Predicting Dust Clouds: Rocket Exhaust Flows in Soil with Triangular Meshes

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Introduction

- Build off of "Numerical Computations For PDE Models Of Rocket Exhaust Flow In Soil"(2007) by Brian Brennan
- When a thruster fires on a surface made of soil, there is a risk of the soil cratering
- Both Apollo 12 and Apollo 15 missions to the moon experienced problematic cratering upon landing [1]

Image Source: nasa.gov

Problem Statement

Navier' s model for volume displacement:

$$\mu\nabla^2\mathbf{u} + (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}) + f = 0$$

Porous Medium Equation (form derived from Darcy's Law):

$$\frac{\partial p}{\partial t} = \frac{k}{2\eta\epsilon}\Delta p^2$$

Body force:

$$f = \rho g + \nabla p$$

Parameters

- p: pressure
- η: viscosity of the gas
- k: permeability of medium
- ϵ: porosity of medium
- μ, λ: material constants
- $\mathbf{u} = (u, v)$: displacement vector field
- g: gravitational acceleration
- ρ: material density

Model General Approach

- Use a finite difference method on a triangular grid for time to solve the porous medium equation
- Use a finite element method (Galerikan method) for Navier's equation (BVP)

Project Goals

- Reproduce results for porous medium equation from Brennan 2007
- Attempt to implement the full model (FEM for Navier's Equation for volume displacement)
- If time permits, examine non-centroidal node refinement of the FEM mesh

Background for Porous Medium Equation

$$\frac{\partial p}{\partial t} = \frac{k}{2\eta\epsilon}\Delta p^2$$

- Popularized by Fourier's essay Theorie Analytique de la Chaleur, published in 1822 [2]
- Also known as the Nonlinear Heat Equation
- Parabolic equation with applications in heat transfer, flow of gas and groundwater among other physical problems [2]
- Combined with Darcy's Law in order to integrate pressure in system, which asserts the rate at which a fluid flows through a permeable substance is the permeability times pressure, divided by viscosity of the fluid

Numerical Method: Crank-Nicolson

- The core of the pressure solving routine uses the Crank-Nicolson method for PDEs:

$$y_{k+1} = y_k + \frac{h}{2}[F(y_k) + F(y_{k+1})]$$

$$\Rightarrow p_{k+1} = p_k + \frac{h\beta}{2}[\Delta_s p_k^2 + \Delta_s p_{k+1}^2]$$

with $\beta = \frac{k}{2\eta\epsilon}$

- Boundary conditions: homogeneous 0-Dirichlet
- Initial conditions: 3-Dimensional Gaussian, with width of curve mimicking rocket nozzle diameter ($\sigma = 0.1$):

$$P_0 = \exp(-\frac{(x_i - 0.5)^2 + (y_i - 0.5)^2}{\sigma^2})$$

Resolving Nonlinearity (1/2)

$$\frac{\partial p}{\partial t} = \frac{k}{2\eta\epsilon}\Delta p^2$$

- To account for the nonlinear term a Chebyshev spectral method (or pseudospectral/ spectral collocation method)
- Can be seen as using the extreme points of the Chebyshev polynomials, $T_N(x) = \cos(N \arccos(x))$, of that has been shifted to the domain of the problem (giving Chebyshev grid points) [3]
- Found by differentiating the polynomial interpolating between the roots of the Chebyshev polynomials

Resolving Nonlinearity (2/2)

- Chebyshev spectral differentiation matrix used is given by [1]:

$$\begin{cases} (D_N)_{00} = \frac{2N^2+1}{6}, & \\ (D_N)_{NN} = -(\frac{2N^2+1}{6}), & \\ (D_N)_{jj} = \frac{-x_j}{2(1-x_j^2)}, & j = 1, 2, ..., N-1 \\ (D_N)_{ij} = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{x_i - x_j}, & i \neq j, i, j = 0, ..., N \end{cases}$$

with

$$c_i = \begin{cases} 2, & i = 0, N \\ 1, & \text{otherwise} \end{cases}$$

- Satisfying

$$\frac{\partial^2}{\partial x^2} \approx I \otimes D^2$$
$$\frac{\partial^2}{\partial y^2} \approx D^2 \otimes I$$
$$\Delta_s \approx C = I \otimes D^2 + D^2 \otimes I$$

Newton's Method

$$z_{i+1} = z_i - F(z)/J(z)$$

with

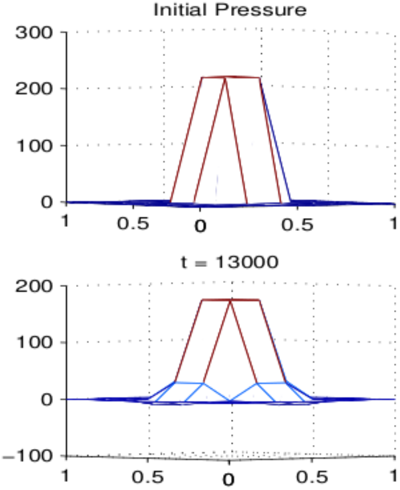
$$F(z) = z - [p_k + \frac{\beta h}{2}(\Delta_s p_k^2 + \Delta_s z^2)]$$

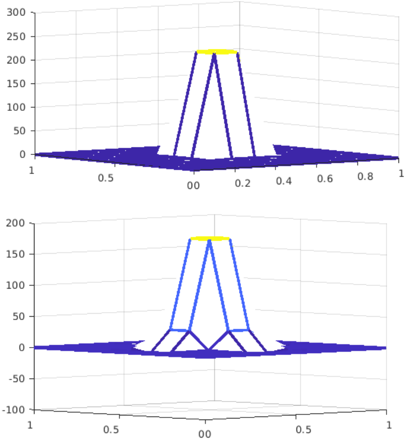
and

$$J(z) = I - h\beta Cz$$

(with initial guess z_0 equal to the pressure Gaussian z_1 found by one step Heun's method)

Results


Initial Pressure
300
200
100
0
1
0.5
0
0.5
1
t = 13000
200
100
0
−100
1
0.5
0
0.5
1

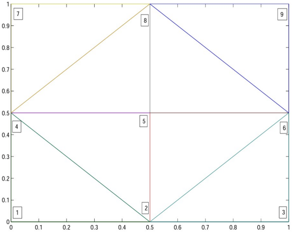

300
250
200
150
100
50
0
1
0.5
00
0.2
0.4
0.6
0.8
1
200
150
100
50
0
-50
-100
1
0.5
00
0.5
1

Background for Navier’s Model for Volume Displacement

- Derived from Navier-Stokes Equations, PDE’s, 1842-1850 by Claude-Louis Navier and George Gabriel Stokes [4]
- For Newtonian fluids (viscosity, how it resists flow, only changes due to pressure or temperature), they can express momentum balance and conservation of mass
- Unlike Euler’s EQ’s, N-S takes into account viscosity, and are parabolic equations
- Navier-Lame equations are a reduction of the above when looking at a homogeneous solid
- Equations of purely mechanical theory of linear elasticity
- Model for volume displacement is a BVP, leading to the use of a finite element method, specifically Galerkin Methods

Galerikan method (FEM) (1/3)

- Finite Element Methods (Galerkin): used to solve Navier's Equation
- Differs from the Finite Differences Method because we solve a weak form of the problem, which we obtain by multiplying the differential equation by a smooth function and integrating over the boundary
- We break up the 2-dimensional domain [0,1] x [0,1] into a finite number of triangles via Delaunay Triangulation, which we call our elements, k
  - These triangles maximize each interior angle



Galerikan method (FEM) (2/3)

- On each triangle, we compute the Local Stiffness matrix A_k and Local Load vector b_k on each triangle element
  - A_k is a symmetric matrix discretization of the left side of the weak form of the equation, the stiffness of the elastic materials
  - b_k is a discretization of the forcing function and the homogeneous Dirichlet boundary conditions
- We then assemble each A_k and b_b into a global stiffness matrix, A, and load vector, b
- This reduces the problem to solving for u in $Au = b$

$$a(w,u) = (w,f)$$

$$a(w,u) = \int_\Omega w_{(i,j)}\sigma_{ij}dx$$

$$(w,f) = \int_\Omega w_i f_i dx$$

Galerikan method (FEM) (3/3)

- Equation for local stiffness matrix

$$A^K = R_K^T C R_K (\text{Area of Triangle K})$$

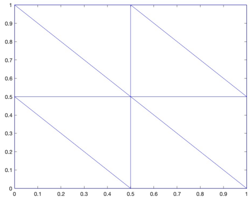
- R_K is the operator which constructs the local strain vector on each element
- Equation for local load vector

$$\int_K \phi(x,y) f(x,y) dx \approx \frac{(\text{Area of K})}{3} \sum_{i=1}^{3} f(x_i, y_i)$$

$$f(x,y) = \rho g + \nabla P$$

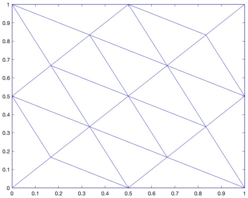
Uniform Mesh of Domain by Delaunay Triangulation

- For our FEM Solver we discretize our domain through Delaunay Triangulation. This is the process of connecting a set of points in a way where no point lies within the circumcircle of any triangle.[1]
- We have created a Delaunay Triangulation for our 2-dimensional domain by cutting up the x-axis into three points and y-axis into three points.
- This configuration is actually the same as the mesh shown in the earlier slide. Both meshes satisfy the conditions set by Delaunay Triangulation.

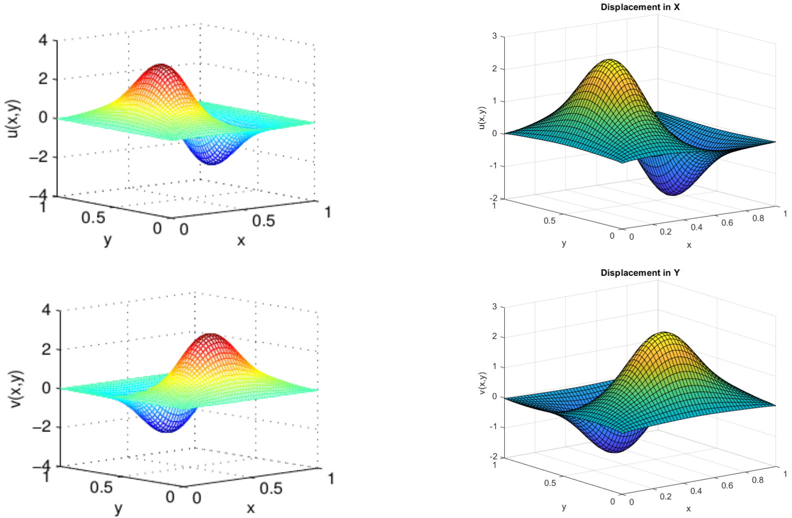


One Step Refinement of Finite Element Method Mesh

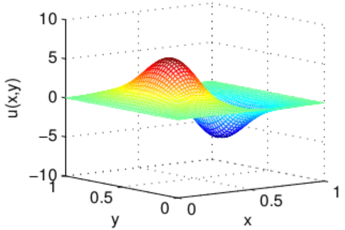
- A possible change to optimize the Finite Element Method is to do an adaptive grid refinement.
- This is done to minimize the error in the approximation. To do this we calculate the error of our approximation to our actual findings and change specific triangles dependent on whether or not their error is above a certain tolerance criterion.
- Here we have calculated the first refinement of our 9 point Delaunay Triangulation in the assumption that all triangles have an error above the tolerance criterion.

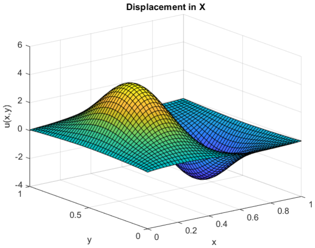


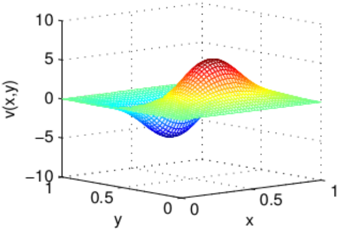
Displacement Solution Results: Max Initial Pressure 100 N

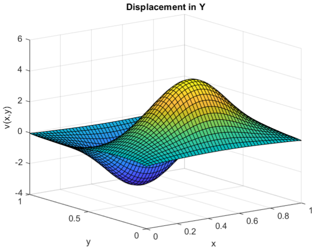


Displacement Solution Results: Max Initial Pressure 200 N









Closing Thoughts

- Pressure from the paper results where successfully reproduced
- Furthermore, we were able to reproduce the displacement field results
- Gained a valuable lesson in numerical methods on non-square and non-uniform grids

Potential Future Work

- Casting pressure as a Gaussian creates issues with negative values of pressure over long time periods.
- To combat this problem we used a centroidal mode replacement and a non-centroidal successive node replacement.
    - This is one solution but another is to transition coordinates to a polar grid potentially creating a finer level of refinement and the possibility for a more accurate model.

References I

[1] Brian Brennan. "Numerical Computations For Pde Models Of Rocket Exhaust Flow In Soil". In: 2010.

[2] Juan Luis Vazquez. *The Porous Medium Equation: Mathematical Theory*. Oxford University Press, Oct. 2006. ISBN: 9780198569039. DOI: 10.1093/acprof:oso/9780198569039.001.0001. URL: https://doi.org/10.1093/acprof:oso/9780198569039.001.0001.

[3] Randall J. LeVeque. *Finite Difference Methods for Ordinary and Partial Differential Equations*. Society for Industrial and Applied Mathematics, 2007. DOI: 10.1137/1.9780898717839. eprint: https://epubs.siam.org/doi/pdf/10.1137/1.9780898717839. URL: https://epubs.siam.org/doi/abs/10.1137/1.9780898717839.

References II

[4] Wikipedia contributors. *Derivation of the Navier–Stokes equations — Wikipedia, The Free Encyclopedia*. [Online; accessed 7-June-2023]. 2023. URL: https://en.wikipedia.org/w/index.php?title=Derivation_of_the_Navier%E2%80%93Stokes_equations&oldid=1151019798.