Computation of Nonlinear Waves

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Goal of Project

Our goal for this project is to study soliton solutions to the Nonlinear Schrödinger equation (NLSE). To do this, we look at how different boundary conditions affect the stability and accuracy of our solutions. We also investigate whether the Modulus Squared Dirichlet (MSD) boundary condition performs better for our solutions that Dirichlet and Neumann boundary conditions. We explore these different boundary conditions and alter the boundary size to discover the best boundary condition.

History of Solitons

Soliton theory emerged from the study of solitary waves by John Scott Russell, a Scottish engineer and naval architect. A solitary wave has a single peak, propagates forward at a constant speed, and preserves its shape over time. Russel was intrigued by solitary waves because typically, waves either exhibit two types of behavior. They can break, like a wave at the beach, or become shallower, wider and dissipate into tiny ripples, as one might observe when skipping a rock in a pond. Solitary waves do not conform to either of these descriptions. Instead, they move like a hypothetical particle, such as an electron or proton. To reflect this particle-like behavior, mathematicians Martin Kruskal and Norman Zabusky coined the term "soliton."

Nonlinear Schrödinger equation

- The Nonlinear Schrödinger (NLS) equation is a notable soliton equation used in the most significant commercial utilization of solitons: the application of solitons of light to communication. The equation should exclusively be utilized in the context of complex numbers, meaning that we must consider both the real and imaginary parts of the solution. In addition, the equation is integrable, meaning that it possesses exact solutions.
- We will be focusing on the one-component time-dependent and one-component time-independent versions of the NLS equation

$$i\partial_t u = -\frac{1}{2}\partial_{xx}u + \gamma|u|^2 u, \quad (1.1) \quad \text{NLS equation}$$

$$u(x,t) = e^{-i\mu t}v(x), \quad (1.2) \quad \text{Complex Solution form}$$

$$-\frac{1}{2}v'' + \left(\gamma|v|^2 - \mu\right)v = 0. \quad (1.3) \quad \text{Time-independent version of NLS}$$

- We will focus on time independent version (eq 1.3)

Soliton solutions

Solitons are solutions for the time independent NLSE. Bright and dark are the two main types of solitons. We graph each soliton with both Dirichlet and Neumann boundary conditions. Dirichlet boundary condition is where boundary points must be 0, and Neumann boundary condition is where the derivative at boundary points must be 0. Using Newton's method, we generate points to graph each soliton with both Dirichlet and Neumann boundary conditions.

$\gamma = -1$: $v(x) = A\,\mathrm{sech}\,(Ax)$ where $\mu = -A^2/2$ (this solution is called a "bright" soliton).
$\gamma = +1$: $v(x) = \sqrt{\mu}\tanh\left(\sqrt{\mu}x\right)$ (this solution is called a "dark" soliton).

Testing Boundary Conditions

- To test soliton solutions with different boundary conditions we need to have a numerical representation of the NLS equation.
- We used a first finite difference scheme, namely, Newton's method for systems of nonlinear equations, to find solutions to the NLS equation numerically.
- Newton's method for systems: $X_{i+1} = X_i - [J(X_i)]^{-1}F(X_i)$
- Using a symmetric spatial domain we introduced a finite number of grid points in the interval, then we chose a particular boundary condition and used Newton's method to find our interior points that solve the NLS equation.
- An important fact to remember is that the NLS equation deals with complex values so there are real and imaginary components to each numerical point that needed to be taken into account when performing Newton's method for systems.

Dirichlet and Neumann Boundary Conditions

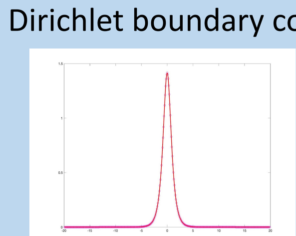
- Dirichlet boundary condition is when the function must equal zero at its boundary points.
- Neumann boundary condition is when the function's derivative at the boundary points is equal to zero.

Parameters

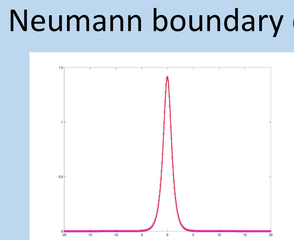
- $A = \sqrt{2}$
- Bright $\gamma = -1$
- Dark $\gamma = 1$
- Bright $\mu = -1$
- Dark $\mu = 1$

Graphs & Points of the Soliton Solutions

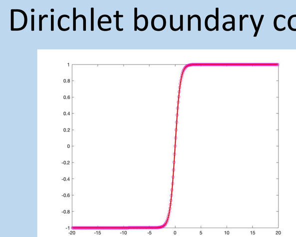
Bright Soliton
- Dirichlet boundary condition



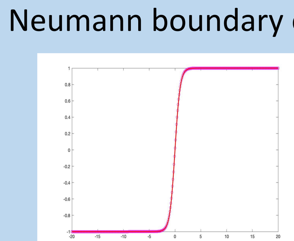
- Neumann boundary condition



Dark Soliton
- Dirichlet boundary condition



- Neumann boundary condition



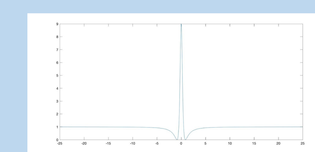
Takeaways

Research:
- The MSD boundary condition performs the best in competition with the Neumann boundary condition when paired with a solution to the time-dependent version of the NLS equation. It outperforms the Neumann boundary condition by staying stable as time passes t = 6s when working with the Peregrine soliton solution.
- We have also found that the MSD boundary condition stays stable as we shrink our boundary.

Personal:
- Throughout this project we learned about solitons and how the Nonlinear Schrödinger equation works. We also gained a deeper understanding about the importance of boundary conditions as well as a new mastery of MATLAB.
- This project taught us that researching is never finished and that there is always something more to explore.
- We have enjoyed the time together working as a team and thank Cal Poly for this great opportunity!

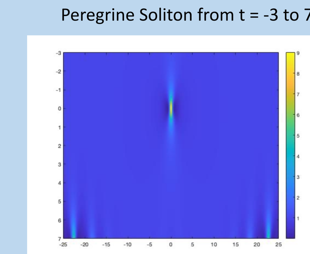
Modulus Squared Dirichlet Boundary Condition

- The Modulus Squared Dirichlet (MSD) boundary condition is used to create time simulations on time-dependent complex partial differential equations. The MSD boundary condition states that at the boundaries the modulus square of the function must be equal to a constant, positive, real value.
- To test the accuracy of the MSD boundary condition we compare time simulations of the Peregrine Soliton solution of the time-dependent NLSE that use the MSD boundary conditions to time simulations of the Peregrine Soliton that use Neumann boundary conditions.
- The Peregrine Soliton is a solution to the time-dependent version of the NLSE. This analytic solution was proposed by Howell Peregrine in 1983.

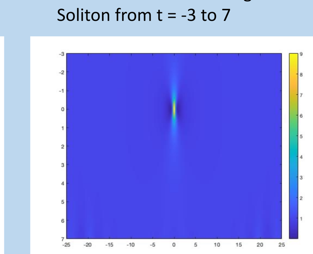


Peregrine Soliton Solution Time Snapshot at t = 0

Neumann version of the Peregrine Soliton from t = -3 to 7



MSD version of the Peregrine Soliton from t = -3 to 7



From the above graphs we see that there are perturbations arising around t = 6 for the Neumann boundary conditions. The MSD boundary condition outperforms the Neumann condition and keeps the stability of the Peregrine Soliton as time continues past t = 7.

Implementing the MSD boundary condition to the Steady-State NLSE

- In our steady-state problem we used Newton's Method for systems to iteratively solve the dark soliton solution.
- This means that we had to compute the second derivative with respect to our spatial variable at the boundaries and compute the partial derivatives for each boundary for the Jacobian.
- We used equation 3.4 to compute the real and imaginary parts of the second derivative for each boundary.

$$(3.2) \qquad \nabla^2\Psi_b \approx \left[\mathrm{Im}\left(i\,\frac{\nabla^2\Psi_{b-1}}{\Psi_{b-1}}\right) + \frac{1}{a}\left(N_{b-1} - N_b\right)\right]\Psi_b$$

$$(3.3) \qquad N_b = s\,|\Psi_b|^2 - V_b, \qquad N_{b-1} = s\,|\Psi_{b-1}|^2 - V_{b-1}.$$

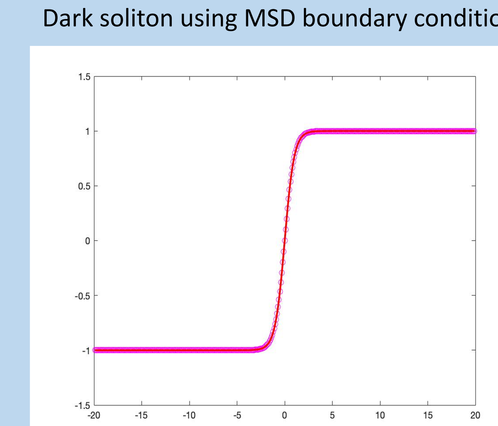
$$(3.4) \qquad \begin{aligned} \nabla^2\Psi_b^R &\approx \left[A + \frac{1}{a}\left(N_{b-1} - N_b\right)\right]\Psi_b^R \\ \nabla^2\Psi_b^I &\approx \left[A + \frac{1}{a}\left(N_{b-1} - N_b\right)\right]\Psi_b^I, \end{aligned}$$

$$(3.5) \qquad A = \frac{\nabla^2\Psi_{b-1}^R\,\Psi_{b-1}^R + \nabla^2\Psi_{b-1}^I\,\Psi_{b-1}^I}{\left(\Psi_{b-1}^R\right)^2 + \left(\Psi_{b-1}^I\right)^2}$$

Applying the MSD boundary condition to the steady state NLS problem

- We now want to apply the MSD boundary condition to our steady-state NLSE.
- We look into the dark soliton solution and see that the MSD boundary condition works well!

Dark soliton using MSD boundary condition



Future Research

- During our research of soliton solutions and testing different boundary conditions we found very small numerical irregularities in computing the eigenvalues for each of the different boundary conditions. Our next goal is to add in a potential term to the NLS equation to see if calculations become more accurate.

Stability Analysis

To study the stability of the solitons, we used the linearization ansatz:

$$u(x,t) = e^{-i\mu t}\left[v(x) + \varepsilon\left(a(x)e^{\lambda t} + b^*(x)e^{\lambda^* t}\right)\right]$$

This results in our eigenvalue problem: $$\tilde{\lambda}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ -A_{12}^* & -A_{11} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix},$$

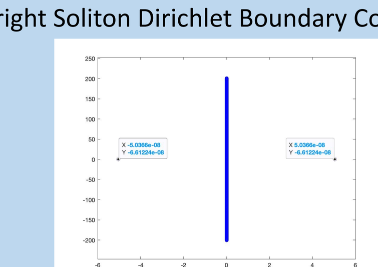
With values: $$\begin{aligned} A_{11} &= -\frac{1}{2}\partial_{xx} + 2\gamma|v|^2 - \mu, \\ A_{12} &= \gamma v^2. \end{aligned}$$

It is important to remember that the eigenvalues are complex meaning they can have real and imaginary parts. Imaginary eigenvalues don't affect the stability of our solution while real eigenvalues affect the stability.

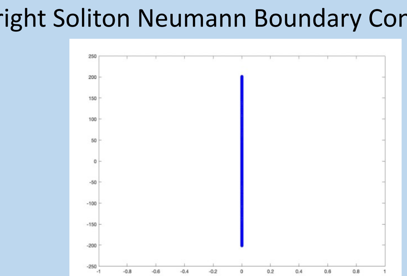
A positive real eigenvalue creates exponential growth and a negative real eigenvalue creates exponential decay; both cause our systems to become unstable.

Stability Graph for Bright Soliton

- Bright Soliton Dirichlet Boundary Condition

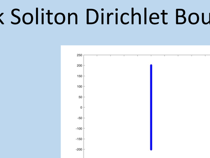


- Bright Soliton Neumann Boundary Condition

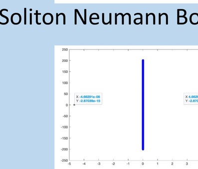


Stability Graph for Dark Soliton

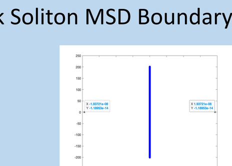
- Dark Soliton Dirichlet Boundary Condition



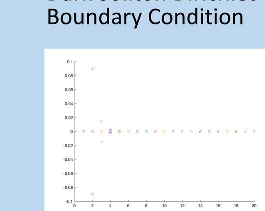
- Dark Soliton Neumann Boundary Condition



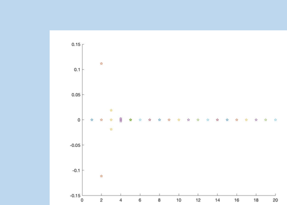
- Dark Soliton MSD Boundary Condition



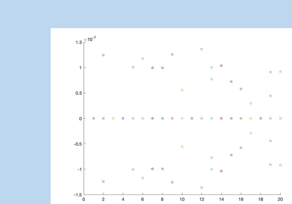
- Change in boundary from (-L,L) where L is decreased from 20 down to 0 and then computing eigenvalues for each numerically generated point.
- Graphed are the real value parts of the eigenvalues.
- Dark Soliton Dirichlet Boundary Condition



- Dark Soliton MSD Boundary Condition



- Dark Soliton Neumann Boundary Condition



- From these graphs we see that the MSD performs better as we shrink the boundary

Acknowledgement and References

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Kasman, A. (2010). *Glimpses of Soliton Theory: The Algebra and Geometry of Nonlinear PDEs*. American Mathematical Society.

Caplan R.M., Carretero-Gonzalez R. (2014). A Modulus-Squared Dirichlet Boundary Condition for Time-Dependent Complex Partial Differential Equations and Its Application to the Nonlinear Schrodinger Equation. *Society for Industrial and Applied Mathematics*, 36(1) A1-A19.