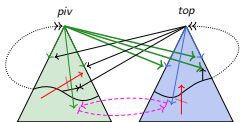


Milner's Proof System for Regular Expressions Modulo Bisimilarity is Complete

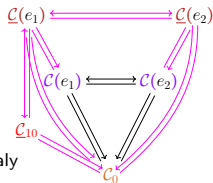
Crystallization: Near-Collapsing Process Graph Interpretations
of Regular Expressions



Clemens Grabmayer



Department of Computer Science, GSSI, L'Aquila, Italy



LICS 2022

Technion, Haifa, Israel

August 4, 2022

Process semantics of regular expressions $\llbracket \cdot \rrbracket_{\mathbf{P}}$ (Milner, 1984)

$0 \xrightarrow{\llbracket \cdot \rrbracket_{\mathbf{P}}} \text{deadlock } \delta, \text{ no termination}$

$1 \xrightarrow{\llbracket \cdot \rrbracket_{\mathbf{P}}} \text{empty-step process } \epsilon, \text{ then terminate}$

$a \xrightarrow{\llbracket \cdot \rrbracket_{\mathbf{P}}} \text{atomic action } a, \text{ then terminate}$

$e + f \xrightarrow{\llbracket \cdot \rrbracket_{\mathbf{P}}} (\text{choice}) \text{ execute } \llbracket e \rrbracket_{\mathbf{P}} \text{ or } \llbracket f \rrbracket_{\mathbf{P}}$

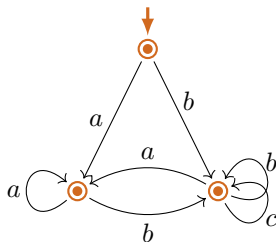
$e \cdot f \xrightarrow{\llbracket \cdot \rrbracket_{\mathbf{P}}} (\text{sequentialization}) \text{ execute } \llbracket e \rrbracket_{\mathbf{P}}, \text{ then } \llbracket f \rrbracket_{\mathbf{P}}$

$e^* \xrightarrow{\llbracket \cdot \rrbracket_{\mathbf{P}}} (\text{iteration}) \text{ repeat (terminate or execute } \llbracket e \rrbracket_{\mathbf{P}})$

$\llbracket e \rrbracket_{\mathbf{P}} := [\mathcal{C}(e)]_{\leftrightarrow} \quad (\text{bisimilarity equivalence class of chart } \mathcal{C}(e))$

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

chart (Milner)



$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart

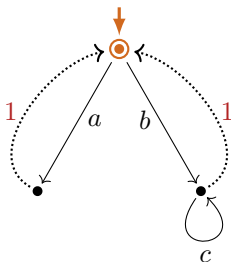
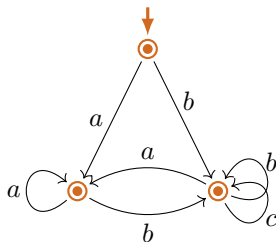


chart (Milner)



$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart

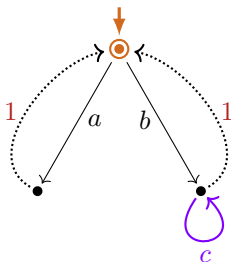
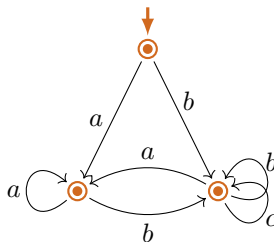


chart (Milner)



$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart

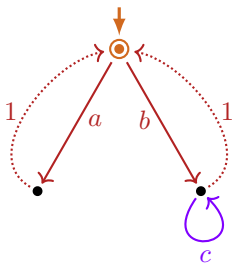
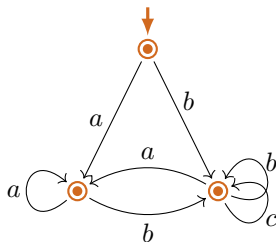


chart (Milner)



$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart

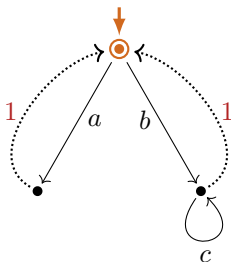
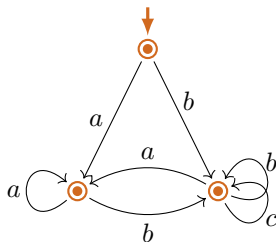


chart (Milner)



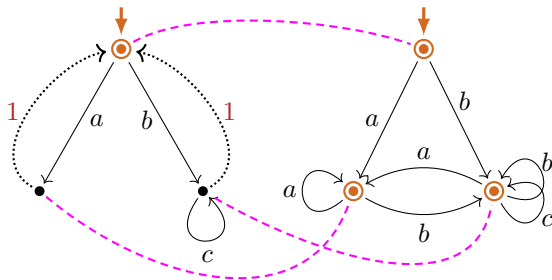
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart

chart (Milner)

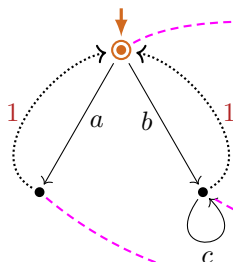


$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

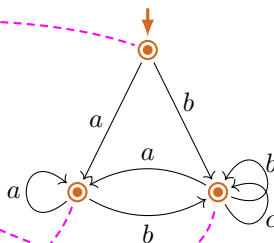
Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart



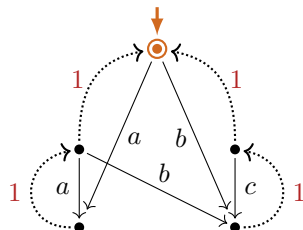
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

chart (Milner)



$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

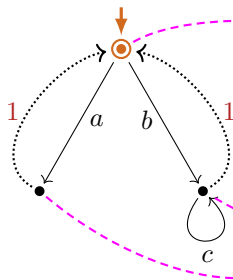
1-chart



$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$

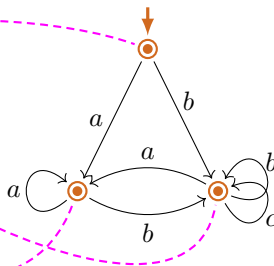
Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart



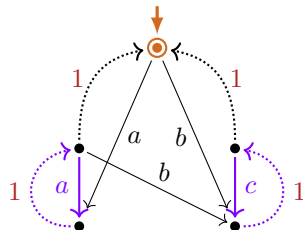
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

chart (Milner)



$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

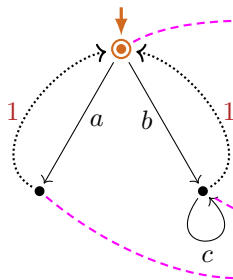
1-chart



$$\begin{aligned} &\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \\ &\quad \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 \\ &\quad + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*) \end{aligned}$$

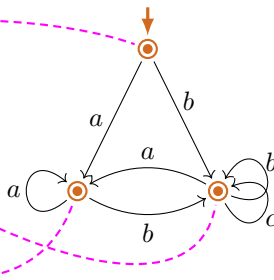
Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart



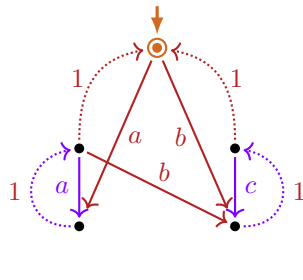
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

chart (Milner)



$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

1-chart



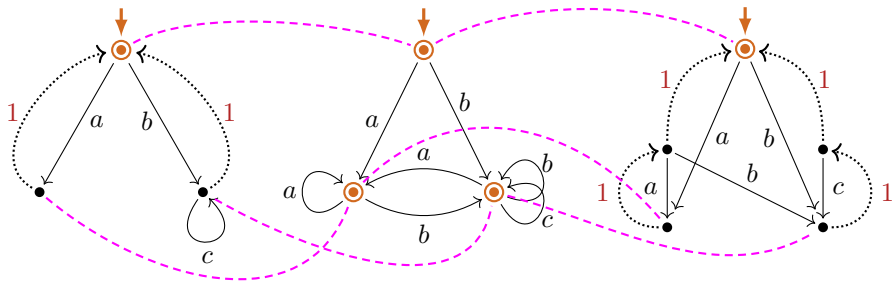
$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart

chart (Milner)

1-chart



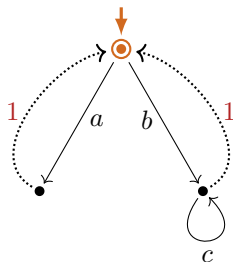
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\begin{aligned} \underline{\mathcal{C}} & \left((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \right. \\ & \quad \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 \\ & \quad \left. + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^* \right) \end{aligned}$$

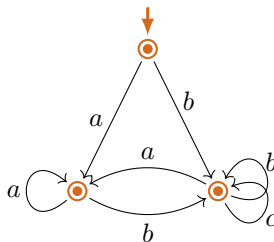
Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart



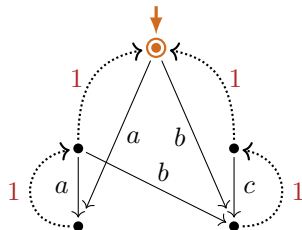
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

chart (Milner)



$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

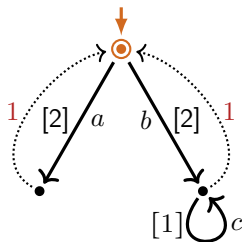
1-chart



$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

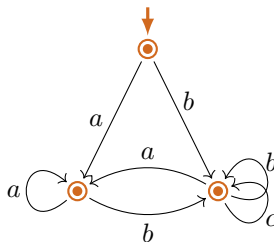
1-chart



LLEE

$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

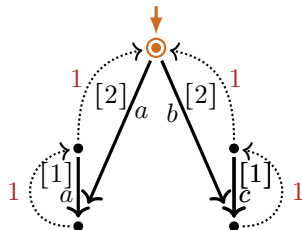
chart (Milner)



LLEE

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

1-chart



LLEE

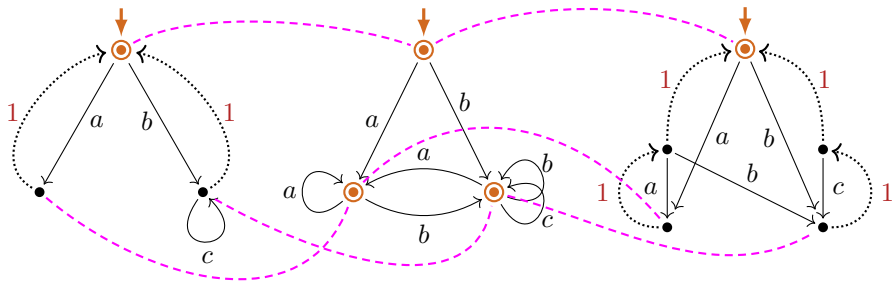
$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart

chart (Milner)

1-chart



LLEE

LLEE

LLEE

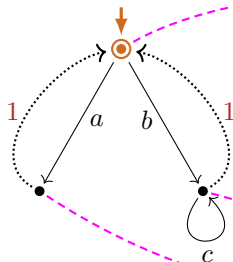
$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

1-chart

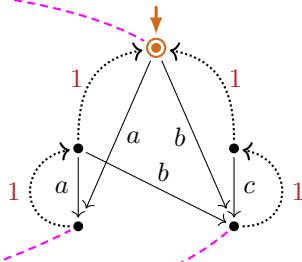


LLEE

$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

\longleftrightarrow

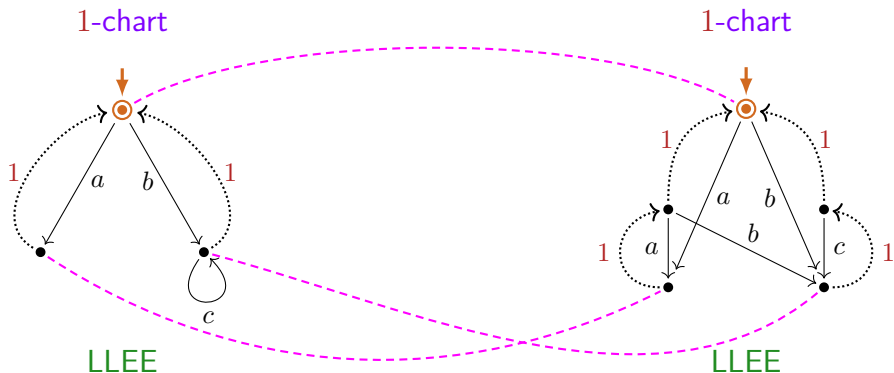
1-chart



LLEE

$$\underline{\mathcal{C}}\left(\left(a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1))\right.\right. \\ \left.\left.\cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1\right.\right. \\ \left.\left.+ b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1\right)^*\right)$$

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)

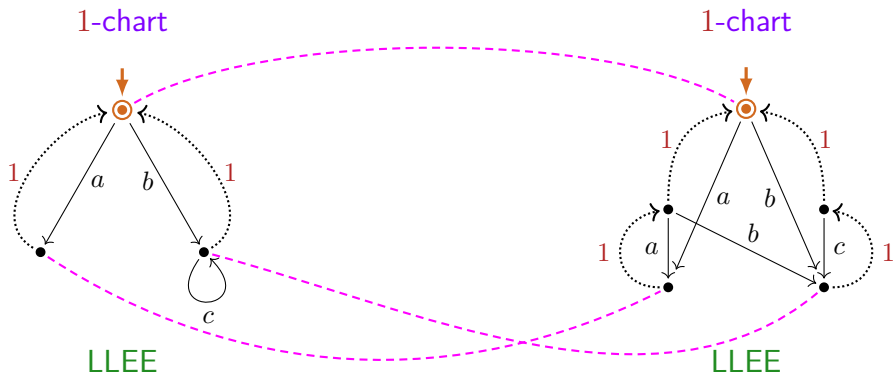


$$\llbracket (a \cdot 1 + b \cdot (c^* \cdot 1))^* \rrbracket_P$$

=

$$\begin{aligned} & \llbracket (a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \\ & \quad \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 \\ & \quad + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^* \rrbracket_P \end{aligned}$$

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)



$$(a \cdot 1 + b \cdot (c^* \cdot 1))^*$$

\equiv_P

$$\begin{aligned} & (a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \\ & \quad \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 \\ & \quad + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^* \end{aligned}$$

Milner's proof system Mil

Axioms:

$$(A1) \quad e + (f + g) = (e + f) + g$$

$$(A2) \quad e + 0 = e$$

$$(A3) \quad e + f = f + e$$

$$(A4) \quad e + e = e$$

$$(A5) \quad e \cdot (f \cdot g) = (e \cdot f) \cdot g$$

$$(A6) \quad (e + f) \cdot g = e \cdot g + f \cdot g$$

$$(A7) \quad e = 1 \cdot e$$

$$(A8) \quad e = e \cdot 1$$

$$(A9) \quad 0 = 0 \cdot e$$

$$(A10) \quad e^* = 1 + e \cdot e^*$$

$$(A11) \quad e^* = (1 + e)^*$$

$$\text{But: } e \cdot (f + g) \neq e \cdot f + e \cdot g$$

$$\text{But: } e \cdot 0 \neq 0$$

Inference rules: rules of equational logic plus

$$\frac{e = f \cdot e + g}{e = f^* \cdot g} \text{RSP}^* \text{ (if } f \text{ does not terminate immediately)}$$

Milner's question (1984)

Is Mil complete with respect to \equiv_P ? (Does $e \equiv_P f \implies e =_{\text{Mil}} f$ hold?)

Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

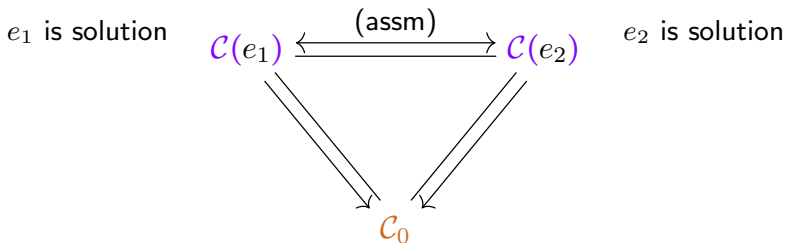
For **1-free** regular expressions e_1 and e_2 :

$$e_1 \text{ is solution} \quad \mathcal{C}(e_1) \xrightleftharpoons{(\text{assm})} \mathcal{C}(e_2) \quad e_2 \text{ is solution}$$

$$e_1 =_{\text{Mil}} e_2$$

Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

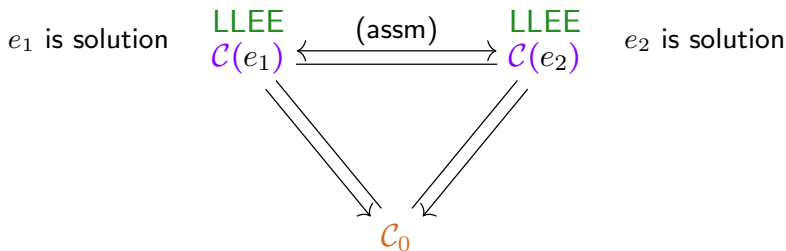
For **1-free** regular expressions e_1 and e_2 :



$$e_1 =_{\text{Mil}} e_2$$

Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

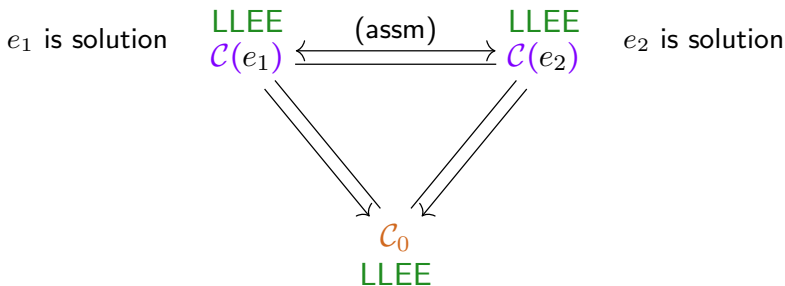
For **1-free** regular expressions e_1 and e_2 :



$$e_1 =_{\text{Mil}} e_2$$

Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

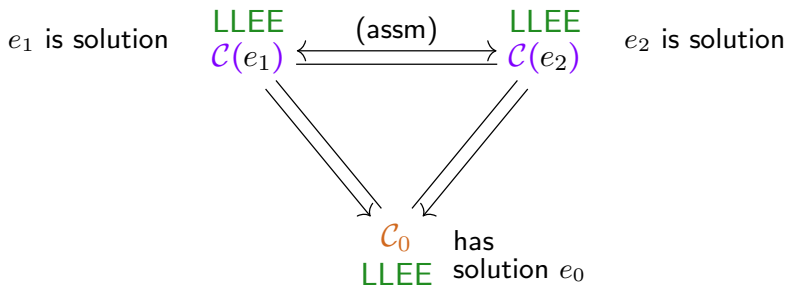
For **1-free** regular expressions e_1 and e_2 :



$$e_1 =_{\text{Mil}} e_2$$

Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

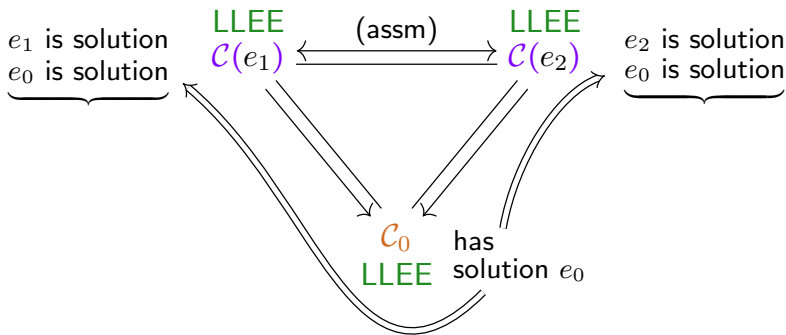
For **1-free** regular expressions e_1 and e_2 :



$$e_1 =_{\text{Mil}} e_2$$

Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

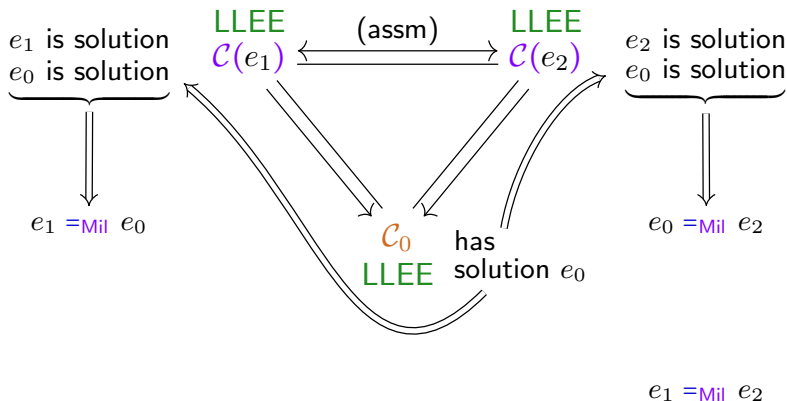
For **1-free** regular expressions e_1 and e_2 :



$$e_1 =_{\text{Mil}} e_2$$

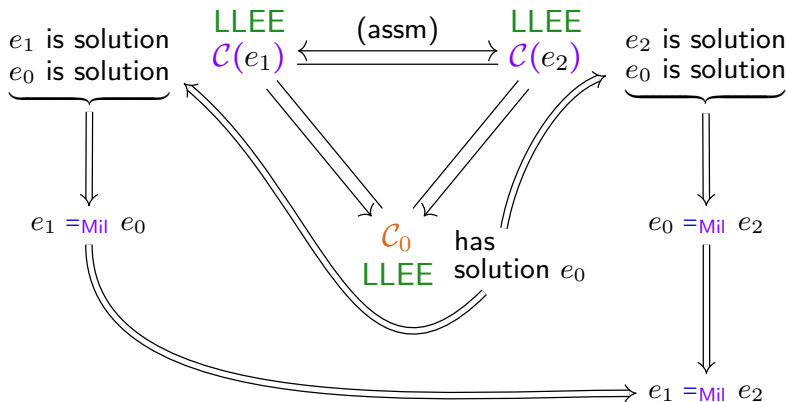
Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

For **1-free** regular expressions e_1 and e_2 :



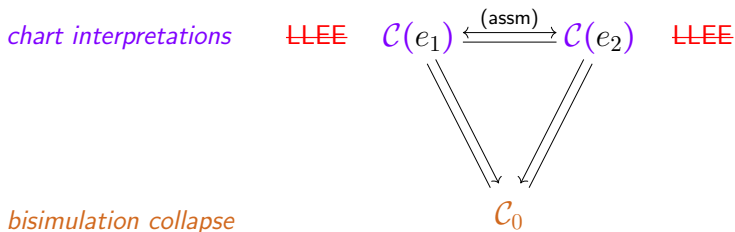
Bisimulation collapse proof strategy (G/Fokkink, LICS'20)

For **1-free** regular expressions e_1 and e_2 :



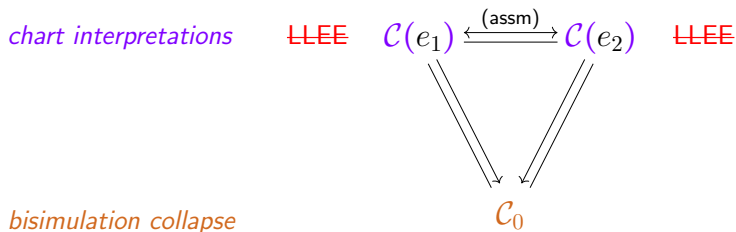
Bisimulation collapse proof strategy (general case)

Problem 1



Bisimulation collapse proof strategy (general case)

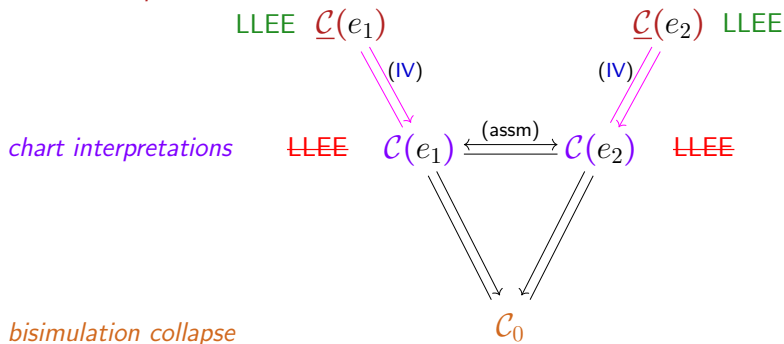
Remedy for Problem 1 (*G, TERMGRAPH 2020*)



Bisimulation collapse proof strategy (general case)

Remedy for Problem 1 (*G, TERMGRAPH 2020*)

1-chart interpretations



Bisimulation collapse proof strategy (general case)

Remedy for Problem 1 (*G, TERMGRAPH 2020*)

1-chart interpretations

LLEE $\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$ LLEE

chart interpretations

$\text{LLEE } C(e_1) \xleftrightarrow{\text{(assm)}} C(e_2) \text{ LLEE}$

bisimulation collapse

C_0

Bisimulation collapse proof strategy (general case)

Remedy for Problem 1 (*G, TERMGRAPH 2020*)

1-chart interpretations

LLEE $\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$ LLEE

chart interpretations

LLEE $C(e_1) \xleftrightarrow{\text{(assm)}} C(e_2)$ LLEE

bisimulation collapse

C_0

Bisimulation collapse proof strategy (general case)

Remedy for Problem 1 (*G, TERMGRAPH 2020*)

1-chart interpretations

LLEE $\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$ LLEE

chart interpretations

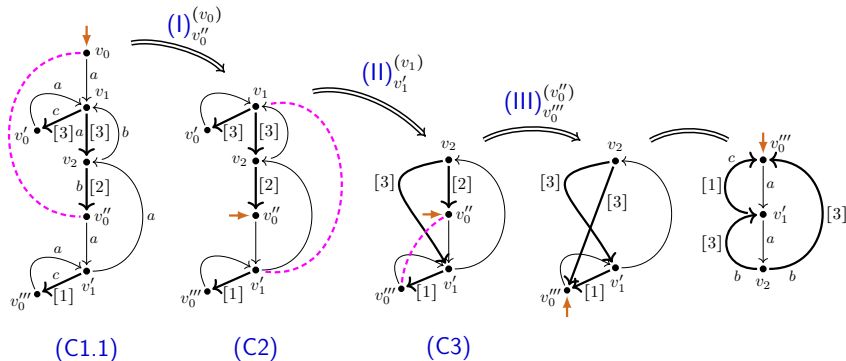
LLEE $C(e_1) \xleftrightarrow{\text{(assm)}} C(e_2)$ LLEE

bisimulation collapse

C_0 LLEE?

LLEE-preserving collapse of LLEE-charts (G/Fokkink, LICS'20)

(no 1-transitions!)

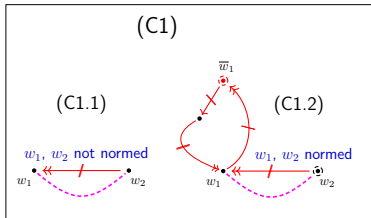


Lemma

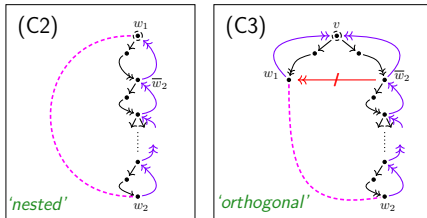
The bisimulation collapse of a LLEE-chart is again a LLEE-chart.

Reduced bisimilarity redundancies in LLEE-charts (no 1-trans.!) (G/Fokkink, LICS'20)

w_1, w_2 in different scc's



w_1, w_2 in the same scc

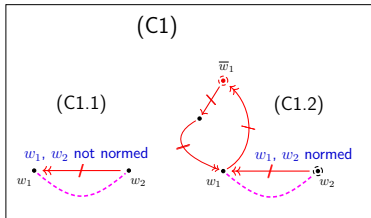


Lemma

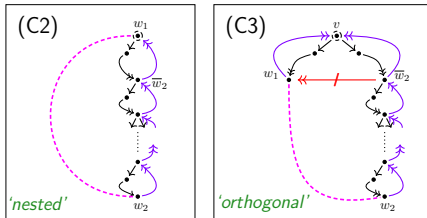
Every *not collapsed* LLEE-chart contains bisimilar vertices $w_1 \neq w_2$ of kind (C1), (C2), or (C3) (a *reduced bisimilarity redundancy* $\langle w_1, w_2 \rangle$):

Reduced bisimilarity redundancies in LLEE-charts (no 1-trans.!) (G/Fokkink, LICS'20)

w_1, w_2 in different scc's



w_1, w_2 in the same scc



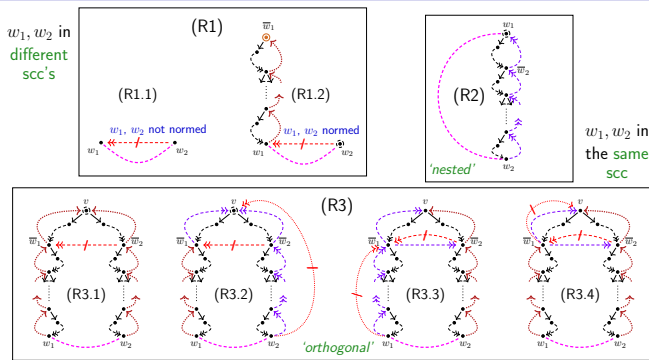
Lemma

Every *not collapsed* LLEE-chart contains bisimilar vertices $w_1 \neq w_2$ of kind (C1), (C2), or (C3) (a *reduced bisimilarity redundancy* $\langle w_1, w_2 \rangle$):

Lemma

Every *reduced bisimilarity redundancy* in a LLEE-chart can be eliminated LLEE-preservingly.

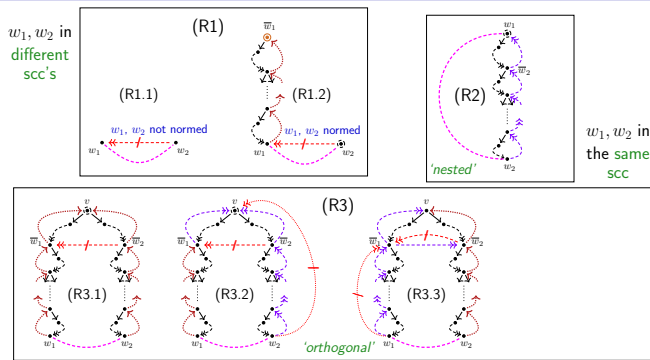
Reduced 1-bisimilarity redundancies in LLEE-1-charts



Lemma

Every *not collapsed* LLEE-1-chart contains 1-bisimilar vertices $w_1 \neq w_2$ (a *reduced 1-bisimilarity redundancy* $\langle w_1, w_2 \rangle$) of kind (R1), (R2), (R3).

Reduced 1-bisimilarity redundancies in LLEE-1-charts



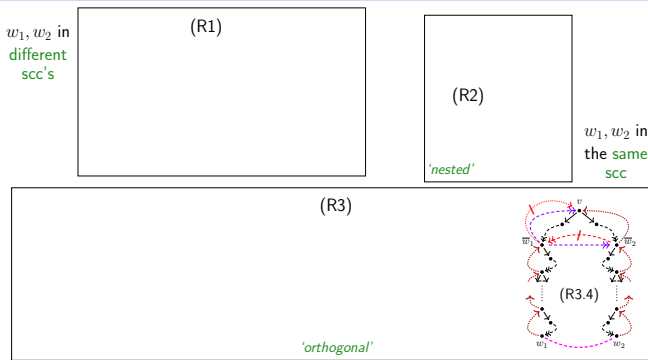
Lemma

Every *not collapsed* LLEE-1-chart contains 1-bisimilar vertices $w_1 \neq w_2$ (a *reduced 1-bisimilarity redundancy* $\langle w_1, w_2 \rangle$) of kind (R1), (R2), (R3).

Lemma

Every *simple* reduced 1-bisimilarity redundancies in a LLEE-1-chart can be eliminated LLEE-preservingly.

Reduced 1-bisimilarity redundancies in LLEE-1-charts



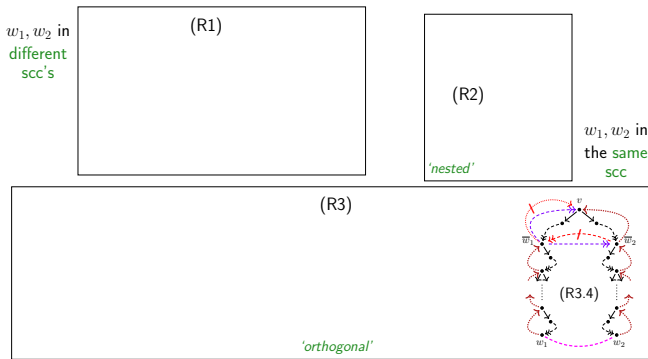
Lemma

Every *not collapsed* LLEE-1-chart contains 1-bisimilar vertices $w_1 \neq w_2$ (a *reduced 1-bisimilarity redundancy* $\langle w_1, w_2 \rangle$) of kind (R1), (R2), (R3).

Stumbling Block

How to LLEE-preservingly eliminate reduced 1-bisimilarity redundancies of kind (R3.4)?

Reduced 1-bisimilarity redundancies in LLEE-1-charts



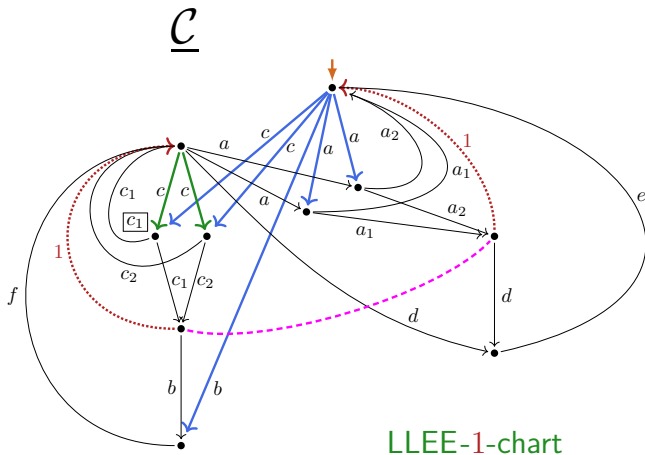
Lemma

Every *not collapsed* LLEE-1-chart contains 1-bisimilar vertices $w_1 \neq w_2$ (a *reduced 1-bisimilarity redundancy* $\langle w_1, w_2 \rangle$) of kind (R1), (R2), (R3).

Stumbling Block

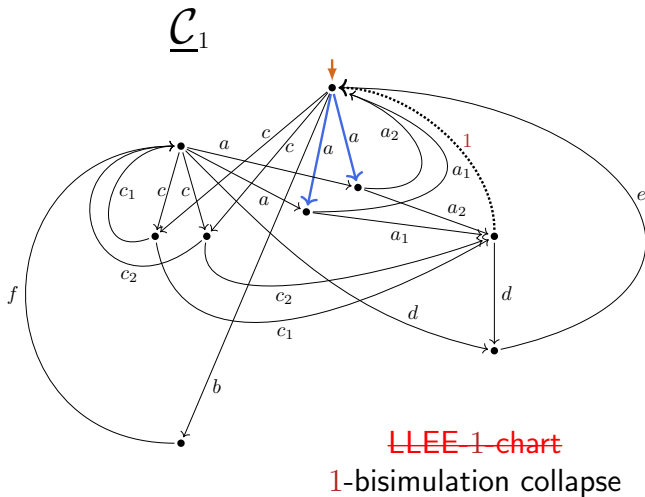
How to LLEE-preservingly eliminate
precrySTALLine reduced 1-bisimilarity redundancies?

Counterexample LLEE-preserving collapse

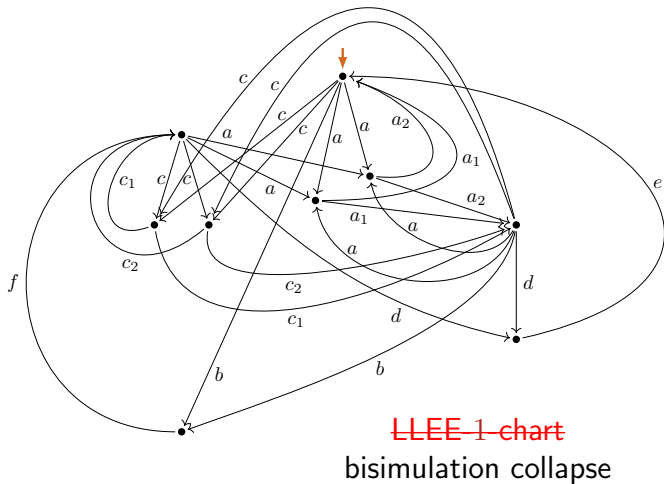


$\langle w_1, w_2 \rangle$ is a reduced
1-bisimilarity redundancy of kind (R3.4)

Counterexample LLEE-preserving collapse



Counterexample LLEE-preserving collapse



Bisimulation collapse proof strategy (general case)

Problem 2: There are regular expressions e_1 and e_2 such that:

1-chart interpretations

LLEE $\underline{C}(e_1)$ \longleftrightarrow $\underline{C}(e_2)$ LLEE

chart interpretations

LLEE $C(e_1)$ $\xleftrightarrow{\text{(assm)}} C(e_2)$ LLEE

bisimulation collapse

C_0
LLEE

Bisimulation collapse proof strategy (general case)

Problem 2: There are regular expressions e_1 and e_2 such that:

1-chart interpretations

LLEE $\underline{\mathcal{C}}(e_1)$ \longleftrightarrow $\underline{\mathcal{C}}(e_2)$ LLEE

chart interpretations

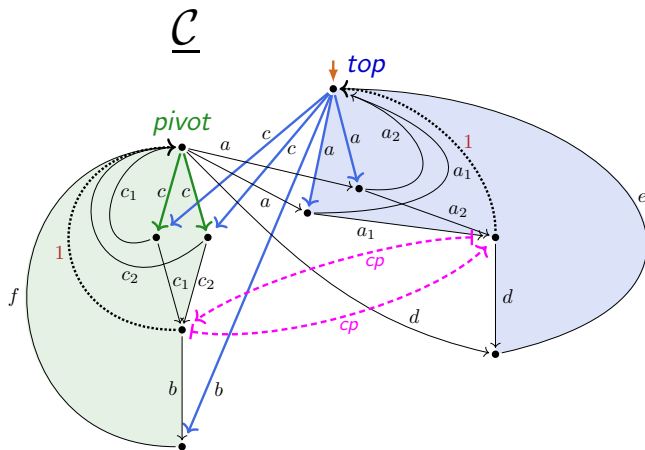
LLEE $\mathcal{C}(e_1)$ $\xleftrightarrow{\text{(assm)}} \mathcal{C}(e_2)$ LLEE

for all \mathcal{C}_0 :

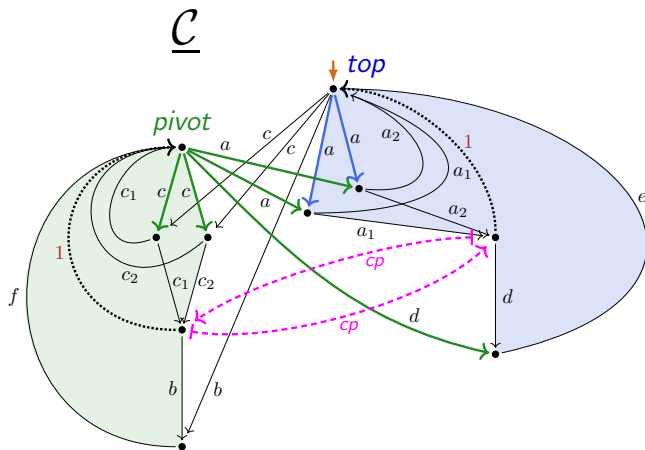
LLEE

$\mathcal{C}(e_1)$, $\mathcal{C}(e_2)$, $\underline{\mathcal{C}}(e_1)$ and $\underline{\mathcal{C}}(e_2)$ are **not** LLEE-preservingly jointly minimizable under bisimilarity.

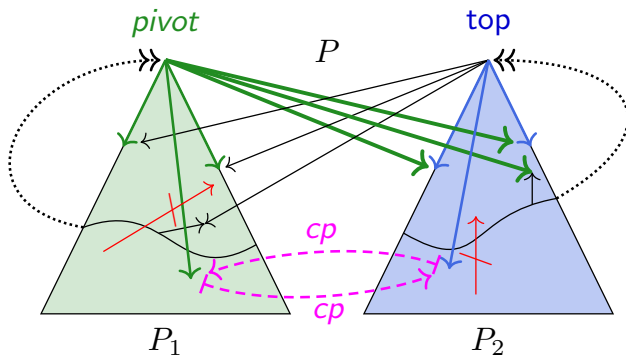
Twin-Crystal



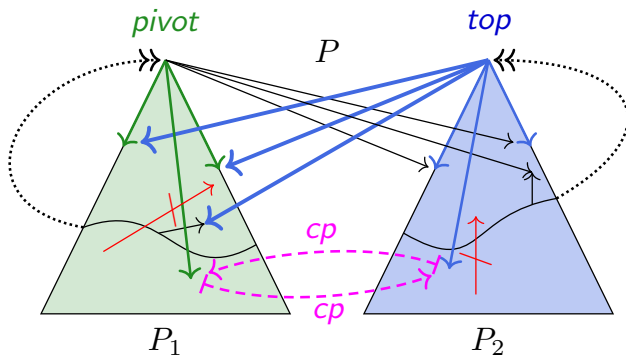
Twin-Crystal



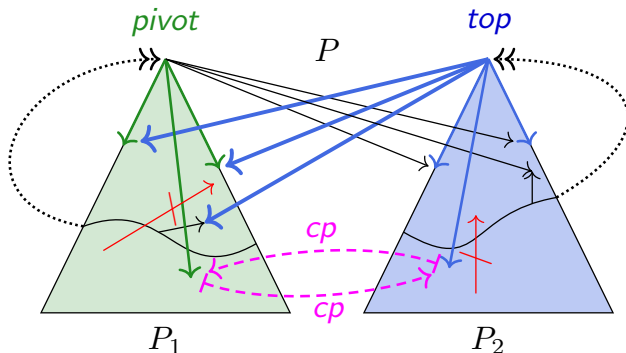
Twin-Crystal



Twin-Crystal

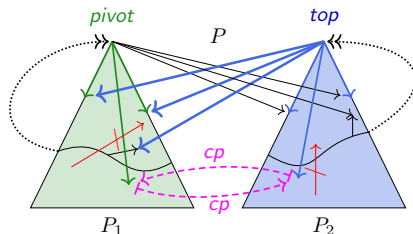


Twin-Crystal



- Provable solutions of **twin-crystals** are **complete**:
they can be transferred to their **bisimulation collapses**

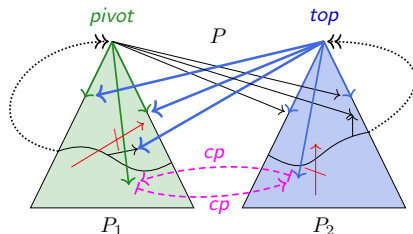
Crystallization



twin-crystal

Crystallized 1-charts = LLEE-1-charts that are collapsed apart from some scc's that are twin-crystals.

Crystallization

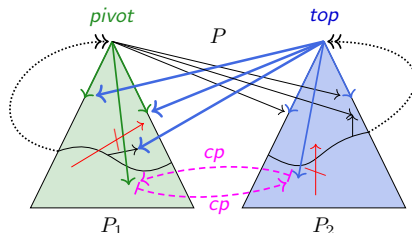


twin-crystal

Crystallized 1-charts = LLEE-1-charts that are collapsed apart from some scc's that are twin-crystals.

(CR) Crystallization: Every LLEE-1-chart can be reduced under bisimilarity to a bisimilar crystallized 1-chart.

Crystallization



twin-crystal

Crystallized 1-charts = **LLEE-1-charts** that are collapsed apart from some scc's that are **twin-crystals**.

(CR) Crystallization: Every **LLEE-1-chart** can be reduced under **bisimilarity** to a bisimilar **crystallized 1-chart**.

(CC) Every provable solution of a **crystallized 1-chart** gives rise to provable solution on the **bisimulation collapse**.

Completeness proof of Mil (structure)

chart interpretations

$$\mathcal{C}(e_1) \xLeftrightarrow{(\text{assm})} \mathcal{C}(e_2)$$

$\stackrel{?}{\Longrightarrow}$

$$e_1 =_{\text{Mil}} e_2$$

Completeness proof of Mil (structure)

chart interpretations

$$\mathcal{C}(e_1) \xLeftrightarrow{(\text{assm})} \mathcal{C}(e_2)$$

Completeness proof of Mil (structure)

1-chart interpretations

LLEE
 e_1 is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

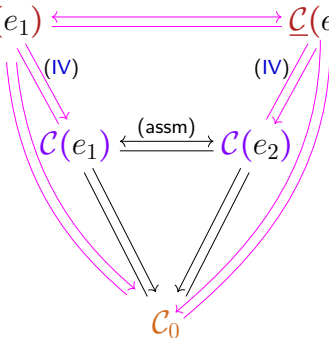
LLEE
 e_2 is solution

chart interpretations

$C(e_1) \xleftrightarrow{\text{(assm)}} C(e_2)$

bisimulation collapse

C_0



Completeness proof of Mil (structure)

1-chart interpretations

LLEE
 e_1 is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE
 e_2 is solution

chart interpretations

(IV) $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$
 (IV) $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$
 (assm) $C(e_1) \longleftrightarrow C(e_2)$
 (CR) $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

crystallized 1-chart

LLEE \underline{C}_{10}

bisimulation collapse

C_0

Completeness proof of Mil (structure)

1-chart interpretations

LLEE
 e_1 is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE
 e_2 is solution

chart interpretations

(IV) $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$
 (IV) $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$
 (CR) $\underline{C}(e_1) \xrightarrow{(CR)} C(e_1)$
 (assm) $C(e_1) \longleftrightarrow C(e_2)$

crystallized 1-chart

LLEE

\underline{C}_{10}

bisimulation collapse

C_0

Completeness proof of Mil (structure)

1-chart interpretations

LLEE
 e_1 is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE
 e_2 is solution

chart interpretations

(IV) $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$
 (IV) $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$
 (assm) $C(e_1) \longleftrightarrow C(e_2)$
 (CR) $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

crystallized 1-chart

LLEE
 e_{10} is complete solution

\underline{C}_{10}

bisimulation collapse

C_0

Completeness proof of Mil (structure)

1-chart interpretations

LLEE
 e_1 is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE
 e_2 is solution

chart interpretations

(IV) $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$
 (IV) $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$
 (assm) $C(e_1) \longleftrightarrow C(e_2)$
 (CR) $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

crystallized 1-chart

LLEE
 e_{10} is complete solution

\underline{C}_{10}

bisimulation collapse

C_0 e_{10} is solution (by (CC))

Completeness proof of Mil (structure)

1-chart interpretations

LLE
 e_1 is solution
 (by (T)) e_{10} is solution

LEE

$$\underline{C}(e_1)$$

(IV)

$$\Rightarrow \underline{C}(e_2)$$

LEE

e_2 is solution
 e_{10} is solution

(by

chart interpretations

(CR)

$$\mathcal{C}(e_1) \xleftarrow{\text{(assm)}}$$

(assm)

crystallized 1-chart

LEE
 e_{10} is complete solution

LEE

$$\underline{\mathcal{C}}_{10}$$
$$C_0$$

e_{10} is solution (by (CC))

bisimulation collapse

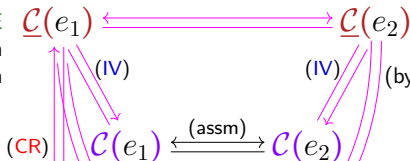
Completeness proof of Mil (structure)

1-chart interpretations

LLEE $\underline{C}(e_1)$
 e_1 is solution
 (by (T)) e_{10} is solution

LLEE $\underline{C}(e_2)$
 e_2 is solution
 (by (T)) e_{10} is solution

chart interpretations



crystallized 1-chart

LLEE \underline{C}_{10}
 e_{10} is complete solution

bisimulation collapse

C_0 e_{10} is solution (by (CC))

$C(e_1)$ LLEE
 e_1 is solution
 e_{10} is solution

$C(e_2)$ LLEE
 e_2 is solution
 e_{10} is solution

Completeness proof of Mil (structure)

1-chart interpretations

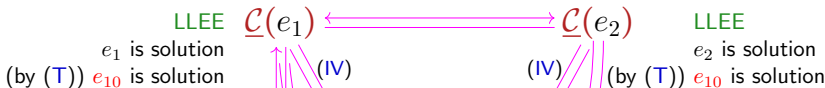
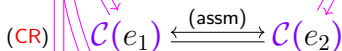


chart interpretations



crystallized 1-chart

LLEE \underline{C}_{10}
 e_{10} is complete solution

bisimulation collapse

C_0 e_{10} is solution (by (CC))

$\left. \begin{array}{l} C(e_1) \text{ LLEE} \\ e_1 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right\} \xRightarrow{\text{(SE)}} e_1 =_{\text{Mil}} e_{10}$

$\left. \begin{array}{l} C(e_2) \text{ LLEE} \\ e_2 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right\}$

Completeness proof of Mil (structure)

1-chart interpretations

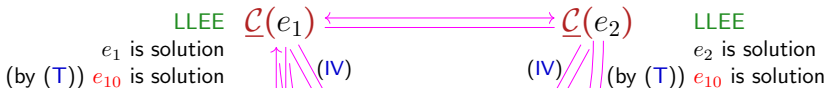


chart interpretations

crystallized 1-chart

LLEE \underline{C}_{10}
 e_{10} is complete solution

bisimulation collapse

C_0 e_{10} is solution (by (CC))

$$\left. \begin{array}{l} C(e_1) \text{ LLEE} \\ e_1 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right\} \xRightarrow{(SE)} e_1 =_{\text{Mil}} e_{10} \quad e_{10} =_{\text{Mil}} e_2 \xleftarrow{(SE)} \left\{ \begin{array}{l} C(e_2) \text{ LLEE} \\ e_2 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right.$$

Completeness proof of Mil (structure)

1-chart interpretations

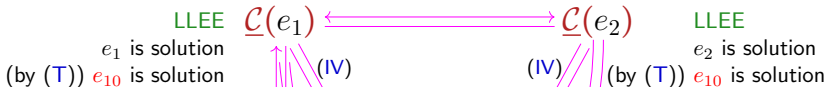


chart interpretations

crystallized 1-chart

bisimulation collapse

$$\left. \begin{array}{l} C(e_1) \text{ LLEE} \\ e_1 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right\} \xRightarrow{(SE)} e_1 =_{\text{Mil}} e_{10} \quad e_{10} =_{\text{Mil}} e_2 \xleftarrow{(SE)} \left\{ \begin{array}{l} C(e_2) \text{ LLEE} \\ e_2 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right.$$

$$\implies e_1 =_{\text{Mil}} e_2$$

Completeness proof of Mil (structure)

1-chart interpretations

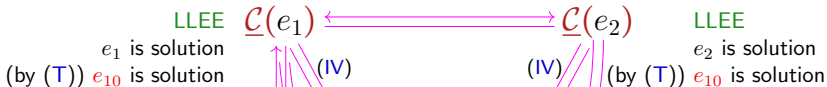
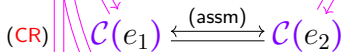


chart interpretations



crystallized 1-chart

LLEE \underline{C}_{10}
 e_{10} is complete solution

bisimulation collapse

C_0 e_{10} is solution (by (CC))

Theorem

Milner's proof system Mil is complete
for process semantics equivalence \equiv_P of regular expressions.

Since: $e_1 \equiv_P e_2 \implies \llbracket e_1 \rrbracket_P = \llbracket e_2 \rrbracket_P \implies C(e_1) \leftrightarrow C(e_2) \implies e_1 =_{\text{Mil}} e_2$.

Outlook

poster presentation

- ▶ tomorrow, 10–10.30

next steps and projects

- ▶ monograph project: proof in fine-grained detail
- ▶ computation/animation tool for crystallization
- ▶ use crystallization for [recognition problem](#)

resources on Github:

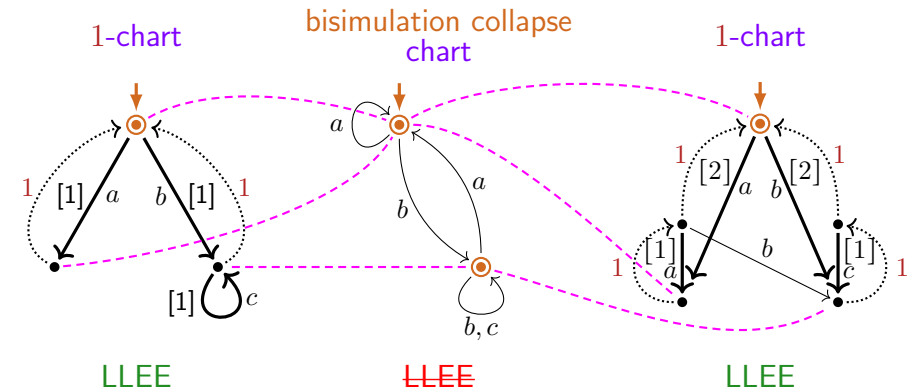
- ▶ <https://github.com/clegra/crystallization/blob/main>
 - ▶ article (after rebuttal): [/cryst-article.pdf](#)
 - ▶ poster: [/poster-lics2022.pdf](#)
 - ▶ presentation: [/presentation-lics2022.pdf](#)

acknowledgment & thanks to:

- ▶ [Wan Fokkink](#) (for long collaboration)

Thank you for your attention!

Process semantics $\llbracket \cdot \rrbracket_P$ (examples, bisimulation collapse)



$$\underline{\mathcal{C}}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

$$\mathcal{C}((a \cdot 1 + b \cdot (c^* \cdot 1))^*)$$

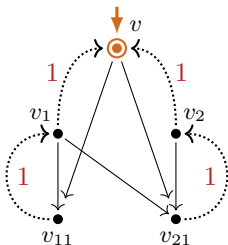
$$\underline{\mathcal{C}}((a \cdot (1 \cdot ((a \cdot 1)^* \cdot 1)) \cdot b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1 + b \cdot ((1 \cdot (c \cdot 1)^*)) \cdot 1)^*)$$

$$\underline{\mathcal{C}}((a + b \cdot c^*)^*)$$

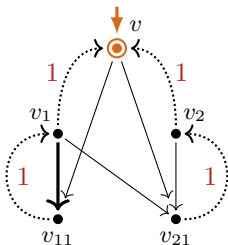
$$\mathcal{C}((a + b \cdot c^*)^*)$$

$$\underline{\mathcal{C}}((a^* \cdot (1 + b \cdot c^*))^*)$$

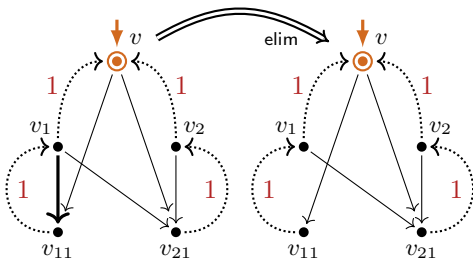
Layered loop existence/elimination and LLEE-witnesses



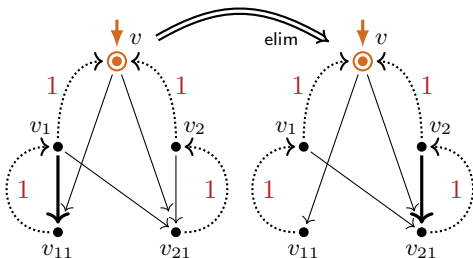
Layered loop existence/elimination and LLEE-witnesses



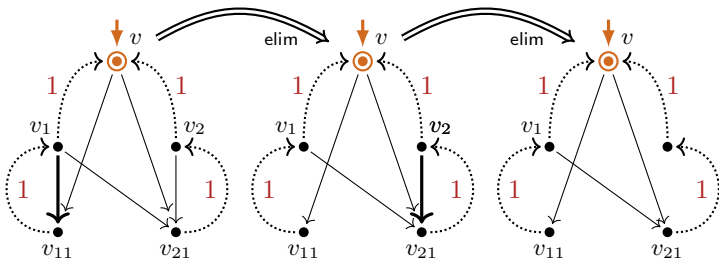
Layered loop existence/elimination and LLEE-witnesses



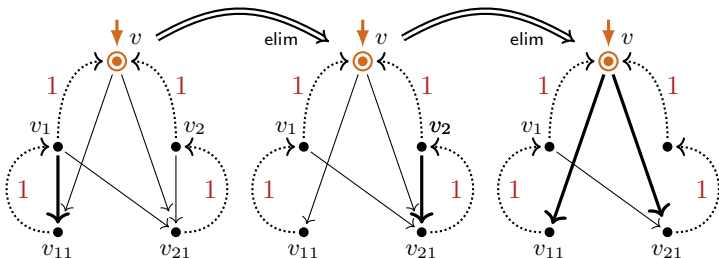
Layered loop existence/elimination and LLEE-witnesses



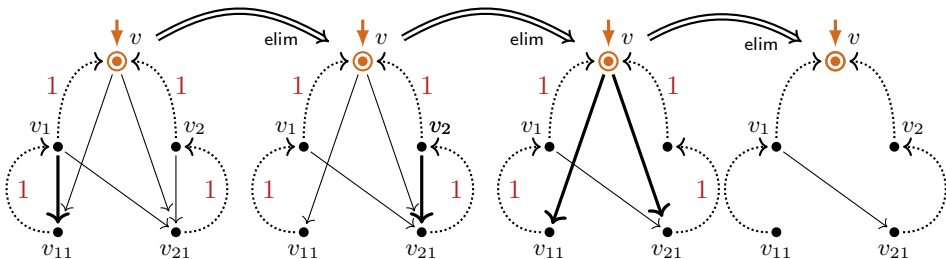
Layered loop existence/elimination and LLEE-witnesses



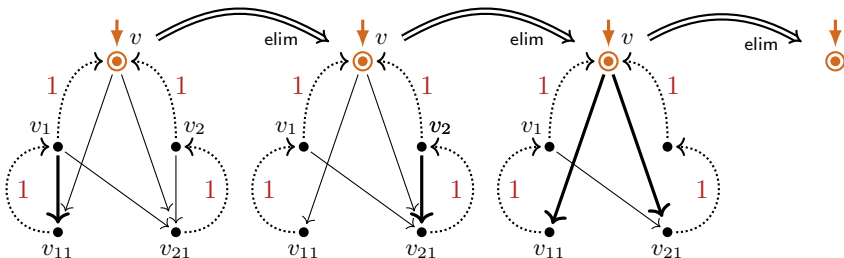
Layered loop existence/elimination and LLEE-witnesses



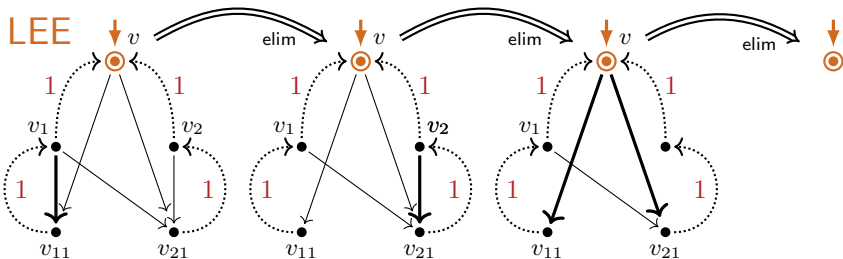
Layered loop existence/elimination and LLEE-witnesses



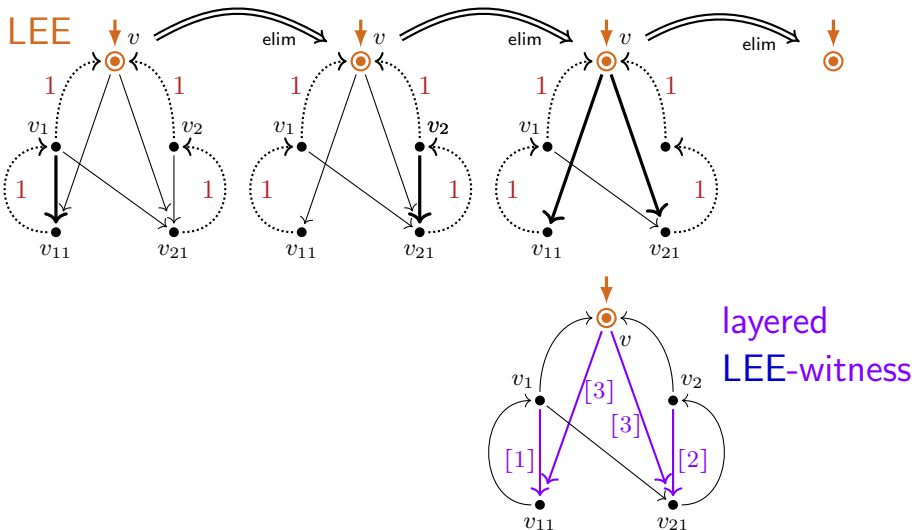
Layered loop existence/elimination and LLEE-witnesses



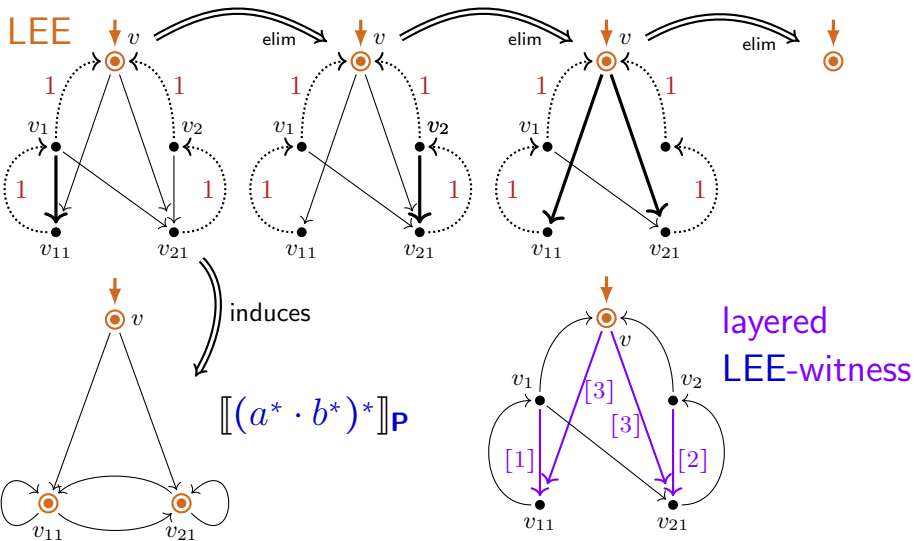
Layered loop existence/elimination and LLEE-witnesses



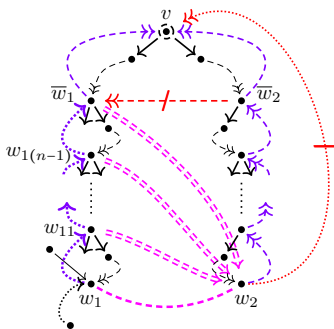
Layered loop existence/elimination and LLEE-witnesses



Layered loop existence/elimination and LLEE-witnesses

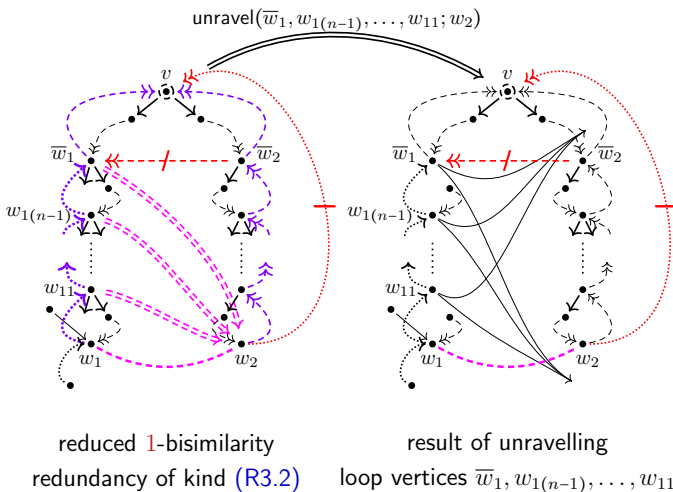


Eliminating reduced 1-bisimilarity redundancies (example)

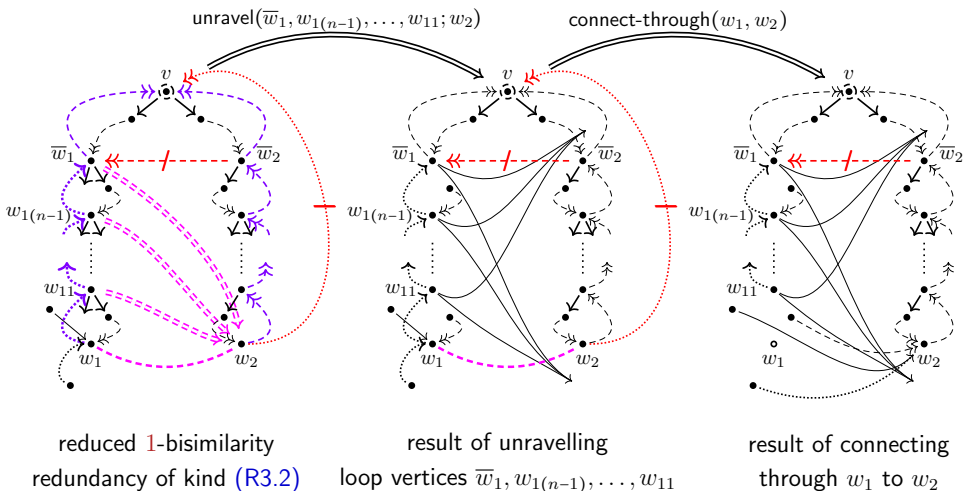


reduced 1-bisimilarity
redundancy of kind (R3.2)

Eliminating reduced 1-bisimilarity redundancies (example)



Eliminating reduced 1-bisimilarity redundancies (example)



Completeness proof of Mil (structure)

chart interpretations

$$\mathcal{C}(e_1) \xLeftrightarrow{(\text{assm})} \mathcal{C}(e_2)$$

$\xRightarrow{?}$

$$e_1 =_{\text{Mil}} e_2$$

Completeness proof of Mil (structure)

chart interpretations

$$\mathcal{C}(e_1) \xLeftrightarrow{(\text{assm})} \mathcal{C}(e_2)$$

Completeness proof of Mil (structure)

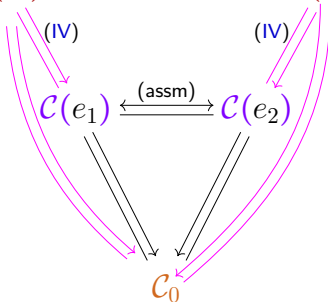
1-chart interpretations

guarded, LLEE
 e_1 is solution

$$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$$

LLEE, guarded
 e_2 is solution

chart interpretations



bisimulation collapse

Completeness proof of Mil (structure)

1-chart interpretations

guarded, LLEE
 e_1 is solution

$\underline{C}(e_1)$

$\underline{C}(e_2)$

LLEE, guarded
 e_2 is solution

chart interpretations

(CR)

$C(e_1)$

$C(e_2)$

(assm)

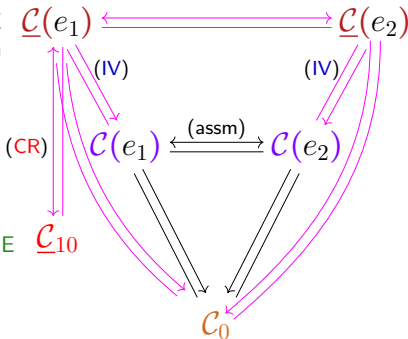
crystallized 1-chart

guarded, LLEE

\underline{C}_{10}

bisimulation collapse

C_0



Completeness proof of Mil (structure)

1-chart interpretations

guarded, LLEE
 e_1 is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE, guarded
 e_2 is solution

chart interpretations

(IV) $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$ $C(e_1) \xrightarrow{(IV)} \underline{C}(e_2)$
(CR) $\underline{C}(e_1) \xrightarrow{(CR)} C(e_1)$ $C(e_2) \xrightarrow{(CR)} \underline{C}(e_2)$
(assm) $C(e_1) \xrightarrow{(assm)} C(e_2)$

crystallized 1-chart

guarded, LLEE

\underline{C}_{10}

bisimulation collapse

C_0

Completeness proof of Mil (structure)

1-chart interpretations

guarded, LLEE
 e_1 is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE, guarded
 e_2 is solution

chart interpretations

(IV) $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$
 (IV) $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$
 (assm) $C(e_1) \longleftrightarrow C(e_2)$
 (CR) $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

crystallized 1-chart

guarded, LLEE
 e_{10} is complete solution

\underline{C}_{10}

bisimulation collapse

C_0

Completeness proof of Mil (structure)

1-chart interpretations

guarded, LLEE
 e_1 is solution

$\underline{C}(e_1)$

$\underline{C}(e_2)$

LLEE, guarded
 e_2 is solution

chart interpretations

(CR)

$C(e_1)$

$C(e_2)$

(assm)

crystallized 1-chart

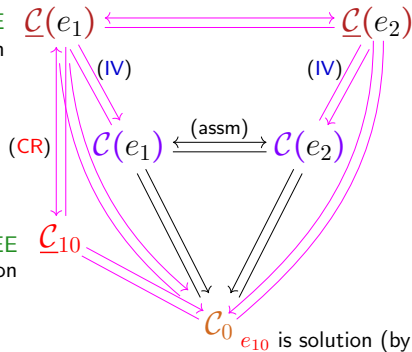
guarded, LLEE
 e_{10} is complete solution

\underline{C}_{10}

bisimulation collapse

C_0

e_{10} is solution (by (CC))



Completeness proof of Mil (structure)

1-chart interpretations

guarded, LLEE
 e_1 is solution
 (by (T)) e_{10} is solution

$\underline{C}(e_1)$

$\underline{C}(e_2)$

LLEE, guarded
 e_2 is solution
 (by (T)) e_{10} is solution

chart interpretations

(CR)

$C(e_1)$

(assm)

$C(e_2)$

crystallized 1-chart

guarded, LLEE
 e_{10} is complete solution

\underline{C}_{10}

bisimulation collapse

C_0

e_{10} is solution (by (CC))

Completeness proof of Mil (structure)

1-chart interpretations

guarded, LLEE $\underline{C}(e_1)$
 e_1 is solution
 (by (T)) e_{10} is solution

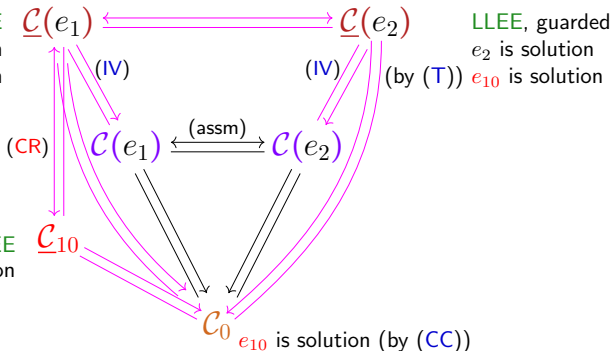
chart interpretations

crystallized 1-chart

guarded, LLEE \underline{C}_{10}
 e_{10} is complete solution

bisimulation collapse

$C(e_1)$ guarded, LLEE
 e_1 is solution
 e_{10} is solution



$C(e_2)$ guarded, LLEE
 e_2 is solution
 e_{10} is solution

Completeness proof of Mil (structure)

1-chart interpretations

guarded, LLEE $\underline{C}(e_1)$
 e_1 is solution
 (by (T)) e_{10} is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE, guarded $\underline{C}(e_2)$
 e_2 is solution
 (by (T)) e_{10} is solution

chart interpretations

(IV) $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$
 (IV) $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$
 (assm) $C(e_1) \longleftrightarrow C(e_2)$
 (CR) $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

crystallized 1-chart

guarded, LLEE C_{10}
 e_{10} is complete solution

bisimulation collapse

C_0 e_{10} is solution (by (CC))

$C(e_1)$ guarded, LLEE
 e_1 is solution
 e_{10} is solution
 $\left. \vphantom{\begin{matrix} C(e_1) \text{ guarded, LLEE} \\ e_1 \text{ is solution} \\ e_{10} \text{ is solution} \end{matrix}} \right\} \xRightarrow{(SE)} e_1 =_{\text{Mil}} e_{10}$

$C(e_2)$ guarded, LLEE
 e_2 is solution
 e_{10} is solution

Completeness proof of Mil (structure)

1-chart interpretations

guarded, LLEE $\underline{C}(e_1)$
 e_1 is solution
 (by (T)) e_{10} is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE, guarded $\underline{C}(e_2)$
 e_2 is solution
 (by (T)) e_{10} is solution

chart interpretations

(IV) $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$
 (IV) $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$
 (assm) $C(e_1) \longleftrightarrow C(e_2)$
 (CR) $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

crystallized 1-chart

guarded, LLEE \underline{C}_{10}
 e_{10} is complete solution

bisimulation collapse

$C_{10} \xrightarrow{} C_0$
 $C(e_1) \xrightarrow{} C_0$
 $C(e_2) \xrightarrow{} C_0$
 e_{10} is solution (by (CC))

$$\left\{ \begin{array}{l} C(e_1) \text{ guarded, LLEE} \\ e_1 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right\} \xRightarrow{(SE)} e_1 =_{\text{Mil}} e_{10} \quad e_{10} =_{\text{Mil}} e_2 \xleftarrow{(SE)} \left\{ \begin{array}{l} C(e_2) \text{ guarded, LLEE} \\ e_2 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right.$$

Completeness proof of Mil (structure)

1-chart interpretations

guarded, LLEE $\underline{C}(e_1)$
 e_1 is solution
 (by (T)) e_{10} is solution

$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$

LLEE, guarded $\underline{C}(e_2)$
 e_2 is solution
 (by (T)) e_{10} is solution

chart interpretations

(IV) $\underline{C}(e_1) \xrightarrow{(IV)} C(e_1)$
 (IV) $\underline{C}(e_2) \xrightarrow{(IV)} C(e_2)$
 (assm) $C(e_1) \longleftrightarrow C(e_2)$
 (CR) $\underline{C}(e_1) \xrightarrow{(CR)} C_{10}$

crystallized 1-chart

guarded, LLEE \underline{C}_{10}
 e_{10} is complete solution

bisimulation collapse

$C_{10} \xrightarrow{} C_0$
 $C(e_1) \xrightarrow{} C_0$
 $C(e_2) \xrightarrow{} C_0$
 C_0 e_{10} is solution (by (CC))

$\left. \begin{array}{l} C(e_1) \text{ guarded, LLEE} \\ e_1 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right\} \xRightarrow{(SE)} e_1 =_{\text{Mil}} e_{10} \quad e_{10} =_{\text{Mil}} e_2 \xleftarrow{(SE)} \left\{ \begin{array}{l} C(e_2) \text{ guarded, LLEE} \\ e_2 \text{ is solution} \\ e_{10} \text{ is solution} \end{array} \right.$
 $\xRightarrow{\quad} e_1 =_{\text{Mil}} e_2$

Completeness proof of Mil (structure)

1-chart interpretations

guarded, LLEE
 e_1 is solution
 (by (T)) e_{10} is solution

$$\underline{C}(e_1) \longleftrightarrow \underline{C}(e_2)$$

LLEE, guarded
 e_2 is solution
 (by (T)) e_{10} is solution

chart interpretations

$$\begin{array}{ccc} & \text{(IV)} & \\ & \swarrow \quad \searrow & \\ \text{(CR)} & \underline{C}(e_1) & \xleftrightarrow{\text{(assm)}} \underline{C}(e_2) \\ & \swarrow \quad \searrow & \end{array}$$

crystallized 1-chart

guarded, LLEE
 e_{10} is complete solution

$$\underline{C}_{10}$$

bisimulation collapse

$$C_0$$

e_{10} is solution (by (CC))

Theorem

Milner's proof system Mil is complete
 for process semantics equivalence \equiv_P of regular expressions.

Since: $e_1 \equiv_P e_2 \implies \llbracket e_1 \rrbracket_P = \llbracket e_2 \rrbracket_P \implies \underline{C}(e_1) \leftrightarrow \underline{C}(e_2) \implies e_1 =_{\text{Mil}} e_2$.