Modeling Epidemics with Seasonal Components Gives Rise to Chaos

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Introduction

- Annual rates of endemic diseases vary largely from steady values to high oscillatory behavior creating cycles of repeated endemics.

- This cyclic behavior leads to the question of seasonal variation affecting the dynamics of the endemic.

Establishing the SEIR Model

We are working with a given population. The population can be divided into 4 groups:

- **Susceptibles**, those able to contract the disease
- **Exposed**, those who have been infected but are not yet infectious
- **Infectives**, those capable of transmitting the disease
- **Recovered**, those who have become immune to the disease

These groups are denoted as S, E, I, R respectively. This model is made to assume that total population, P, is equal to these four groups and to the value 1. $S + E + I + R = P = 1$.

SEIR Model continued

Parameters:

- μ = birth/death rate
- $1/\alpha$ = mean latent period (exposed - infected)
- $1/\gamma$ = mean infectious period (infected - recovered)
- $\beta(t)$ = contact rate
- $\beta(t)S(t)$ = rate of total # S infected by one I
- $\beta(t)S(t)I(t)$ = rate of infection of all S by all I
- Recovered individuals are permanently immune

System of ODEs:

$$S'(t) = \mu - \beta(t)S(t)I(t) - \mu S(t)$$

$$E'(t) = \beta(t)S(t)I(t) - (\mu + \alpha)E(t)$$

$$I'(t) = \alpha E(t) - (\mu + \gamma)I(t)$$

Note: This model is well known for $\beta(t) = \beta_0$, a constant.

Basic Reproductive Rate of Infection and Effective Infectee Number (EIN)

Another important piece is the basic reproductive rate of infection:

$Q = \beta_0\alpha/[(\mu+\alpha)(\mu+\gamma)]$

If $Q > 1$, the model has an endemic equilibrium state and a trivial equilibrium point, making the equilibrium point unstable.

We define the *effective infectee number* (EIN) to be the average number of cases produced per average infective in one infectious period.

We find that the infectee number approaches unity (value 1) if the system approaches equilibrium.

Introduction to Seasonal Component

- Investigating the relationship between Q and inter-epidemic periods
- Varying $\beta(t)$ and keeping the other parameters constant
- We use the function:

$$\beta(t) = \beta_0 (1 + \beta_1 cos(2\pi t))$$

With $0 < \beta_1 < 1$ and $\beta_0 > 0$

- It is important to note that for diseases which confer permanent immunity it is necessary for seasonality in order to have recurrent epidemic

Equilibrium States

We now look at concepts from the equilibrium case that can be applied to recurrent epidemics with seasonal contact rates

Parameters:

- $\eta = 1/\gamma$
- $\lambda = 1/\alpha$

Define the following:

$$C[a, d] = \hat{\eta} \frac{\int_a^d \beta(t) S(t) I(t)\, dt}{\int_a^d I(t)\, dt}$$

with $\hat{\eta} = \eta/[(1 + \mu\eta)(1 + \mu\lambda)]$ seeing that $\hat{\eta} < \eta$.

Breakdown with Periodicity

$C[a,d]/\hat{\eta}$ is the ratio of the average number of susceptibles becoming infected (i.e. becoming exposed) per unit time to the average number of infectives in the time interval $[a, d]$

If $(S(t), E(t), I(t)) = (S(t+p), E(t+p), I(t+p))$ for $t \geq 0$, where p is an integer of greater than or equal to unity (1), then $C[0, p]$ is the effective infectee number (EIN) along a periodic orbit having period p.

We can simplify our $C[0, p]$ by substituting in pieces from our ODE system.

(Continued)

$$C[0,p] = \hat{\eta}\frac{\int_0^p \beta(t)S(t)I(t)\,dt}{\int_0^p I(t)\,dt}$$

$$\text{Note: } E'(t) = \beta(t)S(t)I(t) - (\mu+\alpha)E \rightarrow \beta(t)S(t)I(t) = E' + (\mu+\alpha)E$$

$$\text{and } I'(t) = \alpha E(t) - (\mu+\gamma)I(t) \rightarrow I(t) = \frac{\alpha E(t) - I'(t)}{(\mu+\gamma)}$$

and $\hat{\eta} = \eta/[(1+\mu\eta)(1+\mu\lambda)]$ and $\gamma = \frac{1}{\eta}$ and $\alpha = \frac{1}{\lambda}$

(Continued)

Giving us:

$$C[0,p] = \frac{\eta}{(1+\mu\eta)(1+\mu\lambda)} \frac{\int_0^p E'(t) + (\mu+\alpha)E(t)\,dt}{(\frac{1}{\mu+\gamma})\int_0^p \alpha E(t) - I'(t)\,dt}$$

Note: $\int_0^p E'(t)\,dt = \int_0^p I'(t)\,dt = 0$ since we get $E(p) - E(0) = 0 = L(p) - L(0)$.

So we get $C[0,p] = \dfrac{\eta(\mu+\frac{1}{\eta})(\mu+\frac{1}{\lambda})\int_0^p E(t)\,dt}{(1+\mu\eta)(1+\mu\lambda)\frac{1}{\lambda}\int_0^p E(t)\,dt}$

Giving us $C[0,p] = 1$

Results (Equilibrium States)

From this we see that if:

$$\lim_{t\to\infty}[S(t), E(t), I(t)] = \lim_{t\to\infty}[S(t+p), E(t+p), I(t+p)]$$

then we have that $\lim_{t\to\infty} C[t, t+p] = 1.$

Therefore, a periodic orbit having period p is asymptotically stable, and the EIN approaches 1. Since the EIN measures the average number of cases produced per infective we can infer some scenarios.

- If EIN < 1 then the disease will be less prevalent.
- If EIN > 1 then the disease will become more prevalent.
- If a recurrent epidemic has a high level of incidence a certain year, then there is a high probability that a low level of incidence will happen the following year.

Relation between susceptible fraction at equilibrium and basic reproductive rate for general periodic solutions

Since $C[0,p] = 1$ we have that:

$$\frac{\eta}{(1+\mu\eta)(1+\mu\lambda)} \int_0^p \beta(t)S(t)I(t)\,dt = \int_0^p I(t)\,dt$$

along a periodic orbit.

By MVT we have two positive values denoted as follows:

$$\hat{\beta}\hat{S} = \frac{\int_0^p \beta(t)S(t)I(t)\,dt}{\int_0^p I(t)\,dt}$$

Giving us:

$$\hat{\eta}\hat{\beta}\hat{S} = 1 \text{ or } \hat{S} = \frac{1}{\hat{\eta}\hat{\beta}}$$

Showing that there exist mean values of susceptible fraction and the basic reproductive rate which are inversely related.

Relationship between Exposed and Infectives

This is a simplification process reducing # of ODEs from 3 to 2. To do this we let $\epsilon = \mu(Q - 1)$ and $\epsilon > 0$ *and* $\epsilon \ll 1$. Also let Δ_2 and $\Delta_3 > 0$ such that: $(\mu + \alpha) = \frac{\Delta_2}{\epsilon}$ and $(\mu + \gamma) = \frac{\Delta_3}{\epsilon}$

Next we do a change of variables:

$$S = S_0(1 + x),\ E = E_0(1 + y),\ I = I_0(1 + z)$$

$$(S_0, E0, I0) = \left(\frac{1}{Q}, \frac{\mu + \gamma}{\alpha} I_0, \frac{\epsilon}{\beta_0}\right)$$

which is the endemic equilibrium point

Then we can show:

$$y - z = \frac{\epsilon}{\Delta_3} z' = O(\epsilon),\ y = z + O(\epsilon)$$

Implying:

$$\frac{E}{E_0} = \frac{I}{I_0} + O(\epsilon)$$

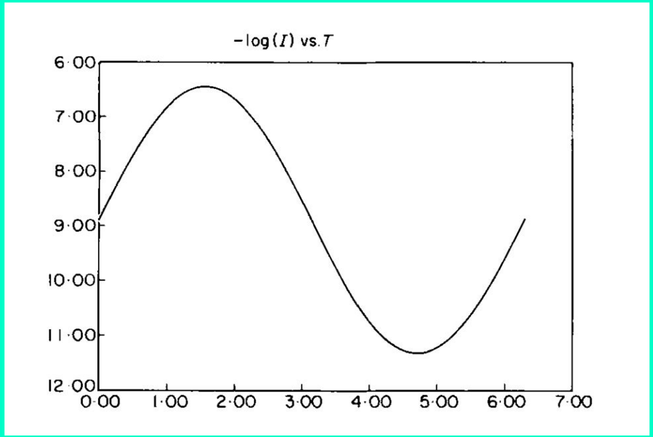
Giving us the ratio of infective to exposed individual to be γ/α to order ϵ, which allows us to restrict to the examination of susceptibles and infectives only.

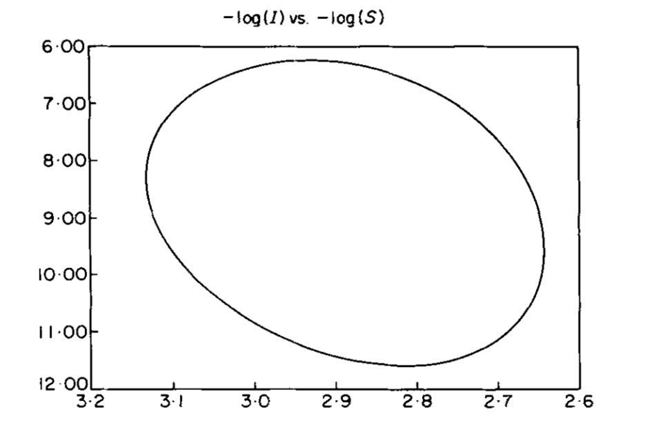
Simulations

Parameters:

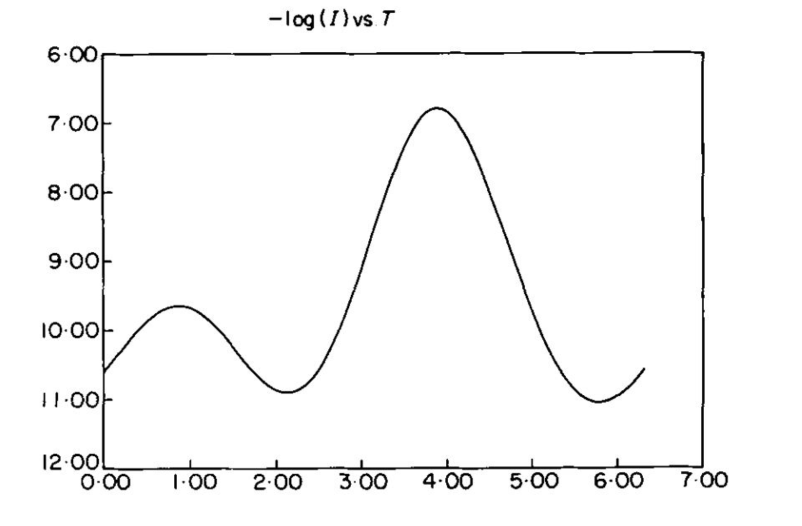
- $\mu = 0.02$/year
- $\alpha = 35.84$/year
- $\gamma = 100$/year
- $\beta_0 = 1800$/year
- Giving $Q \approx 18$

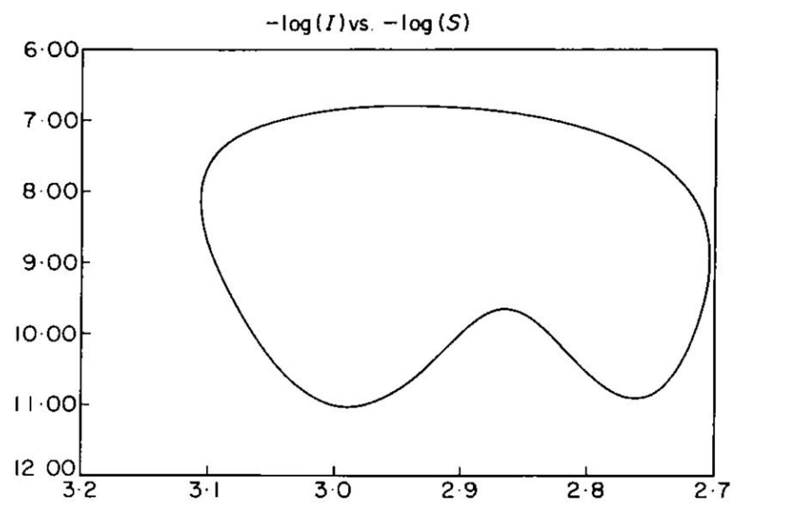
Period 1 , $\beta_1 = 0.05$



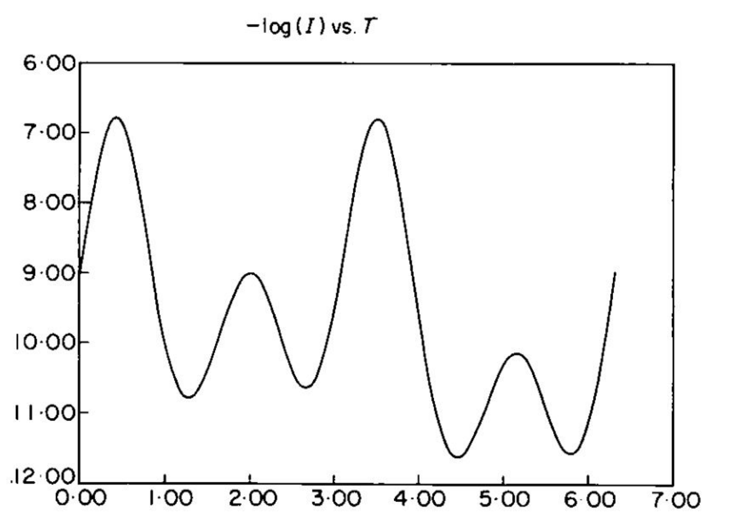


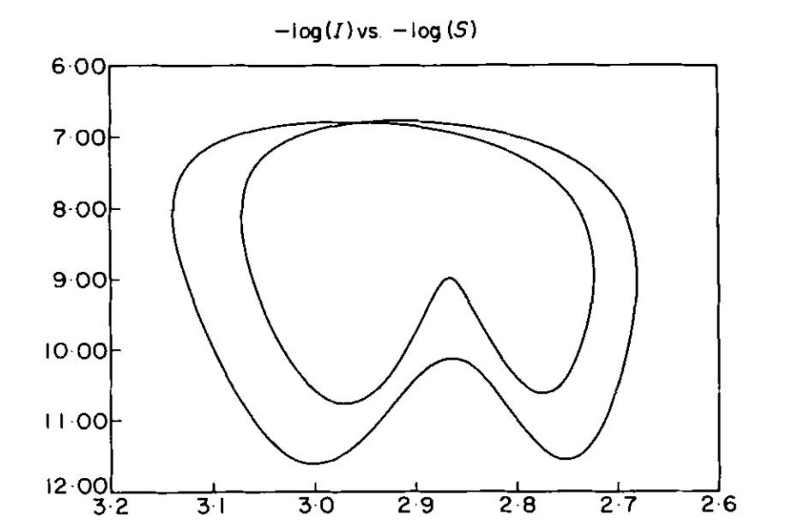
Period 2: $\beta_1 = 0.2$



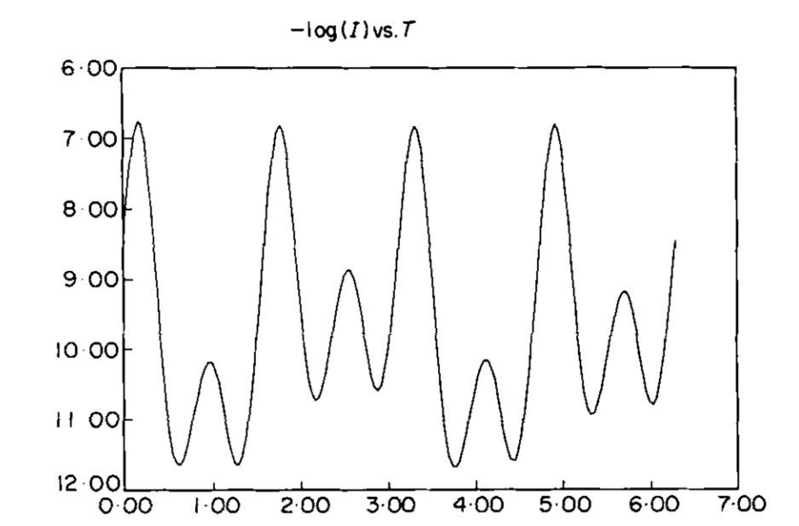


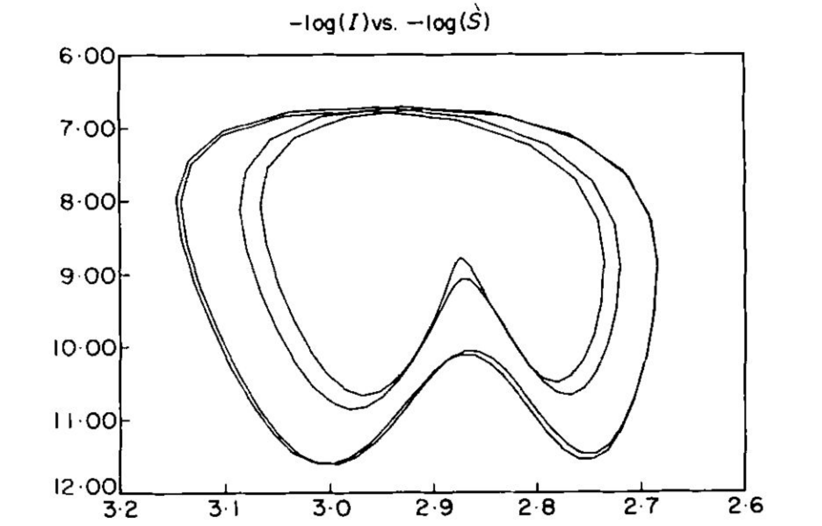
Period 4: $\beta_1 = 0.26$



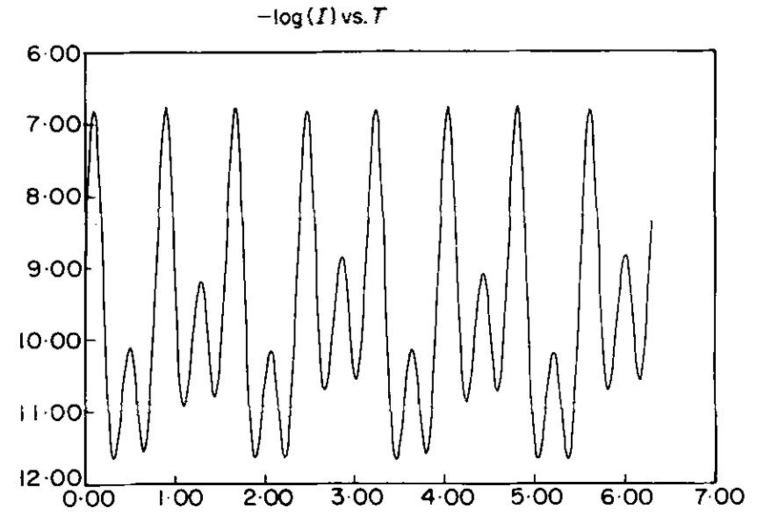


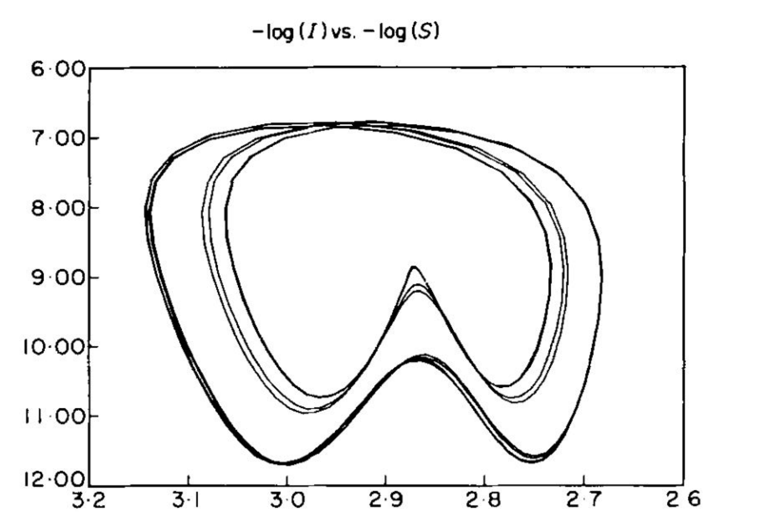
Period 8: $\beta_1 = 0.266$



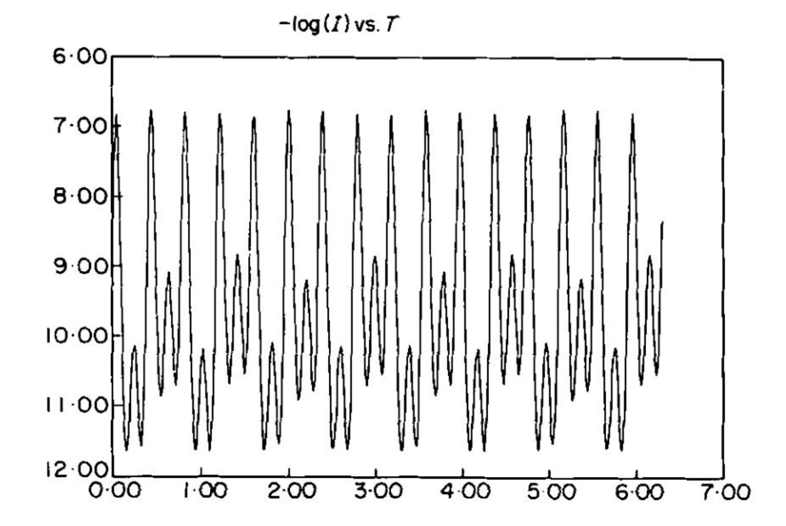


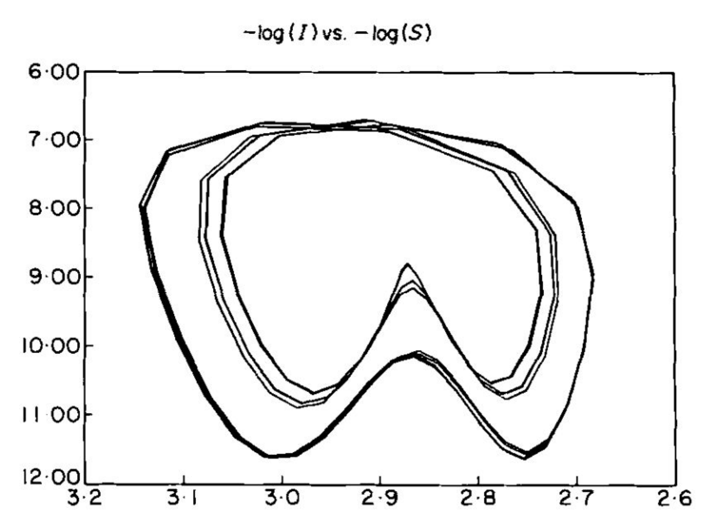
Period 16: $\beta_1 = 0.268$





Period 32: $\beta_1 = 0.2685$





Understanding the Graphs

As β_1 increases, past a critical value of $\beta_{1,c}$, the period 1 orbit becomes unstable and a stable biennial orbit appears.

The bifurcations of period doublings follow the Feigenbaum sequence where an accumulation point is reached at some value of β_1 where every orbit is unstable. At this accumulation point, the epidemic is said to be chaotic.

We see that there is very little variation in the amplitude of the biennial peaks of the infectives over a wide range of β1. The peaks mark the years of high amounts of infective and one can see that the following year the amount is considerably lower and spikes back up the year after that, creating our biennial outbreak.

Relation between bifurcation and contact rate

A key factor to understanding when biennial outbreaks do occur is the basic reproductive rate, which is directly proportional to the contact rate.

In the following graph we see the onset of bifurcations at $\beta_{1,c}$ as a function of average contact rate, β_0 aka $(Q/\hat{\eta})$.

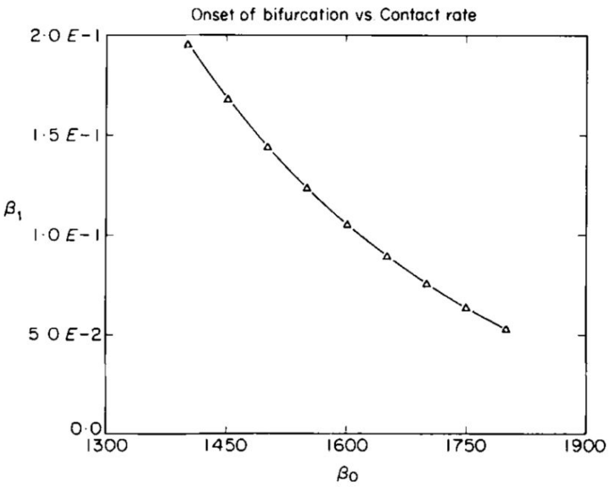


Figure 13: Fig. 7 Onset of bifurcation vs. Contact rate (β_0 vs. β_1), $Q \approx \beta_0/100$.

Low values of β_0 require large values of seasonal forcing (β_1) to produce a biennial cycle and high values of β_0, the contact rate, have biennial outbreaks occurring much more frequently. From this we can conclude that large populations are more susceptible to biennial outbreaks than small populations.

Findings, Meaning, and Future Goals

Through *Seasonality and Period-doubling Bifurcations in an Epidemic Model* it's been numerically shown that small-amplitude periodic solutions arising from the seasonally-forced SEIR epidemic model form a sequence of period-doubling bifurcations.

Interpreting this tells us that as β_1 increases, the period will effectively start to double, going from period 1, to period 2, to period 4, etc. and will lead to chaotic behavior.

Because of the inability to construct small-amplitude orbits of arbitrary periodicity it is difficult to create a unifying theory of recurrent epidemics.

For better modeling we must look into studying epidemics themselves more and look into the possibility of adding in other factors such as the existence of a climactic effect or multiple infectious agents.

Thank you!

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