystallization of LLEE-1-charts

For this central part of the proof we sketch how every weakly guarded LLEE-1-chart can be minimized under 1-bisimilarity far enough to obtain a LLEE-1-chart in 'crystallized' form. By that we mean that the resulting 1-bisimilar LLEE-1-chart is 1-collapsed apart from that some of its scc's may be twin-crystals. A 'groundedness' clause in the definition will ensure that every crystallized 1-chart is also near-collapsed.

The minimization process we describe here is a refinement of the process for LLEE-charts (without 1-transitions) that was defined in [12, 13]. We first find that if a LLEE-1-chart $\underline{C}$ is not 1-collapsed, then $\underline{C}$ contains a '1-bisimilarity redundancy' from one of three 'reduced kinds' (with subkinds). This is stated below by Lem. 7.1, which is a generalization of Prop. 6.4 in [12, 13]. Second, we argue that every reduced 1-bisimilarity redundancy can be eliminated LLEE-preservingly except if it belongs to a subkind which can be found in the not 1-collapsible LLEE-1-chart in Fig. 4. Third, we define a few further transformations for cutting scc's into twin-crystals.

Lemma 7.1 (kinds of reduced 1-bisimilarity redundancies)**.** *Let* $\underline{C}$ *be a* 1*-chart, and let* $\hat{\underline{C}}$ *a* 1*-transition limited* LLEE*-witness of* $\underline{C}$*. Suppose that* $\underline{C}$ *is not a bisimulation collapse.*

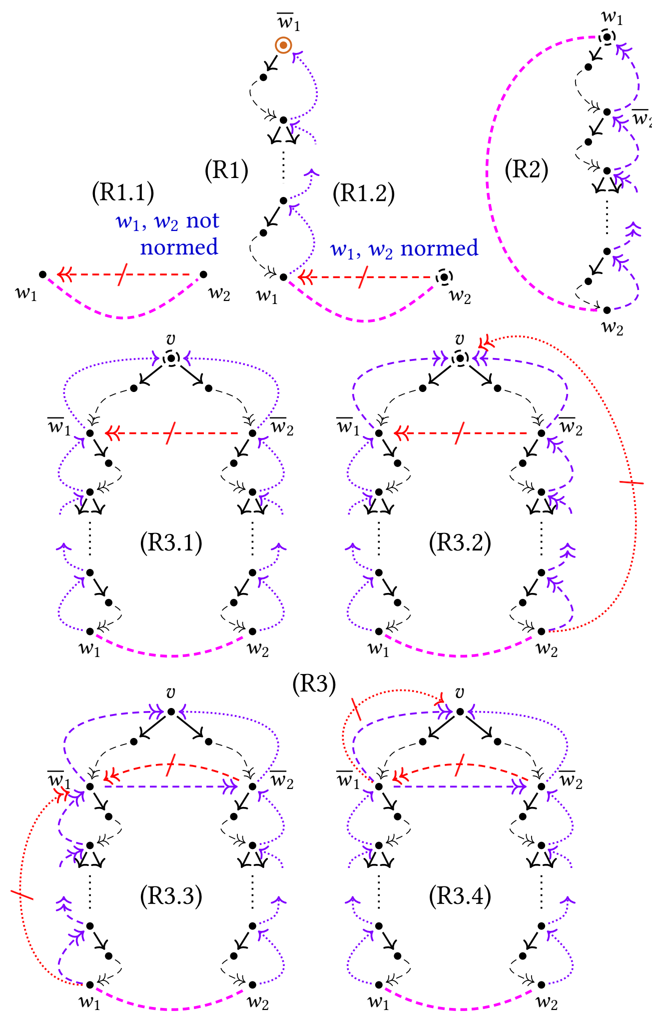


Figure 7. Reduced 1-bisimilarity redundancies, see Lem. 7.1. Upward dotted arrows: body 1-transition backlinks. Upward dashed double arrows: body transition paths of (direct) loops-back-to links. Dashed double arrows: paths of body transitions. Struck out red arrows: prohibited body-transitions and body-tr.-paths. Dashed links, bottom: assumed 1-bisimilarity.

Then $\underline{C}$ *contains a* 1*-bisimilarity redundancy* $\langle w_1, w_2 \rangle$ *(distinct* 1*-bisimilar vertices w_1 and w_2 in* $\underline{C}$*) that satisfies, with respect to* $\hat{\underline{C}}$*, one of the position conditions (kinds)* (R1) *(with subkinds* (R1.1), (R1.2)*)* (R2)*, or* (R3) *(with subkinds* (R3.1), (R3.2), (R3.3), (R3.4)*) that are illustrated in Fig. 7. Note that the vertices w_1 and w_2 are in the same scc for position kinds* (R2) *and* (R3)*, but in different scc's for position kind* (R1)*.*

Definition 7.2. We consider a 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ in $\underline{C}$ under the assumptions of Lem. 7.1 on $\hat{\underline{C}}$ and $\underline{C}$.

We say that $\langle w_1, w_2 \rangle$ is *reduced (with respect to* $\hat{\underline{C}}$*)* if it is of one of the kinds (R1)–(R3) in Fig. 7. We say that $\langle w_1, w_2 \rangle$ is *simple* if it is of (sub-)kind (R1), (R2), (R3.1), (R3.2), or (R3.3) in Fig. 7. We say that $\langle w_1, w_2 \rangle$ is *precrystalline* if it is of subkind

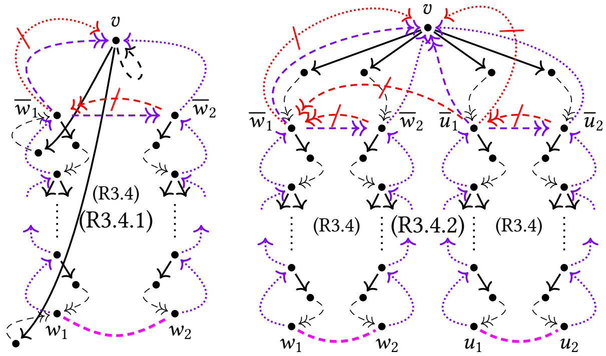


Figure 8. A precrystalline 1-bisimil. redundancy $\langle w_1, w_2\rangle$ is crystalline if it is neither of form (R3.4.1) nor of form (R3.4.2).

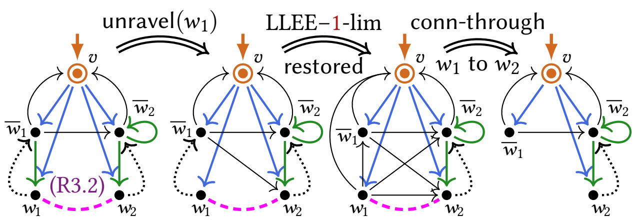


Figure 9. Example for the LLEE-preserving elimination of a red. 1-bisim. red. of kind (R3.2) by redirecting transitions to 1-bisimilar targets. Here all proper action labels are the same.

(R3.4) in Fig. 7. We say that $\langle w_1, w_2\rangle$ is *crystalline* if it is precrystalline, but neither of subkind (R3.4.1) nor (R3.4.2) in Fig. 8.

Example 7.3. In the LLEE-1-charts $\underline{C}$ in Fig. 4, and $\underline{C}_s$ in Fig. 5,the pair $\langle \boxed{abcd_1}, \boxed{abcd_2}\rangle$ of vertices forms a crystalline reduced 1-bisimilarity redundancy, since it is of kind (R3.4), hence precrystalline, but not of kind (R3.4.1) nor (R3.4.2).

From a LLEE-1-chart, every reduced 1-bisimilarity redundancy that is not crystalline can be *eliminated* LLEE*-preservingly* by which we mean that the result is a 1-bisimilar LLEE-1-chart. The transformations needed are adaptations of the connect-w_1-through-to-w_2 operation from [12, 13] in which the incoming transitions at vertex w_1 are redirected to a 1-bisimilar vertex w_2. Here this operation typically requires an unraveling step in which loop levels above w_1 that are reachable by 1-transitions are removed by similar transition redirections. In an example we illustrate the elimination of a reduced 1-bisimilarity redundancy of kind (R3.2) in Fig. 9.

Lemma 7.4. *Every reduced 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ can be eliminated* LLEE*-preservingly from a 1-transition limited* LLEE*-1-chart provided it is of either of the following kinds:*

(i) $\langle w_1, w_2\rangle$ *is simple, or*

(ii) $\langle w_1, w_2\rangle$ *is precrystalline, but not crystalline.*

In order to cut twin-crystals in scc's we also need to safeguard that the joining loop vertices of crystalline reduced 1-bisimilarity redundancies are 'parsimoniously insulated' from above. The top vertices v of 1-bisimilarity redundancies of kind (R3.3) and (R3.4) are, due to the occurring 1-transitions, substates of w_1 and w_2. Therefore any such vertex v can be *insulated from above*, that is, turned into a $\curvearrowleft$-maximal vertex, by redirecting all induced transitions from v into the loops-back-to part of v or below. In the example in Fig. 10, where all proper transitions have the same action label, in an insulation step (first step) the transition from v to u is redirected to the 1-bisimilar target $\overline{w}_2$. In the arising 1-chart v is not yet *parsimonious*, because the loop-entry transition from v to $\overline{w}_1$ can be redirected to 1-bisimilar target w_2 (second step), thereby making less use of the loops-back-to part of v, and eliminating it (and permitting further minimization).

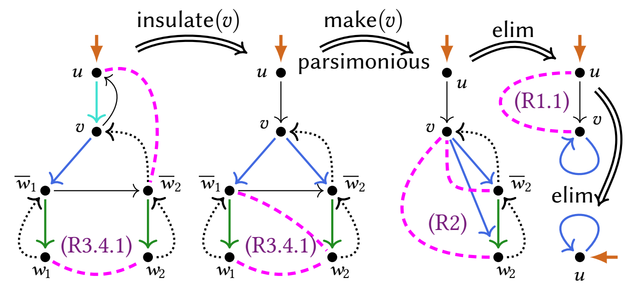


Figure 10. LLEE-Preserving parsimonious insulation from above of a red. 1-bisimilarity redundancy (R3.4.1) that here leads to its elimination, and permits further minimization.

We now define 'crystallized' (LLEE-1-)charts as follows.

Definition 7.5 (crystallized 1-chart (4))**.** Let $\underline{C}$ be a 1-chart.

We say that $\underline{C}$ *is crystallized* if there is a LLEE-witness $\hat{\underline{C}}$ of $\underline{C}$ such that the following four conditions hold:

(cr-1) $\underline{C}$ is a (finite) 1-chart with LLEE−1-lim, and specifically, $\underline{C}$ is 1-transition limited with respect to $\hat{\underline{C}}$.

(cr-2) Every $\hat{\underline{C}}$-reduced 1-bisim. redundancy is crystalline.

(cr-3) Every crystalline $\hat{\underline{C}}$-reduced 1-bisimilarity redundancy in $\underline{C}$ is parsimoniously insulated from above.

(cr-4) Every carrier of an scc in $\underline{C}$ is grounded in $\underline{C}$.

Then we also say that $\underline{C}$ is *crystallized with respect to* $\hat{\underline{C}}$.

The lemma below gathers properties of crystallized 1-charts. The subsequent proposition justifies the term 'crystallized 1-chart' by explaining the connection with twin-crystals.

Lemma 7.6. *Every 1-chart $\underline{C}$ that is crystallized with respect to a* LLEE*-witness $\hat{\underline{C}}$ with loops-back-to relation $\curvearrowleft$, satisfies:*

(i) $\underline{C}$ is 1-collapsed apart from within scc's, i.e. 1-collapsed for loops-back-to parts of $\hat{\underline{C}}$-maximal loop vertices of $\hat{\underline{C}}$.

(ii) $\underline{C}$ is 1-collapsed for every loops-back-to part of a loop vertex of $\hat{\underline{C}}$ that is not $\curvearrowleft$-maximal.

Proposition 7.7 (crystallized $\Rightarrow$ 1-coll./twin-crystal scc's)**.** *For every carrier P of a scc in a 1-chart $\underline{C}$ that is crystallized, either $\underline{C}$ is 1-collapsed for P, or P is the carrier of a twin-crystal.*

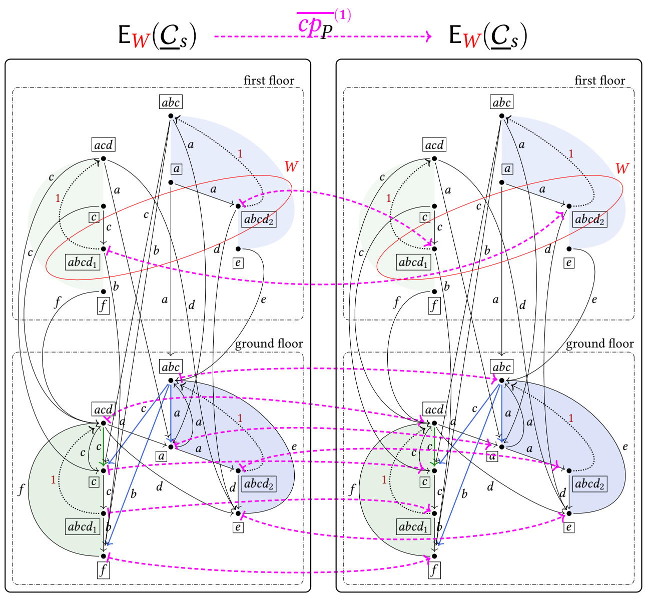


Figure 11. Lifting of the local transfer function cp_P on $\underline{\mathcal{C}_s}$ from Fig. 5 with domain and range W to a transfer function $\overline{cp_P}^{(1)}$ on the elevation $\mathsf{E}_W(\underline{\mathcal{C}_s})$ of W, which is a LLEE-1-LTS.

Lemma 7.8 (④). *Every crystallized* 1*-chart is near-collapsed.*

By combining LLEE-preserving eliminations of precrystalline 1-bisimilarity redundancies, of parsimonious insulation of crystalline 1-bisimilarity redundancies, and of grounding of scc's we are able to prove our main auxiliary statement.

Theorem 7.9 (crystallization, nearcollapse, (③,④)). *Every weakly guarded* LLEE*-*1*-chart $\underline{\mathcal{C}}$ can be transformed, together with a* LLEE*-witness* $\hat{\underline{\mathcal{C}}}$ *of* $\underline{\mathcal{C}}$, *into a* 1*-bisimilar* LLEE*-*1*-chart* $\underline{\mathcal{C}}'$ *with* 1*-transition limited* LLEE*-witness* $\hat{\underline{\mathcal{C}}}'$ *such that* $\underline{\mathcal{C}}$ *is crystallized with respect to* $\hat{\underline{\mathcal{C}}}'$*, and* $\underline{\mathcal{C}}'$ *is near-collapsed.*

8 Near-collapsed LLEE-1-charts have complete solutions

We show that near-collapsed LLEE-1-charts have complete solutions by linking local transfer functions, as in the definition of 'near-collapsed', to transfer functions in order to be able to use the transfer-property **(T)** for provable solutions.

Local-Transfer functions can be linked to transfer functions via the concept of the 1-LTS' $\mathsf{E}_W(\underline{\mathcal{L}})$ that is the 'elevation of a set W of vertices above' a 1-LTS $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow\rangle$, which is constructed as follows. The set of vertices of $\mathsf{E}_W(\underline{\mathcal{L}})$ consists of two copies of its set V of vertices, the 'ground floor' $V \times \{\mathbf{0}\}$, and the 'first floor' $V \times \{\mathbf{1}\}$. These two copies of the set of vertices of $\underline{\mathcal{L}}$ are linked by copies of the corresponding transitions of $\underline{\mathcal{L}}$ with the *exception* that *proper* $\langle v_1, a, v_2\rangle$ of $\underline{\mathcal{L}}$ do not give rise to a proper transition $\langle\langle v_1, \mathbf{1}\rangle, a, \langle v_2, \mathbf{1}\rangle\rangle$ on the first floor *if* the vertex v_2 is not contained in W. Those transitions get redirected as transitions $\langle\langle v_1, \mathbf{1}\rangle, a, \langle v_2, \mathbf{0}\rangle\rangle$ to target the corresponding copy $\langle v_2, \mathbf{0}\rangle$ of v_2 on the ground floor. Note that such redirections from the first floor to the ground floor do not happen for 1-transitions. The sub-1-LTS of $\mathsf{E}_W(\underline{\mathcal{L}})$ that consists of all transitions between vertices on the ground floor is an exact copy of the original 1-LTS $\underline{\mathcal{L}}$. Yet within the elevation $\mathsf{E}_W(\underline{\mathcal{L}})$ of W above $\underline{\mathcal{L}}$, a number of vertices on the ground floor will have additional incoming proper-action transitions from vertices on the first floor.

Example 8.1. Fig. 11 contains two copies of the elevation $\mathsf{E}_W(\underline{\mathcal{C}_s})$ of the set $W := \{\boxed{abcd_1}, \boxed{abcd_2}\}$ of vertices of the 1-chart $\underline{\mathcal{C}_s}$ in Fig. 5 above $\underline{\mathcal{C}_s}$.

Indeed, Fig. 11 also shows how the local transfer function cp_P on 1-chart $\underline{\mathcal{C}_s}$ in Fig. 5 can be lifted to a transfer function $\overline{cp_P}^{(1)}$ on the elevation $\mathsf{E}_W(\underline{\mathcal{C}_s})$ of W above $\underline{\mathcal{C}_s}$: namely by defining $\overline{cp_P}^{(1)}$ as cp_P on the first floor, and as the identity function on the ground floor of $\mathsf{E}_W(\underline{\mathcal{C}_s})$.

In general the following statement holds, which is a generalization to 1-charts of Prop. 2.4 in [10]. Every local transfer function $\phi : V \rightharpoonup V$ on a 1-LTS $\underline{\mathcal{L}}$ with $W := \mathrm{dom}(\phi) \cap \mathrm{ran}(\phi)$ lifts to a transfer function $\overline{\phi}^{(1)}$ on the elevation $\mathsf{E}_W(\underline{\mathcal{L}})$ of W over $\underline{\mathcal{L}}$, via the projection transfer function π_1 from $\mathsf{E}_W(\underline{\mathcal{L}})$ to $\underline{\mathcal{L}}$, such that the diagram below commutes for vertices on the first floor of $\mathsf{E}_W(\underline{\mathcal{L}})$:

$$\begin{tikzcd} \underline{\mathcal{L}} \arrow[r, "\phi"] & \underline{\mathcal{L}} \\ \mathsf{E}_W(\underline{\mathcal{L}}) \arrow[u, "\pi_1"] \arrow[r, "\overline{\phi}^{(1)}"] & \mathsf{E}_W(\underline{\mathcal{L}}) \arrow[u, "\pi_1"'] \end{tikzcd} \qquad \begin{array}{c} \text{for all } w \in \mathrm{dom}(\phi): \\ (\pi_1 \circ \phi)(\langle w, \mathbf{1}\rangle) \\ = \\ (\overline{\phi}^{(1)} \circ \pi_1)(\langle w, \mathbf{1}\rangle) \end{array} \tag{1}$$

Together with preservation of LLEE for elevations, the possibility to lift local transfer functions to transfer functions on elevations facilitates us to use invariance of provable solutions under transfer functions between LLEE-1-LTSs for proving invariance of provable solutions under local transfer functions on LLEE-1-LTSs. Then we can use this fact to show complete solvability of near-collapsed LLEE-1-charts.

Lemma 8.1. *Let $\phi : V \rightharpoonup V$ be a local transfer function on a w.g.* LLEE*-*1*-LTS* $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow\rangle$*. Then every* Mil*-provable solution s of $\underline{\mathcal{L}}$ is* Mil*-provably invariant under ϕ:*

$$s(w) =_{\mathsf{Mil}} s(\phi(w)) \quad (\textit{for all } w \in \mathrm{dom}(\phi))\ . \tag{2}$$

Proof (Sketch). We use the diagram in (1), and that $\mathsf{E}_W(\underline{\mathcal{L}})$ is also a LLEE-1-LTS. Since π_1 and $\overline{\phi}^{(1)} \circ \pi_1$ are transfer functions, by Lem. 3.6 $(s \circ \pi_1)$ and $(s \circ \pi_1 \circ \overline{\phi}^{(1)})$ are Mil-provably equal. Then by using we diagram commutativity on the first floor in (1) we get: $s(w) = (s \circ \pi_1)(\langle w, \mathbf{1}\rangle) =_{\mathsf{Mil}} (s \circ \pi_1 \circ \overline{\phi}^{(1)})(\langle w, \mathbf{1}\rangle) = (s \circ \phi \circ \pi_1)(\langle w, \mathbf{1}\rangle) = s(\phi(w))$, for all $w \in \mathrm{dom}(\phi)$. In this way we have obtained (2). ◻

Lemma 8.2 (⑥). *Every w.g.* LLEE*-*1*-chart that is near-collapsed has a* Mil*-complete* Mil*-provable solution.*

9 Conclusion

As a consequence of the crystallization process for LLEE-1-charts and of Thm. 7.9 we also obtain a new characterization of expressibility of finite process graphs in the process semantics. For this purpose we say that a 1-chart $\underline{C}$ is *expressible by a regular expression modulo 1-bisimilarity* if $\underline{C}$ is 1-bisimilar to the chart interpretation of a star expression.

Corollary 9.1. *A 1-chart $\underline{C}$ is expressible modulo 1-bisimilarity if and only if $\underline{C}$ is 1-bisimilar to a crystallized, and hence to a near-collapsed,* LLEE-1*-chart.*

Since the size of a crystallized 1-chart is bounded by at most double the size of an 1-bisimulation collapse (as every vertex is 1-bisimilar to at most one other vertex in twin-crystals, and crystallized 1-charts), this characterization raises the hopes for a polynomial algorithm for recognizing expressibility of finite process graphs. Such a recognition algorithm would substantially improve on the superexponential algorithm for deciding expressibility in [1].

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A Appendix

In this appendix we provide additional material for the sections of the article, for more scrutiny of our argumentation. In particular, we give details for most of the formal statements of our proof. However, the material in this appendix is still under development for a monograph that we are writing on the completeness proof for Mil. Here we collected the parts that are most crucial for our argumentation. We do not repeat proofs of results that have been published already; in those cases we place references, and explain the formal connections. – Please find below a table of contents of the article, and of this appendix.

Contents

A.1 Supplements for Section 1

A.1.1 Obstacle for the bisimulation-chart proof strategy (more detail).

In Sect. 1, starting on page 1, we argued that a bisimulation-chart proof strategy that would operate in analogy with Salomaa's proof in [16] of completeness of $\mathbf{F_1}$ for language equivalence of regular expressions does not work for showing completeness of Milner's system with respect to the process interpretation. In particular, we argued that an extraction procedure *EP* of a solution from any given guarded linear system of recursion equations as in Salomaa's proof is not possible, not even from systems that are solvable. The reason is as follows. An extraction procedure *EP* that were a completely analogous to the one used by Salomaa would have the following two properties:

(*EP*-1) *EP* extracts a solution from any guarded linear system $\mathcal{S}$ of recursion equations.

(*EP*-2) *EP* proceeds 'data-obliviously' by mechanically combining the actions in the equations of a recursion system $\mathcal{S}$ according to a fixed way of traversing it that, while copying and transporting specific letters from the equations over to the solution, never compares different letters and never takes decisions on the basis of the specific letter at hand. In particular *EP* also never takes a decision that the system would not be solvable, but always produces a result.

Now property (*EP*-1) cannot be obtained, because as Milner showed, see Example A.1 below, there are guarded linear systems of recursion equations that are unsolvable by star expressions in the process semantics. While this shows that an extraction procedure *EP* with (*EP*-1) and (*EP*-2) is impossible, it leaves open the possibility of an extraction procedure EP' that satisfies (*EP*-2) except for that it may sometimes not terminate (see (EP'-2) below), and the restriction (EP'-1) of (*EP*-1) to solvable recursion equations (see (EP'-1), (a)) together with a soundness condition (see (EP'-1), (b)):

(EP'-1) (a) EP' extracts a solution from any guarded linear system $\mathcal{S}$ of recursion equations <u>that is solvable</u>.

(b) Whenever EP' obtains a result for a guarded linear system $\mathcal{S}$ of recursion equations, then that is a solution of $\mathcal{S}$.

(EP'-2) EP' proceeds data-obliviously in the same way as (*EP*-2) requires for *EP*. However, the extraction process is not required to be terminating.

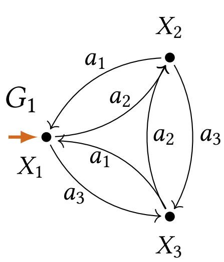
However, such a restricted extraction procedure EP' is not possible, either. The reason is that there are unsolvable specifications $\mathcal{S}_{uso}$ that are 'data-obliviously' the same (that is, the underlying process graph has the same structure when action names are ignored) with a (respectively) corresponding solvable specification $\mathcal{S}_{so}$. Then, in order to safeguard solvability for $\mathcal{S}_{so}$, see (EP'-1), (a), EP' would need to produce a result for both $\mathcal{S}_{uso}$ and $\mathcal{S}_{so}$; but this then leads to a contradiction with (EP'-1), (b), because the result extracted from $\mathcal{S}_{uso}$ cannot be a solution, as $\mathcal{S}_{uso}$ is unsolvable. Such pairs of unsolvable and solvable specifications that are data-obliviously the same arise from Example A.1 and Example A.1 below, where the specifications in Example A.2 are solvable counterparts of unsolvable specifications in Example A.1.

Such pairs of unsolvable and solvable process graphs that are the same 'data-obliviously' the same are not only artificial counterexamples, but do indeed occur from bisimulation charts that link bisimilar expressible graphs, see Example 4.1 in [12, 13].

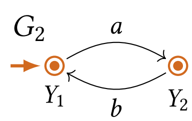
Example A.1 (not expressible process graphs, unsolvable recursive specifications)**.** As mentioned in Sect. 1 on page 2, Milner noticed that guarded systems of recursion equations cannot always be solved by star expressions under the process semantics:

> "[In] contrast with the case for languages—an arbitrary system of guarded equations in [star]-behaviours cannot in general be solved in star expressions" [15].

In fact, Milner showed in [15] that the linear specification $\mathcal{S}(G_1)$ defined by the process graph G_1 below does not have a star expression solution modulo bisimilarity. He conjectured that that holds also for the easier specification $\mathcal{S}(G_2)$ defined by the process graph G_2 below. That that is indeed the case was confirmed and proved later by Bosscher [**?**].



$$\mathcal{S}(G_1) = \begin{cases} X_1 = a_2 \cdot X_2 + a_3 \cdot X_3 \\ X_2 = a_1 \cdot X_1 + a_3 \cdot X_3 \\ X_3 = a_1 \cdot X_1 + a_2 \cdot X_2 \end{cases}$$



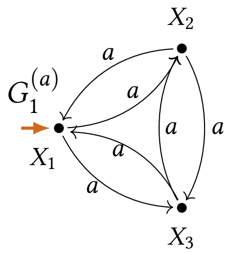
$$\mathcal{S}(G_2) = \begin{cases} Y_1 = 1 + a \cdot Y_2 \\ Y_2 = 1 + b \cdot Y_1 \end{cases}$$

Here the start vertex of a process graph is again highlighted by a brown arrow, and a vertex v with immediate termination is emphasized in brown as ⦿ including a boldface ring.

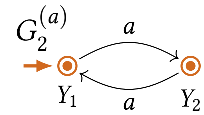
G_1 and G_2 are finite process graphs that are not bisimilar to the process interpretation of any star expression. In this sense, G_1 and G_2 are *not* expressible by a regular expression (under the process semantics). This sets the process semantics of regular expressions apart from the standard language semantics, with respect to which every language that is accepted by a finite-state automaton is the interpretation of some regular expression.

Furthermore it is easy to see that both of G_1 and G_2 do not satisfy LEE. Namely, both of these process graphs do not contain loop subcharts, but each of them represents an infinite behavior. Therefore the loop elimination procedure stops immediately on either of them, unsuccessfully.

Example A.2. While the specifications $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ in Example A.1 are not solvable by star expressions in the process semantics, this situation changes drastically if all actions in $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ are replaced by a single action a. Then we obtain the following process graphs $G_1^{(a)}$ and $G_2^{(a)}$ with appertaining specifications $\mathcal{S}(G_1^{(a)})$ and $\mathcal{S}(G_2^{(a)})$:



$$\mathcal{S}(G_1^{(a)}) = \begin{cases} X_1 = a \cdot X_2 + a \cdot X_3 \\ X_2 = a \cdot X_1 + a \cdot X_3 \\ X_3 = a \cdot X_1 + a \cdot X_2 \end{cases}$$



$$\mathcal{S}(G_2^{(a)}) = \begin{cases} Y_1 = 1 + a \cdot Y_2 \\ Y_2 = 1 + a \cdot Y_1 \end{cases}$$

These specifications are solvable by setting $X_1 := X_2 := X_3 := a^* \cdot 0$ in $\mathcal{S}(G_1^{(a)})$, and by setting $Y_1 := Y_2 := a^*$ in $\mathcal{S}(G_2^{(a)})$, because the following identities are provable in Milner's system Mil (which in any case is sound for the process semantics):

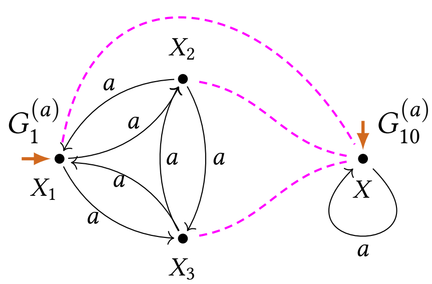
$$a^* \cdot 0 =_{\mathsf{Mil}} (1 + a \cdot a^*) \cdot 0 =_{\mathsf{Mil}} 1 \cdot 0 + (a \cdot a^*) \cdot 0 =_{\mathsf{Mil}} 0 + a \cdot (a^* \cdot 0), \qquad a^* =_{\mathsf{Mil}} 1 + a \cdot a^*$$
$$=_{\mathsf{Mil}} a \cdot (a^* \cdot 0) =_{\mathsf{Mil}} a \cdot (a^* \cdot 0) + a \cdot (a^* \cdot 0).$$

From these Mil-provable identities the Mil-provable correctness conditions for these settings follow directly, for example for X_1 in $\mathcal{S}(G_1^{(a)})$, and for Y_1 in $\mathcal{S}(G_2^{(a)})$:

$$X_1 = a^* \cdot 0 =_{\mathsf{Mil}} a \cdot (a^* \cdot 0) + a \cdot (a^* \cdot 0) = a \cdot X_2 + a \cdot X_3, \qquad Y_1 = a^* =_{\mathsf{Mil}} 1 + a \cdot a^* = 1 + a \cdot Y_2.$$

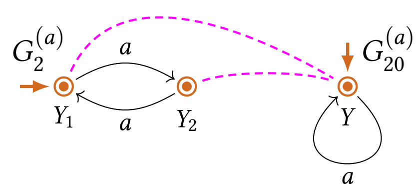
These Mil-provable identities show, together with the analogous ones for X_2, X_3, and Y_2 that the settings $X_1 := X_2 := X_3 := a^* \cdot 0$ and $Y_1 := Y_2 := a^*$ define solutions of $\mathcal{S}(G_1^{(a)})$, and $\mathcal{S}(G_2^{(a)})$, respectively, because Mil-provable identities are also identities with respect to the process semantics (as Milner's system is sound for the process semantics).

The crucial reason why we have obtained solvable specifications $\mathcal{S}(G_1^{(a)})$ and $\mathcal{S}(G_2^{(a)})$ from unsolvable specifications $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$, respectively, is that the underlying process graphs $\mathcal{S}(G_1^{(a)})$ and $\mathcal{S}(G_2^{(a)})$ are not bisimulation collapses, in contrast with the process graphs G_1 and G_2 from which they arose by renaming all actions to a single one. Indeed the bisimulation collapses G_{10} of G_1, and G_{20} of G_2 are of particularly easy form that are obviously solvable by the star expressions $a^* \cdot 0$, and a^*, respectively. These solutions can be transferred backwards over the functional bisimulations $G_1^{(a)} \rightrightarrows G_{10}^{(a)}$, and $G_2^{(a)} \rightrightarrows G_{20}^{(a)}$ (indicated by magenta links in the pictures below) according to Lem. 2.8, (i).



$a^* \cdot 0$ is provable solution

$$\mathcal{S}(G_{10}^{(a)}) = \{ X = a \cdot X \}$$



a^* is provable solution

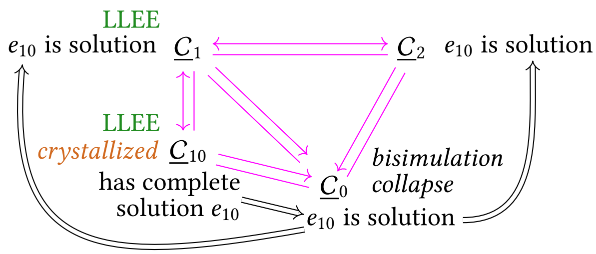
$$\mathcal{S}(G_{20}^{(a)}) = \{ Y = 1 + a \cdot Y \}$$

in order to obtain the Mil-provable solutions described above.

This example witnesses the result from [12, 13] that a process graph G without 1-transitions is expressible by a 1-free star expression in the process semantics if the bisimulation collapse of G has the property LEE. Here the bisimulation collapses $G_{10}^{(a)}$ and $G_{20}^{(a)}$ satisfy LEE, although the process graphs from which they arise by collapse, $G_1^{(a)}$ and $G_2^{(a)}$, do not satisfy LEE.

A.1.2 Recovered bisimulation collapse strategy (explained directly).

With the conceptual tools (2)–(5) on page 3 in place, the collapse proof strategy can be recovered for LLEE-1-charts. The central part of our completeness proof for Mil can be illustrated as follows:



Here we start from the assumption of 1-bisimilar 1-charts $\underline{C}_1$ and $\underline{C}_2$ of which at least $\underline{C}_1$ satisfies LLEE. Then the LLEE-1-chart $\underline{C}_1$ can be crystallized with as result a 1-bisimilar LLEE-1-chart $\underline{C}_{10}$. Due to LLEE, a provable solution s_{10} can be extracted from $\underline{C}_0$. Let e_{10} be the principal value of s_{10}. Since crystallized 1-charts are near-collapsed LLEE-1-charts, the solution s_{10} of $\underline{C}_0$ is complete. As such it defines a provable solution with principal value e_0 on the joint bisimulation collapse $\underline{C}_0$ of $\underline{C}_{10}$, $\underline{C}_1$, and $\underline{C}_2$. This solution can be pulled back conversely over the functional 1-bisimulations from $\underline{C}_1$ and from $\underline{C}_2$ to $\underline{C}_0$. In this way we obtain provable solutions s_1 of $\underline{C}_1$ and s_2 of $\underline{C}_2$, both with the same principal value e_{10}.

This is the central argument of our completeness proof for Mil that we give in Sect. 4 based on lemmas. Its illustration appears in the illustration of the completeness proof in Fig. 3.

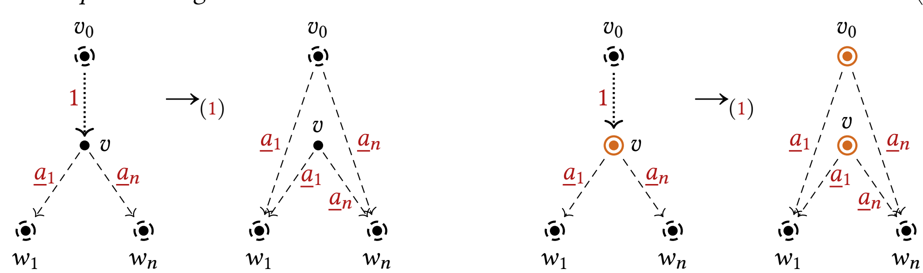
A.2 Supplements for Section 2: Preliminaries

A.2.1 From 1-charts to induced charts via 1-transition elimination.

The induced chart $\underline{C}_{(\cdot]}$ of a 1-chart $\underline{C}$ is a (1-transition free) chart that results from $\underline{C}$ by adding all induced transitions, and induced termination, and by removing all 1-transitions. This process of '1-transition elimination' in a 1-chart $\underline{C}$ in order to construct its induced chart $\underline{C}_{(\cdot]}$ can also be defined by applying local elimination rules for 1-transitions. We define these rules below, and then establish properties of this concept of local elimination of 1-transitions in Lem. A.5 below.

While this alternative way of constructing the induced chart $\underline{C}_{(\cdot]}$ of a 1-chart $\underline{C}$ may seen redundant at this point, we will crucially need 1-transition elimination for the purpose of minimizing 1-charts with LLEE such that LLEE is preserved. This will be important, because induced charts of 1-charts with LLEE do not in general have the property LLEE themselves, as was already noticed in [7, 11]. In other words, for 1-charts with LLEE we will frequently only be able to approximate the induced chart LLEE-*preservingly*. LLEE-preserving 1-transition elimination will be defined in Sect. A.3.2.

Definition A.3. Let $\underline{C}_i = \langle V_i, A, 1, v_s, \rightarrow_i, \downarrow_i \rangle$ for $i \in \{1, 2\}$ be 1-charts. We denote by $\underline{C}_1 \rightarrow_{(1)} \underline{C}_2$ that $\underline{C}_2$ arises from $\underline{C}_1$ by a 1-*transition elimination step* according to one of two local rules each of which removes the 1-transition $\langle v_0, 1, v \rangle$:



and adds, for each transition $\langle v, \underline{a}_i, w_i \rangle$ with $i \in \{1, \ldots, n\}$ from v, a cofinal transition $\langle v_0, \underline{a}_i, w_i \rangle$ with the same label from v. Note that $n = 0$ is possible if v has outdegree 0. Additionally, garbage collection of v is permitted if v becomes unreachable. That immediate termination is permitted (but not required) in a vertex is indicated by a dashed outer ring. While v does not permit immediate termination in the rule on the left, immediate termination at v in the rule on the right is transferred in the step to v_0. We used dashed arrows for the outgoing transitions from v to indicate that they can be either proper transitions or 1-transitions.

We say that $\underline{C}_1$ *(1-transition) refines* $\underline{C}_2$, that $\underline{C}_1$ *is a (1-transition) refinement of* $\underline{C}_2$, and that $\underline{C}_2$ *is* or *can be (1-transition) refined by* $\underline{C}_1$, if $\underline{C}_1 \rightarrow^*_{(1)} \underline{C}_2$, that is, $\underline{C}_2$ arises from $\underline{C}_1$ by a sequence of 1-transition elimination steps.

Remark A.4. We will only be concerned with the elimination of 1-transitions from finite, weakly guarded 1-charts. Therefore we could have restricted the two rules in Def. A.3 by demanding (i) $v_0 \neq v$ (since weakly guarded 1-charts do not have 1-transition self-loops), and (ii) that all outgoing transitions from v are proper transitions (because all 1-transitions in finite weakly guarded 1-charts can be eliminated in a bottomup manner). We left out restriction (i) for simplicity, thereby accepting that it introduces a cyclic reduction for 1-transition selfloops for not weakly guarded 1-charts. But we needed to avoid restriction (ii), because in the proof of Lem. 3.2 in Sect. A.3.2 we will use annotated versions of these rules for situations when it is not possible to eliminate all 1-transitions while keeping the property LEE.

Lemma A.5. *The 1-transition elimination rewrite relation $\rightarrow_{(1)}$ has the following properties, for all 1-charts $\underline{C}, \underline{C}_1, \underline{C}_2$:*

(i) If $\underline{C}_1$ is weakly guarded, and $\underline{C}_1 \rightarrow^*_{(1)} \underline{C}_2$, then C_2 is finite and weakly guarded, and $|V(C_1)| - 1 \leqslant |V(C_2)| \leqslant |V(C_1)|$.

(ii) $\rightarrow_{(1)}$ is terminating from every finite weakly guarded 1-chart.

(iii) $\rightarrow_{(1)}$ normal forms are 1-free 1-chart. $\rightarrow_{(1)}$ normal forms of finite, weakly guarded 1-charts are finite 1-free 1-charts.

(iv) $\underline{C}_1 \rightarrow^*_{(1)} \underline{C}_2 \implies (\underline{C}_1)_{(\cdot]} = (\underline{C}_2)_{(\cdot]}$.

(v) If $\underline{C}$ is finite and weakly guarded, and C is 1-free, then:

$$\underline{C} \rightarrow^*_{(1)} C \iff (\underline{C})_{(\cdot]} = C\,.$$

(vi) If $\underline{C}$ is finite, and weakly guarded, then $\underline{C} \rightarrow^*_{(1)} \underline{C}^{(\mathrm{sc})}_{(\cdot]}$, that is, $\underline{C}$ refines its induced chart $\underline{C}^{(\mathrm{sc})}_{(\cdot]}$, and $\underline{C}^{(\mathrm{sc})}_{(\cdot]}$ is the unique $\rightarrow_{(1)}$ normal form of $\underline{C}$.

Proof. For (i) it suffices to note: every $\rightarrow_{(1)}$ step can only shorten 1-transition paths, but it cannot introduce a 1-transition cycle; in every $\rightarrow_{(1)}$ step at most one vertex, the target v of the 1-transition that is removed, can become unreachable.

For (ii) it suffices to obtain a measure $m(\underline{C})$ on finite w.g. 1-charts $\underline{C}$ that decreases properly in every $\rightarrow_{(1)}$ step from $\underline{C}$. We can use $m(\underline{C}) := \sum_{v \in V(\underline{C})} s_{\underline{C}}(v)$ where $s_{\underline{C}}(v)$ denotes the sum of the lengths of all maximal 1-transition paths from v. Note that, for both rules in Def. A.3, the length of every maximal 1-transition path from v_0 decreases by 1, whereas maximal 1-transition paths from other vertices are either preserved, or are also shortened by 1 if they pass through v_0.

For (iii) we note that every 1-transition gives rise to a $\rightarrow_{(1)}$ step, so normal forms do not contain 1-transitions, and that by (i), (ii) every finite w.g. 1-chart rewrites via finitely many $\rightarrow_{(1)}$ steps to a normal form that is finite and w.g..

Statement (iv) can be shown by induction on the length of $\rightarrow_{(1)}$ paths by using that every $\rightarrow_{(1)}$ step preserves induced transitions and induced termination. The direction "$\Rightarrow$" in (v) follows from (iv) and that $C_{(\cdot]} = C$ for 1-free 1-charts C. For showing the direction "$\Leftarrow$" in (v), suppose that $(\underline{C})_{(\cdot]} = C$ for finite w.g. 1-charts $\underline{C}$, C where C is 1-free. We have to show $\underline{C}'_{(\cdot]} = C$. Then $\underline{C} \rightarrow^*_{(1)} \underline{C}'$ for an $\rightarrow_{(1)}$ normal form $\underline{C}'$ by (ii), which is finite, w.g., and 1-free by (iii). Then we get $\underline{C}'_{(\cdot]} = \underline{C}_{(\cdot]} = C$ by using (iv), and thus obtain $\underline{C}'_{(\cdot]} = C$.

Last, (vi) follows from "$\Leftarrow$" in (v), letting $C := \underline{C}_{(\cdot]}$, and by recalling (iii). ◻

A.2.2 Provable, and complete provable, solutions.

Here we explain how Lem. 2.8 can be proved. We will first focus on its first statement, transfer of provable solutions conversely over functional 1-bisimulations, and then demonstrate its two other statements that concern transfer of complete provable solutions to bisimulation collapses, and to 1-bisimilar 1-charts.

Lemma (= Lem. 2.8)**.** *On finite weakly guarded 1-charts, the following preservation statements hold for (*Mil*-complete)* Mil*-provable solvability, for all star expressions $e \in StExp$:*

(i) Mil*-Provable solvability with principal value e is preserved under converse functional 1-bisimilarity.*

(ii) Mil*-Complete* Mil*-provable solvability with principal value e of a weakly guarded 1-chart $\underline{C}$ implies* Mil*-provable solvability with principal value e of the bisimulation collapse of $\underline{C}$.*

(iii) Mil*-Complete* Mil*-provable solvability with principal value e is preserved under 1-bisimilarity.*

Item (i) of this lemma states that Mil-provable solutions can be pulled back over functional 1-bisimulations between 1-charts. In order to prove this, we will utilize and combine the statements of three lemmas. The first statement (Lem. A.6) guarantees that Mil-provable solutions can be pulled back over bisimulations between charts. This fact has essentially already been proved by Grabmayer and Fokkink in [12, 13], albeit for '1-free' star expressions with binary star iteration instead of with unary star iteration. The second lemma (Lem. A.7) explains, as one of its statements, that functional 1-bisimulations between weakly guarded 1-charts project to functional bisimulations between the induced charts, via 1-transition elimination steps. Eventually the third lemma (Lem. A.8) states that Mil-provable solutions can be transferred forward and backward over (single or multiple) 1-transition elimination steps.

Lemma A.6 (transfer of provable solutions between charts backward over functional bisimulations, $\sim$ Prop. 5.1 in [12, 13]). *On charts, Mil-provable solutions can be pulled back over functional bisimulations. More explicitly, if ϕ is a functional bisimulation from a chart C_1 to a chart C_2 (that is, if $C_1 \rightrightarrows_{\phi} C_2$ holds), and if s_2 is a Mil-provable solution of C_2 with principal value e, then $s_1 := s_2 \circ \phi$ is a Mil-provable solution of C_1 with the same principal value e.*

Proof. This lemma can be proved analogously to Prop. 5.1 in [12, 13]. ◻

Lemma A.7 (projection of 1-bisimulations to bisimulations via 1-transition elimination). *Let $\underline{C}_i = \langle V_i, A, 1, v_{\mathrm{s},i}, \rightarrow_i, \downarrow_i \rangle$, for $i \in 1, 2$, be finite, weakly guarded 1-charts. Let $V_{1,0} \subseteq V_1$ and $V_{2,0} \subseteq V_2$ be the set of vertices in the induced chart $(\underline{C}_1)_{(\cdot)}$ of $\underline{C}_1$, and in the induced chart $(\underline{C}_2)_{(\cdot)}$ of $\underline{C}_2$, respectively.*

Then the following two statements hold:

(i) Every 1-bisimulation $B \subseteq V_1 \times V_2$ between $\underline{C}_1$ and $\underline{C}_2$ projects, via 1-transition elimination steps, to the 1-bisimulation $B_0 := B \cap (V_{1,0} \times V_{2,0})$ between the induced chart $(\underline{C}_1)_{(\cdot)}$ of $\underline{C}_1$ and the induced chart $(\underline{C}_2)_{(\cdot)}$ of $\underline{C}_2$, as witnessed by the diagram:

$$\begin{tikzcd}
\underline{C}_1 \arrow[r, leftrightarrow, "B"] \arrow[d, "(1)"', "*"] & \underline{C}_2 \arrow[d, "(1)"', "*"] \\
(\underline{C}_1)_{(\cdot)} \arrow[r, leftrightarrow, "B_0"] & (\underline{C}_2)_{(\cdot)}
\end{tikzcd} \tag{3}$$

(ii) For every function $\phi : V_1 \rightarrow V_2$ that defines a 1-bisimulation between $\underline{C}_1$ and $\underline{C}_2$, the restriction $\phi_0 := \phi|_{V_{1,0}} : V_{1,0} \rightarrow V_{2,0}$ of ϕ to the set $V_{1,0}$ of $(\underline{C}_1)_{(\cdot)}$ (note that the range of this restriction of ϕ is contained in $V_{2,0}$, because 1-bisimilarity preserves reachability of vertices) defines a functional 1-bisimulation between the induced chart $(\underline{C}_1)_{(\cdot)}$ of $\underline{C}_1$ and the induced chart $(\underline{C}_2)_{(\cdot)}$ of $\underline{C}_2$ such that with 1-transition elimination steps to the induced charts the following diagram holds:

$$\begin{tikzcd}
\underline{C}_1 \arrow[r, Rightarrow, "\phi"] \arrow[d, "(1)"', "*"] & \underline{C}_2 \arrow[d, "(1)"', "*"] \\
(\underline{C}_1)_{(\cdot)} \arrow[r, Rightarrow, "\phi_0"] & (\underline{C}_2)_{(\cdot)}
\end{tikzcd} \tag{4}$$

Proof (Idea). Both items of the lemma can be proved by induction

, for Mil-provable solutions and for Mil-complete Mil-provable solutions, can be proved by induction on the number of $\rightarrow_{(1)}$ steps in a rewrite sequence $\underline{C}_1 \rightarrow_{(1)}^{*} \underline{C}_2$. Hereby the induction step uses the auxiliary statement that Mil-provable solutions, and Mil-complete Mil-provable, can be transferred forward and backward over single 1-transition elimination steps $\underline{C}_1 \rightarrow_{(1)}^{*} \underline{C}_2$. The latter statement can be proved by arguing about the local changes that applications of either of the two rules for 1-transition elimination in Def. A.3 can perform to a 1-chart. ◻

Lemma A.8 (transfer of solutions and complete solutions over 1-transition elimination or 1-transition refinement). *Mil-Provable solvability of finite, weakly guarded 1-charts is preserved under 1-transition elimination and refinement. Moreover, and more explicitly, for all weakly guarded 1-charts $\underline{C}_1 = \langle V_1, A, 1, v_{\mathrm{s},1}, \rightarrow_1, \downarrow_1 \rangle$ and $\underline{C}_2 = \langle V_2, A, 1, v_{\mathrm{s},2}, \rightarrow_2, \downarrow_2 \rangle$ with $\underline{C}_1 \rightarrow_{(1)}^{*} \underline{C}_2$, (that is, $\underline{C}_1$ is a 1-transition refinement of $\underline{C}_2$, and $\underline{C}_2$ arises by 1-transition elimination from $\underline{C}_1$) the following statement holds:*

(i) If s_1 is a Mil-provable solution of $\underline{C}_1$ with principal value $s_1(v_{\mathrm{s},1}) = e$, then there is also a Mil-provable solution s_2 of $\underline{C}_2$ with principal value $s_2(v_{\mathrm{s},2}) = e$.

(ii) If s_2 is a Mil-provable solution of $\underline{C}_2$ with principal value $s_2(v_{\mathrm{s},2}) = e$, then there is also a Mil-provable solution s_1 of $\underline{C}_1$ with principal value $s_1(v_{\mathrm{s},1}) = e$.

Furthermore statements above hold for Mil-complete Mil-provable solutions as well: in particular, statements (i) and (ii) hold also if 'Mil-provable solution' is replaced everywhere by 'Mil-complete Mil-provable solution'.

Proof (Idea). Both statements of the lemma, for Mil-provable solutions and for Mil-complete Mil-provable solutions, can be proved by induction on the number of $\rightarrow_{(1)}$ steps in a rewrite sequence $\underline{C}_1 \rightarrow_{(1)}^{*} \underline{C}_2$. Hereby the induction step uses the auxiliary statement that Mil-provable solutions, and Mil-complete Mil-provable solutions, can be transferred forward and backward over single 1-transition elimination steps $\underline{C}_1 \rightarrow_{(1)} \underline{C}_2$. The latter statement can be proved by arguing about the local changes that applications of either of the two rules for 1-transition elimination in Def. A.3 can perform to a 1-chart. ◻

Now we are in a position to prove item (i) of Lem. 2.8 by utilizing and combining the statements of Lem. A.6, Lem. A.7, and Lem. A.8 above.

Proof of Lem. 2.8, (i). Let $\phi : V_1 \to V_2$ be a function that defines a 1-bisimulation between $\underline{C}_1$ and $\underline{C}_2$, and hence more formally $\underline{C}_1 \rightrightarrows_{\phi} \underline{C}_2$, where $\underline{C}_i = \langle V_i, A, 1, v_{s,i}, \to_i, \downarrow_i\rangle$ for $i \in \{1,2\}$. Let $s_2 : V_2 \to \mathit{StExp}(A)$ be a Mil-provable solution of $\underline{C}_2$ with principal value $s_2(v_{s,2}) = e$ for some $e \in \mathit{StExp}(A)$. We have to show that there is also a Mil-provable solution $s_2 : V_2 \to \mathit{StExp}(A)$ of $\underline{C}_1$ with principal value $s_1(v_{s,1}) = e$.

We start by using tha Lem. A.7, (ii), yields that $\underline{C}_1 \rightrightarrows_{\phi} \underline{C}_2$ projects to $(\underline{C}_1)_{(\cdot)} \rightrightarrows_{\phi_0} (\underline{C}_2)_{(\cdot)}$ via 1-transition elimination steps, for the restriction ϕ_0 of ϕ to the set of vertices of $(\underline{C}_1)_{(\cdot)}$, such that the diagram (4) holds. Now we use the solution transfer lemmas Lem. A.6 and Lem. A.8 in order to use the Mil-provable solution s_2 of $\underline{C}_2$ to obtain Mil-provable solutions of $(\underline{C}_2)_{(\cdot)}$, then of $(\underline{C}_1)_{(\cdot)}$, and finally of $\underline{C}_1$. First, due to Lem. A.8, (i), we obtain from s_2 a Mil-provable solution s_{20} of $(\underline{C}_2)_{(\cdot)}$ with principal value e. Second, we can pull back s_{20} backward over ϕ_0, by using Lem. A.6, in order to obtain the Mil-provable solution $s_{10} := s_{20} \circ \phi_0$ of $(\underline{C}_1)_{(\cdot)}$ with principal value e. Finally third, we can transfer s_{10} backward over the 1-transition elimination steps from $\underline{C}_1$ to $(\underline{C}_1)_{(\cdot)}$, by using Lem. A.8, (ii), to obtain a Mil-provable solution s_1 of $\underline{C}_1$ with principal value e. ◻

For the transfer of complete provable solutions between 1-bisimilar 1-charts we will again use Lem. A.7 for relating 1-bisimulations, and functional 1-bisimulations via 1-transition elimination steps to bisimulations, and functional bisimulation on the induced charts, and Lem. A.8 for transferring complete solutions forward and backward over 1-transition elimination steps. For transferring complete provable solutions between bisimilar induced charts we formulate a lemma (Lem. A.11) that permits to transfer 'strongly complete' solutions backwards not only over functional bisimulations, but also over arbitrary bisimulations. For this purpose we define 'strongly complete' solutions as a tightened concept that, however, has the property that from every complete solution a strongly complete solution can always be obtained rather easily.

Definition A.9. Let $\mathcal{S} \in \{\mathsf{Mil}^-, \mathsf{Mil}\}$. We say that an $\mathcal{S}$-provable solution s of a 1-LTS $\underline{\mathcal{L}}$ is *strongly $\mathcal{S}$-complete* if for all $w_1, w_2 \in V$ it holds:

$$w_1 \leftrightarrows_{\underline{\mathcal{L}}} w_2 \implies s(w_1) = s(w_2),$$

that is, if values of the Mil-provable solution s at 1-bisimilar vertices of $\underline{\mathcal{L}}$ are syntactically equal.

Lemma A.10 (complete solvability ⇒ strongly complete solvability). *If a 1-chart $\underline{C}$ has a* Mil*-complete* Mil*-provable solution with principal value e, then $\underline{C}$ has also a strongly* Mil*-complete* Mil*-provable solution with principal value e.*

Proof. Suppose that s is a Mil-complete Mil-provable solution with principal value e of a 1-chart $\underline{C} = \langle V, A, 1, v_s, \to, \downarrow\rangle$.

As an easy consequence of the definition of Mil-complete solutions we recognize that Mil-complete Mil-provable solutions of $\underline{C}$ are preserved by changes of their values up to provability in Mil. Now since for the Mil-complete Mil-provable solution s of $\underline{C}$ it holds that values of s at 1-bisimilar vertices in $\underline{C}$ are Mil-provably equal, it is possible to change s by picking a joint value for all vertices that are in the same 1-bisimilarity class of vertices of $\underline{C}$, in order obtain another Mil-complete Mil-provable solution s' of $\underline{C}$. By making sure that we pick the value $s(v_s)$ of the start vertex v_s of $\underline{C}$ for all vertices in the 1-bisimilarity equivalence class of the start vertex v_s of $\underline{C}$, we guarantee that $s'(v_s) = s(v_s) = e$.

By this construction, 1-bisimilar vertices of $\underline{C}$ have the same values under s', and s' also has the principal value e. It follows that s' is indeed a strongly Mil-complete Mil-provable solution of $\underline{C}$, which also has e as its principal value. ◻

Lemma A.11 (transfer of strongly complete provable solutions <u>between charts</u> over bisimulations, ∼ Prop. 5.1 in [12, 13]). *On charts, strongly* Mil*-complete* Mil*-provable solutions can transferred over functional bisimulations.*

More specifically, if B is a bisimulation between charts C_1 and C_2 (that is, if $C_1 \leftrightarrows_B C_2$ holds), if $\phi : \mathrm{dom}(B) \to \mathrm{ran}(B)$ is a function with $\mathit{graph}(\phi) \subseteq B$, and if s_2 is a Mil*-complete* Mil*-provable of C_2, then $s_1 := s_2 \circ \phi$ is a* Mil*-complete* Mil*-provable solution of C_1 with the same principal value e.*

Now we can prove items (ii), and (iii) of Lem. 2.8 by combining the statements of Lem. A.7, Lem. A.8, Lem. A.10, and Lem. A.11.

Proof of Lem. 2.8, (ii), and (iii). In order to show item (ii), let $\underline{C} = \langle V, A, 1, v_s, \to, \downarrow\rangle$ be a 1-chart with bisimulation collapse $C_0 = \langle V_0, A, 1, v_{s,0}, \to_0, \downarrow_0\rangle$. Then there is a functional 1-bisimulation $\phi : V \to V_0$ from $\underline{C}$ to C_0, that is formally, $\underline{C} \rightrightarrows_{\phi} C_0$. Since C_0 is, as a bisimulation collapse, a 1-free chart, it follows that $(C_0)_{(\cdot)} = C_0$ (that is, C_0 equals its own induced chart), the diagram that is guaranteed due to $\underline{C} \Rightarrow C_0$ by Lem. A.7, (ii), is of the special form:

$$\begin{tikzcd}
\underline{C} \arrow[d, "(1)\,*"'] \arrow[dr, Rightarrow, "\phi"] & \\
\underline{C}_{(\cdot)} \arrow[r, Rightarrow, "\phi_0"'] & C_0
\end{tikzcd}$$

for a functional bisimulation $\phi_0 : V(\underline{C}_{(\cdot)}) \to V_0$ from $\underline{C}_{(\cdot)}$ to C_0.

Now suppose that s is a Mil-complete Mil-provable solution of $\underline{C}$ with principal value $s(v_s) = e$. We have to show that there is also a Mil-complete Mil-provable solution s_0 of C_0.

By applying Lem. A.8, (i), the version for complete provable solutions, to the Mil-complete Mil-provable solution of $\underline{C}$, it follows that there is a Mil-complete Mil-provable solution $s_{(\cdot)}$ of the induced chart $\underline{C}_{(\cdot)}$ of $\underline{C}$ with principal value e. Due to Lem. A.10, we may assume that $s_{(\cdot)}$ is a strongly Mil-complete Mil-provable solution of $\underline{C}_{(\cdot)}$. Now we can apply Lem. A.11 to $s_{(\cdot)}$ in order to transfer it from the chart $\underline{C}_{(\cdot)}$ over to the chart $\underline{C}_0$. We thereby obtain a Mil-complete Mil-provable solution s_0 of C_0 with the same principal value e, as desired.

For proving item (iii) of the lemma, we suppose that $\underline{C}_1$ and $\underline{C}_2$ are 1-bisimilar 1-charts, that is, $\underline{C}_1 \leftrightarrow \underline{C}_2$ holds, and that s_1 is a Mil-complete Mil-provable solution of $\underline{C}_1$ with principal value e. We have to show that there is also a Mil-complete Mil-provable solution s_2 of $\underline{C}_2$ with principal value e.

Since $\underline{C}_1$ and $\underline{C}_2$ are 1-bisimilar, they have the same bisimulation collapse up to isomorphism. Let $\underline{C}_0$ be a bisimulation collapse of $\underline{C}_1$, and therefore also of $\underline{C}_2$. Then $\underline{C}_1 \Rightarrow C_0 \Leftarrow \underline{C}_2$ holds. Since again $C_0 = (C_0)_{(\cdot)}$ holds for the 1-transition free bisimulation collapse C_0, we get, due to Lem. A.7, (i), the following connection between $\underline{C}_1$ and $\underline{C}_2$ via C_0, and the induced charts $(\underline{C}_1)_{(\cdot)}$ of $\underline{C}_1$, and $(\underline{C}_2)_{(\cdot)}$ of $\underline{C}_2$:

$$\begin{tikzcd}
\underline{C}_1 \arrow[rr, leftrightarrow] \arrow[d, "(1)\,*"'] \arrow[dr, Rightarrow] & & \underline{C}_2 \arrow[d, "(1)\,*"] \arrow[dl, Rightarrow] \\
(\underline{C}_1)_{(\cdot)} \arrow[r, Rightarrow] & C_0 & (\underline{C}_2)_{(\cdot)} \arrow[l, Rightarrow]
\end{tikzcd}$$

Now by using item (ii) of the lemma, which we have established above, we obtain a Mil-complete Mil-provable solution s_0 with principal value e of the bisimulation collapse $\underline{C}_0$ from the Mil-complete Mil-provable solution s_1 of $\underline{C}_1$ with principal value e. As is any Mil-provable solution of a bisimulation collapsed chart, s_0 is also strongly Mil-complete. Therefore we can apply Lem. A.11 to the bottom right corner of the diagram above, the functional bisimilarity $C_0 \Leftarrow (\underline{C}_2)_{(\cdot)}$ between the charts $\underline{C}_0$ and $(\underline{C}_2)_{(\cdot)}$. We obtain a Mil-complete Mil-provable solution $(s_2)_{(\cdot)}$ of $(\underline{C}_2)_{(\cdot)}$ with the same principal value e. Finally we can use Lem. A.8, (ii), the version for complete solvability, to transfer $(s_2)_{(\cdot)}$ backwards over the 1-transition elimination steps $\underline{C}_2 \to_{(1)}^{*} (\underline{C}_2)_{(\cdot)}$ from $(\underline{C}_2)_{(\cdot)}$ to $\underline{C}_2$. In this way we obtain a Mil-complete Mil-provable solution s_2 of $\underline{C}_2$ with the same principal value e as $(s_2)_{(\cdot)}$, and hence also the same principal value as s_1, as desired. ◻

A.3 Supplements for Section 3: LLEE-1-charts

A.3.1 Definitions of LEE and LLEE for 1-charts.

In this subsection we recall principal definitions and statements from [7, 11–13]. We keep formalities to a minimum as necessary for our purpose (in particular for 'LLEE-witnesses').

A 1-chart $\underline{\mathcal{LC}} = \langle V, A, 1, v_s, \to, \downarrow \rangle$ is called a *loop 1-chart* if it satisfies three conditions:

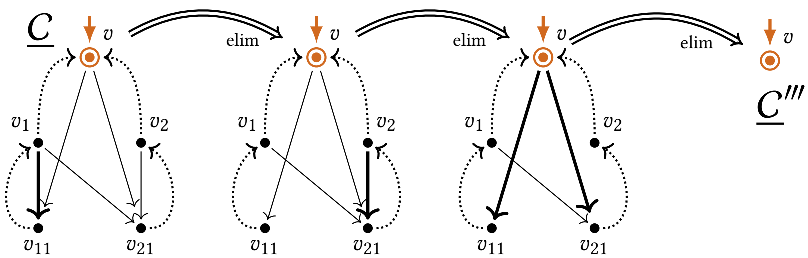
(L1) There is an infinite path from the start vertex v_s.

(L2) Every infinite path from v_s returns to v_s after a positive number of transitions.

(L3) Immediate termination is only permitted at the start vertex, that is, $\downarrow \subseteq \{v_s\}$.

We call the transitions from v_s *loop-entry transitions*, and all other transitions *loop-body transitions*. A *loop sub-1-chart of* a 1-chart $\underline{C}$ is a loop 1-chart $\underline{\mathcal{L}C}$ that is a sub-1-chart of $\underline{C}$ with some vertex $v \in V$ of $\underline{C}$ as start vertex, such that $\underline{\mathcal{L}C}$ is constructed, for a nonempty set U of transitions of $\underline{C}$ from v, by all paths that start with a transition in U and continue onward until v is reached again (so the transitions in U are the loop-entry transitions of $\underline{\mathcal{L}C}$).

The result of *eliminating a loop sub-1-chart $\underline{\mathcal{L}C}$ from a 1-chart $\underline{C}$* arises by removing all loop-entry transitions of $\underline{\mathcal{L}C}$ from $\underline{C}$, and then also removing all vertices and transitions that become unreachable. We say that a 1-chart $\underline{C}$ has the *loop existence and elimination property* (LEE) if the procedure, started on $\underline{C}$, of repeated eliminations of loop sub-1-charts results in a 1-chart without an infinite path. If, in a successful elimination process from a 1-chart $\underline{C}$, loop-entry transitions are never removed from the body of a previously eliminated loop sub-1-chart, then we say that $\underline{C}$ satisfies *layered* LEE (LLEE), and is a LLEE -1*-chart*. While the property LLEE leads to a formally easier concept of 'witness', it is equivalent to the property LEE, see Prop. A.14 below. (For an example of a LEE-witness that is not layered, see further below on page 22.)
The picture below shows a successful run of the loop elimination procedure for the 1-chart $\underline{C}$.

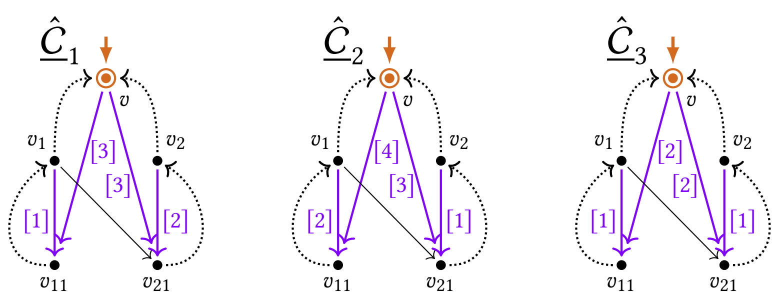


In brown we highlight start vertices by ➛ , and immediate termination with a boldface ring. The loop-entry transitions of loop sub-1-charts that are eliminated in the next step are marked in bold. We have neglected action labels here, except for indicating 1-transitions by dotted arrows. Since the graph $\underline{C}'''$ that is reached after three loop-subgraph elimination steps from the 1-chart $\underline{C}$ does not have an infinite path, and no loop-entry transitions have been removed from a previously eliminated loop sub-1-chart, we conclude that $\underline{C}$ satisfies LEE and LLEE.

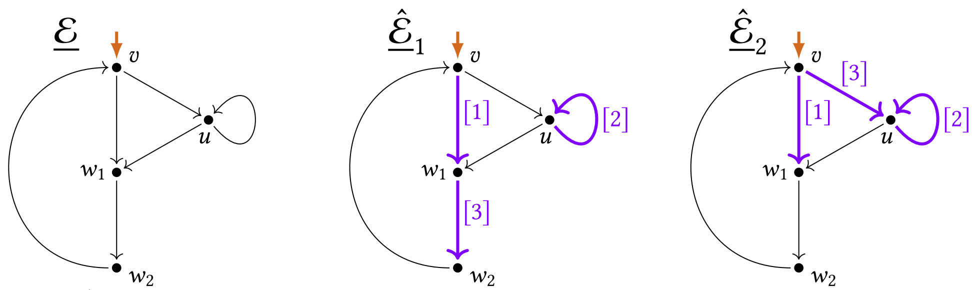
Before giving formal definitions of 'entry/body-labeling' and 'LLEE-witness' below, we explain these concepts more informally.

Intuitively, a LLEE-*witness* $\hat{\underline{C}}$ *of* a 1-chart $\underline{C}$ is the recording of a LLEE-guaranteeing, successful run of the loop elimination procedure by attaching to a transition τ of $\underline{C}$ the marking label n for $n \in \mathbb{N}^+ = \{1, 2, 3, \ldots\}$ (in pictures indicated as $[n]$, in steps as $\rightarrow_{[n]}$) forming a *loop-entry transition* if τ is eliminated in the n-th step, and by attaching marking label 0 to all other transitions of $\underline{C}$ (in pictures neglected, in steps indicated as $\rightarrow_{\mathrm{bo}}$) forming a *body transition*. Formally, LLEE-witnesses arise as *entry/body-labelings* from 1-charts, and are charts in which the transition labels are pairs of action labels over A, and marking labels in $\mathbb{N}$. We say that a LLEE-witness $\hat{\underline{C}}$ *is guarded* if all loop-entry transitions are proper, which means that they have a proper-action transition label. We say that a vertex v is a *loop vertex* of a LLEE-witness $\hat{\underline{C}}$, denoted by $v \in LV(\underline{C})$ if a loop-entry transition departs from v in $\hat{\underline{C}}$.

The entry/body-labeling $\hat{\underline{C}}_1$ below of the 1-chart $\underline{C}$ is a LLEE-witness that arises from the run of the loop elimination procedure earlier above.



The entry/body-labelings $\hat{\underline{C}}_2$ and $\hat{\underline{C}}_3$ of $\underline{C}$ record two other successful runs of the loop elimination procedure of length 4 and 2, respectively, where for $\hat{\underline{C}}_3$ we have permitted to eliminate two loop subcharts at different vertices together in the first step. The 1-chart $\underline{C}$ only has layered LEE -witnesses. But that is not the case for the 1-chart $\underline{\mathcal{E}}$ below:



The entry/body-labeling $\hat{\underline{\mathcal{E}}}_1$ of $\underline{\mathcal{E}}$ as above is a LLEE-witness that is not layered: in the third loop sub-1-chart elimination step that is recorded in $\hat{\underline{\mathcal{E}}}_1$ the loop-entry transition from w_1 to w_2 is removed that is in the body of the loop sub-1-chart at v with loop-entry transition from v to w_1, which (by that time) has already been removed in the first loop-elimination step as recorded in $\hat{\underline{\mathcal{E}}}_1$. But the entry/body-labeling $\hat{\underline{\mathcal{E}}}_2$ of $\underline{\mathcal{E}}$ above is a layered LEE-witness. It can be shown that LEE-witnesses that are not layered can always be transformed into LLEE-witnesses of the same underlying 1-chart. Indeed, the step from $\hat{\underline{\mathcal{E}}}_1$ to $\hat{\underline{\mathcal{E}}}_2$ in the example above, which transfers the loop-entry transition marking label [3] from the transition from w_1 to w_2 over to the transition from v to u, hints at the proof of this general statement. We do, however, not need this result, because we will be able to use the guaranteed existence of LLEE-witnesses (see Lem. 3.7, (i)) for the 1-chart interpretation below (see Def. A.25).

Definition A.12. By an *entry/body-labeling of a* 1*-chart* $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ we mean a chart $\hat{\underline{C}} = \langle V, A \times \mathbb{N}, v_s, \hat{\rightarrow}, \downarrow\rangle$ where $\hat{\rightarrow} \subseteq V \times (A \times \mathbb{N}) \times V$ arises from $\rightarrow$ by adding, for each transition $\tau = \langle v_1, a, v_2\rangle \in \rightarrow$, to the action label a of τ a *marking label* $n \in \mathbb{N}$, yielding $\hat{\tau} = \langle v_1, \langle a, n\rangle, v_2\rangle \in \hat{\rightarrow}$. In an entry/body-labeling we call transitions with marking label 0 *body transitions*, and transitions with marking labels in $\mathbb{N}^+$ *entry transitions*.

Let $\hat{\underline{C}}$ be an entry/body-labeling of $\underline{C}$, and let v and w be vertices of $\underline{C}$ and $\hat{\underline{C}}$. We denote by $v \rightarrow_{\mathrm{bo}} w$ that there is a body transition $v \xrightarrow{\langle a,0\rangle} w$ in $\hat{\underline{C}}$ for some $a \in A$, and by $v \rightarrow_{[n]} w$, for $n \in \mathbb{N}^+$ that there is an entry transition $v \xrightarrow{\langle a,n\rangle} w$ in $\hat{\underline{C}}$ for some $a \in A$. By $E(\hat{\underline{C}})$ of *entry transition identifiers* we denote the set of pairs $\langle v, n\rangle \in V \times \mathbb{N}^+$ such that an entry transition $\rightarrow_{[n]}$ departs from v in $\hat{\underline{C}}$. For $\langle v, n\rangle \in E(\hat{\underline{C}})$, we define by $\underline{C}_{\hat{\underline{C}}}(v, n)$ the subchart of $\underline{C}$ with start vertex v_s that consists of the vertices and transitions which occur on paths in $\underline{C}$ as follows: any path that starts with a $\rightarrow_{[n]}$ entry transition from v, continues with body transitions only (thus does not cross another entry transition), and halts immediately if v is revisited. By a *back-binding* of $\hat{\underline{C}}$ we mean a transition of $\hat{\underline{C}}$ back to the start vertex v of an induced sub-1-chart $\underline{C}_{\hat{\underline{C}}}(v, n)$ of $\hat{\underline{C}}$, for some $\langle v, n\rangle \in E(\hat{\underline{C}})$.

Definition A.13. Let $\underline{C} = \langle V, A, v_s, 1, \rightarrow, \downarrow\rangle$ be a 1-chart. A LLEE*-witness* (a *layered* LEE*-witness*) *of* $\underline{C}$ is an entry/body-labeling $\hat{\underline{C}}$ of $\underline{C}$ with the following three properties:

(W1) *Body-step termination:* There is no infinite path of $\rightarrow_{\mathrm{bo}}$ transitions in $\underline{C}$.

(W2) *Loop condition:* For all $\langle v, n\rangle \in E(\hat{\underline{C}})$, the 1-chart $\underline{C}_{\hat{\underline{C}}}(v, n)$ is a loop 1-chart.

(W3) *Layeredness:* For all $\langle v, n\rangle \in E(\hat{\underline{C}})$, if an entry transition $w \rightarrow_{[m]} w'$ departs from a state $w \neq v$ of $\underline{C}_{\hat{\underline{C}}}(v, n)$, then its marking label m satisfies $m < n$.

The condition (W2) justifies to call an entry transition in a LLEE-witness a *loop-entry transition*. For a loop-entry transition $\rightarrow_{[m]}$ with $m \in \mathbb{N}^+$, we call m its *loop level*.

Proposition A.14. *Every* LLEE*-*1*-chart satisfies* LEE*.*

Proof. Let $\hat{\underline{C}}$ be a LLEE-witness of a 1-chart $\underline{C}$. Repeatedly pick an entry transition identifier $\langle v, n\rangle \in E(\hat{\underline{C}})$ with $n \in \mathbb{N}^+$ minimal, remove the loop sub-1-chart $\underline{C}_v(n)$ that is generated by loop-entry transitions of level n from v (it is indeed a loop by condition (W2) on $\hat{C}$, noting that minimality of n and condition (W3) on $\hat{\underline{C}}$ ensure the absence of departing loop-entry transitions of lower level), and perform garbage collection. Eventually the part of C that is reachable by body transitions from the start vertex is obtained. This sub-1-chart of $\underline{C}$ does not have an infinite path due to condition (W1) on $\hat{\underline{C}}$. Therefore $\underline{C}$ satisfies LEE. □

Definition A.15 (descends-in-loop-to relation $\curvearrowright$, and loops-back-to relations $\circlearrowright$, ${}_d\circlearrowright$, as defined by a LLEE-witness)**.** Let $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a LLEE-1-chart, and let $\hat{\underline{C}}$ be a LLEE-witness of $\underline{C}$. We define the binary relations $\curvearrowright, \circlearrowright, {}_d\circlearrowright \subseteq V \times V$ on the set V of $\underline{C}$ as follows:

(i) Let $v, w \in V$. We denote by $v \curvearrowright w$, and by $w \curvearrowleft v$, that v *descends in a loop down to* w by which we mean that w is in the body of the loop sub-1-chart at v, in the sense that there is a path:

$$v \rightarrow_{[n]} v' \rightarrow^*_{\mathrm{bo}} w$$

from v via a loop-entry transition and subsequent body transitions <u>without</u> encountering v again.

(ii) Let $v, w \in V$. We denote by $v \curvearrowright w$, and by $w \curvearrowleft v$, that v *loops back to* w by which we mean that w is in the body of the loop sub-1-chart at v and that v is reachable from w again by body transition steps, in the sense that there is a path:

$$v \rightarrow_{[n]} v' \rightarrow^{*}_{\mathrm{bo}} w \rightarrow^{+}_{\mathrm{bo}} v$$

from v via a loop-entry transition and subsequent body transitions from <u>without</u> encountering v again to w, and from w via one or more body transition steps back to v.

(iii) The loops-back-to relation $\curvearrowright$ totally orders its successors (see Lem. A.16, (ix)) below. Therefore we define the *direct successor relation* ${}_{d}\!\curvearrowright$ of $\curvearrowright$ on $\underline{C}$ as follows, for all $v, w \in V$:

$$w \;{}_{d}\!\curvearrowright v \;\; :\Longleftrightarrow \;\; w \curvearrowright v \wedge \forall u \in V \big(w \curvearrowright u \implies u = v \vee v \curvearrowright u \big) \, .$$

(We will see in Lem. A.16, (xi) below that the directly loops-back-to relation ${}_{d}\!\curvearrowright$ refines the loops-back-to relation $\curvearrowright$.)

Lemma A.16 (= refinement of Lem. 3.9 in [12, 13], including Lem. 3.1 in [9]). *Let* $\underline{C} = \langle V, A, v_s, 1, \rightarrow, \downarrow \rangle$ *be a* 1*-chart, and let* $\hat{\underline{C}}$ *be a* LLEE*-witness of* $\underline{C}$*. Then the relations* $\rightarrow_{\mathrm{bo}}$, $\rightharpoonup$, $\curvearrowright$, ${}_{d}\!\curvearrowright$ *that are defined with respect to* $\hat{\underline{C}}$ *have the following properties, for all* $w, w_1, w_2, v, v_1, v_2 \in V$*:*

(i) There are no infinite $\rightarrow_{\mathrm{bo}}$ *paths (therefore* $\hat{\underline{C}}$ *does not contain* $\rightarrow_{\mathrm{bo}}$ *cycles).*

(ii) $\leftharpoondown^{+}$ *is a well-founded, strict partial order on* V*.*

(iii) $\leftarrow^{+}_{\mathrm{bo}}$ *is a well-founded strict partial order on* V*.*

(iv) If $scc(w) = scc(v)$*, then* $v \rightharpoonup^{*} w$ *implies* $w \curvearrowright^{*} v$*.*

(v) If $v \rightharpoonup w$ *and* $\neg(w \curvearrowright)$*, then* w *is not normed.*

(vi) $scc(w_1) = scc(w_2)$ *if and only if* $w_1 \curvearrowright^{*} u \curvearrowleft^{*} w_2$ *for some* $u \in V$*.*

(vii) $\exists u \in V \big(u$ *is* $\curvearrowright$*-maximal* $\wedge\; scc(w) = (\curvearrowright^{*} u) \big)$*, that is, every scc of* $\underline{C}$ *is a* $\curvearrowright$*-maximal loops-back-to part with respect to* $\underline{C}$*. Note that loops-back-to part of vertices that are not loop vertices are trivial, that is* $(\curvearrowright^{*} u) = \{u\}$ *for* $u \notin LV(\hat{\underline{C}})$*).*

(viii) $\curvearrowright^{*}$ *is a partial order with the least-upper-bound property: if a nonempty set of vertices has an upper bound with respect to* $\curvearrowright^{*}$*, then it has a least upper bound.*

(ix) $\curvearrowright$ *is a total order on* $\curvearrowright$*-successor vertices: if* $w \curvearrowright v_1$ *and* $w \curvearrowright v_2$*, then* $v_1 \curvearrowright v_2$ *or* $v_1 = v_2$ *or* $v_2 \curvearrowright v_1$*.*

(x) $\curvearrowright^{*}$ *is a total order on* $\curvearrowright^{*}$*-successor vertices: if* $w \curvearrowright^{*} v_1$ *and* $w \curvearrowright^{*} v_2$*, then* $v_1 \curvearrowright^{*} v_2$ *or* $v_1 = v_2$ *or* $v_2 \curvearrowright^{*} v_1$*.*

(xi) $\curvearrowright \subseteq {}_{d}\!\curvearrowright^{+}$*, that is, the directly loops-back-to relation refines the loops-back-to relation: if* $w \curvearrowright v$*, then* $w = u_0 \;{}_{d}\!\curvearrowright u_1 \;{}_{d}\!\curvearrowright \ldots {}_{d}\!\curvearrowright u_{n-1} \;{}_{d}\!\curvearrowright u_n = v$

(xii) ${}_{d}\!\curvearrowright^{+} = \curvearrowright^{+}$*, and* ${}_{d}\!\curvearrowright^{*} = \curvearrowright^{*}$*.*

(xiii) If $v_1 \;{}_{d}\!\curvearrowright v \wedge v_2 \;{}_{d}\!\curvearrowright v \wedge v_1 \neq v_2$*, then* $(\curvearrowright^{*} v_1) \cap (\curvearrowright^{*} v_2) = \varnothing$ *holds, that is, there is no vertex* $u \in V$ *such that both* $u \curvearrowright^{*} v_1$ *and* $u \curvearrowright^{*} v_2$*.*

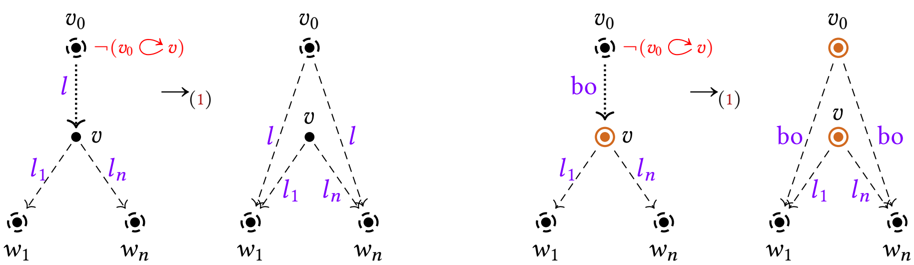
(xiv) If $u_1 \curvearrowright^{*} \overline{u}_1 \;{}_{d}\!\curvearrowright v \;{}_{d}\!\curvearrowleft \overline{u}_2 \curvearrowleft^{*} u_2$ *and* $u_1 \neq u_2$ *hold, then* v *is the least upper bound of* u_1 *and* u_2 *with respect to* $\curvearrowright^{*}$*.*

A.3.2 LLEE-1-charts: 1-transition limited LLEE-1-charts.

In Subsect. A.2.1 we defined 1-transition elimination by local steps. While at first that concept only provides an alternative manner for defining the induced chart $\underline{C}_{(\cdot)}$ of a 1-chart $\underline{C}$, we crucially need it here for defining LLEE-preserving 1-transition elimination. As mentioned before, induced charts of 1-charts with LLEE do not in general have the property LLEE themselves, as noticed in [7, 11]. Therefore we will work 1-transition limited LLEE-1-charts as defined in Def. 3.1, which are optimal LLEE-preserving approximations of induced charts of LLEE-1-charts.

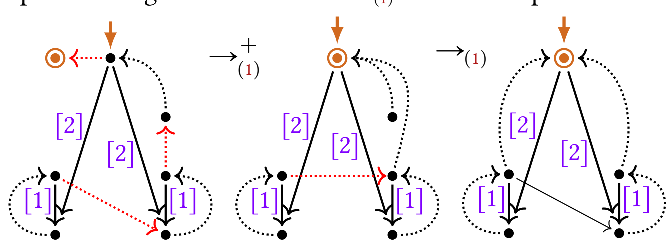
Lemma A.17. *Every (finite)*[1] *weakly guarded* LLEE-1*-chart reduces via a sequence of* $\rightarrow_{(1)}$ *steps to a (finite)* 1*-chart with* LLEE−1-lim.

Proof. It suffices to show that in a LLEE-witness $\hat{\underline{C}}$ of a weakly guarded 1-chart $\underline{C}$ every 1-transition that is not a backlink can be eliminated. For this we annotate the two rules in Def. A.3 with marking labels $l, l_1, \ldots, l_n \in \mathbb{N}$ if the 1-transition from v_0 to v is not a backlink (denoted by $\neg(v_0 \circlearrowright v)$) as follows (we neglect the action labels):



Applications of these rules transform LLEE-witnesses and underlying 1-charts, and their repeated use leads to the elimination of all 1-transitions that are not backlinks. ◻

Example A.18. The LLEE-witness on the left below is not 1-transition limited. It can be transformed into a 1-transition limited LLEE-witness by four steps according to the annotated $\rightarrow_{(1)}$ rules in the proof of Lem. 3.2:



where in the first transformation above three $\rightarrow_{(1)}$ steps are performed in parallel to the 1-transitions highlighted in red. Here and below we drop the marking labels of body transitions, but instead contrast them by emphasizing the loop-entry transitions as thick arrows together with their levels.

Lemma (= Lem. 3.2)**.** *Every weakly guarded* LLEE-1*-chart is* 1*-bisimilar to a* 1*-chart with* LLEE−1-lim.

Proof. Let $\underline{C}$ be a weakly guarded LLEE-1-chart. Then by Lem. A.17 there is a weakly guarded $\underline{C}'$ with LLEE−1-lim such that $\underline{C} \rightarrow^{*}_{(1)} \underline{C}'$ holds. Then $\underline{C} \leftrightarrow \underline{C}'$ follows from the fact that the process semantics of 1-charts is preserved along 1-transition elimination steps. In particular, for 1-charts $\underline{C}_1$ and $\underline{C}_2$, $\underline{C}_1 \rightarrow_{(1)} \underline{C}_2$, implies $\underline{C}_1 \leftrightarrow \underline{C}_2$. This can be show that verifying that the identity relation on the vertices of $\underline{C}_1$ is a 1-bisimulation across any 1-transition elimination step from $\underline{C}_1$ (when garbage collection after the step is ignored, otherwise vertex pairs in the 1-bisimulation to unreachable vertices in the target 1-charts can be deleted). ◻

We will need the following lemma later in Sect. A.4 for the proof of (1). Namely, we will need it in the proof of Lem. A.37, a statement that will help us to show that the LLEE-1-chart in Fig. 4 is not LLEE-preservingly collapsible.

Lemma A.19. *Let* C *be a (finite)*[1] 1*-free* 1*-chart. If* C *can be refined into a w.g.* LLEE-1*-chart, then* C *can be refined into a* 1*-chart with* LLEE−1-lim.

[1] Note that we defined 1-charts to be finite, since we defined them to have only finitely many vertices and actions (which implies that they can only have finitely many transitions).

Proof. Suppose that $\underline{C} \to^*_{(1)} C$ for some w.g. LLEE-1-chart $\underline{C}$. Then $\underline{C}$ is finite, due to Lem. A.5, (i). Then $\underline{C}_{(\cdot)} = C$ by Lem. A.5, (ii), since C is 1-free. Now we can also apply Lem. 3.2 to $\underline{C}$ to find a 1-chart $\underline{C}'$ with LLEE−1-lim such that $\underline{C} \to^*_{(1)} \underline{C}'$. Then we get $\underline{C}'_{(\cdot)} = \underline{C}_{(\cdot)} = C$ by Lem. A.5, (iv), and subsequently $\underline{C}' \to^*_{(1)} C$ by Lem. A.5, (v). In this way we have shown that C can be refined into the 1-chart $\underline{C}'$ with LLEE−1-lim. ◻

A.3.3 LLEE-1-charts: Extraction and unique solvability.

In this subsection we provide references for the proofs of the statements of Lem. 3.3 and Lem. 3.4 in Section A.3.3. These lemmas correspond to, or follow easily from, statements established by Grabmayer in [8, 9]. We recall the definition of the 'extraction function' of a guarded LLEE-witness from [8, 9], and repeat the formulations of crucial lemmas that entail Lem. 3.3 and Lem. 3.4. The definition and the statements from [8, 9] listed below are generalizations to LLEE-1-charts of analogous definitions, and statements for LLEE-charts as formulated and proved by Grabmayer and Fokkink in [12, 13]. Indeed the proofs of these statements in [8, 9] are straightforward adaptations to guarded LLEE-1-charts of proofs for LLEE-charts in [12, 13]. For all of these reasons, we do not repeat the proofs here, but refer to [8, 9], and for the underlying ideas to [12, 13].

We start by repeating from [8, 9] the central definition of the 'relative extraction function' and the 'extraction function' of a guarded LLEE-witness of a (weakly guarded) LLEE-1-chart.

Definition A.20 ((relative) extraction function, see Def. 5.1 in [8, 9] (analogous to Def. 5.3 in [12, 13]))**.** Let $\underline{C} = \langle V, A, 1, v_s, \to, \downarrow\rangle$ be a 1-chart with guarded LLEE-witness $\underline{\hat{C}}$ (therefore $\underline{C}$ is weakly guarded).

We define the extraction function $s_{\underline{\hat{C}}}$ of $\underline{\hat{C}}$ in (ii) below from the relative extraction function $t_{\underline{\hat{C}}}$ of $\underline{\hat{C}}$ as defined in (i). The definitions of both functions presuppose for all $w \in V$, list representations:

$$Tr_{\underline{\hat{C}}}(w) = \big\{ w \xrightarrow{a_i}_{[l_i]} w_i \;\big|\; l_i \in \mathbb{N}^+,\; i \in \{1,\ldots,n\} \big\} \cup \big\{ w \xrightarrow{\underline{b_i}}_{[\mathrm{bo}]} u_i \;\big|\; i \in \{1,\ldots,m\} \big\}, \tag{5}$$

of the set of transitions $Tr_{\underline{\hat{C}}}(w)$ from w in the guarded LLEE-witness $\underline{\hat{C}}$. Note that loop-entry transitions are proper transitions, because $\underline{\hat{C}}$ is a guarded LLEE-witness.

The specific choice of these list representations, which may contain repetitions, can change the definition of these functions up to provability in ACI. Therefore the relative extraction function, and the extraction function of $\underline{\hat{C}}$ are defined, in the clauses below, only up to provability in ACI.

(i) The *relative extraction function* $t_{\underline{\hat{C}}}$ *of* $\underline{\hat{C}}$ is defined by:

$$t_{\underline{\hat{C}}} : \big\{ \langle w, v\rangle \mid w, v \in V(\underline{C}),\; w \curvearrowleft^{=} v \big\} \longrightarrow StExp(A)$$

$$\langle w, v\rangle \longmapsto t_{\underline{\hat{C}}}(w, v) := \begin{cases} 1 & \text{if } w = v, \\ \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{C}}(w_i, w)\Big)^* \cdot \Big(\sum_{i=1}^{m} \underline{b_i} \cdot t_{\underline{C}}(u_i, v)\Big) & \text{if } w \curvearrowleft v, \text{ using the list representation (5)} \\ & \quad\text{of the set } Tr_{\underline{\hat{C}}}(w), \end{cases}$$

where this recursive definition uses induction on the relation $<_{\mathrm{lex}}$:

$$\langle w_1, v_1\rangle <_{\mathrm{lex}} \langle w_2, v_2\rangle \;:\Longleftrightarrow\; v_1 \curvearrowleft^{+} v_2 \,\vee\, (\, v_1 = v_2 \,\wedge\, w_1 \leftarrow^{+}_{\mathrm{bo}} w_2 \,),$$

which is a well-founded strict partial order due to Lem. A.16, (ii), and (iii). The choice of the list representations of the transition sets $Tr_{\underline{\hat{C}}}(w)$ of $\underline{\hat{C}}$ underlying the definition of $t_{\underline{\hat{C}}}$ changes the definition of $t_{\underline{\hat{C}}}$ only up to provability in ACI.

(ii) The *extraction function* $s_{\underline{\hat{C}}}$ *of* $\underline{\hat{C}}$ is defined, for all $w \in V$ on the basis of the list representation (5) of the sets $Tr_{\underline{\hat{C}}}(w)$, by:

$$s_{\underline{\hat{C}}} : V \longrightarrow StExp(A), \quad \langle w, v\rangle \longmapsto s_{\underline{\hat{C}}}(w) := \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{C}}}(w_i, w)\Big)^* \cdot \Big(\tau_{\underline{C}}(w) + \sum_{i=1}^{m} \underline{b_i} \cdot s_{\underline{\hat{C}}}(u_i)\Big),$$

with induction on the strict partial order $\leftarrow^{+}_{\mathrm{bo}}$ (see Lem. A.16, (iii)). Again, the choice of the list representations of the transition sets of $\underline{\hat{C}}$ underlying this definition changes these definition of $s_{\underline{\hat{C}}}$ only up to provability in ACI.

Lem. A.21 below explains the relationship between the extraction function of a guarded LLEE-witness and the relative extraction function for pairs of vertices in a guarded LLEE-1-chart $\underline{C}$ that are related by the descends-in-loop-to relation $\curvearrowright$ of a guarded LLEE-witness of $\underline{C}$. This lemma is crucial for showing that the extraction function of a guarded LLEE-witness provides a Mil^{-}-provable solution, as formulated by Lem. A.22 below.

Lemma A.21 (= Lem. 5.3 in [8, 9], (analogous to Lem. 5.4 in [12, 13])). *Let $\underline{C}$ be a weakly guarded 1-chart $\underline{C}$ with guarded* LLEE*-witness $\hat{\underline{C}}$. Then for all vertices $w, v \in V(\underline{C})$ it holds:*

$$w \curvearrowleft^{=} v \implies s_{\hat{\underline{C}}}(w) =_{\mathsf{Mil}^-} t_{\hat{\underline{C}}}(w, v) \cdot s_{\hat{\underline{C}}}(v)\,. \tag{6}$$

Lemma A.22 (= Lem. 5.4 in [8, 9] (analogous to Lem. 5.5 in [12, 13])). *Let $\underline{C}$ be a weakly guarded 1-chart $\underline{C}$ with guarded* LLEE*-witness $\hat{\underline{C}}$. Then the extraction function $s_{\hat{\underline{C}}}$ of $\hat{\underline{C}}$ is a* Mil^-*-provable solution of $\underline{C}$.*

Now Lem. A.22 clearly demonstates Lem. 3.3.

Lemma (= Lem. 3.3). *From every guarded* LLEE*-witness $\hat{\underline{C}}$ of a (weakly guarded)* LLEE*-1-chart $\underline{C}$ a* Mil^-*-provable solution $s_{\hat{\underline{C}}}$ of $\underline{C}$ can be extracted effectively.*

Lem. A.23 below explains the relationship between an arbitrary Mil-provable solution of a guarded LLEE-1-chart $\underline{C}$ and the relative extraction function of an appertaining guarded LLEE-witness of $\underline{C}$, for pairs of vertices in a guarded LLEE-1-chart that are related by the descends-in-loop-to relation $\curvearrowright$. That lemma in turn is crucial for showing that an arbitrary Mil-provable solution of a guarded LLEE-1-chart $\underline{C}$ is Mil-provably equal to the solution extracted from a guarded LLEE-witness of $\underline{C}$. This statement entails Lem. 3.4, which states that any two Mil-provable solutions of a guarded LLEE-1-chart are Mil-provably equal.

Lemma A.23 (implied by Lem. 5.5 in [8, 9] (which is analogous to Def. 5.7 in [12, 13])). *Let $\underline{C}$ be a 1-chart with guarded* LLEE*-witness $\hat{\underline{C}}$. Let $s : V(\underline{C}) \to StExp(A)$ be a* Mil*-provable solution of $\underline{C}$. Then for all vertices $w, v \in V(\underline{C})$ it holds:*

$$w \curvearrowleft^{=} v \implies s(w) =_{\mathsf{Mil}} t_{\hat{\underline{C}}}(w, v) \cdot s(v)\,. \tag{7}$$

Lemma (= Lem. 3.4, implied by Lem. 5.6 in [8, 9], (which is analogous to Def. 5.8 in [12, 13])). *For every guarded* LLEE*-1-chart $\underline{C}$ it holds that any two* Mil*-provable solutions of $\underline{C}$ are* Mil*-provably equal.*

A.3.4 LLEE-1-charts: 1-Chart interpretation of star expressions.

In this subsection we recall the definition of the '1-chart interpretation' of star expressions as guarded LLEE-1-charts from [7, 11]. We recall the definitions, and we link to results as proved in [7, 11]. Finally we combine these results, and some additional preparatory lemmas in order to obtain a proof of Lem. A.30.

As a remedy for the failure, in general, of LEE for the chart interpretation of star expressions, in [7, 11] the syntax of star expressions is extended by a 'stacked product' symbol $*$, see Def. A.24 below. This symbol is used for tracking transitions that arise by descending into the body of an iteration subexpression f^*. The 1-chart interpretation is then defined (see Def. A.25 below) by means of a transition system specification (TSS) that forces 1-transition backlinks from expressions $g * f^*$ to the iterations f^* in case that g permits immediate termination. In this way the property LEE is restored for 1-chart interpretations (see Lem. A.28), while obtaining 1-bisimilar 1-charts (see Lem. A.26).

Definition A.24. Let A be a set whose members we call *actions*. The set $StExp^{(*)}(A)$ of *stacked star expressions over (actions in) A* is defined by the following grammar:

$$E \ ::= \ e \mid E \cdot e \mid E * e^* \qquad (\text{where } e \in StExp(A))\,.$$

Note that the set $StExp(A)$ of star expressions would arise again if the clause $E * e^*$ were dropped.

The *star height* $|E|_*$ of stacked star expressions E is defined by adding the clauses:

$$|E \cdot e|_* := \max\{|E|_* + |e|_*\}\,, \qquad |E * e^*|_* := \max\{|E|_* + |e^*|_*\}\,,$$

to the defining clauses for star height of star expressions.

The *projection function* $\pi : StExp^{(*)}(A) \to StExp(A)$ is defined by interpreting $*$ as $\cdot$ by the clauses:

$$\pi(e) := e\,, \qquad \pi(E * e^*) := \pi(E) \cdot e^*\,, \qquad \pi(E \cdot e) := \pi(E) \cdot e\,,$$

for all $E \in StExp^{(*)}(A)$, and $e \in StExp(A)$.

Definition A.25. By the 1*-chart interpretation* $\underline{C}(e)$ of a star expression e we mean the 1-chart that arises together with the entry/body-labeling $\widehat{\underline{C}(e)}$ as the e-rooted sub-LTS generated by $\{e\}$ according to the following TSS:

$$\frac{}{1{\downarrow}} \qquad \frac{e_i{\downarrow}}{(e_1+e_2){\downarrow}}\;(i\in\{1,2\}) \qquad \frac{e_1{\downarrow} \qquad e_2{\downarrow}}{(e_1\cdot e_2){\downarrow}} \qquad \frac{}{(e^*){\downarrow}}$$

$$\frac{}{a \xrightarrow{a}_{\mathrm{bo}} 1} \qquad \frac{e_i \xrightarrow{a}_{l} E_i' \quad (i\in\{1,2\})}{e_1+e_2 \xrightarrow{a}_{\mathrm{bo}} E_i'} \qquad \frac{e \xrightarrow{a}_{l} E' \quad (\text{if } nd^+(e))}{e^* \xrightarrow{a}_{[|e^*|_*]} E' * e^*} \qquad \frac{e \xrightarrow{a}_{l} E' \quad (\text{if } \neg nd^+(e))}{e^* \xrightarrow{a}_{\mathrm{bo}} E' * e^*}$$

$$\frac{E_1 \xrightarrow{\underline{a}}_{l} E_1'}{E_1\cdot e_2 \xrightarrow{\underline{a}}_{l} E_1'\cdot e_2} \qquad \frac{E_1 \xrightarrow{\underline{a}}_{l} E_1'}{E_1 * e_2^* \xrightarrow{\underline{a}}_{l} E_1' * e_2^*} \qquad \frac{e_1{\downarrow} \qquad e_2 \xrightarrow{a}_{l} E_2'}{e_1\cdot e_2 \xrightarrow{a}_{\mathrm{bo}} E_2'} \qquad \frac{e_1{\downarrow}}{e_1 * e_2^* \xrightarrow{1}_{\mathrm{bo}} e_2^*}$$

where $l \in \{\mathrm{bo}\} \cup \{[n] \mid n \in \mathbb{N}^+\}$, and star expressions using a 'stacked' product operation $*$ are permitted, which helps to record iterations from which derivatives originate. The condition $nd^+(e)$ means that e permits a positive length path to an expression f with $f{\downarrow}$. Furthermore $|f|_*$ means the syntactical star-height of a star expression f.

The central results in [7, 11] are Lem. A.26, which links 1-chart interpretations via bisimilarity to chart interpretations, and Lem. A.28 below, which guarantees LEE for 1-chart interpretations. For the purpose of showing Lem. A.30, we need to extend Lem. A.26 to a link via 1-bisimulations, see Lem. A.27 below.

Lemma A.26 (= Thm. 5.9 in [7, 11]). *$\underline{C}(e)_{(\cdot]} \rightrightarrows C(e)$, that is, there is a functional bisimulation from the induced chart $\underline{C}(e)_{(\cdot]}$ of the 1-chart interpretation $\underline{C}(e)$ of e to the chart interpretation $C(e)$ of e, for every star expression $e \in StExp(A)$.*

Lemma A.27. *$\underline{C}(e) \rightrightarrows C(e)$, that is, there is a functional 1-bisimulation from the 1-chart interpretation $\underline{C}(e)$ of e to the chart interpretation $C(e)$ of e, for every star expression $e \in StExp(A)$.*

Proof. The statement of the lemma follows from Lem. A.26 and the following diagram that links, for every star expression $e \in StExp(A)$, the 1-chart interpretation $\underline{C}(e)$, its induced chart $\underline{C}(e)_{(\cdot]}$, and the chart interpretation $C(e)$:

$$\begin{tikzcd}
\underline{C}(e) \arrow[d, shift left, Rightarrow] \arrow[dr, dashed, Rightarrow] & \\
\underline{C}(e)_{(\cdot]} \arrow[r, "\text{Lem. A.26}"'] & C(e)
\end{tikzcd} \tag{8}$$

Hereby $\underline{C}(e) \rightrightarrows \underline{C}(e)_{(\cdot]}$ is witnessed by the restriction of the identity function on the set of vertices of $\underline{C}(e)$ to the set of vertices of $\underline{C}(e)_{(\cdot]}$. The composition of the functional 1-bisimulations from $\underline{C}(e)$ to $\underline{C}(e)_{(\cdot]}$, and the functional bisimulation from $\underline{C}(e)_{(\cdot]}$ to $C(e)$ (which is also a functional 1-bisimulation) yields a functional 1-bisimulation from $\underline{C}(e)$ to $C(e)$. ◻

Lemma A.28 (= Thm. 5.14 in [7, 11]). *Let $e \in StExp(A)$ be a star expression.*

The entry/body-labeling $\widehat{\underline{C}(e)}$ of the 1-chart interpretation $\underline{C}(e)$ as defined in Def. A.25 is a LLEE*-witness. Therefore the 1-chart interpretation $\underline{C}(e)$ of a star expression e is a* LLEE *-1-chart.*

For establishing the third statement in Lem. 3.7, we refer to Prop. 2.9 in [12, 13], whose demonstration can be adapted in a straightforward manner to a proof of the following lemma for the chart intepretation of star expressions.

Lemma A.29 ($\sim$ Prop. 2.9 in [12, 13]). *e is the principal value of a* Mil^-*-provable solution of $C(e)$, for every star expression $e \in StExp(A)$.*

Proof (reference). This lemma is the generalization of Prop. 2.9 in [12, 13] from '1-free' star expressions (without 1, but with binary star iteration $(\cdot)^{\circledast}(\cdot)$ instead of unary star iteration $(\cdot)^*$) to all star expressions. The lemma can be established analogously as Prop. 2.9 in [12, 13]. ◻

The statement of this lemma can be transferred to the 1-chart interpretation by using Lem. A.27, and the possibility to transfer provable solutions conversely over functional 1-bisimulations as stated by Lem. 2.8, (i). In this way we can prove the following lemma.

Lemma A.30. *e is the principal value of a* Mil*-provable solution of $\underline{C}(e)$, for every star expression $e \in StExp(A)$.*

Proof. Let $e \in StExp(A)$ be a star expression over set A of actions. Then due to Lem. A.30, there is a Mil^--provable solution s of $C(e)$ with principal value e. It follows that s is also a Mil-provably solution of $C(e)$. Then due to $\underline{C}(e) \rightrightarrows C(e)$, as guaranteed by Lem. A.27, we can apply Lem. 2.8, (i), to s in order to obtain a Mil-provably $\underline{s}$ solution of $\underline{C}(e)$ with principal value e. ◻

Finally we are in a position to combine the results above to a proof of Lem. 3.7.

Lemma (= Lem. 3.7)**.** *For every star expression $e \in StExp(A)$ the 1-chart interpretation $\underline{C}(e)$ of e has the following properties:*

(i) $\underline{C}(e)$ is a guarded, and indeed 1-transition limited, LLEE*-1-chart,*

(ii) $\underline{C}(e) \leftrightarrow C(e)$, that is, $\underline{C}(e)$ is 1-bisimilar to the chart interpretation $C(e)$ of e,

(iii) e is the principal value of a Mil*-provable solution of $\underline{C}(e)$.*

Proof. Let $e \in StExp(A)$ be a star expression. We need to show that the 1-chart interpretation $\underline{C}(e)$ of e as defined in Def. A.25 above satisfies the properties (i), (ii), and (iii).

For showing property (i), we first show that $\underline{C}(e)$ is a guarded LLEE-1-chart. First we find by using Lem. A.28 above that $\underline{C}(e)$ is a LLEE-1-chart. Second, we need to show that $\widehat{\underline{C}(e)}$ is also guarded, that is, that all loop-entry transitions in the entry/body-labeling $\widehat{\underline{C}(e)}$ of $\underline{C}(e)$ as defined in Def. A.25 above are proper transitions. This, however, is guaranteed as a property of transitions that are derivable from the TSS in Def. A.25, which can be proved by induction on the depth of derivations: every loop-entry transition (that is, with marking label greater or equal to 1) that is derivable in this TSS is a proper transition. We conclude that $\underline{C}(e)$ is a indeed a guarded LLEE-1-chart, and therefore (i) holds for $\underline{C}(e)$.

For showing that $\widehat{\underline{C}(e)}$ is in fact also a 1-transition limited LLEE-witness of $\widehat{\underline{C}(e)}$, for every $e \in StExp(A)$, the following auxiliary statement can be used: $F * f^*$ is a vertex of $\underline{C}(e)$ if and only if there is a vertex f^* of $\underline{C}(e)$ such that $f^* \rightarrow_{[|f|_*]} \cdot \rightarrow^*_{\text{bo}} F * f^*$ holds. Then if $F = g$ and $g{\downarrow}$, the arising 1-transition $F * f^* \xrightarrow{1} f^*$ is indeed a backlink due to $f^* \rightharpoondown F * f^* \xrightarrow{1} f^*$, which shows $F * f^* \circlearrowleft f^*$. For showing that all 1-transitions are backlinks, this auxiliary statement needs to be generalized by putting both $F * f^*$ and f^* into an applicative context.

Statement (ii), that $\underline{C}(e)$ is 1-bisimilar to the chart interpretation $C(e)$ of e, for all $e \in StExp(A)$, is guaranteed by Lem. A.27, which states that there is actually a functional 1-bisimulation from $\underline{C}(e)$ to $C(e)$, for all $e \in StExp(A)$.

Finally, statement (iii) is guaranteed by Lem. A.30. ◻

A.3.5 Characterization of transfer functions.

Here we provide a characterization of transfer functions between 1-LTSs by specializing the forth, back, and termination conditions to functional 1-bisimulations, that is, by reformulating the forth, back, and termination conditions for functional 1-bisimulations for functions that induce 1-bisimulations. This characterization prepares for a similar characterization of local transfer functions that we will give later in Sect. A.5.5.

We note that transfer functions as defined in Def. 3.5 are non-empty, that is, they have a non-empty domain and image: this is because 1-bisimulations between 1-LTSs are non-empty relations. Furthermore, since the graph of a transfer function ϕ between 1-LTSs $\underline{\mathcal{L}}_1$ and $\underline{\mathcal{L}}_2$ is a 1-bisimulation, such a function ϕ maps states to 1-bisimilar states, which means more formally that $v_1 \leftrightarrow_{\underline{\mathcal{L}}_1,\underline{\mathcal{L}}_2} v_2$ follows from $\phi(v_1) = v_2$.

Lemma A.31 (characterization of transfer function)**.** *We consider two 1-LTSs $\underline{\mathcal{L}}_i = \langle V_i, A, 1, \rightarrow_i, \downarrow_i \rangle$ where $i \in \{1, 2\}$. Let $\phi : V_1 \rightharpoonup V_2$ be a partial function from the state set V_1 of $\underline{\mathcal{L}}_1$ to the state set V_2 of $\underline{\mathcal{L}}_2$ with domain $W_1 := \text{dom}(\phi)$.*

Then ϕ is a transfer function between $\underline{\mathcal{L}}_1$ and $\underline{\mathcal{L}}_2$ if and only if ϕ has non-empty domain $W_1 \neq \varnothing$, and for all $w_1, w_1', w_2' \in V$ and $a \in A$ the following conditions hold:

$$w_1 \in W_1 \wedge w_1 \xrightarrow{(a)}_1 w_1' \quad \Longrightarrow \quad \phi(w_1) \xrightarrow{(a)}_2 \phi(w_1'), \tag{9}$$

$$\exists w_1' \in V_1 \big(w_1 \xrightarrow{(a)}_1 w_1' \wedge \phi(w_1') = w_2' \big) \quad \Longleftarrow \quad w_1 \in W_1 \wedge \phi(w_1) \xrightarrow{(a)}_2 w_2', \tag{10}$$

$$w_1 \in W_1 \quad \Longrightarrow \quad \big(w_1 {\downarrow}_1^{(1)} \quad \Longleftrightarrow \quad \phi(w_1) {\downarrow}_2^{(1)} \big) . \tag{11}$$

Proof. The partial function $\phi : V_1 \rightarrow V_2$ is a transfer function between $\underline{\mathcal{L}}_1$ and $\underline{\mathcal{L}}_2$ if and only if the graph relation $B := graph(\phi) \subseteq V_1 \times V_2$ of ϕ is a non-empty 1-bisimulation between $\underline{\mathcal{L}}_1$ and $\underline{\mathcal{L}}_2$. Now B is a 1-bisimulation between $\underline{\mathcal{L}}_1$ and $\underline{\mathcal{L}}_2$ if and only if $B \subseteq V \times V$ is a bisimulation between the induced-transition LTSs $(\underline{\mathcal{L}}_1)_{(\cdot)}$ of $\underline{\mathcal{L}}_1$ and $(\underline{\mathcal{L}}_2)_{(\cdot)}$ of $\underline{\mathcal{L}}_2$, which means that the forth, back, and termination conditions hold for B with respect to induced transitions in $(\underline{\mathcal{L}}_1)_{(\cdot)}$ and $(\underline{\mathcal{L}}_2)_{(\cdot)}$. These forth, back, and termination conditions, however, amount to (9), (10), and (11), respectively, quantified over all $w_1, w_1', w_2' \in V$, and $a \in A$. Put together, this chain of equivalences demonstrates the statement of the lemma. ◻

A.4 Proofs in Section 5: Failure of LLEE-preserving collapse

In this subsection we prove observation (1), which is crucial for our completeness proof for two reasons: First, it leads us to the recognition that the bisimulation collapse proof strategy from [12, 13] cannot be used for a completeness proof of Mil, at least not directly, when arguing with LLEE-1-charts; indeed it also shows that this strategy cannot be refined into a '1-bisimulation collapse strategy'. And second, the concrete counterexample that we provide for observation (1) is able to guide us to an adaptation of the bisimulation collapse strategy for proving completeness of Milner's system Mil.

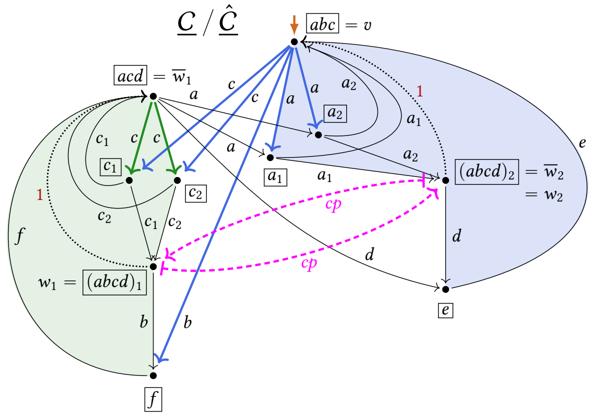
Here we specifically establish Prop. 5.1, which justifies the restriction to weakly guarded LLEE-1-charts of the statement:

(1) LLEE-1-*charts are not in general collapsible to (collapsed)* LLEE-1-*charts.* Nor do 1-bisimilar LLEE-1-charts always have a joint (1-bisimilarity) minimization.

We develop a proof of Prop. 5.1 in a sequence of steps that consist of examples and lemmas. We start with the concrete example of a weakly guarded LLEE-1-chart on which the counterexamples mentioned in the proposition are based.

A.4.1 LLEE-1-Chart used as counterexample to LLEE-preserving 1-collapsibility.

Example A.32. We consider the 1-chart $\underline{C}$ with actions in $\{a, a_1, a_2, b, c, c_1, c_2, d, e, f\}$, and the LLEE-witness $\hat{\underline{C}}$ of $\underline{C}$ that is indicated by colored loop-entry transitions from vertex v (of level 2, blue) and from vertex $\overline{w}_1$ (of level 1, green):



The framebox name of a vertex u symbolizes the behavior from u in $\underline{C}$ by listing the actions of induced transitions from u. This is meaningful because all transitions with the same label from $\{b, d\}$ are cofinal, respectively, transitions with the same label in $\{a_1, a_2, c_1, c_2, e, f\}$ depart from just a single vertex, respectively, and transitions with the same label in $\{a, c\}$ occur as pairs of co-initial transitions such that the transitions of such a pair from a vertex u_1 can be joined in their targets by the transitions of the corresponding pair from a vertex u_2. It follows that vertices with the same label list are 1-bisimilar. In particular, the vertex pair $w_1 = \boxed{(abcd)_1}$ and $w_2 = \boxed{(abcd)_2}$ (their action lists are indexed to distinguish them) are 1-bisimilar (indicated by dashed magenta correspondences $\vdash\dashrightarrow$ that induce a grounded 1-bisimulation slice): every induced transition from w_1 can be joined by a cofinal induced transition from w_2, and vice versa.

Since $\underline{C}$ contains two 1-bisimilar vertices, this LLEE-1-chart is not a 1-bisimulation collapse.

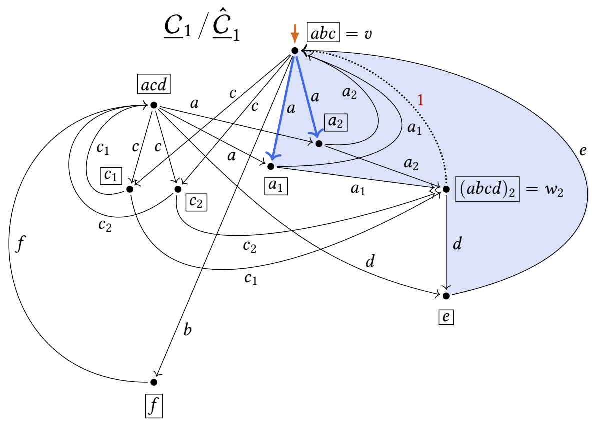
Lemma A.33. *The* 1*-chart* $\underline{C}$ *in Fig. 4 (see also Ex. A.32 above) is a* LLEE-1*-chart. The induced chart* $\underline{C}_{(\cdot]}$ *of* $\underline{C}$ *is isomorphic to the chart interpretation of a star expression.*

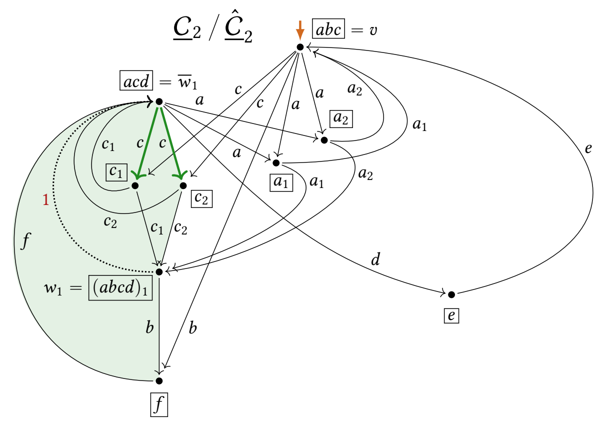
Proof. It is easy to check that the entry/body-labeling $\hat{\underline{C}}$ indicated in Ex. A.32 is a LLEE-witness of $\underline{C}$. It is also routine to verify that $\underline{C}_{(\cdot]} \simeq C(g_v)$ for the star expression $g_v \in StExp(\{a, a_1, a_2, b, c, c_1, c_2, d, e, f\})$ that is defined as:

$$
\begin{aligned}
g_v &:= (1 \cdot ((C + b \cdot f) \cdot g_{\overline{w}_1} + A)^*) \cdot 0 \\
\text{where:}\quad C &:= c \cdot (c_1 + c_1 \cdot g_{w_1}) + c \cdot (c_2 + c_2 \cdot g_{w_1}) \\
g_{w_1} &:= 1 + b \cdot f \\
g_{\overline{w}_1} &:= (1 \cdot C^*) \cdot \big(A + d \cdot e\big) \\
A &:= a \cdot (a_1 + a_1 \cdot g_{w_2}) + a \cdot (a_2 + a_2 \cdot g_{w_2}) \\
g_{w_2} &:= 1 + d \cdot e
\end{aligned}
$$

Hereby the meaning of a star expression g_u for some vertex u of $\underline{C}$ is to express the behavior in $\underline{C}$ from vertex u until for the first time a vertex is reached to which u directly loops back (which for $u = v$ coincides with the behavior from u as v does not loop back). A and C express the behavior in $\underline{C}$ starting from a vertex with the pair of a- and c-transitions, respectively, until the vertex is reached to which both of their targets loop back directly ($\overline{w}_1$ and v, respectively). ◻

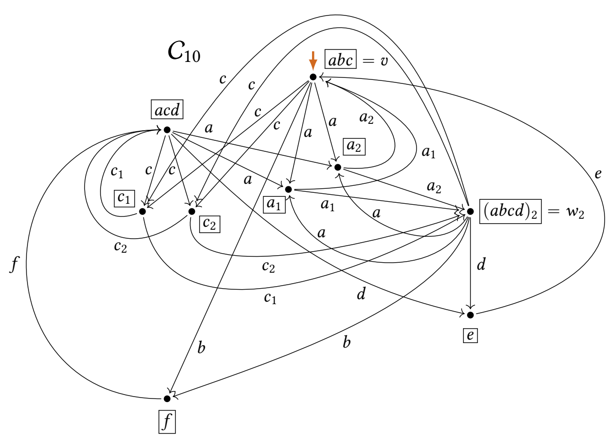
Example A.34. From the 1-chart $\underline{C}$ in Ex. A.32 the 1-chart $\underline{C}_1$ below arises by eliminating the 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ through redirecting both of the incoming transitions at w_1 over to w_2. And similarly, the 1-chart $\underline{C}_2$ below arises by eliminating the 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ through redirecting both of the incoming transitions at w_2 over to w_1:



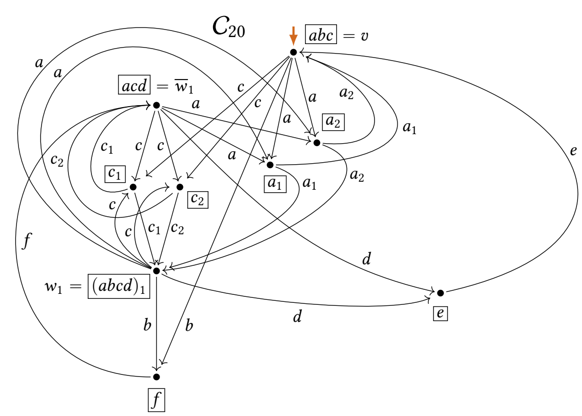


We kept in color only those loop-entry transitions of C that are loop-entry transitions still in $\underline{C}_1$, and respectively, in $\underline{C}_2$. The 1-charts $\underline{C}_1$ and $\underline{C}_2$ are 1-collapsed (see Def. 2.4) because no two different vertices of $\underline{C}_1$, and respectively of $\underline{C}_2$, are 1-bisimilar. Therefore both $\underline{C}_1$ and $\underline{C}_2$ are 1-bisimulation collapses of $\underline{C}$.

The induced charts $C_{10} := (\underline{C}_1)_{(\cdot)}$ of $\underline{C}_1$, and $C_{20} := (\underline{C}_2)_{(\cdot)}$ of $\underline{C}_2$ are of the following forms:



$\simeq$



These two 1-transition free 1-charts are isomorphic. Both of these 1-charts are collapsed (see Def. 2.4). Therefore each of them can be viewed, modulo isomorphism $\simeq$, as *the* bisimulation collapse of $\underline{C}$.

Lemma A.35. *Neither of the 1-charts $\underline{C}_1$, $\underline{C}_2$, $\underline{C}_{10}$, and $\underline{C}_{20}$ in Ex. A.34 is a* LLEE*-1-chart. Each of the 1-charts $\underline{C}_1$, $\underline{C}_2$, $\underline{C}_{10}$, and $\underline{C}_{20}$ is 1-collapsed, and both of $\underline{C}_{10}$ and $\underline{C}_{20}$ are collapsed.*

Proof. We have already noticed in Ex. A.34 that $\underline{C}_1$ and $\underline{C}_2$ are 1-collapsed, and that $\underline{C}_{10}$ and $\underline{C}_{20}$ are collapsed. We need to show that neither of them is a LLEE-1-chart.

The 1-chart $\underline{C}_1$ only contains two transitions that induce loop subcharts: those that are marked in blue. For every other transition $\tau = \langle u_1, \underline{a}, u_2\rangle$ of $\underline{C}_1$ that is not contained in one of these induced loop subcharts it is easy to check that τ does not induce a loop subchart: because there is an infinite path from u_1 starting along τ that avoids the two blue marked loop-entry transitions, and does not return to the path's source vertex u_1. It follows that $\underline{C}_1$ does not satisfy LEE, the loop existence and elimination condition: after eliminating the two blue marked loop-entry transitions, no other loop-entry transitions exist, yet infinite paths are still possible. Since LLEE implies LEE by Prop. A.14, it follows that $\underline{C}_1$ cannot be a LLEE-1-chart.

Similarly, the 1-chart $\underline{C}_2$ only contains two transitions that induce loop subcharts: those that are marked in green. Neither of the other transitions of $\underline{C}_2$ induces a loop subchart. Then we can argue similarly as above for $\underline{C}_1$ that $\underline{C}_2$ is not a LLEE-1-chart.

The 1-chart C_{10} does not contain a loop-entry transition, which can be checked separately for each transition. Since it expresses infinite behavior, but does not satisfy LEE, C_{10} is not a LLEE-1-chart. Analogously, it is also straightforward to check that the 1-chart C_{10} does not contain a loop-entry transition. As it clearly expresses an infinite behavior, but does not satisfy LEE, C_{20} cannot be a LLEE-1-chart, either. ◻

A.4.2 The counterexample LLEE-1-chart is not LLEE-preservingly 1-collapsible.

To show that neither $\underline{C}_1$ nor $\underline{C}_{10}$ above can be refined into a LLEE-1-chart, we will use the lemma below.

Lemma A.36. *For every finite 1-chart $\underline{\mathcal{D}}$ with* LLEE−1-lim *there is a finite 1-chart $\underline{\mathcal{D}}'$ with the same induced chart, and with a 1-transition limited* LLEE-*witness $\hat{\underline{\mathcal{D}}}'$ such that:*

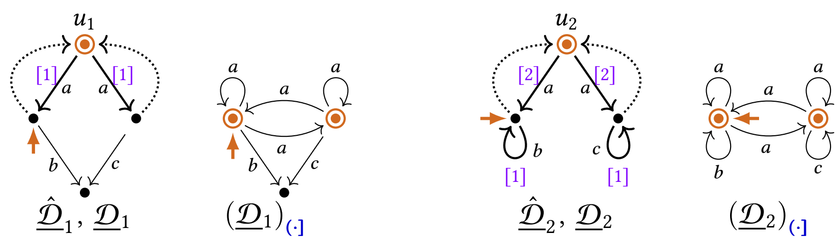
(ptt) *If for a loop-entry identifier $\langle v, n\rangle \in E(\hat{\underline{C}}')$ the loop sub-1-chart $\underline{C}_{\hat{\underline{\mathcal{D}}}'}(v, n)$ of $\underline{\mathcal{D}}'$*

(a) consists of a single scc (strongly connected component),

(b) contains no other loop vertex than v (is innermost),

then it contains only proper-transition targets in $\underline{\mathcal{D}}'$.

Before proving this lemma, we want to demonstrate that it does not hold without either of the restrictions (a), (b) in (ptt). The 1-charts $\underline{\mathcal{D}}_1$ and $\underline{\mathcal{D}}_2$ below with 1-transition limited LLEE-witnesses $\hat{\underline{\mathcal{D}}}_1$ and $\hat{\underline{\mathcal{D}}}_2$ show that for (a) and (b).



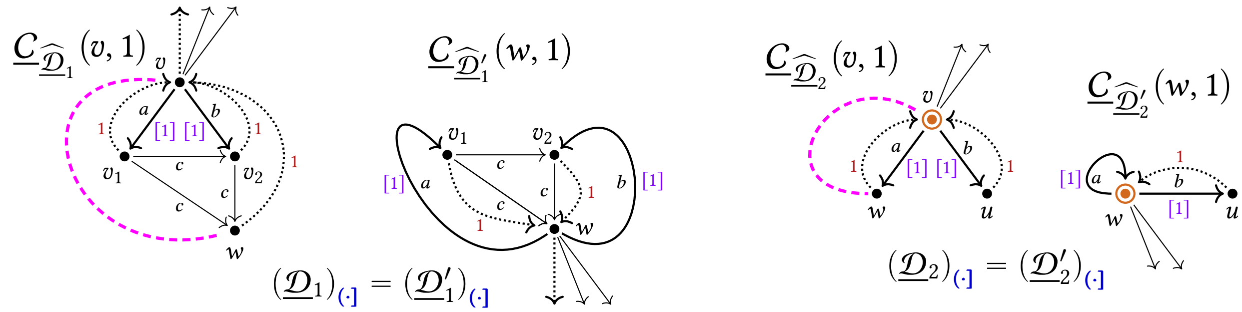
While the loop sub-1-chart at u_1 in $\hat{\underline{\mathcal{D}}}_1$ is not an scc, the loop sub-1-chart at u_2 in $\hat{\underline{\mathcal{D}}}_2$ is not innermost. The induced charts of $\underline{\mathcal{D}}_1$ and $\underline{\mathcal{D}}_2$ do not satisfy LEE, and cannot be refined into LLEE-1-charts that only contain proper-transition targets.

Proof of Lem. A.36. It suffices to show that in a finite 1-chart $\underline{\mathcal{D}}'$ with LLEE−1-lim, and with a 1-transition limited LLEE-witness $\hat{\underline{\mathcal{D}}}'$ every violation of (ptt) for a loop-entry identifier $\langle v, n\rangle \in E(\hat{\underline{C}}')$ can be removed with as result a finite 1-chart $\underline{\mathcal{D}}'$ with a 1-transition limited LLEE-witness $\hat{\underline{\mathcal{D}}}'$ that has one violation of (ptt) less than $\underline{\mathcal{D}}$ and $\hat{\underline{\mathcal{D}}}$, and that has the same induced chart as $\underline{\mathcal{D}}$, that is, $\underline{\mathcal{D}}'_{(\cdot)} = \underline{\mathcal{D}}_{(\cdot)}$. This is because repeated application of this transformation statement to a 1-chart with LLEE−1-lim $\underline{\mathcal{D}}$ then yields a LLEE-1-chart $\underline{\mathcal{D}}'$ and a LLEE-witness $\hat{\underline{\mathcal{D}}}'$ with (ptt) and with the same induced chart, as stated by the lemma.

Suppose that $\underline{\mathcal{D}}$ is a 1-chart with 1-transition limited LLEE-witness $\hat{\underline{\mathcal{D}}}$ such that there is a violation of (ptt) in the form of a loop-entry identifier $\langle v, n\rangle \in E(\hat{\underline{C}}')$ such that $\underline{C}_{\hat{\underline{\mathcal{D}}}}(v, n)$ (a) consists of a single scc, (b) is innermost, but (c) it contains vertices that are not proper-transition targets in $\underline{\mathcal{D}}$.

We first show that v is the single vertex of $\underline{C}_{\hat{\underline{\mathcal{D}}}}(v, n)$ that is not a proper-transition target in $\underline{\mathcal{D}}$. For this, let u be a vertex of $\underline{C}_{\hat{\underline{\mathcal{D}}}}(v, n)$ that is not a proper-transition target. Then u must be the target of a 1-transition. But as 1-transitions are backlinks in $\hat{\underline{\mathcal{D}}}$ since $\hat{\underline{\mathcal{D}}}$ is 1-transition limited, it follows that u must be a loop vertex of $\hat{\underline{\mathcal{D}}}$. However, as v is an innermost loop vertex of $\hat{\underline{\mathcal{D}}}$ by assumption (b), we conclude that $u = v$. Since $\underline{C}_{\hat{\underline{\mathcal{D}}}}(v, n)$ contains vertices that are not proper-transition targets by assumption (c), it follows that v must be such a vertex. Consequently it is the single vertex in $\underline{C}_{\hat{\underline{\mathcal{D}}}}(v, n)$ that is not a proper-transition target.

Next we note that $\underline{C}_{\hat{\underline{\mathcal{D}}}}(v, n)$ cannot contain a 1-transition self-loop at v, because $\underline{\mathcal{D}}$ is weakly guarded as a 1-chart with LLEE−1-lim. Therefore all transitions from v in $\underline{C}_{\hat{\underline{\mathcal{D}}}}(v, n)$ are proper transitions. We choose a maximal path π of proper transitions from v in $\underline{C}_{\hat{\underline{\mathcal{D}}}}(v, n)$. This path π departs from v in the first step, and does not return to v, because otherwise v would be a proper-transition target. The path π must be finite, due to loop 1-chart condition (L2) on the loop sub-1-chart $\underline{C}_{\hat{\underline{\mathcal{D}}}}(v, n)$ of $\underline{\mathcal{D}}$. Therefore π stops at a vertex w from which no further proper transition departs. However, since $\underline{C}_{\hat{\underline{\mathcal{D}}}}(v, n)$ is an scc by assumption (a), there must be an outgoing 1-transition from w. As $\underline{\mathcal{D}}$ is 1-transition limited, this 1-transition must be a backlink to v. In the example below left, for 1-chart $\underline{\mathcal{D}} = \underline{\mathcal{D}}_1$ with LLEE-witness $\hat{\underline{\mathcal{D}}}_1$, this vertex w is unique, but there are three possible paths from v to w. In the example below right, for 1-chart $\underline{\mathcal{D}} = \underline{\mathcal{D}}_2$ with LLEE-witness $\hat{\underline{\mathcal{D}}}_2$, there are two possible choices for w (of which only one is drawn).



Since the only outgoing transition from w in $\underline{\mathcal{D}}$ is a 1-transition to v (by construction of w), the vertices w and v are 1-bisimilar in $\underline{\mathcal{D}}$ (as indicated by the magenta links in the pictures).

We can now transform $\underline{\mathcal{D}}$ and $\hat{\underline{\mathcal{D}}}$ by removing v, and by letting the loop sub-1-chart start at w instead, transferring the loop-entry transitions in $\hat{\underline{\mathcal{D}}}$ from v to w, directing 1-transition backlinks in $\hat{\underline{\mathcal{D}}}$ to v now over to w, and moving possible termination at v over to w (see the second example above), as well as also changing the source of transitions that depart from v to w. In this way we obtain from $\underline{\mathcal{D}}$ and $\hat{\underline{\mathcal{D}}}$ a 1-chart $\underline{\mathcal{D}}'$ with a 1-transition limited LLEE-witness $\hat{\underline{\mathcal{D}}}'$ such that:

- ▻ induced transitions in $\underline{\mathcal{D}}$ are preserved as induced transitions of $\hat{\underline{\mathcal{D}}}$,
- ▻ therefore $\underline{\mathcal{D}}'_{(\cdot)} = \underline{\mathcal{D}}_{(\cdot)}$ follows,
- ▻ now $\underline{C}_{\widehat{\underline{\mathcal{D}}'}}(v, n)$ consists only of proper-transition targets,
- ▻ other innermost loop sub-1-charts of $\underline{\mathcal{D}}$ with respect to $\hat{\underline{\mathcal{D}}}$ are not changed.

In this way we have shown the proof obligation for the lemma. □

Lemma A.37. *Neither of the 1-charts $\underline{C}_1$ and $\underline{C}_{10}$ in Ex. A.32 can be refined into a weakly guarded* LLEE-1*-chart.*

Proof. We show the statement of the lemma for C_{10} by an argument that will demonstrate it also for $\underline{C}_1$.

We argue indirectly, towards a contradiction. Suppose that C_{10} can be refined into a w.g. LLEE-1-chart. Then C_{10} can also be refined, due to Lem. A.19, into a 1-chart with LLEE–1-lim. Furthermore by Lem. A.36 it follows, in view of Lem. A.5, (iv) and (v), that C_{10} can even be refined into a 1-chart with LLEE–1-lim and (ptt).

Therefore there is a 1-chart $\underline{\mathcal{D}}$ with 1-transition limited LLEE-witness $\hat{\underline{\mathcal{D}}}$ and (ptt) such that $\underline{\mathcal{D}} \to^*_{(1)} C_{10}$. Since C_{10} is not a LLEE-1-chart by Lem. A.35, it follows that $\underline{\mathcal{D}} \neq C_{10}$, and therefore also that $\underline{\mathcal{D}} \to^+_{(1)} C_{10}$.

We now let $\underline{\mathcal{D}}_1$ be the 1-chart that results from $\underline{\mathcal{D}}$ by eliminating, via $\to_{(1)}$ steps, all 1-transition backlinks in loop sub-1-charts as delimited by $\hat{\underline{\mathcal{D}}}$ that are *not* innermost. Then $\underline{\mathcal{D}} \to^*_{(1)} \underline{\mathcal{D}}_1 \to^+_{(1)} C_{10}$ follows by Lem. A.5, (vi), yielding $\underline{\mathcal{D}}_1 \to^*_{(1)} C_{10}$, and where $\underline{\mathcal{D}}_1 \to^+_{(1)} C_{10}$ holds, because the innermost loop sub-1-charts in $\underline{\mathcal{D}}$ must still be present in $\underline{\mathcal{D}}_1$, but C_{10} does not contain any loop sub-1-charts. We will show, for a 1-chart $\underline{C}_2$ that will be illustrated below and is similar to $\underline{C}_1$, the following contradictory statements:

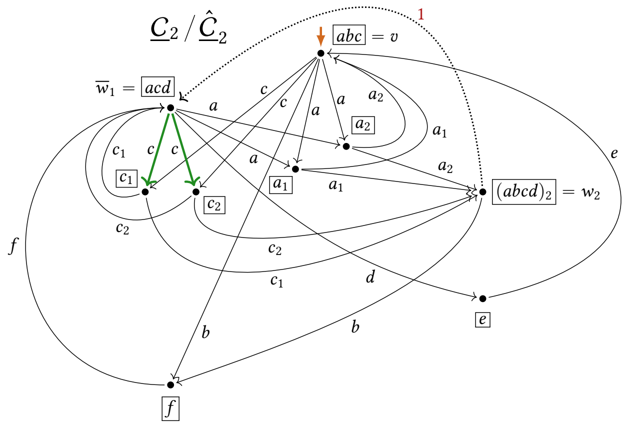
(S1) $\underline{\mathcal{D}} \to^+_{(1)} \underline{\mathcal{D}}_1 \to_{(1)} C_{10}$ for $\underline{\mathcal{D}}_1 \in \{\underline{C}_1, \underline{C}_2\}$, with $\underline{\mathcal{D}} \neq \underline{\mathcal{D}}_1$.

(S2) Neither $\underline{C}_1$ nor $\underline{C}_2$ can be refined into a 1-chart with LLEE–1-lim.

The contradiction arises between (S2) and the part of (S1) that states that either $\underline{C}_1$ or $\underline{C}_2$ can be refined into $\underline{\mathcal{D}}$ which we assumed is a 1-chart with LLEE–1-lim. Therefore it remains to define $\underline{C}_2$ and to prove (S1) and (S2).

Turning to defining $\underline{C}_2$ and showing (S1), we use that $\underline{\mathcal{D}}$ and $\hat{\underline{\mathcal{D}}}$ satisfy (ptt). Therefore all innermost loop sub-1-charts of $\underline{\mathcal{D}}$ delimited by $\hat{\underline{\mathcal{D}}}$ consist only of proper-transition targets in $\underline{\mathcal{D}}$. Since backlinks in these innermost loop sub-1-charts of $\underline{\mathcal{D}}$ are only removed in the steps $\underline{\mathcal{D}}_1 \to^+_{(1)} C_{10}$, it follows that $\underline{\mathcal{D}}_1$ has the same vertices as C_{10}. Conversely, these innermost loop sub-1-charts of $\underline{\mathcal{D}}$ can be made visible starting from C_{10} by adding 1-transition backlinks between existing vertices of C_{10}, and by removing transitions that become superfluous in the sense that they correspond to newly formed induced transitions. However, to obtain a 1-bisimilar refinement any such 1-transition backlink $\langle u, 1, u' \rangle$ can only be added to C_{10} if u' is a substate of u already in C_{10}, and that is, if u' has part of the behavior of u. There are only two possibilities for such an added backlink: one from w_2 to v (due to $v \sqsubseteq_{C_{10}} w_2$), and another one from w_2 to $\overline{w}_1$ (due to $\overline{w}_1 \sqsubseteq_{C_{10}} w_2$). It is not possible to add them both as 1-transitions, because as such they correspond to backlinks in the 1-transition limited LLEE-witness $\hat{\underline{\mathcal{D}}}$, and no vertex can loop back directly to two different loop vertices in a LLEE-witness. It follows that $\hat{\underline{\mathcal{D}}}$ can contain only one innermost loop vertex, namely either v or $\overline{w}_1$.

Therefore in order to make visible, starting from C_{10}, the innermost loop subchart of $\underline{\mathcal{D}}$ defined by $\hat{\underline{\mathcal{D}}}$, we either obtain the 1-chart $\underline{C}_1$ in Ex. A.34, if v is a loop vertex and $\langle w_2, 1, v \rangle$ is a backlink of $\hat{\underline{\mathcal{D}}}$, or the 1-chart $\underline{C}_2$ below:

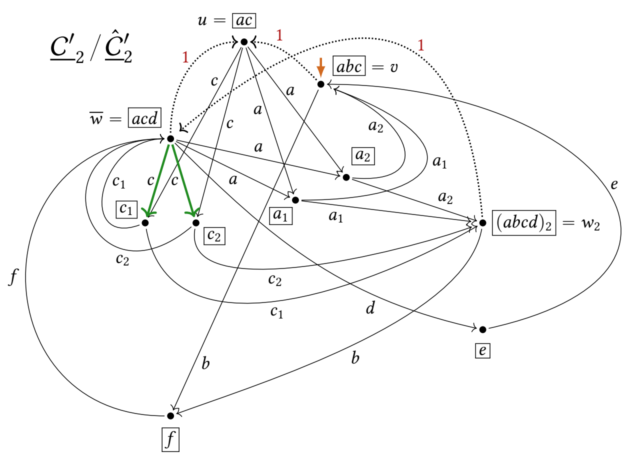


if $\overline{w}_1$ is a loop vertex, and $\langle w_2, 1, \overline{w}_1 \rangle$ is a backlink of $\underline{\hat{\mathcal{D}}}$. It is straightforward to check that only the two transitions marked in green induce loop sub-1-charts of $\underline{\mathcal{C}}_2$, and that after eliminating those, no further loop-entry transitions arise, but infinite paths remain. Therefore $\underline{\mathcal{C}}_2$ cannot be a LLEE-chart. It follows that $\underline{\mathcal{D}}$ must be a further refinement either of $\underline{\mathcal{C}}_1$ or of $\underline{\mathcal{C}}_2$. In this way we have established (S1).

For (S2) we focus on $\underline{\mathcal{C}}_2$, because the argument for $\underline{\mathcal{C}}_1$ is closely analogous. We assume that a LLEE-1-chart $\underline{\mathcal{D}}$ with 1-transition limited LLEE-witness $\underline{\hat{\mathcal{D}}}$ is a refinement of $\underline{\mathcal{C}}_2$. Then $\underline{\mathcal{D}} \to^{+}_{(1)} \underline{\mathcal{C}}_2$ follows, because $\underline{\mathcal{C}}_2$ is not a LLEE-1-chart.

Now we note that $\underline{\mathcal{C}}_2$ cannot be refined in the direction of the 1-transition limited LLEE-witness $\underline{\hat{\mathcal{D}}}$ by adding 1-transition backlinks to vertices that are already present in $\underline{\mathcal{C}}_2$. This is because, to satisfy the layeredness condition (W3) for the LLEE-witness $\underline{\hat{\mathcal{D}}}$, any such backlink needed to start from $\overline{w}_1$, or from a vertex outside of $\underline{\mathcal{C}}_{\widehat{\mathcal{D}}}(\overline{w}_1, 1)$ (the sub-1-chart of $\underline{\mathcal{C}}_2$ delimited by $\underline{\hat{\mathcal{D}}}$ and by its entry/body-labeling $\underline{\hat{\mathcal{C}}}_2$), and target a different of these vertices that furthermore needs to be a substate. But there is not such a vertex in $\underline{\mathcal{C}}_2$: neither $\overline{w}_1$ nor one of v, $\boxed{a_1}$, $\boxed{a_2}$, $\boxed{e}$ outside of $\underline{\mathcal{C}}_{\widehat{\mathcal{D}}}(\overline{w}_1, 1)$ has a substate among the other 4 of these vertices.

Consequently, the only option to refine $\underline{\hat{\mathcal{C}}}_2$ towards the 1-transition limited LLEE-witness $\underline{\hat{\mathcal{D}}}$ is to add a vertex u that is not a proper-transition target, with 1-transition backlinks directed to it from one of the five vertices $\overline{w}_1$, v, $\boxed{a_1}$, $\boxed{a_2}$, or $\boxed{e}$ such that proper transitions in $\underline{\mathcal{C}}_2$ arise as induced transitions via u. The only option to simplify the structure is to share, via the new vertex u proper transitions from two of these five vertices. Such a non-trivial sharing of transitions is possible only between $\overline{w}_1$ and v, because none of other pairs of vertices have an action label of an outgoing transition in common. The maximal option is to share, via the new vertex u, all outgoing a- and c-transitions from $\overline{w}_1$ and v. This leads to the 1-chart $\underline{\mathcal{C}'}_2$:



But we note now that no loop-entry transition is created at u in $\underline{\mathcal{C}'}_2$, and so the added 1-transitions are not backlinks. So this refinement step is not one in the direction of the (assumed) 1-transition limited LLEE-witness $\underline{\hat{\mathcal{D}}}$. While it would be possible to share only a non-empty subset of the four transitions from u, that would not give rise to a loop-entry transition at u, either.

Hence we must recognize that it is not possible to refine $\underline{C}_2$ into the LLEE-1-chart $\underline{\mathcal{D}}$ as assumed. Thus we have shown the part of (S2) for $\underline{C}_2$.

Analogously it can be argued that neither can $\underline{C}_1$ be refined into a LLEE-1-chart, thereby obtaining (S2).

Having obtained a contradiction, we conclude that neither C_{10} nor $\underline{C}_1$ can be refined into a w.g. LLEE-1-chart. □

With the tools provided by this preparation, we can now prove Prop. 5.1.

Proposition (= Prop. 5.1)**.** *The following two statements hold:*

(i) Weakly guarded LLEE-1*-charts are not in general* LLEE*-preservingly* 1*-collapsible, hence not in general* LLEE*-preservingly collapsible. This statement is witnessed by the* LLEE-1*-chart* $\underline{C}$ *in Fig. 4 (see also Ex. A.32 above):* $\underline{C}$ *is not* LLEE*-preservingly* 1*-collapsible.*

(ii) Two given 1*-bisimilar, weakly guarded* LLEE-1*-charts are not in general* LLEE*-preservingly jointly minimizable under* $\Rightarrow$*. This statement is witnessed by the* 1*-bisimilar generated sub-*LLEE-1*-charts* $\underline{C}{\downarrow}^{w_1}_{*}$ *and* $\underline{C}{\downarrow}^{w_2}_{*}$ *of the* LLEE-1*-chart* $\underline{C}$ *in Fig. 4 (and also in Ex. A.32 above):* $\underline{C}{\downarrow}^{w_1}_{*}$ *and* $\underline{C}{\downarrow}^{w_2}_{*}$ *are not jointly minimizable under* $\Rightarrow$*.*

Proof of Prop. 5.1. We demonstrate statement (i) in Prop. 5.1 by showing that the LLEE-1-chart $\underline{C}$ in Ex. A.32 is not LLEE-preservingly 1-collapsible, making use of the fact stated by Lem. A.37 that the (1-transition free) bisimulation collapse C_{10}, see Ex. A.34, of $\underline{C}$ cannot be 1-transition refined into a LLEE-1-chart.

For showing this, suppose, towards a contradiction, that $\underline{C}$ has a 1-bisimulation collapse $\underline{C}_0$ that is a weakly guarded LLEE-1-chart. Then $\underline{C} \Rightarrow_{\phi} \underline{C}_0$ holds via a function ϕ that induces a functional 1-bisimulation by mapping the start vertex of $\underline{C}$ to the start vertex of $\underline{C}_0$, and every vertex of $\underline{C}$ that is a proper-transition target to its unique 1-bisimilar counterpart in $\underline{C}_0$. Now as $\underline{C}_0$ is a 1-bisimulation collapse of $\underline{C}$ it follows that the induced chart $(\underline{C}_0)_{(\cdot)}$ of $\underline{C}_0$ is a (1-transition free) bisimulation collapse of $\underline{C}$. Since bisimulation collapses are unique modulo isomorphism, and C_{10} in Ex. A.34 is also a bisimulation collapse of $\underline{C}_1$, it follows that $(\underline{C}_0)_{(\cdot)} \simeq C_{10}$, that is, $(\underline{C}_0)_{(\cdot)}$ and C_{10} are isomorphic. Since the isomorphism between $(\underline{C}_0)_{(\cdot)}$ and C_{10} permits to rename the vertices of $(\underline{C}_0)_{(\cdot)}$ to those of C_{10}, and then the vertices of $\underline{C}_0$ accordingly, we may assume that $(\underline{C}_0)_{(\cdot)} = C_{10}$. But now from this Lem. A.5, (vi), implies $\underline{C}_0 \to^{*}_{(1)} (\underline{C}_0)_{(\cdot)} = C_{10}$. This means that the LLEE-1-chart $\underline{C}_0$ 1-transition refines C_{10}, and conversely, that the bisimulation collapse C_{10} of $\underline{C}$ can be 1-transition refined into the LLEE-1-chart $\underline{C}_0$. This, however, contradicts Lem. A.37 which states that C_0 cannot be refined into a weakly guarded LLEE-1-chart.

We conclude that $\underline{C}$ does not have a weakly guarded LLEE-1-chart as a 1-bisimulation collapse. This shows that weakly guarded LLEE-1-charts are not in general LLEE-preservingly 1-collapsible, the statement (i) in Prop. 5.1.

In order to demonstrate (ii) in Prop. 5.1 we show that $\underline{C}{\downarrow}^{w_1}_{*}$ and $\underline{C}{\downarrow}^{w_2}_{*}$ are not LLEE-preservingly jointly minimizable under $\Rightarrow$. Again we argue indirectly. Suppose, towards a contradiction, that $\underline{C}{\downarrow}^{w_1}_{*}$ and $\underline{C}{\downarrow}^{w_2}_{*}$ are LLEE-preservingly jointly minimizable under $\Rightarrow$. Then there exists a LLEE-1-chart $\underline{C}_0$ such that $\underline{C}{\downarrow}^{w_1}_{*} \Rightarrow_{\phi_1} \underline{C}_0 \Leftarrow_{\phi_2} \underline{C}{\downarrow}^{w_2}_{*}$ holds via functional 1-bisimulations that are defined by functions ϕ_1 and ϕ_2, respectively.

We first note that there is no functional 1-bisimulation between $\underline{C}{\downarrow}^{w_1}_{*}$ and $\underline{C}{\downarrow}^{w_2}_{*}$. For example, a functional 1-bisimulation B from $\underline{C}{\downarrow}^{w_1}_{*}$ to $\underline{C}{\downarrow}^{w_2}_{*}$ needed to relate the start vertex w_1 of $\underline{C}{\downarrow}^{w_1}_{*}$ to the start vertex w_2 of $\underline{C}{\downarrow}^{w_2}_{*}$, and it needed to related every vertex $w \notin \{w_1, w_2\}$ to w itself because such vertices are unique up to 1-bisimilarity in $\underline{C}{\downarrow}^{w_1}_{*}$, in $\underline{C}{\downarrow}^{w_1}_{*}$, and in $\underline{C}$; then it is easy to see (see the analogous, albeit simpler situation in Fig. 5) that in order to be a 1-bisimulation B also needs to relate w_1 to w_1 itself, and hence it cannot be defined by function. Note that Fig. 5 can explain, although it concerns the slightly simplified version $\underline{C}_s$ of $\underline{C}$, that the function on $\underline{C}$ that maps w_1 to w_2, and w_2 to w_1 can only be extended into a 1-bisimulation on $\underline{C}$ that is not a functional.

Second, we note that it follows that both of the functional 1-bisimulations $\underline{C}{\downarrow}^{w_1}_{*} \Rightarrow_{\phi_1} \underline{C}_0$ and $\underline{C}{\downarrow}^{w_2}_{*} \Rightarrow_{\phi_1} \underline{C}_0$ need to identify different vertices of $\underline{C}{\downarrow}^{w_1}_{*}$, and of $\underline{C}{\downarrow}^{w_2}_{*}$, respectively. This is because if either of ϕ_1 or ϕ_2 does not identify any different vertices, then it is an isomorphism, and therefore the other of these two functions would give rise to a functional 1-bisimulation between $\underline{C}{\downarrow}^{w_1}_{*}$ and $\underline{C}{\downarrow}^{w_2}_{*}$, in contradiction with our first observation above.

Third, we note that, as both of ϕ_1 and ϕ_2 must identify some vertices of $\underline{C}{\downarrow}^{w_1}_{*}$ and $\underline{C}{\downarrow}^{w_2}_{*}$, they both must identify w_1 and w_2, because these are the only two different vertices in $\underline{C}$ (and hence in $\underline{C}{\downarrow}^{w_1}_{*}$ and $\underline{C}{\downarrow}^{w_2}_{*}$) that are 1-bisimilar. As a consequence the mutual target 1-chart $\underline{C}_0$ must be 1-bisimulation collapsed, because in it the only pairs of non-1-bisimilar vertices in $\underline{C}{\downarrow}^{w_1}_{*}$ and $\underline{C}{\downarrow}^{w_2}_{*}$ have been collapsed to a single vertex by ϕ_1, and by ϕ_2, respectively. Now as $\underline{C}_0$ was assumed to be a LLEE-1-chart, it follows that $\underline{C}{\downarrow}^{w_1}_{*}$ and $\underline{C}{\downarrow}^{w_2}_{*}$ have a joint 1-bisimulation collapse that is a LLEE-1-chart. In particular it follows that we are in the situation $\underline{C}{\downarrow}^{w_1}_{*} \Rightarrow_{\phi_1} \underline{C}_0{\downarrow}^{\phi_1(w_1)}_{*} = \underline{C}_0{\downarrow}^{\phi_2(w_2)}_{*} \Rightarrow_{\phi_2} \underline{C}{\downarrow}^{w_2}_{*}$ with $\underline{C}_0{\downarrow}^{\phi_1(w_1)}_{*} = \underline{C}_0{\downarrow}^{\phi_2(w_2)}_{*}$ a 1-bisimulation collapse.

Since $v = \boxed{abc}$ in $\underline{C}{\downarrow}^{w_1}_{*}$ is not 1-bisimilar to any other vertex in $\underline{C}{\downarrow}^{w_1}_{*}$, and similarly in $\underline{C}{\downarrow}^{w_1}_{*}$, and since $\underline{C}_0$ is a 1-bisimulation collapse, it follows that $\phi_1(v) = \phi_2(v)$. Then by changing the start vertex in $\underline{C}_0$ from $\phi_1(w_1) = \phi_2(w_2)$ to $\phi_1(v) = \phi_2(w)$, we also obtain: $\underline{C} = \underline{C}{\downarrow}^{v}_{*} \Rightarrow_{\phi_1} \underline{C}_0{\downarrow}^{\phi_1(v)}_{*} = \underline{C}_0{\downarrow}^{\phi_2(v)}_{*} \Rightarrow_{\phi_2} \underline{C}{\downarrow}^{v}_{*} = \underline{C}$. Furthermore, since the property of a 1-chart to be a LLEE-1-chart is

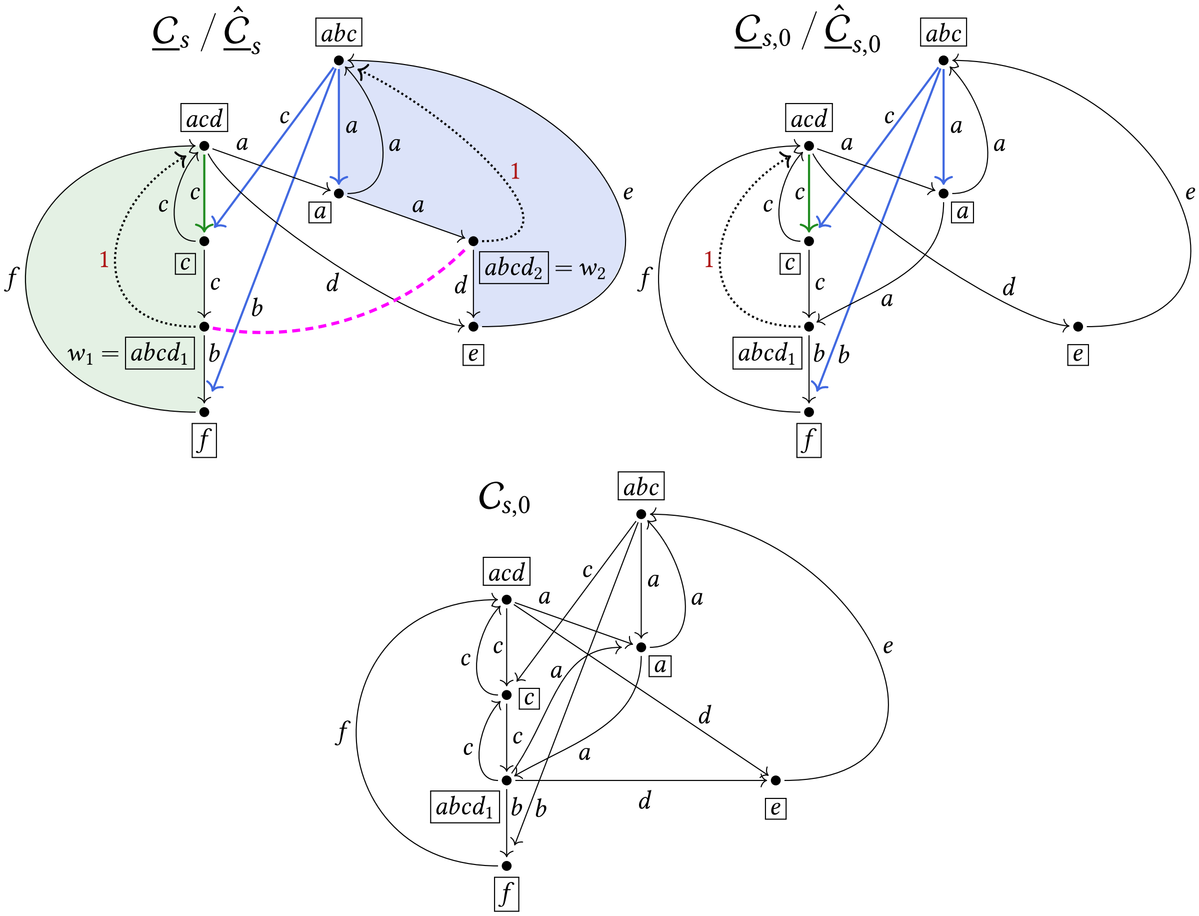


Figure 12. The 1-chart $\underline{C}_s$ from Fig. 4 (see here left in the top row with its indicated LLEE-witness $\underline{\hat{C}}_s$) is LLEE-preservingly 1-collapsible, because it has a 1-bisimulation collapse $\underline{C}_{s,0}$ (see right in the top row with its indicated LLEE-witness $\underline{\hat{C}}_{s,0}$) that is still a LLEE-1-chart. But $\underline{C}_s$ is not LLEE-preservingly collapsible, because the 1-transition free bisimulation collapse $C_{s,0}$ (see bottom row in the middle) of $\underline{C}_s$ and $\underline{C}_{s,0}$ does not satisfy LEE.

preserved under changing the start vertex, it follows that $\underline{C}$ has the LLEE-1-chart $\underline{C}' := \underline{C}_0{\downarrow}_*^{\phi_1(v)} = \underline{C}_0{\downarrow}_*^{\phi_2(v)}$ as 1-bisimulation collapse. But this contradicts the fact we have shown above for statement (i) of the proposition, namely that $\underline{C}$14 is not LLEE-preservingly 1-collapsible.

We conclude that $\underline{C}{\downarrow}_*^{w_1}$ and $\underline{C}{\downarrow}_*^{w_2}$ are not LLEE-preservingly jointly minimizable under $\Rightarrow$. This shows that pairs of 1-bisimilar weakly guarded LLEE-1-charts are not in general LLEE-preservingly 1-collapsible, and that is the statement (ii) in Prop. 5.1. ◻

A.4.3 Simplified version of the counterexample LLEE-1-chart .

Remark A.38 (remarked at the end of Sect. 5). The LLEE-1-chart $\underline{C}_s$ from Rem. 5, which is a simplified version of the example LLEE-1-chart $\underline{C}$ in Fig. 4, only witnesses statement (ii) in Prop. 5.1, but not statement (i), nor can it be used to demonstrate (ii).

Proof. That the LLEE-1-chart $\underline{C}_s$ in in Fig. 5 only witnesses statement (ii) in Prop. 5.1, but not the statement (i), is the statement of Lem. A.39 below. Furthermore, due to statement (iii) of Lem. A.39 below, the LLEE-1-chart $\underline{C}_s$ cannot be used in the same way as this is possible for $\underline{C}$ in Fig. 4 in order to show that LLEE-1-charts are not in general LLEE-preservingly minimizable under functional 1-bisimilarity. ◻

Lemma A.39. *The* LLEE*-1-chart $\underline{C}_s$ in Fig. 5 (and above left in Fig. 12) has the following properties:*

(i) $\underline{C}_s$ is LLEE*-preservingly 1-collapsible.*

(ii) $\underline{C}_s$ is not LLEE*-preservingly collapsible.*

(iii) $\underline{\mathcal{C}}_s{\downarrow}^{w_1}_{*} \Rightarrow \underline{\mathcal{C}}_{s,0} \Leftarrow \underline{\mathcal{C}}_s{\downarrow}^{w_2}_{*}$*, that is, the generated subcharts* $\underline{\mathcal{C}}_s{\downarrow}^{w_1}_{*}$ *and* $\underline{\mathcal{C}}_s{\downarrow}^{w_2}_{*}$ *of* $\underline{\mathcal{C}}_s$ *are* LLEE*-preservingly jointly minimizable under functional* 1*-bisimilarity (to the* LLEE*-*1*-chart* $\underline{\mathcal{C}}_{s,0}$ *in Fig. 12).*

Proof. We argue with the illustrations in Fig. 12, starting from the 1-chart $\underline{\mathcal{C}}_s$ of twin-crystal shape in Fig. 5

We first show (i). The LLEE-1-chart $\underline{\mathcal{C}}_s$ has, as connect-w_1-through-to-w_2 1-chart, the 1-bisimilar 1-chart $\underline{\mathcal{C}}_{s,0}$ Fig. 12. Now $\underline{\mathcal{C}}_{s,0}$ is still a LLEE-1-chart, because it has the LLEE-witness $\hat{\underline{\mathcal{C}}}_{s,0}$ with loop-entry transitions as indicated: green loop-entry transitions of level 1, and blue loop-entry transitions of level 2. Furthermore, $\underline{\mathcal{C}}_{s,0}$ is a 1-bisimulation collapse, because no two distinct vertices of $\underline{\mathcal{C}}_{s,0}$ can be 1-bisimilar, because each of the vertices of $\underline{\mathcal{C}}_{s,0}$ facilitates induced transitions for a different set of actions, as witnessed by the vertex name in the framed boxes. Therefore $\underline{\mathcal{C}}_{s,0}$ is a LLEE-1-chart that is a 1-bisimulation collapse of $\underline{\mathcal{C}}_s$. In this way we have shown that $\underline{\mathcal{C}}_s$ is LLEE-preservingly 1-collapsible.

Second, we show (ii). The 1-chart $\mathcal{C}_{s,0}$ without 1-transitions in Fig. 12 is the induced chart of the LLEE-1-chart $\underline{\mathcal{C}}_{s,0}$. As the induced chart of a 1-bisimulation collapsed 1-chart, $\mathcal{C}_{s,0}$ is a bisimulation collapse that is 1-bisimilar to $\underline{\mathcal{C}}_{s,0}$, and hence also to $\underline{\mathcal{C}}_s$. (No two distinct vertices of $\mathcal{C}_{s,0}$ can be bisimilar, because, equally as in $\underline{\mathcal{C}}_{s,0}$, each of the vertices of $\underline{\mathcal{C}}_{s,0}$ facilitates induced transitions for a different set of actions, as witnessed by the vertex name in the framed boxes.) But now $\mathcal{C}_{s,0}$ is not a LLEE-chart, because it does not satisfy LEE: none of its transitions is a loop-entry transition that generates a loop subchart, as can be checked for each transition separately; therefore the loop elimination procedure stops immediately; since $\underline{\mathcal{C}}_{s,0}$ obviously contains infinite paths, $\underline{\mathcal{C}}_{s,0}$ does obviously not satisfy LEE. As $\underline{\mathcal{C}}_{s,0}$ is the (1-transition free) bisimulation collapse of $\underline{\mathcal{C}}_s$ (as well as of $\underline{\mathcal{C}}_{s,0}$), and yet it cannot be a LLEE-chart, we have shown that $\underline{\mathcal{C}}_s$ is not LLEE-preservingly collapsible.

Third, we establish (iii). The function ϕ on the set of vertices of $\underline{\mathcal{C}}_s$ that maps w_2 to w_1, and acts as the identity function on the other vertices of $\underline{\mathcal{C}}_s$ is a functional 1-bisimulation from $\underline{\mathcal{C}}_s$ to its bisimulation collapse $\mathcal{C}_{s,0}$. Since the 1-bisimulation collapse $\underline{\mathcal{C}}_{s,0}$ of $\underline{\mathcal{C}}_s$ has the same set of vertices as the bisimulation collapse $\mathcal{C}_s$, it follows that ϕ is also a functional 1-bisimulation from $\underline{\mathcal{C}}_s$ to the 1-bisimulation collapse $\underline{\mathcal{C}}_{s,0}$. As a consequence, ϕ is a functional 1-bisimulation from both of the generated subcharts $\underline{\mathcal{C}}_s{\downarrow}^{w_1}_{*}$ and $\underline{\mathcal{C}}_s{\downarrow}^{w_2}_{*}$ to $\underline{\mathcal{C}}_{s,0}$. Thus we have concluded that, more formally, $\underline{\mathcal{C}}_s{\downarrow}^{w_1}_{*} \Rightarrow_{\phi} \underline{\mathcal{C}}_{s,0} \Leftarrow_{\phi} \underline{\mathcal{C}}_s{\downarrow}^{w_2}_{*}$ holds. This entails (iii). ◻

A.5 Supplements for Section 6: Twin-Crystals

A.5.1 Terminology needed.

Here we define concepts that we need for the formulation of the definition of 'twin-crystal'. We stipulate when a 1-LTS or a 1-chart is '1-collapsed apart from' a family of sets (or scc's). Then we define when we call a set of vertices in a 1-chart or in a 1-LTS 'squeezed' and/or 'grounded'. Finally, we define when proper transitions from some vertex can be said to 'favor' a set of vertices.

Before giving the definition of '1-collapsed apart from' we reformulate the definition of when a 1-LTS or a 1-chart is 1-collapsed.

Lemma A.40. *The following two statements holds:*

(i) A 1*-LTS* $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow\rangle$ *is* 1*-collapsed, and a* 1*-bisimulation collapse, if and only if it holds:*

$$\underline{\leftrightarrow}_{\underline{\mathcal{L}}} \;\subseteq\; = \,,$$

that is, two vertices of $\underline{\mathcal{L}}$ *are only* 1*-bisimilar if they are identical.*

(ii) A 1*-chart* $\underline{\mathcal{C}} = \langle V, A, v_s, 1, \rightarrow, \downarrow\rangle$ *is* 1*-collapsed, and a* 1*-bisimulation collapse, if and only if it holds:*

$$\underline{\leftrightarrow}_{\underline{\mathcal{C}}} \;\subseteq\; = \,.$$

Let $\{P_i\}_{i\in I}$ be a family of sets $P_i \subseteq A$ for a set A. Let $B \subseteq A$. We say that $\{P_i\}_{i\in I}$ *covers* the set B if $\bigcup_{i\in I} P_i \supseteq B$ holds, that is, if the union of the sets of the family contains B.

Definition A.41 (1-collapsed 1-LTS/1-chart apart from within sets/scc's)**.** We consider a 1-LTS $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow\rangle$, and a 1-chart $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ (for accordance in the defining clauses we denote the vertex set of $\underline{\mathcal{C}}$ also by V).

Let $\{P_i\}_{i\in I}$ be a family that covers the state set V of $\underline{\mathcal{L}}$. Then we say that $\underline{\mathcal{L}}$ *is* 1*-collapsed apart from within sets of* $\{P_i\}_{i\in I}$ if it holds that:

$$\underline{\leftrightarrow}_{\underline{\mathcal{L}}} \;\subseteq\; \bigcup_{i\in I} (P_i \times P_i)\,, \tag{12}$$

that is, any two states of $\underline{\mathcal{L}}$ are only 1-bisimilar in $\underline{\mathcal{L}}$ if both are contained in some set P_i for $i \in I$. We say that the 1-chart $\underline{\mathcal{C}}$ *is* 1*-collapsed apart from within sets of* $\{P_i\}_{i\in I}$ if the 1-LTS $\underline{\mathcal{L}}(\underline{\mathcal{C}}) = \langle V, A, 1, \rightarrow, \downarrow\rangle$ that underlies $\underline{\mathcal{C}}$ is 1-collapsed apart from within sets of $\{P_i\}_{i\in I}$.

We say that a 1-LTS $\underline{\mathcal{L}}$ *is 1-collapsed apart from within scc's* if $\underline{\mathcal{L}}$ is 1-collapsed apart from sets of the family $\{\mathrm{scc}(v)\}_{v\in V}$ of carriers of scc's of vertices of $\underline{\mathcal{L}}$. Accordingly we say that $\underline{\mathcal{C}}$ *is 1-collapsed apart from within scc's* if the 1-LTS $\underline{\mathcal{L}}(\underline{\mathcal{C}}) = \langle V, A, 1, \rightarrow, \downarrow\rangle$ that underlies $\underline{\mathcal{C}}$ is 1-collapsed apart from within scc's.

Definition A.42 (squeezed/grounded sets of vertices)**.** We consider a 1-chart $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$, and a subset $P \subseteq V$ of its set of vertices.

We say that P *is squeezed in $\underline{\mathcal{C}}$ (with respect to 1-bisimilarity $\underline{\leftrightarrow}$)* if it holds:

$$\forall w_1, w_2 \in V\, \big[\, w_1 \mathrel{\underline{\leftrightarrow}}_{\underline{\mathcal{C}}} w_2 \implies \big(\, w_1 \in P \implies w_2 \in P \,\big) \big]\,. \tag{13}$$

Due to the symmetry of $\underline{\leftrightarrow}$ the inner implication can be replaced by a bi-implication. Therefore (13) can be rephrased as the requirement that for every pair of 1-bisimilar vertices of P either both of them are contained in P, or none of them is. This is furthermore equivalent to requiring that $\underline{\mathcal{C}}$ is collapsed apart from within sets of the family $\{W_i\}_{i\in\{1,2\}}$ with $W_1 := P$ and $W_2 := V\backslash P$ (see Def. A.41).

We say that P *is grounded in $\underline{\mathcal{C}}$ (with respect to 1-bisimilarity $\underline{\leftrightarrow}$)* if it holds:

$$\begin{aligned} &\forall w_1, w_2, w_1', w_2' \in V\ \forall a_1, a_2 \in A \\ &\quad \big(\, w_1, w_2 \in P \,\wedge\, w_1 \xrightarrow{(a_1)} w_1' \,\wedge\, w_2 \xrightarrow{(a_2)} w_2' \\ &\qquad\qquad \wedge\, w_1', w_2' \notin P \,\wedge\, w_1' \mathrel{\underline{\leftrightarrow}}_{\underline{\mathcal{C}}} w_2' \implies w_1' = w_2' \,\big)\,, \end{aligned} \tag{14}$$

that is, if any two induced transitions that depart from P and target 1-bisimilar vertices outside of P target the same vertex.

Definition A.43 (proper transitions favor a set)**.** Let $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a 1-chart. Let $v \in V$ be a vertex of $\underline{\mathcal{C}}$, and let $W \subseteq V$ be a subset of vertices of $\underline{\mathcal{C}}$.

We say that *proper transitions from v favor (targets in) W* if it holds:

$$\forall v' \in V\ \forall a \in A\, \big(\, v \xrightarrow{a} v' \,\wedge\, \exists u \in W \big(\, v' \mathrel{\underline{\leftrightarrow}}_{\underline{\mathcal{C}}} u \,\big) \implies v' \in W \big)\,, \tag{15}$$

that is, if for every proper transition from v it holds that if its target vertex v' is 1-bisimilar to a vertex in W, then v' is actually contained in W.

Reformulated equivalently by using contraposition, the defining condition (15) for the statement that proper transitions from v favor W is:

$$\forall v' \in V\ \forall a \in A\, \big(\, v \xrightarrow{a} v' \,\wedge\, v' \notin W \implies \neg\exists u \in W \big(\, v' \mathrel{\underline{\leftrightarrow}}_{\underline{\mathcal{C}}} u \,\big)\big)\,,$$

that is, the target vertex v' of a proper transition from v is outside of W only in case that v' is not 1-bisimilar to a vertex in W.

A.5.2 Definition of twin-crystals.

In this subsection we define the central concept of our proof, in formal detail. A 'twin-crystal' as a maximal loops-back-to part in a precrystallized 1-chart that exhibits a symmetry in the form of a local transfer function.

Definition A.44 ((balanced) twin-crystal)**.** Let $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a finite 1-chart that satisfies LLEE-1-lim. Let $P \subseteq V$ be a set of vertices of $\underline{\mathcal{C}}$.

We say that P *carries a twin-crystal in $\underline{\mathcal{C}}$* (short for P *is the carrier set of a twin-crystal shaped scc in $\underline{\mathcal{C}}$*) if there are:

(α) a LLEE-witness $\hat{\underline{\mathcal{C}}} = \langle V, A \times \mathbb{N}, v_s, \hat{\rightarrow}, \downarrow\rangle$ of $\underline{\mathcal{C}}$, with the property that $\underline{\mathcal{C}}$ is 1-transition limited with respect to $\hat{\underline{\mathcal{C}}}$, which we call *the underlying* LLEE-*witness* with respect to which the loops-back-to relations used below are understood, that is, we write $\curvearrowright$ for $\curvearrowright_{\hat{\underline{\mathcal{C}}}}$,

(β) vertices $top, piv \in V$, which we call the *top-vertex*, and piv the *pivot-vertex*, respectively,

(γ) vertex subsets $P_1, P_2 \subseteq V$, where P_1 is the *pivot part*, and P_2 the *top part*,

(δ) a subset $E_2 \subseteq \{\langle top, \langle \underline{a}, n\rangle, v'\rangle \mid \langle top, \langle \underline{a}, n\rangle, v'\rangle \in \hat{\rightarrow},\ n \geqslant 1\}$ of the (marking-labeled, labeled) *top part entry transitions* from top in $\hat{\underline{\mathcal{C}}}$,

such that the following conditions hold:

(tc-1) $P = (\curvearrowright^* top)$ is a maximal loops-back-to part (hence top is $\curvearrowright^*$-maximal, and P is an scc by Lem. A.16, (vi)).

(tc-2) $P = P_1 \uplus P_2$ (that is, $P = P_1 \cup P_2$, and $P_1 \cap P_2 = \varnothing$) for $P_1 := (\curvearrowright^* piv)$ and $P_2 := ({}^{E_2}\curvearrowleft^* top)$. We call piv *the pivot*. (Then $piv \in P_1 \subseteq P$, $top \in P_2 \subseteq P$, and $\{P_i\}_{i\in\{1,2\}}$ is a partition of P).

(tc-3) $\underline{\mathcal{C}}$ is not collapsed for P (hence P contains 1-bisimilarity redundancies).

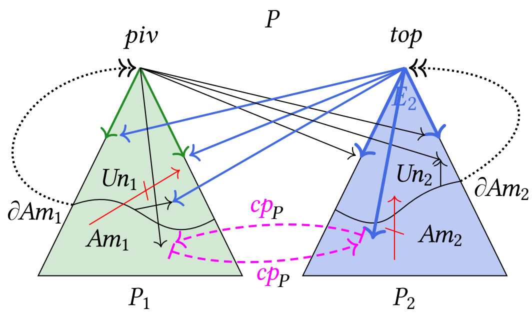


Figure 13. Schematic illustration of a twin-crystal with carrier P with top vertex *top*, and pivot vertex *piv*, and top entry transitions E_2 (thick blue). Hereby P partitions into pivot part P_1 and top part P_2. The pivot part P_1 partitions into the sets Un_1 and Am_1 of unambiguous, and ambiguous vertices, respectively. The top part P_1 partitions into the sets Un_2 and Am_2 of unambiguous, and ambiguous vertices, respectively. The boundary vertices ∂Am_1 of P_1 have 1-transition paths to *piv*, and the boundary vertices ∂Am_2 of P_2 have 1-transition paths to *top*. There are no transitions directly from $Am_1 \backslash \partial Am_1$ to Un_1, and no transitions directly from $Am_2 \backslash \partial Am_2$ to Un_2. 1-Bisimilar vertices in Am_1 and Am_2 are related by the collapse (partial) function cp_P.

(tc-4) All reduced 1-bisimilarity redundancies $\langle w_1, w_2 \rangle$ of $\underline{C}$ that are contained in P are precrystalline and have position $\langle w_1, piv, top, \overline{w}_2, w_2 \rangle$ for some $\overline{w}_2 \in V$.
(tc-5) Proper transitions from *piv* favor P_1. (See Def. A.43.)
(tc-6) Proper transitions from *top* favor P_2. (See Def. A.43.)
(tc-7) P is squeezed in $\underline{C}$. (See Def. A.42.)
(tc-8) P is grounded in $\underline{C}$. (See Def. A.42.)

If it holds additionally:
(btc) $\neg(piv \sqsubseteq top) \wedge \neg(top \sqsubseteq piv)$.

then say that P carries a *balanced* twin-crystal within $\underline{C}$. Making the dependency on the assumed concepts (α), (β), (γ), (δ) explicit, we will also say that $\underline{C}$ carries a twin-crystal (and accordingly, a balanced twin-crystal) *with respect to (the underlying* LLEE*-witness)* $\hat{\underline{C}}$*, with pivot-vertex piv, top-vertex* P_2*, pivot part* P_1*, top part* P_2*, and top entry transitions* E_2.

By a *(balanced) twin-crystal in* $\underline{C}$ we mean the full subchart of a set of vertices $P \subseteq V$ together with all transitions that depart from P (and their target vertices) provided that P carries a (balanced) twin-crystal in $\underline{C}$.

A.5.3 Notation for, and basic properties, of twin-crystals.

Definition A.45 (notation for twin-crystals)**.** Let $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow \rangle$ be a finite 1-chart with LLEE−1-lim. Let $P \subseteq V$ be the carrier of a twin-crystal in $\underline{C}$.

We denote the sets of <u>am</u>biguous, and <u>un</u>ambiguous vertices, of P, by:

$$Am := \left\{ w \in P \;\middle|\; \exists \overline{w} \in P \big(\overline{w} \neq w \wedge \overline{w} \mathrel{\underline{\leftrightarrow}_{\underline{C}}} w \big) \right\} \subseteq P ,$$

$$Un := P \backslash Am = \left\{ w \in P \;\middle|\; \forall \overline{w} \in P \big(\overline{w} \mathrel{\underline{\leftrightarrow}_{\underline{C}}} w \;\Rightarrow\; \overline{w} = w \big) \right\} \subseteq P ,$$

respectively. Note that $Am \neq \emptyset$, since P contains 1-bisimilarity redundancies by (tc-3). We also note that, because P is squeezed in $\underline{C}$ by (tc-7), no vertex in P, and hence no vertex in Am or in Un, has a 1-bisimilar counterpart outside of P in $\underline{C}$.

Furthermore, we introduce notation for the relativizations of the sets Am of ambiguous vertices, and Un of unambiguous vertices in P to P_2, and to P_1, respectively:

$$Am_1 := Am \cap P_1 , \qquad Am_2 := Am \cap P_2 , \tag{16}$$

$$Un_1 := Un \cap P_1 , \qquad Un_2 := Un \cap P_2 . \tag{17}$$

Example A.1. The 1-chart $\underline{C}$ in Figure 14, which we encountered previously in Figure 4, is a twin-crystal. Indeed, its entire set V of vertices is the carrier of a twin-crystal in $\underline{C}$ for the LLEE-witness $\hat{\underline{C}}$ as indicated (by assigning entry level 1 to the transitions marked in green, and entry level 2 to the transitions marked in blue, and all other transitions get level 0, and so are

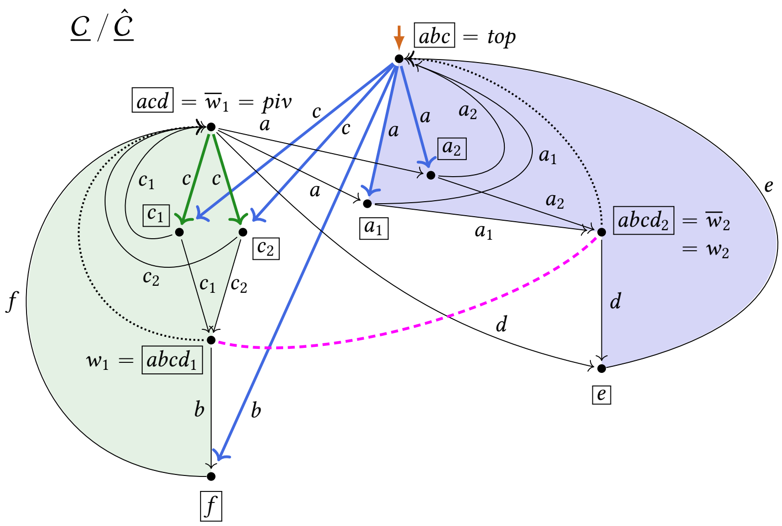
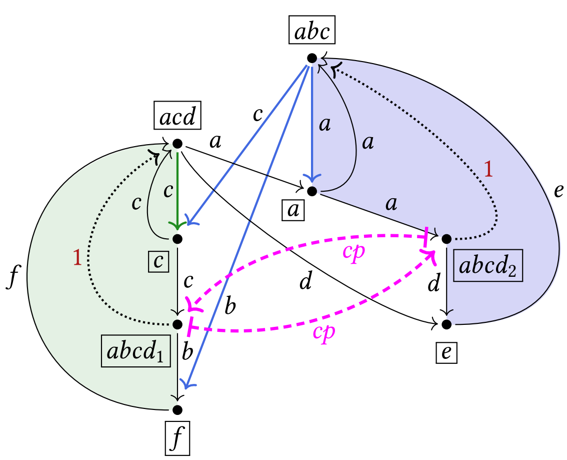


Figure 14. Example of a twin-crystal: the 1-chart $\underline{C}$ from Fig. 4 with LLEE-witness $\underline{\hat{C}}$ which is indicated by colored loop-entry transitions from the perpetual-loop vertex *top* (of level 2, blue) and from the vertex $\overline{w}_1$ (of level 1, green). The reflexive closure of the dashed magenta link that connects the 1-bisimilar vertices w_1 and w_2 defines a self-1-bisimulation on $\underline{C}$. The bisimilarity redundancy $\langle w_1, w_2\rangle$ is reduced, and precrystalline: it satisfies the position condition (R3.4) in Lem. 7.1, a situation for which we have not found a LLEE-preserving transformation that eliminates w_1 or w_2.

body transitions of $\underline{\hat{C}}$), and for $P, P_1, P_2, Am, Am_1, Am_2 \subseteq V$, $top, piv \in V$, and set E_2 of transitions from *top* in P as follows:

$$\begin{aligned}
top &= \boxed{abc}\,,\\
piv = \overline{w}_1 &= \boxed{acd}\,,\\
P &= (\circlearrowright^* top) = \{\boxed{abc}, \boxed{acd}, \boxed{c_1}, \boxed{c_2}, \boxed{a_1}, \boxed{a_2}, \boxed{abcd_1}, \boxed{abcd_2}, \boxed{e}, \boxed{f}\}\,,\\
P_1 &= (\circlearrowright^* piv) = \{\boxed{acd}, \boxed{c_1}, \boxed{c_2}, \boxed{abcd_1}, \boxed{f}\}\,,\\
E_2 &= \{\langle top, \langle a, 2\rangle, \boxed{a_1}\rangle, \langle top, \langle a, 2\rangle, \boxed{a_2}\rangle\}\,,\\
P_2 &= ({}^{E_2}\circlearrowright^* top) = \{\boxed{acd}, \boxed{a_1}, \boxed{a_2}, \boxed{abcd_2}, \boxed{e}\}\,,
\end{aligned}$$

$$\begin{aligned}
Am &= \{\boxed{abcd_1}, \boxed{abcd_2}\}\,, & Un_1 &= \{\boxed{acd}, \boxed{c_1}, \boxed{c_2}, \boxed{f}\}\,,\\
Am_1 &= \{\boxed{abcd_1}\}\,, & Un_2 &= \{\boxed{abc}, \boxed{a_1}, \boxed{a_2}, \boxed{e}\}\,,\\
Am_2 &= \{\boxed{abcd_2}\}\,, & Un &= \{\boxed{acd}, \boxed{abc}, \boxed{a_1}, \boxed{a_2}, \boxed{c_1}, \boxed{c_2}, \boxed{e}, \boxed{f}\}\,.
\end{aligned}$$



Lemma A.46 (properties of subsets in a twin-crystal)**.** *We consider a* 1*-chart* $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \to, \downarrow \rangle$ *with* LLEE-1-lim. *Let* $\hat{\underline{\mathcal{C}}}$ *be a* LLEE*-witness of* $\underline{\mathcal{C}}$.

Let $P \subseteq V$ *be the carrier of a twin-crystal in* $\underline{\mathcal{C}}$ *with respect to* $\hat{\underline{\mathcal{C}}}$*. Then* P *has the following properties, also in relation to the concepts defined for it in Def. A.45, the loops-back-to relation* $\curvearrowleft$ *of* $\hat{\underline{\mathcal{C}}}$*, and the concepts of ambiguous, unambiguous, and counterpart vertices defined in Def. A.56:*

(i) $P_1 \neq \varnothing$*, and* $P_2 \neq \varnothing$*, as a consequence of* $piv \in P_1$*, and* $top \in P_2$*, respectively.*
(ii) $\{P_i\}_{i\in\{1,2\}}$ *is a partition of* P.
(iii) $P = Am \uplus Un$.
(iv) $P_2 = Am_2 \uplus Un_2$.
(v) $P_1 = Am_1 \uplus Un_1$.
(vi) $Am = Am_1 \uplus Am_2$.
(vii) $Am \neq \varnothing$, $Am_1 \neq \varnothing$*, and* $Am_2 \neq \varnothing$.
(viii) $Un = Un_1 \uplus Un_2$.
(ix) $\forall w \in Am\, \exists! \overline{w} \in Am\, (\, \overline{w} \;\underline{\leftrightarrow}_{\underline{\mathcal{C}}}\; w \wedge \overline{w} \neq w\,)$*. More verbally, every ambiguous vertex in* P *has precisely one ambiguous vertex in* P *as its counterpart.*
(x) Counterparts of ambiguous vertices in P *are contained in different of the sets* Am_2 *and* Am_1*, or more formally:*

$$\forall w_1, w_2 \in P \left[\, w_1 \;\underline{\leftrightarrow}_{\underline{\mathcal{C}}}\; w_2 \wedge w_1 \neq w_2 \implies \begin{array}{l} (\, w_1 \in Am_2 \wedge w_2 \in Am_1\,) \\ \vee\, (\, w_2 \in Am_2 \wedge w_1 \in Am_1\,)\,\big]\,. \end{array}\right.$$

(xi) $\underline{\mathcal{C}}$ *is collapsed for* P_1*, for* P_2*, and consequently also for* Am_1*, for* Un_1*, for* Am_2*, and for* Un_2.
(xii) $\underline{\mathcal{C}}$ *is collapsed for every loops-back-to part of a loop vertex* $u \in LV(\hat{\underline{\mathcal{C}}}) \cap P$ *in* P *that is* <u>*not*</u> $\curvearrowleft$*-maximal.*
(xiii) 1*-bisimilarity redundancies* $\langle w_1, w_2\rangle$ *in* P *with* $w_1 \in P_2$ *and* $w_2 \in P_1$ *are* balanced *in the sense that* w_1 *has a* 1*-transition path to* v *if and only if* w_2 *has a* 1*-transition path to* piv*, more formally:*

$$\forall w_1 \in P_1\, \forall w_2 \in P_2 \big(\, w_1 \;\underline{\leftrightarrow}_{\underline{\mathcal{C}}}\; w_2 \implies \big(\, w_1 \xrightarrow{1}{}^{*}\, piv \iff w_2 \xrightarrow{1}{}^{*}\, v \,\big)\big)\,.$$

Lemma A.47 (properties of twin-crystals: loops-back relation/(induced) transitions)**.** *Let* $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \to, \downarrow \rangle$ *be a finite* 1*-chart with* LLEE-1-lim.

Let $P \subseteq V$ *be the carrier of a twin-crystal in* $\underline{\mathcal{C}}$ *with underlying* LLEE*-witness* $\hat{\underline{\mathcal{C}}}$*,* $top, piv \in P$ *the top and the pivot vertex of* P*, respectively, with pivot part* P_1*, top part* P_2*, and subsets* Am*,* Am_1*,* Am_2*,* Un*,* Un_1*,* Un_2 *as defined in Def. A.45. Then the following statements hold:*

(i) Every transition between P_1 *and* P_2 *in* $\underline{\mathcal{C}}$ *(that is, with its source and target in different of these sets) is of either of the following two kinds with respect to its marking label in the* LLEE*-witness* $\hat{\underline{\mathcal{C}}}$*:*
  - ▻ *a loop-entry transition from* $top \in P_2$ *to a vertex in* P_1.
  - ▻ *a body transition from* $piv \in P_1$ *to a vertex in* P_2.

(ii) All transitions from top *are proper transitions.*
(iii) All transitions from piv *are proper transitions.*
(iv) $piv \;{}_{d}\!\curvearrowleft\; v \wedge piv \neq v \wedge \neg(piv \xrightarrow{1}{}^{*}\, v)$.
(v) There is <u>*no transition*</u> *from* top *to a vertex in* Am_1*. Hence every transition from* top *to* P_1 *targets a vertex in* Un_1.
(vi) There is <u>*no transition*</u> *from* piv *to a vertex in* Am_2*. Hence every transition from* piv *to* P_2 *targets a vertex in* Un_2.
(vii) P_1 *is closed under transitions from vertices* $\neq piv$ *that target vertices in* P.
(viii) Every transition from a vertex in P_1 *to a vertex in* P_2 *departs from* piv.
(ix) P_2 *is closed under transitions from vertices* $\neq top$ *that target vertices in* P.
(x) Every transition from a vertex in P_2 *to a vertex in* P_1 *departs from* top.
(xi) Every induced transition from a vertex in Am_1 *to a vertex in* P *targets a vertex in* $Am_1 \uplus Un$*. There is* <u>*no induced transition*</u> *from a vertex in* Am_1 *to a vertex in* Am_2.
(xii) Every induced transition from a vertex in Am_2 *to a vertex in* P *targets a vertex in* $Am_2 \uplus Un$*. Hence there is* <u>*no induced transition*</u> *from a vertex in* Am_2 *to a vertex in* Am_1.
(xiii) For every induced transition τ *with source and target in* Am *there is an induced transition from the counterpart of the source of* τ *to the counterpart of the target of* τ *in* Am*. More specifically, for all* $w, w', \overline{w} \in V$*, where by* $\overline{w}$ *we mean the counterpart*

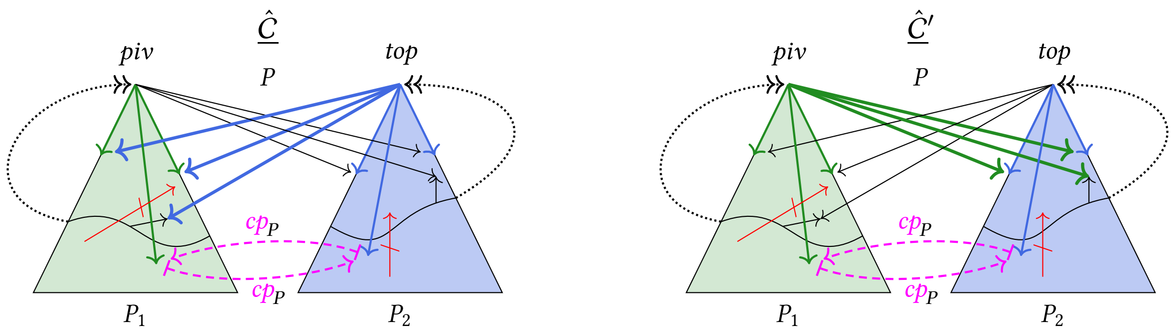


Figure 15. Interchanging the roles of top vertex and pivot vertex in a twin-crystal by modifying the underlying LLEE-witness, see Prop. A.48. The idea is that loop-entry transitions from *top* to P_1 (drawn very thick blue on the left) are 'pushed through' to give rise to loop-entry transitions from *piv* to P_2 (drawn very thick green on the right).

of w *if* $w \in Am$*, for all* $a \in A$*, and* $i \in \{1, 2\}$ *it holds:*

$$Am_i \ni w \xrightarrow{(a]} w' \in Am \implies w' \in Am_i \wedge \exists \bar{w}' \in Am_{3-i} \big(\overline{w} \xrightarrow{(a]} \bar{w}' \wedge \overline{w} \mathrel{\underline{\leftrightarrow}_{\underline{\mathcal{C}}}} \bar{w}' \big) . \tag{18}$$

(xiv) *For every induced transition from a source* w *in* Am *to a target* w' *outside of* Am *there is an induced transition from the image of* w *under* cp_P *to the same target* w'*. More formally, for all* $w, w', \overline{w} \in V$*, where by* $\overline{w}$ *we mean the counterpart of* w *in* P*, and for all* $a \in A$ *it holds:*

$$Am \ni w \xrightarrow{(a]} w' \notin Am \implies \overline{w} \xrightarrow{(a]} w' . \tag{19}$$

A.5.4 Symmetrical nature of twin-crystals.

At the end of Sect. 6 on page 10 we remarked on the 'symmetric nature of twin-crystals'. This remark can be made precise by the statement of the following proposition.

Proposition A.48 (interchangeability of the roles of top and pivot vertex in a twin-crystal)**.** *Let* $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \to, \downarrow \rangle$ *be a* 1*-chart with* LLEE−1-lim*, and let* $\hat{\underline{\mathcal{C}}}$ *be a* 1*-transition limited* LLEE*-witness of* $\underline{\mathcal{C}}$*. Let* $P \subseteq V$*, and let* $top, piv \in P$*.*

If P *is the carrier of a twin-crystal in* $\underline{\mathcal{C}}$ *with respect to* $\hat{\underline{\mathcal{C}}}$*, with top vertex* top*, and with pivot vertex* piv*, then there is another* LLEE*-witness* $\hat{\underline{\mathcal{C}}}'$ *of* $\underline{\mathcal{C}}$ *such that* P *is also carrier of a twin-crystal in* $\underline{\mathcal{C}}$ *with respect to* $\hat{\underline{\mathcal{C}}}'$*, but with top vertex* piv*, and with pivot vertex* top*.*

Proof (Sketch). Suppose that P is the carrier of a twin-crystal in $\underline{\mathcal{C}}$ with respect to $\hat{\underline{\mathcal{C}}}$, with top vertex *top* and with pivot vertex *piv*. Let m be the maximum level of a loop-entry transition from *top* in $\hat{\underline{\mathcal{C}}}$.

In order to define an entry/body-labeling $\hat{\underline{\mathcal{C}}}'$ of $\underline{\mathcal{C}}$ that is different from $\hat{\underline{\mathcal{C}}}$, we change the entry/body-labeling $\hat{\underline{\mathcal{C}}}$ of $\underline{\mathcal{C}}$ as indicated in Fig. 15: We turn all loop-entry transitions that depart from *top* and target vertices in P_1 into body transitions. Furthermore, we turn all body transitions that depart from *piv* (these transitions target vertices in P_2) into loop-entry transitions of level m. For all other transitions of $\underline{\mathcal{C}}$ we copy the marking label from $\hat{\underline{\mathcal{C}}}$ over to the entry/body-labeling $\hat{\underline{\mathcal{C}}}'$.

Then it is not difficult to verify that $\hat{\underline{\mathcal{C}}}'$ is also a LLEE-witness of $\underline{\mathcal{C}}$. As for the definition of $\hat{\underline{\mathcal{C}}}'$ only transitions from *top* and *piv* have been relabeled, and because none of the bodies of other loop vertices of $\underline{\mathcal{C}}$ are affected by this change, the LLEE-witness conditions only have to be checked for *top* and *piv* in $\hat{\underline{\mathcal{C}}}'$. For *top*, the LLEE-witness conditions of $\hat{\underline{\mathcal{C}}}$ imply the LLEE-witness conditions for $\hat{\underline{\mathcal{C}}}'$ as well, because in $\hat{\underline{\mathcal{C}}}'$ there are fewer LLEE-witness conditions for loops at *top* to check. For *piv*, the LLEE-witness conditions of $\hat{\underline{\mathcal{C}}}'$ follow from those of $\hat{\underline{\mathcal{C}}}$ at *top*: hereby the idea is that the loop sub-1-charts in $\hat{\underline{\mathcal{C}}}$ that have been induced on $\underline{\mathcal{C}}$ by the loop-entry transitions from *top* to vertices in P_1 have been 'pushed through', in the construction of $\hat{\underline{\mathcal{C}}}'$ from *top* to *piv* to now be anchored at *piv*, forming loop sub-1-charts at *piv* that are induced by the loop-entry transitions from *piv* to vertices in P_2.

Finally it is routine to check that P is also a twin-crystal with respect to $\hat{\underline{\mathcal{C}}}'$, with top vertex *piv*, and pivot vertex *top*. For this purpose we can use the twin-crystal properties (tc-1)–(tc-8) that hold for P as a twin-crystal with respect to $\hat{\underline{\mathcal{C}}}$, with top vertex *top*, and pivot vertex *piv*. ◻

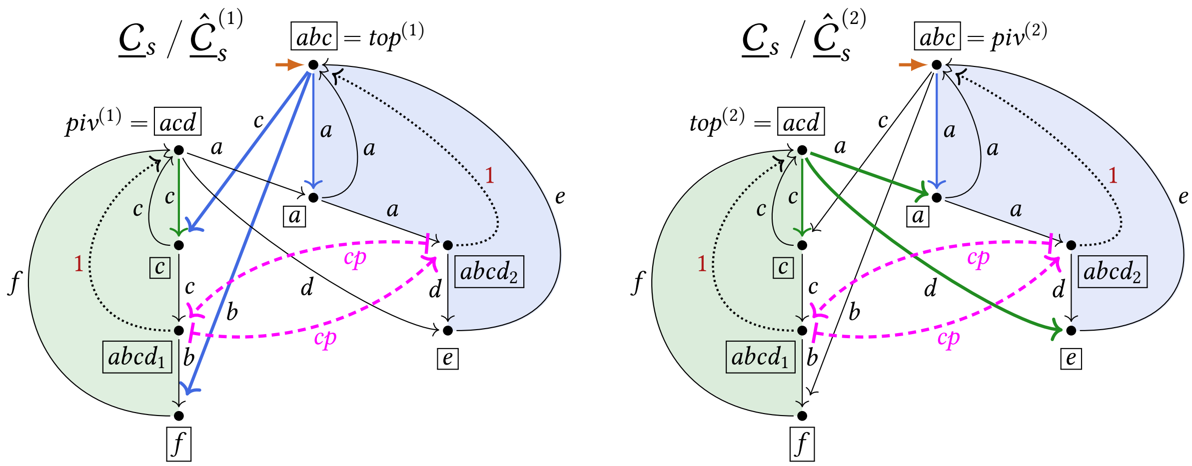


Figure 16. Example for the interchangeability of the roles of top vertex and pivot vertex in a 1-chart $\underline{C}_s$ of twin-crystal shape: The vertex set P of $\underline{C}_s$ is the carrier of a twin-crystal with top vertex $top^{(1)} = \boxed{abc}$ and pivot vertex $piv^{(1)} = \boxed{acd}$ with respect to the LLEE-witness $\hat{\underline{C}}_s^{(1)}$, in which the blue loop-entry transitions have level 2, and the green loop-entry transition has level 1. Also, P is the carrier of a twin-crystal with top vertex $top^{(2)} = \boxed{acd}$ and pivot vertex $piv^{(2)} = \boxed{abc}$ with respect to the LLEE-witness $\hat{\underline{C}}_s^{(2)}$, in which the green loop-entry transitions have level 2, and the blue loop-entry transition has level 1.

Example A.49. We consider again the LLEE-1-chart $\underline{C}_s$ from Fig. 5 that we recognized to be twin-crystal-shaped in Ex. 6.6. In Fig. 16 we illustrate the interchanging of the roles, as stated by Prop. A.48, of *piv* and *top* LLEE-1-chart $\underline{C}_s$ from Fig. 5.

A.5.5 Characterization of local-transfer functions.

Similar to the characterization of transfer functions between 1-LTSs in Sect. A.3.5, we here develop characterizations of local transfer functions, and of self-inverse local transfer functions, on 1-LTSs in the two lemmas below. We will use the characterization of self-inverse local transfer function in the next section in order to verify that the counterpart function on a twin-crystal is a local transfer function.

Lemma A.50 (characterization of local-transfer function)**.** *We consider a 1-LTS $\underline{\mathcal{L}} = \langle V, A, 1, \to, \downarrow\rangle$. Let $\phi : V \rightharpoonup V$ be a partial function on the state set V of $\underline{\mathcal{L}}$ with domain $W_1 := \mathrm{dom}(\phi)$, and range $W_2 := \mathrm{ran}(\phi)$.*

Then ϕ is a local-transfer function on $\underline{\mathcal{L}}$ if and only if ϕ has a non-empty domain $W_1 \neq \emptyset$, and if, for all $w_1, w_1', w_2' \in V$, and $a \in A$ the following five conditions hold:

$$w_1 \in W_1 \wedge w_1 \xrightarrow{(a]} w_1' \wedge w_1' \in W_1 \implies \phi(w_1) \xrightarrow{(a]} \phi(w_1'), \tag{20}$$

$$w_1 \in W_1 \wedge w_1 \xrightarrow{(a]} w_1' \wedge w_1' \notin W_1 \implies \phi(w_1) \xrightarrow{(a]} w_1' \wedge w_1' \notin W_2, \tag{21}$$

$$\exists w_1' \in V_1\big(\, w_1 \xrightarrow{(a]} w_1' \wedge w_1' \in W_1 \wedge \phi(w_1') = w_2' \,\big) \impliedby \big(\, w_1 \in W_1 \wedge \phi(w_1) \xrightarrow{(a]} w_2' \wedge w_2' \in W_2 \big), \tag{22}$$

$$w_1 \xrightarrow{(a]} w_2' \wedge w_2' \notin W_1 \impliedby \big(\, w_1 \in W_1 \wedge \phi(w_1) \xrightarrow{(a]} w_2' \wedge w_2' \notin W_2 \big), \tag{23}$$

$$w_1 \in W_1 \implies \big(\, w_1{\downarrow^{(1)}} \iff \phi(w_1){\downarrow^{(1)}} \big)\,. \tag{24}$$

Proof. The partial function $\phi : V \rightharpoonup V$ is a local-transfer function on $\underline{\mathcal{L}}$ if and only if the graph relation $B := graph(\phi) \subseteq V \times V$ of ϕ is a non-empty grounded 1-bisimulation slice on $\underline{\mathcal{L}}$. Now B is a grounded 1-bisimulation slice on $\underline{\mathcal{L}}$ if and only if B is a grounded bisimulation slice on the induced-transition LTS $\underline{\mathcal{L}}_{(\cdot]}$ of $\underline{\mathcal{L}}$, which means that the forth, back, and termination conditions $(\mathrm{forth})_{\mathrm{s}}$, $(\mathrm{back})_{\mathrm{s}}$, and $(\mathrm{termination})_{\mathrm{s}}$ for B to be a bisimulation slice on $\underline{\mathcal{L}}_{(\cdot]}$, and the forth, and back conditions $(\mathrm{forth})_{\mathrm{g}}$, and $(\mathrm{back})_{\mathrm{g}}$ for B to be a grounded bisimulation slice on $\underline{\mathcal{L}}_{(\cdot]}$ hold for B with respect to induced transitions in $\underline{\mathcal{L}}_{(\cdot]}$. Now the conditions $(\mathrm{forth})_{\mathrm{s}}$, $(\mathrm{back})_{\mathrm{s}}$, and $(\mathrm{termination})_{\mathrm{s}}$ coincide with (20), (22), and (24), respectively, and the conditions $(\mathrm{forth})_{\mathrm{g}}$, and $(\mathrm{back})_{\mathrm{g}}$ with (21), and (23), respectively, when quantified over all $w_1, w_1', w_2' \in V$ and $a \in A$. Put together, this chain of equivalences demonstrates the statement of the lemma. ◻

Definition A.51 (self-inverse partial function)**.** We say that a partial function $\phi : V \rightharpoonup V$ on a set V is *self-inverse*terminology]self-inverse partial function if the following two statements hold:

(i) $\mathrm{ran}(\phi) \subseteq \mathrm{dom}(\phi)$, that is, its domain and range coincide,

(ii) $\phi(\phi(v)) = v$ holds for all $v \in \mathrm{dom}(\phi)$ (which is a well-defined requirement due to (i), and then also implies the strengthening $\mathrm{ran}(\phi) = \mathrm{dom}(\phi)$ of (i)).

Lemma A.52 (characterization of self-inverse local-transfer function)**.** *Let $\underline{\mathcal{L}} = \langle V, A, 1, \to, \downarrow \rangle$ be a 1-LTS. Let $\phi : V \rightharpoonup V$ be a self-inverse partial function on the state set V of $\underline{\mathcal{L}}$ with domain $W := \mathrm{dom}(\phi)$.*

Then ϕ is a local-transfer function on $\underline{\mathcal{L}}$ if and only if ϕ has non-empty domain $W \neq \varnothing$, and the following three conditions hold, for all $w, w' \in V$ and $a \in A$:

$$w \in W \wedge w \xrightarrow{(a]} w' \wedge w' \in W \implies \phi(w) \xrightarrow{(a]} \phi(w'), \tag{25}$$

$$w \in W \wedge w \xrightarrow{(a]} w' \wedge w' \notin W \implies \phi(w) \xrightarrow{(a]} w', \tag{26}$$

$$w \in W \implies \big(w\downarrow^{(1)} \iff \phi(w)\downarrow^{(1)} \big). \tag{27}$$

Proof. For showing the directions "⇒" and "⇐" in the statement of the lemma, we consider a self-inverse partial function $\phi : V \rightharpoonup V$ with $W := \mathrm{dom}(\phi) = \mathrm{ran}(\phi)$.

For showing "⇒", we assume that ϕ is a local-transfer function on $\underline{\mathcal{L}}$. Then ϕ is a local-transfer function between $\underline{\mathcal{L}}$ and $\underline{\mathcal{L}}$, and hence, by Lem. A.50, its domain $W := \mathrm{dom}(\phi)$ is non-empty, and the conditions (20)–(24) in Lem. A.50, hold with $W_1 := \mathrm{dom}(\phi) = W$, $W_2 := \mathrm{ran}(\phi) = W$ (note that ϕ is self-inverse), and for all $w_1, w_1', w_2' \in W$ and $a \in A$. Now the universally quantified statement (20) coincides, under the mentioned settings, with (25), universally quantified. Also, the universally quantified statements (10) coincides, under the mentioned settings for W_1 and W_2, with (26), universally quantified. Furthermore (24), universally quantified under $W_1 = W$ coincides with (27), universally quantified. Therefore we conclude that $W \neq \varnothing$, and that (25), (26), and (27) hold for all $w, w' \in V$ and $a \in A$. In this way we have demonstrated "⇒" in the statement of the lemma.

For showing "⇐", we assume that ϕ is such that $W \neq \varnothing$, and the conditions (25), (26), and (27) are satisfied, for all $w, w' \in V$ and $a \in A$. In order to show that ϕ is a local-transfer function on $\underline{\mathcal{L}}$, it suffices to demonstrate, by Lem. A.50, the universally quantified conditions (20)–(24) in that lemma for $W_1 := W_2 := W$. Considered under these assumptions, of these conditions (20), universally quantified, is demonstrated by (25), universally quantified; also (21), universally quantified, is demonstrated by (26), universally quantified; and (24), universally quantified, is demonstrated by (27), universally quantified. What are left over to demonstrate are conditions (22) and (23), universally quantified, which here take the form that, for all $w_1, w_2' \in V$ and $a \in A$, it holds, respectively:

$$\exists w_1' \in V \big(w_1 \xrightarrow{(a]} w_1' \wedge w_1' \in W \wedge \phi(w_1') = w_2' \big) \impliedby \big(w_1 \in W \wedge \phi(w_1) \xrightarrow{(a]} w_2' \wedge w_2' \in W \big), \tag{28}$$

$$w_1 \xrightarrow{(a]} w_2' \wedge w_2' \notin W \impliedby \big(w_1 \in W \wedge \phi(w_1) \xrightarrow{(a]} w_2' \wedge w_2' \notin W \big). \tag{29}$$

In order to show (28), we pick $w_1, w_2' \in V$ and $a \in A$ arbitrary, but such that:

$$w_1 \in W \wedge \phi(w_1) \xrightarrow{(a]} w_2' \wedge w_2' \in W. \tag{30}$$

Now by applying the universally quantified statement (25), and by using that ϕ is self-inverse, we obtain:

$$w_1 \in W \wedge w_1 = \phi(\phi(w_1)) \xrightarrow{(a]} \phi(w_2') \wedge w_2' \in W.$$

By letting $w_1' := \phi(w_2') \in \mathrm{ran}(\phi) = W$, for which then $\phi(w_1') = \phi(\phi(w_2')) = w_2'$ since ϕ is self-inverse, we obtain:

$$w_1 \xrightarrow{(a]} w_1' \wedge w_1' \in W \wedge \phi(w_1') = w_2'.$$

This, however, yields:

$$\exists w_1' \in V \big(w_1 \xrightarrow{(a]} w_1' \wedge w_1' \in W \wedge \phi(w_1') = w_2' \big). \tag{31}$$

Now since for arbitrarily picked $w_1, w_2' \in V$, and $a \in A$ with (30) we have shown (31), we have demonstrated (28) for all $w_1, w_2' \in V$ and $a \in A$.

The universally quantified statement (29) can be shown very similarly, by using (26), and that ϕ is self-inverse.

By having shown the directions "⇒" and "⇐" in its statement, we have established the lemma. □

A.5.6 Twin-Crystals are locally near-collapsed. Here we define the 'counterpart function' on a twin-crystal, gather important properties that we will need in a lemma, and then show the crucial statement that this counterpart function on a twin-crystal is a self-inverse local transfer function.

Definition A.53 (counterpart function on a twin-crystal)**.** We consider a 1-chart $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ with LLEE-1-lim. We suppose that $P \subseteq V$ is the carrier of a twin-crystal in $\underline{C}$ with set Am of ambiguous vertices.

By *the counterpart function on P* we mean the partial function on V:

$$cp_P : V \longrightarrow V\,, \;\; w \longmapsto \begin{cases} \overline{w} & \text{if } w \in Am\text{, and} \\ & \quad \overline{w} \text{ the counterpart of } w \text{ in } P\,, \\ \text{undefined} & \text{if } w \notin Am\,. \end{cases}$$

Both of these functions are well-defined due to the uniqueness of counterparts in the ambiguous part Am of the carrier P of a twin-crystal, see Lem. A.46, (ix).

Lemma A.54 (properties of the counterpart function)**.** *We consider a* 1*-chart* $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ *with* LLEE-1-lim. *We suppose that $P \subseteq V$ is the carrier of a twin-crystal in $\underline{C}$ with set Am of ambiguous vertices.*

The counterpart function on P satisfies the following properties:

(i) $\mathrm{dom}(cp_P) = \mathrm{ran}(cp_P) = Am$.

(ii) cp_P *is self-inverse.*

(iii) cp_P *relates vertices in Am_i, for $i \in \{1,2\}$, with their counterparts in Am_{3-i}, that is, for all $w, u \in V$ with $w \in \mathrm{dom}(cp_P) = Am$ it holds:*

$$u = cp_P(w) \iff u \;\underline{\leftrightarrow}_{\underline{C}}\; w \;\wedge\; \exists i \in \{1,2\}\,\big(\, w \in Am_i \wedge u \in Am_{3-i}\,\big)\,. \tag{32}$$

(iv) Two vertices in P are 1*-bisimilar if and only if both are contained in Am, and either of them is the value of cp_P of the other:*

$$\forall w_1, w_2 \in P\big(\, w_1 \;\underline{\leftrightarrow}_{\underline{C}}\; w_2 \iff w_1 = w_2 \vee \big(\, w_1, w_2 \in Am \wedge cp_P(w_1) = w_2 \,\big)\big)\,.$$

(v) $cp_P(Am_1) = Am_2$, *and* $cp_P(Am_2) = Am_1$.

(vi) $cp_P(w) \;\underline{\leftrightarrow}_{\underline{C}}\; w$, *for all* $w \in Am$.

(vii) For every induced transition τ with source and target in Am there is an induced transition between the images under cp_P of the source and the target of τ. That is, for all $w, w' \in V$ and $a \in A$ it holds:

$$Am \ni w \xrightarrow{(a)} w' \in Am \implies cp_P(w) \xrightarrow{(a)} cp_P(w')\,. \tag{33}$$

(viii) Vertices w_1 and w_2 in P are 1*-bisimilar if and only if they either coincide or w_1 is Am and w_2 is the value of w_1 under cp_P, that is more formally:*

$$\begin{aligned} \forall w_1, w_2 \in P\big(\, & w_1 \;\underline{\leftrightarrow}_{\underline{C}}\; w_2 \\ & \iff w_1 = w_2 \;\vee\; \big(\, w_1 \in Am \wedge cp_P(w_1) = w_2 \,\big)\big) \end{aligned} \tag{34}$$

(ix) For every induced transition from a source w in Am to a target w' outside of Am there is an induced transition from the image of w under cp_P to the same target w'. More formally, for all $w, w' \in V$ and $a \in A$ it holds:

$$Am \ni w \xrightarrow{(a)} w' \notin Am \implies cp_P(w) \xrightarrow{(a)} w'\,. \tag{35}$$

Lemma A.55 (counterpart function is local transfer function)**.** *We consider a* 1*-chart* $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ *with* LLEE-1-lim. *Let $P \subseteq V$ be the carrier of a twin-crystal in $\underline{C}$. Then the counterpart function on P is a self-inverse local transfer function.*

Proof. Let $cp_P : P \rightharpoonup P$ be the counterpart function on the carrier P of a twin-crystal in $\underline{C}$. Then cp_P is self-inverse (due to its definition) by Lem. A.54, (ii).

It remains to show that cp_P is a local transfer function on $\underline{C}$. Since cp_P is self-inverse, we can apply Lem. A.52. It suffices to show the three conditions (25), (26), and (27) of that Lem. A.52 for $\phi := cp_P$. We repeat these conditions here as the following statements that are universally quantified over all $w, w' \in V$ and $a \in A$:

$$w \in Am \wedge w \xrightarrow{(a)} w' \wedge w' \in Am \implies cp_P(w) \xrightarrow{(a)} cp_P(w')\,, \tag{36}$$

$$w \in Am \wedge w \xrightarrow{(a)} w' \wedge w' \notin Am \implies cp_P(w) \xrightarrow{(a)} w'\,, \tag{37}$$

$$w \in Am \implies \big(\, w{\downarrow^{(1)}} \iff cp_P(w){\downarrow^{(1)}} \,\big)\,. \tag{38}$$

In order to demonstrate (36), we let $w, w' \in V$, and $a \in A$ be such that:

$$w \in Am \wedge w \xrightarrow{(a]} w' \wedge w' \in Am\,. \tag{39}$$

We have to show:

$$cp_P(w) \xrightarrow{(a]} cp_P(w')\,. \tag{40}$$

Since $w \in Am = Am_1 \uplus Am_2$ by Lem. A.46, (v), we distinguish the two possible cases of whether w is contained in Am_1 or in Am_2.

Case 1: $w \in Am_1$.

Then $cp_P(w) \in Am_2$ holds by Lem. A.54, (v). Since there is no induced transition from Am_1 to Am_2 by Lem. A.47, (xi) , and $Am = Am_1 \uplus Am_2$ by Lem. A.46, (vi), $w' \in Am$ entails $w' \in Am_1$. Hence $cp_P(w') \in Am_2$, by Lem. A.54, (v). Now since $w \mathrel{\underline{\leftrightarrow}}_{\underline{C}} cp_P(w)$ holds by Lem. A.54, (v), we can conclude by the forth condition of the 1-bisimulation $\mathrel{\underline{\leftrightarrow}}_{\underline{C}}$ on $\underline{C}$ that there is $u \in V$ such that:

$$cp_P(w) \xrightarrow{(a]} u \;\wedge\; w' \mathrel{\underline{\leftrightarrow}}_{\underline{C}} u\,. \tag{41}$$

Since $w' \in Am_1 \subseteq Am \subseteq P$, it follows from $w' \mathrel{\underline{\leftrightarrow}}_{\underline{C}} u$ by squeezedness of P in $\underline{C}$, which is due to (tc-7) for the carrier P of a twin-crystal, that also $u \in P$. Since $cp_{\underline{C}}(w) \in Am_2$, and there is no induced transition from Am_2 to Am_1 due to Lem. A.47, (xii) , we conclude that $u \notin Am_1$, and hence:

$$u \in P \backslash Am_1 = (Un \uplus Am) \backslash Am_1 = (Un \uplus (Am_1 \uplus Am_2)) \backslash Am_1 = Un \uplus Am_2\,.$$

Since $w' \in Am_1$, and $u \notin Am_1$ is 1-bisimilar to w' by (41), it follows that u is a counterpart of w'. Hence $u \in Un$ is not possible (as the vertices in Un do not have counterparts). Therefore $u \in Am_2$. Then "$\Leftarrow$" in (32) of Lem. A.54, (iii), implies that $cp_P(w') = u$. Together with (41) this implies (40).

Case 2: $w \in Am_2$.

In this case we can argue analogously to show (40). Here we use that there is no induced transition from Am_2 to Am_1 by Lem. A.47, (xii) .

Therefore we have shown that in each of these two cases (40) holds.

Now since we have shown that (40) always follows from (39), for all $w, w' \in V$ and $a \in A$, we have demonstrated (28), for all $w, w' \in V$ and $a \in A$.

In order to establish (37), we let $w, w' \in V$, and $a \in A$ be such that:

$$w \in Am \wedge w \xrightarrow{(a]} w' \wedge w' \notin Am\,. \tag{42}$$

We have to show:

$$cp_P(w) \xrightarrow{(a]} w'\,. \tag{43}$$

Since $w \mathrel{\underline{\leftrightarrow}}_{\underline{C}} cp_P(w)$ holds by Lem. A.54, (v), we can conclude by the forth condition of the 1-bisimulation $\mathrel{\underline{\leftrightarrow}}_{\underline{C}}$ on $\underline{C}$ that there is $u \in V$ such that:

$$cp_P(w) \xrightarrow{(a]} u \;\wedge\; w' \mathrel{\underline{\leftrightarrow}}_{\underline{C}} u\,. \tag{44}$$

We pick u accordingly. We distinguish the two cases of whether w' is contained in P or not.

Case 1: $w' \in P$. Then $w' \in P \backslash Am = Un$ by Lem. A.46, (ii).

Due to $w' \in P$, and $w' \mathrel{\underline{\leftrightarrow}}_{\underline{C}} u$ it follows from squeezedness of $\underline{C}$ for P, which is part of (tc-7) for P, that $u \in P$ holds as well. Now since u is 1-bisimilar to w', but $w' \in Un$ and thus w' has no counterpart in P, it must be the case that $u = w'$. Then (44) implies (43).

Case 2: $w' \notin P$.

Then $cp_P(w) \in Am \subseteq P$ by Lem. A.54, (i), because $w \in Am$. Since $w' \notin P$, and $w' \mathrel{\underline{\leftrightarrow}} u$ by (44), it follows from squeezedness of P in $\underline{C}$ that also $u \notin P$. So we have obtained, together with $cp_P(w) \in Am \subseteq P$ (due to $w \in Am$, and the definition of cp_P), (42) and (44):

$$w, cp_P(w) \in P \wedge w \xrightarrow{(a]} w' \wedge cp_P(w) \xrightarrow{(a]} u \wedge w' \mathrel{\underline{\leftrightarrow}} u \wedge w', u \notin P\,.$$

Now since P is grounded in $\underline{C}$, due to (tc-7) for P, it follows that $u = w'$. Then (44) again implies (43).

Therefore we have shown that in each of these two cases (43) holds.

Now since (42) always follows from (41), for all $w, w' \in V$ and $a \in A$, we have demonstrated (37), for all $w, w' \in V$ and $a \in A$.

That statement (30) holds for all $w \in Am$ is an immediate consequence of the fact that $cp_P(w) \underline{\leftrightarrow}_{\underline{C}} w$ holds for all $w \in Am$ due to Lem. A.54, (vi), and the termination condition for the 1-bisimulation $\underline{\leftrightarrow}_{\underline{C}}$ on $\underline{C}$.

Now since we have shown (28), (29), and (30), for all $w, w' \in V$ and $a \in A$, we can indeed conclude by Lem. A.52 that the counterpart function cp_P on P is a local transfer function on $\underline{C}$. In this way we have established the statement of the lemma. □

Lemma A.2 (twin-crystals are locally near-collapsed)**.** *A 1-chart $\underline{C}$ with* LLEE−1-lim *is near-collapsed for every carrier of a twin-crystal in $\underline{C}$.*

Proof. Let $\underline{C} = \langle V, A, v_s, 1, \rightarrow, \downarrow\rangle$ be a 1-chart with LLEE−1-lim, and let $P \subseteq V$ be the carrier of a twin-crystal in $\underline{C}$. We have to show that $\underline{C}$ is near-collapsed for P.

By Lem. A.55 the counterpart function $\phi : V \rightharpoonup V$ on P is a local transfer function. Hence in order to prove that $\underline{C}$ is near-collapsed for P it suffices to show:

$$\underline{\leftrightarrow}_{\underline{C}} \cap (P \times P) \quad \subseteq \quad \leftrightarrow^{=}_{graph(\psi)} . \tag{45}$$

For doing so, we let $w_1, w_2 \in V$ be arbitrary, and argue as follows:

$$
\begin{array}{lll}
 & \langle w_1, w_2\rangle \in \underline{\leftrightarrow} \cap \ (P \times P) & \\
\Longrightarrow & w_1 = w_2 \vee cp_P(w_1) = w_2 & \text{(by Lem. A.54, (iv))} \\
\Longrightarrow & w_1 = w_2 \vee w_1 \rightarrow_{graph(cp_P)} w_2 & \text{(by using rewrite notation} \\
 & & \text{ for the graph relation of } cp_P\text{)} \\
\Longrightarrow & w_1 = w_2 \vee w_1 \leftrightarrow_{graph(cp_P)} w_2 & \text{(by using rewrite notation} \\
 & & \text{ for symmetric closure)} \\
\Longrightarrow & w_1 \leftrightarrow^{=}_{graph(cp_P)} w_2 & \text{(by using rewrite notation} \\
 & & \text{ for reflexive–symmetric closure)}
\end{array}
$$

Since $w_1, w_2 \in V$ were arbitrary in this argument, we have demonstrated (45).

As that statement was what had remained to show, we have indeed established that $\underline{C}$ is near-collapsed for P. □

A.6 Supplements for Section 7: Crystallization of LLEE-1-charts

A.6.1 1-Bisimilarity redundancies.

In this subsection we define '1-bisimilarity redundancies' based on 1-bisimilarity on 1-charts, and the terminology of (1-bisimilar) counterparts of vertices, and vertices that are 'ambiguous' or 'unambiguous' with respect to 1-bisimilarity in a 1-chart. For the definitions below, we recall that in Def. 2.4 we defined 1-bisimilarity $\underline{\leftrightarrow}_{\underline{C}}$ on a 1-chart $\underline{C}$ as the largest 1-bisimulation between $\underline{C}$ and $\underline{C}$ (that is, on $\underline{C}$).

Definition A.56 (counterparts, ambiguous and unambiguous vertices)**.** Let $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a 1-chart. Let $W \subseteq V$ be a subset of the set of vertices of $\underline{C}$.

Let $w, \overline{w} \in V$. We say that $\overline{w}$ *is a counterpart of w in W* (short for that $\overline{w}$ *is a 1-bisimilar counterpart of w in W*) if $\overline{w}$ is contained in W, $\overline{w}$ is 1-bisimilar to w in $\underline{C}$, and different from w, that is more formally, if:

$$\overline{w} \in W \wedge \overline{w} \underline{\leftrightarrow}_{\underline{C}} w \wedge \overline{w} \neq w \, .$$

We say that $\overline{w}$ *is a counterpart of w (in $\underline{C}$)* if $\overline{w}$ is a counterpart of w in V. We say that w *has a counterpart in W* if there is a counterpart of w in W. We say that w *has a counterpart (in $\underline{C}$)* if there is a counterpart of w in $\underline{C}$.

Let $w \in V$. We say that w *is ambiguous in W* if w is contained in W, and it has a counterpart in W, that is more explicitly:

$$w \in W \wedge \exists \overline{w} \in W \left(w \underline{\leftrightarrow}_{\underline{C}} \overline{w} \wedge w \neq \overline{w} \right) .$$

We say that w *is unambiguous in W* if w is contained in W, and w does not have a counterpart in W. We say that w *is ambiguous (unambiguous) (in $\underline{C}$* if v is ambiguous (unambiguous) in V.

Definition A.57 ((unordered/ordered) 1-bisimilarity redundancies)**.** We consider a 1-chart $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$.

By an *unordered 1-bisimilarity redundancy in $\underline{C}$* (*unordered redundancy in $\underline{C}$ with respect to 1-bisimilarity*) we mean a two-element set $\{u_1, u_2\}$ of vertices of $\underline{C}$ where u_1 and u_2 are each other's counterpart with respect to 1-bisimilarity $\underline{\leftrightarrow}_{\underline{C}}$ on $\underline{C}$.

By an *(ordered) 1-bisimilarity redundancy in $\underline{C}$* we mean a pair $\langle w_1, w_2\rangle$ such that $\{w_1, w_2\}$ is an unordered 1-bisimilarity redundancy in $\underline{C}$.

By $\mathit{1BR}(\underline{C})$ and by $\langle \mathit{1BR}\rangle(\underline{C})$, we define the set of all unordered 1-bisimilarity redundancies in $\underline{C}$, and the set of all ordered 1-bisimilarity redundancies in $\underline{C}$:

$$\mathit{1BR}(\underline{C}) := \big\{ \{w_1, w_2\} \,\big|\, w_1, w_2 \in V,\ w_1 \neq w_2,\ w_1 \underline{\leftrightarrow}_{\underline{C}}\, w_2 \big\}, \qquad \langle \mathit{1BR}\rangle(\underline{C}) := \big\{ \langle w_1, w_2\rangle \,\big|\, \{w_1, w_2\} \in \mathit{1BR}(\underline{C}) \big\}.$$

We say that an ordered 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ is a *representation of* an unordered 1-bisimilarity redundancy $\langle u_1, u_2\rangle$ if $\{w_1, w_2\} = \{u_1, u_2\}$.

By a '1-bisimilarity redundancy' we will mean an ordered 1-bisimilarity redundancy without stressing the attribute 'ordered'. We will explicitly refer to 'unordered 1-bisimilarity redundancies' in cases in which we mean the unordered concept.

When checking that a relation B is a 1-bisimulation between two 1-LTSs or two 1-charts, and specifically, when checking that the forth and back conditions hold for a pair $\langle w_1, w_2\rangle \in B$, we need to match, for the forth condition, induced transitions $w_1 \xrightarrow{(a]} w_1'$ by an induced transition $w_2 \xrightarrow{(a]} w_2'$ such that $\langle w_1', w_2'\rangle \in B$ holds; a similar check has to be done for the back condition. If in such a check for a pair $\langle w_1, w_2\rangle \in B$ a pair $\langle w_1', w_2'\rangle \in B$ can arise, then we say that $\langle w_1, w_2\rangle$ 'propagates to' $\langle w_1', w_2'\rangle$ in B. We now define this 'propagation relation' with respect to a 1-bisimulation on pairs of vertices that are 1-bisimilarity redundancies.

Definition A.58 (propagation of 1-bisimilarity redundancies)**.** We consider a 1-chart $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$.

We define the *propagation relation* $\longrightarrow_{prop} \subseteq \langle \mathit{1BR}\rangle(\underline{C}) \times \langle \mathit{1BR}\rangle(\underline{C})$ on ordered 1-bisimilarity redundancies of $\underline{C}$ by stipulating, for all $\langle w_1, w_2\rangle, \langle w_1', w_2'\rangle \in \langle \mathit{1BR}\rangle(\underline{C})$:

$$\langle w_1, w_2\rangle \longrightarrow_{prop} \langle w_1', w_2'\rangle \;\; :\Longleftrightarrow \;\; \exists a \in A \big(w_1 \xrightarrow{(a]} w_1' \wedge w_2 \xrightarrow{(a]} w_2' \big)\,. \tag{46}$$

We say that $\langle w_1, w_2\rangle$ *propagates to* $\langle w_1', w_2'\rangle$ if $\langle w_1, w_2\rangle \longrightarrow_{prop} \langle w_1', w_2'\rangle$ holds for 1-bisimilarity redundancies $\langle w_1, w_2\rangle$, and $\langle w_1', w_2'\rangle$ of $\underline{C}$.

A.6.2 Reducing 1-bisimilarity redundancies (proof of Lem. 7.1).

In this section we justify Lem. 7.1. We will restate it more formally as Lem. A.59, but then also provide an operational refinement in Lem. A.63 whose proof we will explain, and from which Lem. A.59 and Lem. 7.1 follow.

First we formulate a lemma that characterizes the three possible kinds of positions that any 1-bisimilarity redundancy in a LLEE-1-chart may have. This lemma has been shown, for charts without 1-transitions and with a single sink vertex with immediate termination, as part of the proof of Prop. 6.4 in [12, 13].

Lemma A.59 (kinds of 1-bisimilarity redundancies in a LLEE-**1**-chart (from the proof of Prop. 6.4 in [12, 13]).)**.** *We consider a 1-chart $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ and a* LLEE*-witness $\hat{\underline{C}}$ of $\underline{C}$, and the loops-back-to relations $\curvearrowright$ and ${}_d\!\curvearrowright$ on $\underline{C}$ as defined by $\hat{\underline{C}}$.*

Every unordered 1-bisimilarity redundancy $\{u_1, u_2\}$ in $\underline{C}$ has an ordered representation $\langle w_1, w_2\rangle$ in $\underline{C}$ that satisfies precisely one of the following three statements:

(K1) $\neg(w_1 \leftarrow^* w_2)$, *which implies $scc(w_1) \neq scc(w_2)$;*

(K2) $w_1 \circlearrowleft^+ w_2$, *which implies $scc(w_1) = scc(w_2)$;*

(K3) *There are $v, \overline{w}_1, \overline{w}_2 \in V$ such that it holds:*
(K3-body) $w_1 \curvearrowright^* \overline{w}_1 \;{}_d\!\curvearrowright\; v \;{}_d\!\curvearrowleft\; \overline{w}_2 \circlearrowleft^* w_2 \;\wedge\; \neg(\overline{w}_1 \leftarrow^*_{\mathrm{bo}} \overline{w}_2)$,
which also implies $scc(w_1) = scc(w_2)$.

Consequently, if $\underline{C}$ is not a 1-bisimulation collapse, then it contains an ordered 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ such that either of (K1), (K2)*, or* (K3) *holds.*

Based on this lemma, we introduce the following terminology.

Definition A.60 (kinds of 1-bisimilarity redundancies in a LLEE-**1**-chart, positions/joining loop vertex for kind (K3))**.** Let $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a 1-chart with LLEE-witness $\hat{\underline{C}}$ of $\underline{C}$, with respect to which the loops-back-to relations $\curvearrowright$ and ${}_d\!\curvearrowright$ on $\underline{C}$ in the conditions (K1), (K2), (K3) in Lem. A.59 are defined.

Let $\langle w_1, w_2\rangle$ be a 1-bisimilarity redundancy in $\underline{C}$. We say that a 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ *is of kind* (K1), (K2), or (K3) if the condition (K1), (K2), or (K3) holds, respectively. If $\langle w_1, w_2\rangle$ is of kind (K2), then we say that w_1 is the *the joining loop vertex of* w_1 *and* w_2. If $\langle w_1, w_2\rangle$ is of kind (K3) and satisfies (K3-body) for some $\overline{w}_1, \overline{w}_2, v \in V$, then we say that v is *the joining loop vertex of* w_1 *and* w_2, and that $\langle w_1, w_2\rangle$ *has position* $\langle w_1, \overline{w}_1, v, \overline{w}_2, w_2\rangle$ *in* $\underline{C}$.

As a preparation of the formal restatement of Lem. 7.1 in Lem. A.62 below we also define the following notation for reachability via different kinds of steps in 1-transition limited LLEE-witnesses.

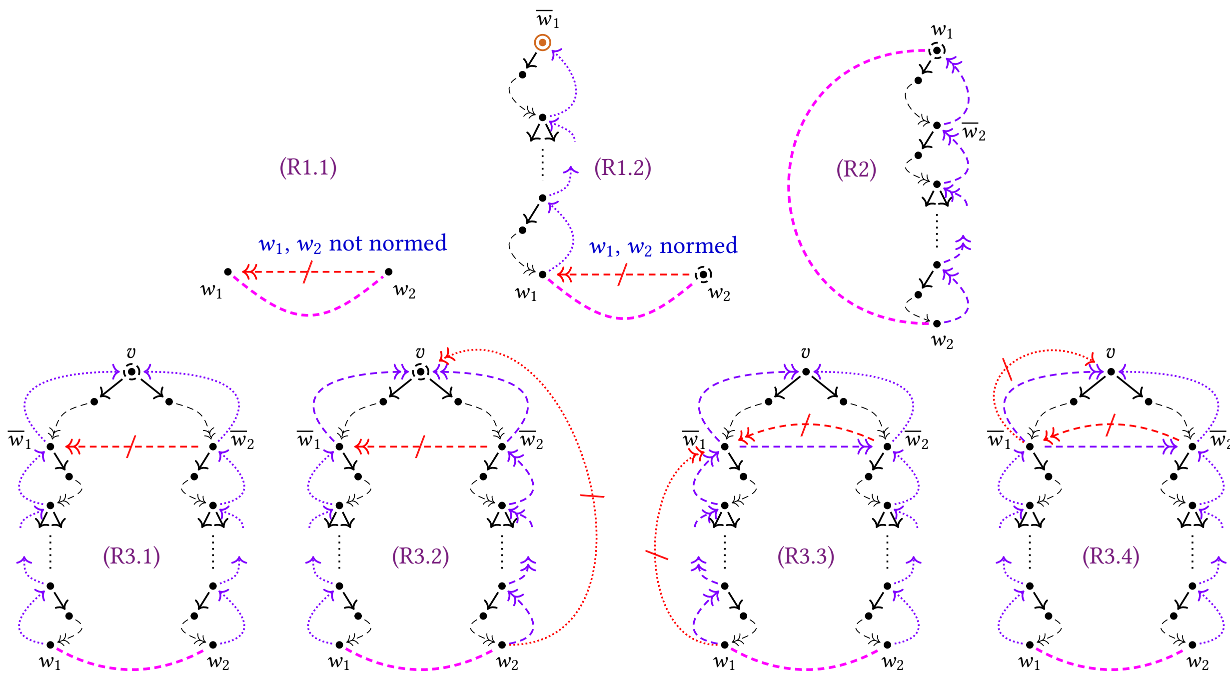


Figure 17. Illustrations of the kinds (R1) (with subkinds (R1.1) and (R1.2)), (R2), and (R3) (with subkinds (R3.1), (R3.2), (R3.3), (R3.4)) in Lemma 7.1 that define reduced 1-bisimilarity redundancies $\langle w_1, w_2 \rangle$ (indicated by dashed magenta links) in a LLEE-1-chart. We call those of kind (R3.4) precrystalline, and all other ones simple. These illustrations take into account that for the kinds (R3.3) and (R3.4) it follows that $v \not\downarrow$ holds[2] (indicated by the absence of a double ring that would indicate immediate termination at v) and that this does not hold in general for (R3.1) and (R3.2) (by the presence of a dashed outer ring that indicates the possible presence of immediate termination at v). Similarly, $w_1 \not\downarrow$ and $w_2 \not\downarrow$ holds for kind (R1.1) (by definition of 'normed'), but $w_2\downarrow$ is possible for kind (R1.2), and $w_1\downarrow$ for kind (R2).

Notation A.61. Let $\underline{C} = \langle V, A, 1, v_s, \to, \downarrow \rangle$ be a 1-chart with LLEE−1-lim. Let $\underline{\hat{C}}$ be a LLEE-witness of $\underline{C}$ such that $\underline{C}$ is 1-transition limited with respect to $\underline{\hat{C}}$.

We define the following relations, based on the entry/body-labeling given by $\underline{\hat{C}}$:

$$\begin{aligned}
\cdots\!\rightarrow \;&:=\; \xrightarrow{1}_{\mathrm{bo}} && \subseteq V \times V\,,\\
\dashrightarrow \;&:=\; \bigcup_{\underline{a} \in \underline{A}} \xrightarrow{\underline{a}}_{\mathrm{bo}} \;=\; \cdots\!\rightarrow \cup \bigcup_{a \in A} \xrightarrow{a}_{\mathrm{bo}} && \subseteq V \times V\,,\\
\Rrightarrow \;&:=\; \dashrightarrow^{+} \setminus \cdots\!\rightarrow^{*} && \subseteq V \times V\,.
\end{aligned}$$

Note that, since $\underline{C}$ is 1-transition limited with respect to $\underline{\hat{C}}$, $\cdots\!\rightarrow$ steps denote steps across 1-transition backlinks of $\underline{\hat{C}}$; steps across arbitrary body transitions are $\dashrightarrow$ steps; and a sequence of steps across body transitions are $\Rrightarrow$ steps if there is no sequence of 1-transition backlinks with the same source and target.

Based on this notation, we restate Lem. 7.1 more formally, as an existence lemma for reduced 1-bisimilarity redundancies in 1-charts with LLEE−1-lim. In doing so, Lem. A.62 below refines Lem. A.59 above, which separated representations of arbitrary unordered 1-bisimilarity redundancies in LLEE-charts into three different groups (K1), (K2), and (K3). More precisely, Lem. A.62 below states that every not collapsed 1-charts contains a 1-bisimilarity redundancy that belongs to one of three condition groups (R1), (R2), or (R3), which refine the conditions (K1), (K2), and (K3) in Lem. A.59, respectively.

[2] In both cases there is a 1-transition path from w_2 to v. Hence w_2 is a substate of v. Then $v\downarrow$ would imply $w_2 \downarrow^{(1)}$, and by $w_1 \leftrightarrow w_2$ (since $\langle w_1, w_2 \rangle$ is 1-bisimilarity redundancy) also $w_1 \downarrow^{(1)}$. But the latter is not possible: (R3.3) and (R3.4) exclude 1-transition paths from w_1 to v. Yet such a path would be needed to induce termination at w_1, because immediate termination is not permitted in LLEE-witnesses at vertices in loop bodies, but only at vertices that are outside of all loops, or that are outermost loop vertices.

Lemma A.62 (= formal statement of Lem. 7.1)**.** *Let $\underline{C} = \langle V, A, 1, v_s, \to, \downarrow\rangle$ be a 1-chart with* LLEE−1-lim. *Let $\hat{\underline{C}}$ be a* LLEE*-witness of $\underline{C}$ such that $\underline{C}$ is 1-transition limited with respect to $\hat{\underline{C}}$.*

If $\underline{C}$ is not a bisimulation collapse, then it contains a 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ that satisfies one of the following position conditions with respect to $\hat{\underline{C}}$:

(R1) *comprised of subkinds:*
 (R1.1) *w_1 and w_2 are not normed,*
 (R1.2) *w_1 and w_2 are normed $\wedge$ $\exists\, \overline{w}_1 \in V\,(\, \overline{w}_1 \rightsquigarrow^* w_1 \mathrel{\cdots\!\!\rightarrow^*} \overline{w}_1 \wedge \overline{w}_1\!\downarrow\,)$.*
 Here w_1 and w_2 are in different scc's. In case that w_2 is normed, this condition requires that w_1 is equal to, or loops back to via 1-transitions to, some vertex $\overline{w}_1$ that admits immediate termination.

(R2) *$w_2 \curvearrowright^{+} w_1$ holds (and hence, due to Lem. A.16, (vi), $scc(w_1) = scc(w_2)$).*

(R3) *There are $v, \overline{w}_1, \overline{w}_2 \in V$ such that: $w_1 \curvearrowright^{*} \overline{w}_1 \;{}_{d}\!\curvearrowright v \;{}_{d}\!\curvearrowleft \overline{w}_2 \circlearrowleft^{*} w_2 \;\wedge\; \neg(\overline{w}_1 \dashleftarrow^{*} \overline{w}_2)$, (hence $scc(w_1) = scc(w_2)$), and* <u>*additionally*</u> *one of the following four sub-conditions holds (please see Figure 17 for a guiding illustration):*
 (R3.1) $w_1 \mathrel{\cdots\!\!\rightarrow^{*}} \overline{w}_1 \mathrel{\cdots\!\!\rightarrow} v \mathrel{\leftarrow\!\!\cdots} \overline{w}_2 \mathrel{\leftarrow\!\!\cdots^{*}} w_2$,
 (R3.2) *$w_1 \mathrel{\cdots\!\!\rightarrow^{*}} \overline{w}_1 \dashrightarrow^{+} v \dashleftarrow^{+} \overline{w}_2 \dashleftarrow^{*} w_2$ with $\neg(v \mathrel{\leftarrow\!\!\cdots^{*}} w_2)$, hence $v \Lleftarrow w_2$.*
 (R3.3) *$w_1 \Rrightarrow \overline{w}_1 \dashrightarrow^{+} v \mathrel{\leftarrow\!\!\cdots} \overline{w}_2 \mathrel{\leftarrow\!\!\cdots^{*}} w_2$, and $\overline{w}_1 \dashrightarrow^{+} \overline{w}_2$,*
 (R3.4) *$w_1 \mathrel{\cdots\!\!\rightarrow^{+}} \overline{w}_1 \Rrightarrow v \mathrel{\leftarrow\!\!\cdots} \overline{w}_2 \mathrel{\leftarrow\!\!\cdots^{*}} w_2$, and $\overline{w}_1 \dashrightarrow^{+} \overline{w}_2$.*

This lemma follows from the next lemma, which is a refinement of its statement. This is because Lem. A.63 below also explains how 1-bisimilarity redundancies of the exhaustive kinds (K1), (K2), and (K3) as stated by Lem. A.59 propagate to the reduced kinds (R1), (R2), and (R3) as stated by Lem A.63 above. The 'operational' propagation statements in the refined lemma below reflect its proof that we will sketch by using argumentation with pictures.

Lemma A.63 (reducing 1-bisimilarity redundancies by propagation)**.** *Let $\underline{C} = \langle V, A, 1, v_s, \to, \downarrow\rangle$ be a (finite) 1-chart with* LLEE−1-lim. *Let $\hat{\underline{C}}$ be a* LLEE*-witness of $\underline{C}$ such that $\underline{C}$ is 1-transition limited with respect to $\hat{\underline{C}}$.*

Every 1-bisimilarity redundancy $\langle u_1, u_2\rangle$ in $\underline{C}$ propagates by $\longrightarrow_{prop}$ steps, modulo symmetry, to a 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ of one of the kinds (R1), (R2), (R3) in Lem. 7.1. More precisely, for every ordered 1-bisimilarity redundancy $\langle u_1, u_2\rangle \in \langle 1BR\rangle(\underline{C})$ the following three statements hold:

(i) If $\langle u_1, u_2\rangle$ satisfies (K1), that is, $\neg(\,u_1 \leftarrow^{*} u_2\,)$ (and hence $scc(u_1) \neq scc(u_2)$), then $\langle u_1, u_2\rangle$ propagates to a 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ of kind (R1).

(ii) If $\langle u_1, u_2\rangle$ satisfies (K2), that is, $u_2 \curvearrowright^{+} u_1$ (and hence $scc(u_1) = scc(u_2)$), then $\langle u_1, u_2\rangle$ propagates trivially (by zero $\longrightarrow_{prop}$ steps) to the 1-bisimilarity redundancy $\langle w_1, w_2\rangle := \langle u_1, u_2\rangle$ of kind (R2).

(iii) If $\langle u_1, u_2\rangle$ satisfies (K3), that is, $u_1 \curvearrowright^{*} \overline{u}_1 \;{}_{d}\!\curvearrowright v \;{}_{d}\!\curvearrowleft \overline{u}_2 \circlearrowleft^{*} u_2 \wedge \neg(\overline{u}_1 \leftarrow^{*}_{\mathrm{bo}} \overline{u}_2)$ holds for some $\overline{u}_1, \overline{u}_2, v \in V$ (hence also $scc(u_1) = scc(u_2)$), then $\langle u_1, u_2\rangle$ propagates to a 1-bisimilarity redundancy $\langle u_1', u_2'\rangle$ such that there is a 1-bisimilarity redundancy $\langle w_1, w_2\rangle \in \{\langle u_1', u_2'\rangle, \langle u_2', u_1'\rangle\}$ (equal to $\langle u_1', u_2'\rangle$ or its converse), for which (precisely) one of the following statements holds:
 (a) $\langle w_1, w_2\rangle$ satisfies (R1),
 (b) $\langle w_1, w_2\rangle$ satisfies (R2) with $w_1 = v$ (thus $w_2 \curvearrowright^{+} w_1 = v$),
 (c) $\langle w_1, w_2\rangle$ satisfies (R3), and has a position $\langle w_1, \overline{w}_1, v, \overline{w}_2, w_2\rangle$ for some $\overline{w}_1, \overline{w}_2 \in V$ (thus $w_1 \curvearrowright^{*} \overline{w}_1 \;{}_{d}\!\curvearrowright v \;{}_{d}\!\curvearrowleft \overline{w}_2 \circlearrowleft^{*} w_2$ and $\neg(\overline{w}_1 \dashleftarrow^{*} \overline{w}_2)$).
 If actually $\overline{u}_1 \rightarrow^{*}_{\mathrm{bo}} \overline{u}_2$ holds (as a strengthening of $u_1 \curvearrowright^{*} \overline{u}_1 \;{}_{d}\!\curvearrowright v \;{}_{d}\!\curvearrowleft \overline{u}_2 \circlearrowleft^{*} u_2 \wedge \neg(\overline{u}_1 \leftarrow^{*}_{\mathrm{bo}} \overline{u}_2)$), then the following more specialized statement holds: $\langle w_1, w_2\rangle$ satisfies (R3), and has a position $\langle w_1, \overline{u}_1, v, \overline{w}_2, w_2\rangle$ for some $\overline{w}_2 \in V$ (note $\overline{u}_1$ instead of $\overline{w}_1$ as above, and thus $w_1 \curvearrowright^{*} \overline{u}_1 \;{}_{d}\!\curvearrowright v \;{}_{d}\!\curvearrowleft \overline{w}_2 \circlearrowleft^{*} w_2$ and $\neg(\overline{w}_1 \dashleftarrow^{*} \overline{w}_2)$).

Proof (with schematic illustrations). Due to Lem. A.59, and as we have to show a propagation result modulo symmetry of 1-bisimilarity redundancies, we may assume without loss of generality that $\langle u_1, u_2\rangle$ is a 1-bisimilarity redundancy of kind (K1), (K2), or (K3).

Here we focus on the most demanding part, the propagation of 1-bisimilarity redundancies of kind (K3). This is because:

- The argumentation for a 1-bisimilarity redundancy $\langle u_1, u_2\rangle$ of kind (K1) is analogous, but much simpler. It consists of an argumentation that if both of the 1-bisimilar vertices u_1 and u_2 are normed, then induction on the norms of u_1 and u_2 can be used to propagate along induced transitions from u_1 that reduce the norm, in order to show that always a 1-bisimilarity redundancy $\langle u_1, u_2\rangle$ of subkind (R1.2) is obtained. In case that u_1 and u_2 are not normed, then $\langle u_1, u_2\rangle$ is already a 1-bisimilarity redundancy of subkind (R1.1).

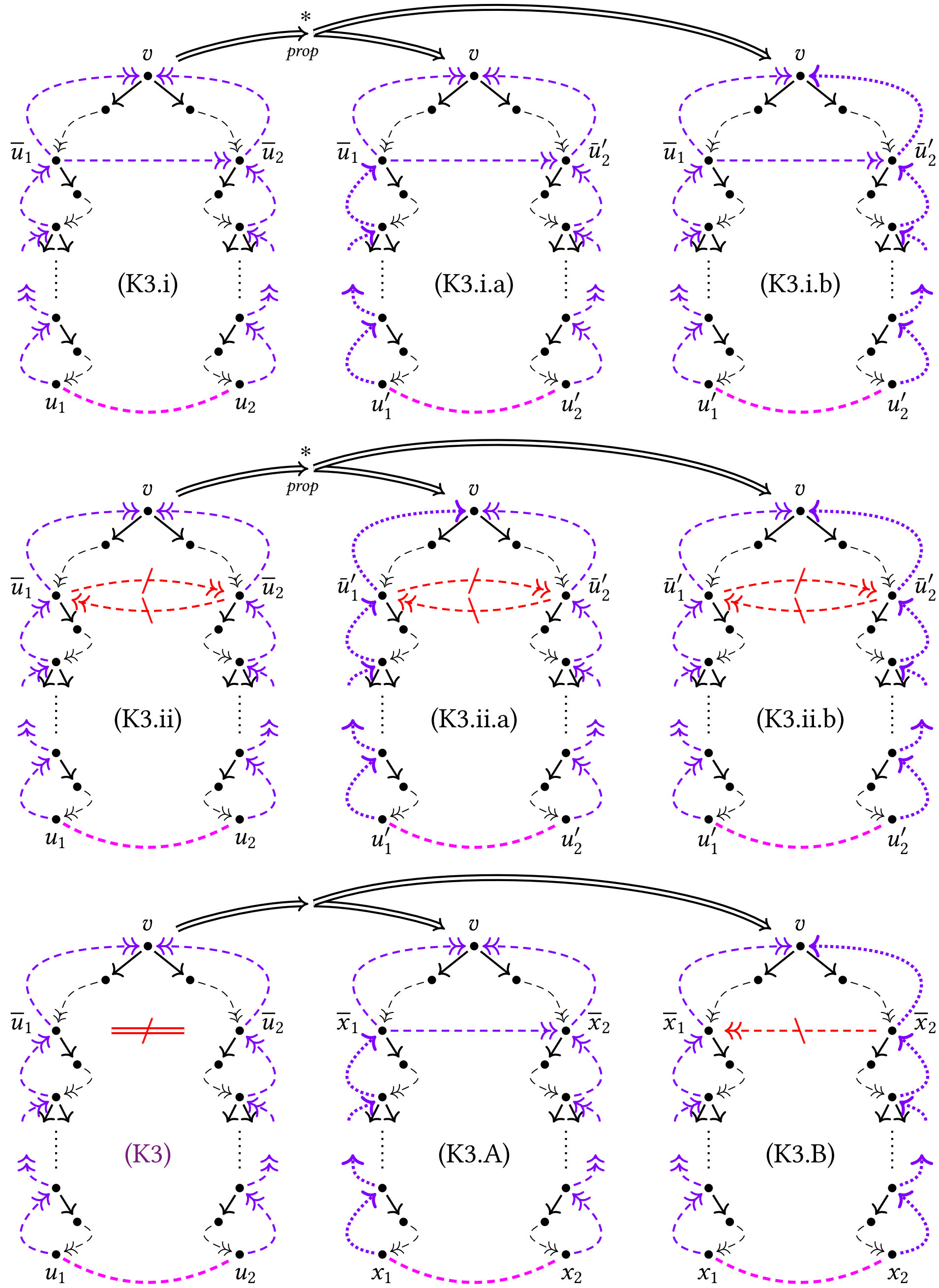


Figure 18. Propagation $\langle u_1, u_2 \rangle \rightarrow^*_{prop} \langle x_1, x_2 \rangle$ of a 1-bisimilarity redundancy $\langle u_1, u_2 \rangle$ at orthogonal positions (K3) in a loops-back-to part $(\circlearrowright^* v)$, thereby assuming that propagation stays within $(\circlearrowright^* v)$, to a 1-bisimilarity redundancy $\langle x_1, x_2 \rangle$ that will be recognized as reduced in Figure 19. The first two lines describe $\longrightarrow_{prop}$ steps for the cases (K3.i) and (K3.ii) of situation (K3). The third lines summarizes them to a $\longrightarrow_{prop}$ step that may be followed by a mirroring step, noting that case (K3.ii.a) can be mirrored to case (K3.B), and that case (K3.i.b) is also of form (K3.B) because the body transition path from u'_1 to u'_2 prevents a body transition path in the opposite direction (otherwise a $\rightarrow_{\text{bo}}$ cycle would arise, contradicting Lem. A.16,(i)).

Propagation is done by induction on the minimum of the induced-transition distances of w_1 to $\overline{w}_1$, and of w_2 to v (first line), on the minimum of the induced-transition distances of w_1 to v, and of w_2 to v (second line). In the induction steps an induced transition from w_1 or w_2 is chosen that decreases this minimum distance properly, and then matched via 1-bisimilarity by an induced transition from the other vertex.

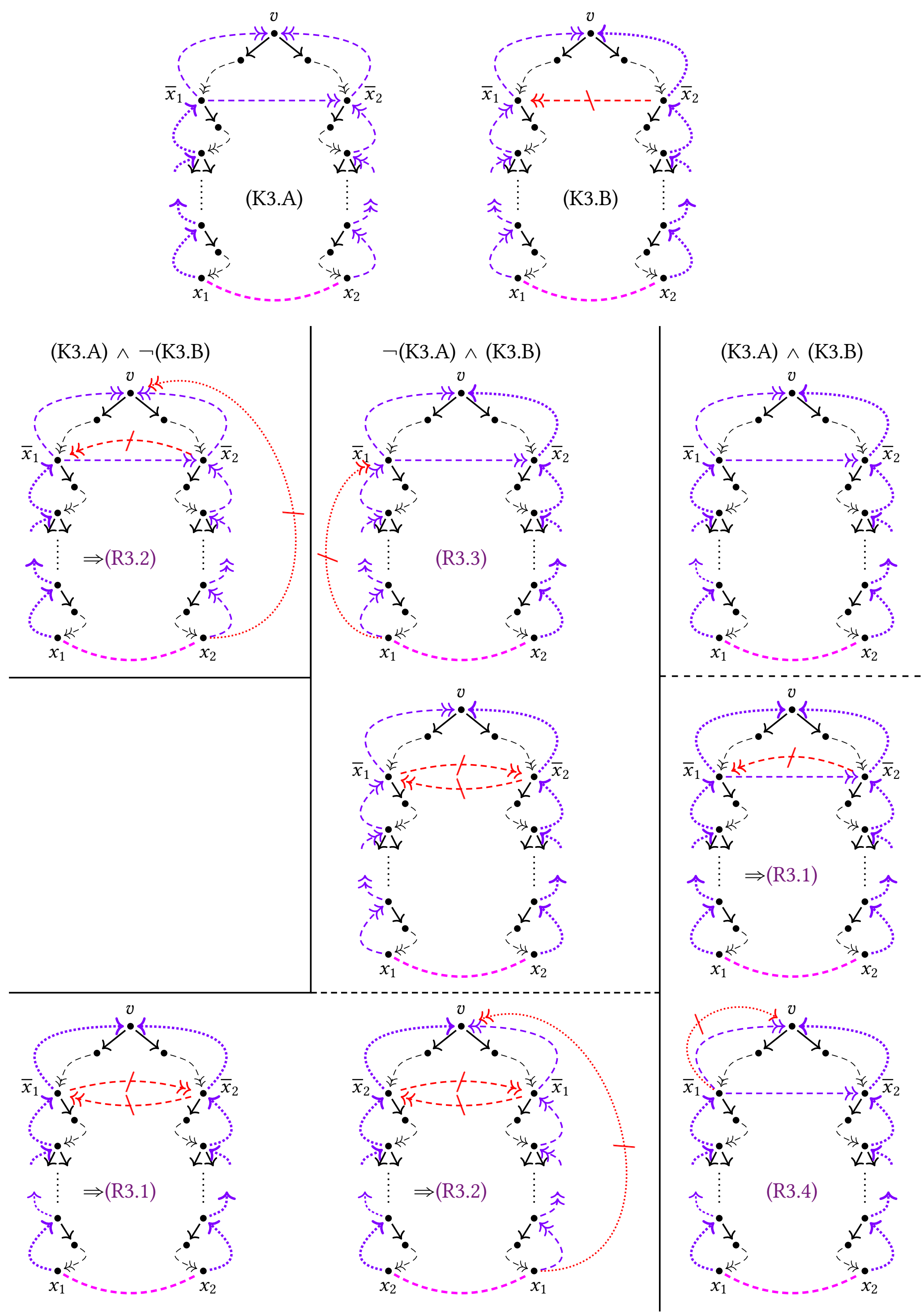


Figure 19. Exhaustive case analysis of the overlap (K3.A) ∧ (K3.B), and the differences (K3.A) ∧ ¬(K3.B) and ¬(K3.A) ∧ (K3.B) of the cases (K3.A) and (K3.B) of 1-bisimilarity redundancies $\langle x_1, x_2 \rangle$ obtained in Figure 18. The cases are reduced to (R3.1)–(R3.4) either directly, or (in the middle and right columns) with an intermediate step until the dashed separators, where a case that is not recognized as one of (R3.1)–(R3.4) is exhaustively analyzed further in order to finally obtain a case of (R3.1), (R3.2), or (R3.4).

- For 1-bisimilarity redundancies of kind (K2) the propagation step to be shown is trivial.

For the case of 1-bisimilarity redundancies $\langle w_1, w_2 \rangle$ at orthogonal positions (K3) in a loops-back-to part, the argumentation proceeds according to Figure 18 and Figure 19.

In particular, every 1-bisimilarity redundancy of kind (K1) can be propagated, and possibly mirrored, according to the illustration in Fig. 18, to a 1-bisimilarity redundancy of an already somewhat reduced kind. Propagation is done by induction on the minimum of the induced-transition distance of w_1 to $\overline{w}_1$, and the induced-transition distance of w_2 to v (in the first line of Fig. 18), on the minimum of the induced-transition distance of w_1 to v, and the induced-transition distance of w_2 to v (in the second line of Fig. 18). In the induction steps an induced transition from w_1 or w_2 is chosen that decreases this minimum distance properly, and then matched via 1-bisimilarity by an induced transition from the other vertex.

Then the exhaustive case analysis of Figure 19 is used to recognize the obtained reduced 1-bisimilarity redundancy as one of one of the cases (R3.1), (R3.2), (R3.3), or (R3.4), and thus of case (R3) in the lemma. ◻

A.6.3 Elimination of all but crystalline reduced 1-bisimilarity redundancies.

This section is devoted to the elimination of reduced 1-bisimilarity redundancies in a LLEE-preserving manner. We proceed by lemmas each of which concerns the elimination of a reduced 1-bisimilarity redundancy of the kinds (R1), (R2), and the subkinds (R3.1), (R3.2), (R3.3), (R3.4.1), (R3.4.2), respectively.

Lemma A.64. *Every reduced 1-bisimilarity redundancy of kind* (R1) *can be eliminated* LLEE*-preservingly from a 1-transition limited* LLEE *-1-chart.*

Proof (by reference to transformation I in [12, 13] for (R1.1)*, and adaptation for* (R1.2) *of transformation for* (R3.2)*).* Let $\langle w_1, w_2 \rangle$ be a 1-bisimilarity redundancy of kind (R1).

If $\langle w_1, w_2 \rangle$ is of subkind (R1.1), elimination of w_1 can take place LLEE-preservingly by connecting through (all incoming transitions) at w_1 over to w_2, just like transformation (I) in [12, 13].

If $\langle w_1, w_2 \rangle$ is of subkind (R1.2), then there are vertices $w_{1,1}, \ldots, w_{1,n-1}, \overline{w}_1$ such that $w_1 \mathrel{{}_{d}\!\curvearrowleft} w_{1,1} \mathrel{{}_{d}\!\curvearrowleft} \cdots \mathrel{{}_{d}\!\curvearrowleft} w_{1,n-1} \mathrel{{}_{d}\!\curvearrowleft} \overline{w}_1$ via 1-transition backlinks $w_1 \xrightarrow{1}_{\mathrm{bo}} w_{1,1} \xrightarrow{1}_{\mathrm{bo}} \cdots \xrightarrow{1}_{\mathrm{bo}} w_{1,n-1} \xrightarrow{1}_{\mathrm{bo}} \overline{w}_1$, and additionally $\overline{w}_1{\downarrow}$ holds. In this case it holds that: $w_1 \sqsupseteq w_{1,1} \sqsupseteq \cdots \sqsupseteq w_{1,n-1} \sqsupseteq \overline{w}_1$, that is, $w_1, w_{1,1}, \ldots, w1, n-1, \overline{w}_1$ are related by the converse $\sqsupseteq$ of the substate relation $\sqsubseteq$. Now since $w_1 \mathrel{\underline{\leftrightarrow}} w_2$ as $\langle w_1, w_2 \rangle$ is a 1-bisimilarity redundancy, it follows also that $\overline{w}_1 \sqsubseteq w_{1,n-1} \sqsubseteq \cdots \sqsubseteq w_{1,1} \sqsubseteq w_2 \mathrel{\underline{\leftrightarrow}} w_1$. Therefore every loop-entry transition from one of $\overline{w}_1, w_{1,n-1}, \ldots w_{1,1}$ can be redirected, in this order, to the 1-bisimilar target of an induced transitions from w_2.

By such redirections the loop vertices $\overline{w}_1, w_{1,n-1}, \ldots w_{1,1}$ above w_1 can be 'unraveled', in this order, similarly as we will do so again below in several cases, but most typically for kind (R3.2). After such unravelings, and elimination of 1-transitions, w_1 will be a $\curvearrowleft$-maximal vertex with termination. It can then be eliminated by connecting through w_1 to w_2 according to the LLEE-preserving and process-semantics preserving transformation I in [12, 13]. ◻

Lemma A.65. *Every reduced 1-bisimilarity redundancy of kind* (R2) *can be eliminated* LLEE*-preservingly from a 1-transition limited* LLEE *-1-chart.*

Proof (by reference to transformation II in [12, 13], and by an adaptation of this transformation). Let $\langle w_1, w_2 \rangle$ be a reduced 1-bisimilarity redundancy of kind (R2), that is, $w_2 \curvearrowleft^{+} w_1$ holds.

We distinguish the two possible cases in which either there is a chain of 1-transition backlinks from w_1 to w_2 (while there may also be other body transition paths from w_2 to w_1), or there is no such chain of 1-transition backlinks from w_1 to w_2.

In case that there is no such chain of 1-transition backlinks from w_1 to w_2 we can use transformation II from [12, 13] to show that the result of connecting (incoming transitions at) w_1 through to w_2 is a 1-bisimilar 1-chart that again satisfies LLEE.

If, however, there is a chain $w_2 \xrightarrow{1} w_{2,1} \xrightarrow{1} \cdots \xrightarrow{1} w_{2,n-1} \xrightarrow{1} w_1$ of 1-transitions that are backlinks witnessing $w_2 \mathrel{{}_{d}\!\curvearrowleft} w_{2,1} \mathrel{{}_{d}\!\curvearrowleft} \cdots \mathrel{{}_{d}\!\curvearrowleft} w_{2,n-1} \mathrel{{}_{d}\!\curvearrowleft} w_1$, then it holds that $w_2 \sqsupseteq w_{2,1} \sqsupseteq \cdots \sqsupseteq w_{2,n-1} \sqsupseteq w_1$ (that is, $w_2, w_{2,1}, \ldots, w_{2,n-1}, w_1$ are related by the converse $\sqsupseteq$ of the substate relation $\sqsubseteq$). Then due to $w_2 \mathrel{\underline{\leftrightarrow}} w_1$ it follows that $w_2 \mathrel{\underline{\leftrightarrow}} w_{2,1} \mathrel{\underline{\leftrightarrow}} \cdots \mathrel{\underline{\leftrightarrow}} w_{2,n-1} \mathrel{\underline{\leftrightarrow}} w_1$, which means that all vertices (which are loop vertices) on this loops-back-to path are 1-bisimilar. Then all of the vertices $w_2, w_{2,1}, \ldots, w_{2,n-1}$ can be connected through to w_1 LLEE-preservingly, making all of $w_2, w_{2,1}, \ldots, w_{2,n-1}$ unreachable. In this way the reduced 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ has been eliminated LLEE-preservingly and preserving the process semantics (by making, in any case, w_2 unreachable). ◻

Lemma A.66. *Every reduced 1-bisimilarity redundancy of subkind* (R3.1) *can be eliminated* LLEE*-preservingly from a 1-transition limited* LLEE *-1-chart.*

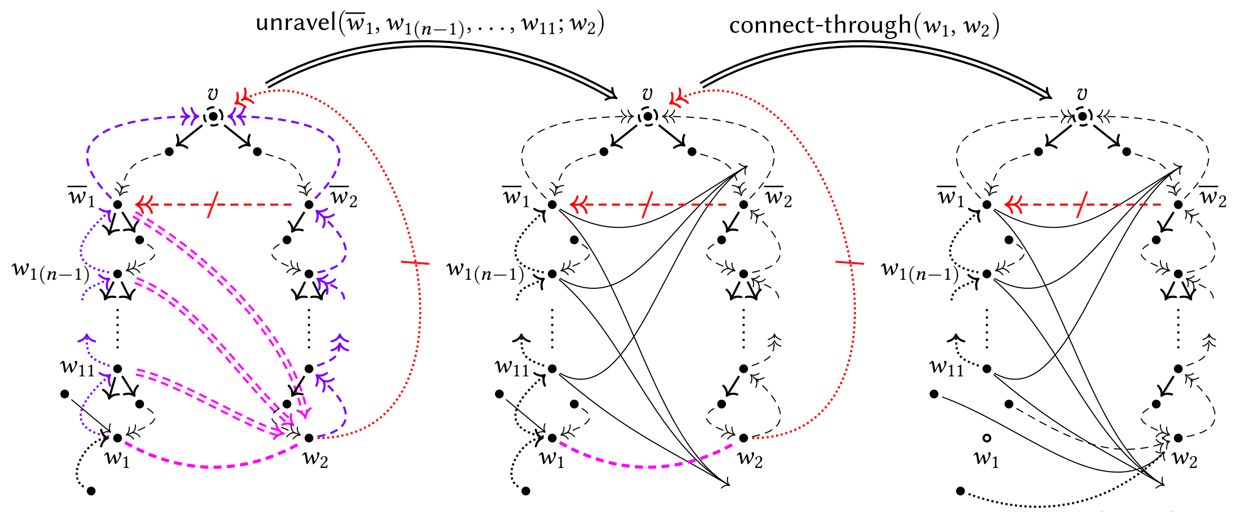


Figure 20. Illustration of the elimination from a 1-chart with LLEE−1-lim of a 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ (indicated as dashed magenta link) of kind (R3.2). By its form, $w_1, w_{11}, \ldots, w_{1(n-1)}, \overline{w}_1$ are substates of w_2 (indicated in magenta), that is, $\overline{w}_1 \sqsubseteq w_{1(n-1)} \sqsubseteq \cdots \sqsubseteq w_{11} \sqsubseteq w_1$ holds, and therefore proper transitions from any of these vertices can be mimicked by induced transitions from w_2. Therefore, in the first step, the loop vertices between w_1 and v can be unraveled by redirecting departing loop-entry transitions from each of $w_{11}, \ldots, w_{1(n-1)}, \overline{w}_1$ to targets of induced transitions from w_2. If, in the LLEE-1-chart obtained, w_1 still loops back to v, then w_1 directly loops back to v. In the second step, w_1 is connected through to w_2 by redirecting incoming transitions at w_1 over to w_2. The result is a 1-bisimilar LLEE-1-chart (not yet necessarily 1-transition limited) in which w_1 is now unreachable, and so the 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ has been eliminated.

Proof (Sketch of adaptation necessary for kind (R3.1) of transformation for kind (R3.2)). A reduced 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ of kind (R3.1) with position $\langle w_1, \overline{w}_1, v, \overline{w}_2, w_2 \rangle$ can be eliminated analogously to the elimination that we describe below for a reduced 1-bisimilarity redundancy of kind (R3.2). (Please see below.) Namely, by unraveling all loop vertices from $\overline{w}_1$ down to w_1 (see the proof of Lem. A.67 below), and then by connecting (incoming transitions at) w_1 through to w_2. However, in order to be able to perform the unravelling step, a preprocessing step is necessary here: we have to remove every loop-entry transition $u \xrightarrow{a}_{[m]} u'$ (which is a proper transition because we work on 1-charts with LLEE−1-lim) that can be mimicked by an induced transition $u \xrightarrow{1}{}^{+}_{\text{bo}} x \xrightarrow{a} x'$ with 1-bisimilar target $x' \leftrightarrow v'$ (where the mimicking induced transition first uses one or more 1-transitions backlinks from u). Such transitions are indeed superfluous with respect to the process semantics in the sense that we obtain a 1-bisimilar 1-chart by removing them. ◻

Lemma A.67. *Every reduced 1-bisimilarity redundancy of subkind (R3.2) can be eliminated* LLEE*-preservingly from a 1-transition limited* LLEE*-1-chart.*

Proof (by a graphical illustration of the transformation). In Fig. 20 we graphically represent the elimination of a reduced 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ of kind (R3.2).

The crucial step is the connect-w_1-through-to-w_2 operation that redirects all incoming transitions at w_1 over to the 1-bisimilar vertex w_2. However, performing this step immediately may create a 1-chart that does not satisfy LLEE any more, because w_1 may not loop back directly to the joining loop vertex v of w_1 and w_2, but w_1 may loop back via a non-empty sequence of 1-transitions to $\overline{w}_1$ (as indicated in (R3.2) and in Fig. 20). An example that explains this problem is Ex. A.13 in [12].

Therefore an intermediate preparation step is necessary in case that $w_1 \;{}_{d}\!\circlearrowright\; w_{11} \;{}_{d}\!\circlearrowright \cdots {}_{d}\!\circlearrowright\; w_{1(n-1)} \;{}_{d}\!\circlearrowright\; \overline{w}_1$ holds via 1-transition backlinks and $\overline{w}_1 \;{}_{d}\!\circlearrowright\; v$. Then the loop vertices $\overline{w}_1, w_{1(n-1)}, \ldots, w_{11}$ above w_1 have to be 'unraveled', in that order, by redirecting all loop-entry transitions over to 1-bisimilar vertices that are reachable by an induced transition from w_1, thereby turning them into body transitions, and turning $\overline{w}_1, w_{1(n-1)}, \ldots, w_{11}$ into 'common' vertices that are not loop vertices. This is possible, because due to the 1-transition backlinks from w_1 to $w_{11}, \ldots, w_{1(n-1)}, \overline{w}_1$ it follows that $w_{11}, \ldots, w_{1(n-1)}, \overline{w}_1$ are substates of w_1, and hence also of w_2.

As a consequence of these unraveling operations, w_1 may not any longer be reachable from v. In that case v has either become unreachable itself, and can be removed by garbage collection, or it just has fallen out of the scc of v, in which case the 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ now is a 1-bisimilarity redundancy of kind (R1) (which can be removed by an appeal to Lem. A.64).

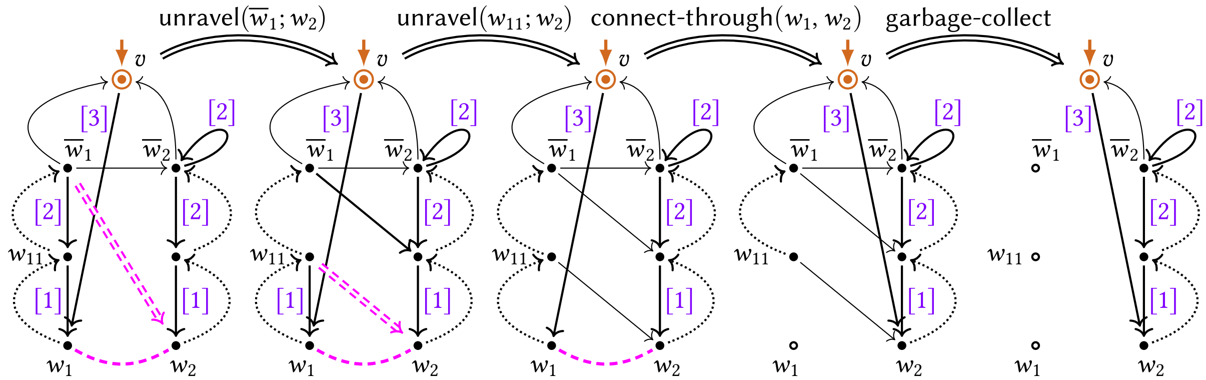


Figure 21. Example of the elimination of a 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ (indicated as dashed magenta link) of kind (R3.2) from a LLEE-1-chart according to the transformation illustrated in Figure 20. All transitions are assumed to carry the same action label. Here the two substeps of the step of unraveling the loop vertices $\overline{w}_1$ and w_{11} between w_1 and v are illustrated separately. These two steps are then succeeded by a connect-through step of w_1 to w_2, in which w_1 becomes unreachable, thereby eliminating the 1-bisimilarity redundancy $\langle w_1, w_2\rangle$, followed by garbage collection.

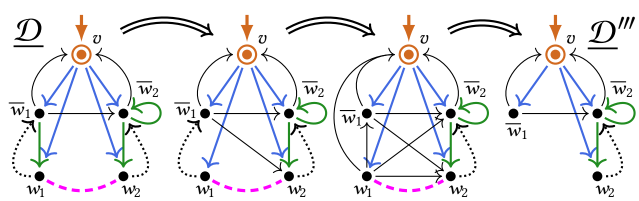
If, however, $\langle w_1, w_2\rangle$ is still a 1-bisimilarity redundancy of kind (R3.2) after the unravellings, then w_1 now loops back directly to v. In this situation the connect-w_1-through-to-w_2 operation from [12, 13] (Transformation III in the proof of Prop. 6.8) can be used to eliminate the reduced 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ by redirecting all incoming transitions at w_1 over to the 1-bisimilar vertex w_2, and thereby making w_1 unreachable.

In Fig. 20 the subsequent unraveling steps are pictured together as the first step, which then is followed by the connect-w_1-through-to-w_2 step. ◻

Example A.68. In Fig. 21 we give an example of the elimination of a 1-bisimilarity redundancy of form (R3.2) from a LLEE-1-chart according to the transformation illustrated in Fig. 20.

Example A.69. Also the example in Fig. 9 from the main text of the article is an example for the transformation described above for the elimination of a reduced 1-bisimilarity redundancy of kind (R3.2).

In the LLEE-1-charts below we assume that all action labels are the same, and indicate a LLEE-witness by green loop-entry transitions of level 1, and blue ones of level 2. The LLEE-1-chart $\underline{\mathcal{D}}$ on the left contains the pair $\langle w_1, w_2\rangle$ of 1-bisimilar vertices that is of reduced form (R3.2). Here w_1 can be eliminated by three steps two of which redirect transitions to 1-bisimilar targets:



The first step removes the loop at $\overline{w}_1$, the second restores LLEE−1-lim, and in the third w_1 is eliminated. The result is the 1-chart $\underline{\mathcal{D}}'''$ with LLEE−1-lim that is 1-bisimilar to $\underline{\mathcal{D}}$.

Next we focus on reduced 1-bisimilarity redundancies of kind (R3.3), and show that they, too, can be eliminated LLEE-preservingly from LLEE-1-charts. In order to eliminate such 1-bisimilarity redundancies $\langle w_1, w_2\rangle$ from a LLEE-1-chart $\underline{C}$ we will proceed indirectly: by transforming $\underline{C}$ into a 1-bisimilar LLEE-1-chart $\underline{C}'$ with the same or fewer reachable vertices in which the converse pair $\langle w_2, w_1\rangle$ is a 1-bisimilarity redundancy of kind (R3.2). Before detailing this transformation below in the proof of Lem. A.71, we start by explaining the transformation for a suggestive example.

Example A.70. See Figure 22 for an example of the transformation of a 1-bisimilarity redundancy of kind (R3.3) into a 1-bisimilarity redundancy of kind (R3.2).

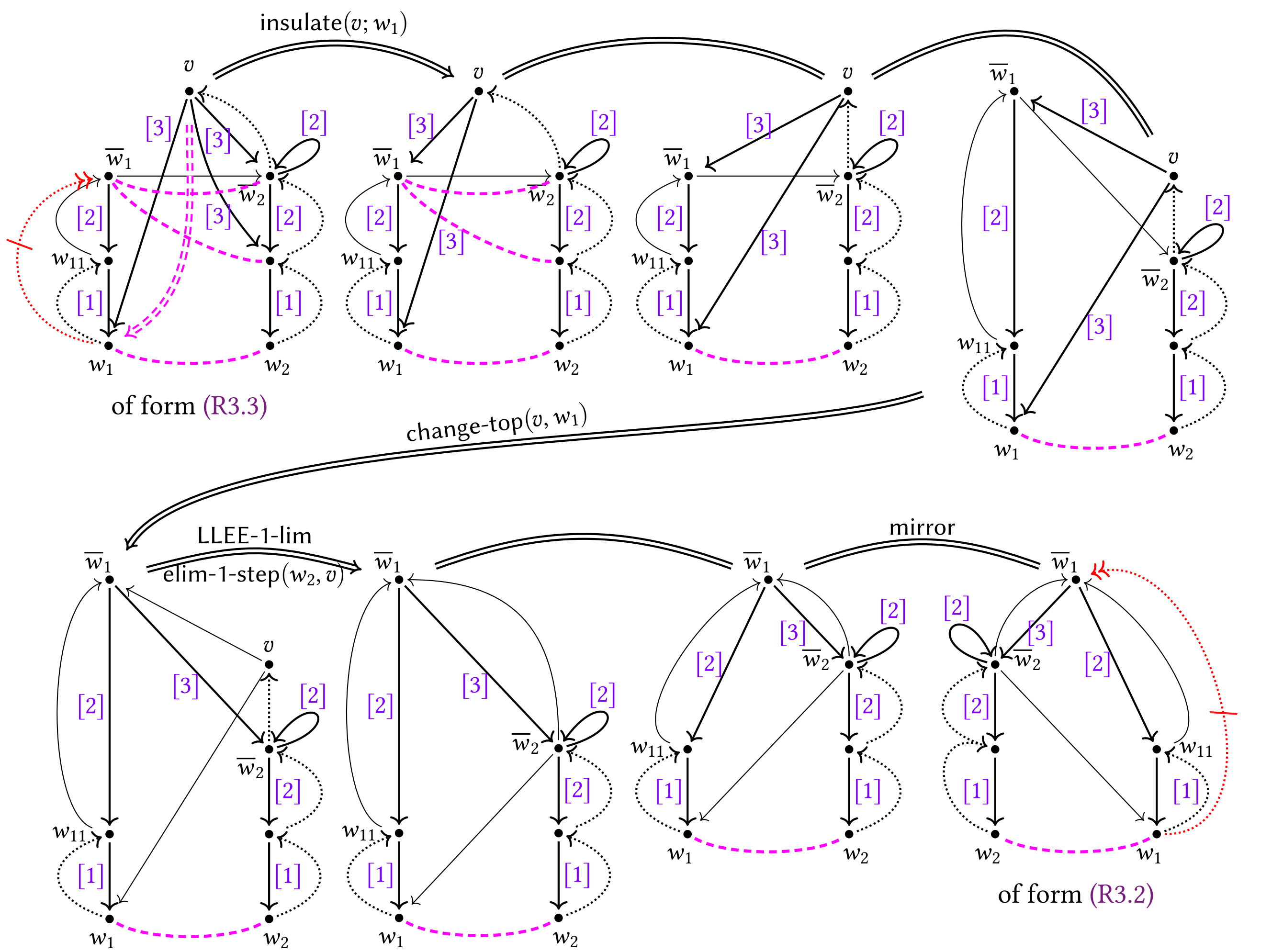


Figure 22. Example of the transformation steps for a 1-bisimilarity redundancy of kind (R3.3) in order to obtain a 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ of kind (R3.2). We assume that all transitions to carry the same action label. Transformation steps that do not change the 1-chart or its accompanying LLEE-witness but only deform it are indicated without arrow heads (as overlong $=$ signs). The form of (R3.3) implies that v is a substate of w_1. Therefore v can be insulated with respect to w_1, in the first step, with all transitions from v now directed to targets of induced transitions from w_1. The result is again a loops-back-to part of v without termination (which here it was all along, but the form has changed). After a rearrangement deformation that puts $\overline{w}_1$ on top, the form of the loops-back-to of v permits to change the entry/body-labeling such that the entry transitions at v are transferred over to $\overline{w}_1$. In this way the 1-chart becomes the loops-back-to part of $\overline{w}_1$. Then after eliminating 1-transitions and mirroring the 1-chart at a vertical line, a reduced 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ of kind (R3.2) is obtained.

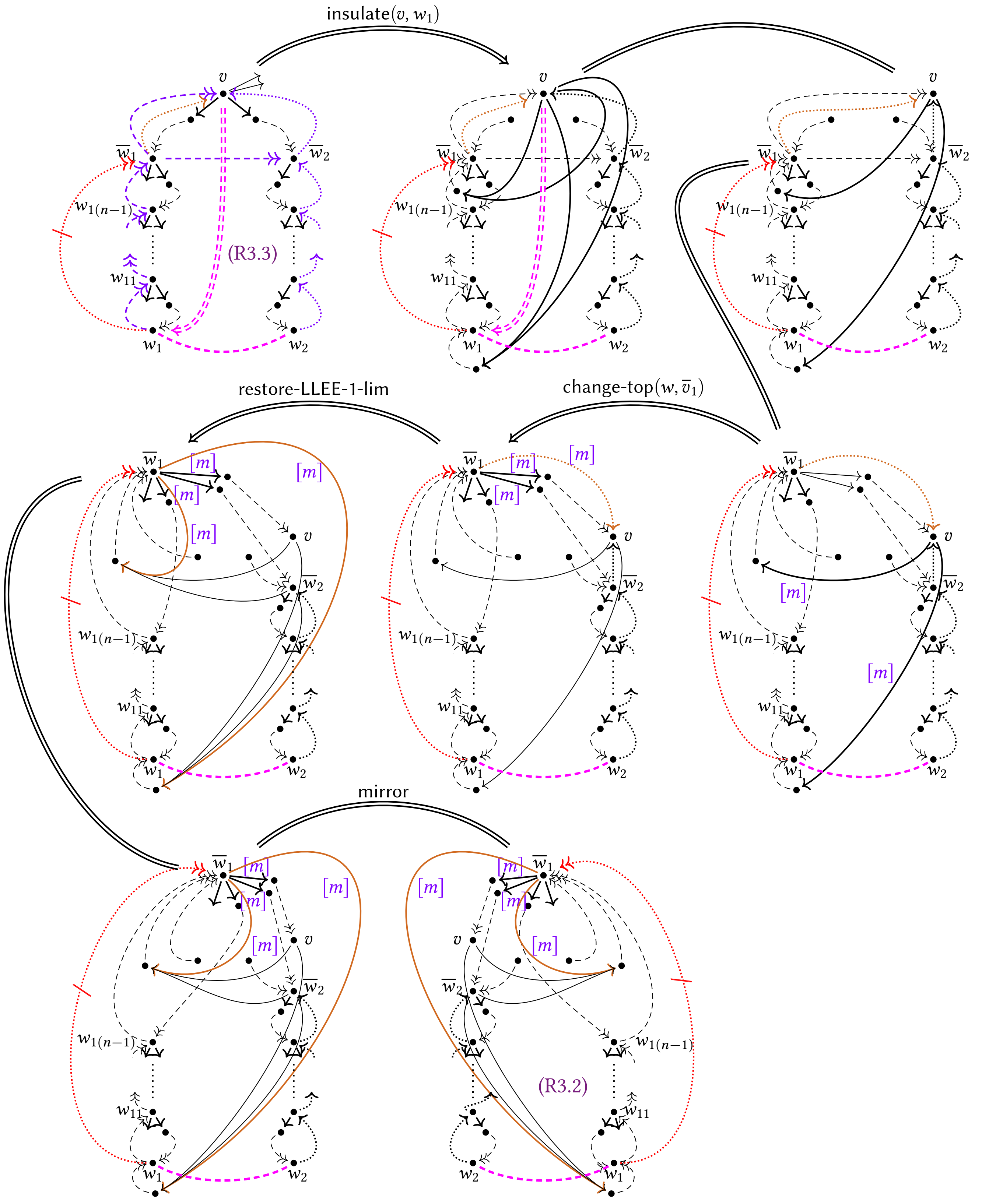


Figure 23. Illustration of the transformation of a 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ of kind (R3.3) into a 1-bisimilarity redundancy $\langle w_2, w_1\rangle$ of kind (R3.2). Transformation steps that do not change the 1-chart or its accompanying LLEE-witness but only deform it are indicated without arrow heads (as overlong = signs).

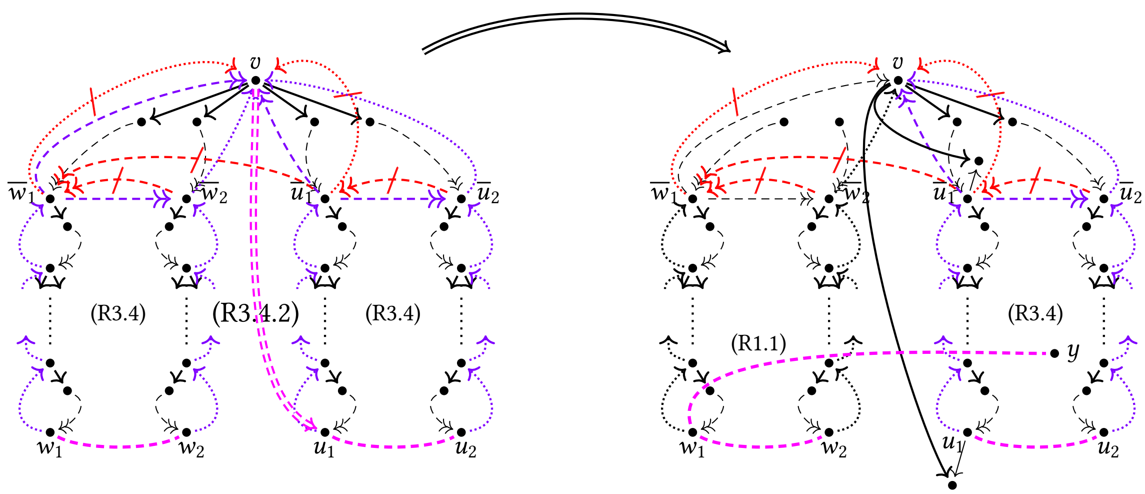


Figure 24. Illustration of the crucial step for eliminating a precrystalline reduced 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ of kind (R3.4.2): by transforming the underlying 1-chart so that w_1 then falls outside of the scc of v (which is the loops-back-to part of v). For this we use that, due to the form of the given 1-bisimilarity redundancy of kind (R3.4.2), v is a substate of u_1 (indicated as magenta dashed arrow left). This fact can be used to redirect every loop-entry transition from v to vertices that are reachable from u_1 by an induced transition. This step reduces the size of the scc of v, because now at least w_1 falls outside of the scc of v, and is in particular no longer reachable from v. As a consequence, w_1 must be 1-bisimilar to a vertex y in the decreased scc. In this way a 1-bisimilarity redundancy $\langle w_1, y \rangle$ and $\langle w_2, y \rangle$ of kind (R1.1) arises, which can be eliminated by Lem. 7.4. However, $\langle u_1, u_2 \rangle$ remains as a 1-bisimilarity redundancy of kind (R3.4) in the decreased scc.

Lemma A.71. *Every reduced 1-bisimilarity redundancy of subkind* (R3.3) *can be eliminated* LLEE*-preservingly from a 1-transition limited* LLEE *-1-chart.*

Proof. Rather than eliminating reduced 1-bisimilarity redundancies of subkind (R3.3) directly, we describe, in Figure 23, a transformation of a 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ of kind (R3.3) into a 1-bisimilarity redundancy $\langle w_2, w_1 \rangle$ of kind (R3.2) (which then can be eliminated according to Lem. A.67). ☐

Lemma A.72. *Every reduced 1-bisimilarity redundancy of subkind* (R3.4.1) *can be eliminated* LLEE*-preservingly from a 1-transition limited* LLEE *-1-chart.*

Proof. We note that the transformation process we described by illustrations in the proof of Lem. A.72 transforms a given reduced 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ of kind (R3.3) in the first step into a reduced 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ of kind (R3.4.1), which is then turned, by needed 1-transition eliminations and according and appropriate change of the accompanying LLEE-witness, into a reduced 1-bisimilarity redundancy of kind (R3.2). Therefore we can use this process described there also for the elimination of reduced 1-bisimilarity redundancies of kind (R3.4.1) by first transforming it into kind (R3.2), and then using Lem. A.67 in order to eliminate that reduced 1-bisimilarity redundancy of kind (R3.2). ☐

Lemma A.73. *Every reduced 1-bisimilarity redundancy of subkind* (R3.4.2) *can be eliminated* LLEE*-preservingly from a 1-transition limited* LLEE *-1-chart.*

Proof. A reduced 1-bisimilarity redundancy consists of two reduced 1-bisimilarity redundancies $\langle w_1, w_2 \rangle$ and $\langle u_1, u_2 \rangle$ of kind (R3.4) with overlapping joining loop vertex v, and thus with positions $\langle w_1, \overline{w}_1, v, \overline{w}_2, w_2 \rangle$ and $\langle u_1, \overline{u}_1, v, \overline{u}_2, u_2 \rangle$, respectively, and furthermore such that $\overline{w}_1$ is not reachable by a body transition path from $\overline{u}_1$.

A schematic illustration of a transformation for eliminating such a reduced 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ of kind (R3.4.2) with position $\langle w_1, \overline{w}_1, v, \overline{w}_2, w_2 \rangle$ is described by the picture in Fig. 24. We add the following additional explanation.

The asymmetry between the two reduced 1-bisimilarity redundancies $\langle w_1, w_2 \rangle$ and $\langle u_1, u_2 \rangle$ of kind (R3.4) that underlie the considered reduced 1-bisimilarity redundancy of kind (R3.4.2) can be used, together with the fact that $v \sqsubseteq u_1$ holds, that is, v is be a substate of u_2 (due to the form of 1-bisimilarity redundancies of kind (R3.4)), to redirect loop-entry transitions from v over to 1-bisimilar targets of induced transitions from u_2. In this way the loops-back-to part of v is decreased, as well as the scc of v, in such a way that w_1 now falls outside of it, and therefore $\langle w_1, w_2 \rangle$ is no longer a 1-bisimilarity redundancy of kind (R3.4.2).

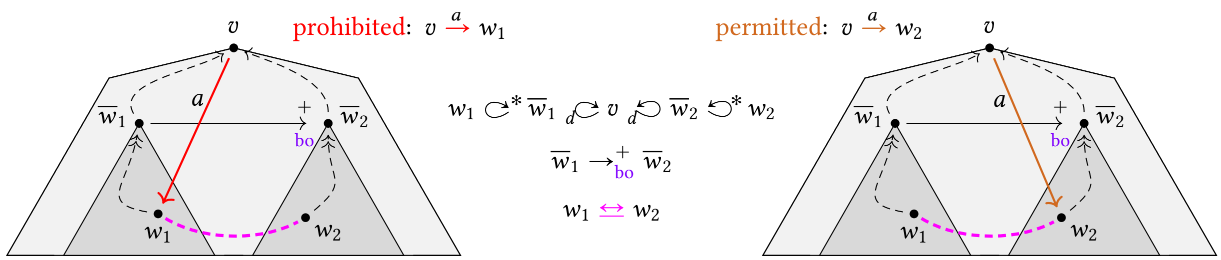


Figure 25. Illustration, on the left, of a situation that is prohibited for a parsimonious loop vertex v in a LLEE-1-chart: a proper transition from v would target the left vertex w_1 in a 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ of kind (K3) with position $\langle w_1, \overline{w}_1, v, \overline{w}_2, w_2\rangle$, thus with joining vertex v such that $\overline{w}_2$ is furthermore reachable via a body transition path from $\overline{w}_1$. The a-transition from v to w_1 does not make 'parsimonious use' of the loops-back-to part ($\curvearrowright^* v$) of v, because it could be replaced, see on the right, by a permitted a-transition to the 1-bisimilar target w_2, whose behavior makes less use, compared with w_1, of the loops-back-to part ($\curvearrowright^* v$) of v until v is reached again first. This is because whereas there is a path from w_1 to w_2 that does not visit v first (namely one via $\overline{w}_1$ and $\overline{w}_2$), there is no path from w_2 to w_1 that does not visit v first (otherwise a body transition path from $\overline{w}_2$ to $\overline{w}_1$ would arise, which would create a $\rightarrow_{\mathrm{bo}}$ cycle with the body transition path from $\overline{w}_1$ to $\overline{w}_2$, contradicting the property that LLEE-witnesses do not contain $\rightarrow_{\mathrm{bo}}$ cycles).

Indeed, the transformed LLEE-1-chart must then contain a 1-bisimilarity redundancy $\langle w_1, y\rangle$ with y in the decreased scc. This is because u in the decreased scc is 1-bisimilar to u in the original scc, and since v_1 is reachable from u in the original scc it follows that the decreased scc must contain a vertex y that is 1-bisimilar to v_1. This 1-bisimilarity redundancy $\langle w_1, y\rangle$ is of kind (K1). Indeed, it is of kind (R1.1), because w_1 and w_2 are not normed as vertices of a reduced 1-bisimilarity redundancy of kind (R3.4), and therefore also y is not normed. The 1-bisimilarity redundancy $\langle w_1, y\rangle$ can then be eliminated by Lem. A.64. ◻

Lemma A.74 (= Lem. 7.4)**.** *Every reduced 1-bisimilarity redundancy can be eliminated* LLEE*-preservingly from a 1-transition limited* LLEE*-1-chart provided it is of either of the following kinds:*

(i) $\langle w_1, w_2\rangle$ is simple, or

(ii) $\langle w_1, w_2\rangle$ is precrystalline, but not crystalline.

Proof. Statement (i) of the lemma follows from the cumulative statements of Lem. A.64, Lem. A.65, Lem. A.66, Lem. A.67, and Lem. A.71, which guarantee that every simple reduced 1-bisimilarity redundancy (hence of the kinds (R1), (R2), and the subkinds (R3.1), (R3.2), (R3.3)) can be eliminated LLEE-preservingly.

Statement (ii) of the lemma follows from Lem. A.72, and Lem. A.73, which guarantee that every precrystalline reduced 1-bisimilarity redundancy that is not crystalline (hence of the kinds (R3.4.1) or (R3.4.2)) can be eliminated LLEE-preservingly. ◻

A.6.4 Parsimonious insulation of precrystalline 1-bisimilarity redundancies.

In this section we are concerned with 'parsimonious insulation from above' of 1-bisimilarity redundancies of reduced kinds (R3.3) and the precrystalline ones of kind (R3.4). We first have to define formally when we call a loop vertex in a LLEE-witness 'insulated from above', and when we call it 'parsimonious'. Also, we extend define when reduced 1-bisimilarity redundancies of kinds (R3.3) and (R3.4) are 'parsimoniously insulated' from above. After providing an example, we show in Lem. A.78 that 1-bisimilarity redundancies of reduced kinds (R3.3) and (R3.4) can indeed always be parsimoniously insulated from above.

Definition A.75 (insulated (from above), and parsimonious loop vertices)**.** Let $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a 1-chart with LLEE−1-lim, and specifically, let $\hat{\underline{C}}$ be a LLEE-witness such that $\underline{C}$ is 1-transition limited with respect to $\hat{\underline{C}}$. Let $\curvearrowright^*$ be the loops-back-to relation defined on $\underline{C}$ by $\hat{\underline{C}}$. For loop vertices $v \in LV(\hat{\underline{C}})$ we define the following two properties:

(i) We say that the loop vertex v *is insulated from above in* $\hat{\underline{C}}$ if v is $\curvearrowright$-maximal.

(ii) We say that v *is parsimonious* if there is no proper transition from v to the vertex w_1 in a 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ in $\underline{C}$ for which v is the joining vertex, and there are $\overline{w}_1, \overline{w}_2 \in V$ such that:

$$w_1 \curvearrowright^* \overline{w}_1 \,{}_d\!\curvearrowright v \,{}_d\!\curvearrowleft \overline{w}_2 \curvearrowleft^* w_2 \;\wedge\; \overline{w}_1 \rightarrow^+_{\mathrm{bo}} \overline{w}_2\,,$$

that is, $\langle w_1, w_2\rangle$ is of kind (K3) with v as joining loop vertex, and with an additional body-step transition reachability condition. See Fig. 25 on the left for an illustration of the situation of a loop vertex v for which a transition violates the defining clause for 'v is parsimonious loop vertex of $\hat{\underline{C}}$'.

Based on this terminology we now explain what we mean by saying that a reduced 1-bisimilarity redundancy of kind (R3) is '(parsimoniously) insulated from above'.

Definition A.76 ((parsimoniously) insulated from above 1-bisimilarity redundancies of kind (R3))**.** $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a 1-chart with LLEE–1-lim, and let $\hat{\underline{C}}$ be a 1-transition limited LLEE-witness of $\underline{C}$ with loops-back-to relation $\curvearrowright$. Let $\langle w_1, w_2\rangle$ be a 1-bisimilarity redundancy of kind (R3) with respect to $\hat{\underline{C}}$.

We say that $\langle w_1, w_2\rangle$ in $\underline{C}$ *is parsimoniously insulated from above* if its joining loop vertex is parsimonious, and insulated from above.

Example A.77. We consider the three LLEE-1-charts in Fig. 26, in which we assume that all action labels of proper transition labels are the same (but the dotted arrows still represent 1-transitions as usual).

In the LLEE-1-chart on the left, the loop vertex v is not insulated from above, because v loops-back-to u, and therefore v is not $\curvearrowright$-maximal. However, in the middle and right LLEE-1-charts, v is insulated from above, because there v is $\curvearrowright$-maximal.

Since v is the joining loop vertex of the reduced 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ in all three of these LLEE-1-charts, $\langle w_1, w_2\rangle$ is not insulated from above in the left LLEE-1-chart, but insulated from above in the middle and right LLEE-1-chart.

In the LLEE-1-chart on the middle, the loop vertex v is not parsimonious, because the loop-entry transition from v to $\overline{w}_1$ can be redirected to 1-bisimilar target w_2, thereby making less use of the loops-back-to part of v (and reducing it). The same holds for v in the LLEE-1-chart on the left. In the LLEE-1-chart on the right, however, v is parsimonious.

Consequently, $\langle w_1, w_2\rangle$ is not parsimoniously insulated from above in the left and middle LLEE-1-chart, but it is indeed parsimoniously insulated from above in the right LLEE-1-chart.

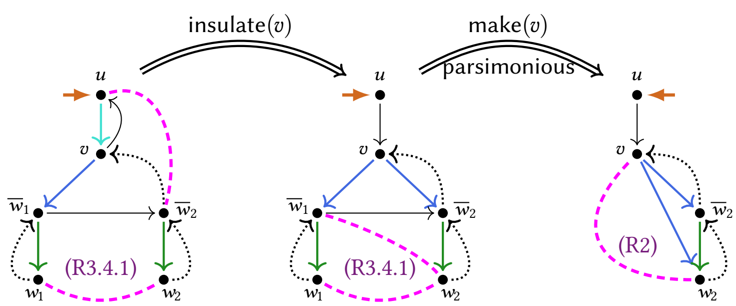


Figure 26. Example of a parsimonious-insulation transformation for a reduced 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ of kind (R3.4.1) in two steps. In the first step v is insulated from above, but not yet made parsimonious. The second step here turns v also into a parsimonious loop vertex. As a consequence the reduced 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ is eliminated here in two steps, and, in this specific situation, a reduced 1-bisimilarity redundancy $\langle v, w_2\rangle$ of kind (R2) arises.

The example in Fig. 26 is also a simple example for the transformation that turns a reduced 1-bisimilarity redundancy of kind (R3.3) or (R3.4) into one that is parsimoniously insulated from above, or eliminates it.

Lemma A.78 (LLEE–1-lim preserving parsimonious insulation for reduced 1-bisimilarity redundancies of kind (R3.3) and (R3.4))**.** *Let $\underline{C}$ be a* LLEE-1*-chart with 1-transition limited* LLEE*-witness $\hat{\underline{C}}$.*

Then every $\hat{\underline{C}}$-reduced 1-bisimilarity redundancy of kind (R3.3) *or* (R3.4) *can be* LLEE*-preservingly parsimoniously insulated from above, or eliminated, with a 1-bisimilar* LLEE-1*-chart $\underline{C}'$ and* LLEE*-witness $\hat{\underline{C}}'$ of $\underline{C}'$ as the result.*

Consequently, every precrystalline $\hat{\underline{C}}$-reduced 1-bisimilarity redundancy in $\underline{C}$ can be LLEE*-preservingly insulated from above, or eliminated.*

Proof (Sketch). We consider a reduced 1-bisimilarity redundancy $\langle w_1, w_2\rangle$ in $\underline{C}$ with respect to $\hat{\underline{C}}$ of kind (R3.3) or (R3.4) with position $\langle w_1, \overline{w}_1, v, \overline{w}_2, w_2\rangle$. This implies $v \sqsubseteq w_2$ (v is a substate of w_2) due to $w_2 \xrightarrow{1}{}^{+}_{\text{bo}} v$ in kinds (R3.3) and (R3.4), and, because $w_1 \leftrightarrow w_2$ holds as $\langle w_1, w_2\rangle$ is a 1-bisimilarity redundancy, also $v \sqsubseteq w_1$ (v is also a substate of w_1). Now $v \sqsubseteq w_1$ implies, together with $\neg(w_1 \xrightarrow{1}{}^{*} v)$ that holds due to the form of (R3.3) and (R3.4) (there is no 1-transition path from w_1 to v), that for every induced transition from v there is a 1-bisimilar vertex that is reachable by an induced transition from w_1, which, due to $\neg(w_1 \xrightarrow{1}{}^{*} v)$ and $w_1 \curvearrowright^{+} v$, must be contained in $(v {\rightharpoonup}^{*})_{\hat{\underline{C}}}$.

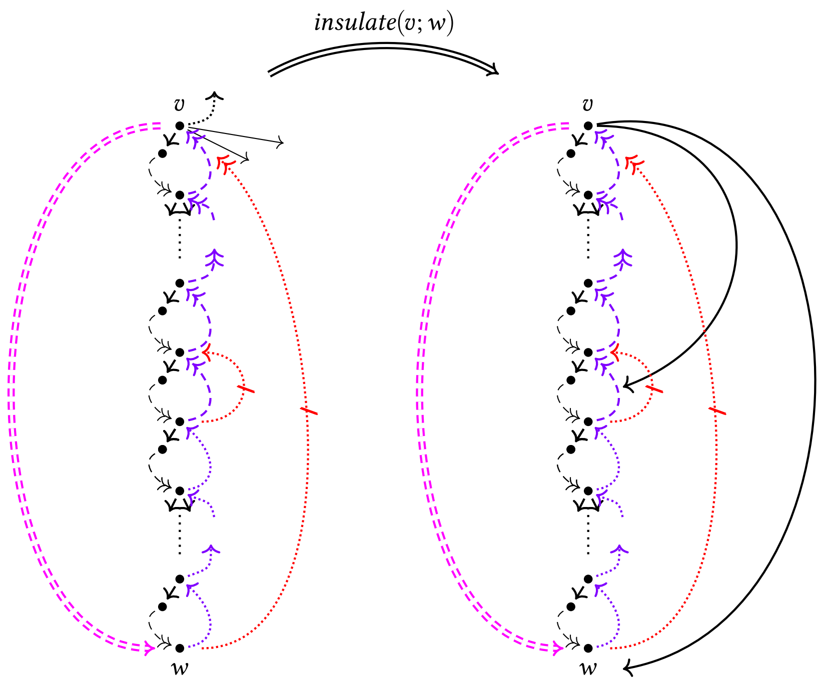


Figure 27. Illustration of an insulation step for a loop vertex v that is a substate of a vertex w in its loops-back-to part $(\circlearrowleft^{*} v)$ (that is, $v \sqsubseteq w$ holds, indicated by $\Rightarrow$, and $w \circlearrowleft^{*} v$), but not insulated from above (that is, v is not $\circlearrowleft$-maximal). The transformation redirects induced transitions from v that leave the descends-in-loop-to part $(v \curvearrowright^{*})$ of v as proper transitions to 1-bisimilar targets of induced transitions from w in $(v \curvearrowright^{*})$. In this way, v is turned into a loop vertex that is $\circlearrowleft$-maximal, and that therefore is insulated from above.

Therefore every induced transition from v that starts along some body transition from v and targets a vertex outside of $(v \curvearrowright^{*})_{\hat{\underline{C}}}$ can be mimicked by a proper transition from v to a 1-bisimilar target inside of $(v \curvearrowright^{*})_{\hat{\underline{C}}}$. We can redirect such induced transitions accordingly, thereby also turning them into loop-entry transitions from v (see the more schematic situation in Fig. 27). In this way v is turned into a loop vertex that is $\circlearrowleft$-maximal, and hence is insulated from above. Additionally, care can be taken to add redirected induced transitions from v only in a parsimonious manner, and to also redirect existing proper loop-entry transitions from v that violate the defining condition for v to be parsimonious. This transformation may decrease the loops-back-to part $(\circlearrowleft^{*} v)_{\hat{\underline{C}}}$ of v just like that is the case in the two steps of this transformation in the example in Fig. 25.

The result of this transformation is a LLEE-1-chart $\underline{C}'$ with LLEE-witness $\hat{\underline{C}}'$ in which $\langle w_1, w_2 \rangle$, if still $w_1 \in (\circlearrowleft^{*} v)_{\widehat{\underline{C}'}}$ holds, is a 1-bisimilarity redundancy of the same kind (R3.3) or (R3.4) that is parsimoniously insulated from above; in case that $w_1 \notin (\circlearrowleft^{*} v)_{\widehat{\underline{C}'}}$ holds, w_1 is either unreachable now, and then the 1-bisimilarity redundancy has been removed altogether, or otherwise $\langle w_1, w_2 \rangle$ is now a 1-bisimilarity redundancy of kind (K1), which can be removed by repeated elimination of reduced 1-bisimilarity redundancies of kind (R1). ◻

A.6.5 Grounding of scc's.

The transformation of LLEE-preservingly grounding scc's in a LLEE-1-chart is rather straightforward, as we explain in the proof of the following lemma.

Lemma A.79. *Every* LLEE-1-*chart in which P is the carrier of an scc can be transformed, by redirecting transitions that depart from P, into a 1-bisimilar* LLEE-1-*chart that is grounded for P.*

Proof. Let $\underline{C}$ be a LLEE-1-chart with LLEE-witness $\hat{\underline{C}}$, and where P is the carrier of an scc.

We may assume that transitions in $\hat{\underline{C}}$ that leave P are body transitions. This is because any such transition cannot induce a loop sub-1-chart, as its target does not have a path back to its source. Therefore such transitions can be relabeled as body transitions while preserving that the entry/body-labeling remains a LLEE-witness of $\underline{C}$.

In order to obtain the property that P is grounded in $\underline{C}$ for a LLEE-preserving and process-semantics preserving transformed 1-chart we have to ensure that any two transitions with sources in P to 1-bisimilar vertices outside of P have the same target.

This can be reached by a finite number of appropriate redirections of transitions of $\underline{C}$ and $\hat{\underline{C}}$. Any changes of marking labels of transitions are no longer necessary. Since the transitions that leave P are body transitions that are not parts of cyclic parts of loop sub-1-charts of $\underline{C}$, these transitions can stay body transitions also after redirections while preserving a LLEE-witness. □

A.6.6 Construction of crystallized LLEE-1-charts.

Here we repeat the definition of crystallized 1-charts, we justify the main crystallization statement on the basis of transformations that we have developed, and we prove and argue for important properties of crystallized 1-charts.

Definition (= Def. 7.5, crystallized 1-chart ((4))). Let $\underline{C}$ be a 1-chart.

We say that $\underline{C}$ *is crystallized* if there is a LLEE-witness $\hat{\underline{C}}$ of $\underline{C}$ such that the following four conditions hold:

(cr-1) $\underline{C}$ is a finite 1-chart with LLEE−1-lim, and specifically, $\underline{C}$ is 1-transition limited with respect to $\hat{\underline{C}}$.

(cr-2) Every $\hat{\underline{C}}$-reduced 1-bisimilarity redundancy in $\underline{C}$ is crystalline.

(cr-3) Every crystalline $\hat{\underline{C}}$-reduced 1-bisimilarity redundancy in $\underline{C}$ is parsimoniously insulated from above.

(cr-4) Every carrier of an scc in $\underline{C}$ is grounded in $\underline{C}$.

Then we also say that $\underline{C}$ is *crystallized with respect to* $\hat{\underline{C}}$.

Lemma A.80 (= preparation for Thm. 7.9, crystallization). *Every finite, guarded* LLEE-1*-chart* $\underline{C}$ *can be transformed, together with a* LLEE*-witness* $\hat{\underline{C}}$ *of* $\underline{C}$*, by repeated operations of the following four kinds:*

(Cr-1) *transformation of a finite, w.g.* LLEE-1*-chart into a* 1*-chart with* LLEE−1-lim*,*

(Cr-2) *elimination of a reduced* 1*-bisimilarity redundancy that is simple, or precrystalline (but not crystalline),*

(Cr-3) *parsimonious insulation from above of a crystalline reduced* 1*-bisimilarity redundancy,*

(Cr-4) *grounding of an scc,*

where each one preserves the process semantics, (Cr-1) *preserves* LLEE*, and* (Cr-2)–(Cr-4) *preserve* LLEE−1-lim*, into a* 1*-chart* $\underline{C}'$ *that is* 1*-bisimilar to* $\underline{C}$*, and a* 1*-transition limited* LLEE*-witness* $\hat{\underline{C}}'$ *such that* $\underline{C}$ *is crystallized with respect to* $\hat{\underline{C}}'$*.*

Proof (Sketch). The transformations stated by the steps (Cr-1), (Cr-2), (Cr-3), (Cr-4) in the theorem are justified by the statements of Lem. A.17, Lem. 7.4, Lem. A.78, and Lem. A.79, respectively. Termination of applications of transformations (Cr-2)–(Cr-4) in an arbitrary order can be argued as follows: every application of (Cr-2) properly reduces the number of reachable vertices; every application of (Cr-3) splits at least one scc (loops-back-to part), without merging others, and without increasing the number of reachable vertices; every application of (Cr-4) removes a groundedness violation for the considered scc, without affecting the measures for the other transformations. It is therefore not difficult to prove termination of these rules by using a lexicographic order of the mentioned measures as well-founded order that is decreased in any transformation step. □

Lemma (= Lem. 7.6). *Every* 1*-chart* $\underline{C}$ *that is crystallized with respect to a* LLEE*-witness* $\hat{\underline{C}}$ *with loops-back-to relation* $\curvearrowleft$ *satisfies:*

(i) $\underline{C}$ *is* 1*-collapsed apart from within scc's, hence* 1*-collapsed for loops-back-to parts of* $\hat{\underline{C}}$*-maximal loop vertices of* $\hat{\underline{C}}$*.*

(ii) $\underline{C}$ *is* 1*-collapsed for every loops-back-to part of a loop vertex of* $\hat{\underline{C}}$ *that is not* $\curvearrowleft$*-maximal.*

Proof. Statement (i) of the lemma is an easy consequence of the absence of 1-bisimilarity redundancies of kind (K1) in the crystallized 1-chart $\underline{C}$: this fact is in turn a consequence of the absence of reduced 1-bisimilarity redundancies of kind (R1) in the crystallized 1-chart $\underline{C}$ due to condition (cr-2) on $\underline{C}$, and the fact that 1-bisimilarity redundancies of kind (K1) propagate to 1-bisimilarity redundancies of kind (R1) due to Lem. A.63, (i).

Statement (ii) of the lemma is a consequence of the fact that crystalline reduced 1-bisimilarity redundancies, which are of kind (R3.4) (but not of kind (R3.4.1) or (R3.4.2)), can be insulated from above by Lem. A.78, and of the propagation statement (iii) in Lem. A.63.

We prove statement (ii) of the lemma indirectly. We assume that there is a vertex $v \in LV(\hat{\underline{C}})$ in $\underline{C}$ that is not $\curvearrowleft$-maximal, and such that $\underline{C}$ is not 1-collapsed for its loops-back-to part $(\curvearrowleft^* v)$. From this assumption we will derive a contradiction.

Since $\underline{C}$ is not 1-collapsed for $(\curvearrowleft^* v)$, we can pick an unordered 1-bisimilarity redundancy in $(\curvearrowleft^* v)$ that, due to Lem. A.59, has an ordered representation $\langle u_1, u_2 \rangle$ of kind (K1), (K2), or (K3) (with u_1 for w_1, and u_2 for w_2). We note that, as it contains the two distinct vertices u_1 and u_2 of the 1-bisimilarity redundancy $\langle u_1, u_2 \rangle$, the loops-back-to part $(\curvearrowleft^* v)$ cannot be trivial.

Now since $u_1, u_2 \in (\curvearrowleft^* v)$ we have $u_1 \curvearrowleft^* v$ and $u_2 \curvearrowleft^* v$. Then it follows from Lem. A.16, (vi), that u_1 and u_2 are in a joint scc of $\underline{C}$. Consequently $\langle u_1, u_2 \rangle$ cannot be of kind (K1), but must be of kind (K2) or (K3). We distinguish these two cases:

Case 1: $\langle u_1, u_2 \rangle$ is of kind (K2), that is, $u_1 \curvearrowleft^{+} u_2$ holds.

Then $\langle u_1, u_2 \rangle$ is also a $\hat{\underline{C}}$-reduced 1-bisimilarity redundancy of kind (R2) in Lem. 7.1. Hence $\underline{C}$ contains a simple $\hat{\underline{C}}$-reduced 1-bisimilarity redundancy. But this contradicts the condition (cr-2) that holds for $\underline{C}$ as a crystallized 1-chart with respect to $\hat{\underline{C}}$, according to which $\underline{C}$ may contain only crystalline, but neither simple nor precrystalline $\hat{\underline{C}}$-reduced 1-bisimilarity redundancies.

Case 2: $\langle u_1, u_2 \rangle$ is of kind (K3), that is, $u_1 \curvearrowright^{*} \overline{u}_1 \;{}_{d}\!\curvearrowright v_0 \;{}_{d}\!\curvearrowleft \overline{u}_2 \curvearrowleft^{*} u_2$ and $\neg(\overline{u}_1 \leftarrow^{*}_{\mathrm{bo}} \overline{u}_2)$ (and hence $\overline{u}_1 \neq \overline{u}_2$) holds from some $\overline{u}_1, \overline{u}_2, v_0 \in V$.

Then it follows by Lem. A.16, (xiv), that v_0 is the l.u.b. of u_1 and u_2 with respect to $\curvearrowright^{+}$. This implies, due to $u_1 \curvearrowright^{*} v \curvearrowleft^{*} u_2$, that $v_0 \curvearrowright^{*} v$. Since v is not $\curvearrowright$-maximal, it follows that v_0 is not $\curvearrowright$-maximal, either.
To the 1-bisimilarity redundancy $\langle u_1, u_2 \rangle$ we can apply Lem. A.63, (iii), We obtain that $\langle u_1, u_2 \rangle$ propagates to a $\hat{\underline{C}}$-reduced 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ of kind (R1), (R2), or (R3). Now since $\underline{C}$ is crystallized with respect to $\hat{\underline{C}}$, and hence by (cr-2) contains only reduced 1-bisimilarity redundancies that are crystalline, it follows that $\langle w_1, w_2 \rangle$ must be crystalline, and thus of subkind (R3.4). From the application of Lem. A.63, (iii), to $\langle u_1, u_2 \rangle$ that we have performed, we then also obtain that $\langle w_1, w_2 \rangle$ is a crystalline $\hat{\underline{C}}$-reduced 1-bisimilarity redundancy with position $\langle w_1, \overline{w}_1, v_0, \overline{w}_2, w_2 \rangle$ and thus with joining loop vertex v_0 for some $\overline{w}_1, \overline{w}_2 \in V$.
But then, as v_0 is not $\curvearrowright$-maximal, the crystalline reduced 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ is not insulated from above. Hence $\langle w_1, w_2 \rangle$ cannot be parsimoniously insulated from above, either. This, however, contradicts condition (cr-3) that holds for $\underline{C}$ and $\hat{\underline{C}}$ (as $\underline{C}$ is crystallized with respect to $\hat{\underline{C}}$), and guarantees that every crystalline reduced 1-bisimilarity redundancy in $\underline{C}$ is parsimoniously insulated from above.

In both possible cases we have obtained a contradiction. Therefore the assumption that $\underline{C}$ is not 1-collapsed for a loop vertex v of $\hat{\underline{C}}$ that is not $\curvearrowright^{*}$-maximal cannot be sustained. We conclude that $\underline{C}$ is 1-collapsed for every loops-back-to parts of a loop vertex that is not $\curvearrowright$-maximal. ◻

Proposition (= Prop. 7.7, crystallized ⇒ 1-collapsed/twin-crystal scc's)**.** *For every carrier P of a scc in a 1-chart $\underline{C}$ that is crystallized, either $\underline{C}$ is 1-collapsed for P, or P is the carrier of a twin-crystal.*

Sketch of the Proof. Let $\underline{C} = \langle V, A, 1, v_s, \to, \downarrow \rangle$ by a crystallized 1-chart. Then there is a LLEE-witness $\hat{\underline{C}}$ of $\underline{C}$ such that $\underline{C}$ is crystallized with respect to $\hat{\underline{C}}$. Then $\hat{\underline{C}}$ is 1-transition limited by (cr-1). Let $\curvearrowright$ be the loops-back-to relation on $\underline{C}$ that is defined according to $\hat{\underline{C}}$.

Let P be the carrier of an scc in $\underline{C}$. Then by Lem. A.16, (vii), $P = (\curvearrowright^{*} v)$ for some loop vertex $v \in LV(\underline{C}) \subseteq V$, that is, P is the loops-back-to part of a $\curvearrowright$-maximal part. In order to prove the statement of the proposition for P, we assume that $\underline{C}$ is not collapsed for P. We have to show that P is the carrier of a twin-crystal in $\underline{C}$. This is sufficient, because if $\underline{C}$ is collapsed for P, then the statement of the proposition holds trivially for P.

Therefore we assume that $\underline{C}$ is not collapsed for P. This means that P contains a 1-bisimilarity redundancy $\langle u_1, u_2 \rangle$. Then P must be a non-trivial scc of $\underline{C}$, and hence, again by Lem. A.16, (vii), a non-trivial loops-back-to part of $\hat{\underline{C}}$. Therefore we may assume, without loss of generality (since we could exchange the roles of u_1 and u_2), that $\langle u_1, u_2 \rangle$ is a 1-bisimilarity redundancy of kind (K3).

By Lem. A.63, (iii), we obtain that $\langle u_1, u_2 \rangle$ then propagates to reduced 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ of kind (R1), kind (R2), or kind (R3) <u>within</u> $P = (\curvearrowright^{*} v)$. Since due to (cr-2), the crystallized 1-chart $\underline{C}$ may only contain crystalline reduced 1-bisimilarity redundancies, it follows that $\langle w_1, w_2 \rangle$ must be a crystalline reduced 1-bisimilarity redundancy <u>within</u> P. Consequently, $\langle w_1, w_2 \rangle$ must have position $\langle w_1, \overline{w}_1, v_0, \overline{w}_2, w_2 \rangle$ for some $\overline{w}_1$, $\overline{w}_2$, and v_0 such that $w_1 \curvearrowright^{*} \overline{w}_1 \;{}_{d}\!\curvearrowright v_0 \;{}_{d}\!\curvearrowleft \overline{w}_2 \curvearrowleft^{*} w_2$, $v_0 \curvearrowright v$, and $\overline{w}_1 \to^{*}_{\mathrm{bo}} \overline{w}_2$ holds. Since $\underline{C}$ is collapsed for every loops-back-to part of a loop vertex that is not $\curvearrowright$-maximal by Lem. 7.6, (ii), the joining loop vertex v_0 of $\langle w_1, w_2 \rangle$ cannot be different from v: otherwise $v_0 \curvearrowright^{+} v$ would hold, and thus v_0 would not be $\curvearrowright$-maximal, and its loops-back-to part $(\curvearrowright^{*} v_0)$ would not be collapsed as it contains a 1-bisimilarity redundancy; this would contradict Lem. 7.6, (ii) for the crystallized 1-chart $\underline{C}$. Therefore the crystalline reduced 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ in P has position $\langle w_1, \overline{w}_1, v, \overline{w}_2, w_2 \rangle$.

We now ascertain five additional properties of P in $\underline{C}$:

(A) Every reduced 1-bisimilarity redundancy $\langle x_1, x_2 \rangle$ in P is crystalline, and has position $\langle x_1, \overline{w}_1, v, \overline{x}_2, x_2 \rangle$ for some vertex $\overline{x}_2$ relative to the position of $\langle w_1, w_2 \rangle$, which is $\langle w_1, \overline{w}_1, v, \overline{w}_2, w_2 \rangle$. (Hence x_1 also is part of $(\curvearrowright^{*} \overline{w}_1)$ like w_1.)

This is because of the following two reasons. First, as argued above for $\langle w_1, w_2 \rangle$, the reduced 1-bisimilarity redundancy $\langle x_1, x_2 \rangle$ is crystalline by (cr-2) on $\underline{C}$, and it must have position $\langle x_1, \overline{x}_1, v, \overline{x}_2, x_2 \rangle$ with also v as joining loop vertex, for some vertices $\overline{x}_1$ and $\overline{x}_2$, because loop vertices below v have collapsed loops-back-to parts and therefore cannot contain a 1-bisimilarity redundancy. Second, $\overline{x}_1 \neq \overline{w}_1$ is not possible, because otherwise either $\langle w_1, w_2 \rangle$ or $\langle x_1, x_2 \rangle$ would be a reduced 1-bisimilarity redundancy of kind (R3.4.2), in contradiction with the fact that $\underline{\mathcal{C}}$ only contains crystalline reduced 1-bisimilarity redundancies due to condition (cr-2) on $\underline{\mathcal{C}}$ as being a crystallized 1-chart.

(B) P is grounded in $\underline{\mathcal{C}}$.

This is guaranteed for the scc P by the condition (cr-4) on $\underline{\mathcal{C}}$ as a crystallized 1-chart.

(C) Every (not necessarily reduced) 1-bisimilarity redundancy in P has an ordered representation $\langle u_1, u_2 \rangle$ of kind (K3), and furthermore has position $\langle u_1, \overline{w}_1, v, \overline{u}_2, u_2 \rangle$ (hence also $u_1 \in (\circlearrowleft^* \overline{w}_1)$).

This follows from the propagation statement of Lem. A.63, in particular by using items (iii), (c), in view of the fact that a 1-bisimilarity redundancy $\langle u_1, u_2 \rangle$ of kind (K3) with assumed position $\langle u_1, \overline{u}_1, v_0, \overline{u}_{20}, u_2 \rangle$, for some $\overline{u}_1, v_0, \overline{u}_{20}$, can only propagate to a crystalline reduced 1-bisimilarity redundancy with position $\langle u_1', \overline{u}_1, v_0, \overline{u}_2, u_2' \rangle$, for some $\overline{u}_2$, in P, by Lem. A.63, (iii), (c); then $v_0 = v$ must hold due to property (A).

(D) Not all loop-entry transitions from v target vertices in $(\circlearrowleft^* \overline{w}_1)$ or v itself.

This is because otherwise $\langle w_1, w_2 \rangle$ would be a reduced 1-bisimilarity redundancy of kind (R3.4.1), contradicting the condition (cr-3) on $\underline{\mathcal{C}}$ to only contain crystalline reduced 1-bisimilarity redundancies.

(E) If the target v' of a loop-entry transition from v is in $(\circlearrowleft^* v)$ but not in $(\circlearrowleft^* \overline{w}_1)$, then v' is not 'upstream' from $\overline{w}_1$, that is, there is no body transition path from v' to $\overline{w}_1$. More formally:

$$\forall v' \forall m \in \mathbb{N}^+ \forall a \in A \big(v \xrightarrow{a}_{[m]} v' \wedge v' \notin (\circlearrowleft^* \overline{w}_1) \implies \neg (v' \rightarrow^*_{\mathrm{bo}} \overline{w}_1) \big)$$

The reason is that in case that $v \xrightarrow{a}_{[m]} v' \wedge v' \notin (\circlearrowleft^* \overline{w}_1) \wedge (v' \rightarrow^+_{\mathrm{bo}} \overline{w}_1)$ holds (note that $v' = \overline{w}_1$ is excluded by $v' \notin (\circlearrowleft^* \overline{w}_1)$), a 1-bisimilarity redundancy $\langle v', w_1' \rangle$ would arise (since $v \sqsubseteq w_2 \underline{\leftrightarrow} w_1$ by the form of the reduced 1-bisimilarity redundancy $\langle w_1, w_2 \rangle$ of kind (R3.4) with joining loop vertex v, the transition $v \xrightarrow{a}_{[m]} v'$ must be able to be matched by an induced transition $w_1 \xrightarrow{(a)} w_1'$ with $v' \underline{\leftrightarrow} w_1'$, for some w_1' such that $w_1' \in (\circlearrowleft^* \overline{w}_1)$ or $\overline{w}_1 \rightarrow_{\mathrm{bo}} w_1'$) that does not conform to the property found in (C), because v' will be 'upstream' from $\overline{w}_1$ due to $v' \rightarrow^*_{\mathrm{bo}} \overline{w}_1$.

On the basis of this properties it can be verified that for the definitions:

$$\begin{aligned} &\mathit{piv} := \overline{w}_1, && \mathit{top} := v, && E_2 := \{ \langle \mathit{top}, \langle \underline{a}, n \rangle, v' \rangle \mid \langle \mathit{top}, \langle \underline{a}, n \rangle, v' \rangle \in \hat{\rightarrow},\ n \geqslant 1,\ v' \notin (\circlearrowleft^* \overline{w}_1) \}, \\ &P_1 := (\circlearrowleft \mathit{piv}), && P_2 := ({}^{E_2}\!\circlearrowleft^* \mathit{top}), \end{aligned}$$

the scc P is the carrier of a twin-crystal with respect to LLEE-witness $\hat{\underline{\mathcal{C}}}$, top vertex top, pivot vertex piv, top part P_2, pivot part P_1, and set E_2 of top part entry transitions. ☐

A.6.7 Crystallized 1-charts are near-collapsed, and the crystallization theorem.

Here we establish that crystallized 1-charts are also near-collapsed, on the basis of knowledge we gathered about crystallized 1-charts, and by using the following lemma. Its statement will permit us to infer that a 1-LTS is near-collapsed provided that it is near-collapsed for all of its scc's, and that it is collapsed apart from within scc's.

Lemma A.81 (from locally near-collapsed 1-LTSs/1-charts to near-collapsed 1-LTSs/1-charts)**.** *We consider a 1-LTS $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow \rangle$. Suppose that for a family $\{P_i\}_{i \in I}$ of sets $P_i \subseteq V$ that covers the state set, that is, for which $V = \bigcup_{i \in I} P_i$ holds, the 1-LTS $\underline{\mathcal{L}}$ satisfies the following two conditions:*

(i) $\underline{\mathcal{L}}$ is near-collapsed for each P_i where $i \in I$,

(ii) $\underline{\mathcal{L}}$ is 1-collapsed apart from within sets of the family $\{P_i\}_{i \in I}$.

Then $\underline{\mathcal{L}}$ is (globally) near-collapsed. This statements also hold for 1-charts.

Proof. We let $\underline{\mathcal{L}}$, and the family $\{P_i\}_{i \in I}$ be as assumed in the lemma, and suppose that $\underline{\mathcal{L}}$ is near-collapsed for each P_i where $i \in I$, and that $\underline{\mathcal{L}}$ is 1-collapsed apart from within sets of $\{P_i\}_{i \in I}$. Then for every $i \in I$ there is a family $\{\phi_{ij}\}_{j \in J_i}$ of local transfer functions on $\underline{\mathcal{L}}$ that witnesses that $\underline{\mathcal{L}}$ is near-collapsed for P_i, and hence for all $i \in I$:

$$\underline{\leftrightarrow}_{\underline{\mathcal{L}}} \cap (P_i \times P_i) = \leftrightarrow^{=}_{R_i} \cap (P_i \times P_i), \qquad \text{for } R_i := \textstyle\bigcup_{j \in J_i} \mathit{graph}(\phi_{ij}). \tag{47}$$

Then we form the union of these families of local transfer functions, obtaining the family $\{\phi_{ij}\}_{\langle i,j \rangle \in IJ}$ of local transfer functions on $\underline{\mathcal{L}}$ with $IJ := \{\langle i, j \rangle \mid i \in I, j \in J_i\}$. We set and find:

$$R := \bigcup_{\langle i,j \rangle \in IJ} \mathit{graph}(\phi_{ij}) = \bigcup_{i \in I} \bigcup_{j \in J_i} \mathit{graph}(\phi_{ij}) = \bigcup_{i \in I} R_i. \tag{48}$$

Now we can argue as follows:

$$\begin{aligned}
\underline{\leftrightarrow}_{\underline{\mathcal{L}}} &= \underline{\leftrightarrow}_{\underline{\mathcal{L}}} \cap \bigcup_{i\in I}(P_i \times P_i) && \text{(due to (12), since } \{P_i\}_{i\in I} \text{ covers } V, \text{ and } \underline{\mathcal{L}} \text{ is 1-collapsed apart from within sets of the family } \{P_i\}_{i\in I}) \\
&= \bigcup_{i\in I}\big(\underline{\leftrightarrow}_{\underline{\mathcal{L}}} \cap (P_i \times P_i)\big) && \text{(by a distributive set algebra law)} \\
&= \bigcup_{i\in I}\big(\leftrightarrow^{=}_{R_i} \cap (P_i \times P_i)\big) && \text{(by (47), for each } i \in I) \\
&\subseteq \bigcup_{i\in I} \leftrightarrow^{=}_{R_i} && \text{(by dropping intersections)} \\
&= \leftrightarrow^{=}_{\bigcup_{i\in I} R_i} && \text{(since the union of symmetric–transitive closures of relations is the symmetric–transitive closure of the union of the relations)} \\
&= \leftrightarrow^{=}_{R} && \text{(by (48)).}
\end{aligned}$$

This chain of set inclusions demonstrates that the family $\{\phi_{ij}\}_{\langle i,j\rangle\in IJ}$ of local transfer functions witnesses that $\underline{\mathcal{L}}$ is (globally) near-collapsed. ◻

Lemma (= Lem. 7.8, (4)). *Every crystallized 1-chart is near-collapsed.*

Proof. Let $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a crystallized 1-chart. We have to show that $\underline{\mathcal{C}}$ is (globally) near-collapsed.

We first show that $\underline{\mathcal{C}}$ is near-collapsed for every carrier P of an scc of $\underline{\mathcal{C}}$. Let $P \subseteq V$ be the carrier set of an scc of $\underline{\mathcal{C}}$. Then by applying Prop. 7.7 to P and the crystallized 1-chart $\underline{\mathcal{C}}$ we find that either $\underline{\mathcal{C}}$ is 1-collapsed for P, or P is the carrier of a twin-crystal in $\underline{\mathcal{C}}$. In the first case, $\underline{\mathcal{C}}$ is also near-collapsed for P, because if $\underline{\mathcal{C}}$ is 1-collapsed for P, then $\underline{\mathcal{C}}$ is also near-collapsed for P. In the second case, $\underline{\mathcal{C}}$ is near-collapsed for P by Lem. A.2. Hence in both cases $\underline{\mathcal{C}}$ is near-collapsed for P.

Second, by applying Lem. 7.6, (i), to the crystallized 1-chart $\underline{\mathcal{C}}$ we obtain that $\underline{\mathcal{C}}$ is 1-collapsed apart from within scc's of $\underline{\mathcal{C}}$.

From the two facts we have thus obtained we conclude that $\{\mathrm{scc}(v)\}_{v\in V}$, the family of scc's of vertices of $\underline{\mathcal{C}}$, which covers the set V of vertices of $\underline{\mathcal{C}}$, satisfies the assumptions (i) and (ii) of Lem. A.81. Therefore we can apply that lemma, and conclude that $\underline{\mathcal{C}}$ then is also (globally) near-collapsed. ◻

Theorem (= Thm. 7.9, crystallization, nearcollapse, ((3),(4))). *Every weakly guarded* LLEE*-1-chart $\underline{\mathcal{C}}$ can be transformed, together with a* LLEE*-witness $\hat{\underline{\mathcal{C}}}$ of $\underline{\mathcal{C}}$, into a 1-bisimilar* LLEE*-1-chart $\underline{\mathcal{C}}'$ with 1-transition limited* LLEE*-witness $\hat{\underline{\mathcal{C}}}'$ such that $\underline{\mathcal{C}}$ is crystallized with respect to $\hat{\underline{\mathcal{C}}}'$, and $\underline{\mathcal{C}}'$ is near-collapsed.*

Proof. The statement of the theorem now follows directly from Lem. 3.2, Lem. A.80 and Lem. 7.8. ◻

A.7 Proofs in Section 8: Near-Collapsed LLEE-1-charts have complete solutions

We introduce the concept of elevation $\mathsf{E}_W(\underline{\mathcal{L}})$ of a given set W of vertices above a 1-LTS $\underline{\mathcal{L}}$. Hereby $\mathsf{E}_W(\underline{\mathcal{L}})$ is a 1-LTS that consists of the elevation of, actually, the full subchart of W in $\underline{\mathcal{L}}$, extended by 1-transitions, above $\underline{\mathcal{L}}$, and together with $\underline{\mathcal{L}}$. Its construction proceeds as follows. The set of vertices of $\mathsf{E}_W(\underline{\mathcal{L}})$ consists of two copies of its set V of vertices, the 'ground floor' $V \times \{\mathbf{0}\}$, and the 'first floor' $V \times \{\mathbf{1}\}$. These two copies of the set of vertices of $\underline{\mathcal{L}}$ are linked by copies of the corresponding transitions of $\underline{\mathcal{L}}$ with the exception that proper transitions $\langle v_1, a, v_2\rangle$ of $\underline{\mathcal{L}}$ do not give rise to a proper transition $\langle\langle v_1, \mathbf{1}\rangle, a, \langle v_2, \mathbf{1}\rangle\rangle$ on the first floor if the vertex v_2 is not contained in W. Those transitions get redirected as transitions $\langle\langle v_1, \mathbf{1}\rangle, a, \langle v_2, \mathbf{0}\rangle\rangle$ to target the corresponding copy $\langle v_2, \mathbf{0}\rangle$ of v_2 on the ground floor. Note that such redirections from the first floor to the ground floor are not defined for 1-transitions. The sub-1-LTS of $\mathsf{E}_W(\underline{\mathcal{L}})$ that consists of all transitions between vertices on the ground floor is an exact copy of the original 1-LTS $\underline{\mathcal{L}}$. Yet within the elevation $\mathsf{E}_W(\underline{\mathcal{L}})$ of W above $\underline{\mathcal{L}}$, a number of vertices on the ground floor will have additional incoming proper-action transitions from vertices on the first floor.

Definition A.82 (elevation of a vertex set above a **1-LTS** ((6)), (refinement of Def. 2.2 in [10] to 1-LTSs)). Let $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow\rangle$ be a 1-LTS, and let $W \subseteq V$ be a subset of the set of vertices of $\underline{\mathcal{L}}$.

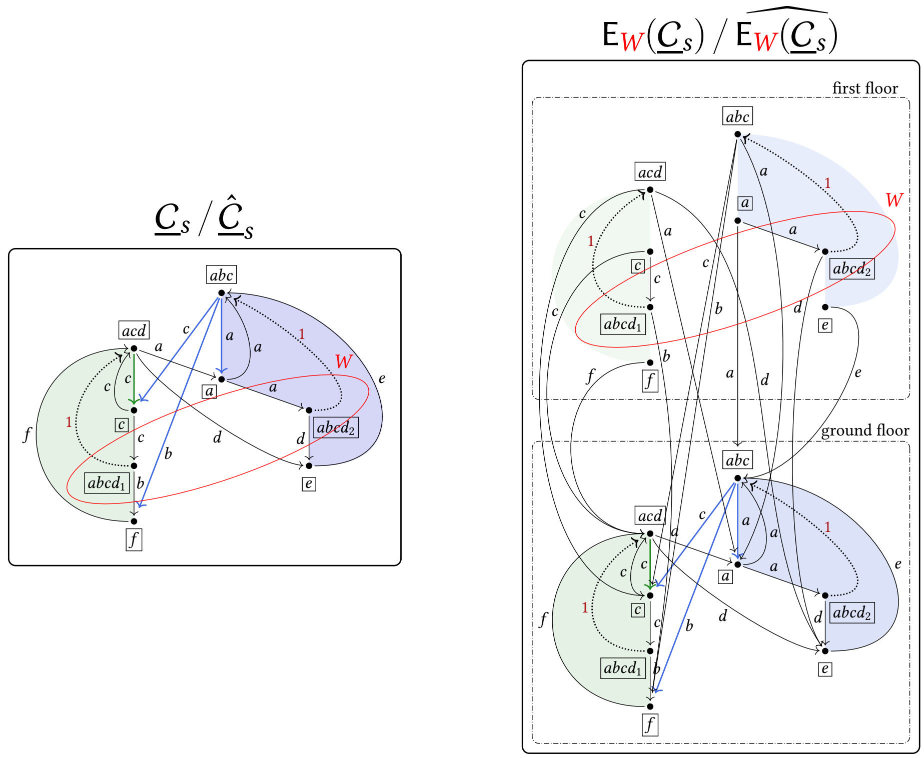


Figure 28. Left: the LLEE-1-chart $\underline{\mathcal{C}}_s$ from Fig. 5 with, as indicated, the LLEE-witness $\hat{\underline{\mathcal{C}}}_s$. Right: The elevation $\mathsf{E}_W(\underline{\mathcal{C}}_s)$ of the set of vertices $W := \{\boxed{abcd_1}, \boxed{abcd_2}\}$ of $\underline{\mathcal{C}}_s$ above $\underline{\mathcal{C}}_s$, together with a LLEE-witness $\widehat{\mathsf{E}_W(\underline{\mathcal{C}}_s)}$ that is derived $\hat{\underline{\mathcal{C}}}_s$.

By *the elevation of W above $\underline{\mathcal{L}}$* (short for *the elevation of the full sub-1-LTS of W in $\underline{\mathcal{L}}$, extended by 1-transitions, above $\underline{\mathcal{L}}$*) we mean the 1-LTS that is of the form $\mathsf{E}_W(\underline{\mathcal{L}}) = \langle V_{\mathsf{E}_W}, A, 1, \to_{\mathsf{E}_W}, \downarrow_{\mathsf{E}_W}\rangle$ where:

$$
\begin{aligned}
V_{\mathsf{E}_W} &:= V \times \{\mathbf{0},\mathbf{1}\}\,, \quad \to_{\mathsf{E}_W} := \left\{ \langle\langle w_1,\mathbf{1}\rangle, a, \langle w_2,\mathbf{1}\rangle\rangle \;\middle|\; \begin{array}{l}\langle w_1, a, w_2\rangle \in \to \\ \wedge\ a\in A \ \wedge\ w_2 \in W\end{array} \right\} \quad \downarrow_{\mathsf{E}_W} := \left\{ \langle w, i\rangle \;\middle|\; w\in V,\ i\in\{\mathbf{0},\mathbf{1}\},\ w{\downarrow} \right\},\\
&\qquad \cup \left\{ \langle\langle w_1,\mathbf{1}\rangle, 1, \langle w_2,\mathbf{1}\rangle\rangle \;\middle|\; \langle w_1, 1, w_2\rangle \in \to \right\}\\
&\qquad \cup \left\{ \langle\langle w_1,\mathbf{1}\rangle, a, \langle w_2,\mathbf{0}\rangle\rangle \;\middle|\; \begin{array}{l}\langle w_1, a, w_2\rangle \in \to \\ \wedge\ a\in A \ \wedge\ w_2 \notin W\end{array} \right\}\\
&\qquad \cup \left\{ \langle\langle w_1,\mathbf{0}\rangle, \underline{a}, \langle w_2,\mathbf{0}\rangle\rangle \;\middle|\; \langle w_1, \underline{a}, w_2\rangle \in \to \right\},
\end{aligned}
$$

Example A.83. Fig. 28 shows the elevation $\mathsf{E}_W(\underline{\mathcal{C}}_s)$ of the set $W := \{\boxed{abcd_1}, \boxed{abcd_2}\}$ of vertices of the 1-chart $\underline{\mathcal{C}}_s$ in Fig. 5 above $\underline{\mathcal{C}}_s$.

Lemma A.84 (elevation above weakly guarded 1-LTS is weakly guarded)**.** *Let $\underline{\mathcal{L}} = \langle V, A, 1, \to, \downarrow\rangle$ be a 1-LTS, and let $W \subseteq V$ be a subset of its set of states. If $\underline{\mathcal{L}}$ is weakly guarded, then the elevation $\mathsf{E}_W(\underline{\mathcal{L}})$ of W above $\underline{\mathcal{L}}$ is weakly guarded.*

The following lemma gathers crucial properties concerning correspondences between proper transitions, 1-transitions, and induced transitions in a 1-LTS $\underline{\mathcal{L}}$, and proper transitions, 1-transitions, and induced transitions in the unfolding $\mathsf{E}_W(\underline{\mathcal{L}})$ of $\underline{\mathcal{L}}$ with respect to the set W of vertices.

Lemma A.85 (properties of an elevation over a 1-LTS)**.** *Let $\underline{\mathcal{L}} = \langle V, A, 1, \to, \downarrow\rangle$ be a 1-LTS. Let $W \subseteq V$ be a subset of the set of vertices of $\underline{\mathcal{L}}$.*

The following three statements hold for transitions, and induced transitions, and for termination, and induced termination on the elevation $\mathsf{E}_W(\underline{\mathcal{L}})$ of W above $\underline{\mathcal{L}}$:

(i) Transitions in $\mathsf{E}_W(\underline{\mathcal{L}})$ correspond to transitions in $\underline{\mathcal{L}}$ as follows, where $w, w' \in V$, proper actions $a \in A$, and floors $\mathbf{i}, \mathbf{i}' \in \{\mathbf{0},\mathbf{1}\}$ are arbitrary:

$$\langle w, \mathbf{i}\rangle \xrightarrow{1}_{\mathsf{E}_W(\underline{\mathcal{L}})} \langle w', \mathbf{i}'\rangle \iff \mathbf{i} = \mathbf{i}' \ \wedge\ w \xrightarrow{1}_{\underline{\mathcal{L}}} w'\,, \tag{49}$$

$$\langle w, \mathbf{1}\rangle \xrightarrow{a}_{\mathsf{E}_W(\underline{\mathcal{L}})} \langle w', \boldsymbol{i}'\rangle \iff w \xrightarrow{a}_{\underline{\mathcal{L}}} w' \wedge \big((w' \in W \wedge \boldsymbol{i}' = \mathbf{1}) \vee (w' \notin W \wedge \boldsymbol{i}' = \mathbf{0})\big), \tag{50}$$

$$\langle w, \mathbf{0}\rangle \xrightarrow{a}_{\mathsf{E}_W(\underline{\mathcal{L}})} \langle w', \boldsymbol{i}'\rangle \iff w \xrightarrow{a}_{\underline{\mathcal{L}}} w' \wedge \boldsymbol{i} = \mathbf{0}. \tag{51}$$

(ii) *Induced transitions in $\mathsf{E}_W(\underline{\mathcal{L}})$ correspond to induced transitions in $\underline{\mathcal{L}}$ as follows, where $w, w' \in V$, proper actions $a \in A$, and floors $\boldsymbol{i}, \boldsymbol{i}' \in \{\mathbf{0}, \mathbf{1}\}$ are arbitrary:*

$$\langle w, \mathbf{1}\rangle \xrightarrow{(a]}_{\mathsf{E}_W(\underline{\mathcal{L}})} \langle w', \boldsymbol{i}'\rangle \iff w \xrightarrow{(a]}_{\underline{\mathcal{L}}} w' \wedge \big((w' \in W \wedge \boldsymbol{i}' = \mathbf{1}) \vee (w' \notin W \wedge \boldsymbol{i}' = \mathbf{0})\big), \tag{52}$$

$$\langle w, \mathbf{0}\rangle \xrightarrow{(a]}_{\mathsf{E}_W(\underline{\mathcal{L}})} \langle w', \boldsymbol{i}'\rangle \iff w \xrightarrow{(a]}_{\underline{\mathcal{L}}} w' \wedge \boldsymbol{i}' = \mathbf{0}. \tag{53}$$

(iii) *Termination, and induced termination on $\mathsf{E}_W(\underline{\mathcal{L}})$ correspond to termination, and induced termination on $\underline{\mathcal{L}}$, respectively, that is, the following two equivalence statements hold for all $w \in V$ and $\boldsymbol{i} \in \{\mathbf{0}, \mathbf{1}\}$:*

$$\langle w, \boldsymbol{i}\rangle{\downarrow}_{\mathsf{E}_W(\underline{\mathcal{L}})} \iff w{\downarrow}_{\underline{\mathcal{L}}}, \tag{54}$$

$$\langle w, \boldsymbol{i}\rangle{\downarrow}^{(1)}_{\mathsf{E}_W(\underline{\mathcal{L}})} \iff w{\downarrow}^{(1)}_{\underline{\mathcal{L}}}. \tag{55}$$

As an easy consequence of this lemma with properties of elevation 1-LTSs we can show that the projection function from elevations of a 1-LTS $\underline{\mathcal{L}}$ back to $\underline{\mathcal{L}}$ define functional 1-bisimulations.

Lemma A.86 (elevation above **1-LTS** is functionally 1-bisimilar with **1-LTS**)**.** *We consider a 1-LTS $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow\rangle$, and a subset $W \subseteq V$ of its set of states.*

Then $\mathsf{E}_W(\underline{\mathcal{L}}) \rightrightarrows_{\pi_1} \underline{\mathcal{L}}$ holds, that is, the projection function $\pi_1 : V \times \{\mathbf{0}, \mathbf{1}\} \to V, \langle v, \boldsymbol{i}\rangle \mapsto v$ defines a functional 1-bisimulation from $\mathsf{E}_W(\underline{\mathcal{L}})$ to $\underline{\mathcal{L}}$, and therefore is a surjective transfer function from $\mathsf{E}_W(\underline{\mathcal{L}})$ to $\underline{\mathcal{L}}$.

Furthermore, Lem. A.85 now also facilitates us to show that every grounded 1-bisimulation slice B on a 1-LTS $\underline{\mathcal{L}}$ can be lifted to a 1-bisimulation on the elevation $\mathsf{E}_{\mathrm{field}(B)}(\underline{\mathcal{L}})$ of the field $\mathrm{field}(B) := \mathrm{dom}_{\mathrm{act}}(B) \cup \mathrm{cod}_{\mathrm{act}}(B)$ of B above $\underline{\mathcal{L}}$.

Lemma A.87 (grounded 1-bisim. slice $\Rightarrow$ 1-bisimulation on the elevation)**.** *Let $\underline{\mathcal{L}} = \langle V, 1, A, \rightarrow, \downarrow\rangle$ be a 1-LTS. Then for every grounded 1-bisimulating slice $B \subseteq V \times V$ with $\mathrm{field}(B) = \mathrm{dom}_{\mathrm{act}}(B) \cup \mathrm{cod}_{\mathrm{act}}(B)$ the relation $\widehat{B}$ defined by:*

$$\begin{aligned} \widehat{B} \;&:=\; \widehat{B}^{(1)} \cup \widehat{B}_{(\mathbf{0})} \\ &\subseteq\; (V \times \{\mathbf{0}, \mathbf{1}\}) \times (V \times \{\mathbf{0}, \mathbf{1}\}) \;\subseteq\; V_{\mathsf{E}_{\mathrm{field}(B)}} \times V_{\mathsf{E}_{\mathrm{field}(B)}}, \\ \textit{where: } \widehat{B}^{(1)} \;&:=\; \{\langle\langle w_1, \mathbf{1}\rangle, \langle w_2, \mathbf{1}\rangle\rangle \mid \langle w_1, w_2\rangle \in B\}, \\ \widehat{B}_{(\mathbf{0})} \;&:=\; \underset{(V \times \{\mathbf{0}\})}{=}, \end{aligned} \tag{56}$$

is a 1-bisimulation on the elevation $\mathsf{E}_{\mathrm{field}(B)}(\underline{\mathcal{L}})$ of $\mathrm{field}(B)$ above $\underline{\mathcal{L}}$.

The specialization of this result for grounded 1-bisimulation slices to graphs of local transfer functions, which are functional, grounded 1-bisimulation slices, leads us to the following central result. It states that local transfer functions can be lifted, via the projection transfer functions, to transfer functions on elevations.

Lemma A.88 (local transfer function $\Rightarrow$ transfer function on elevation (refinement of Prop. 2.4 in [10] to 1-LTSs))**.** *Let $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow\rangle$ be a 1-LTS. Every local transfer function $\phi : V \rightharpoonup V$ on $\underline{\mathcal{L}}$ with $W := \mathrm{dom}(\phi) \cup \mathrm{ran}(\phi)$ lifts to a transfer function $\overline{\phi}^{(1)}$ on the elevation $\mathsf{E}_{\underline{\mathcal{L}}}(W)$ of W over $\underline{\mathcal{L}}$, via the projection transfer function π_1 from $\mathsf{E}_{\underline{\mathcal{L}}}(W)$ to $\underline{\mathcal{L}}$, where $\overline{\phi}^{(1)} : (V \times \{\mathbf{0}, \mathbf{1}\}) \rightharpoonup (V \times \{\mathbf{0}, \mathbf{1}\})$ is defined by:*

$$\overline{\phi}^{(1)}(\langle w, \boldsymbol{i}\rangle) \;:=\; \begin{cases} \langle \phi(w), \boldsymbol{i}\rangle & \textit{if } \boldsymbol{i} = \mathbf{1} \wedge w \in \mathrm{dom}(\phi), \\ \text{undefined} & \textit{if } \boldsymbol{i} = \mathbf{1} \wedge w \notin \mathrm{dom}(\phi), \\ \langle w, \boldsymbol{i}\rangle & \textit{if } \boldsymbol{i} = \mathbf{0}, \end{cases}$$

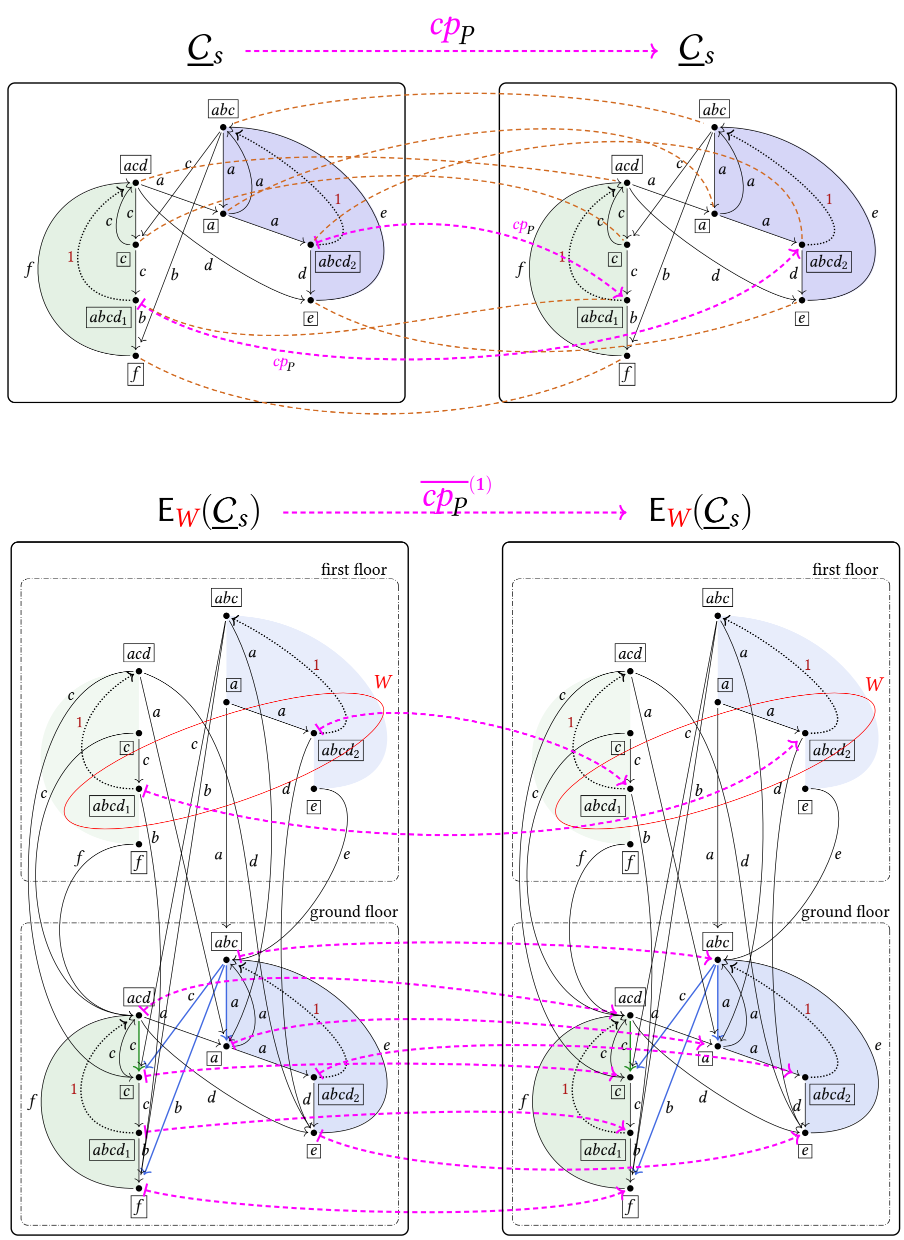


Figure 29. Illustration of the lifting, as stated by Lem. A.88, of the local transfer function cp_P from Fig. 5 (see also above) to a transfer function $\overline{cp_P}^{(1)}$ on the elevation $\mathsf{E}_W(\underline{\mathcal{C}}_s)$ of $W = \{\boxed{abcd_1}, \boxed{abcd_2}\}$ above the 1-chart $\underline{\mathcal{C}}_s$ from Fig. 5.

such that the diagram below commutes for vertices in $V(\mathsf{E}_W(\underline{\mathcal{L}})) \cap (V \times \mathbf{1})$ *on the first floor of* $\mathsf{E}_W(\underline{\mathcal{L}})$*:*

$$\begin{tikzcd} \underline{\mathcal{L}} \arrow[r, "\phi"] & \underline{\mathcal{L}} \\ \mathsf{E}_W(\underline{\mathcal{L}}) \arrow[u, "\pi_1"] \arrow[r, "\overline{\phi}^{(1)}"] & \mathsf{E}_W(\underline{\mathcal{L}}) \arrow[u, "\pi_1"'] \end{tikzcd} \tag{57}$$

$$(\pi_1 \circ \overline{\phi}^{(1)})(\langle w, \mathbf{1}\rangle) = (\phi \circ \pi_1)(\langle w, \mathbf{1}\rangle) \quad (w \in \mathrm{dom}(\phi)). \tag{58}$$

Example A.89. Fig. 29 above is an illustration of the lifting of the local transfer function cp_P in Fig. 5 on the 1-chart $\underline{C}_s$ in Fig. 5 to a transfer function $\overline{cp_P}^{(1)}$ on the elevation $\mathsf{E}_W(\underline{C}_s)$ of $W := \{\boxed{abcd_1}, \boxed{abcd_2}\}$ above $\underline{C}_s$.

An important additional property is described by the following lemma: elevations of LLEE-1-LTSs are again LLEE-1-LTSs.

Lemma A.90 (elevation above LLEE-1-LTS is LLEE-1-LTS). *We consider a 1-LTS $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow\rangle$, and a subset $W \subseteq V$ of its set of states. Then the following two statements hold:*

(i) If $\underline{\mathcal{L}}$ has a LLEE*-witness (thus $\underline{\mathcal{L}}$ is a* LLEE*-1-LTS), then also the elevation $\mathsf{E}_W(\underline{\mathcal{L}})$ of W above $\underline{\mathcal{L}}$ has a* LLEE*-witness (hence it is a* LLEE*-1-LTS).*

(ii) Moreover, if $\underline{\mathcal{L}}$ is has a guarded LLEE*-witness, then also the elevation $\mathsf{E}_W(\underline{\mathcal{L}})$ of W above $\underline{\mathcal{L}}$ has a guarded* LLEE*-witness.*

By using provable invariance of provable solutions under transfer functions (see Lem. 3.6), we can now use Lem. A.88, the lifting of local transfer functions to transfer functions on suitable elevations, to show that provable solutions one LLEE-1-LTSs are provably invariant under local transfer functions.

Lemma (= Lem. 8.1). *Let $\phi : V \rightharpoonup V$ be a local transfer function on a weakly guarded* LLEE*-1-LTS $\underline{\mathcal{L}} = \langle V, A, 1, \rightarrow, \downarrow\rangle$. Then every* Mil*-provable solution s of $\underline{\mathcal{L}}$ is* Mil*-provably invariant under ϕ:*

$$s(w) =_{\mathsf{Mil}} s(\phi(w)) \quad (\textit{for all } w \in \mathrm{dom}(\phi))\;. \tag{59}$$

Proof. We use the diagram in (57), and that $\mathsf{E}_W(\underline{\mathcal{L}})$ is also a LLEE-1-LTS. Since π_1 and $\overline{\phi}^{(1)} \circ \pi_1$ are transfer functions, by Lem. 3.6 $(s \circ \pi_1)$ and $(s \circ \pi_1 \circ \overline{\phi}^{(1)})$ are Mil-provably equal. Then by using we diagram commutativity on the first floor in (57) we get: $s(w) = (s \circ \pi_1)(\langle w, \mathbf{1}\rangle) =_{\mathsf{Mil}} (s \circ \pi_1 \circ \overline{\phi}^{(1)})(\langle w, \mathbf{1}\rangle) = (s \circ \phi \circ \pi_1)(\langle w, \mathbf{1}\rangle) = s(\phi(w))$, for all $w \in \mathrm{dom}(\phi)$. In this way we have obtained (59). ◻

In the next lemma we extend the statement of Lem. A.91 concerning provable invariance of provable solutions under local transfer functions on LLEE-1-charts to families of local transfer functions.

Lemma A.91. *Let $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a finite, weakly guarded 1-chart with* LLEE*. Let $\{\phi_i\}_{i\in I}$ be a family of local transfer functions on $\underline{C}$, and define $R := \bigcup_{i\in I} graph(\phi) \subseteq V \times V$. Let $s : V \to StExp(A)$ be a* Mil*-provable solution of $\underline{C}$.*

Then s is Mil*-provably invariant under $\leftrightarrow_R^*$, that is, for all $w_1, w_2 \in V$ it holds:*

$$w_1 \leftrightarrow_R^* w_2 \implies s(w_1) =_{\mathsf{Mil}} s(w_2)\,. \tag{60}$$

Proof. We assume $\underline{C}$, $\{\phi_i\}_{i\in I}$, R, and s as in the assumption of the lemma. We have to show (60) for all $w_1, w_2 \in V$.

For this we will use that, for all $w_1, w_2 \in V$, the following implications hold:

$$w_1 \rightarrow_R w_2 \implies s(w_1) =_{\mathsf{Mil}} s(w_2)\,, \tag{61}$$

$$w_1 \leftrightarrow_R w_2 \implies s(w_1) =_{\mathsf{Mil}} s(w_2)\,. \tag{62}$$

Indeed, (60), for all $w_1, w_2 \in V$, can be shown from (62) by a straightforward induction on the length n of a conversion $w_1 = w_{10} \leftrightarrow_R w_{11} \leftrightarrow_R \ldots \leftrightarrow_R w_{1n} = w_2$ between w_1 and w_2. Furthermore, since $\leftrightarrow_R \,=\, \leftarrow_R \cup \rightarrow_R$ holds, statement (62), for all $w_1, w_2 \in V$, follows immediately from statement (61), for all $w_1, w_2 \in V$. Therefore it remains to show (61), for all $w_1, w_2 \in V$. For this, we let $w_1, w_2 \in V$ be arbitrary, and argue as follows:

$$\begin{aligned}
w_1 \rightarrow_R w_2 &\implies \exists\, i \in I\big(\, w_1 \rightarrow_{graph(\phi_i)} w_2 \,\big) && \text{(by def. of } R \text{ and } \rightarrow_R\text{)}\\
&\implies \exists\, i \in I\big(\, w_2 = \phi_i(w_1) \,\big) && \text{(by def. of } \rightarrow_{graph(\phi_i)}\text{)}\\
&\implies \exists\, i \in I\big(\, s(w_2) =_{\mathsf{Mil}} s(\phi_i(w_1)) = s(w_2) \,\big) && \text{(by Lem. 8.1 for the}\\
& && \;\text{local transfer funct. } \phi_i\text{)}\\
&\implies s(w_1) =_{\mathsf{Mil}} s(w_2)\,.
\end{aligned}$$

Since $w_1, w_2 \in V$ had been picked arbitrarily, we have shown (61), for all vertices $w_1, w_2 \in V$. Since we have already argued that this statement implies (60), for all vertices $w_1, w_2 \in V$, we have shown the statement of the lemma. ◻

Now we can finally show that finite, near-collapsed, weakly guarded LLEE-1-charts have complete solutions.

Lemma A.92 ((= extended version of Lem. 8.2, 6)). *Every finite, weakly guarded* LLEE*-1-chart that is near-collapsed has a* Mil*-complete* Mil*-provable solution. In particular, the solution extracted from any near-collapsed 1-chart $\underline{C}$ that is finite, and has a guarded* LLEE*-witness is a* Mil*-complete* Mil*-provable solution of $\underline{C}$.*

Proof. Let $\underline{C} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a finite, weakly guarded LLEE-1-chart that is near-collapsed. Then there is a family $\{\phi_i\}_{i\in I}$ of local transfer functions on $\underline{C}$ such that for the binary relation $R := \bigcup_{i\in I} graph(\phi) \subseteq V \times V$ it holds:

$$\underline{\leftrightarrow}_{\underline{C}} \subseteq \leftrightarrow^*_R . \tag{63}$$

We pick $\{\phi_i\}_{i\in I}$, and define R, in this way. Statement (63), however, implies:

$$\underline{\leftrightarrow}_{\underline{C}} = \leftrightarrow^*_R . \tag{64}$$

because also $\leftrightarrow^*_R \subseteq \underline{\leftrightarrow}_{\underline{C}}$ holds, since the graphs of local transfer functions are contained in 1-bisimilarity on $\underline{C}$.

Since $\underline{C}$ is a LLEE-1-chart, it has a LLEE-witness $\hat{\underline{C}}$. Then the from $\hat{\underline{C}}$ extracted solution $s_{\hat{\underline{C}}} : V \rightarrow StExp(A)$ is a Mil-provable solution of $\underline{C}$ by Lem. A.22. We will show that $s_{\hat{\underline{C}}}$ is also a Mil-complete solution of $\underline{C}$.

For this we let $w_1, w_2 \in V$ be arbitrary, and argue as follows:

$$\begin{aligned}
\underline{C}\downarrow^{w_1}_* \underline{\leftrightarrow} \underline{C}\downarrow^{w_2}_* &\iff w_1 \underline{\leftrightarrow}_{\underline{C}} w_2 && \text{(by definition of } \underline{\leftrightarrow}_{\underline{C}}\text{)} \\
&\implies w_1 \leftrightarrow^*_R w_2 && \text{(by (64))} \\
&\implies s_{\hat{\underline{C}}}(w_1) =_{\mathsf{Mil}} s_{\hat{\underline{C}}}(w_2) && \text{(by Lem. A.91 for the Mil-provable solution } s_{\hat{\underline{C}}} \text{ of } \underline{C}\text{).}
\end{aligned}$$

Since $w_1, w_2 \in V$ had been chosen arbitrarily for this argument, we have established that solution values of the from $\hat{\underline{C}}$ extracted Mil-provable solution $s_{\hat{\underline{C}}}$ at 1-bisimilar vertices are themselves Mil-provably equal. We have shown that $s_{\hat{\underline{C}}}$ is a Mil-complete Mil-provable solution of $\underline{C}$.

In this way we have established that every near-collapsed LLEE-1-charts has a Mil-complete Mil-provable solution. ◻