Probabilistic Methods

Faustino Palmero Ramos, Connor Leipelt, Marco Calabrese

1 Introduction

We aim to introduce a multi-linear large deviation estimate for Gaussian series, known as Wiener chaos, and an L^4 Strichartz estimate on the square torus. To do so, we will first introduce the Ornstein-Uhlenbeck semigroup and use it to obtain L^p bounds on Hermite polynomials.

1.1 Ornstein-Uhlenbeck semigroup

Let $d \geq 1$, and let us define the Ornstein-Uhlenbeck operator:

$$\begin{aligned} L &:= \Delta - x \cdot \nabla, \\ &= \sum_{j=1}^{d} \left(\frac{\partial^2}{\partial x_j^2} - x_j \frac{\partial}{\partial x_j} \right). \end{aligned}$$

This is a self-adjoint operator on $\mathcal{K} := L^2(\mathbb{R}^d, d\rho_d)$, where $d\rho_d = e^{-|x|^2/2}dx$ is the Gaussian measure in $\mathbb{R}^d$, and with domain:

$$D := \{u : u(x) = e^{|x|^2/4}v(x), v \in \mathcal{H}^2\},$$

where $\mathcal{H}^2 := \{v \in L^2(\mathbb{R}^d) : x^\alpha \partial_x^\beta v(x) \in L^2(\mathbb{R}^d) \, \forall (\alpha, \beta) \in \mathbb{Z}_{\geq 0}^{2d}, |\alpha| + |\beta| \leq 2\}$. We now define the Hermite polynomials so that for each $n \geq 0$ we have:

$$P_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

and so P_n is an eigenfunction of the one-dimensional Ornstein-Uhlenbeck operator ($d = 1$) with eigenvalue $-n$. For $d \geq 1$, let $k = k_1 + \cdots + k_d$. Then, the Hermite polynomial:

$$P_k(x_1, \ldots, x_d) = P_{k_1}(x_1) \cdots P_{k_d}(x_d)$$

is an eigenfunction of L with eigenvalue k. We now define the eigenspaces:

$$W_k := \{f \in \mathcal{K} : Lf = -kf\},$$

so that we may write $\mathcal{K}$ as:

$$\mathcal{K} = \bigoplus_{k \geq 0} W_k.$$

Finally, we define the *Ornstein-Uhlenbeck semigroup* $\{\sigma_t\}_{t\geq 0}$ as the semigroup generated by L, i.e. for any $f \in D$ and any $t \geq 0$ we have:

$$\sigma_t f = e^{tL} f.$$

In particular, we note that if $\tilde{P}_k \in W_k$, then:

$$\sigma_t \tilde{P}_k = e^{-kt} \tilde{P}_k.$$

1.2 L^p bound on Hermite polynomials

We now present a proof of the following theorem [3, Lemma 2.3].

Theorem 1.1. *Let $d \geq 1$, $k \in \mathbb{Z}_{\geq 0}$. Then, for any $\tilde{P}_k \in W_k$, and any $p \geq 2$:*

$$\|\tilde{P}_k\|_{L^p(\mathbb{R}^d, d\rho_d)} \leq (p-1)^{k/2} \|\tilde{P}_k\|_{L^2(\mathbb{R}^d, d\rho_d)}.$$

Proof. This Theorem is a consequence of the following Corollary to the more general Gross Theorem:

Corollary 1.2. *For any $f \in D$, and for any $q(t) = 1 + e^{2t}(q(0) - 1)$, with $q(0) > 1$, we have:*

$$\|\sigma_t f\|_{L^{q(t)}(\mathbb{R}^d, d\rho_d)} \leq \|\tilde{P}_k\|_{L^{q(0)}(\mathbb{R}^d, d\rho_d)}.$$

We now choose $f = \tilde{P}_k \in W_k$, so that applying Corollary 1.2 we obtain the inequality:

$$\|\sigma_t \tilde{P}_k\|_{L^{q(t)}(\mathbb{R}^d, d\rho_d)} \leq \|\tilde{P}_k\|_{L^{q(0)}(\mathbb{R}^d, d\rho_d)}.$$

Since $\tilde{P}_k \in W_k$, we have $\sigma_t \tilde{P}_k = e^{-kt} \tilde{P}_k$, hence we may rewrite the inequality as:

$$\begin{aligned} & e^{-kt} \|\tilde{P}_k\|_{L^{q(t)}(\mathbb{R}^d, d\rho_d)} \leq \|\tilde{P}_k\|_{L^{q(0)}(\mathbb{R}^d, d\rho_d)}, \\ \Rightarrow \ & \|\tilde{P}_k\|_{L^{q(t)}(\mathbb{R}^d, d\rho_d)} \leq e^{kt} \|\tilde{P}_k\|_{L^{q(0)}(\mathbb{R}^d, d\rho_d)}. \end{aligned}$$

Finally, since $q(t) = 1 + e^{2t}(q(0) - 1)$, we may choose $q(t) = p$, $q(0) = 2$ so that $t = \frac{1}{2}\log(p-1)$, and therefore we obtain:

$$\begin{aligned} & \|\tilde{P}_k\|_{L^p(\mathbb{R}^d, d\rho_d)} \leq e^{\frac{k}{2}\log(p-1)} \|\tilde{P}_k\|_{L^2(\mathbb{R}^d, d\rho_d)}, \\ \Rightarrow \ & \|\tilde{P}_k\|_{L^p(\mathbb{R}^d, d\rho_d)} \leq (p-1)^{k/2} \|\tilde{P}_k\|_{L^2(\mathbb{R}^d, d\rho_d)}, \end{aligned}$$

which is what we wanted to get. □

2 Wiener Chaos

We now present and prove the main multi-linear large deviation estimate, commonly referred to as Wiener chaos.

Theorem 2.1 (Wiener chaos). *Let $d \geq 1$, $c(n_1, \dots, n_k) \in \mathbb{C}$. Let $(g_n)_{1 \leq n \leq d} \in \mathcal{N}_{\mathbb{C}}(0,1)$ be complex, one-dimensional, L^2-normalized independent Gaussians.*
For $k \geq 1$, let us denote $A(k,d) := \{(n_1, \dots, n_k) \in \{1, \dots, d\}^k, n_1 \leq n_2 \leq \dots \leq n_k\}$, and:

$$S_k(\omega) := \sum_{A(k,d)} c(n_1, \dots, n_k) g_{n_1}(\omega) \cdots g_{n_k}(\omega).$$

Then, for all $d \geq 1$ and $p \geq 2$:

$$\|S_k\|_{L^p(\Omega)} \leq \sqrt{k+1}(p-1)^{k/2} \|S_k\|_{L^2(\Omega)}.$$

Proof. First of all, we note that for any $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$ we may decompose it into real and imaginary parts, i.e.:

$$g_n = \gamma_n + i\tilde{\gamma}_n,$$

where $\gamma_n, \tilde{\gamma}_n \in \mathcal{N}_{\mathbb{C}}(0,1)$ are mutually independent. Thus, up to an index change (sum up to $2d$ instead of d) and a relabelling of the coefficients we may assume without loss of generality that the g_n are real, independent, Gaussian random variables.
Now, define the following:

$$\Sigma_k(x_1, \dots, x_d) = \sum_{A(k,d)} c(n_1, \dots, n_k) x_{n_1} \cdots x_{n_k}.$$

It is important to note that since we are summing over $A(k,d)$ that each x_{n_i}, for $i = 1, \dots, k$, must be from the collection $x_1, \dots, x_d$. From here we see that since g_{n_i} are real Gaussians we have the following equality:

$$\|S_k\|_{L^p(\Omega)} = \|\Sigma_k\|_{L^p(\mathbb{R}^d, d\rho_d)}.$$

This equality is due to the fact that the RHS is in terms of $d\rho_d$ making the norm equal to taking the norm of real Gaussians (which is what is happening on the LHS).
We now investigate all of the terms of summing over $A(k,d)$ by grouping up the x_{n_i} ($i = 1, \dots, k$) into their corresponding $x_1, \dots, x_d$ for each term of the sum as follows:

$$x_{n_1} \cdots x_{n_k} = x_{m_1}^{p_1} \cdots x_{m_l}^{p_l},$$

where we have $l \leq k$, $n_1 = m_1 < \dots < m_l \leq n_k$. This breaks up $x_{n_1} \cdots x_{n_k}$ into distinct monomials of the form $x_{m_j}^{p_j}$. We know that Hermite Polynomials form a Hilbertian basis of L on $L^2(\mathbb{R}^d, d\rho_d)$ from earlier due to $L^2(\mathbb{R}^d, d\rho_d) = \bigoplus_{k \geq 0} W_k$. Since each $x_{m_j}^{p_j}$ is made up of $x_{n_1}, \dots, x_{n_k} \in L^2(\mathbb{R}^d, d\rho_d)$ we see that $x_{m_j}^{p_j}$ can be written as a linear combination of basis functions which are our Hermite Polynomials for each W_k. We shall denote these polynomials as $P_j \in W_j$. From this we have the following break down:

$$x_{m_1}^{p_1} \cdots x_{m_l}^{p_l} = \sum_{j=0}^{k} \beta_j(x_{m_1}^{p_1}, \dots, x_{m_l}^{p_l}) P_j(x_{m_1}^{p_1}, \dots, x_{m_l}^{p_l}).$$

From here we look at the summation of these under the set $A(k,d)$ and define $\tilde{P}_j$ as follows:

$$\sum_{A(k,d)} \sum_{j=0}^{k} c(n_1,\ldots,n_k)\beta_j(x_{m_1}^{p_1},\ldots,x_{m_l}^{p_l})P_j(x_{m_1}^{p_1},\ldots,x_{m_l}^{p_l}),$$

$$\tilde{P}_j := \sum_{A(k,d)} c(n_1,\ldots,n_k)\beta_j(x_{m_1}^{p_1},\ldots,x_{m_l}^{p_l})P_j(x_{m_1}^{p_1},\ldots,x_{m_l}^{p_l}).$$

Since we are summing up over finite sets we can interchange the two sums and see that this is in fact Σ_k and equal to the following:

$$\Sigma_k(x_1,\ldots,x_d) = \sum_{j=0}^{k} \tilde{P}_j(x_1,\ldots,x_d).$$

Note that $(x_{m_1}^{p_1},\ldots,x_{m_l}^{p_l})$ are made up of $(x_1,\ldots,x_d)$ and that for each P_j the break down of the monomials must fall into the jth category.
Now we recall Lemma 2.3:
For $d \geq 1$ and $k \in \mathbb{Z}_{\geq 0}$, assuming $\tilde{P}_k$ is an eigenfunction of L with eigenvalue $-k$, then for $p \geq 1$:

$$\|\tilde{P}_k\|_{L^p(\mathbb{R}^d,d\rho_d)} \leq (p-1)^{\frac{k}{2}}\|\tilde{P}_k\|_{L^2(\mathbb{R}^d,d\rho_d)}.$$

This means that for each $\tilde{P}_j$ we have:

$$\|\tilde{P}_j\|_{L^p(\mathbb{R}^d,d\rho_d)} \leq (p-1)^{\frac{j}{2}}\|\tilde{P}_j\|_{L^2(\mathbb{R}^d,d\rho_d)}.$$

We now have by the triangle inequality:

$$\|\Sigma_k\|_{L^p(\mathbb{R}^d,d\rho_d)} = \|\tilde{P}_1 + \cdots + \tilde{P}_k\|_{L^p(\mathbb{R}^d,d\rho_d)} \leq \|\tilde{P}_1\|_{L^p(\mathbb{R}^d,d\rho_d)} + \cdots + \|\tilde{P}_k\|_{L^p(\mathbb{R}^d,d\rho_d)},$$

and by Lemma 2.3:

$$\|\tilde{P}_1\|_{L^p(\mathbb{R}^d,d\rho_d)} + \cdots + \|\tilde{P}_k\|_{L^p(\mathbb{R}^d,d\rho_d)} \leq (p-1)^{\frac{1}{2}}\|\tilde{P}_1\|_{L^2(\mathbb{R}^d,d\rho_d)} + \cdots + (p-1)^{\frac{k}{2}}\|\tilde{P}_k\|_{L^2(\mathbb{R}^d,d\rho_d)}.$$

From here since $(p-1)^{\frac{i}{2}} \leq (p-1)^{\frac{k}{2}}$ for $i = 1,\ldots,k$ we have the following inequality:

$$\|\Sigma_k\|_{L^p(\mathbb{R}^d,d\rho_d)} \leq (p-1)^{\frac{k}{2}} \sum_{j=0}^{k} \|\tilde{P}_j\|_{L^2(\mathbb{R}^d,d\rho_d)}$$

From here we use Cauchy-Schwarz for summations $(\sum_{i=1}^{n} u_i v_i)^2 \leq (\sum_{i=1}^{k} u_i^2)(\sum_{i=1}^{k} v_i^2)$ with $u_i = 1$ and $v_i = \|\tilde{P}_i\|_{L^2(\mathbb{R}^d,d\rho_d)}$:

$$\sum_{j=0}^{k} 1 \cdot \|\tilde{P}_j\|_{L^2(\mathbb{R}^d,d\rho_d)} \leq \left(\sum_{j=0}^{k} 1^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{k} \|\tilde{P}_j\|_{L^2(\mathbb{R}^d,d\rho_d)}^2\right)^{\frac{1}{2}}.$$

From this we see $(\sum_{j=0}^{k} 1^2)^{\frac{1}{2}} = (k+1)^{\frac{1}{2}}$ and we have by the orthogonality of the Hermite Polynomials, $\tilde{P}_j$, that $(\sum_{j=0}^{k} \|\tilde{P}_j\|^2_{L^2(\mathbb{R}^d, d\rho_d)})^{\frac{1}{2}} \leq \|\Sigma_k\|_{L^2(\mathbb{R}^d, d\rho_d)}$. Putting this together we have:

$$\|\Sigma_k\|_{L^p(\mathbb{R}^d, d\rho_d)} \leq \sqrt{k+1}(p-1)^{\frac{k}{2}} \|\Sigma_k\|_{L^2(\mathbb{R}^d, d\rho_d)}.$$

Lastly, we recall that $\|\Sigma_k\|_{L^p(\mathbb{R}^d, d\rho_d)} = \|S_k\|_{L^p(\Omega)}$ so we have the final inequality:

$$\|S_k\|_{L^p(\Omega)} \leq \sqrt{k+1}(p-1)^{\frac{k}{2}} \|\Sigma_k\|_{L^2(\mathbb{R}^d, d\rho_d)},$$

as desired. $\square$

3 L^4 Strichartz estimate on torus

Theorem (Bourgain ('93), Bourgain-Demeter ('14)).
For $N \geq 1$ let $\phi \in L^2(\mathbb{T}^d)$ be a smooth function such that $supp(\hat{\phi}) \subset \mathbb{Z}^d$. Then for any $\epsilon > 0$ the following estimates hold:

- $\|S(t)\phi\|_{L^q_t L^q_x(\mathbb{T}^{d+1})} \lesssim C_q \|\phi\|_{L^2_x(\mathbb{T}^d)}$ *if* $q < \frac{2(d+2)}{d}$
- $\|S(t)\phi\|_{L^q_t L^q_x(\mathbb{T}^{d+1})} \ll N^{\epsilon} \|\phi\|_{L^2_x(\mathbb{T}^d)}$ *if* $q = \frac{2(d+2)}{d}$
- $\|S(t)\phi\|_{L^q_t L^q_x(\mathbb{T}^{d+1})} \lesssim C_q N^{\frac{d}{2} - \frac{d+2}{q}} \|\phi\|_{L^2_x(\mathbb{T}^d)}$ *if* $q > \frac{2(d+2)}{d}$

Observation. *In the periodic setting, proving these was nontrivial and required new ideas that those used on $\mathbb{R}^d$. Bourgain proved some of them for rational torus and in 2014 Bourgain-Demeter obtained the full range for rational and irrational tori.*

In these notes we present the original proof of Bourgain for the square torus $\mathbb{T}^2$ and $q = 4$, in this case the inequality is:

$$\|S(t)\phi\|_{L^4_t L^4_x([0,1]\times\mathbb{T}^2)} \ll N^{\epsilon} \|\phi\|_{L^2_x(\mathbb{T}^2)}$$

Proof. Let $N \geq 1$ (dyadic) be fixed. Let $\phi_N \in L^2(\mathbb{T}^2)$ such that $supp(\hat{\phi}_N) \subset B_N(0)$ where $B_N(0) = \{n \in \mathbb{Z} \ s.t. \ |n| \leq N\}$. We write:

$$S(t)\phi_N(x) = \sum_{k\in\mathbb{Z}^2, |k|\leq N} a_k e^{i(x\cdot k - |k|^2 t)}$$

where a_k are the Fourier coefficients of ϕ_N. Now we will abbreviate $L^4_t L^4_x([0,1]\times\mathbb{T}^2)$ with $L^4([0,1]\times\mathbb{T}^2)$, observe that:

$$\left\| \sum_{|k|\leq N} a_k e^{i(x\cdot k - |k|^2 t)} \right\|^4_{L^4([0,1]\times\mathbb{T}^2)} = \left\| \left(\sum_{|k|\leq N} a_k e^{i(x\cdot k - |k|^2 t)} \right)^2 \right\|^2_{L^2([0,1]\times\mathbb{T}^2)}$$

And now observe that:

$$\left(\sum_{|k|\le N} a_k e^{i(x\cdot k-|k|^2 t)}\right)^2 = \sum_{|k_1|\le N, |k_2|\le N} a_{k_1} a_{k_2} e^{i(x\cdot(k_1+k_2)-(|k_1|^2+|k_2|^2)t)} = \sum_{k,m} b_{k,m} e^{i(x\cdot k - mt)}$$

where $b_{k,m} = \sum_{\mathcal{S}_{k,m}} a_{k_1} a_{k_2}$ and $\mathcal{S}_{k,m} = \{(k_1,k_2)\ s.t.\ k = k_1+k_2, m = |k_1|^2+|k_2|^2, |k_i| \le N\ for\ i=1,2\}$. So now using Plancherel I have:

$$\left\| \left(\sum_{|k|\le N} a_k e^{i(x\cdot k-|k|^2 t)}\right)^2 \right\|^2_{L^2([0,1]\times\mathbb{T}^2)} = \sum_{k,m} |b_{k,m}|^2$$

Now using Cauchy-Schwarz I obtain:

$$\sum_{k,m} |b_{k,m}|^2 = \sum_{k,m} \left| \sum_{\mathcal{S}_{k,m}} a_{k_1} a_{k_2} \right|^2 \le \sum_{|k|\le N, m\le N^2} \sum_{\mathcal{S}_{k,m}} |\mathcal{S}_{k,m}| |a_{k_1}|^2 |a_{k_2}|^2$$

Now for concluding the proof we need to estimate $|\mathcal{S}_{k,m}|$, so we need to count the couples $(k_1,k_2) \in \mathcal{S}_{k,m}$, for doing this we write:

$$\begin{aligned} & m = |k_1|^2 + |k-k_1|^2 = 2|k_1|^2 - 2(k_1\cdot k) + |k|^2 \\ & \Rightarrow \frac{m-|k|^2}{2} = |k_1|^2 - k_1\cdot k \\ & \Rightarrow \frac{m}{2} - \frac{|k|^2}{4} = \left| k_1 - \frac{k}{2} \right|^2 \end{aligned}$$

Observation. $2m - |k|^2 \ge 0$ *since the set is empty otherwise.*

From these last passages we notice that $k_1 \in \mathbb{Z}^2$ lies on the circle having center at $\frac{k}{2}$ and radius $R^2 = \frac{m}{2} - \frac{|k|^2}{4}$. From a well know results of analytic number theory we can say:

$$|\mathcal{S}_{k,m}| \lesssim \sup_R \frac{\log R}{\log\log R}$$

Now using that $m \le 2N^2$ we have that $R \le N^2$ and so:

$$|\mathcal{S}_{k,m}| \lesssim N^{\epsilon}$$

Now we can conclude our proof since:

$$\begin{aligned} \sum_{|k|\le N, m\le N^2} \sum_{\mathcal{S}_{k,m}} |\mathcal{S}_{k,m}| |a_{k_1}|^2 |a_{k_2}|^2 &\lesssim N^{\epsilon} \sum_{|k|\le N, m\in\mathbb{Z}} \left(\sum_{k=k_1+k_2, m=|k_1|^2+|k_2|^2} |a_{k_1}|^2 |a_{k_2}|^2 \right) \\ &\lesssim N^{\epsilon} \sum_{|k_i|\le N} \left(|a_{k_1}|^2 |a_{k_2}|^2 \sum_{k=k_1+k_2, m=|k_1|^2+|k_2|^2} (1) \right) \\ &= N^{\epsilon} \sum_{|k_i|\le N} \left(|a_{k_1}|^2 |a_{k_2}|^2 \right) \end{aligned}$$

Now using Plancherel we have:

$$\sum_{|k_i|\leq N}\left(|a_{k_1}|^2|a_{k_2}|^2\right)=\sum_{k_1\leq N}|a_{k_1}|^2\sum_{k_2\leq N}|a_{k_2}|^2=\|\phi_N\|_{L^2}^2\|\phi_N\|_{L^2}^2=\|\phi_N\|_{L^2}^4$$

and so we obtain:

$$\|S(t)\phi\|_{L_t^4L_x^4([0,1]\times\mathbb{T}^2)}\ll N^\epsilon\|\phi\|_{L_x^2(\mathbb{T}^2)}\ \forall\epsilon>0$$

□

References

[1] Ané, C., Blachère, S., Chafaï, D., Fougères, P., Gentil, I., Malrieu, F., Roberto, C., & Scheffer, G. (2000). *Sur les inégalités de Sobolev logarithmiques.* `https://api.semanticscholar.org/CorpusID:117819536`

[2] Bourgain, J. (1996). *Invariant measures for the 2d-defocusing nonlinear Schrödinger Equation.* Communications in Mathematical Physics, 176(2), 421–445. `https://doi.org/10.1007/bf02099556`

[3] Thomann, L., & Tzvetkov, N. (2010). *Gibbs measure for the periodic derivative nonlinear Schrödinger equation.* Nonlinearity, 23(11), 2771–2791. `https://doi.org/10.1088/0951-7715/23/11/003`

[4] Tzvetkov, N. (2009). *Construction of a Gibbs measure associated to the periodic Benjamin–Ono equation.* Probability Theory and Related Fields, 146(3–4). `https://doi.org/10.1007/s00440-008-0197-z`

[5] Bourgain, J. *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations.* Geometric and Functional Analysis 3, 107–156 (1993). `https://doi.org/10.1007/BF01896020`

[6] Bourgain, J. & Demeter, C. (2015). *The proof of the l^2 Decoupling Conjecture.* Annals of Mathematics. 182. 351-389. `https://doi.org/10.4007/annals.2015.182.1.9`