Math 560 Project: Representation Theory

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1 Representations

Definition 1.1: A *representation* of a group G is a pair (V, ρ) where V is a vector space and $\rho : G \to GL(V)$ is a group homomorphism.
Note: Because they are isomorphic, we will often use $GL_n(V)$, the set of n×n invertible matrices, in place of $GL(V)$.

Definition 1.2: The *dimension* of a representation (V, ρ) is the dimension of V.

Definition 1.3: The *order* of a representation (V, ρ) is the order of the image of ρ. In other words, the order of a representation is the number $|im(\rho)|$.

Example 1.4: Let G be any group. We can define a one-dimensional represenation of G by (V, ρ), where V is any one-dimensional vector space and $\rho(g) = (1) \in M_{1\times 1}(\mathbb{R})$ for all $g \in G$. We call this the *trivial representation.*

Example 1.5: Let G $= \langle i \rangle = \{i, -1, -i, 1\} \subseteq \mathbb{C}$. We define a two-dimensional representation of G by $(\mathbb{C}^2, \rho)$, with

$$\rho(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

Example 1.6: Let G be the permutation group S_n. Then, we can define an n-dimensional representation by (V, ρ), where $\rho(g)$ is the corresponding $n \times n$ permutation matrix. We give a more detailed example of this for S_3 in Example 2.2.

Example 1.7: Let $G = D_n$ where D_n is the dihedral group of size $2n$. Take $n = 4$, so $G = D_8$. This can be seen as reflections and symmetries of a square. We will denote a counterclockwise rotation of $\pi/2$ as r and a symmetrical flip across the y-axis as s. We can write

$$D_8 = \langle r, s | r^4 = e, s^2 = e, rs = sr^3 \rangle$$

We can also write out all elements of D_8.

$$D_8 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

We can create a 2-dimensional representation where $\rho(g) = A$ is the matrix such that the map $v \to Av, v \in \mathbb{R}^2$ is the associated reflection or rotation in $\mathbb{R}^2$.
Since reflections and and symmetries are unique to each other, we will have two $\rho(g)$ representations.
For a counterclockwise rotation of θg is:

$$\rho(g) \begin{pmatrix} cos(2\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

For g a reflection about the line which makes angle θ with x-axis is:

$$\begin{pmatrix} cos(2\theta) & sin(2\theta) \\ sin(2\theta) & -cos(2\theta) \end{pmatrix}$$

For our case of D_8, $\theta = \pi n/2$ for rotation.
For symmetric across y-axis: $\theta = \pi/2$.
We will now list all elements of D_8 in their representation form.
Note: $\rho(g) = \rho(s)$ is reflection on $x = 0$, $\rho(g) = \rho(sr)$ is reflection on $y = x$, $\rho(g) = \rho(sr^2)$ is reflection on $y = 0$, $\rho(g) = \rho(s)$ is reflection on $y = -x$.

$$\text{For } g = 1 = e,\ \rho(e) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \rho(r^4) = \rho(s^2)$$

$$\begin{aligned} g &= r, \rho(r) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};\ g &= r^2, \rho(r^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ g &= r^3, \rho(r^3) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\ g &= s, \rho(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ g &= sr, \rho(sr) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ g &= sr^2, \rho(sr^2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

$$\text{For } g = sr^3,\ \rho(sr^3) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Example 1.8: Let $G = S_n$ and consider the one-dimensional representation where $\rho(g) = (\text{sign}(g))$, the sign of permutation g.
Notable Definition:
The *sign* of a permutation σ, denoted $(\text{sign}(g))$, is defined as $+1$ if σ is even and -1 if σ is odd.
Let's look at $n = 4$, so $G = S_4$.
We have listed out all of the permutations of S_4:

$$S_4 =$$
$$\{e, (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23), (123), (124), (132)$$
$$(134), (142), (143), (234), (243), (1234), (1243), (1324), (1342), (1423), (1432)\}$$

The decomposition of the three cycle and four cycle elements are as follows:

$$(123) = (12)(23) \to \text{even } (124) = (12)(24) \to \text{even } (132) = (13)(23) \to \text{even}$$
$$(134) = (13)(34) \to \text{even } (142) = (14)(24) \to \text{even } (143) = (14)(34) \to \text{even}$$
$$(234) = (23)(34) \to \text{even } (243) = (24)(34) \to \text{even}$$

$$(1234) = (12)(23)(34) \to \text{odd } (1234) = (12)(24)(34) \to \text{odd}$$
$$(1234) = (13)(23)(24) \to \text{odd } (1234) = (13)(34)(24) \to \text{odd}$$
$$(1234) = (14)(24)(23) \to \text{odd } (1234) = (14)(34)(23) \to \text{odd}$$

Since all transposition are odd, we have 12 odd elements, and since e and composition of transpositions are even, we have 12 even elements.

Therefore all even elements can be represented as:

$$\begin{gathered}\text{even} = (1) = \{e, (12)(34), (13)(24), (14)(23), (123), (124), (132)\\ (134), (142), (143), (234), (243)\}\end{gathered}$$

And all odd elements can be represented as:

$$\begin{gathered}\text{odd} = (-1) = \{(12), (13), (14), (23), (24), (34), (1234),\\ (1243), (1324), (1342), (1423), (1432)\}\end{gathered}$$

This is a one-dimensional representation of S_4 of order 2.

2 Homomorphisms of Representations

Definition 2.1: A *homomorphism of representations* $T : (V, \rho) \to (V', \rho')$ is a linear map $T : V \to V'$ such that

$$T \circ \rho(g) = \rho'(g) \circ T \tag{1}$$

for all $g \in G$. The *homomorphism of representations* T is an *isomorphism* if the linear map T itself is an isomorphism. We say two representations are *isomorphic* if there exists an isomorphism between them.

Example 2.2: For this example we will be looking S_3.
We will define:

$$S_3 = \{e, (12), (13), (23), (123), (132)\} \tag{2}$$

where the parentheses signify a permutations of the given numbers following from left to right. Our goal is to find a homomorphism of representations for S_3. To do this we define a matrix representation of S_3 as follows:

$$\rho(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \rho(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \rho(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\rho(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \rho(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \rho(132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Now, we define a second representation of S_3 by $\rho'(g) = \rho(13) \circ \rho(g)$ where $\rho(g)$ is defined as above. Suppose T is a homomorphism $T : (V, \rho) \to (V, \rho')$. Since T is a homomorphism, $T \circ \rho(g) = \rho'(g) \circ T$ for all $g \in S_3$

First, let $g = (23)$. From above we wish to determine our matrix T. To do this we look at the formula:

$$T \circ \rho(g) = \rho'(g) \circ T \tag{3}$$

Letting M(T) $= \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ We plug in our values:

Note: $\rho(g)$ is $e(g) = g$ and $\rho'(g) = (13)g$.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

After matrix multiplication we see:

$$\begin{pmatrix} a & c & b \\ d & f & e \\ g & i & h \end{pmatrix} = \begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$$

This means that $a = d = g$ and $b = c = e = f = h = i$.

Therefore, we can write M(T) $= \begin{pmatrix} a & b & b \\ a & b & b \\ a & b & b \end{pmatrix}$

To refine our M(T) let's try this same process by letting $g = (23)$.
For $T \circ \rho(g) = \rho'(g) \circ T$ we get:

$$\begin{pmatrix} a & b & b \\ a & b & b \\ a & b & b \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & b \\ a & b & b \\ a & b & b \end{pmatrix}$$

After multiplication we get: $\begin{pmatrix} b & a & b \\ b & a & b \\ b & a & b \end{pmatrix} = \begin{pmatrix} a & a & a \\ b & b & b \\ b & b & b \end{pmatrix}$

So $a = b$ and M(T) $= \begin{pmatrix} a & a & a \\ a & a & a \\ a & a & a \end{pmatrix}$

This means that our homomorphism of representations is a matrix of the form:

$$M(T) = a \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Note: T is a non-injective, non-surjective homomorphism.

Example 2.3: Let $G = S_2$. Let's define two representations, ρ and ρ', where

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ \rho(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \rho'(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ \rho'(12) = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$$

Then, since $\rho(e) = \rho'(e) = I$, any homomorphism between these representations only need satisfy $T \circ \rho(g) = \rho'(12) \circ T$. So, solving the following gives us

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to M(T) = \begin{pmatrix} \frac{3}{4}c + \frac{1}{4}d & \frac{1}{4}c + \frac{3}{4}d \\ c & d \end{pmatrix}$$

The above $M(T)$ is the general form for *any* homomorphism between representations. If we take $c = 0$ and $d = 1$, we see $M(T) = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ 0 & 1 \end{pmatrix}$, an isomorphism. Thus, the T given by this matrix representation is our first view of an *isomorphism* between representation. Additionally, we can say that these two representations of S_2 are *isomorphic*.

3 Regular and Complex Representations

Definition 3.1: If X is a finite set, define the complex vector space $\mathbb{C}[X]$ of linear combinations of the elements of X:

$$\mathbb{C}[X] := \{\sum_{x \in X} a_x x | a_X \in \mathbb{C}\} \tag{4}$$

Here, addition and scalar multiplication is done coefficientwise:

$$\lambda \sum_{x \in X} a_x x + \mu \sum_{x \in X} b_x x := \sum_{x \in X} (\lambda a_x + \mu b_x) x \tag{5}$$

Definition 3.2: Let G be a finite group. Let X be the set $X := G$ together with the action $G \times X \to X$ given by left multiplication, $g \cdot x = gx$. Then, the resulting representation is $(\mathbb{C}[X], \rho)$. This is known as the *regular representation* of X.
Note: Since $X := G$, we can drop the X and write this representation as $(\mathbb{C}[G], \rho)$.

Example 3.3: Consider the group $G = S_3$. Then, the regular representation of G is $(\mathbb{C}[S_3], \rho)$, where

$$\mathbb{C}[S_3] = \{a_1 e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132) \mid a_i \in \mathbb{C}\} \tag{6}$$

and $\rho(g)(x) = g \cdot x$ for all $g, x \in S_3$, which linearly extends to all of $\mathbb{C}[S_n]$ by:

$$\rho(g)(\sum_{x \in S_3} a_x x) = \sum_{x \in S_3} a_x (g \cdot x) \tag{7}$$

Example 3.4: Consider the previously given regular representation of S_3. Now, examine the elements $g = (12)$ and $x = 2e + (13) - i(132)$. Then, we calculate $\rho(g)(x)$ as follows"

$$\rho(g)x = \rho(12)(2e + (13) - i(132)) = 2(12)e + (12)(13) - i(12)(132) = \\ 2(12) + (132) - i(13)$$

4 Subrepresentations

Definition 4.1 A *subrepresentation* of a representation (V, ρ) is a vector space $W \subseteq V$ such that $\rho(g)(W) \subseteq W$ for all $g \in G$. Such a W is also called a *G-invariant subspace.* Thus, $(W, \rho|_W)$ is a representation, where $\rho|_W : G \to GL(W)$ is defined by $\rho|_W(g) := \rho(g)|_W$. An important note here is that we are not restricting the domain of ρ, but rather the domain of each $\rho(g)$.

Note 4.2: This is the textbook's notation, and it is kinda of clunky and we don't like it. They define a subrepresentation by $\rho(g)W \subseteq W$ for all $g \in G$. What this really means is $Range(\rho(g)) \subseteq W$ for all $g \in G$. In other words, the range of each $\rho(g)$ is a subspace of W. A final way we could rewrite this is $\rho(g)w \in W \ \forall w \in W$.

The best way to think about a subrepresentation is that it is a $\rho(g)$-invariant subspace for each $g \in G$.

Definition 4.3: A subrepresentation is *proper* if $W \subsetneq V$ and *nonzero* if $W \neq \{0\}$.

Example 4.4: Consider the representation of S_2 given by $(\mathbb{R}^2, \rho)$, where

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \rho(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Notice that the eigenvalues of $\rho(e)$ are $\lambda_{1,2} = 1$ and the eigenvalues of $\rho(12)$ are $\lambda_1 = 1, \lambda_2 = -1$. Then, we see $E(\lambda = 1, \rho(e)) = \mathbb{R}^2$, $E(\lambda = 1, \rho(12)) =$ span(1,1), and $E(\lambda = -1, \rho(12)) =$ span(1,-1). Thus, the two non-trivial, proper subrepresentations are $W_1 =$ span(1,1) and $W_2 =$ span(1,-1).

Definition 4.5: A representation is (V, ρ) is *irreducible* (or *simple*) if it is nonzero and there does not exist any proper nonzero subrepresentation of V. It is *reducible* if it has a proper nonzero subrepresentation.

Proposition 4.6: For W finite-dimensional, show that $\rho(g)(W) \subseteq W$ implies $\rho(g)(W) = W$.

Proof: Let's define (V, ρ) for a group G. Suppose $W \subseteq V$ is a finite dimensional subspace and $\rho(g)(W) \subseteq W$. To show $\rho(g)(W) = W$, we need only show $\rho(g)(W) \supseteq W$. This is easy to show since $\rho(g) \in GL(V)$, meaning $\rho(g)$ is bijective. So since $\rho(g)(W) \subseteq W$, we see $dim(\rho(g)|_W) = dim(W)$. Thus, since $\rho(g)|_W$ is surjective, $\rho(g)(W) = W$.

5 Direct Sums

Definition 5.1: Given two vector spaces V_1 and V_2, the *external direct sum* of V_1 and V_2 is defined $V_1 \oplus V_2 := V_1 \times V_2$. **Note:** This is a different idea from the *internal direct sum*, which is the usual notion of direct sum from Linear Algebra. Here, $V_1 \oplus_{\text{int}} V_2 := V_1 + V_2$, where $V_1 \bigcap V_2 = \{0\}$.

Definition 5.2: Given two representations (V_1, ρ_1) and (V_2, ρ_2), the *external direct sum* is the representation $(V_1, \rho_1) \oplus_{\text{ext}} (V_2, \rho_2) := (V, \rho)$ where $V = V_1 \oplus_{\text{ext}} V_2$ and $\rho(g)(v_1, v_2) = (\rho_1(g)(v_1), \rho_2(g)(v_2))$.

Definition 5.3: Given a representation (V, ρ) and subrepresentations $V_1, V_2 \subseteq V$, we say that (V, ρ) is the *internal direct sum* of $(V_1, \rho|_{V_1})$ and $(V_2, \rho|_{V_2})$ if $V = V_1 \oplus_{\text{int}} V_2$.

Definition 5.4: A nonzero representation is *decomposable* if it is a direct sum of two proper nonzero subrepresentations. Otherwise, we say the representation is *indecomposable*.

Example 5.5: Recall Example 4.4, where we have W_1=span(1,1) and $W_2 =$ span$(1, -1)$. Then, we notice that span(1,1) $\bigcap$ span(1,-1) $= \{0\}$ so span$(1, 1) \oplus_{\text{int}}$ span$(1, -1)$. Thus, we see our $(\mathbb{R}^2, \rho)$ is decomposable into (span(1,1)$, \rho|_{\text{span(1,1)}})$ and (span(1,-1)$, \rho|_{\text{span(1,-1)}})$.

Definition 5.6: A representation is *semisimple* or *completely reducible* if it is an (internal) direct sum of irreducible representations.

6 Maschke's Theorem

Definition 6.1: Let (V, ρ) be a representation of G and $W \subseteq V$ a subrepresentation. A *complementary representation* is a subrepresentation $U \subseteq V$ such that $V = W \oplus_{\text{int}} U$.

Theorem 6.2: Maschke's Theorem
Let (V, ρ) be a finite-dimensional representation of a finite group G. Let $W \subseteq V$ be any subrepresentation. Then there exists a complementary subrepresenation $U \subseteq V$.

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