Chapter 3

Mean-field approximation

Symbol "(★)": Questions and exercises indicated with a (★) are optional. No worries if you don't have time to try to solve them, or if you don't manage to solve them on your own.

The symbol "</>": indicates optional questions with numerical simulation.

Toolbox. We recall the following Taylor expansions:

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)\,, \qquad \text{for } x \text{ close to } 0 \tag{3.1}$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)\,, \qquad \text{for } x \text{ close to } 0 \tag{3.2}$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + o(x^4)\,, \qquad \text{for } x \text{ close to } 0 \tag{3.3}$$

We recall the derivative of the hyperbolic tangent:

$$[\tanh(x)]' = \frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)\,. \tag{3.4}$$

3.1 Critical exponents for the Mean-Field Ising model

In the mean-field approximation, the avegare magnetisation per site is given by the self-consistency equation:

$$m = \tanh(\beta(H + Jmq)) \tag{3.5}$$

where H is the external magnetic field, J is the coupling parameter, and q is the number of nearest neighbors. In absence of any external field, $H = 0$, finding the solution of this equation graphically gives the critical temperature $T_c = qJ/k_B$ (see lecture). For clarity, we will denote m_0 the magnetization per spin when $H = 0$, i.e. $m_0(T) = m(H = 0, T)$.

Q1. In absence of any external field ($H = 0$), study the behavior of m_0 close to the critical point ($T \to T_c$) and show that:

$$\text{for } T < T_c\,, \qquad m_0 \sim a\,(T_c - T)^\beta\,, \text{ when } T \to T_c^-\,, \tag{3.6}$$

where β is an exponent to specify and a is a constant coefficient. What is the critical exponent β? What is the multiplicative coefficient a?

Q2. Similarly, can you show that, near the critical point, the magnetic susceptibility behaves as:

$$\chi \sim |T - T_c|^{-\gamma}\,, \tag{3.7}$$

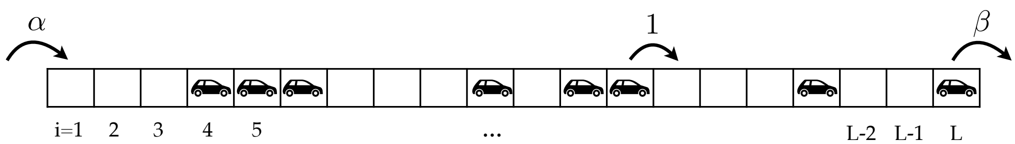
for $T < T_c$, as well as for $T > T_c$.

Q3. (★) Similarly, can you find out what are the critical exponents for: a) the heat capacity close to the critical point T_c for the phase transition at $H = 0$; and b) the behavior of the magnetization at $T = T_c$ for small values of external field H:

$$c \sim |T - T_c|^{-\alpha}\,, \qquad \text{and} \qquad m \sim H^{1/\delta}\,. \tag{3.8}$$

3.2 Back to TASEP!

We recall the TASEP model for traffic that we have discussed in the first tutorial T0.



Cars evolve on a one-dimensional lattice with L sites opened on both sides (which represents a portion of road). Cars randomly hop into the lattice from the left with rate α, jump along the lattice to the right with rate 1, and exit the lattice from the right with rate β (see figure). Each site can only be occupied by one car at a time, and a car can hop to the next site only if it is empty (exclusion interaction).

We denote by $\rho_i(t)$ the probability that there is a car at the site i at time t: $\rho_i(t) = P[n_i(t) = 1]$. The probability that there is no car at site i is then $(1 - \rho_i)$ (as a site can either be occupied or not). We define $J_i(t) = P[n_i(t) = 1; n_{i+1}(t) = 0]$ as the current of cars that exit site i to the right at time t.

We recall that the local density of cars $\rho_i(t)$ follows the equation of evolution:

$$\frac{d\rho_i}{dt}(t) = J_{i-1}(t) - J_i(t)\,, \text{ for all the sites } i \in \{1, \cdots, L\}\,, \tag{3.9}$$

with the current at the left and right boundaries respectively given by:

$$\begin{cases} J_0(t) & = \alpha\,(1 - \rho_1(t))\,, \\ J_L(t) & = \beta\,\rho_L(t)\,. \end{cases} \tag{3.10}$$

Q1. In the stationary state, the local densities and currents become time-independent. Deduce from the previous equations that the current is uniform in the stationary state, i.e. for all i, $J_i = J$ is a constant. Can you give the relation between ρ_1 and J, and between ρ_L and J?

Q2. We are interested in studying the stationary state and understand if the system can be in different phases. Using a mean-field approximation, can you obtain a recurrence relation between ρ_i and ρ_{i+1} in the stationary state?

Q3. Can you find what are the fixed points of the recurrence relation?

Q4. Studying the stability of the fixed points: Depending on the values of α and β: which fixed points are stable?

Q5. From this analysis, can you obtain the phase diagram in (α, β) for the TASEP model?

3.3 Ising model with long-range interactions; Mean-field Ising and Landau theory

In this exercise, we are interested in showing that the mean-field Ising model becomes exact, in the case of an Ising model in which all the spins interact identically (not just the nearest neighbors).

Let us consider N spins ($N \gg 1$). There are $N(N-1)/2$ pairs of spins. We take the coupling parameter to be inversely proportional to N, so that the coupling energy remains proportional to N. The energy of the system take the form:

$$E(\vec{s}) = -\frac{J}{N} \sum_{pair(i,j)} s_i s_j \,, \tag{3.11}$$

in which the sum is over all possible pairs of spins (and not just the nearest neighbors), and J is a coupling constant.

Q1. Can you show that the energy can be re-written under the form:

$$E(\vec{s}) = -\frac{J}{2N} \left[\left(\sum_{i=1}^{N} s_i \right)^2 - N \right] \; ? \tag{3.12}$$

Q2. We denote $M = \sum_{i=1}^{N} s_i$ the total magnetization. We recall that $-N \leq M \leq N$. Can you show that the number of states with the same magnetization M is given by:

$$\Omega(M) = \frac{N!}{\left(\frac{N+M}{2}\right)! \, \left(\frac{N-M}{2}\right)!} \tag{3.13}$$

Q3. Can you show that the partition function Z can be re-written under the form:

$$Z = \sum_{M} \Omega(M) \exp\left(-\beta \, E(M)\right) \,, \tag{3.14}$$

where $E(M)$ is the energy of a state with magnetization M. We introduce $Z(M, T) = \Omega(M) \exp\left(-\beta \, E(M)\right)$ such that:

$$Z = \sum_{M} Z(M, T) \,, \tag{3.15}$$

Q4. Using the Stirling formula[1] for $N!$ for large N, can you show that the function $F(M, T) = -k_B T \ln Z(M)$ can be written under the form:

$$F(M, T) = -\frac{JM^2}{2N} + \frac{J}{2} + k_B T \left(N_+ \ln N_+ + N_- \ln N_- - N \ln N\right) \,, \tag{3.16}$$

where $N_+ = (N + M)/2$ and $N_- = (N - M)/2$. In Eq. (3.15), Z can be approximated by the largest term in the sum, which is the term $Z(M_0, T)$ for which M_0 minimizes $F(M, T)$.

Q5. Let us define $m = M/N$ the magnetization per spin, and $f(m, T) = F(m, T)/N$. Using Eq. (3.16), can you give the expression of $f(m, T)$ as a function of m? Can you show that the values of m that minimize $f(m, T)$ satisfy a self-consistency equation that is similar to the characteristic equation of the mean-field Ising model? Using that $\tanh^{-1}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ will help recover the equation of the mean-field Ising model. Can you show that the system has a phase transition and compute

[1] Stirling formula: $\ln(N!) \simeq N \ln N - N$ as N goes to $+\infty$.

the critical temperature T_c?

Q6. Show that for m close to 0, $f(m, T)$ has the form expected by the Landau theory, i.e.:

$$f(m, T) = f_0 + \frac{1}{2} a_2(T) m^2 + \frac{1}{4} a_4(T) m^4 , \tag{3.17}$$

where $a_2(T_c) = 0$. Can you give the expressions of $a_2(T)$ and $a_4(T)$?

3.1 Solutions

3.1.1 Critical exponents for the Mean-Field Ising model

A1. To find out the behavior of m close to the critical point, we must solve the self-consistency equation Eq. (3.5) with $H = 0$ for T close to T_c. We know already that for all temperature $T > T_c$, there is only one solution that is $m_0 = 0$. We are therefore interested in what happens in the region $T < T_c$. In general, it is not easy to solve Eq. (3.5), even with $H = 0$. However, for temperature smaller than the critical point but close to the critical point, we expect the magnetization per spin m_0 to be very small, i.e. to be close to zero. This is because the value of m evolves continuously through the phase transition at T_c (continuous phase transition) and we know that for $T > T_c$ the magnetization per spin is zero $m_0 = 0$. Thanks to this, we can make an expansion of the non-linear function on the right-hand-side of Eq. (3.5) for small values of m_0.

We start from Eq. (3.5), in which we take $H = 0$ and replace the expression of $T_c = qJ/k_B$:

$$m_0 = \tanh\left(\frac{T_c}{T}\, m_0\right) \tag{3.18}$$

We then expend the hyperbolic tangent around $m_0 = 0$ (see Taylor expansion in Eq. (3.1)), which gives:

$$m_0 = \frac{T_c}{T}\, m_0 - \left(\frac{T_c}{T}\right)^3 \frac{m_0^3}{3} + o(m_0^3)\,. \tag{3.19}$$

Note that an expansion to the 3 order is sufficient to extract a value for m. An expansion to the first order would give us that "$T = T_c$" to first order, which is not very useful. This then yields a quadratic equation for m_0 of the form

$$\left(\frac{T_c - T}{T}\right) = \left(\frac{T_c}{T}\right)^3 \frac{m_0^2}{3} + o(m_0^2) \tag{3.20}$$

We can check that the terms on each part of the equality are both positive (as $T < T_c$). We can then take the square root on both sides and get two solutions, for positive and negative m_0 close to T_c:

$$m_0 \sim \pm\sqrt{3}\left(\frac{T}{T_c}\right)^{3/2}\left(\frac{T_c - T}{T}\right)^{1/2}, \qquad \text{for } T < T_c\,. \tag{3.21}$$

In the limit $T \longrightarrow T_c$, the fraction $\frac{T_c}{T} \longrightarrow 1$, so we obtain:

$$m_0 \sim \pm\sqrt{3}\left(\frac{T_c - T}{T_c}\right)^{1/2}, \text{ for } T \longrightarrow T_c^-\,. \tag{3.22}$$

From the above, we read off that $\beta = 1/2$, and $a = \sqrt{3/T_c}$.

> To summarize, close to the critical point, $T \longrightarrow T_c$, we have that:
>
> $$m_0 \sim \begin{cases} \pm\sqrt{\dfrac{3}{T_c}}\,(T_c - T)^{1/2}\,, & \text{for } T < T_c \\ 0\,, & \text{for } T > T_c \end{cases} \tag{3.23}$$

A2. The magnetic susceptibility is given by $\chi = \lim_{H\to 0}\left(\frac{\partial m}{\partial H}\right)_T$.

We start from Eq. (3.5) in which we replace the expression of $T_c = qJ/k_B$:

$$m = \tanh\left(\frac{T_c}{T}\, m + \frac{H}{k_B T}\right) \tag{3.24}$$

Version 1: Deriving both side of this equation according to H, we obtain:

$$\left(\frac{\partial m}{\partial H}\right)_T = \frac{1}{\cosh^2\left(\frac{T_c}{T}\,m + \frac{H}{k_BT}\right)}\left(\frac{T_c}{T}\left(\frac{\partial m}{\partial H}\right)_T + \frac{1}{k_BT}\right) \tag{3.25}$$

Here we used the first expression in Eq. (3.4) for the derivative of $\tanh(x)$. Note that we also had to keep the derivative of m with respect to H for the derivation of the term inside the hyperbolic tangent. Taking the limit $H \longrightarrow 0$ leads to

$$\chi = \frac{1}{\cosh^2\left(\dfrac{T_c}{T}\,m_0\right)}\left(\frac{T_c}{T}\,\chi + \frac{1}{k_BT}\right), \tag{3.26}$$

where we used the definition of χ and the relabeling $m_0(T) = m(H = 0, T)$. Re-organising the terms to extract χ from this equation gives:

$$\chi = \frac{1}{k_BT}\,\frac{1}{\cosh^2\left(\dfrac{T_c}{T}\,m_0\right) - \dfrac{T_c}{T}} \tag{3.27}$$

• **For** $T > T_c$, we have $m_0 = 0$, which gives (we recall that $\cosh(0) = 1$):

$$\chi = \frac{1}{k_BT}\,\frac{1}{1 - \dfrac{T_c}{T}} = \frac{1}{k_B}(T - T_c)^{-1}\,. \tag{3.28}$$

Therefore, we get the critical exponent $\gamma = 1$ for $T > T_c$.

• **For** $T < T_c$ **and** T **close to** T_c, the value of m_0 is close to 0, and its behavior is given by Eq. (3.22). We can expand $\cosh(T_c m_0/T)$ for small values of m_0 in Eq. (3.27):

$$\cosh^2\left(\frac{T_c}{T}\,m_0\right) = \left[1 + \frac{1}{2}\left(\frac{T_c}{T}\,m_0\right)^2 + o(m_0^2)\right]^2 \tag{3.29}$$

$$= \left[1 + 2\,\frac{1}{2}\left(\frac{T_c}{T}\,m_0\right)^2 + o(m_0^2)\right] = \left[1 + \left(\frac{T_c}{T}\,m_0\right)^2 + o(m_0^2)\right] \tag{3.30}$$

in which we neglected all the terms of order higher than m_0^2 in m_0. Replacing this expansion in Eq. (3.27), we get:

$$\chi = \frac{1}{k_BT}\,\frac{1}{1 + \left(\dfrac{T_c}{T}\right)^2 m_0^2 - \dfrac{T_c}{T}} \tag{3.31}$$

In which we can replace $m_0(T)$ by its behavior close to the critical temperature Eq. (3.22), $m_0^2 \sim 3\,(T_c - T)/T_c$:

$$\chi = \frac{1}{k_BT}\,\frac{1}{1 + \left(\dfrac{T_c}{T}\right)^2 3\left(\dfrac{T_c - T}{T_c}\right) - \dfrac{T_c}{T}} = \frac{1}{k_B}\,\frac{1}{T + \left(\dfrac{T_c}{T}\right) 3\,(T_c - T) - T_c} \tag{3.32}$$

$$\chi = \frac{1}{k_B}\,\frac{1}{\left(\dfrac{T_c}{T}\right) 3\,(T_c - T) - (T_c - T)} \tag{3.33}$$

In the limit $T \longrightarrow T_c^-$, one finally gets:

$$\chi = \frac{1}{k_B}\,\frac{1}{2\,(T_c - T)} = \frac{1}{2k_B}(T_c - T)^{-1} \tag{3.34}$$

Therefore, we also get the critical exponent $\gamma = 1$ for $T < T_c$. This is the same exponent than for $T > T_c$, only the coefficient differ: $1/(2k_B)$ instead of $1/k_B$.

> To summarize, close to the critical point, $T \longrightarrow T_c$, we have that:
>
> $$\chi \sim \begin{cases} \dfrac{1}{2k_B}\,(T_c - T)^{-1}, & \text{for } T < T_c \\ \dfrac{1}{k_B}\,(T - T_c)^{-1}, & \text{for } T > T_c \end{cases} \tag{3.35}$$

Version 2: Note, one can obtain these results "faster" by using another expression for the derivative of $\tanh(x)$ (second expression in Eq. (3.4)). This is much simpler, as it doesn't require to take any additional expansion in small m, because the non-linear part, $\tanh(\cdots)$, is directly equal to m, thanks to the self-consistency equation (3.5). Restarting from Eq. (3.24), taking the derivative according to H:

$$\left(\frac{\partial m}{\partial H}\right)_T = \left[1 - \tanh^2\left(\frac{T_c}{T}\,m + \frac{H}{k_B T}\right)\right]\left(\frac{T_c}{T}\left(\frac{\partial m}{\partial H}\right)_T + \frac{1}{k_B T}\right) \tag{3.36}$$

In this equation, we can directly replace the tanh-term by m using Eq. (3.24) we just started from:

$$\left(\frac{\partial m}{\partial H}\right)_T = \left[1 - m^2\right]\left(\frac{T_c}{T}\left(\frac{\partial m}{\partial H}\right)_T + \frac{1}{k_B T}\right) \tag{3.37}$$

Taking the limit $H \longrightarrow 0$ on both sides, we then get:

$$\chi = \left[1 - m_0^2\right]\left(\frac{T_c}{T}\,\chi + \frac{1}{k_B T}\right) \tag{3.38}$$

from which we can extract χ:

$$\chi = \frac{1}{k_B T}\,\frac{1}{\left(\frac{1}{\left[1 - m_0^2\right]} - \frac{T_c}{T}\right)} = \frac{(1 - m_0^2)}{k_B T}\,\frac{1}{1 - (1 - m_0^2)\frac{T_c}{T}} \tag{3.39}$$

• For $T > T_c$, we know that $m_0 = 0$, which gives back Eq. (3.28).
• For $T < T_c$, we replace in Eq. (3.39) the expression of Eq. (3.22) for the behavior of m_0 close to the critical point, $m_0^2 \sim 3\,(T_c - T)/T_c$, which gives:

$$(1 - m_0^2) = \frac{T_c - 3\,(T_c - T)}{T_c} = \frac{T + 2(T - T_c)}{T_c} \underset{T \to T_c}{\longrightarrow} 1 + 2\frac{(T - T_c)}{T_c} \tag{3.40}$$

and therefore gives:

$$\chi = \frac{1}{k_B T}\left[\frac{T + 2(T - T_c)}{T_c}\right]\frac{1}{1 - \left[\frac{T + 2(T - T_c)}{T_c}\right]\frac{T_c}{T}} \qquad \text{for } T < T_c \tag{3.41}$$

$$= \frac{1}{k_B T}\left[\frac{T + 2(T - T_c)}{T_c}\right]\frac{1}{1 - 1 - \left[\frac{2(T - T_c)}{T}\right]} = \frac{1}{k_B T}\left[\frac{T + 2(T - T_c)}{T_c}\right]\frac{1}{\left[\frac{2(T_c - T)}{T}\right]} \tag{3.42}$$

$$\underset{T \to T_c}{\longrightarrow} \frac{1}{k_B}\,[1]\,\frac{1}{[2(T_c - T)]}\,. \tag{3.43}$$

We recover equation (3.34) in the limit $T \longrightarrow T_c^-$ (coming from below the critical point):

$$\chi = \frac{1}{2k_B}\,(T_c - T)^{-1} \tag{3.44}$$

From Eq. (3.39), one can also simply expand $1/(1-m_0^2)$ in second order in m_0, which gives:

$$\frac{1}{\chi} = k_B T \left(\frac{1}{1-m_0^2} - \frac{T_c}{T}\right) = k_B T \left(1 + m_0^2 + o(m_0^2) - \frac{T_c}{T}\right) \tag{3.45}$$

$$\sim k_B T \left(1 + \frac{3\,(T_c - T)}{T_c} - \frac{T_c}{T}\right) \sim k_B \left(T + \frac{3\,T}{T_c}\,(T_c - T) - T_c\right) \tag{3.46}$$

$$\sim k_B \left(\frac{3\,T}{T_c} - 1\right) (T_c - T) \tag{3.47}$$

$$\underset{T\to T_c^-}{\sim} 2\,k_B\,(T_c - T) \tag{3.48}$$

which gives back eq. (3.44).

A3. (⋆) See lecture slides on Landau theory: In the slides the free energy is expended in small values of m and the steps on how to find out the critical behavior of c and m are then detailed.

3.1.2 Back to TASEP!

A1. In the stationary state, $d\rho_i/dt = 0$ and so

$$0 = J_{i-1}(t) - J_i(t) \implies J_i(t) = J_{i-1}(t) \equiv J \tag{3.49}$$

Setting $J_0(t) = J_L(t) = J$ allows us to find a relation between ρ_1 and ρ_L and J.

$$J = \alpha(1-\rho_1), \quad J = \beta\rho_L \implies \rho_1 = 1 - \frac{J}{\alpha}, \quad \rho_L = \frac{J}{\beta} \tag{3.50}$$

A2. There are two different ways to approach the mean-field approximation.

Version 1: The first approach consists in applying directly the approximation to the joint probability distribution $J_i = P[n_i = 1;\ n_{i+1} = 0]$, by replacing the joint distribution by a product of independent distributions:

$$J_i = P[n_i = 1;\ n_{i+1} = 0] \simeq P[n_i = 1]\ P[n_{i+1} = 0]\,. \tag{3.51}$$

In this approach to the mean-field approximation, we assume that the probability that a given site is occupied is independent of the state of the other sites; each site only "feels" the mean influence of the other sites. Replacing this in the stationary equation (3.49), one gets:

$$J = P[n_i = 1]\ P[n_{i+1} = 0] \tag{3.52}$$

$$= \rho_i(1 - \rho_{i+1})\,, \tag{3.53}$$

where we used that $\rho_i = P[n_i = 1]$ and $P[n_{i+1} = 0] = 1 - \rho_{i+1}$. This finally yields the recurrence relation:

$$\rho_i = \frac{J}{1-\rho_{i+1}} \qquad \text{or equivalently} \qquad \rho_{i+1} = 1 - \frac{J}{\rho_i}\,. \tag{3.54}$$

Version 2: In the second approach to the mean-field approximation, we write the number of cars at site i is equal to the mean number of cars $\langle n_i(t)\rangle$, plus some fluctuations $\delta n_i(t)$ around the mean value:

$$n_i(t) = \langle n_i(t)\rangle + \delta n_i(t)\,. \tag{3.55}$$

We then assume that these fluctuations are negligeable compared to $\langle n_i\rangle$. This version of the mean-field approximation is useful only if we have written down an equation for the evolution of the number of cars in the system. We observe that, as there can only be one car at a time at each site, the number $n_i(t)$ of cars at site i can only be 0 or 1 at any time. This allows to obtain the following expression for the mean number of cars in site i at time t:

$$\begin{aligned}\langle n_i(t)\rangle &= 1\times\rho_i(t) + 0\times[1-\rho_i(t)] &(3.56)\\ &= \rho_i(t)\,, &(3.57)\end{aligned}$$

as $\rho_i(t) = P[n_i(t) = 1]$ is the probability that site i is occupied at time t. Similarly, we observe that the current $J_i(t)$ in i at time t is equal to the ensemble average $\langle n_i(t)(1-n_{i+1}(t))\rangle$. Indeed, $n_i(t)(1-n_{i+1}(t)$ also only takes two values: value 1 if i is occupied AND $(i+1)$ is empty, and value 0 in all other cases, and therefore:

$$\begin{aligned}\langle\, n_i(t)\,(1-n_{i+1}(t))\,\rangle &= 1\times P[n_i=1;\; n_{i+1}=0] + 0\times[1-P[n_i=1;\; n_{i+1}=0]] &(3.58)\\ &= J_i(t)\,, &(3.59)\end{aligned}$$

The equation (3.9) could then be re-written under the form:

$$\frac{d\,\langle n_i\rangle}{dt}(t) = J_{i-1}(t) - J_i(t)\,,\ \text{where } J_i(t) = \langle n_i(t)(1-n_{i+1}(t))\rangle\,, \tag{3.60}$$

and the stationary equation take the form: $J_i = \langle n_i(1-n_{i+1})\rangle = J$ is a constant. Using the mean-field approximation Eq. (3.55) in the stationary equation we get:

$$\begin{aligned}J &= \langle(\langle n_i\rangle + \delta n_i)(1-\langle n_{i+1}\rangle - \delta n_{i+1})\rangle &(3.61)\\ &\simeq \langle n_i\rangle\ (1-\langle n_{i+1}\rangle) &(3.62)\\ &\simeq \rho_i\,(1-\rho_{i+1})\,, &(3.63)\end{aligned}$$

where we have neglected all the terms of order δn_i, δn_{i+1}, or higher. This equation finally yields the recurrence relations Eq. (3.54). Currently, this relation only holds for sites $i\in[2, L-1]$. By defining the auxiliary site

$$\rho_0 \equiv \alpha, \quad \rho_{L+1} \equiv 1-\beta \tag{3.64}$$

we make sure that the recurrence relations holds for all sites on the chain.

A3. For a recurrence relation of the form $a_{i+1} = f(a_i)$, the fixed points are given by the condition $f(a_i) = a_i$. So for our particular recurrence relation we have

$$\rho_i = 1 - \frac{J}{\rho_i} = \frac{\rho_i - J}{\rho_i} \implies (\rho_i)^2 - \rho_i + J = 0\,. \tag{3.65}$$

This quadratic equation has solutions only if its discriminant $\Delta = 1-4J$ is positive. Therefore, we have three cases:

1) $\Delta < 0 \iff J > 1/4$: there are no solution such that $J > 1/4$: the current of cars in the system cannot be larger than $1/4$.

2) $\Delta = 0 \iff J = 1/4$: this is the maximal current reachable, and in that case the fixed point is $\rho^\infty = 1/2$.

3) $\Delta > 0 \iff J < 1/4$: in this case there are two possible fixed points:

$$\rho^\infty_\pm = \frac{1\pm\sqrt{1-4J}}{2}\,. \tag{3.66}$$

A4. For $J > 1/4$ there are no steady state solutions.

For $J = 1/4$ we have the solution $\rho_\infty = \frac{1}{2}$. Starting from $\rho_0 = \alpha$, we find that if

$$\rho_1 = 1 - \frac{1}{4\alpha} < \rho_\infty \tag{3.67}$$

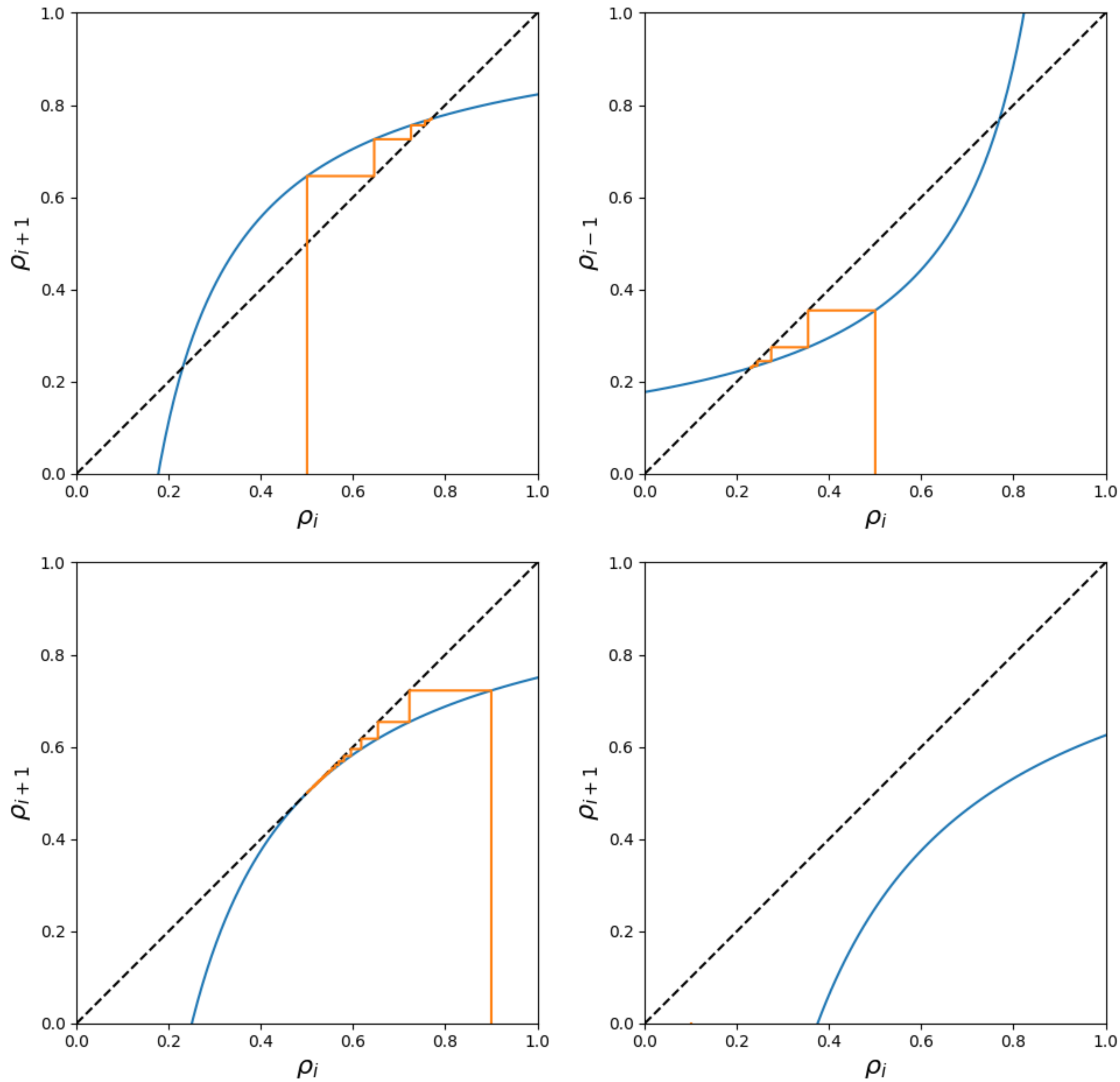


Figure 3.1: Graphical representation of the different steady-solutions. Top left: $J < 1/4$. The iteration converges to ρ_+ as long as $\alpha > \rho_-$. Top right: $J < 1/4$ but starting from the right boundary with initial condition $\rho_{L+1} = 1 - \beta$. In this case the iteration only converges to the steady-state solution if $\beta > \alpha$. Bottom left: $J = 1/4$. The iteration converges as long as $\alpha \geq \frac{1}{2}$. Bottom right: $J > 1/4$. There are no steady-state solutions.

we will move away from the fixed point. Solving for α gives that for $\alpha < \frac{1}{2}$ we will not converge to the steady-state density. So for $J = 1/4$ the steady-state is stable for $\alpha \geq \frac{1}{2}$. Similarly, requiring that

$$\rho_L = \frac{1}{4\beta} \geq \rho_\infty \tag{3.68}$$

gives a stability condition in terms of β. Solving for β, we find $\beta \geq \frac{1}{2}$. So the fixed point $\rho_\infty = \frac{1}{2}$ corresponding to $J = 1/4$ is stable if $\alpha \geq \frac{1}{2}$ and $\beta \geq \frac{1}{2}$.

If $J < 1/4$ we have two steady-state solutions given by

$$\rho_\pm = \frac{1 \pm \sqrt{1-4J}}{2} \tag{3.69}$$

and we can distinguish between the regimes $\alpha \leq \frac{1}{2}$ and $\beta \leq \frac{1}{2}$. For $\alpha \leq \frac{1}{2}$, we find that the solution converges to a steady state as long as (see Fig. 3.1 - top left)

$$\rho_0 = \alpha \geq \rho_- \tag{3.70}$$

Picking $\alpha = \rho_-$ we find that

$$\alpha = \frac{1-\sqrt{1-4J}}{2} \implies J = \alpha(1-\alpha) \tag{3.71}$$

At the other end of the chain, we then find

$$\rho_{i-1} = \frac{\alpha(1-\alpha)}{1-\rho_i} \tag{3.72}$$

We find that for initial condition $\rho_{L+1} = 1-\beta$, the solution only converges to the stable steady-state solution if $1-\beta < 1-\alpha$, or equivalently $\beta > \alpha$ (see Fig. 3.1 - top right). For $\beta \leq \frac{1}{2}$, the argument is essentially the same. In that case we get $J = \beta(1-\beta)$ and $\alpha > \beta$.

A5. From the above analysis, we have found three different phases. The **maximum current phase** is given by $\alpha \geq \frac{1}{2}, \beta \geq \frac{1}{2}$ and $J = 1/4$ with bulk density $\rho_\infty = \frac{1}{2}$. The **low density phase** is characterized by $\alpha \leq \frac{1}{2}, \beta > \alpha$ and $J = \alpha(1-\alpha)$, where $\rho_0 = \rho_\infty = \alpha$ and $\rho_L = \frac{\alpha(1-\alpha)}{\beta}$. The **high density phase** is characterized by $\beta \leq \frac{1}{2}, \alpha > \beta$ and $J = \beta(1-\beta)$. In this case $\rho_{L+1} = \rho_\infty = 1-\beta$ and $\rho_1 = 1 - \frac{\beta(1-\beta)}{\alpha}$.

3.1.3 Ising model with long-range interactions; Mean-field Ising and Landau theory

A1. Expanding out the square of the sum yields

$$\left(\sum_{i=1}^N s_i\right)^2 = \sum_{i=1}^N \sum_{j=1}^N s_i s_j = \sum_{i=1}^N \sum_{j\neq i}^N s_i s_j + \sum_{i=j}^N s_i s_j \tag{3.73}$$

Note that when $s_i = s_j$, we obtain $s_i s_i = (s_i)^2 = 1$ and therefore we find that

$$\left(\sum_{i=1}^N s_i\right)^2 = 2 \sum_{pair(i,j)} s_i s_j + N \tag{3.74}$$

where the factor 2 is due to the fact that we count both the pairs (i,j) and the pairs (j,i). Rearranging the equation above yields the re-written form of the energy.

A2. Since the magnetization is a sum over the spins, the configurations with equal magnetization are those for which we have the same number of spins in the +1 configuration. If N_+ is the number of spins in this configuration, then we can formally write that the number of states is

$$\Omega(M) = \binom{N}{N_+} = \frac{N!}{N_+!(N-N_+)!} \tag{3.75}$$

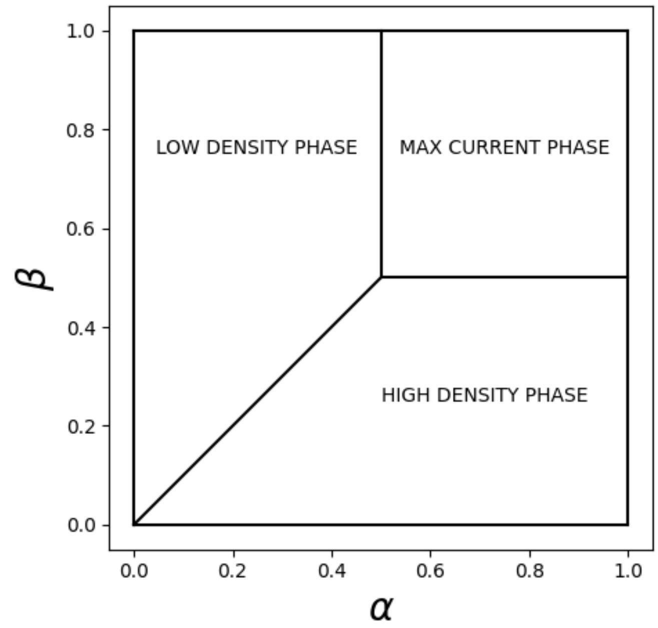


Figure 3.2: The TASEP phase diagram as function of α and β.

To rewrite this in terms of M, we note that

$$M = N_+(+1) + N_-(-1) = N_+ - N_- = N_+ - (N - N_+) = 2N_+ - N \tag{3.76}$$

Using this relation, we can express N_+ in terms of M and find

$$N_+ = \frac{M + N}{2} \tag{3.77}$$

Plugging this relation into our expression for the number of states with the same magnetization yields the requested formula.

A3. Since the partition function is a sum over all the possible states, and because we can write the energy as

$$E(\vec{s}) = -\frac{J}{2N}\left[\left(\sum_{i=1}^{N} s_i\right)^2 - N\right] = -\frac{J}{2N}\left[M^2 - N\right], \tag{3.78}$$

clearly each state with the magnetization has the same energy. We can therefore sum over the states with equal magnetization and account for their multiplicity with the factor $\Omega(N)$ that we have derived.

A4. We take the partition function Z and only keep the term with magnetization M^*, which minimizes the free energy. In this approximation, the partition function is just

$$Z^* = \Omega(M^*) \exp(-\beta E(M^*)) \tag{3.79}$$

We write just M instead of M^* and the logarithm to find

$$-k_B T \ln Z = -k_B T [\ln \Omega(M) - \beta E(M)] \tag{3.80}$$

We already expressed the energy in terms of the magnetization M in our answer to Q3, so we can write

$$-k_B T \ln Z = E(M) - k_B T \ln \Omega(M) = -\frac{J(M^2 - N)}{2N} - k_B T \ln \Omega(M) \tag{3.81}$$

The Stirling formula allows us to rewrite factorials for large values as a more tractable expression. It states

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{3.82}$$

or we can immediately use its logarithmic form, i.e.

$$\ln(n!) \approx n\ln n - n \tag{3.83}$$

Taking the logarithm of the multiplicity, we find

$$\ln\Omega(M) = \ln(N!) - \ln(N_+!) - \ln(N_-!) \approx N\ln N - N - N_+\ln N_+ + N_+ - N_-\ln N_- + N_- \tag{3.84}$$

Since $N_+ + N_- = N$, the non-logarithmic terms cancel and we have

$$F(M,T) = -\frac{JM^2}{2N} + \frac{J}{2} + k_BT(N_+\ln N_+ + N_-\ln N_- - N\ln N) \tag{3.85}$$

A5. Taking the derivative of $F(M,T)$ with respect to M (and remembering that N_+ and N_- depend on M) we get

$$\frac{\partial F(M,T)}{\partial M} = -\frac{JM}{N} + \frac{k_BT}{2}\left(\log\frac{N+M}{N-M}\right) = -\frac{JM}{N} + k_BT\tanh^{-1}\left(\frac{M}{N}\right) = 0 \tag{3.86}$$

Recognizing that $m = M/N$, we find the following transcendental equation for m

$$m = \tanh(\beta Jm) \tag{3.87}$$

If we plot both m and $\tanh(\beta Jm)$ as a function of m, one notices that for $\beta J < 1$, the graphs only intersect at the origin. For $\beta J > 1$, the graph of $\tanh(\beta J)$ is intersected twice at the corresponding values of the magnetization $\pm m$. Therefore, the critical temperature occurs when $\beta J = 1$ and so

$$T_c = \frac{J}{k_B} \tag{3.88}$$

Note that because we scaled our coupling parameter with N, we got rid of the dependence on q in the expression for the critical temperature. (Strictly speaking, we cancelled a factor of N/N.)

A6. We first write the free energy in terms of the per-spin-magnetization m as follows:

$$F(m,T) = -\frac{JNm^2}{2} + \frac{J}{2} + k_BT\left[\frac{N(1+m)}{2}\log\frac{N(1+m)}{2} + \frac{N(1-m)}{2}\log\frac{N(1-m)}{2} - N\log N\right] \tag{3.89}$$

Dividing by N we have

$$f(m,T) = -\frac{Jm^2}{2} + \frac{J}{2N} + k_BT\left[\frac{1+m}{2}\log\frac{1+m}{2} + \frac{1-m}{2}\log\frac{1-m}{2}\right] \tag{3.90}$$

Expanding around $m = 0$ and using the fact that $\log(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$ we find

$$f(m,T) = -\frac{Jm^2}{2} + \frac{J}{2N} + k_BT\left[\frac{m^2}{2} + \frac{m^4}{12} - \log 2\right] \tag{3.91}$$

Collecting like powers of m we have

$$f(m,T) = \frac{J}{2N} - k_BT\log 2 + m^2\left(\frac{k_BT - J}{2}\right) + \frac{k_BT}{12}m^4 \tag{3.92}$$

Since we had $T_c = k_BJ$ we can write

$$f(m,T) = f_0 + \frac{1}{2}m^2k_B(T - T_c) + \frac{1}{4}\frac{k_BT}{3}m^4 \tag{3.93}$$

and so

$$f_0 = \frac{J}{2N} - k_BT\log 2, \quad a_2(T) = k_B(T - T_c), \quad a_4(T) = \frac{k_BT}{3} \tag{3.94}$$