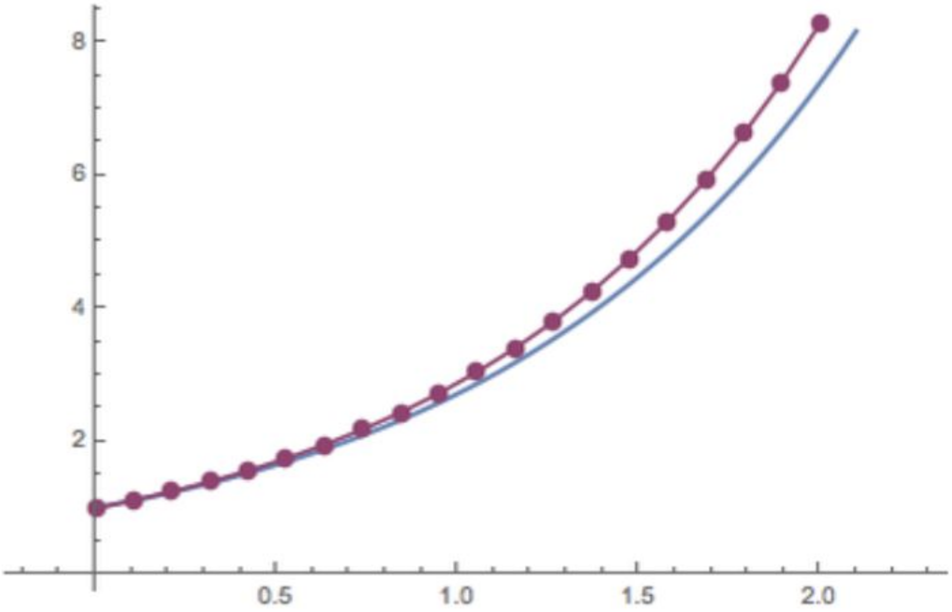
Solving ODE’s Using Numerical Methods

By Kate Davis, Sarah Ellwein, Olivia Hartnett, and Connor Leipelt

$$u'(t) = f(u(t), t), \ t > t_0$$


8
6
4
2
0.5
1.0
1.5
2.0

Euler’s Method

Euler’s Method

$$\begin{cases} u'(t) = f(u,t), \ t > t_0 \\ u(t_0) = \eta \end{cases}$$

Euler's Method

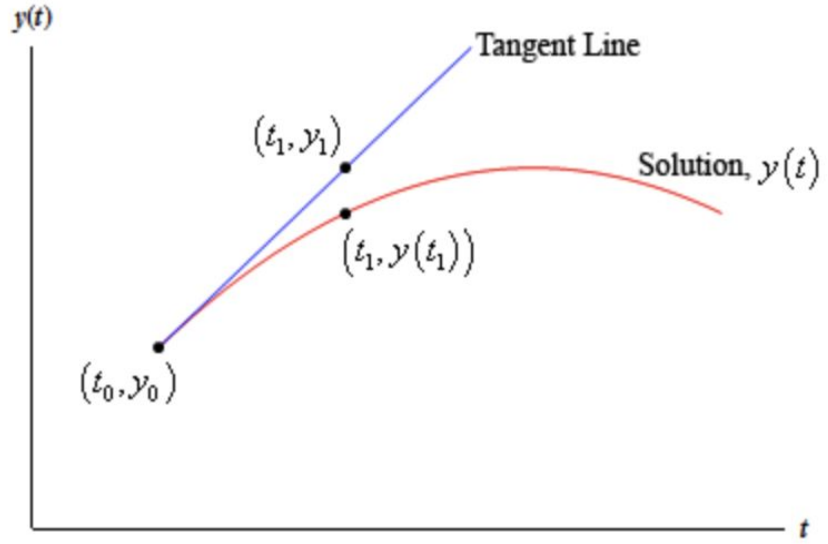
$$u' \approx \frac{\Delta u}{\Delta t}$$

Euler's Method

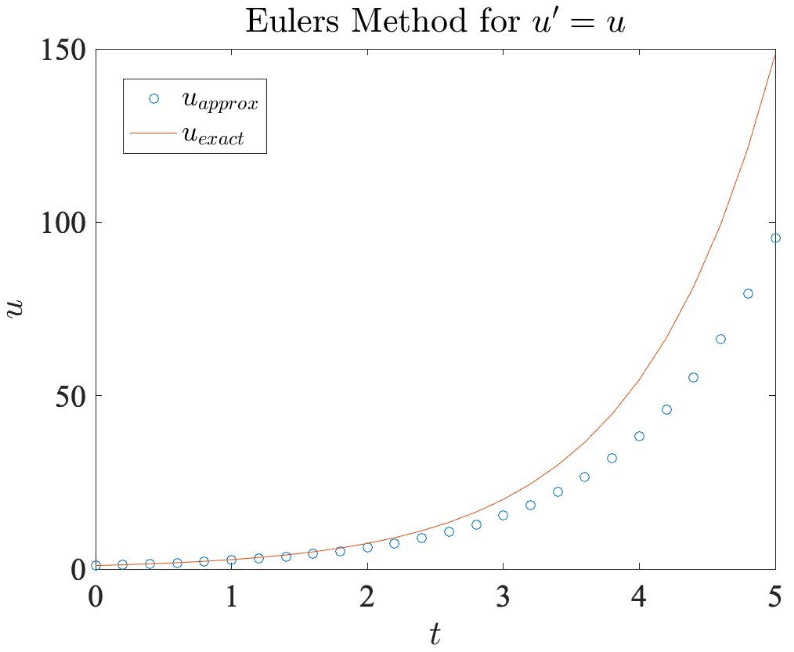
$$u' \approx \frac{u_{j+1} - u_j}{k} \Rightarrow u_{j+1} - u_j \approx ku' \Rightarrow u_{j+1} - u_j \approx kf(u_j, t_j) \Rightarrow u_{j+1} \approx u_j + kf(u_j, t_j)$$

$$\boxed{u_{j+1} \approx u_j + kf(u_j, t_j)}$$

Euler's Method

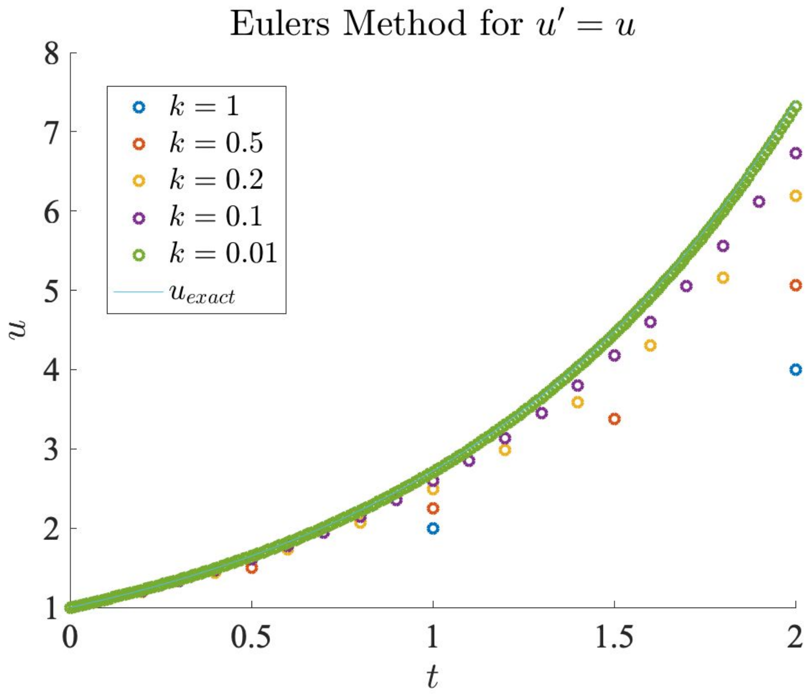


Euler's Method

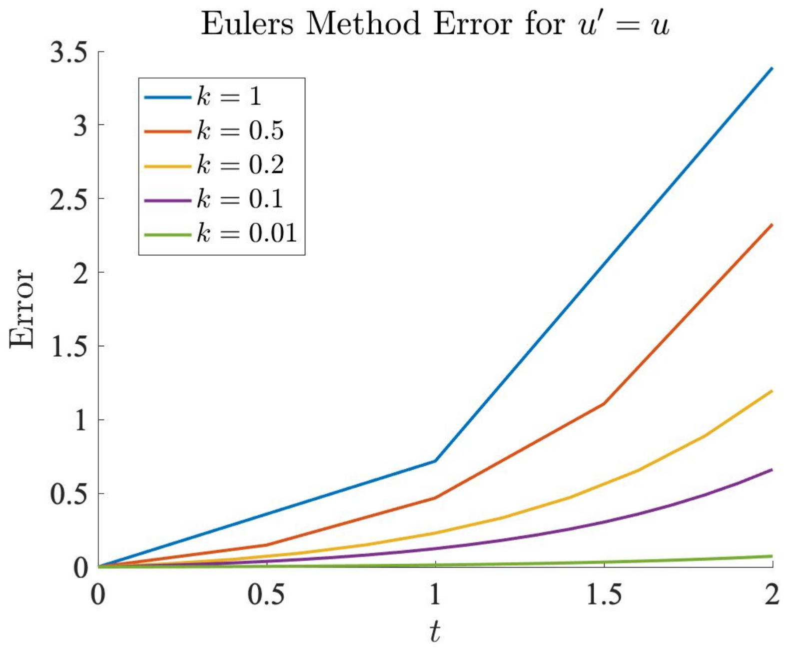


k = 0.2

Euler’s Method



Euler’s Method



Heun’s Method

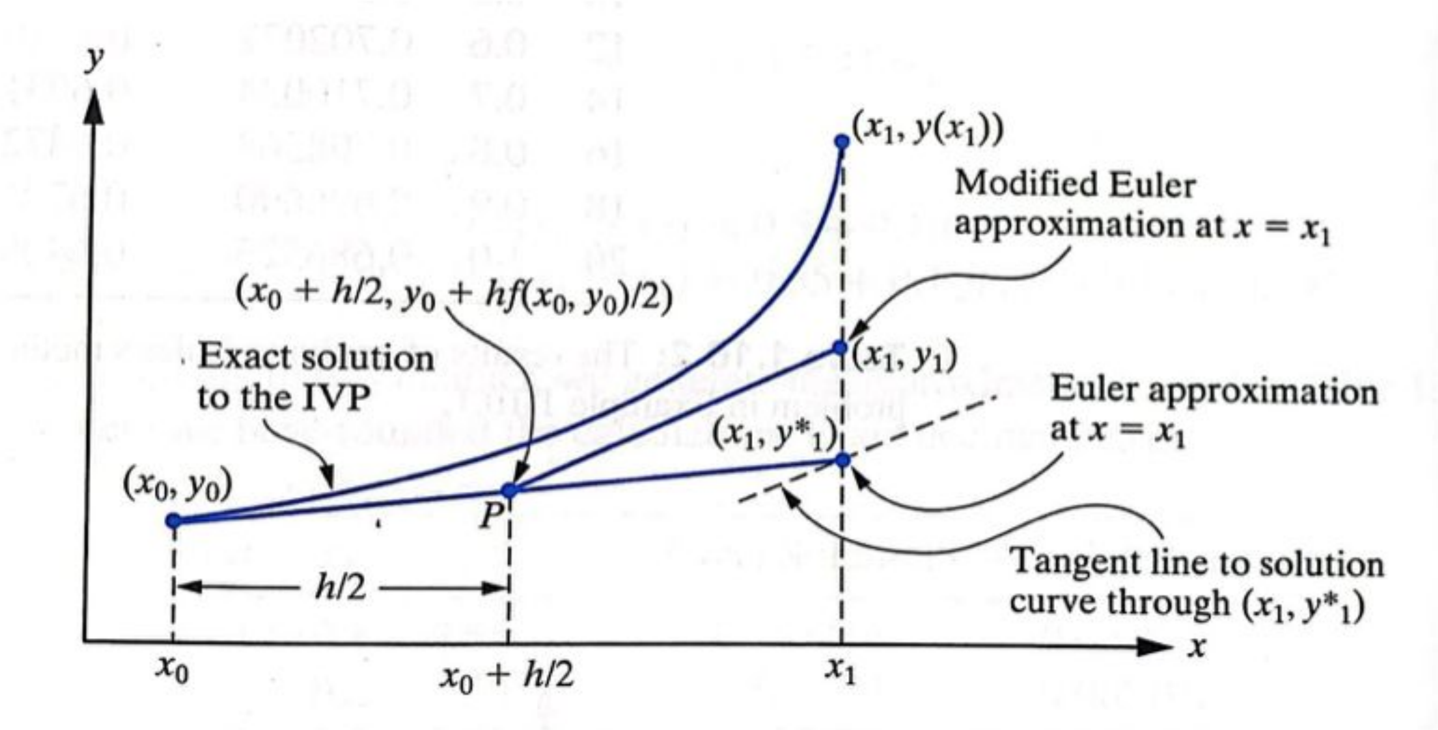
Heun's Method (Modified Euler's Method)

- Example of a predictor-corrector method
- Has two steps:
  - Prediction step- computes a rough approximation of the solution $u(t_{n+1})$ using Euler's method
  - This approximation is denoted as $U^* = U^n + kf(U^n)$
  - Correction step- the first approximation is refined by using Euler's method again, but instead we use the average of the slopes of the solution curves through $(t_n, u(t_n))$ and (t_{n+1}, U^*)

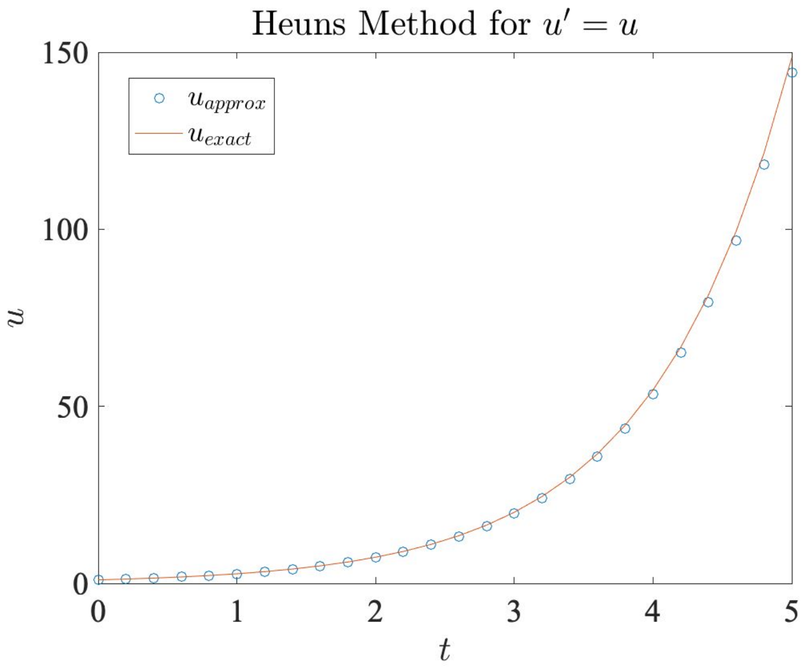
    Then, we have

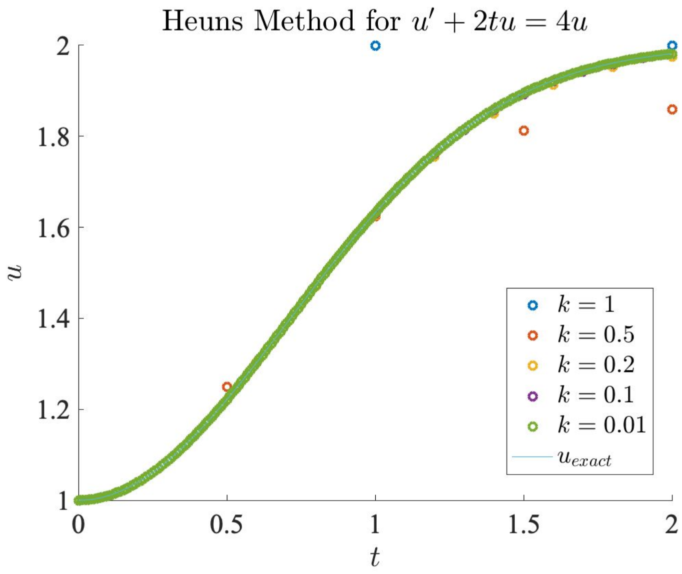
    $$U^{n+1} = U^n + (1/2)k(f(U^n) + f(U^{n+1}))$$

Heun's Method Visual

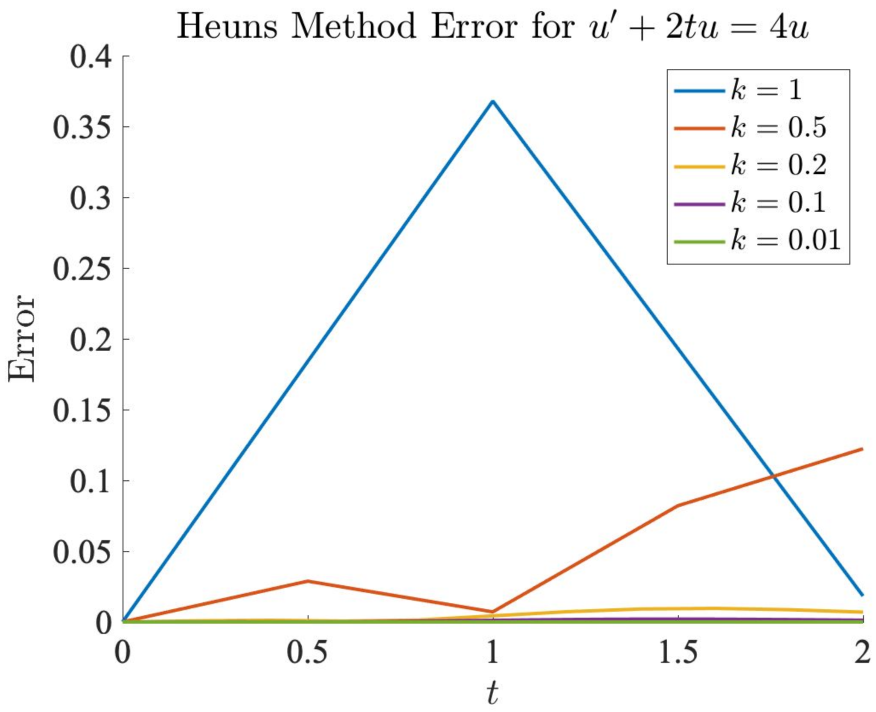


Heun's Method





Heun’s Method



Runge-Kutta Method

Runge-Kutta Method

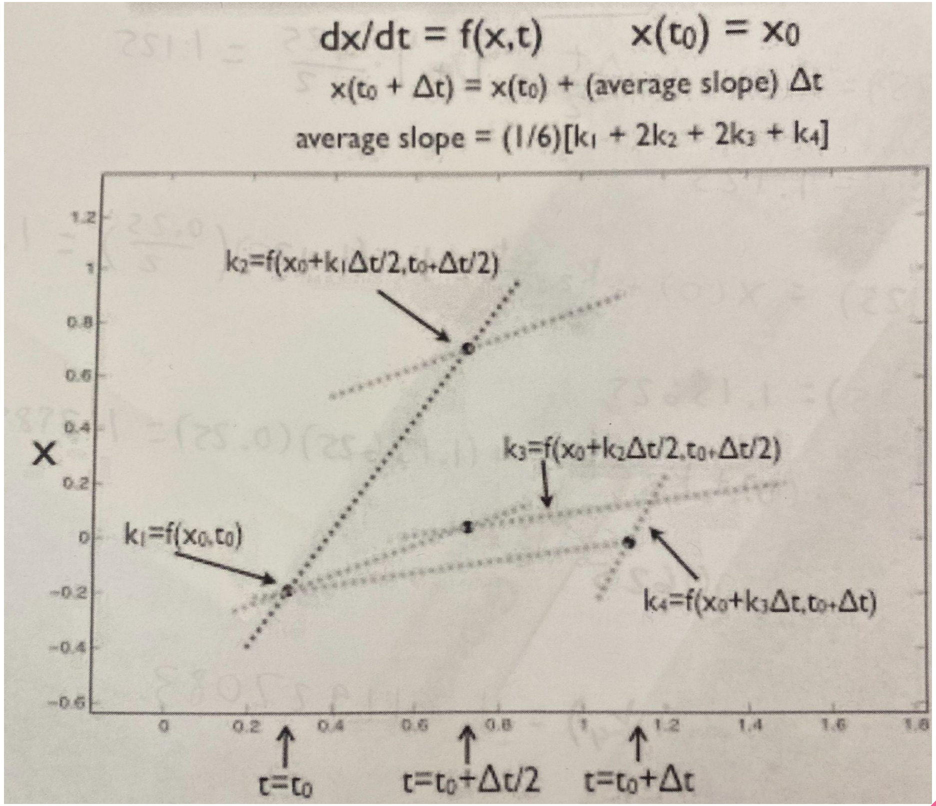
Runge-Kutta Method finds y_n+1, using the initial point and the slope at the initial point.

We will have to find the slope at 4 points, and we will use half steps to get t.

We will then take the weighted average of the 4 k's: (k1+2k2+2k3+k4)/6

- k1: slope at beginning of interval Δt
- k2 and k3: slope at midpoint of interval Δt
- k4: slope at end of the interval Δt

Runge-Kutta Method



Runge-Kutta Method - 4th order, 4-stage method

$$\frac{dx}{dt} = f(x(t), t)$$
$$x(t_0 + \Delta t) = x(t_0) + (\text{average slope}) * \Delta t$$
$$x(t_0) = x_0$$
$$k_1 = f(x^*(t_0), t_0)$$
$$x_1(t_0 + \tfrac{\Delta t}{2}) = x(t_0) + k_1 * \tfrac{\Delta t}{2}$$
$$k_2 = f(x^*(t_0) + k_1 \tfrac{\Delta t}{2}, t_0 + \tfrac{\Delta t}{2})$$
$$x_2(t_0 + \tfrac{\Delta t}{2}) = x(t_0) + k_2 * \tfrac{\Delta t}{2}$$
$$k_3 = f(x^*(t_0) + k_2 \tfrac{\Delta t}{2}, t_0 + \tfrac{\Delta t}{2})$$
$$x_3(t_0 + \Delta t) = x(t_0) + k_1 * \Delta t$$
$$k_4 = f(x^*(t_0) + k_3 \Delta t, t_0 + \Delta t)$$

The previous slope is used to get the next slope.

By using half steps, we get better precision. So it’s better than Euler’s and Heun’s Methods.

Using the weighted average, there is more emphasis on the midpoints.