From Micro to Macro
Example of Percolation

Chapter I. Lecture 3

Monday April 17

From micro to macro
Example of Percolation

Chapter I — Lecture 3

Plan:

1) Introduction to Percolation
   a) Percolation: What is it?
   b) Continuous phase transition
   c) Applications

2) Percolation phase transition
   a) Control parameter and Order parameter
   b) 1d site percolation
   c) Betthe Lattice

3) Properties at the critical point, universality
   a) Different type of percolation problems
   b) Pc is not universal
   c) Critical exponents are universal

4) Universality

Expectations: Participate in the discussions, take notes, try to do the analytical derivations

References: Book "Complexity and Criticality", K. Christensen, N. Moloney, **Chapter 1: Percolation**

"Entropy, Order Parameters, and Complexity", by J. P. Sethna, **12.1 Universality**

Introduction to Percolation problems

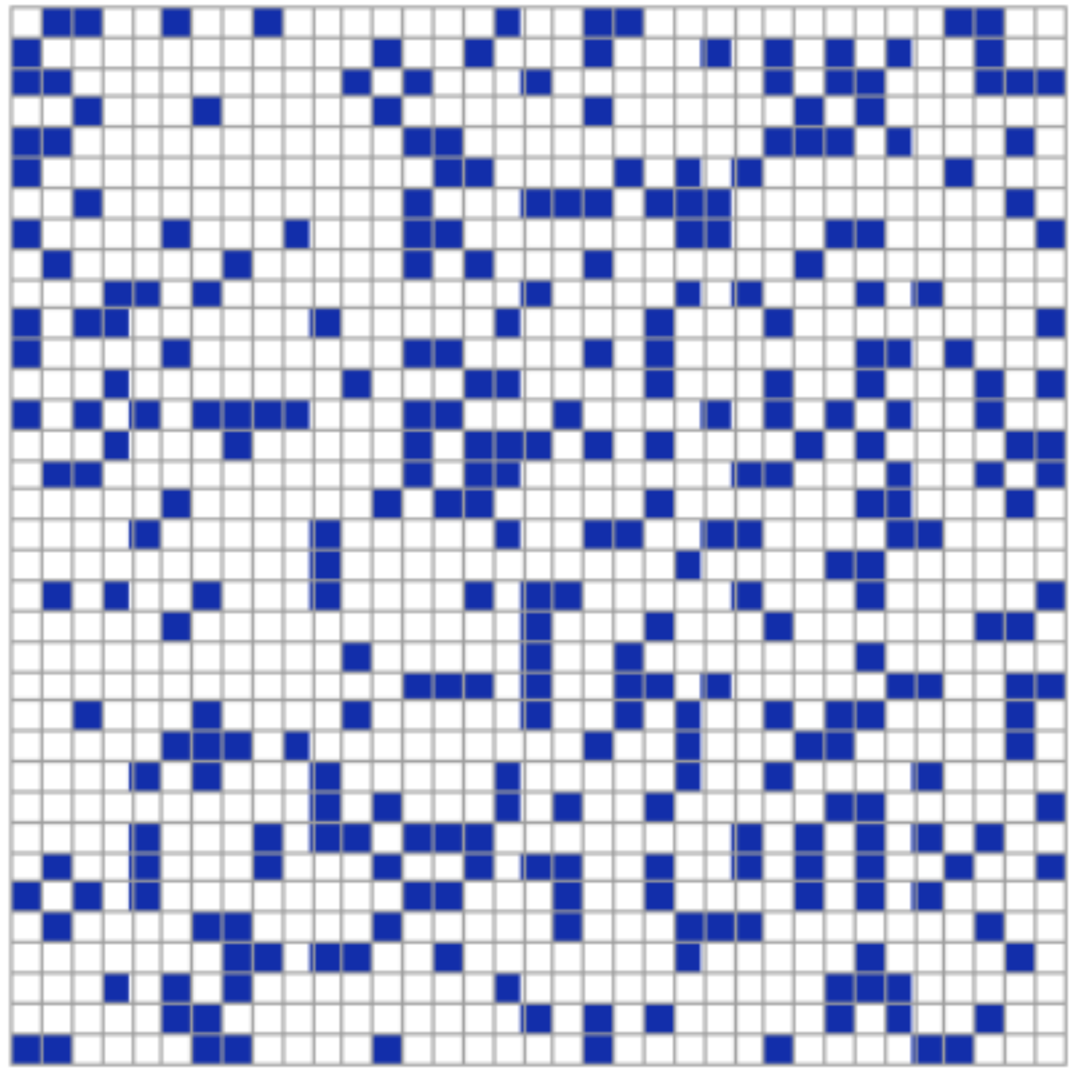
One of the **simplest models** undergoing a **continuous phase transition.**

Yet, relevant in **many applications**!

Percolation

Take a **squared paper** and **randomly black out** a **chosen fraction of the squares.**

Consider the **clusters** of **adjacent black squares**



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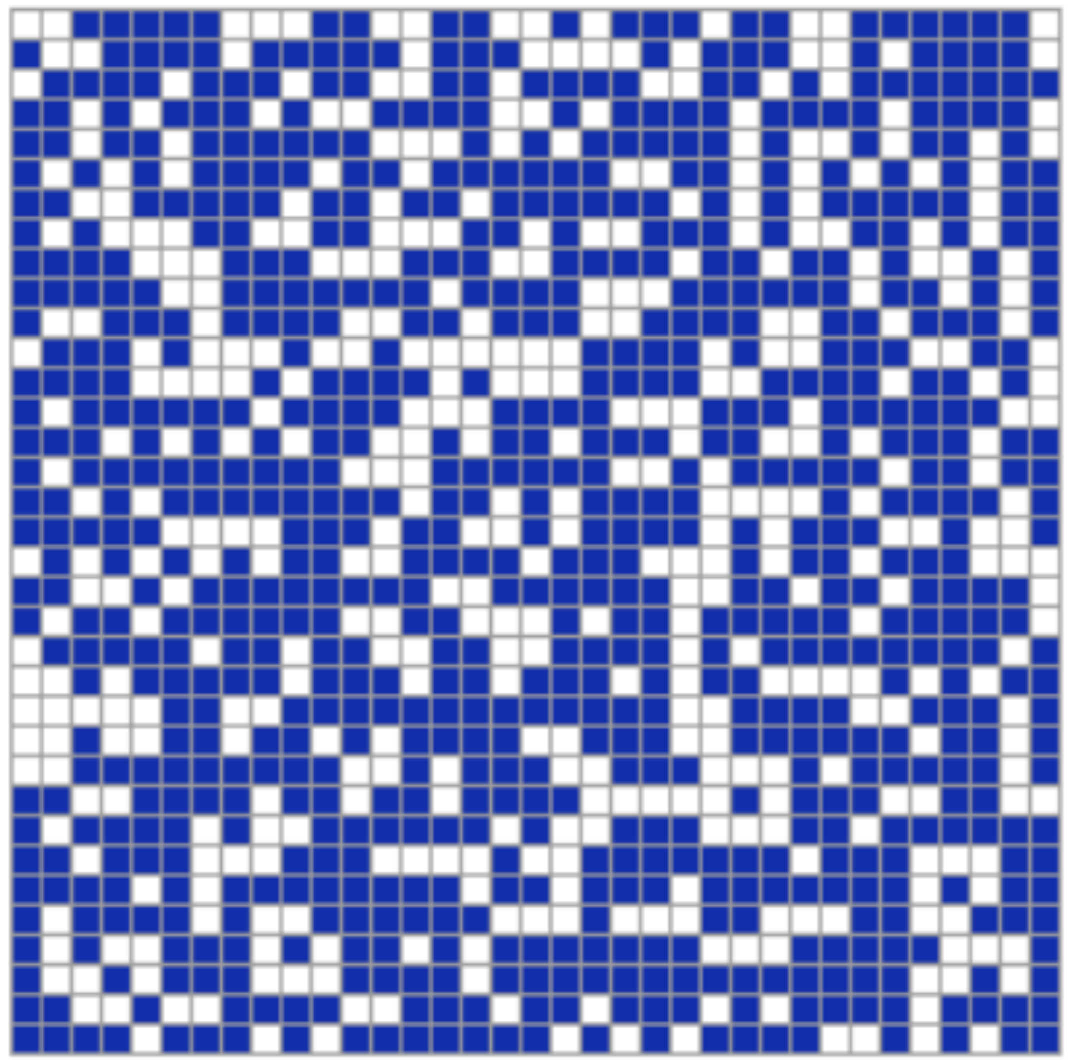

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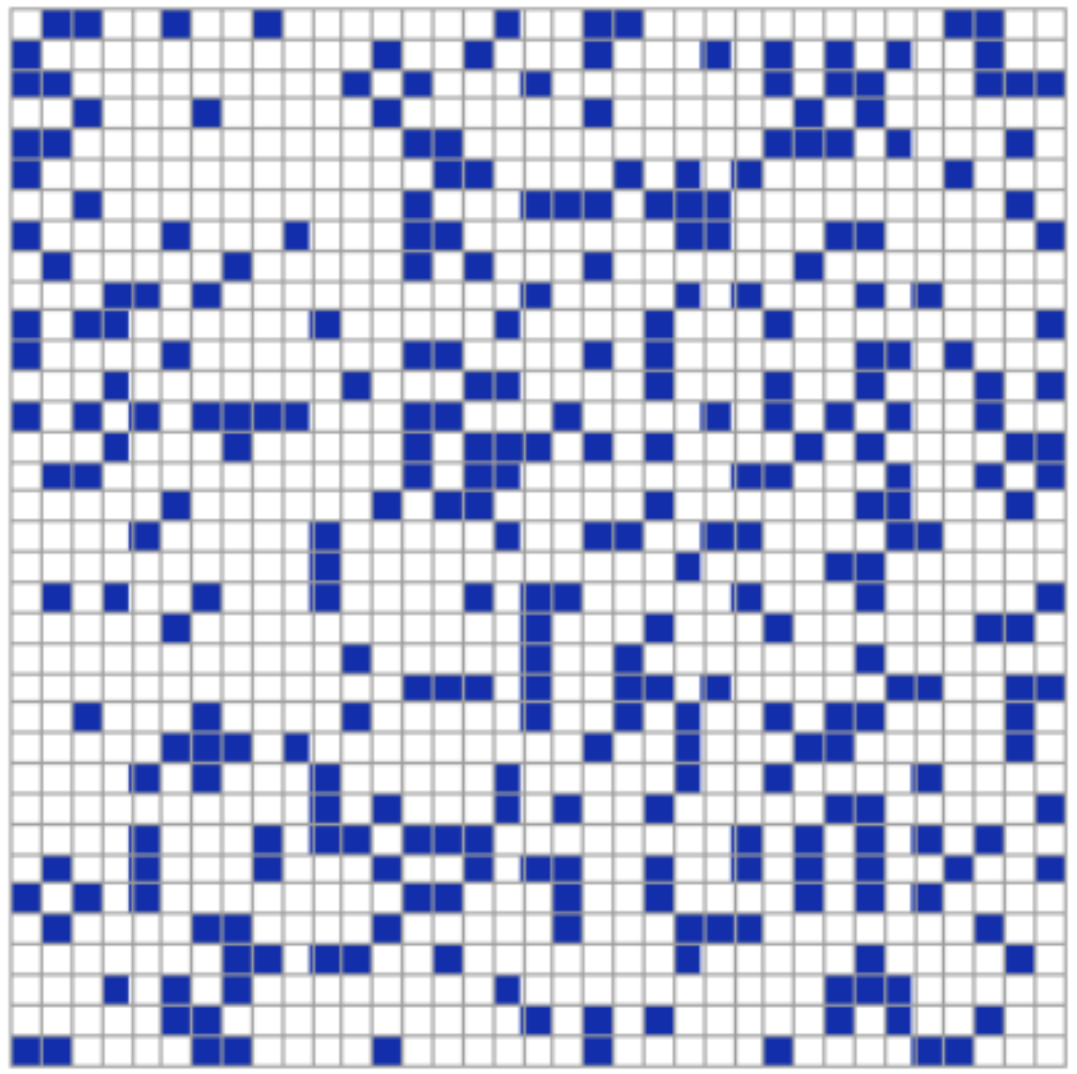
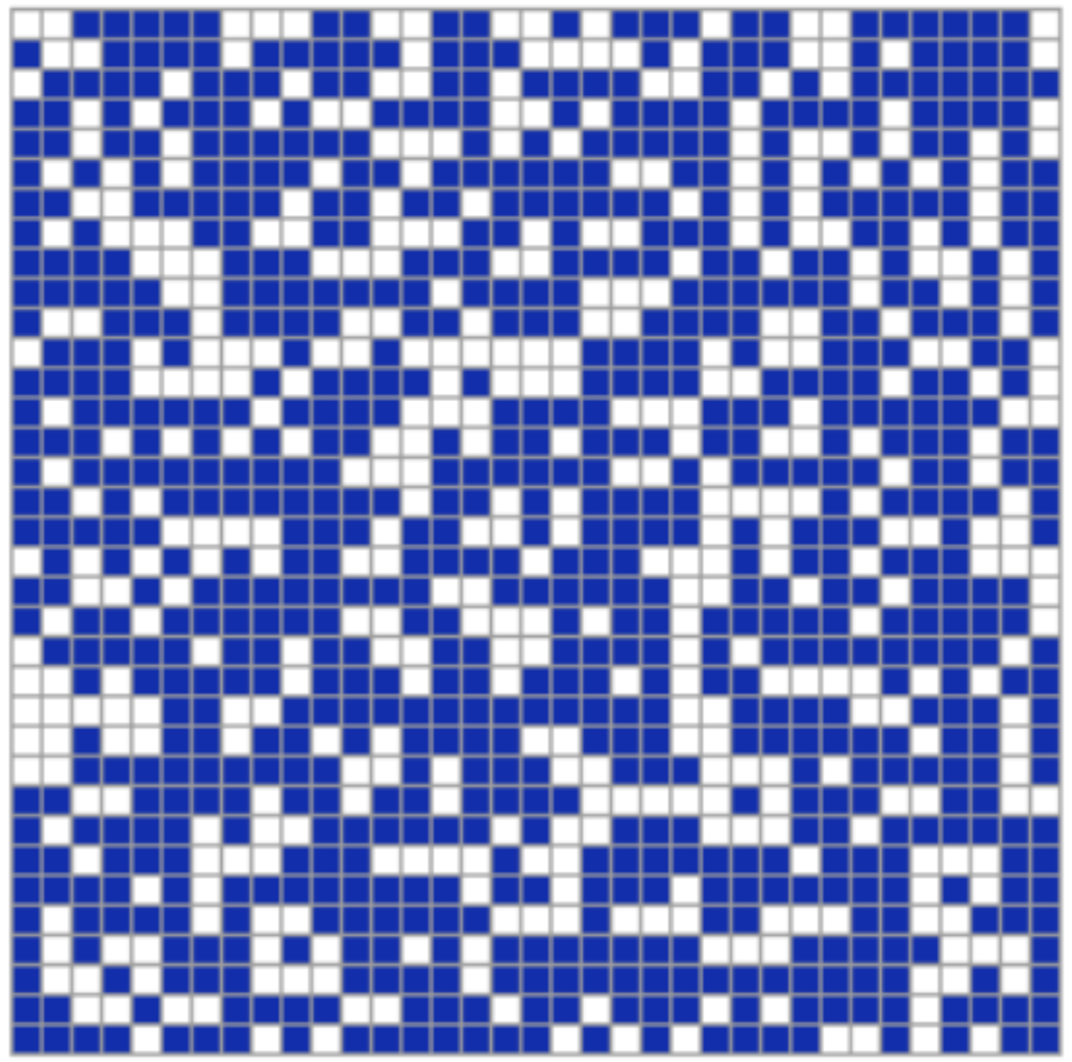


A **small fraction** of the squares are **blacked out**,
it is **unlikely** that the **cluster extends across opposite sides** of the paper



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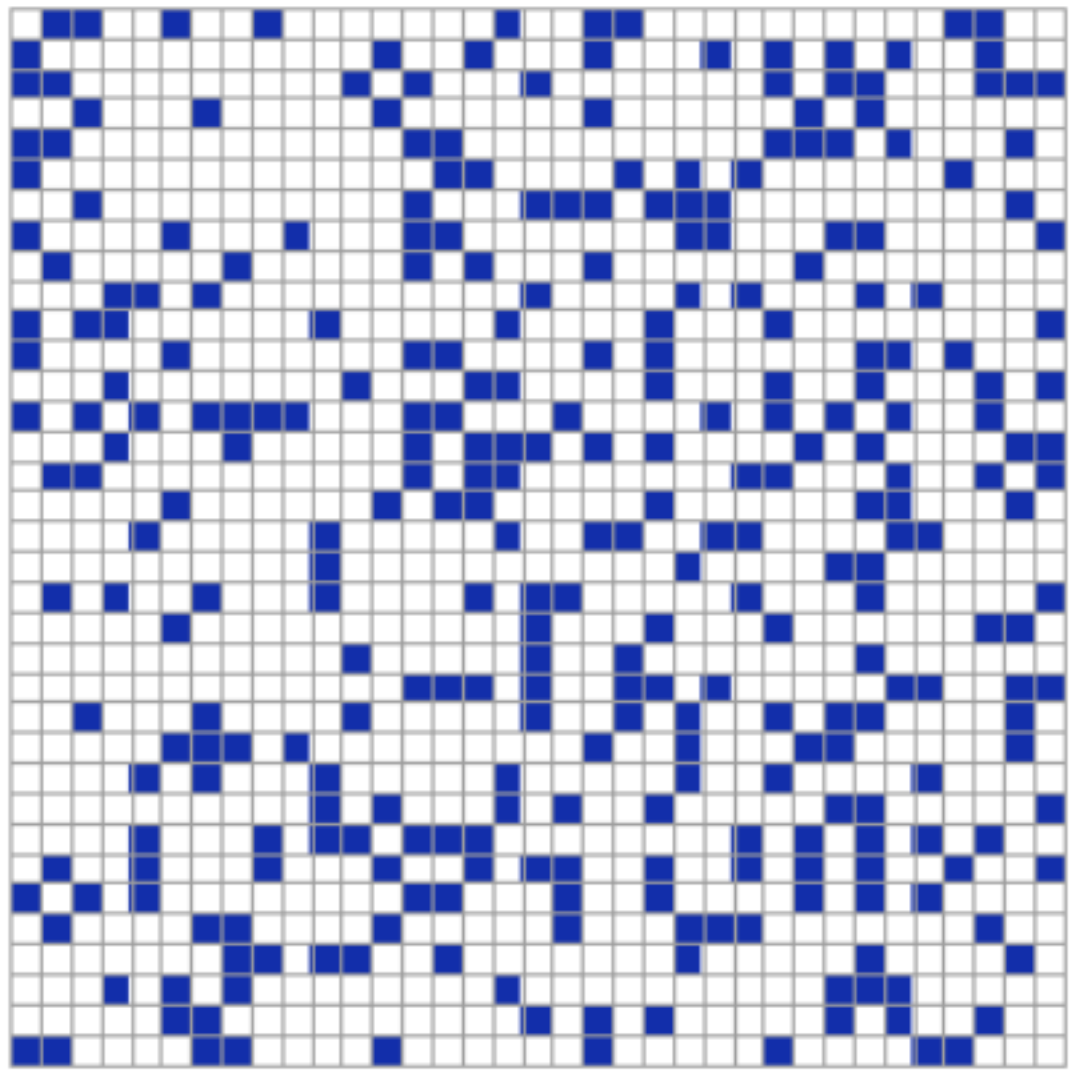

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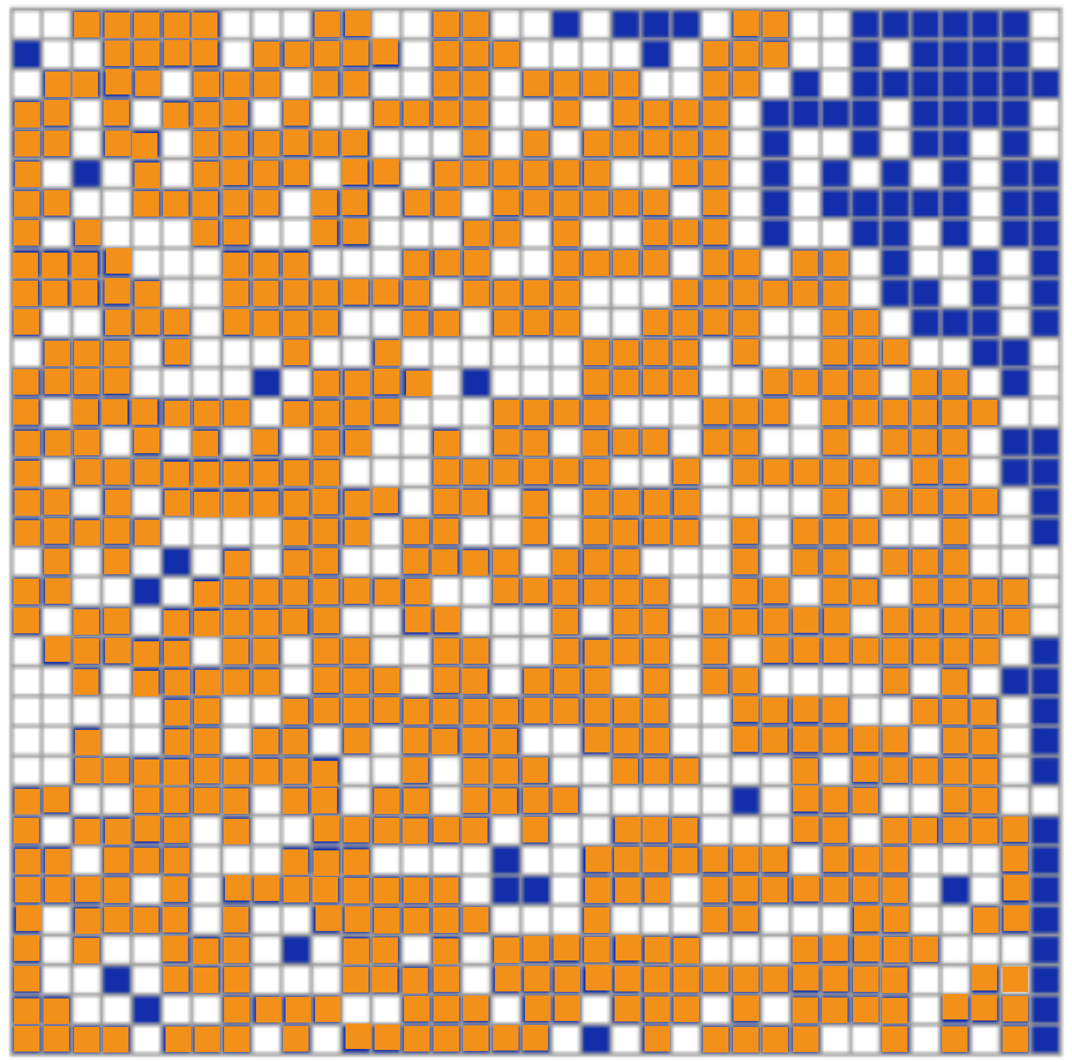

A **large fraction** of the squares are **blacked out**,
it is **likely** that the cluster **"percolates"** across the paper.

Like water *percolates* through the coffee machine

In physics/chemistry: *percolation* is the process of filtering a liquid through a porous material.

Percolation

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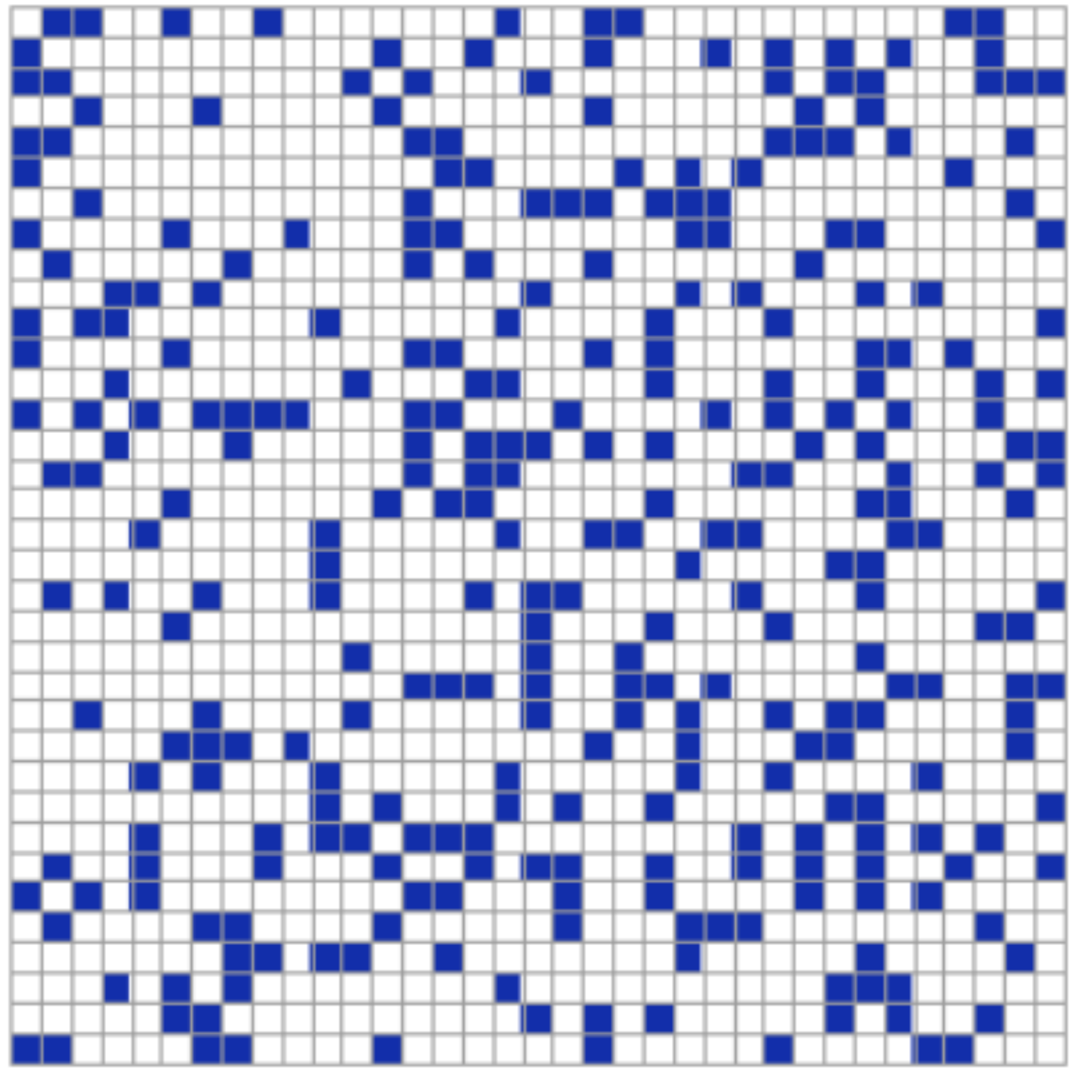
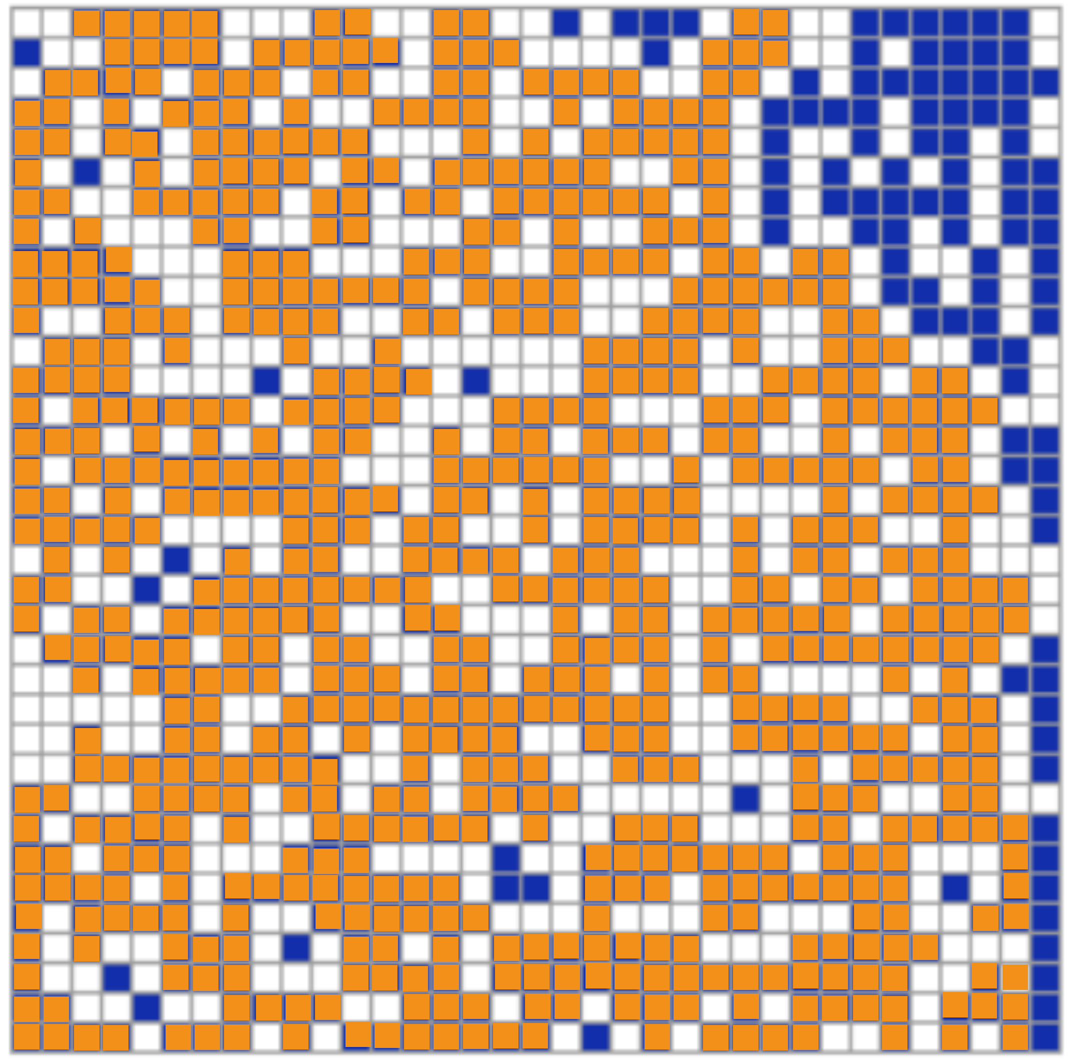


phase transition

Control parameter
=
probability to black out a square



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There is a **fraction of randomly blacked out squares** that **marks** a **phase transition from non-percolating cluster to percolating cluster**.

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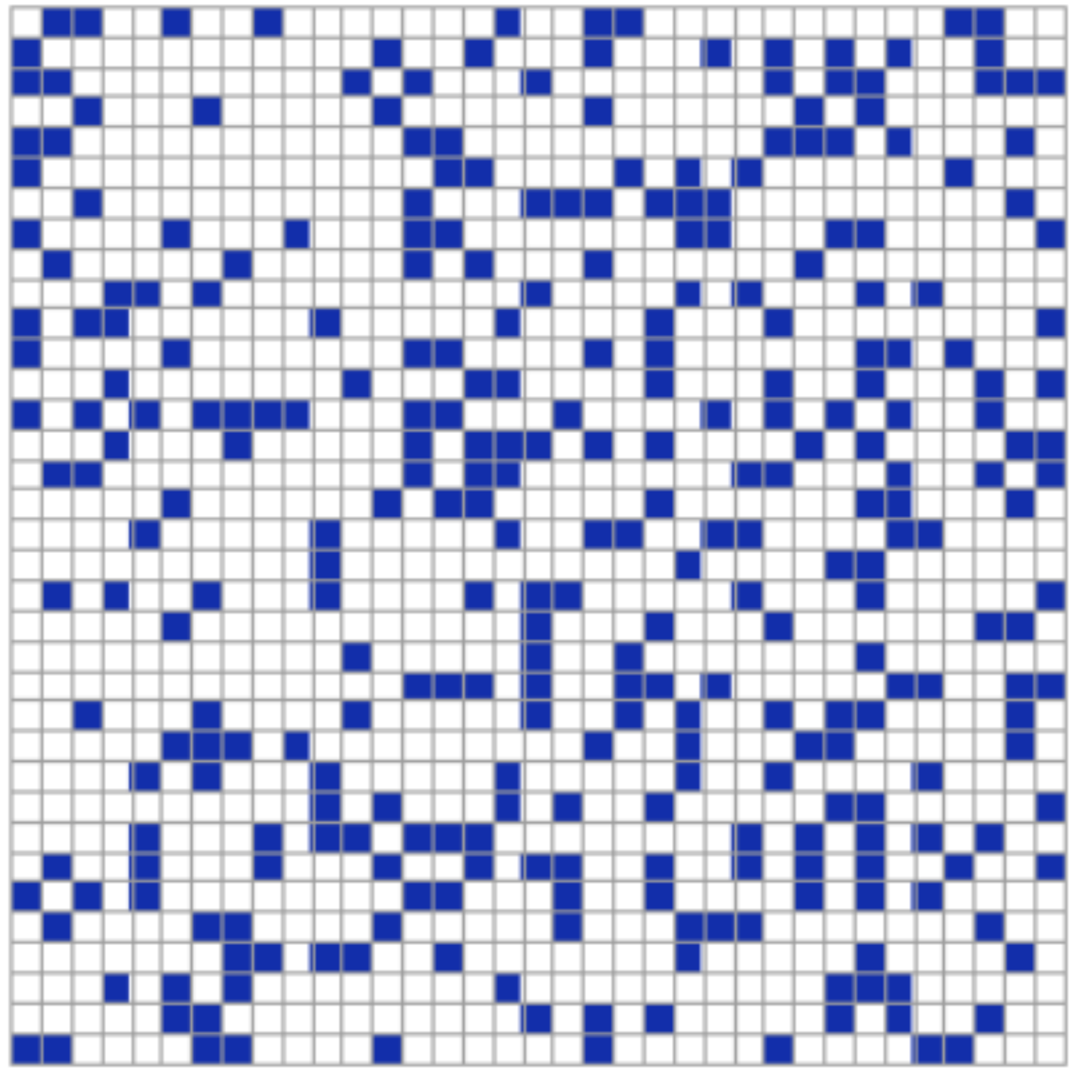

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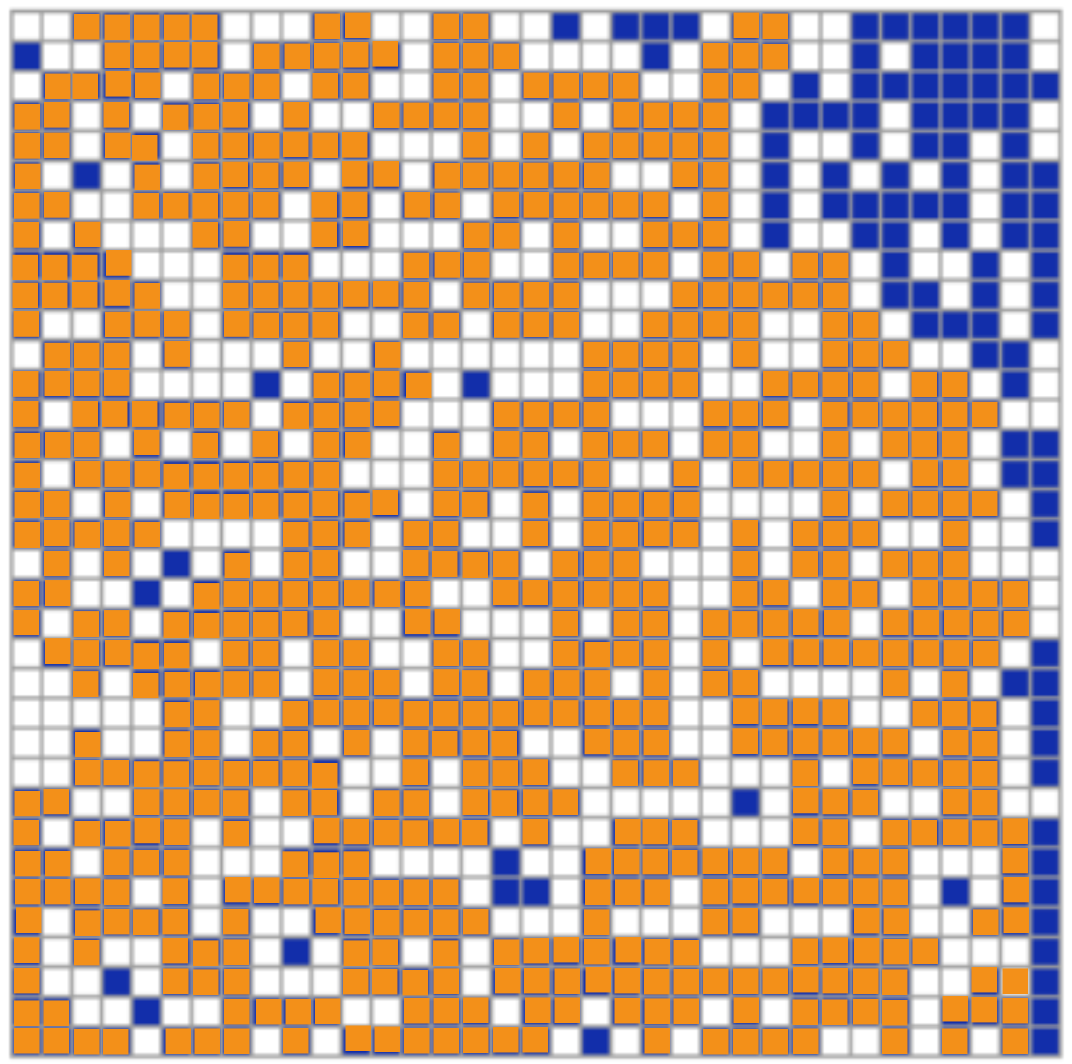

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One of the simplest model undergoing a PHASE TRANSITION

Which type of phase transition?

What happens at the transition?

Experiment:
Percolation transition is a continuous phase transition

Percolation Theory and Electrical Conductivity

B. J. Last and D. J. Thouless
Department of Mathematical Physics, University of Birmingham, Birmingham B15 2TT, England
(Received 4 October 1971)

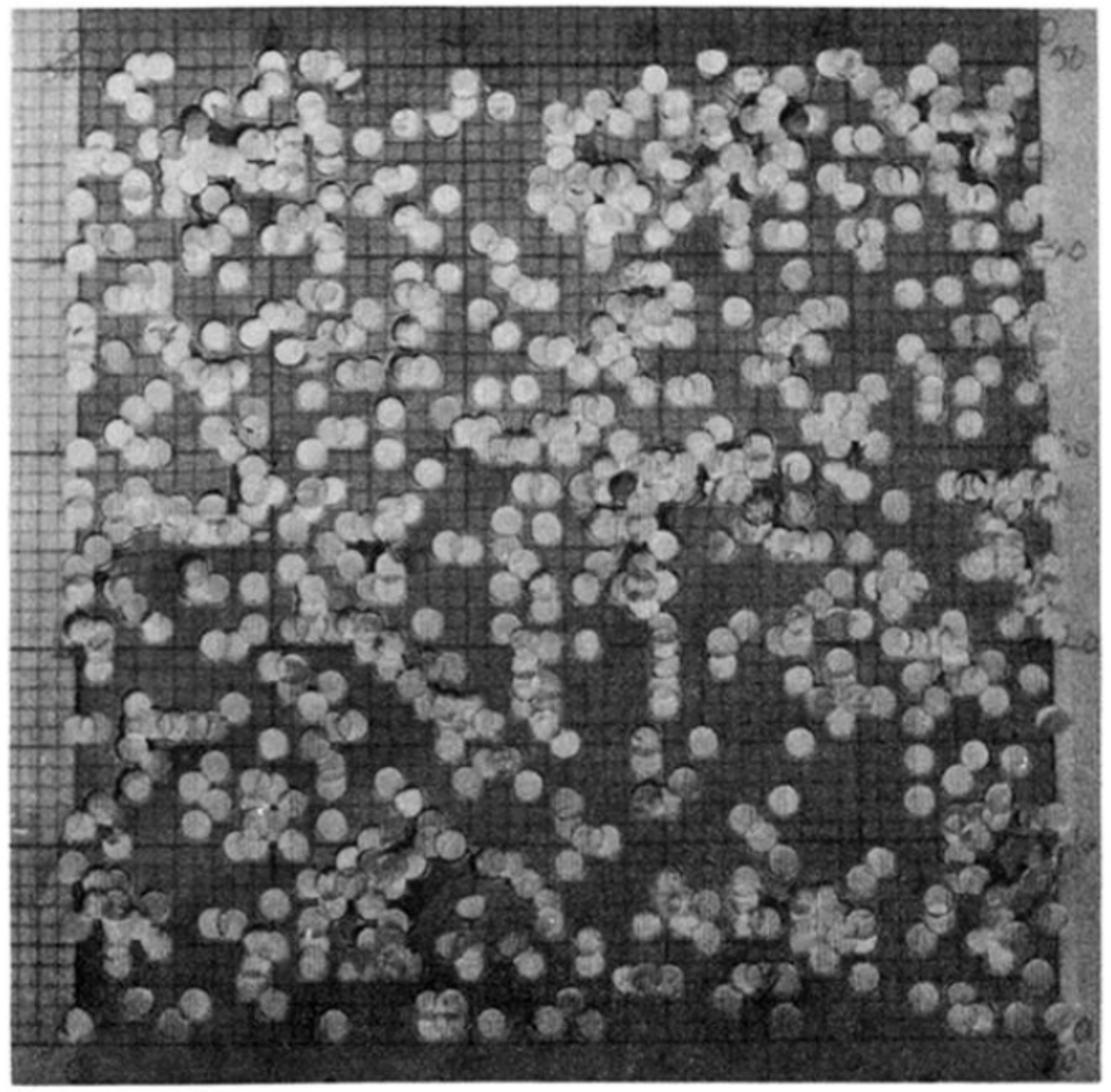


FIG. 1. Photograph of the sheet of conducting paper at the stage where the concentration of holes is 0.268.

Experiment:

punching **holes at random** places **in a conducting sheet** of paper and **measuring the conductance**.

(measures how easily electrical current can pass through a material)

Shows that it is a **continuous transition**:

— **conductance fell to a very small value** as the number of holes approached the **critical concentration**

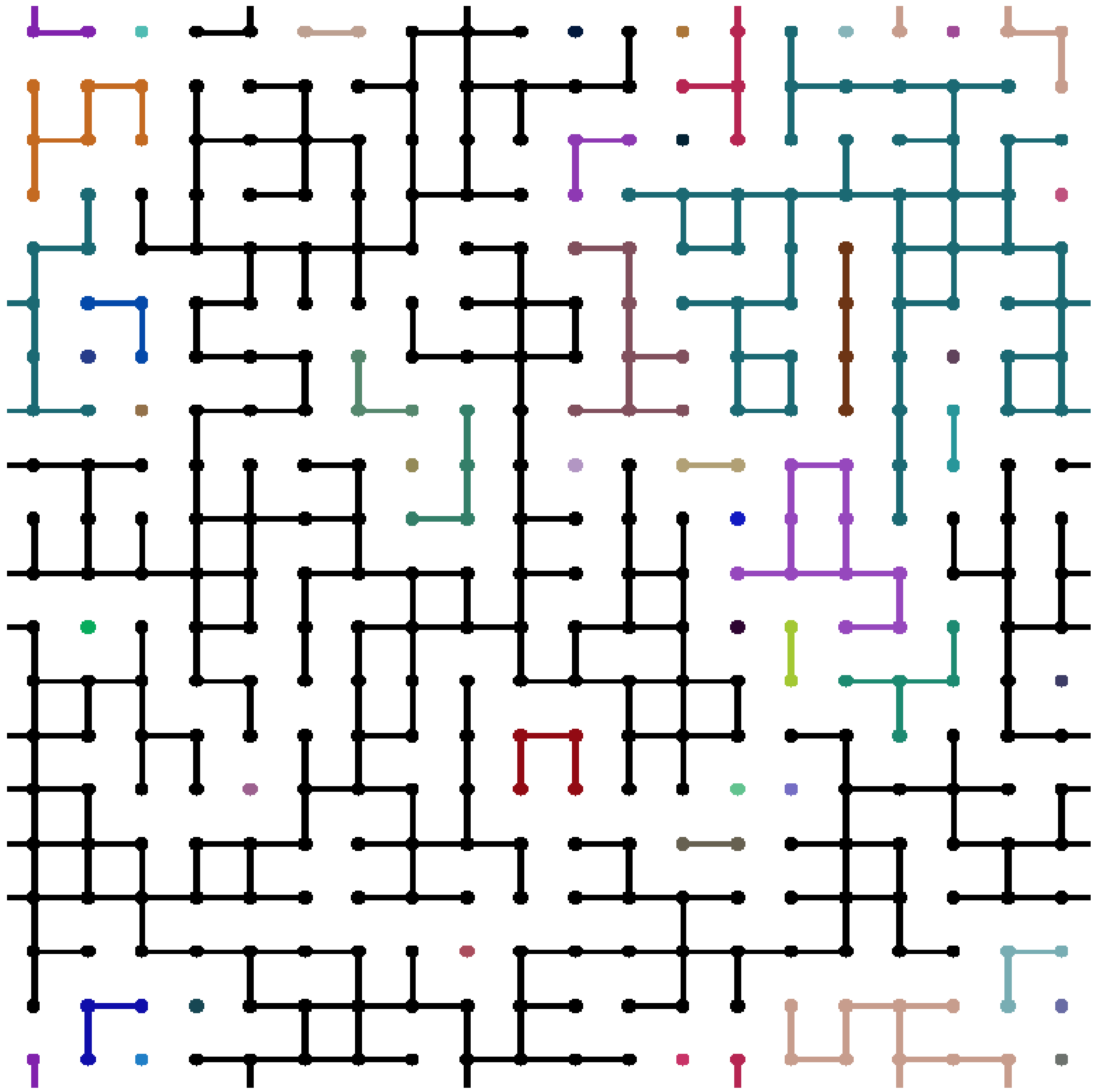
— the conducting **paths** were **few** and **tortuous** just before the sheet fell apart —> singular properties

One of the simplest model undergoing a CONTINUOUS phase transition.

What happens at the transition?

Ex. **Simulation of bond percolation:**

- Start from a grid and remove bonds (instead of making holes)
- **Bonds** between nodes are
  - **kept** with probability **p**
  - **removed** with probability **(1-p)**



In black: the **largest cluster span the entire lattice**

Other colors: highlight smaller clusters

For p close to 1: large conductivity

Conductivity decreases as p decreases.

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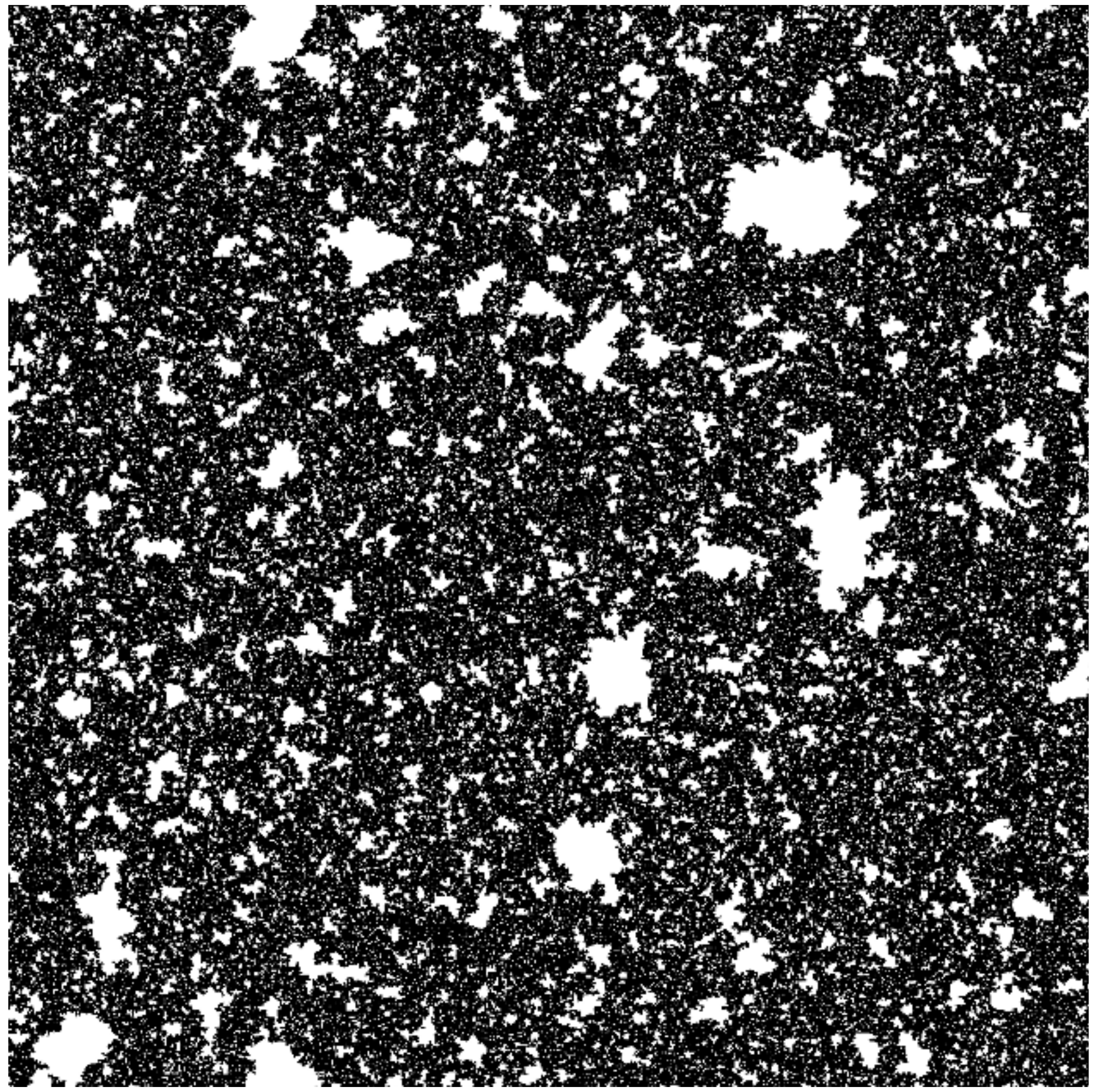

p = 0.51

The largest cluster is intact with only small holes.

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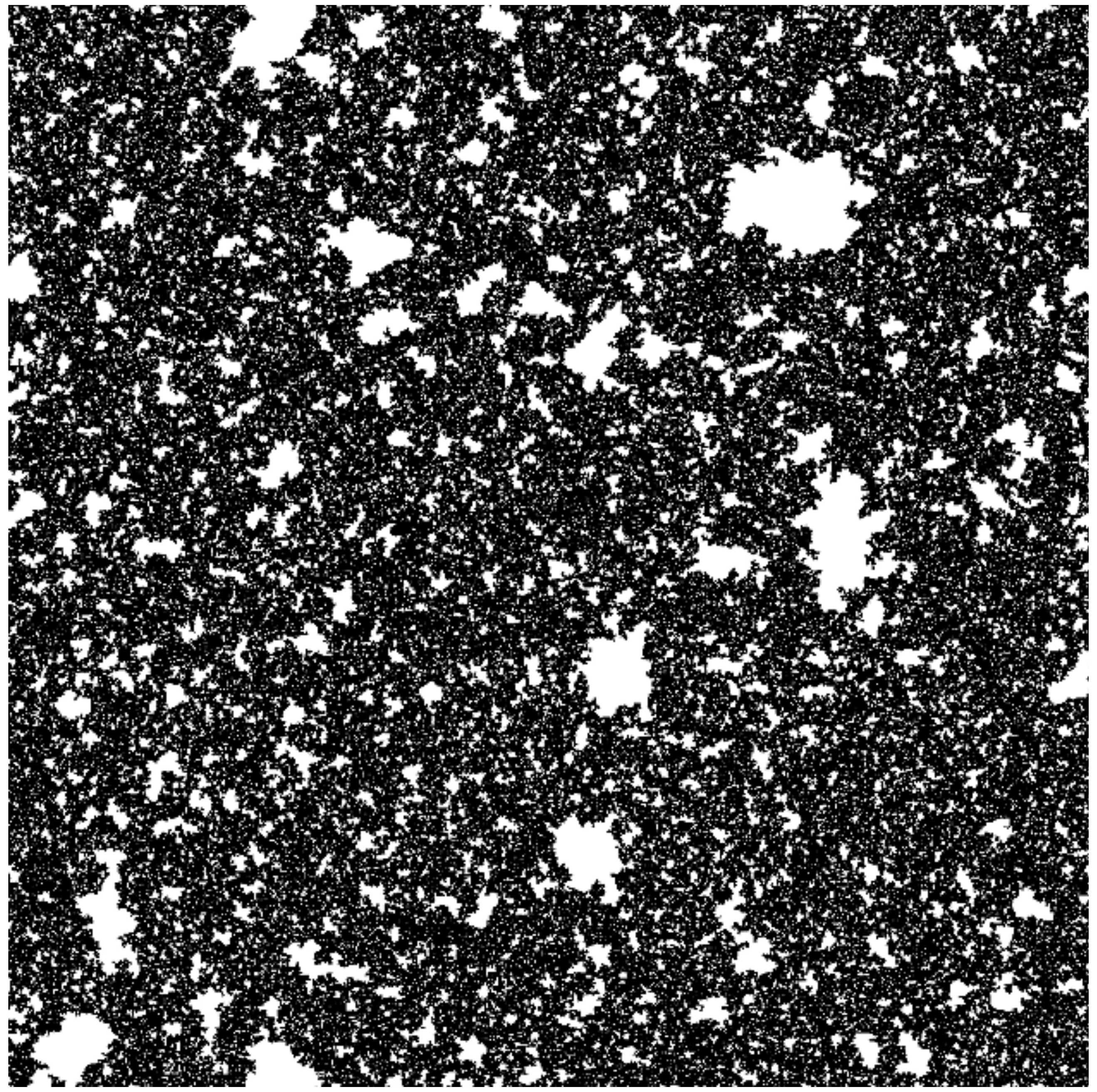

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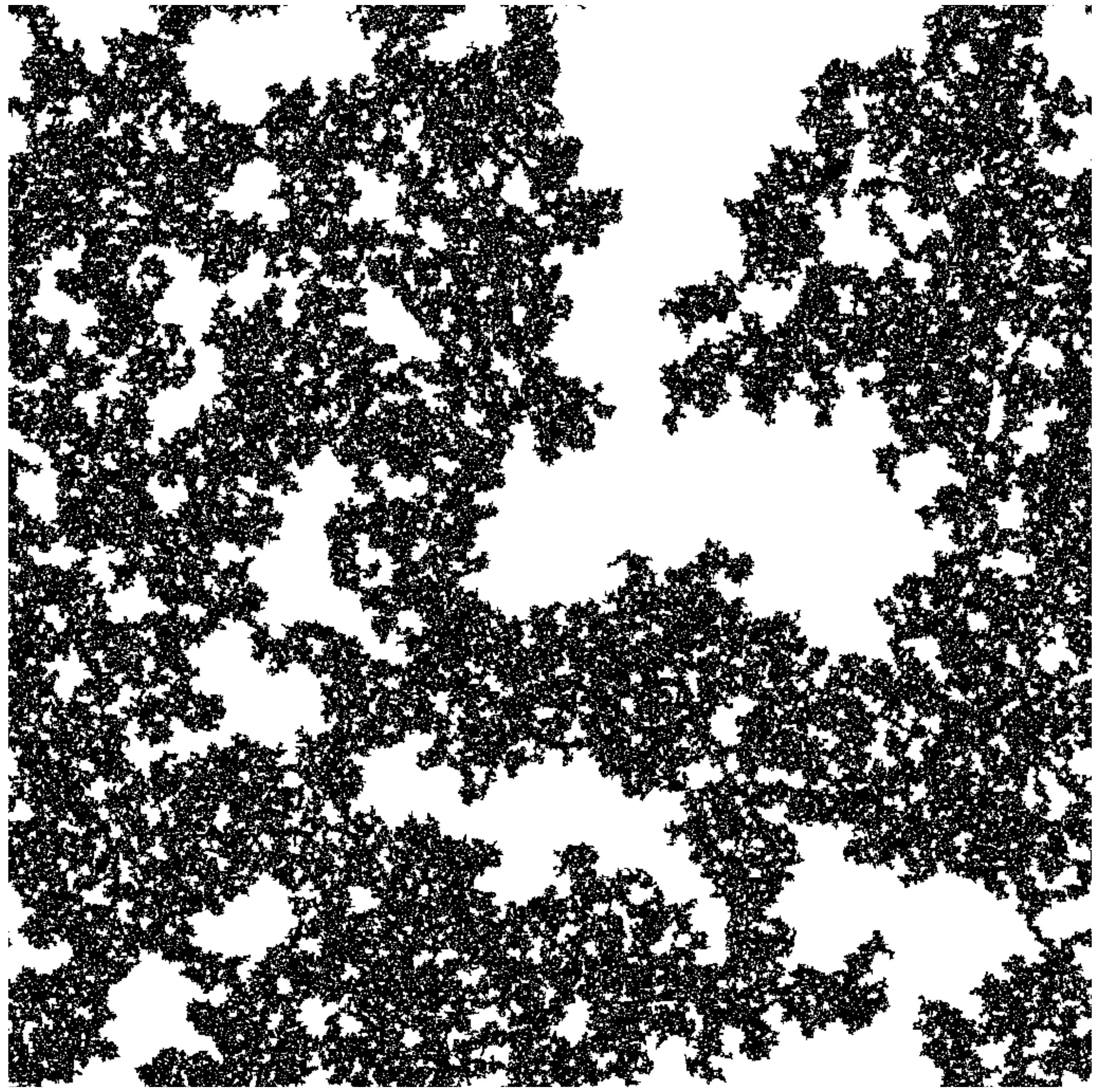

p = pc = 0.5

The biggest cluster barely hangs together, with holes on all length scales.

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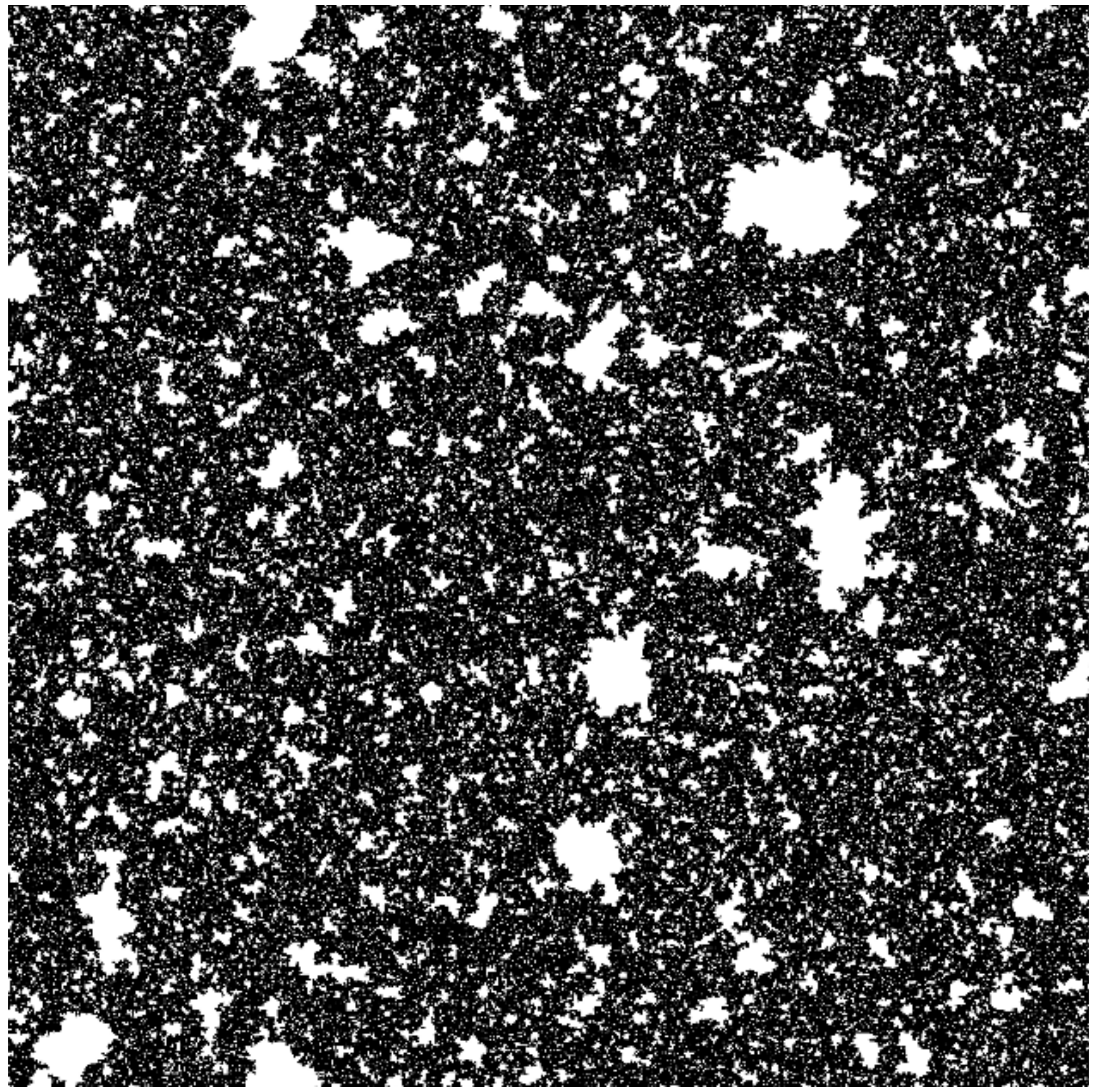

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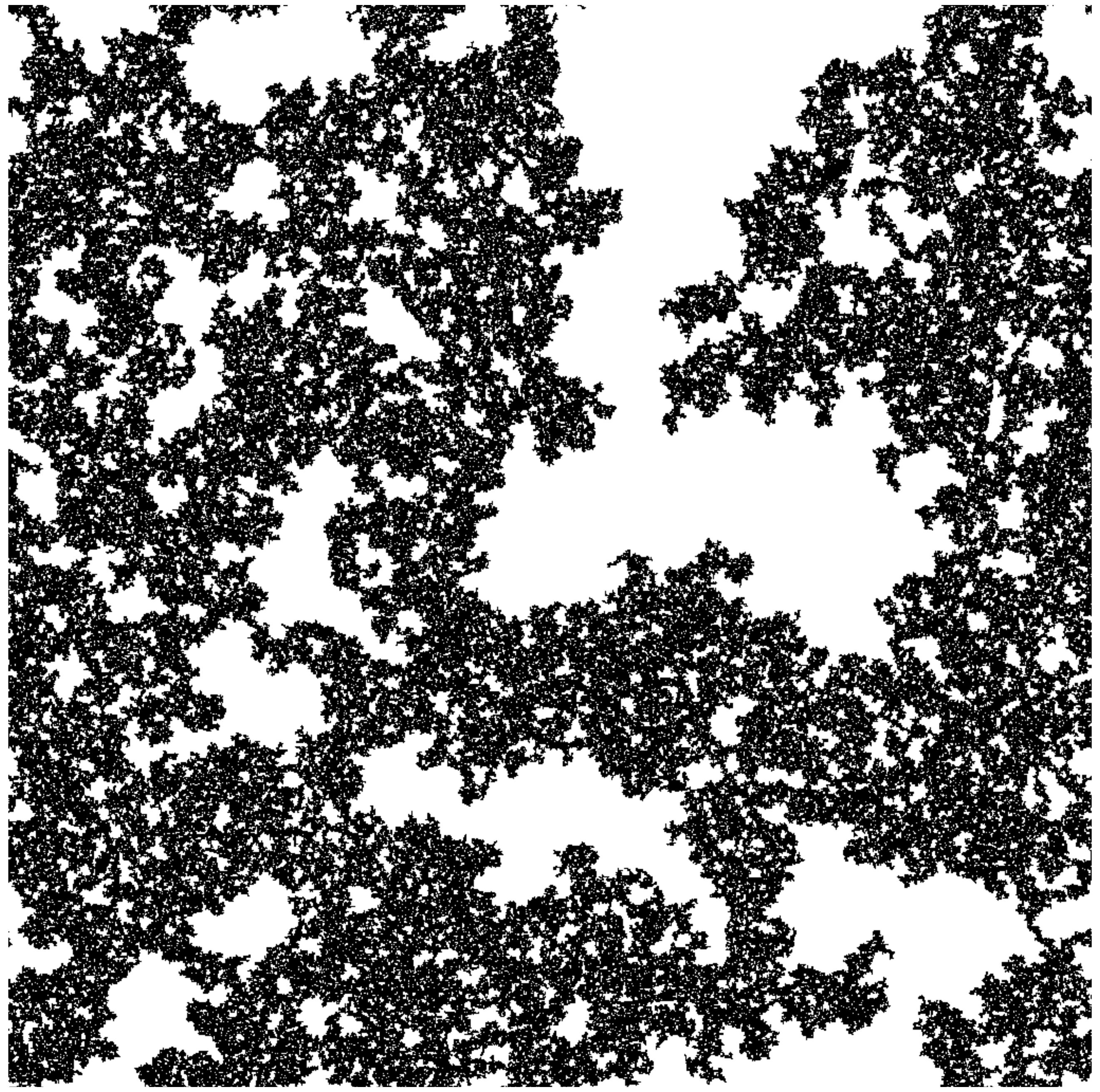

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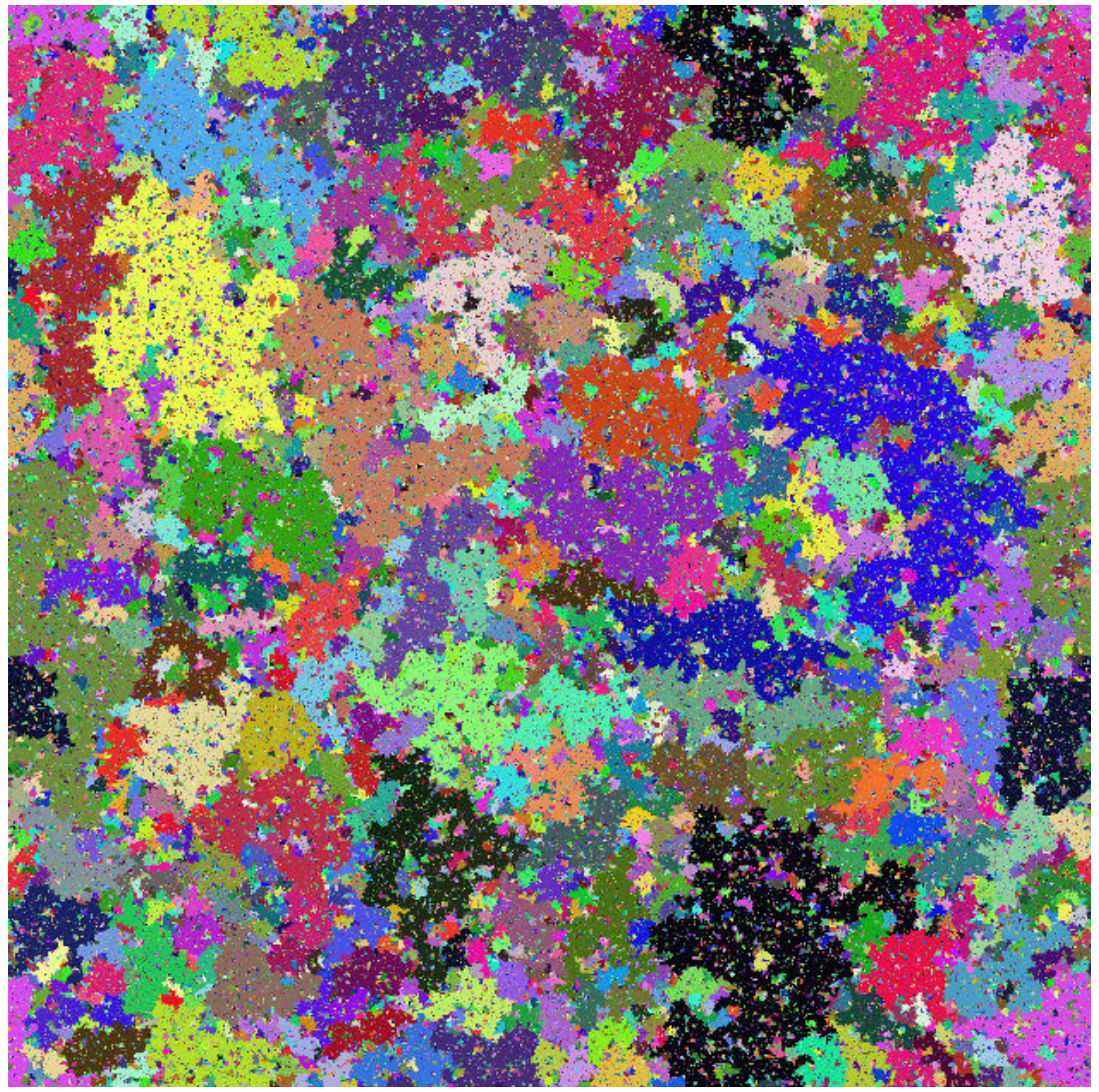

p = 0.49

The sheet falls into small fragments. (colors indicate clusters)

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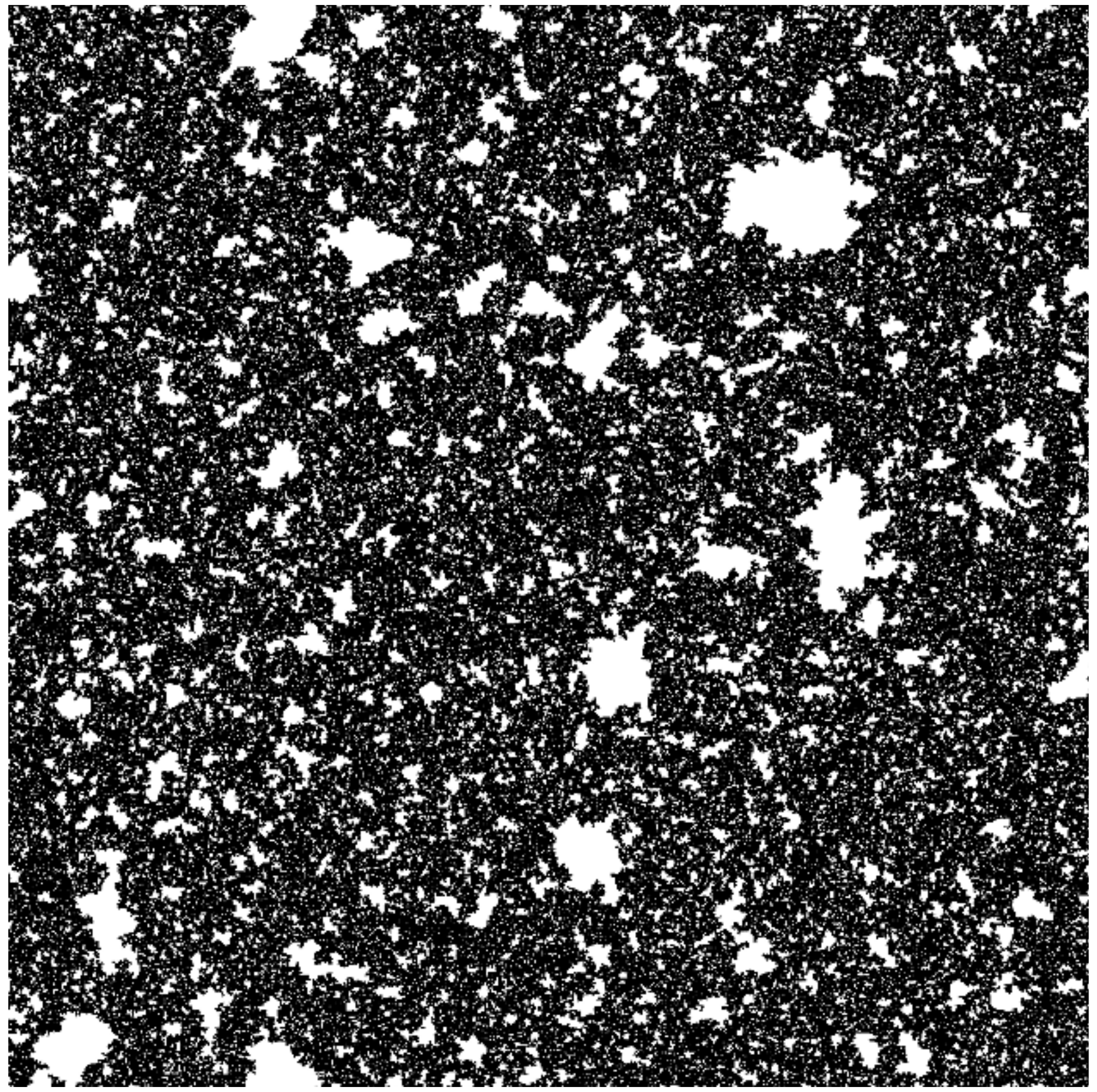

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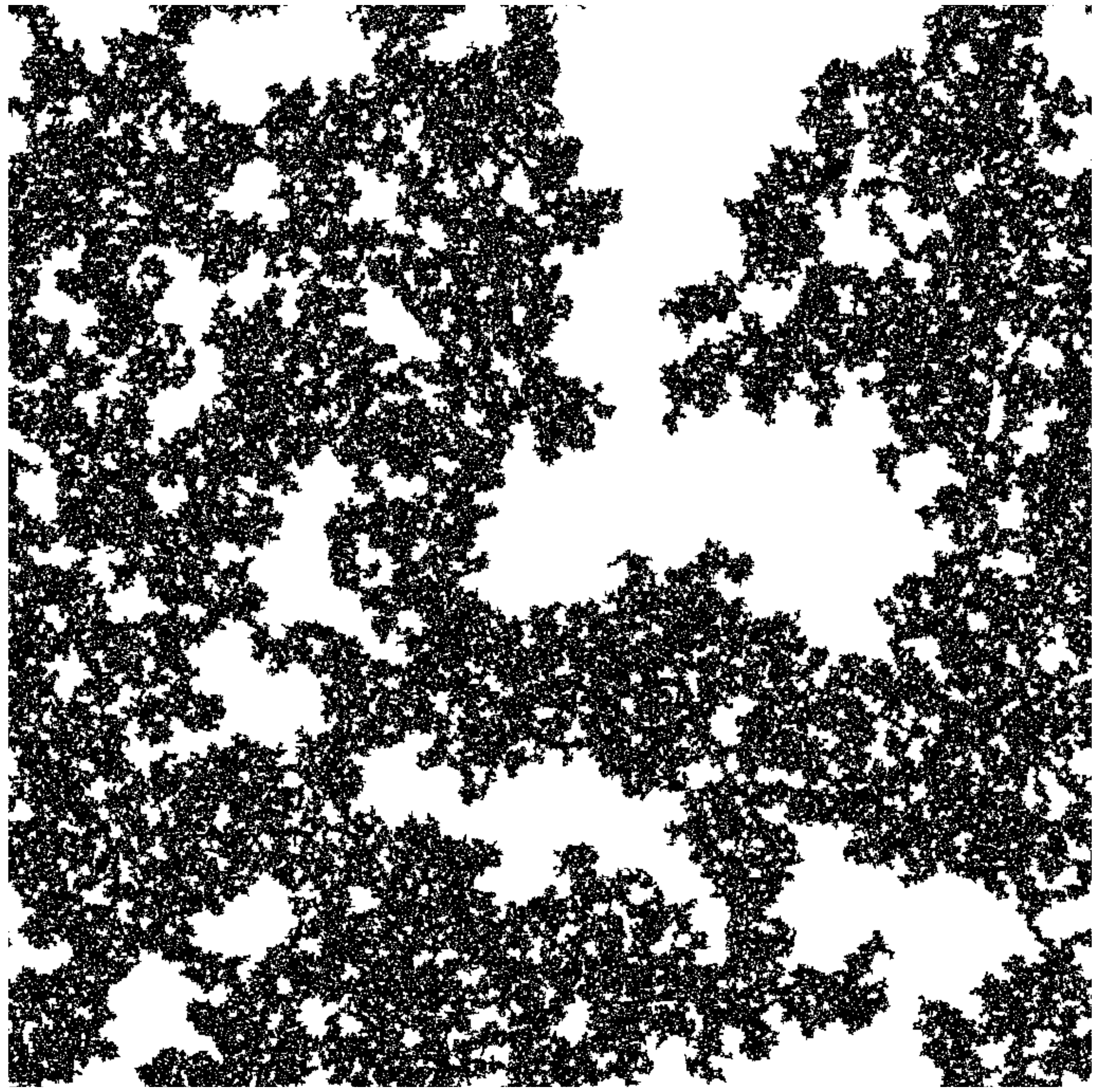

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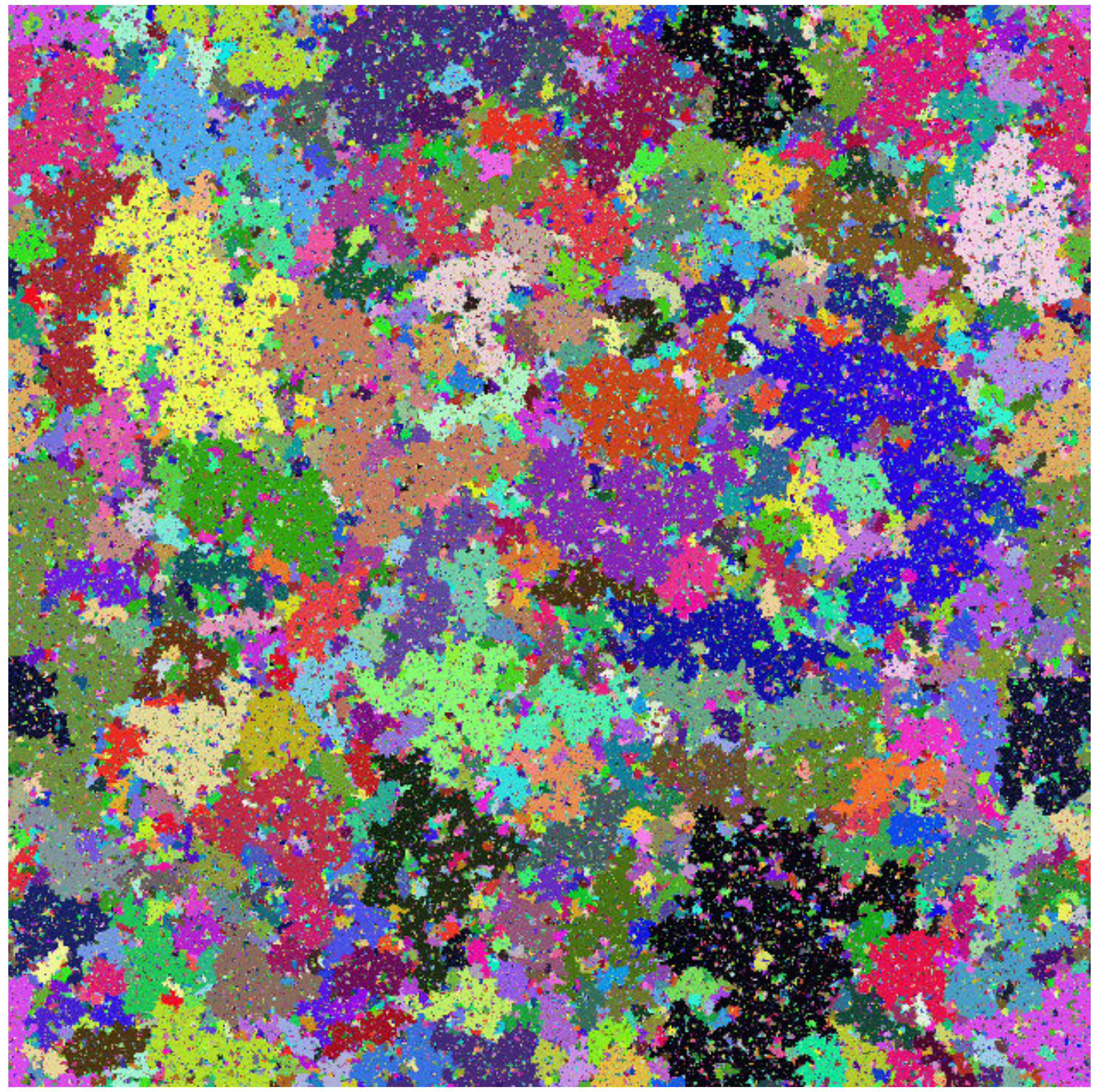

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The sheet falls into small fragments.

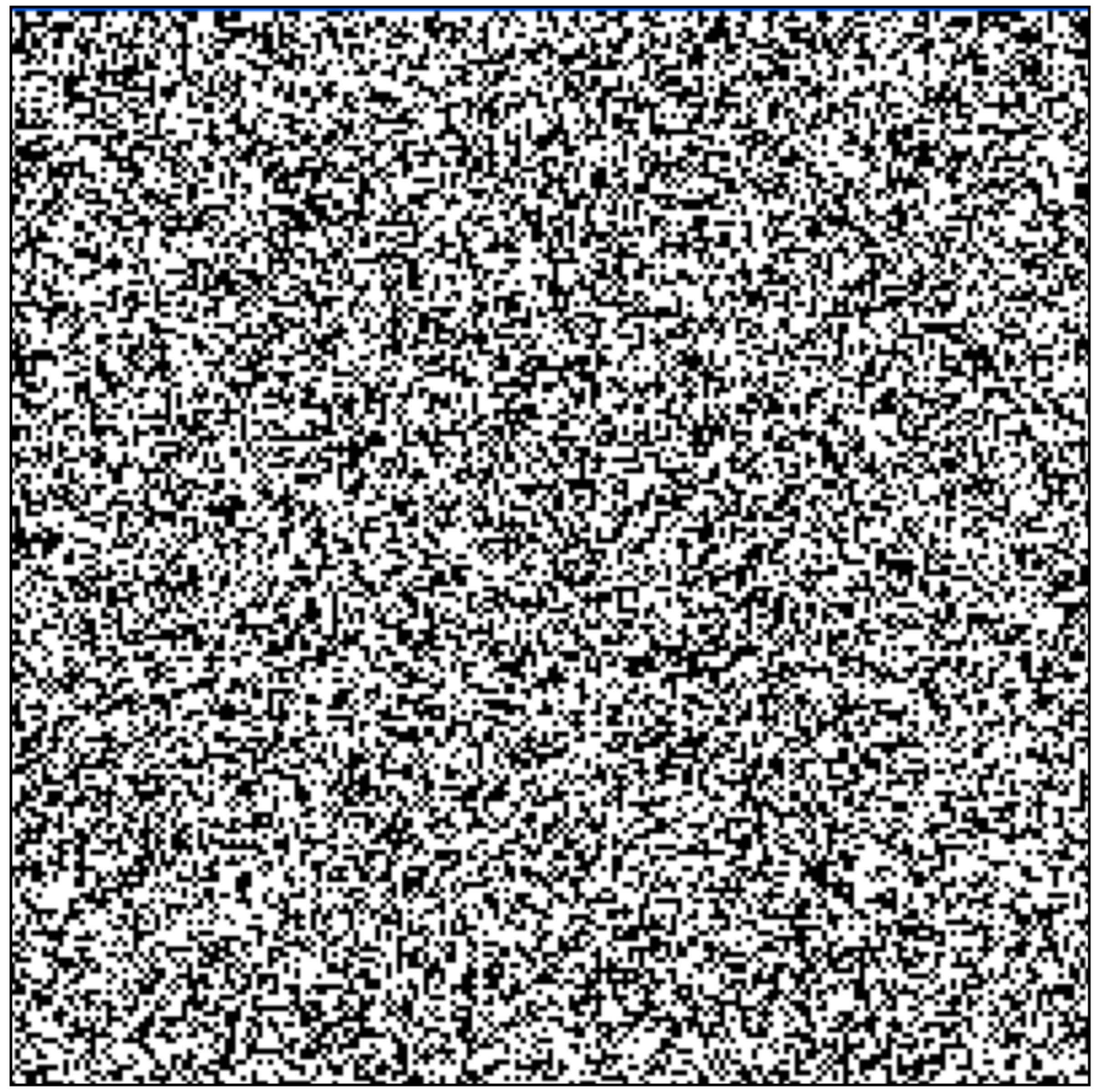
Percolation: **transition between a connected phase and a fragmented/disconnected phase.**

Multiple applications

Ex. **Oil recovery from a porous soil, making coffee** (See simulations here)

Take a **porous medium** (ground coffee, soil)**:** **Can the liquid can make it through the medium?**

How does it depend on the porosity of the medium (i.e. the fraction of material that is empty)?



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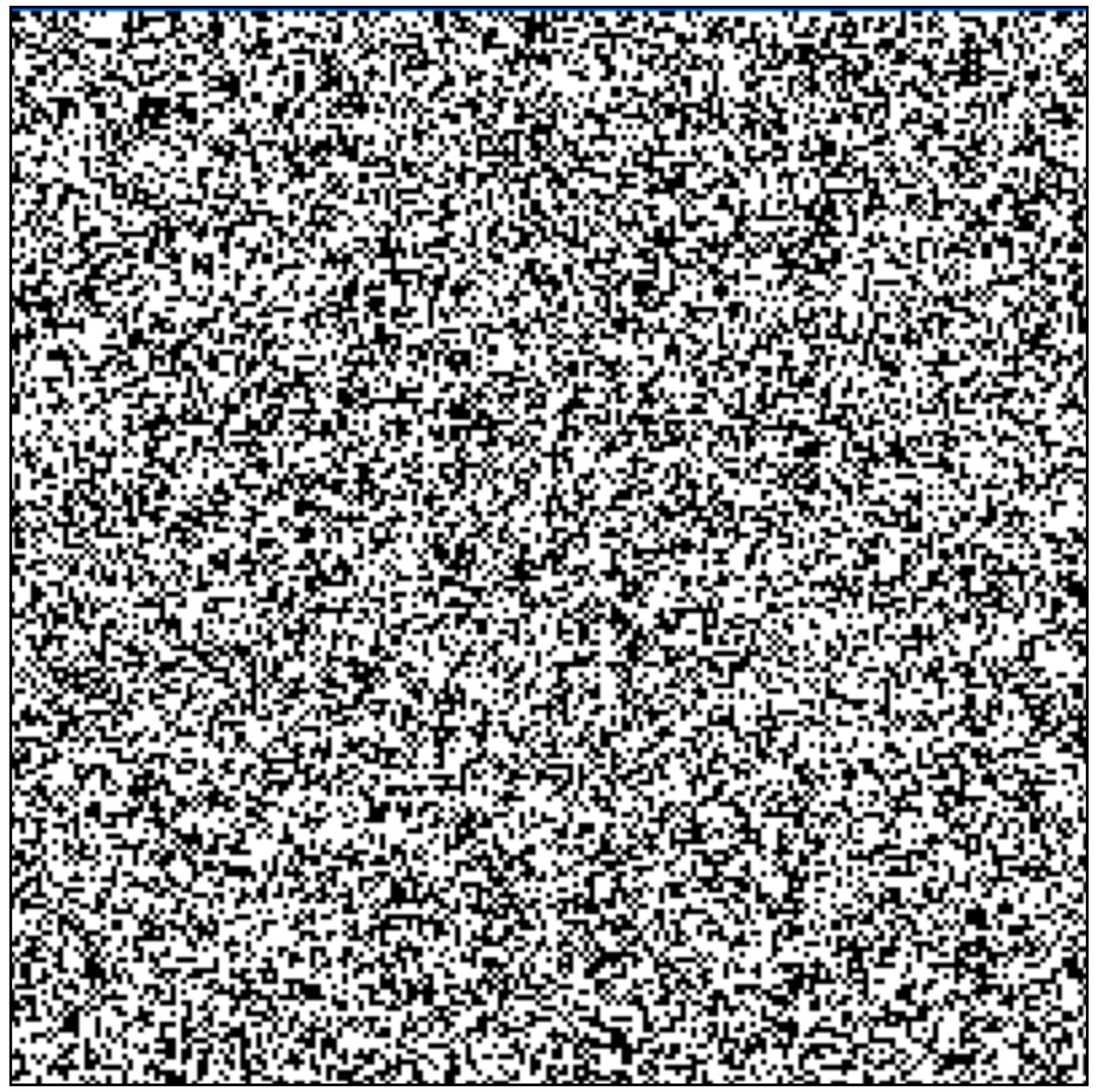

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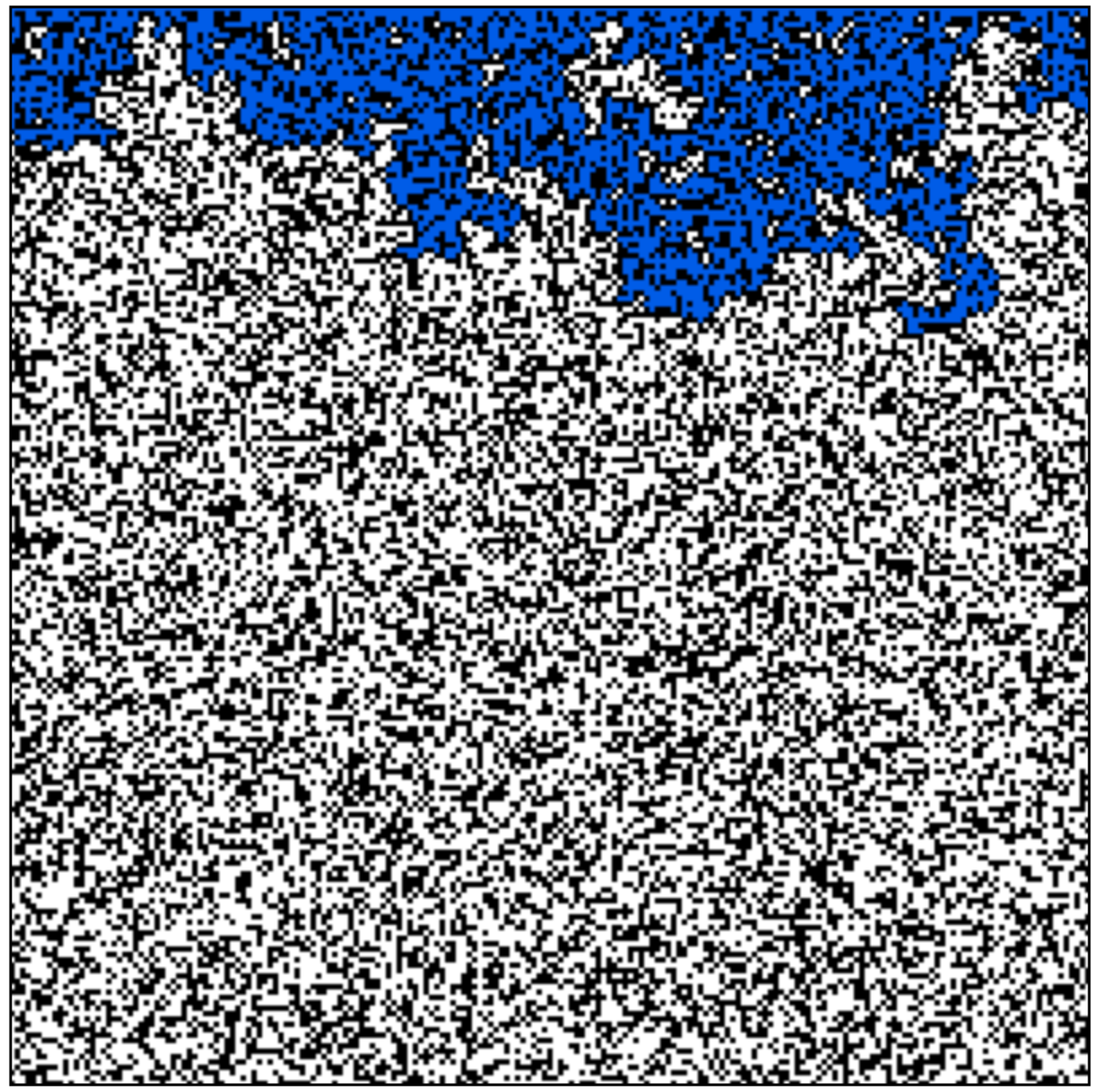
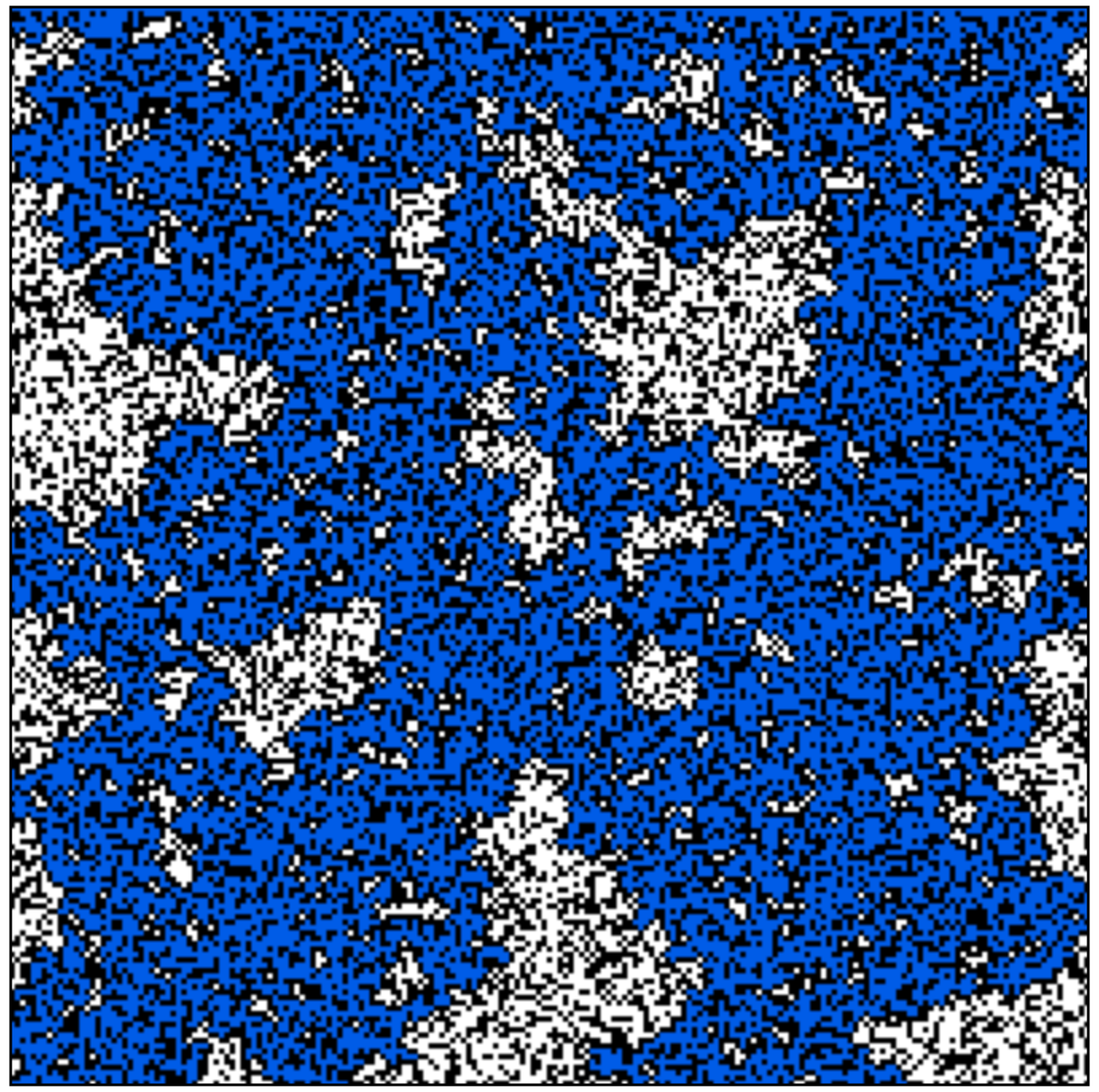


$p = 0.59 < p_c$

liquid almost never makes it through

(never make it through if the system has infinite size)



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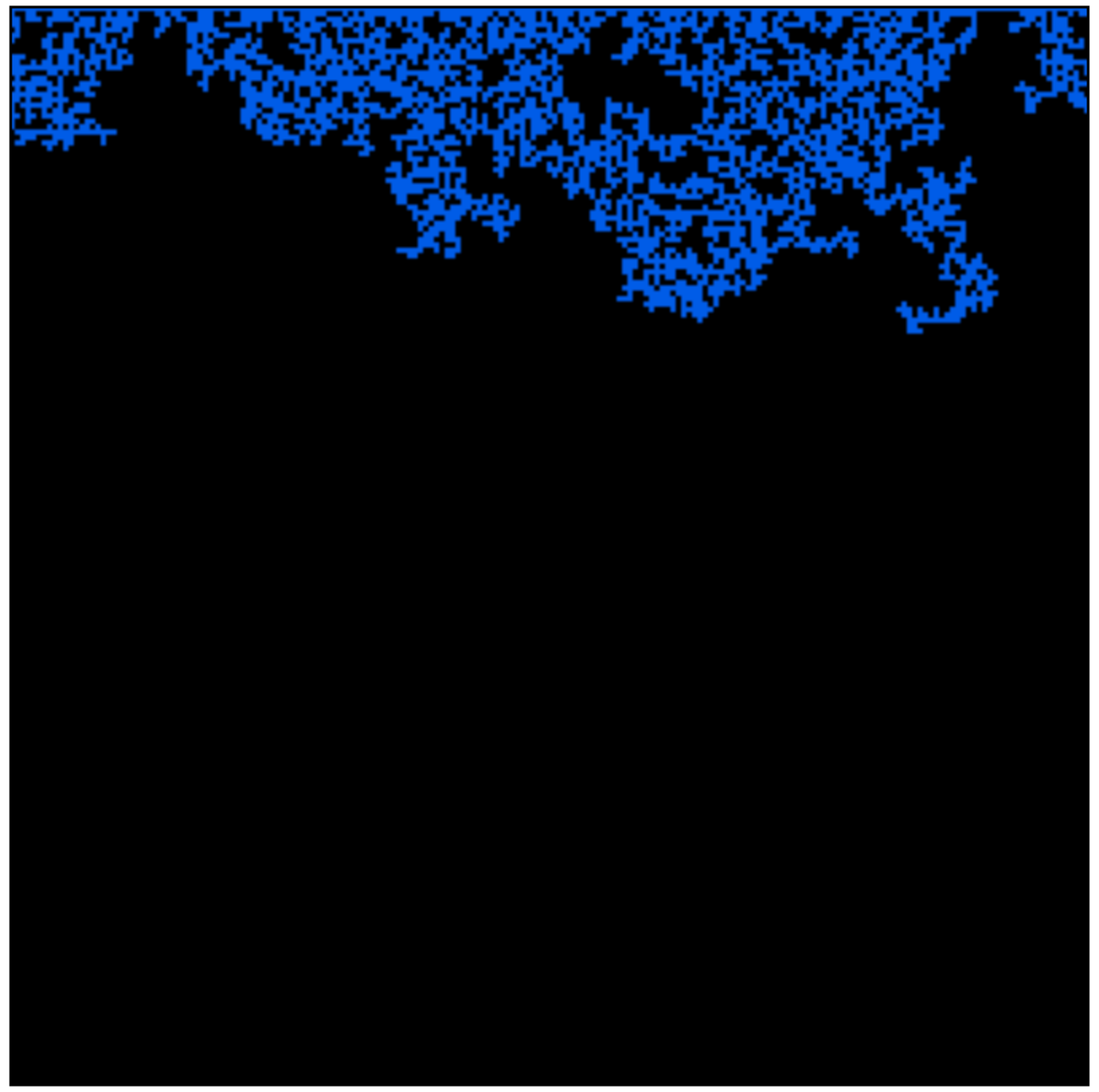

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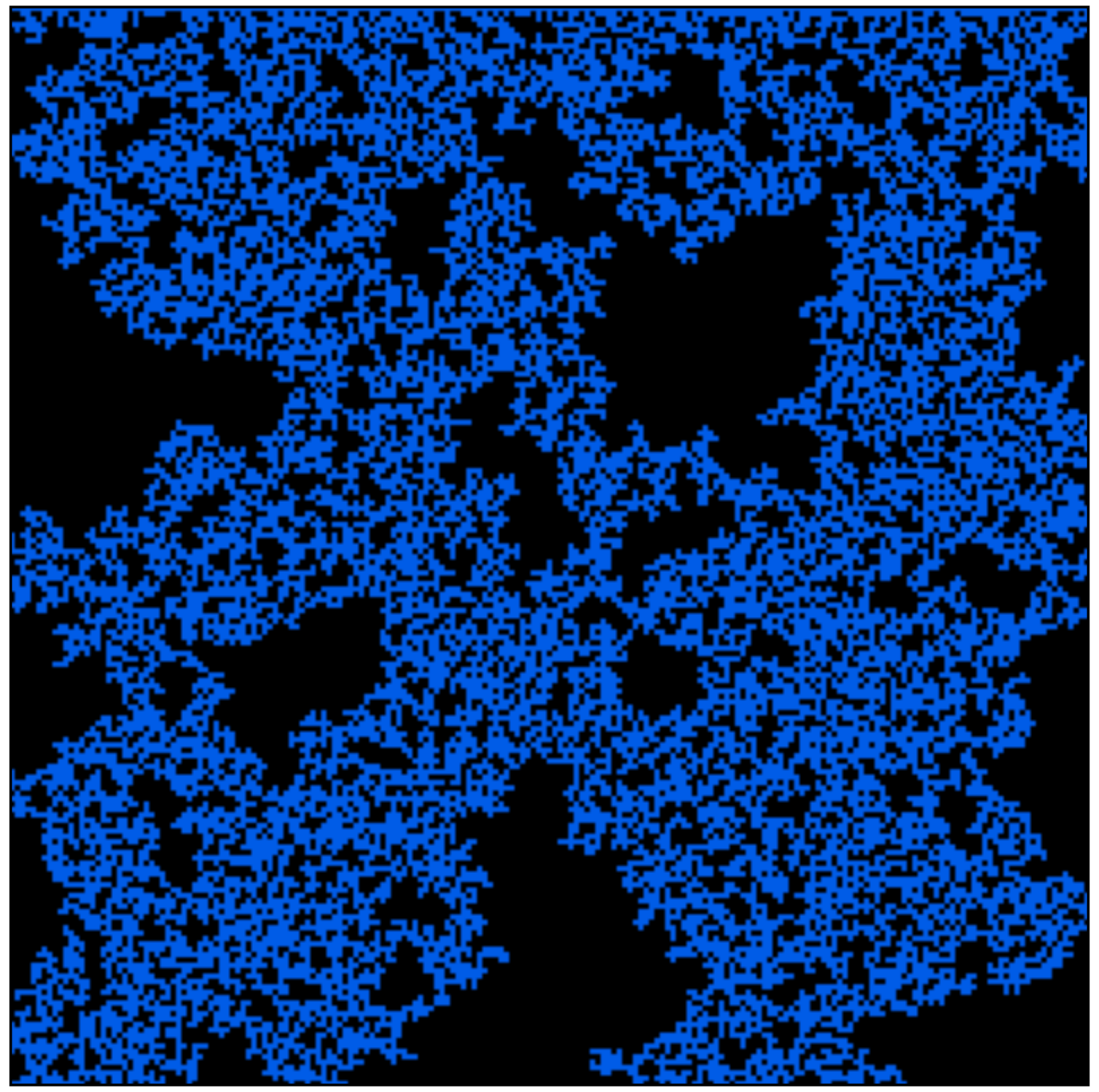

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Model of formation of gels (networks of chemical bonds spanning the whole system, ex. Boiling an egg become solid-like) **—> Flory-Stockmayer theory**

Forest Fire, Epidemic modeling, metal-insulator transition, networks fragmentation, fractures in rocks and earthquakes 'nucleation', …

Cluster growth models (e.g., for growth of a cancer cell, a snowflake, or an iceberg)

2) Percolation phase transition

a) Control parameter and critical value

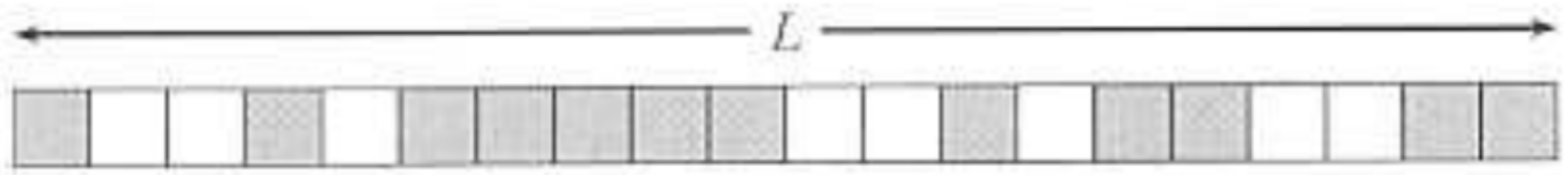
b) Order parameter and quantities of interests

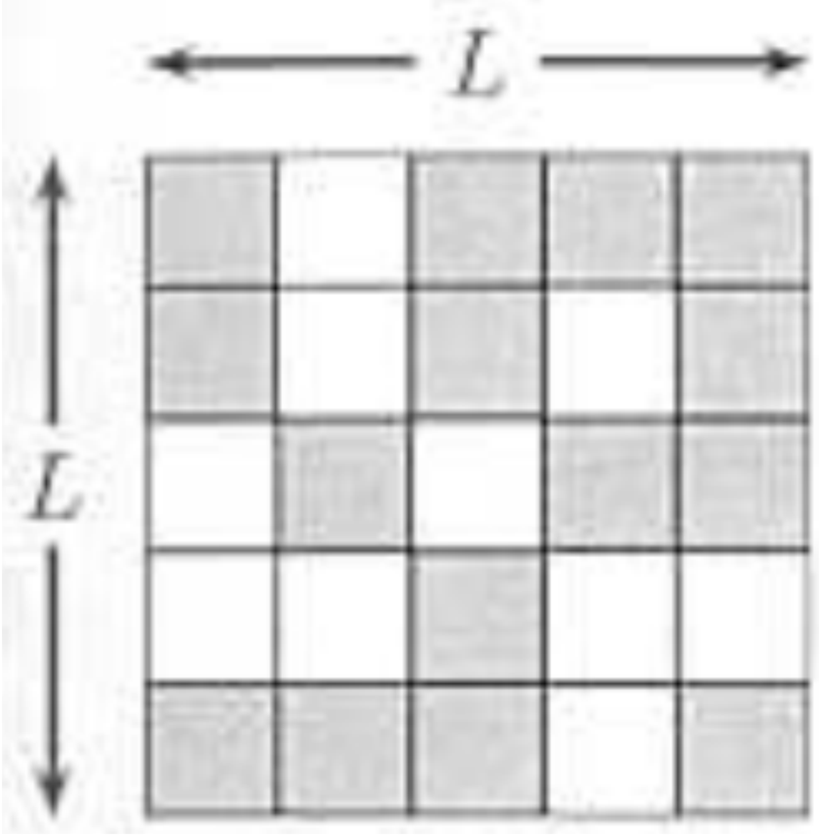
c) 1d site percolation

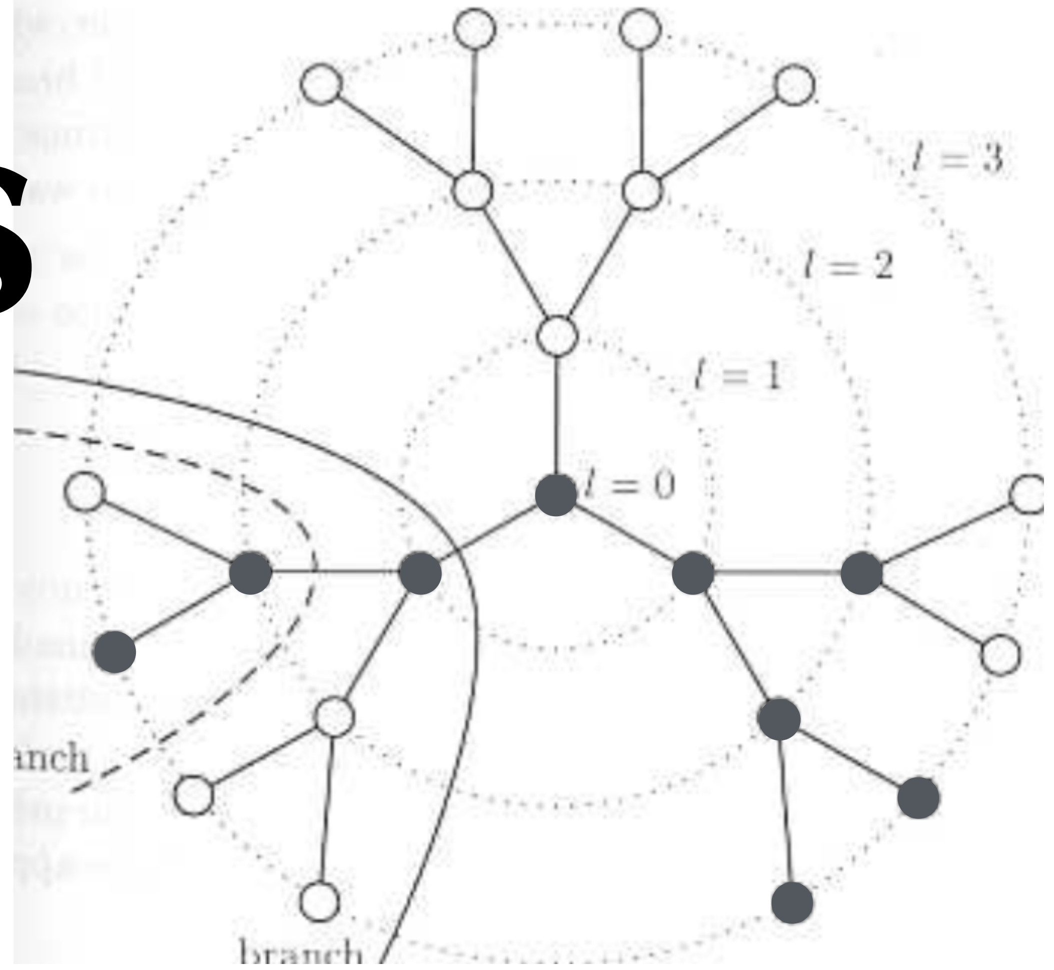
d) Bethe Lattice

Theoretical solutions

Results on Percolation problems



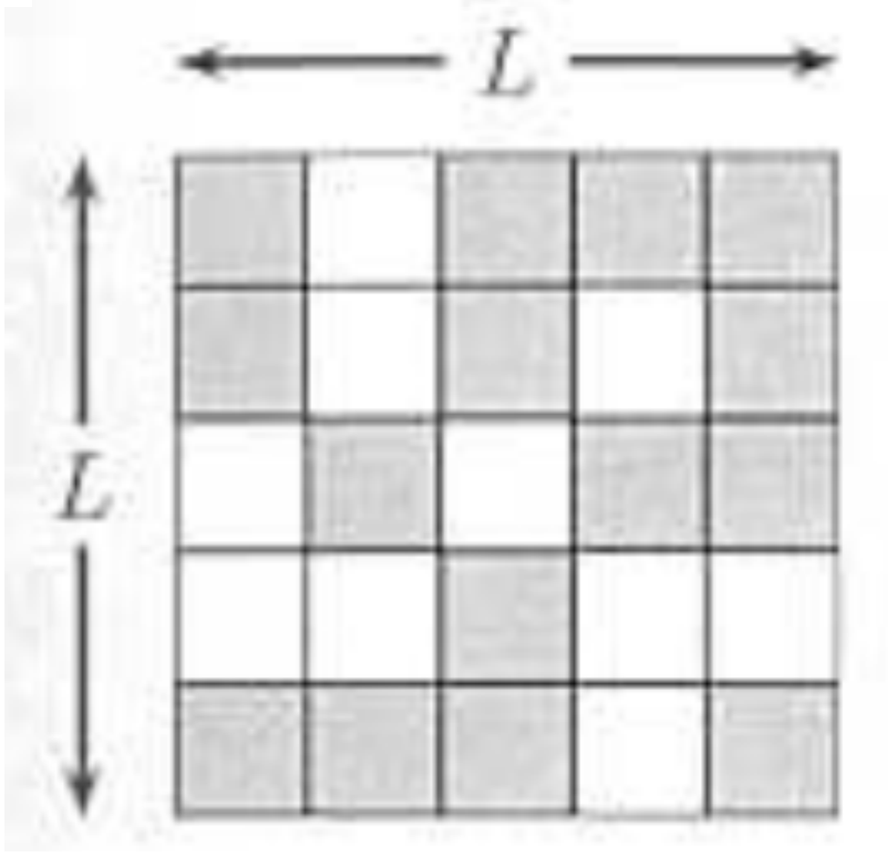




1) Control parameter and Critical value

Working model: Site percolation on a lattice

Ex. **2D lattice of L x L sites**



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Average number of sites that are occupied = **L x L x p**

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Site percolation on a lattice

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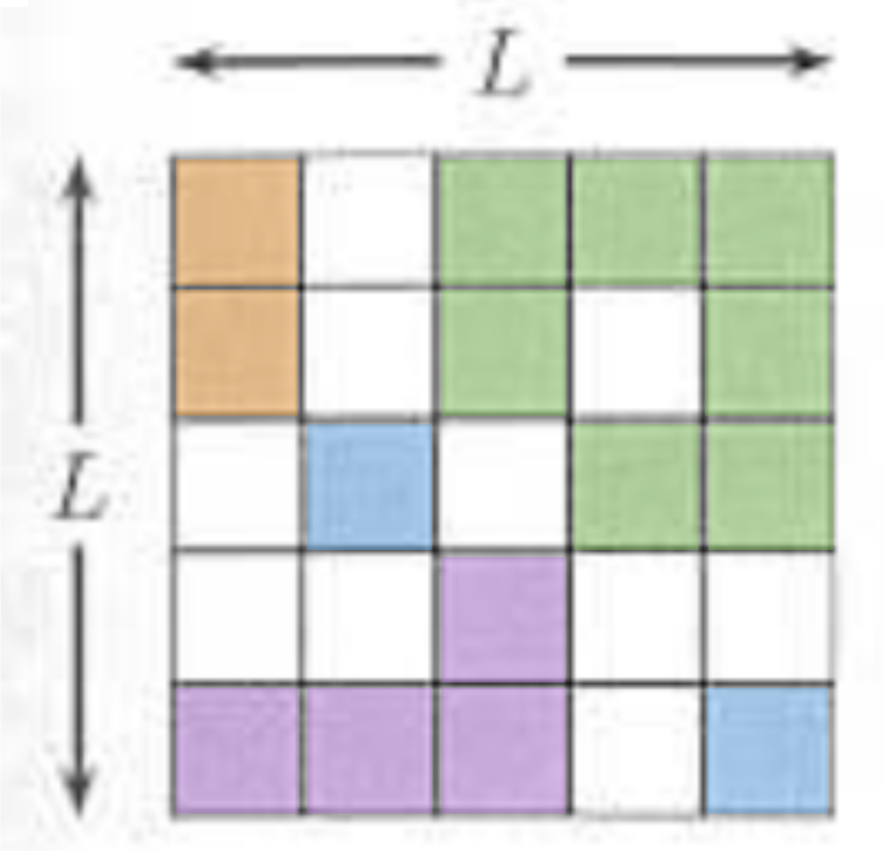

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cluster = group of nearest-neighboring occupied sites

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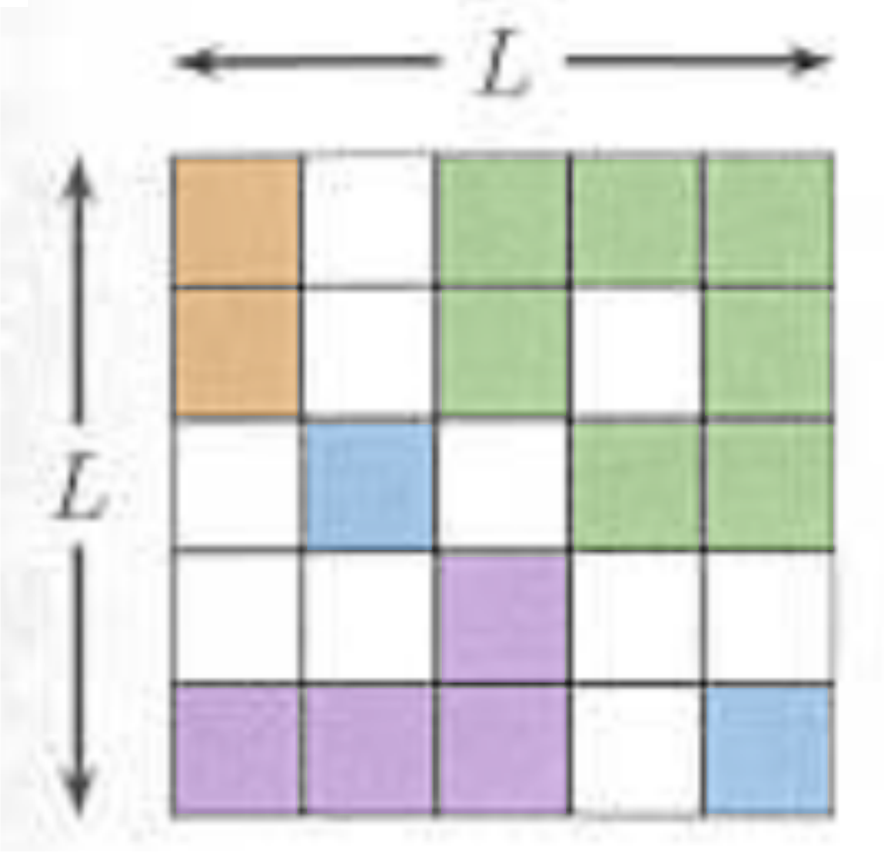

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Percolation theory: studies the **properties of the clusters** formed

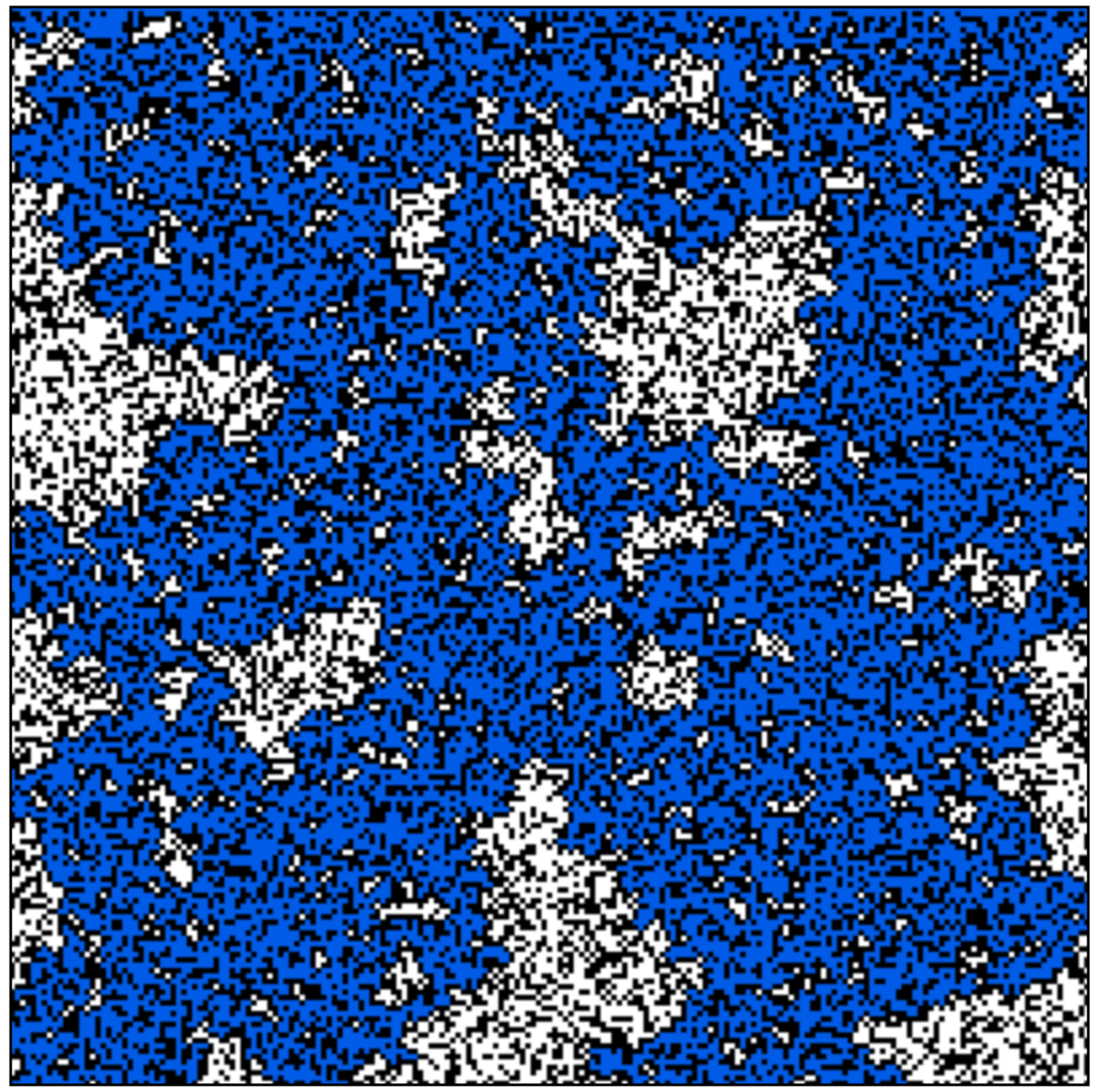
cluster = group of nearest-neighboring occupied sites

Cluster is percolating = **Cluster is infinite!** ⟶ **For finite size systems:**

Clusters that span the entire lattice

from left to right boundaries
or from top to bottom boundaries

are good candidates of percolating clusters

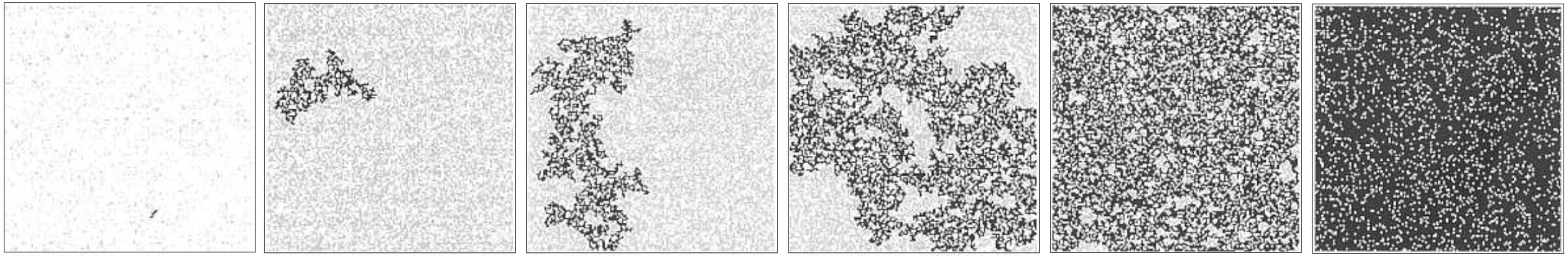


Control parameter

Expected behavior:

- **For p=0:** the lattice is empty
- **For p=1:** there is only 1 cluster of size L^2
- **For intermediate p:** each realization will be different, but we expect the size of the largest cluster to increase with **p**.

L=150



p=0.10 **p=0.55** **p=0.58** **p~pc** **p=0.65** **p=0.90**

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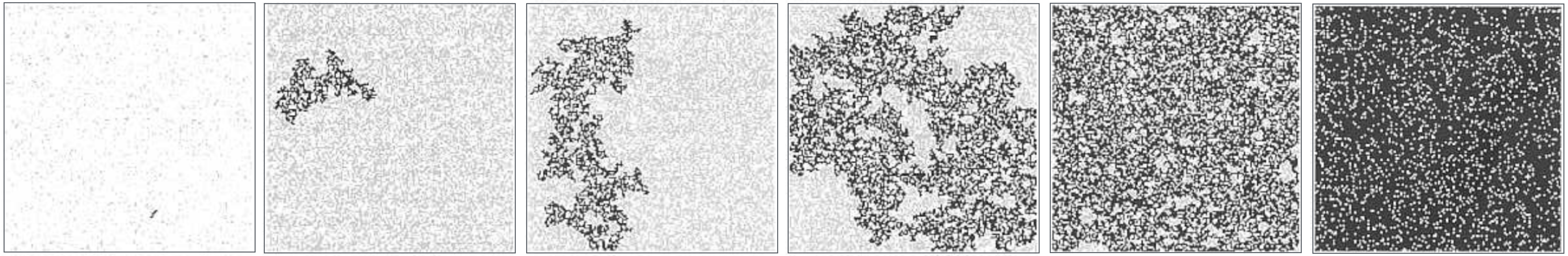
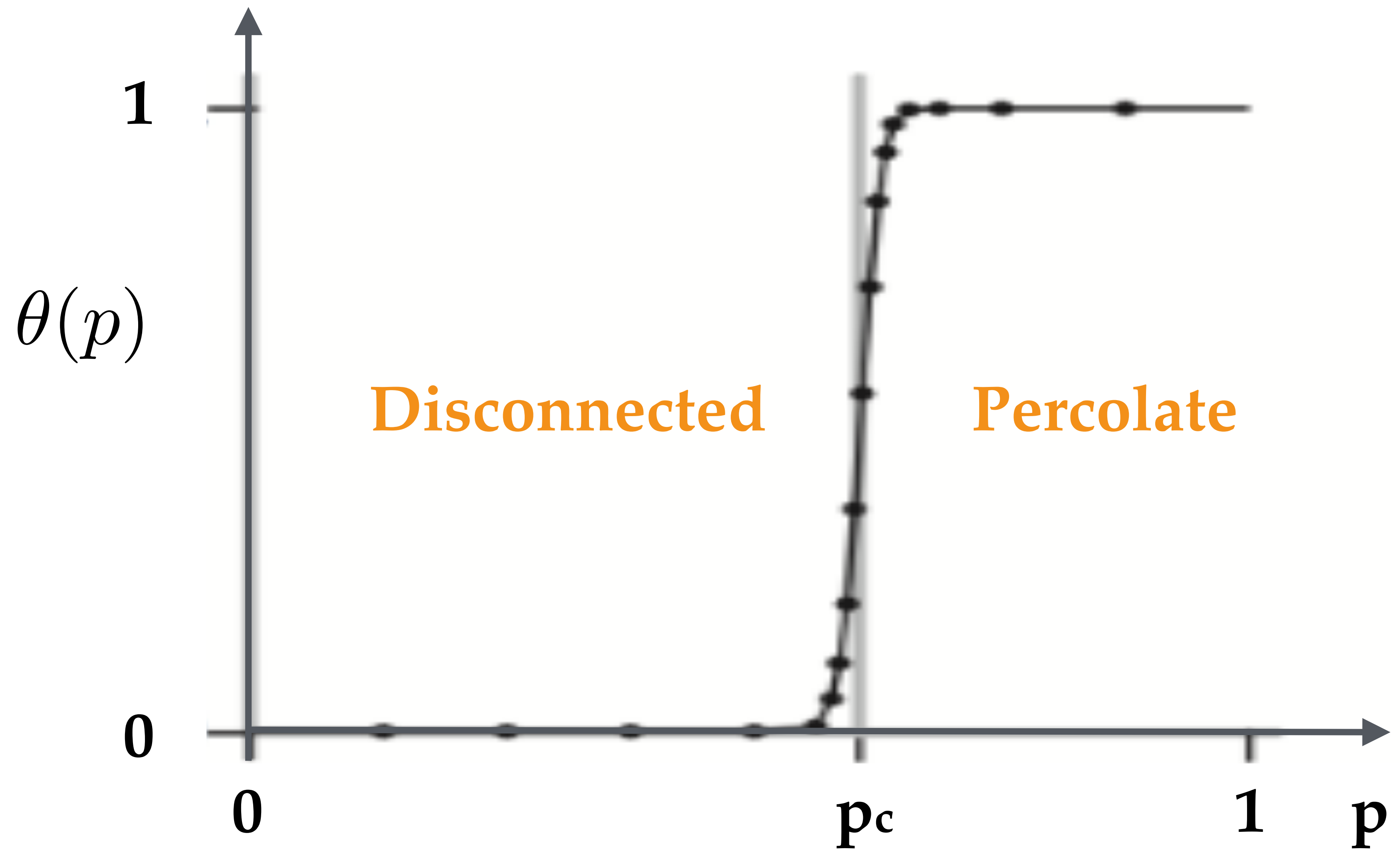


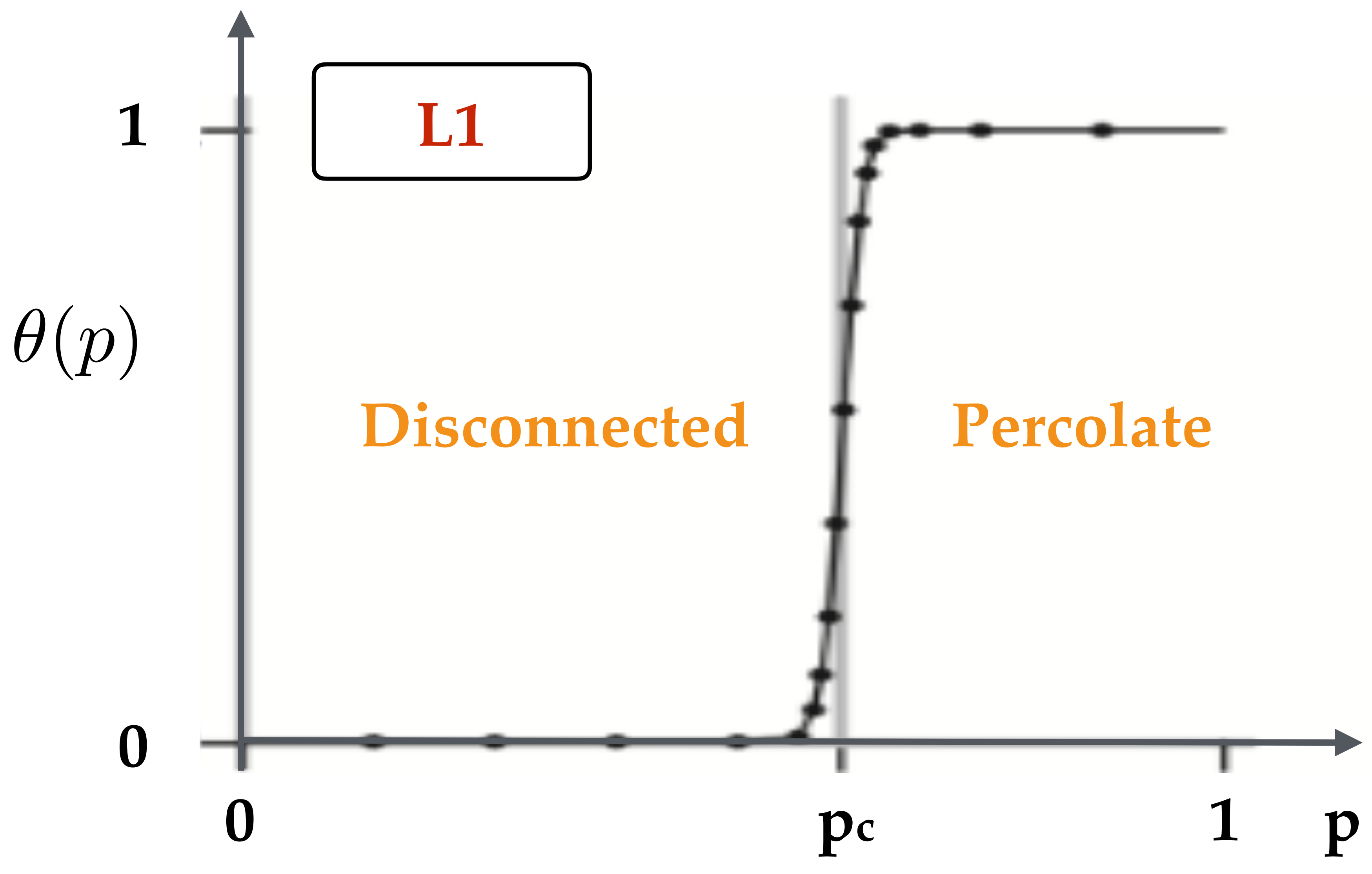
p=0.10 **p=0.55** **p=0.58** **p~p$_c$** **p=0.65** **p=0.90**

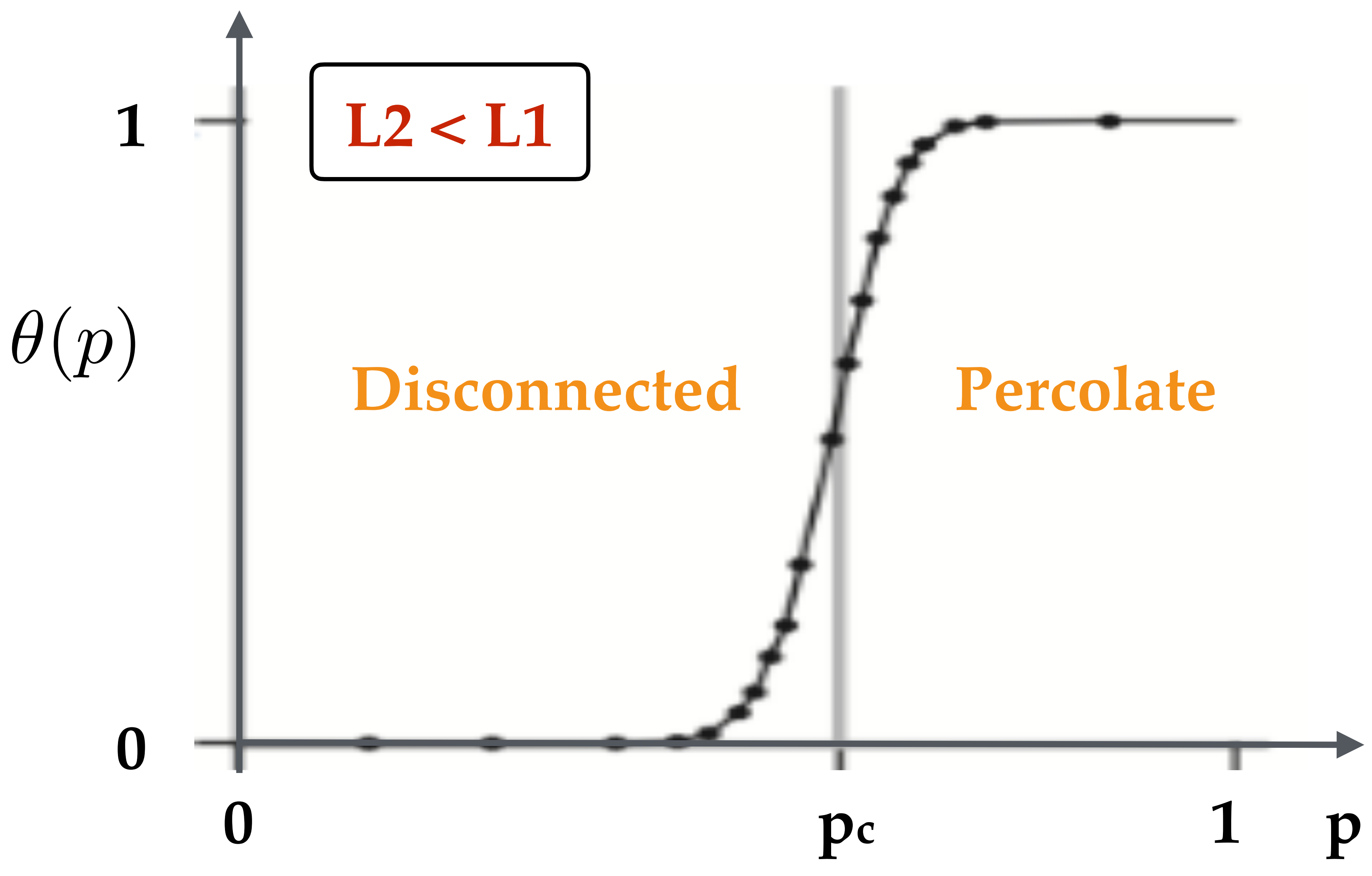
$\theta(p)$ **= percolation probability**

= probability that there is a cluster of infinite size



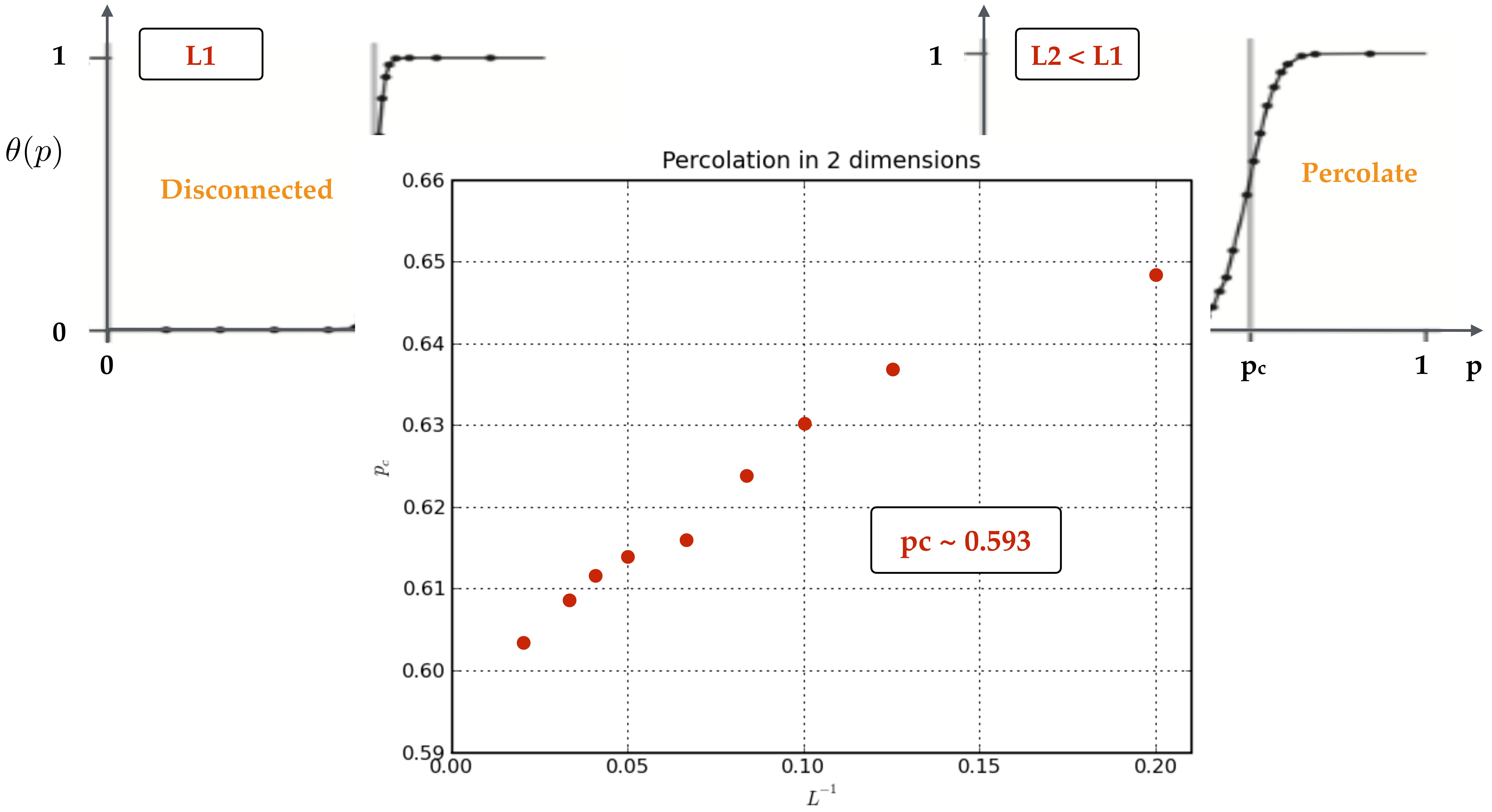
Critical parameter





Larger L: the curve increase at $\mathbf{p_c}$ becomes sharper

Critical parameter



2) Parameters, and interesting quantities for the problem

Percolation Parameters

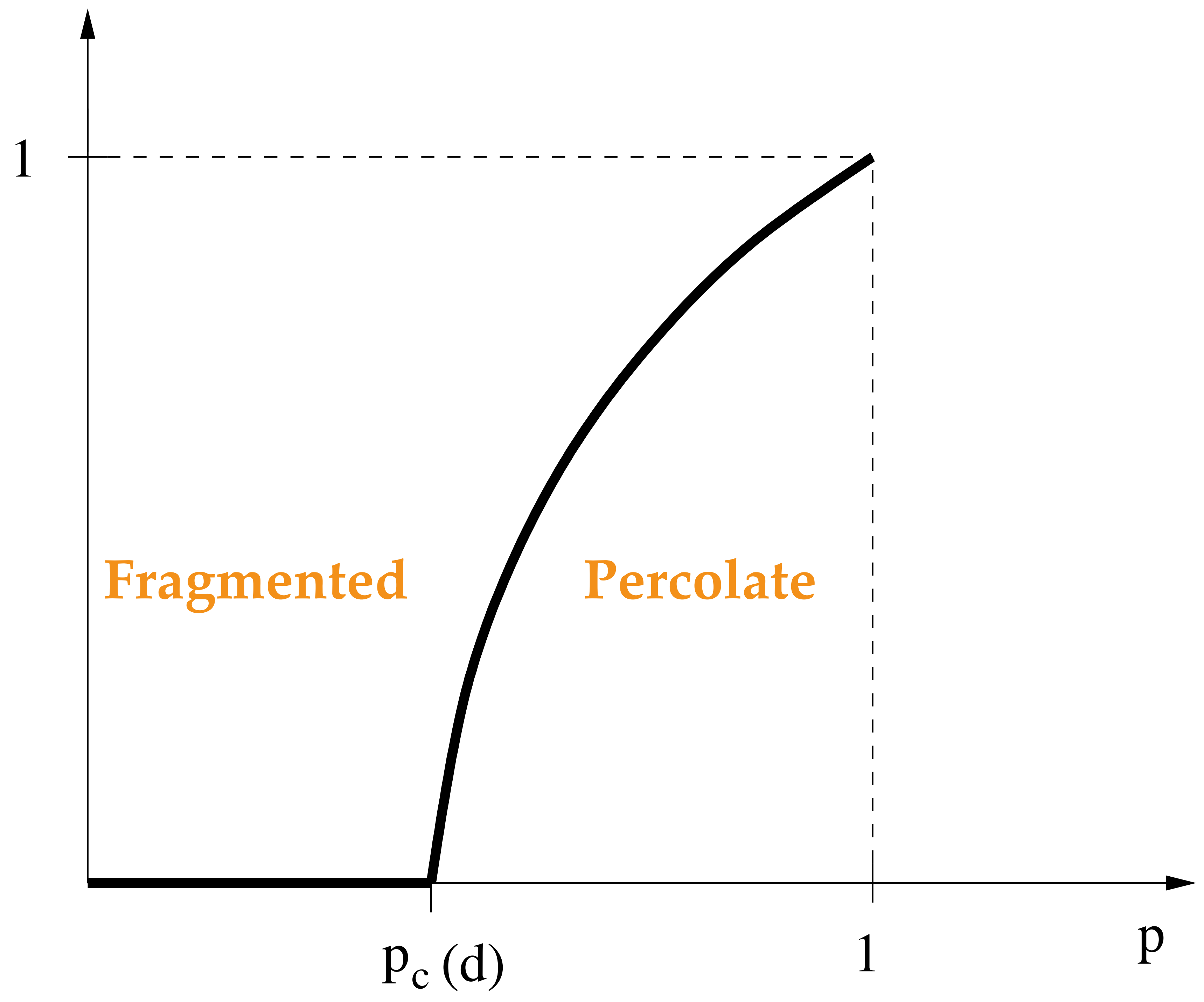
Control parameters:

Order parameter:

Percolation Parameters

Control parameters: p **= occupation probability** = probability that a site is occupied

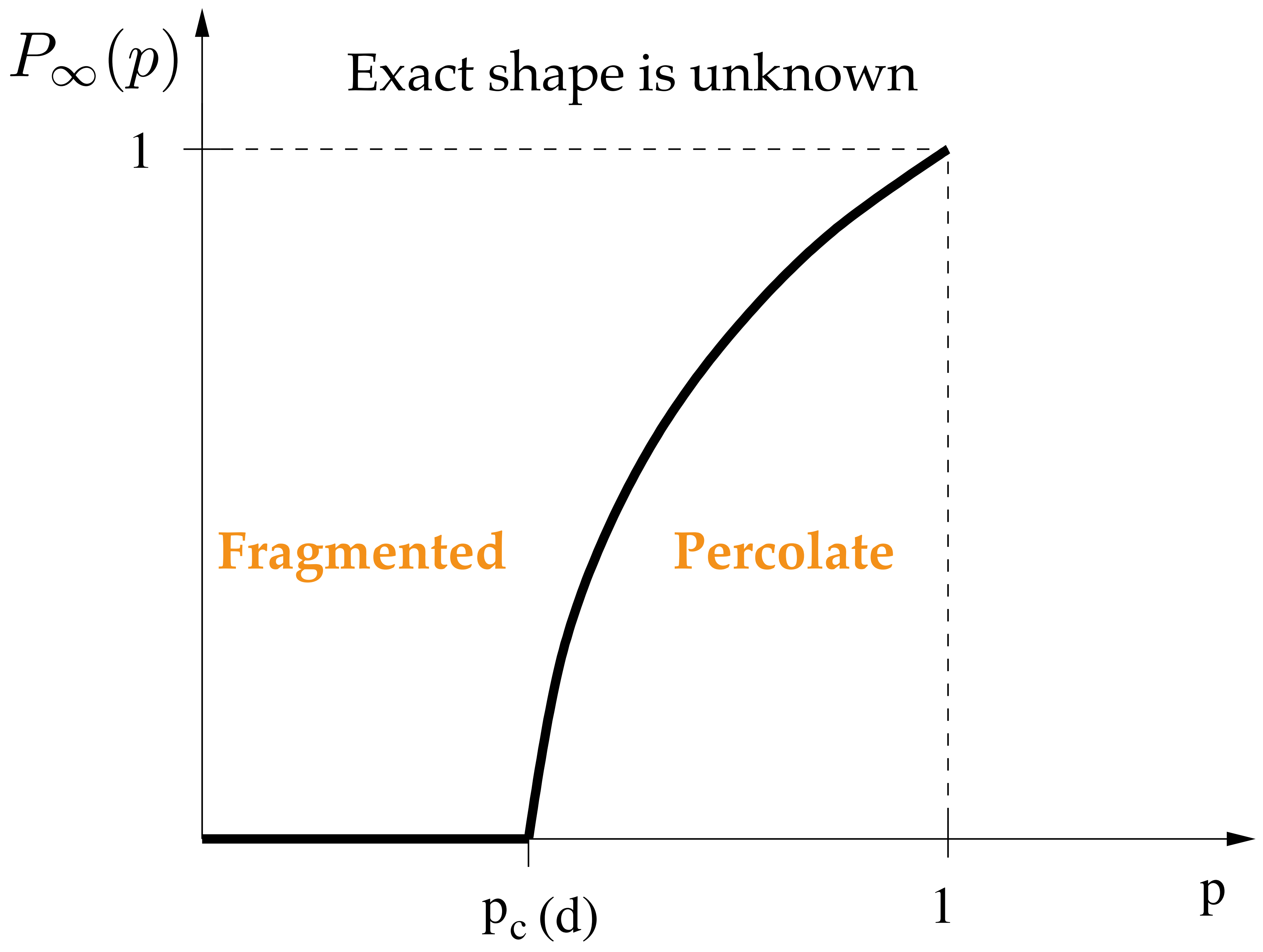
Order parameter:



Percolation Parameters

Control parameters: p **= occupation probability** = probability that a site is occupied

Order parameter: $P_\infty(p)$ **= probability that any given site belongs to the percolating cluster**



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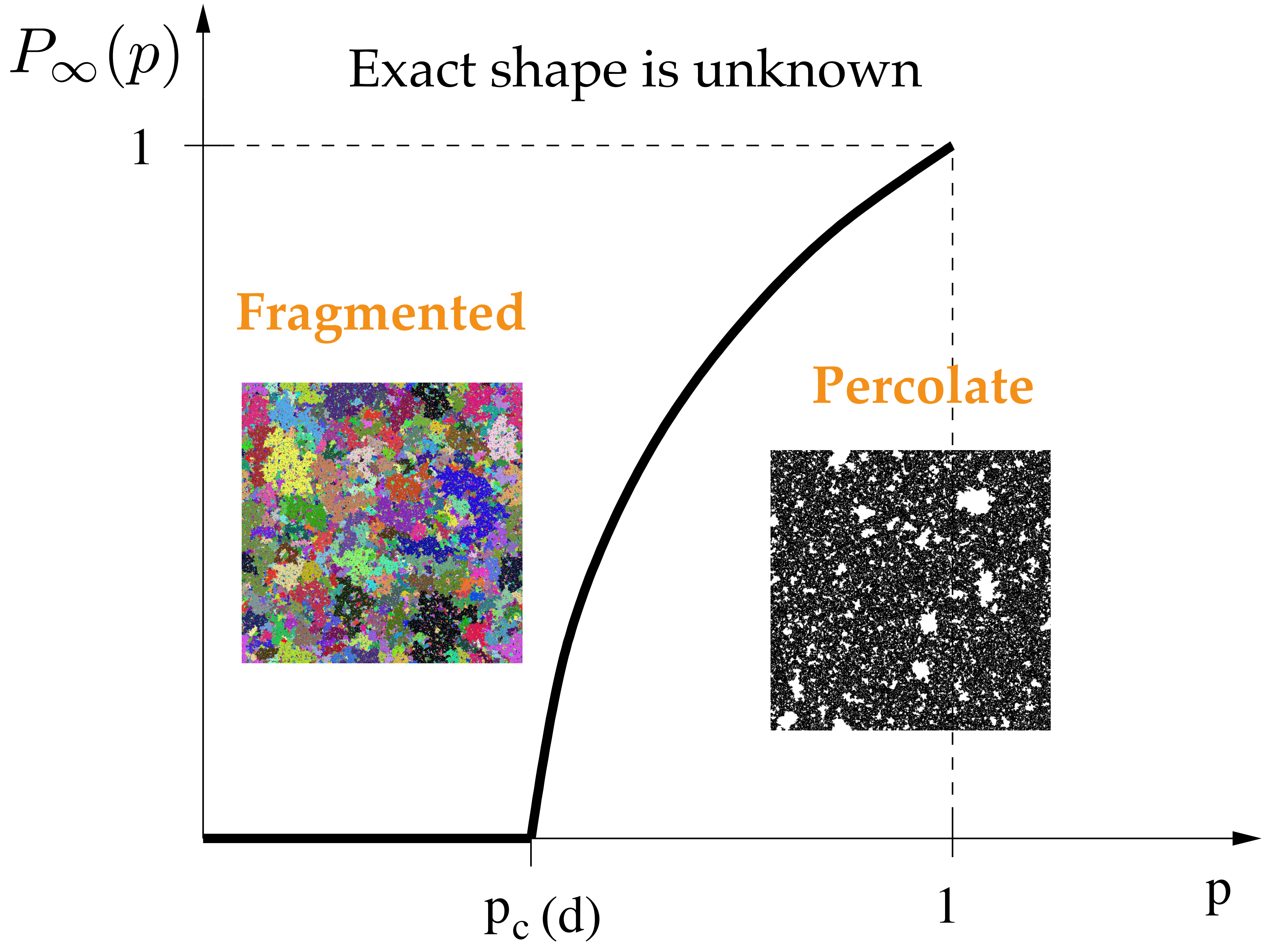

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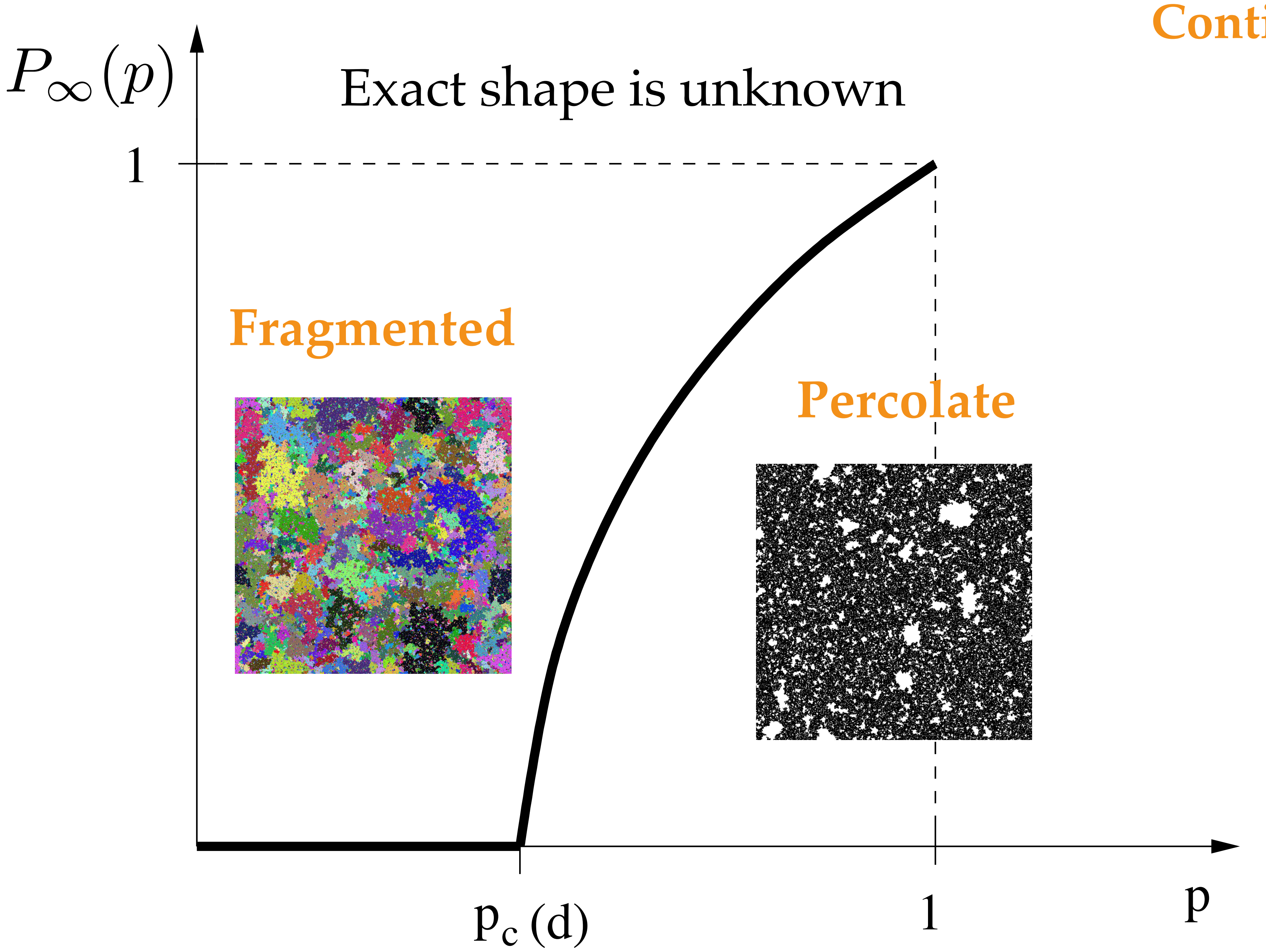
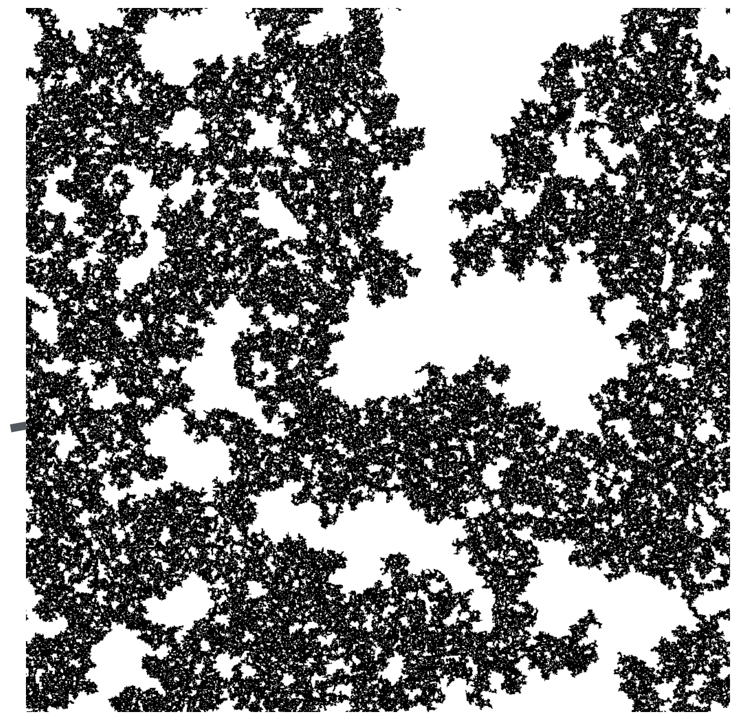


Continuous phase transition: The fact that $P_\infty(p)$ is continuous at $\mathbf{p_c}$ is non-trivial

How to see this intuitively? **At the critical point:**

- the percolating cluster is "very fragile", removing a few occupied site may break percolation
- the percolating cluster is very tortuous, very "thin"; the fraction of sites that belongs to the percolating cluster is very small



Interesting quantities

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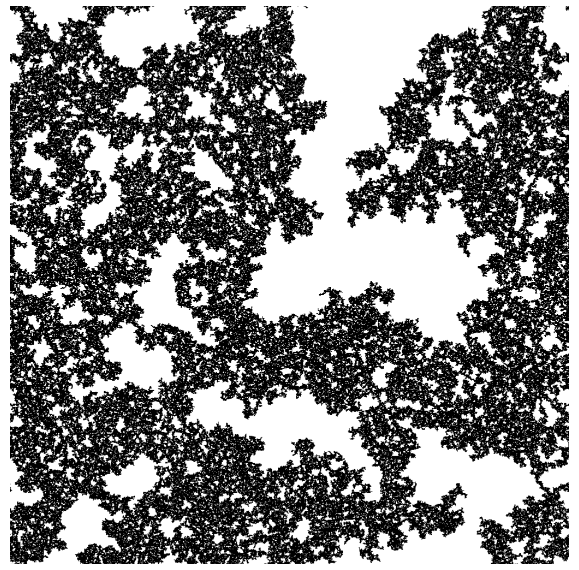
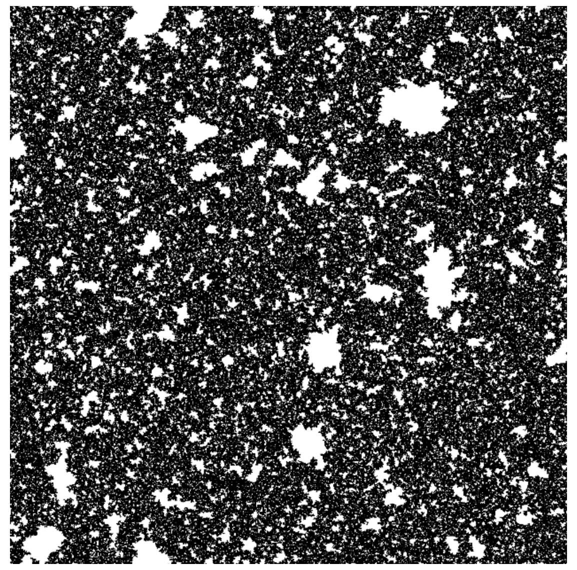


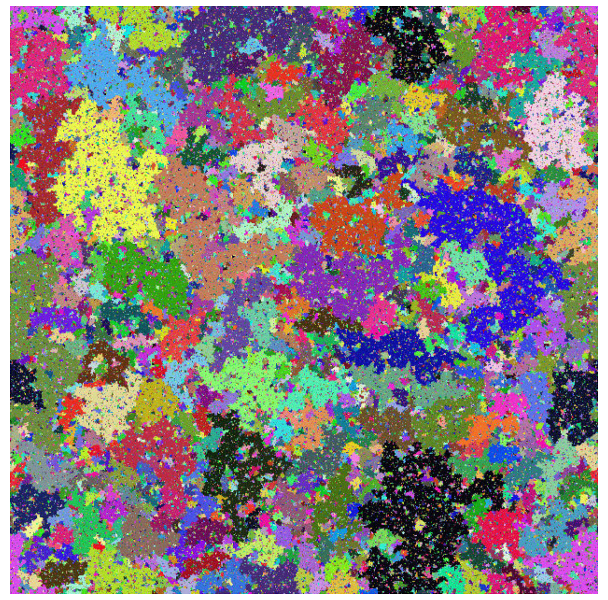


General questions:

Size, Infinite Cluster: How much space does the percolating cluster occupy?

= What is the **probability that a site belong to the percolating cluster?** $P_\infty(p)$

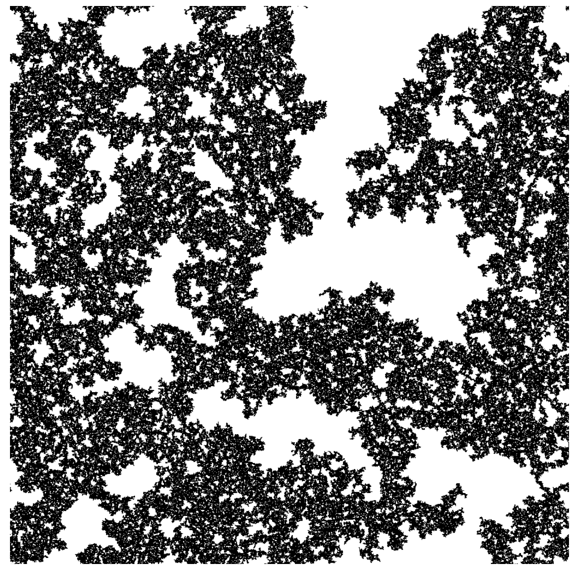
Finite clusters:

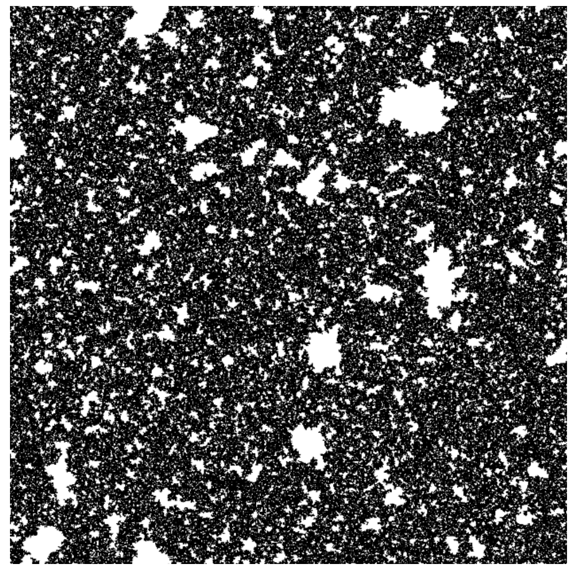


Interesting quantities

Intensif parameters: p **= occupation probability** = probability that a site is occupied

Order parameter: $P_\infty(p)$ **= probability that any given site belongs to the percolating cluster**





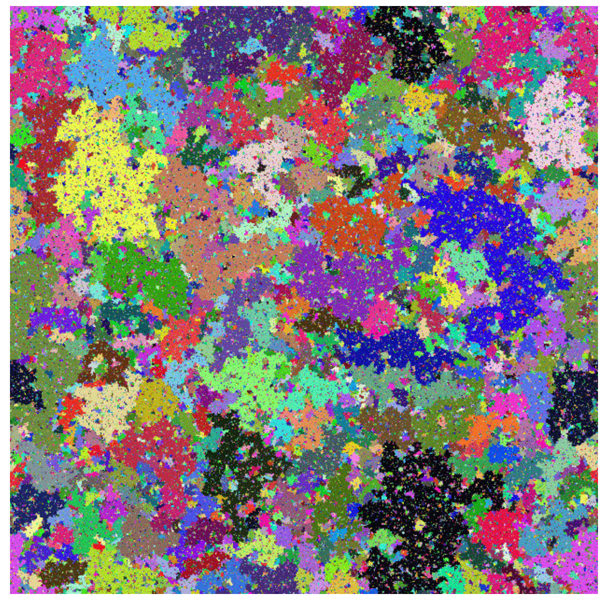
General questions:

Size, Infinite Cluster: How much space does the percolating cluster occupy?

= What is the **probability that a site belong to the percolating cluster?** $P_\infty(p)$

Finite clusters: What is the distribution of sizes of the finite clusters? $n(s,p)$

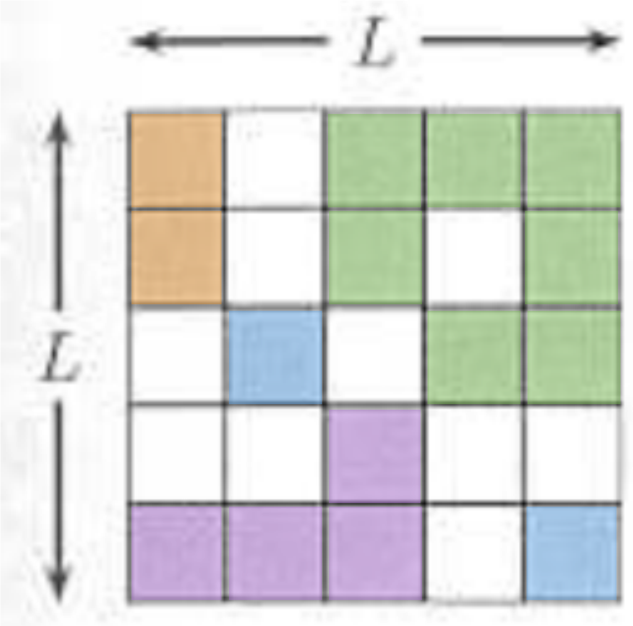
What is the **probability that a given site belongs to a cluster of size s?** $s\,n(s,p)$



s = cluster size

= number of sites in a cluster

Ex.



s	$N(s,p)$
1	2
2	1
4	1
7	1

$N(s,p)$ = **Number of clusters of size** s

At 15h today:

Speaker: Jacopo Grilli

research scientist at the International Center for Theooretical Physics, Trieste, Italy.

Title: Competition without contingency and functional convergence

Abstract: **Microbial communities** are taxonomically diverse and variable: **species presence and their abundances widely fluctuate over time and space, and even across biological replicates in experimental controlled conditions.** On the other hand, environmental conditions exert a strong selection on the traits of community members and the function they perform, and similar environmental conditions are expected to correspond to functionally similar communities. The consequence of this environmental selection, together with taxonomic variability, lead to the influential concept of functional redundancy: the same function is performed by many species, so that one may assemble communities with different species but the same functional profile. The centrality of the concept of functional redundancy in microbial ecology does not parallel with a theoretical understanding of its origin. **In this talk I will describe the eco-evolutionary dynamics of communities interacting through competition and cross-feeding.** I will show that the eco-evolutionary trajectories rapidly converge to a "functional attractor", characterized by a functional composition uniquely determined by environmental conditions. The taxonomic composition instead follows non-reproducible dynamics, constrained by the conservation of the functional composition. This framework provides a deep theoretical foundation to the concepts of functional robustness and redundancy.

Location: Institute for Advanced Study (IAS) second floor library, Oude Turfmarkt 147

Zoom link: https://uva-live.zoom.us/j/89587131703

At 11h today: **There will be the video, if some of you are interested**

Speaker: Stefan Thurner

full professor Science of Complex Systems at the Medical University of Vienna,
president of the Complexity Science Hub Vienna, external faculty at the Santa Fe institute

Title: Towards a statistics of driven complex systems

Abstract: Most complex systems are not in equilibrium but driven. We argue that driven systems can be understood by so-called sample space reducing (SSR) processes. They provide an intuitive understanding of the origin and ubiquity of fat-tailed distributions in complex systems, power-laws in particular. SSR processes are mathematically simple and offer an alternative to Boltzmann-equation based approaches to non-equilibrium systems. We show that in many situations the statistics is determined by the details of the driving process and does not depend on the specific relaxation dynamics. Simple (homogeneous) driving strategies universally lead to Zipf's law and exact power laws. Other driving processes results in exponential, Gamma, normal, Weibull, Gompertz, and Pareto distributions. We discuss a number of examples of SRR processes, including fragmentation processes, language formation, cascading and search processes, as well as a classic in physics: inelastical collisions.

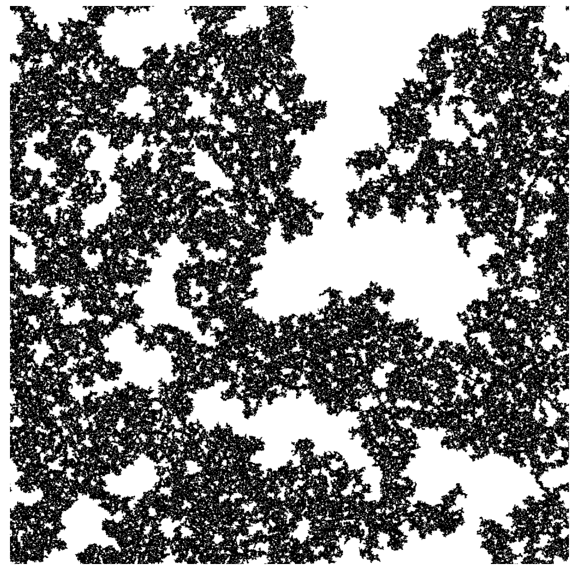
Location: Institute for Advanced Study (IAS) second floor library, Oude Turfmarkt 147

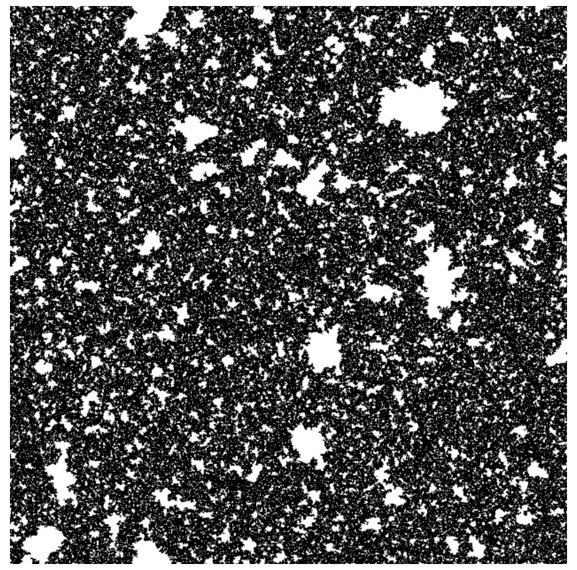
Zoom link: https://uva-live.zoom.us/j/89587131703

Interesting quantities

Intensif parameters: p **= occupation probability** = probability that a site is occupied

Order parameter: $P_\infty(p)$ **= probability that any given site belongs to the percolating cluster**

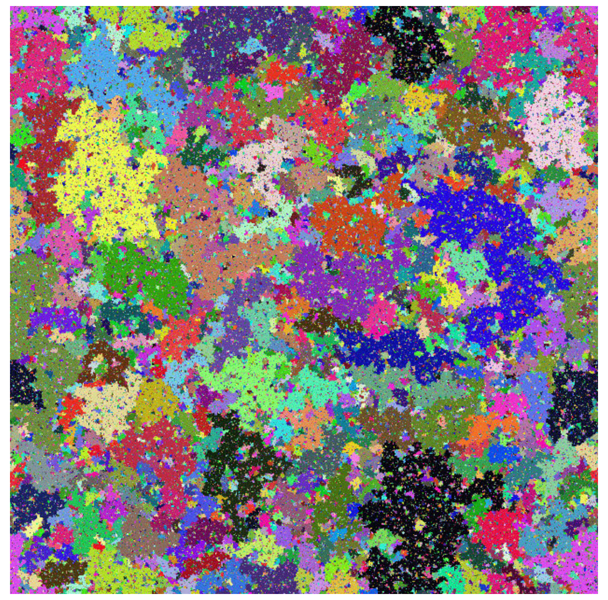




General questions:

Size, Infinite Cluster: How much space does the percolating cluster occupy?

= What is the **probability that a site belong to the percolating cluster?** $P_\infty(p)$



Finite clusters: What is the distribution of sizes of the finite clusters? $n(s,p)$

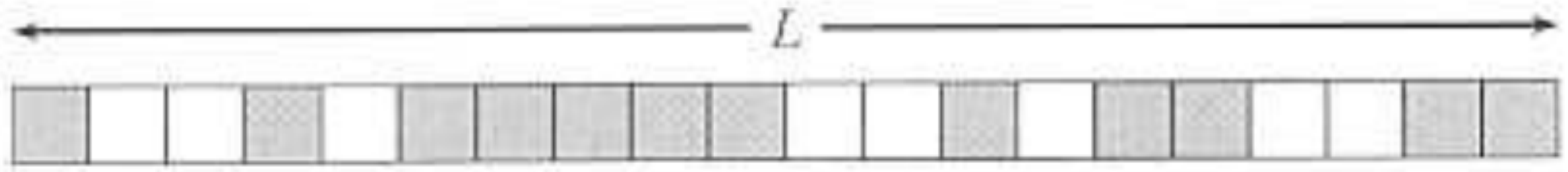
What is the **probability that a given site belongs to a cluster of size s?** $s\,n(s,p)$

$n(s,p)$ $\longrightarrow$ Will allow us to compute all the quantities of interest.

Average cluster size: Take an occupied site, what is the average size of the cluster it belongs to? $\chi(p)$

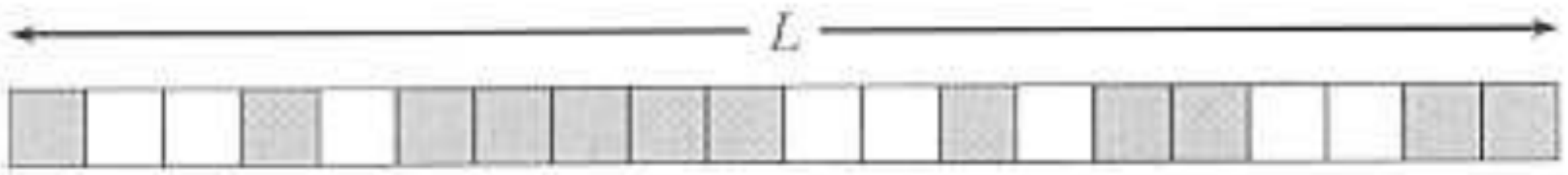
Correlation: Probability two sites at a distance *r* apart belong to the same cluster $g(r)$

3) Site Percolation in d=1



Ex. 1D site percolation

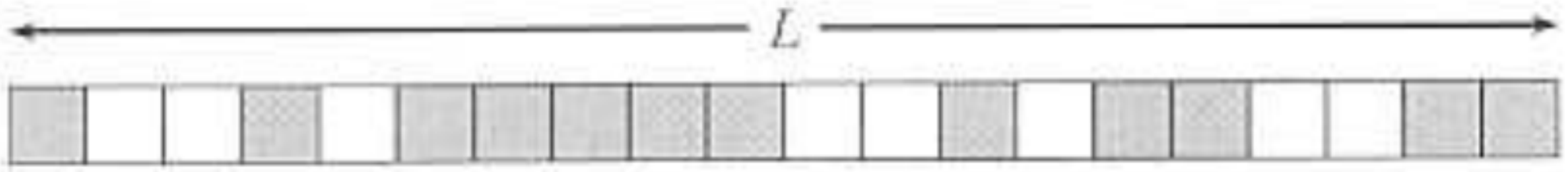
Critical probability



In 1d lattice: **What is p_c?**

Ex. 1D site percolation

Critical probability



In 1d lattice: The percolating cluster must **include every sites in the lattice** to span from left to right

If p< 1: There is no percolating cluster

If p = 1: There is a percolating cluster

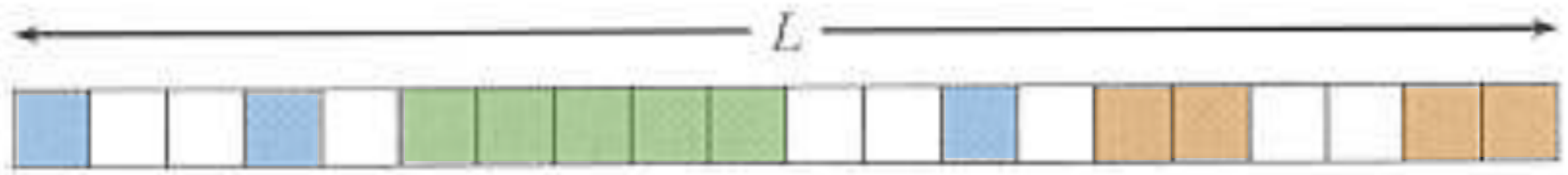
$\longmapsto$ **1D percolation:** $p_c[1D] = 1$

$P_\infty(p)$ **= probability that any given site belongs to the percolating cluster**

$$P_\infty(p) = \begin{cases} 0 & \text{for } p < 1 \\ 1 & \text{for } p = 1 \end{cases}$$

Ex. 1D site percolation

Cluster number density



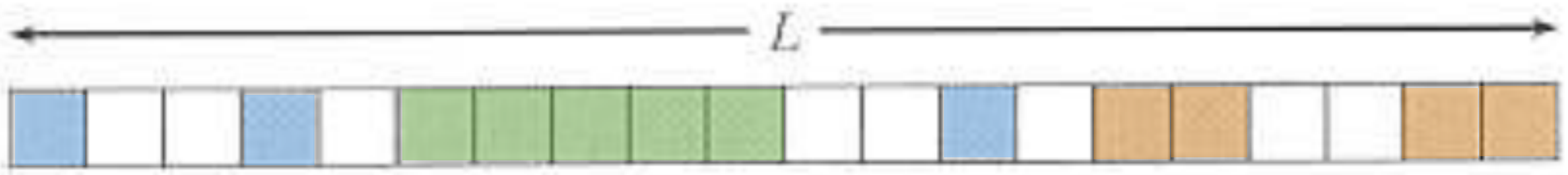
Note: boundary effects are ignored, as they become irrelevant at large L

s	$N(s,p)$
1	3
2	2
5	1

$N(s,p)$ = **cluster size frequency** = number of clusters of size **s** (in a system of size **L**)

Ex. 1D site percolation

Cluster number density



Note: boundary effects are ignored, as they become irrelevant at large L

s	$N(s,p)$
1	3
2	2
5	1

Extensive quantity

$N(s,p)$ = **cluster size frequency** = number of clusters of size **s** (in a system of size **L**)

$n(s,p) = \dfrac{N(s,p)}{L}$ = **cluster size density** = density of clusters of size **s**

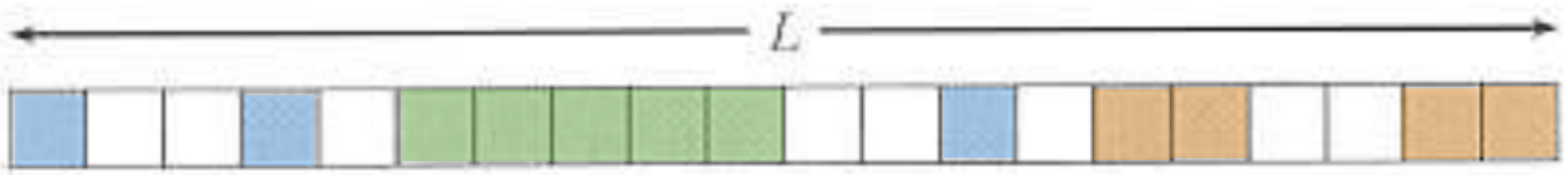
= **number of clusters of size s per unit length**

It is better to consider n(s, p) than N(s,p),
as we want to study what happens as we take L very large
And N depends on L

→ We want to compute $n(s,p)$

Ex. 1D site percolation

Cluster number density



Note: boundary effects are ignored, as they become irrelevant at large L

→ We want to compute $n(s,p)$ = **number of clusters of size s, per unit length**

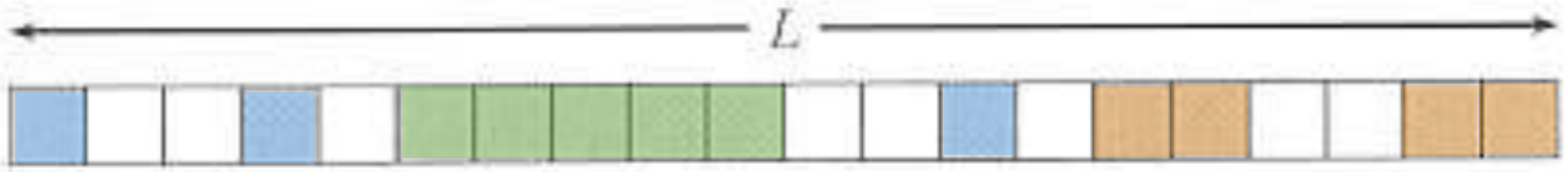
Observe that:

$s\, n(s,p)$ = probability that a given site belong to a cluster of size **s**

$n(s,p)$ = probability that a given site belongs to the LHS of a cluster of size s

Ex. 1D site percolation

Cluster number density



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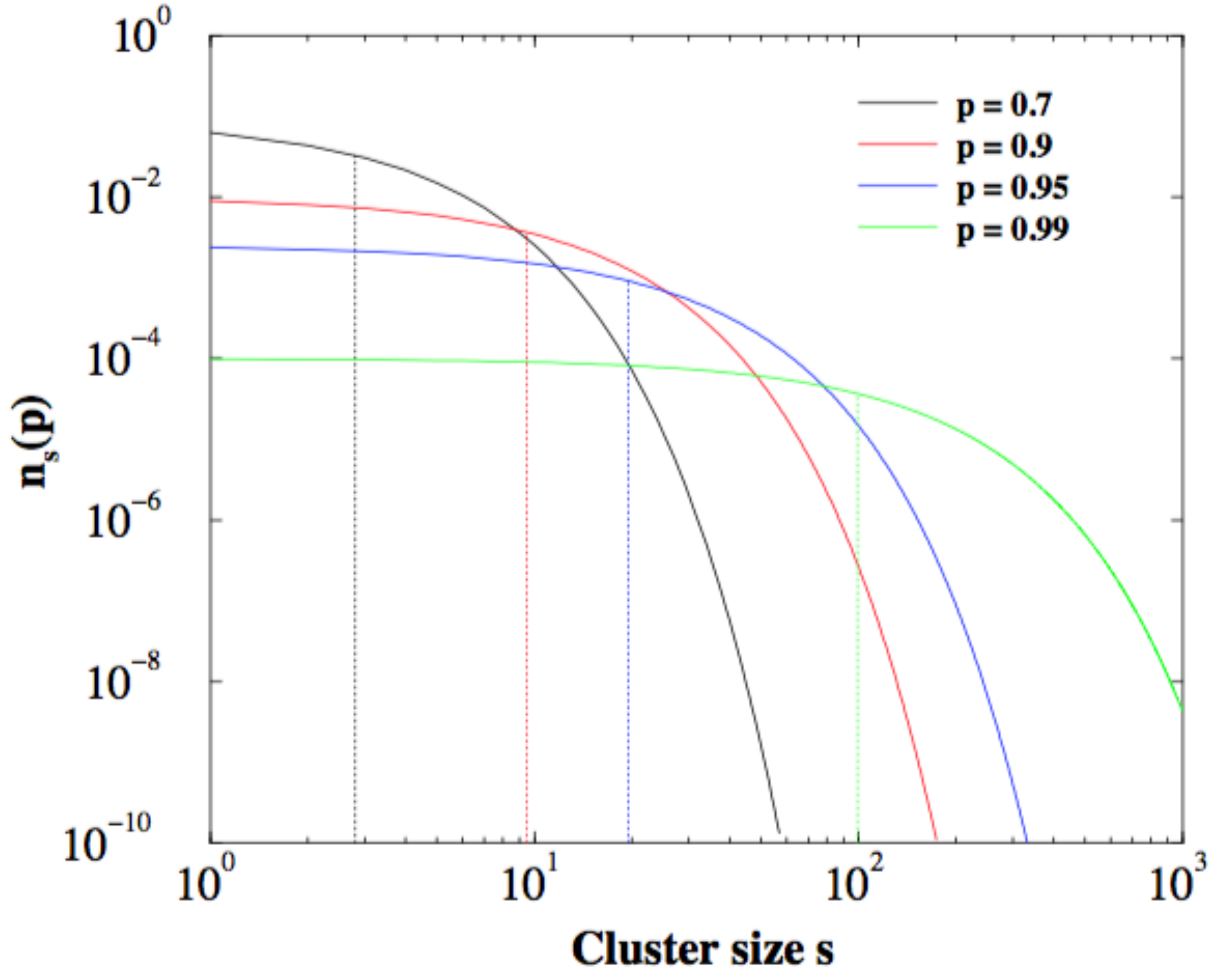


Now n(s,p) is simple to compute: $n(s,p) = p^s(1-p)^2$

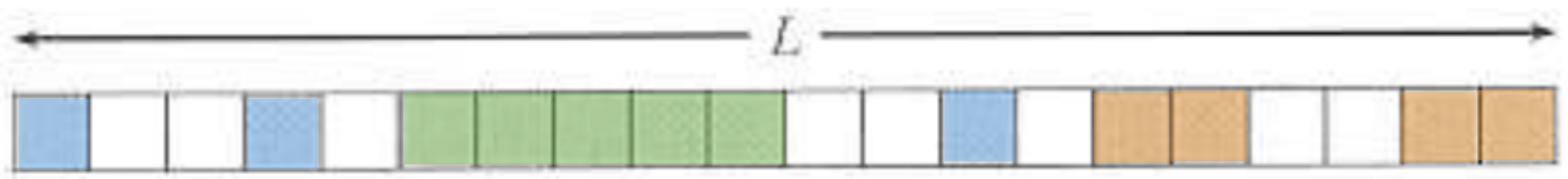
Behavior of n(s,p) for large s: **there is a cutoff!**

very unlikely to find a cluster of size much larger than the cutoff value

Cutoff depends on p: the larger p, the larger the cutoff value

Ex. 1D site percolation

Cluster number density



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⟶ We want to compute $n(s,p)$ = **number of clusters of size s, per unit length**

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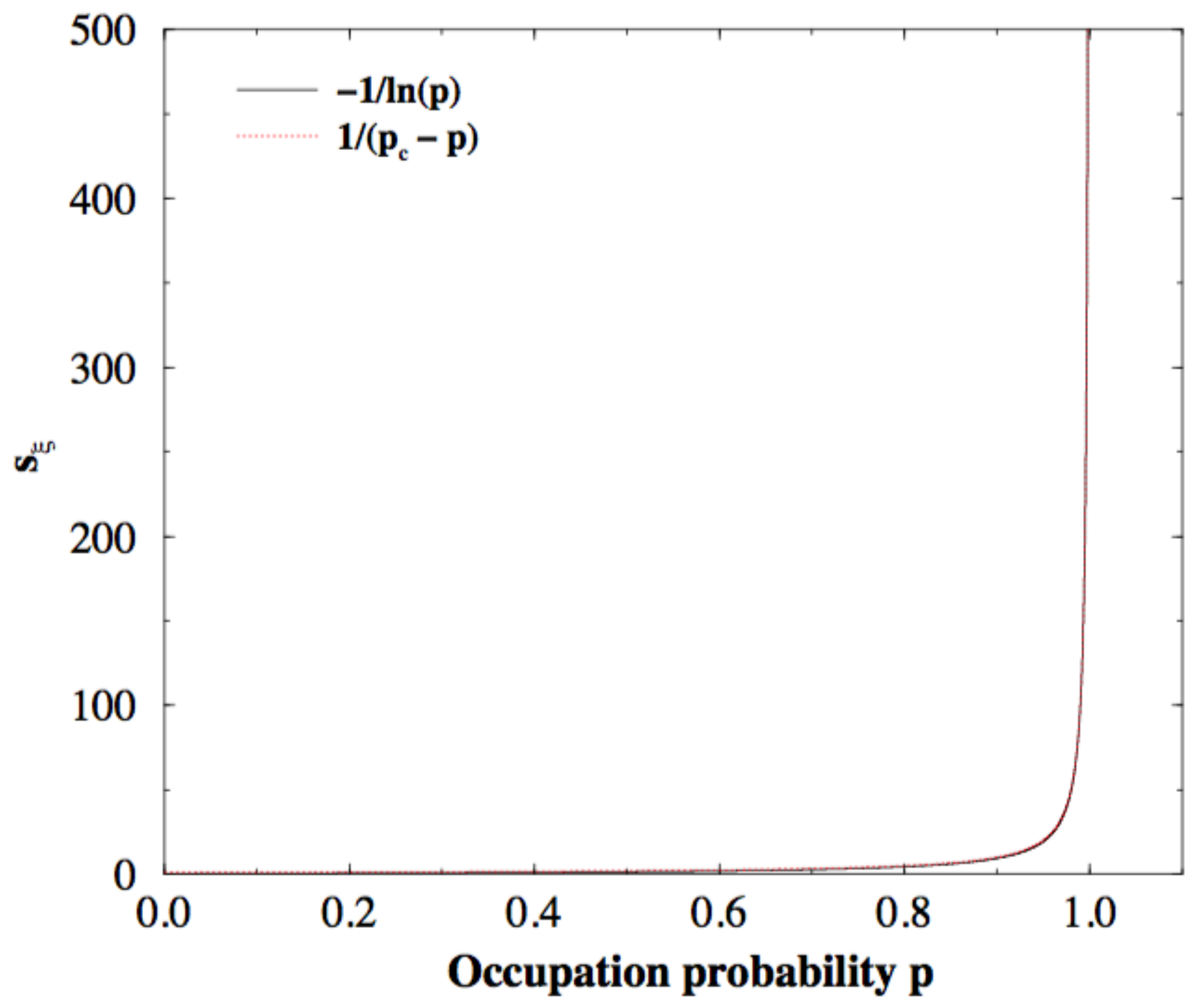
Now n(s,p) is simple to compute: $$n(s,p) = p^s(1-p)^2$$

$$= (p_c - p)^2 \exp(-\frac{s}{s_\xi})$$

s_ξ = **cutoff cluster size** = characteristic cluster size

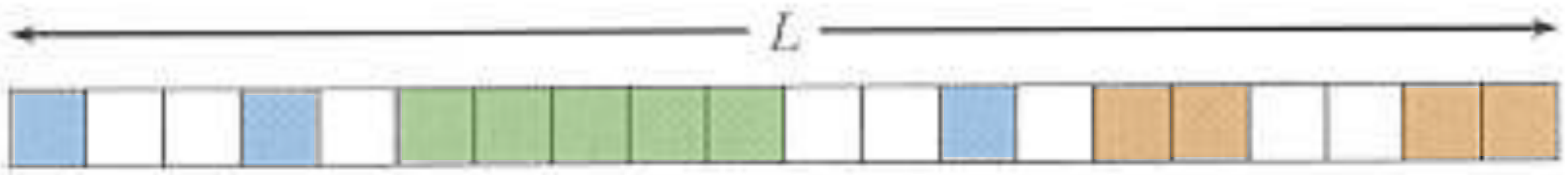
$$s_\xi = \frac{-1}{\ln(p)}$$

s_ξ diverges as p —> pc



Ex. 1D site percolation

Cluster number density



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$\longrightarrow$ We want to compute $n(s,p)$ = **number of clusters of size s, per unit length**

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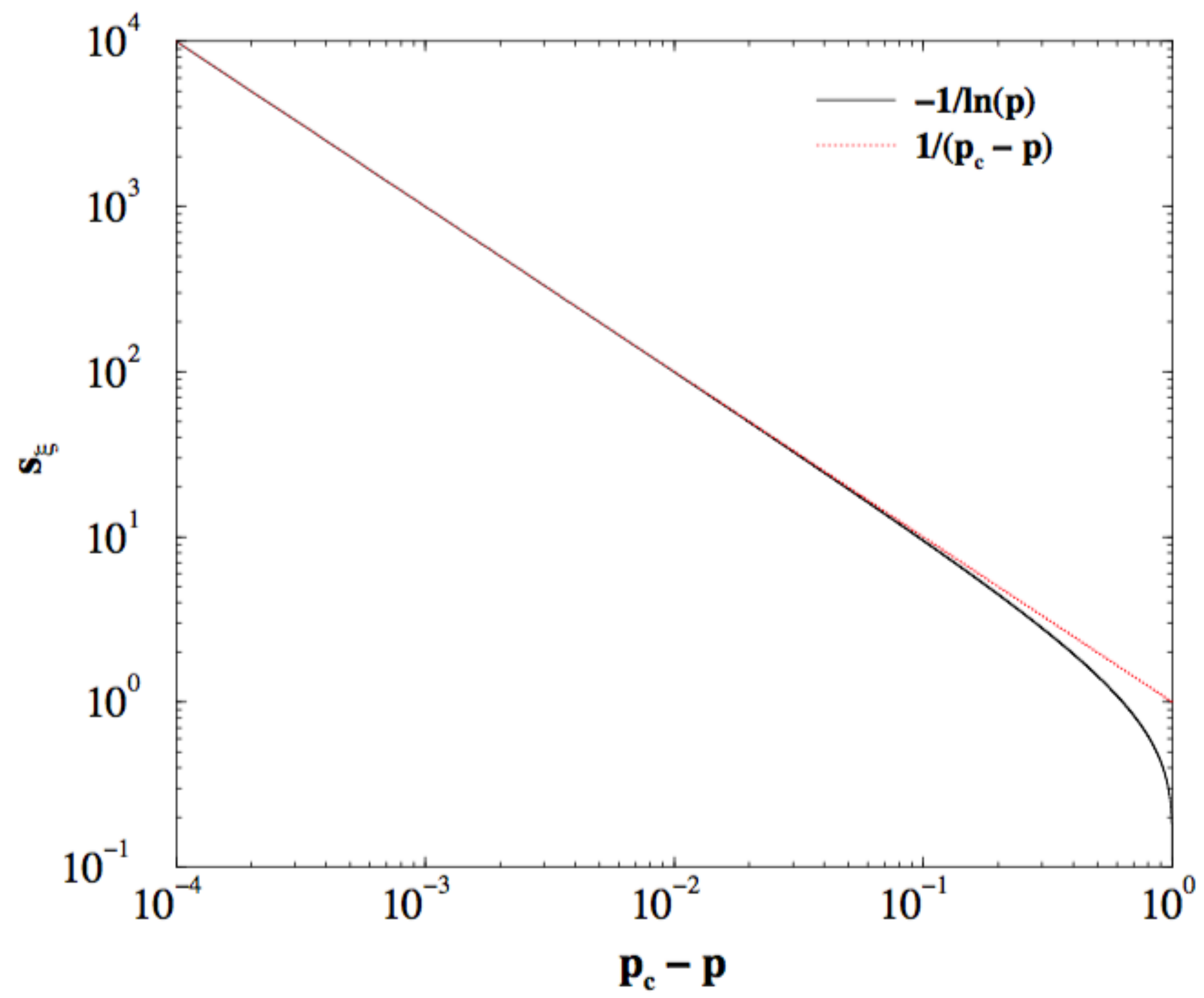

Now n(s,p) is simple to compute:

$$n(s,p) = p^s(1-p)^2$$
$$= (p_c - p)^2 \exp(-\frac{s}{s_\xi})$$

s_ξ = **cutoff cluster size** = characteristic cluster size

$\ln(1-x) = -x + o(x)$

$$s_\xi = \frac{-1}{\ln(p)} \xrightarrow[p\to p_c]{} \frac{1}{p_c - p} = (p_c - p)^{-1}$$

Cutoff cluster size s_ξ diverges for p → p_c as a power law!

$s_\xi \xrightarrow[p\to p_c]{} |p_c - p|^{-1/\sigma}$ Critical exponent: $\sigma = 1$

Ex. 1D site percolation

Useful equality

$s\,n(s,p)$ = probability that a given site belong to a cluster of size **s**

Probability that a site belongs to any cluster is: $$\sum_s s\,n(s,p) = p$$ for $p < p_c$

Valid at any dimension, and for any lattice size

Ex. 1D site percolation

Mean cluster size

We are taking an occupied site, because if not, the mean will be biased by the value of 0 for all the sites that are not occupied

Take a site that is occupied: What is the average size of the cluster it belongs to?

Probability that a site is occupied: $p = \sum_s s\, n(s,p)$

Probability that a site belongs to a cluster of size s: $s\, n(s,p)$

Ex. 1D site percolation

Mean cluster size

Take a site that is occupied: What is the average size of the cluster it belongs to?

Probability that a site is occupied: $p = \sum_s s\, n(s,p)$

Probability that a site belongs to a cluster of size s: $s\, n(s,p)$

Probability that an occupied site belongs to a cluster of size s: $w(s,p) = \frac{s\, n(s,p)}{p} = \frac{s\, n(s,p)}{\sum_s s\, n(s,p)}$

Ex. 1D site percolation

Mean cluster size

Take a site that is occupied: how large is the cluster it belongs to?

Probability that a site is occupied: $p = \sum_s s\, n(s,p)$

Probability that a site belongs to a cluster of size s: $s\, n(s,p)$

Probability that an occupied site belongs to a cluster of size s: $w(s,p) = \frac{s\, n(s,p)}{p} = \frac{s\, n(s,p)}{\sum_s s\, n(s,p)}$

Mean cluster size: $\chi(p) = \sum_{s=1}^{\infty} s\, w(s,p)$

Ex. 1D site percolation

Mean cluster size

Take a site that is occupied: how large is the cluster it belongs to?

Probability that a site is occupied: $p = \sum_s s\, n(s,p)$

Probability that a site belongs to a cluster of size s: $s\, n(s,p)$

Probability that an occupied site belongs to a cluster of size s: $w(s,p) = \frac{s\, n(s,p)}{p} = \frac{s\, n(s,p)}{\sum_s s\, n(s,p)}$

Mean cluster size:

$$\chi(p) = \sum_{s=1}^{\infty} s\, w(s,p) = \sum_{s=1}^{\infty} \frac{s^2\, n(s,p)}{p} = \frac{(1-p)^2}{p} \sum_{s=1}^{\infty} s^2 p^s$$

$$= \frac{(1-p)^2}{p} (p\frac{d}{dp})(p\frac{d}{dp})(\sum_{s=1}^{\infty} p^s)$$

$$n(s,p) = p^s (1-p)^2$$

$$\chi(p) = \frac{1+p}{1-p} = \frac{p_c + p}{p_c - p} \underset{p \to p_c}{\longrightarrow} \frac{2\, p_c}{p_c - p} \propto (p_c - p)^{-1}$$

Ex. 1D site percolation

Mean cluster size

Take a site that is occupied: how large is the cluster it belongs to?

Probability that a site is occupied: $p = \sum_s s\, n(s,p)$

Probability that a site belongs to a cluster of size s: $s\, n(s,p)$

Probability that an occupied site belongs to a cluster of size s: $w(s,p) = \dfrac{s\, n(s,p)}{p} = \dfrac{s\, n(s,p)}{\sum_s s\, n(s,p)}$

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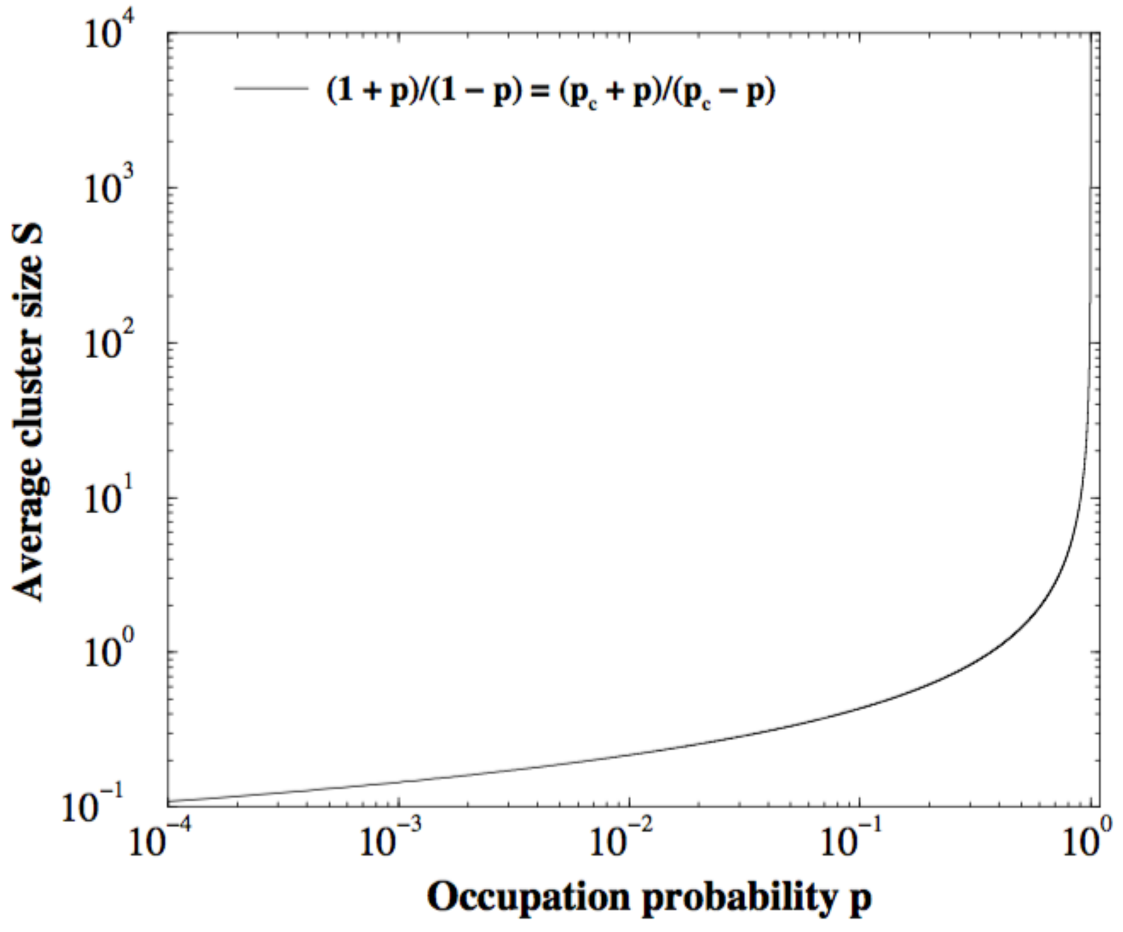

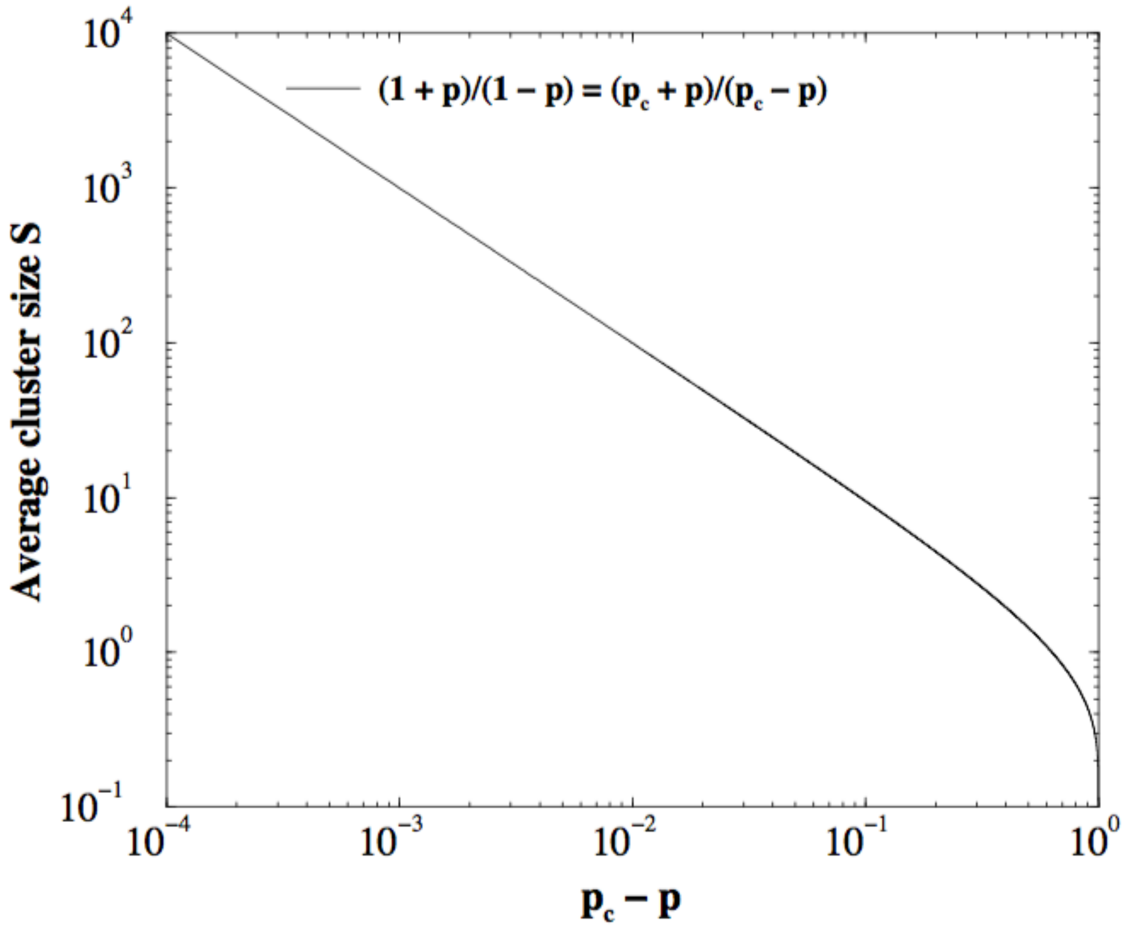

In 1d, the mean cluster size diverges like a power law in the quantity (p_c – p) when p → p_c

$\chi(p) \propto (p_c - p)^{-\gamma}$ Critical exponent: $\gamma = 1$

Ex. **1D site percolation**

Correlation function

Correlation function: probability that a site at a distance r apart from an occupied site belongs to the same cluster.

$$g(0) =$$

In 1d, for a site at position r to be occupied and belong to the same (finite) cluster, this site and the (r – 1) intermediate sites must be occupied:

$$g(r) =$$

Ex. 1D site percolation

Correlation function

Correlation function: probability that a site at a distance r apart from an occupied site belongs to the same cluster.

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$$g(r) = p^r$$

Ex. 1D site percolation

Correlation function

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In 1d, for a site at position r to be occupied and belong to the same (finite) cluster, this site and the (r – 1) intermediate sites must be occupied:

$$g(r) = p^r$$

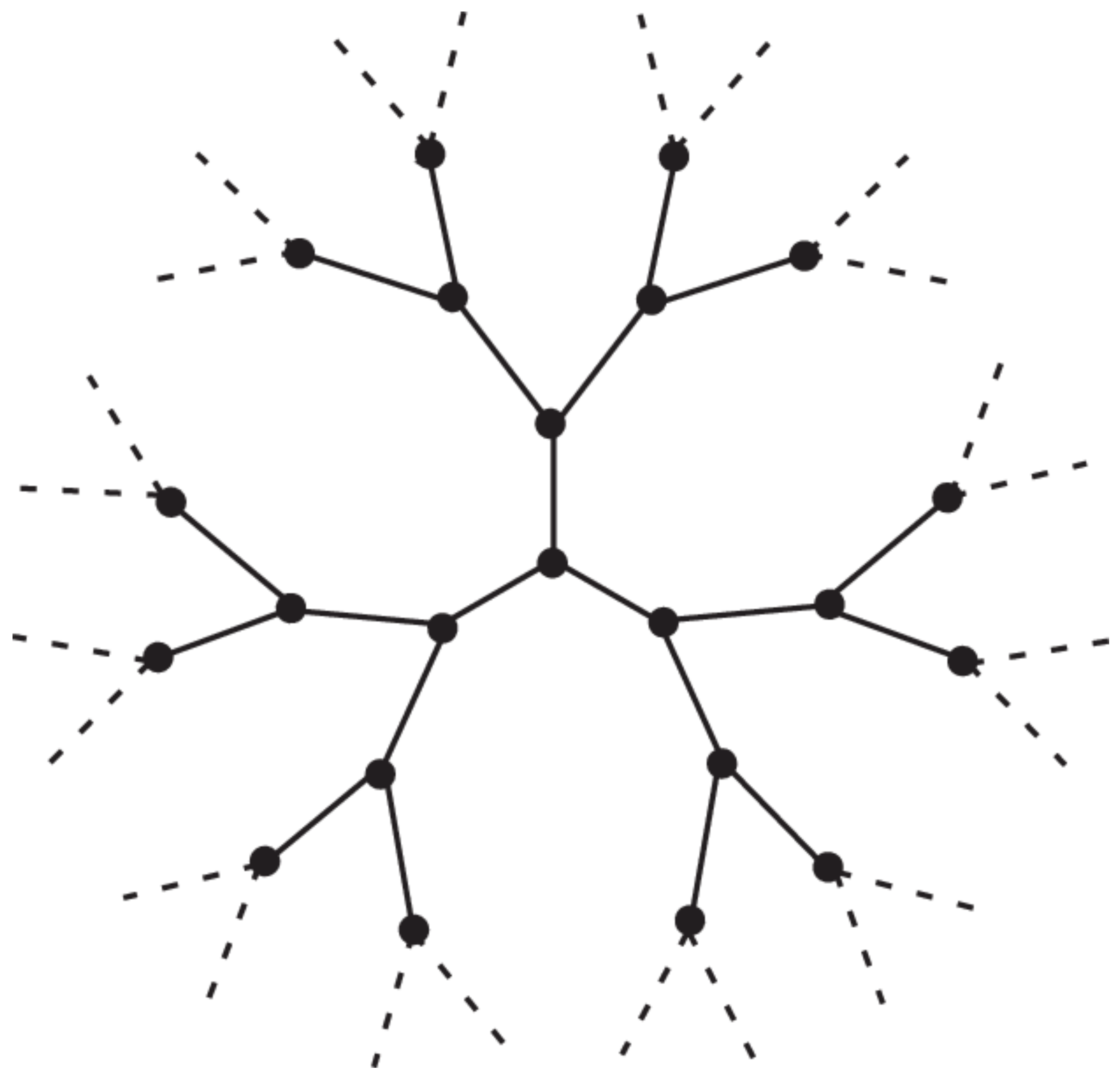
$$g(r) = \exp(r\ln(p)) = \exp(-r/\xi)$$

Correlation length: $\xi = -\frac{1}{\ln(p)} = -\frac{1}{\ln(p_c - (p_c - p))} \longrightarrow \frac{1}{(p_c - p)} = (p_c - p)^{-1}$

Correlation length diverges for p—> pc:

$$\xi = |p_c - p|^{-\nu} \qquad \text{critical exponent} \quad \nu = 1$$

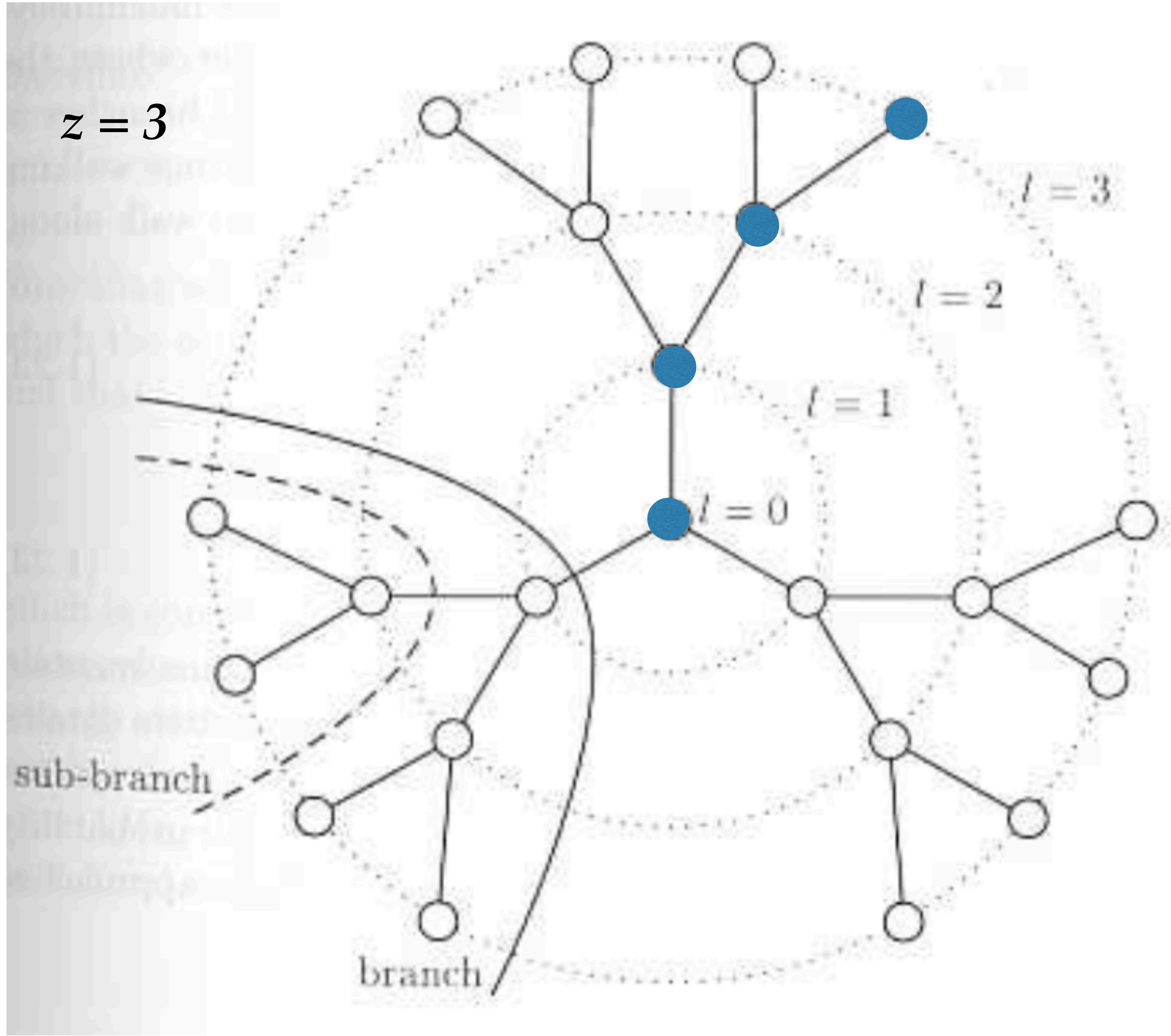
4) Percolation on Bethe Lattice



Bethe Lattice

Bethe Lattice: Infinite tree in which each site has exactly z neighbours

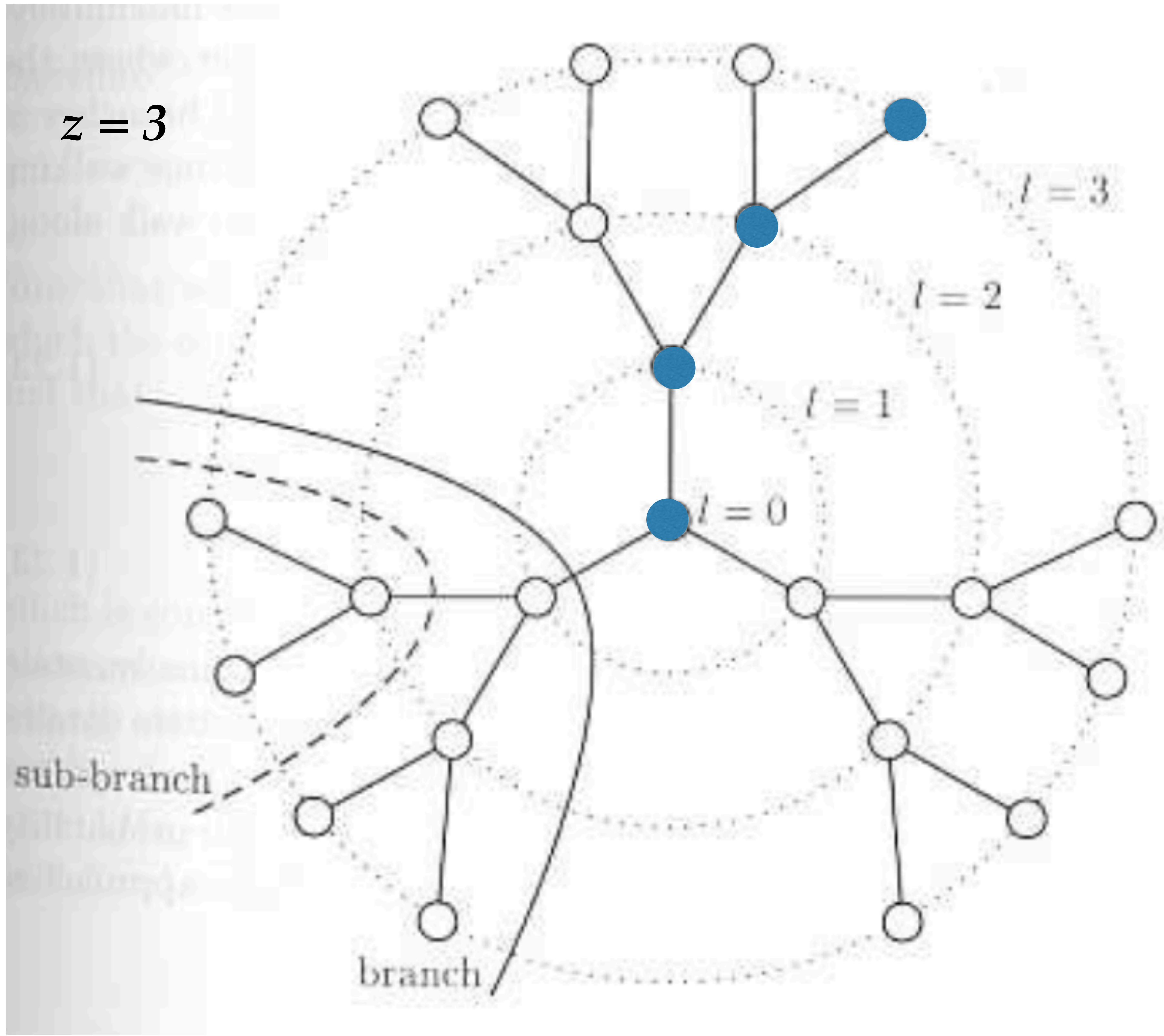
Observation: — it is a cycle free
— all nodes are the same, there is no special root



$z = 2$ = **1d lattice**

Bethe Lattice

Bethe Lattice (Cayley tree): Tree in which each site has exactly z neighbours



Observation: — it is a cycle free
— all nodes are the same, there is no special root

$z = 2$ = 1d lattice

<u>Critical value of p:</u>

Take an occupied site on a path, place it "at the center"

Average number of nodes that can be in the next step of the path = ?

Bethe Lattice

Bethe Lattice (Cayley tree): Tree in which each site has exactly z neighbours

Observation: — it is a cycle free
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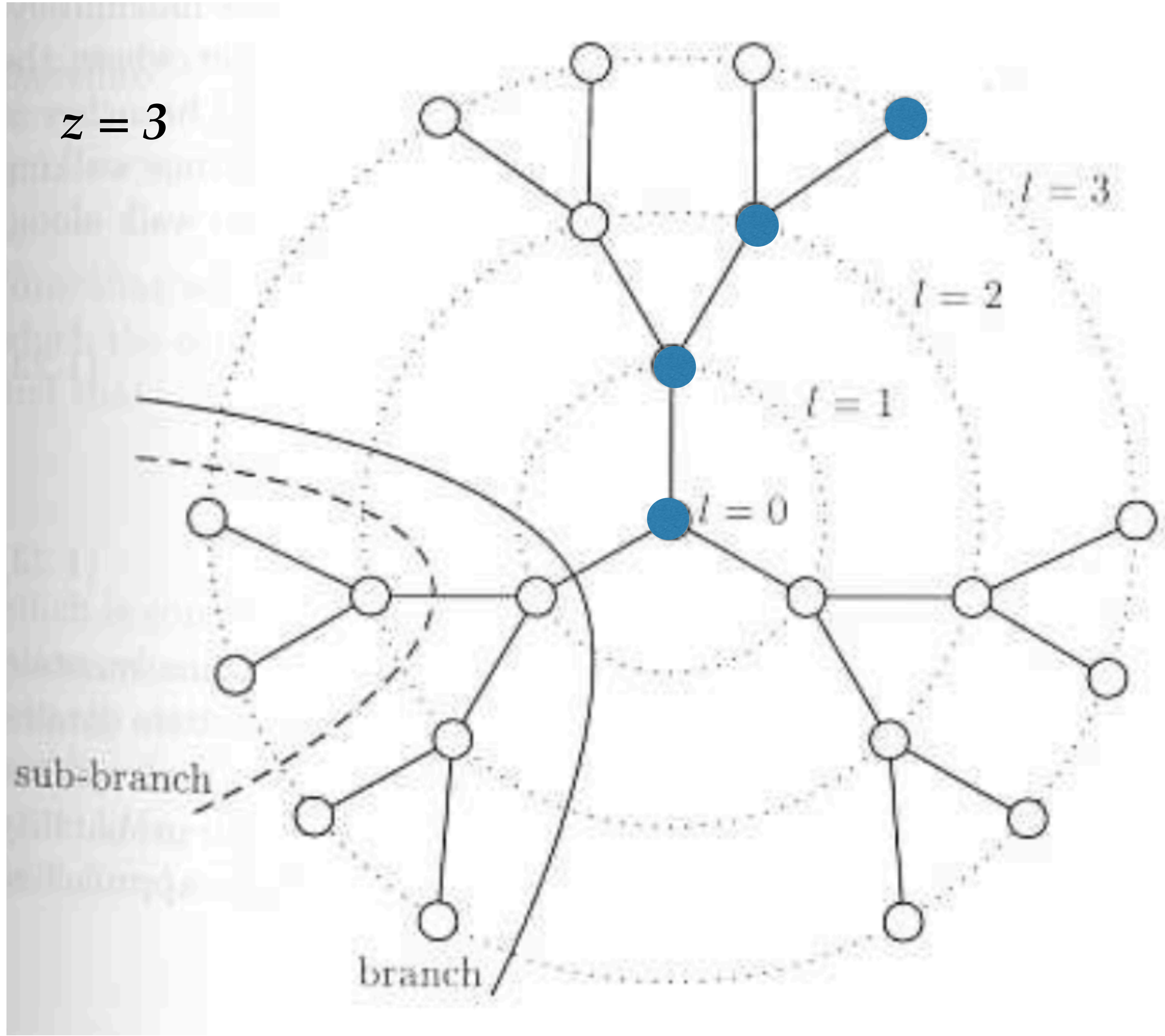

$z = 2$ **= 1d lattice**

Critical value of p:

Take an occupied site on a path, place it "at the center"

Average number of nodes that can be in the next step of the path = ?

$= p\,(z\text{-}1)$

Bethe Lattice

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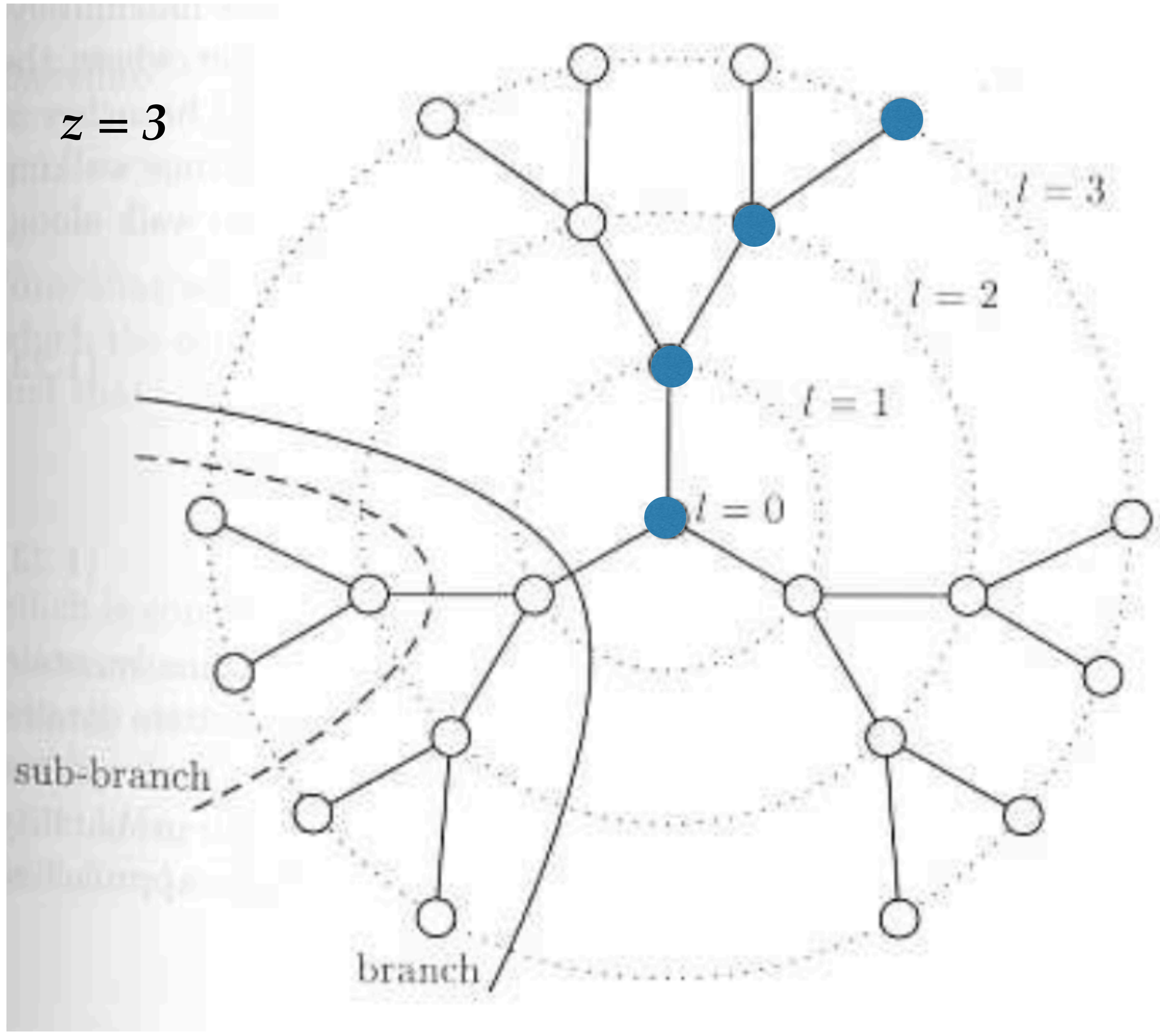


$z = 2$ = 1d lattice

Critical value of p:

Take an occupied site on a path, place it "at the center"

Average number of nodes that can be in the next step of the path = ?

$= p\,(z\text{-}1)$

For the path to percolate we need at least $p\,(z\text{-}1) >= 1$

$p_c = 1 / (z - 1)$

For $z = 2$: $p_c = 1$ —> 1D lattice

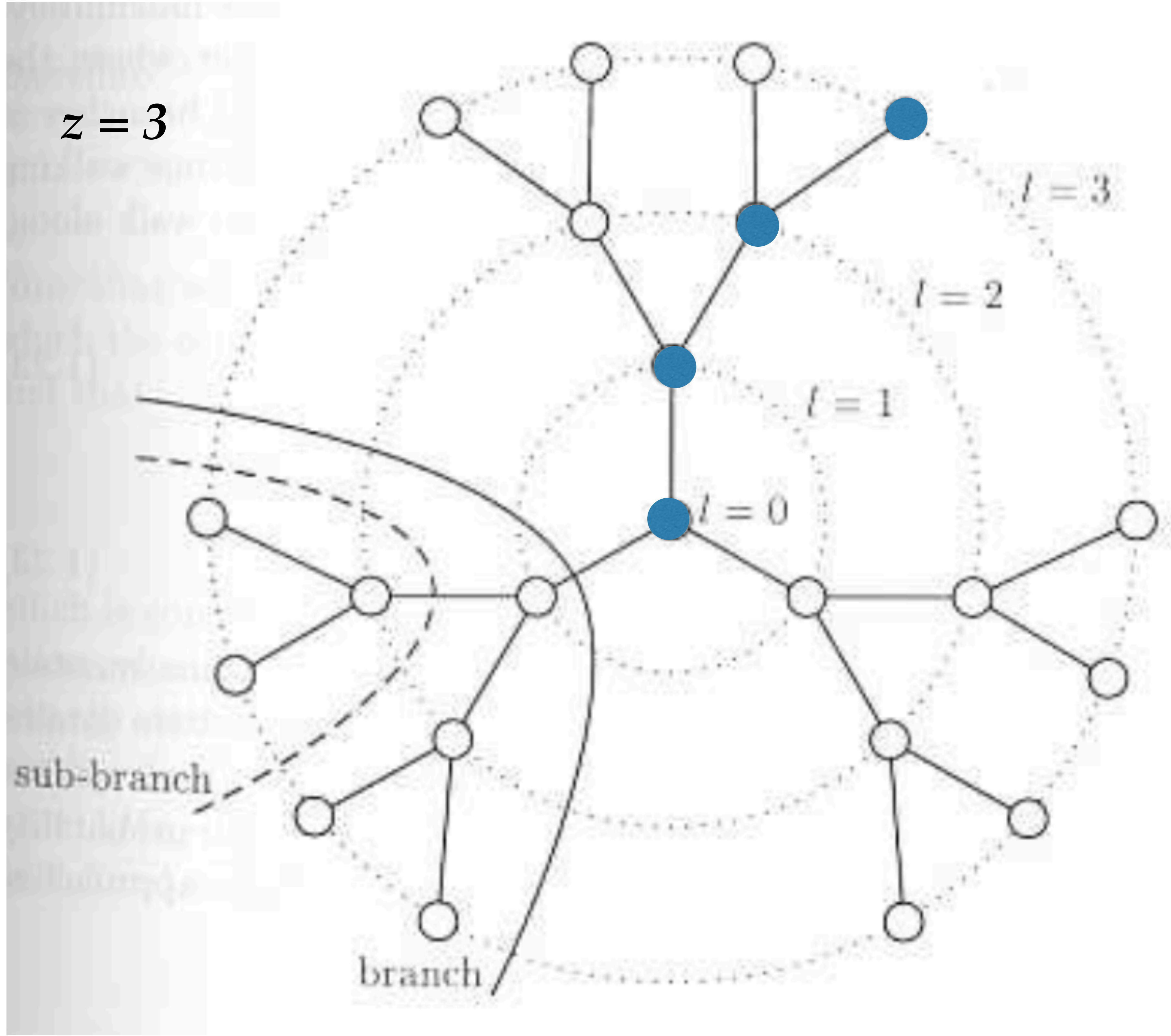
For $z > 2$: $p_c < 1$

The value of p_c depends on the lattice structure

Bethe Lattice

Bethe Lattice (Cayley tree): Tree in which each site has exactly z neighbours

Observation: — it is a cycle free
— all nodes are the same, there is no special root



$z = 2$ **= 1d lattice**

Mean Cluster size: ?

Bethe Lattice

Bethe Lattice (Cayley tree): Tree in which each site has exactly z neighbours

Observation: — it is a cycle free
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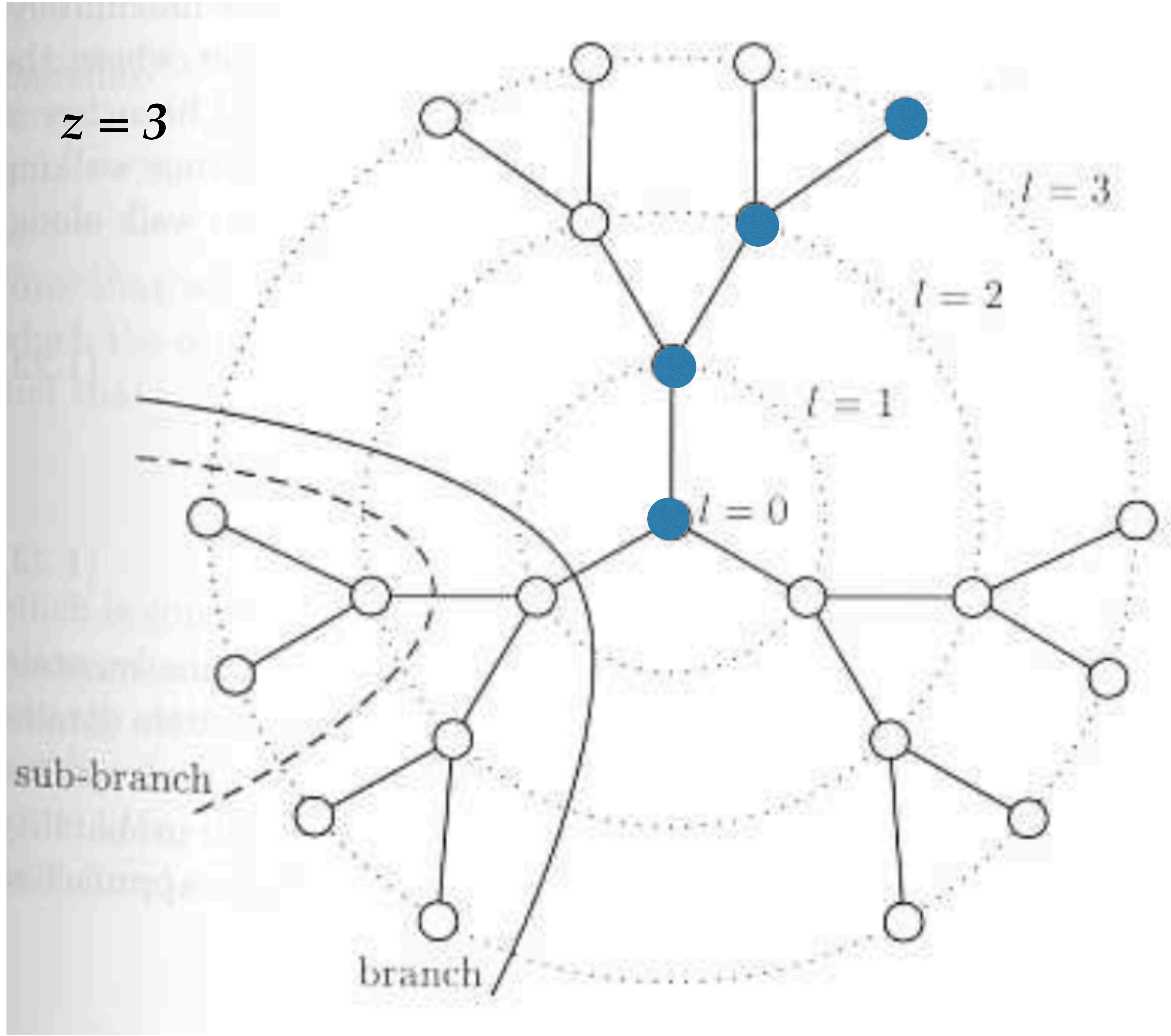
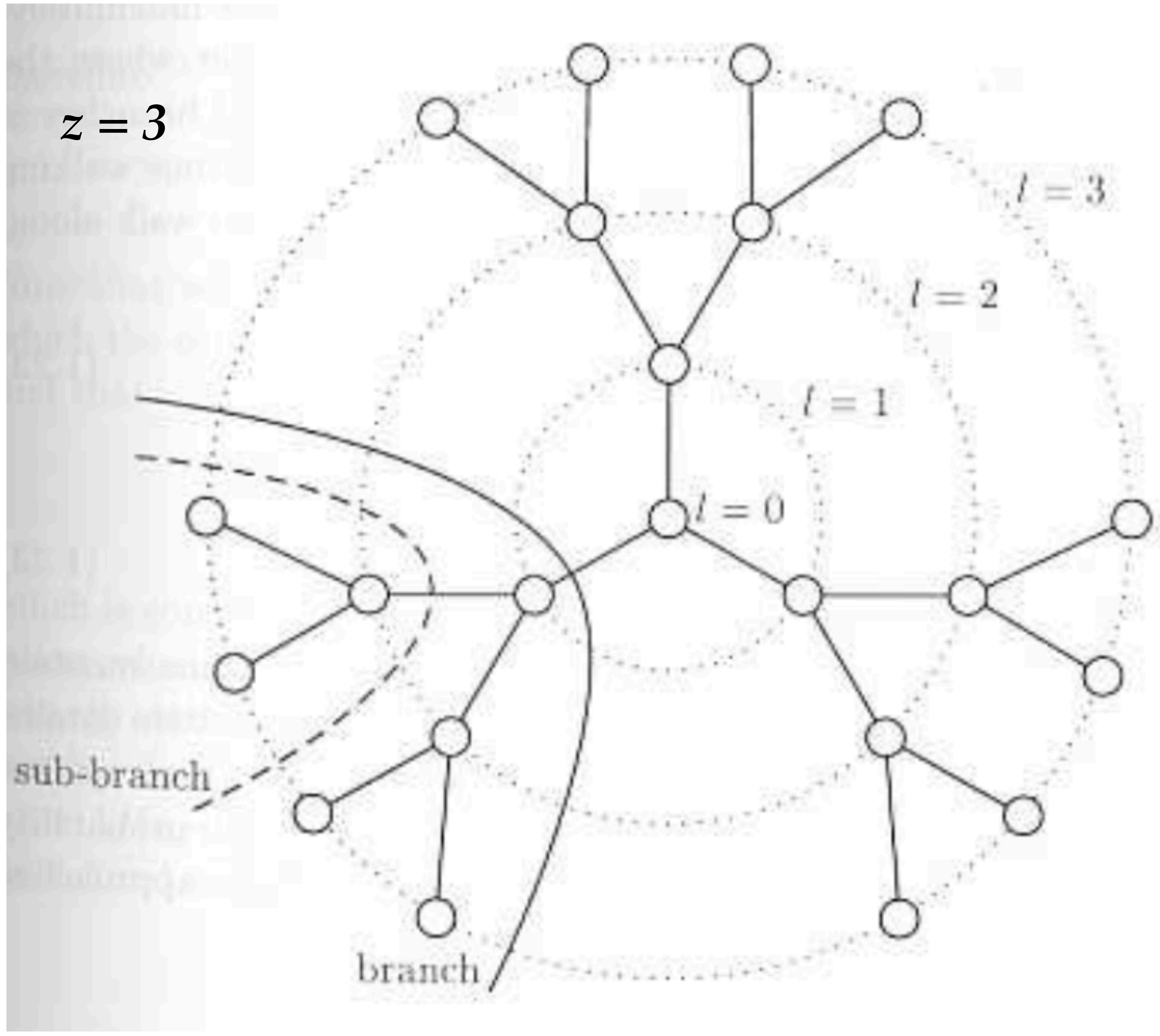


$z = 2$ **= 1d lattice**

Mean Cluster size: ? $\gamma = 1$ $\chi(p) \propto |p - p_c|^{-\gamma}$

Bethe Lattice



The value of p_c depends on the lattice structure

$$p_c = 1 / (z - 1)$$

Interesting quantities are also power laws;

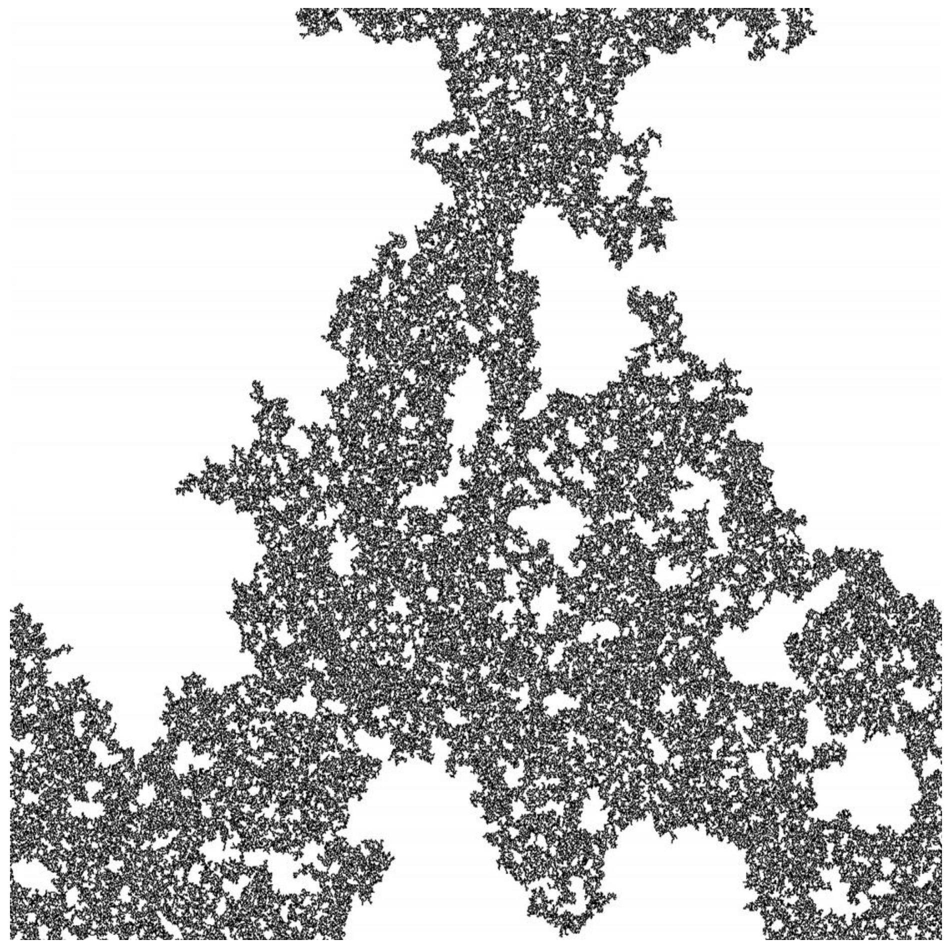
which defines critical exponents:

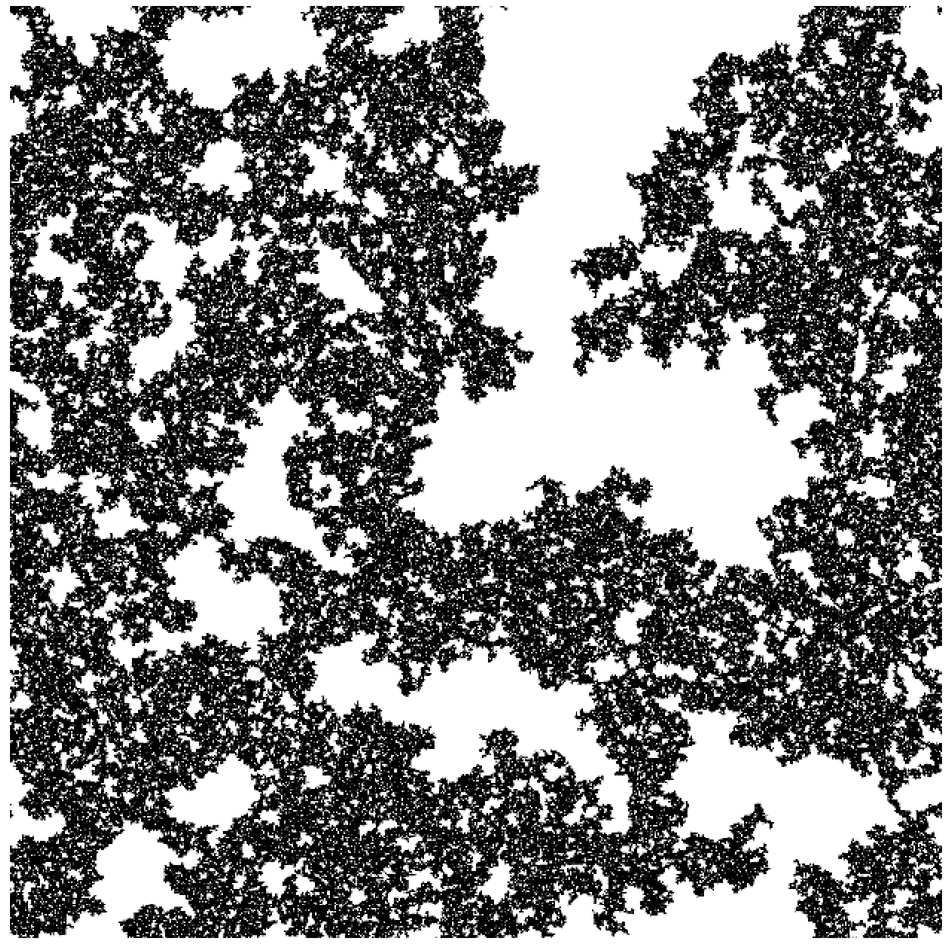
Exponent	Quantity	Bethe	
β	$P_\infty(p) \propto (p - p_c)^\beta$	1	Probability that a site belongs to percolating cluster
γ	$\chi(p) \propto \|p - p_c\|^{-\gamma}$	1	Mean cluster size
ν	$\xi(p) \propto \|p - p_c\|^{-\nu}$	1/2	Correlation length
σ	$\sigma_\xi(p) \propto \|p - p_c\|^{-1/\sigma}$	1/2	Cutoff cluster size
D	$\sigma_\xi \propto \xi^D$	4	

3) Properties at the critical point; Universality
   a) Different type of percolation problems
   b) Pc is not universal
   c) Critical exponents are universal

Properties at the Critical Point

Universality

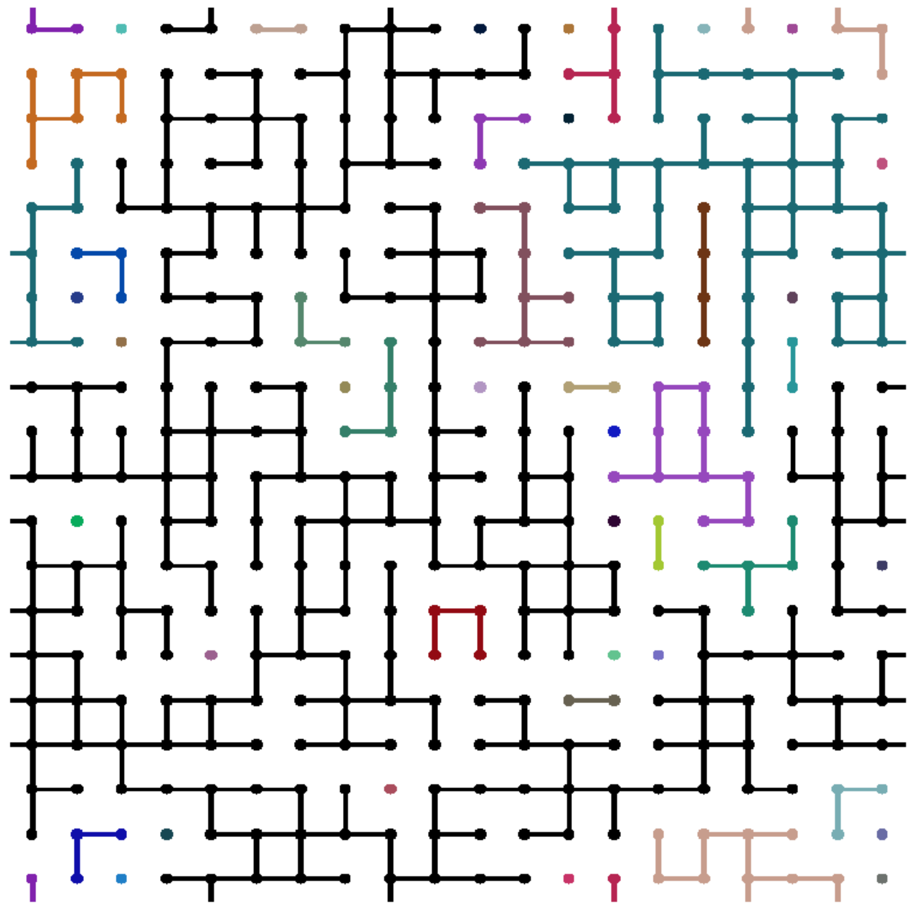




Percolation and Universality

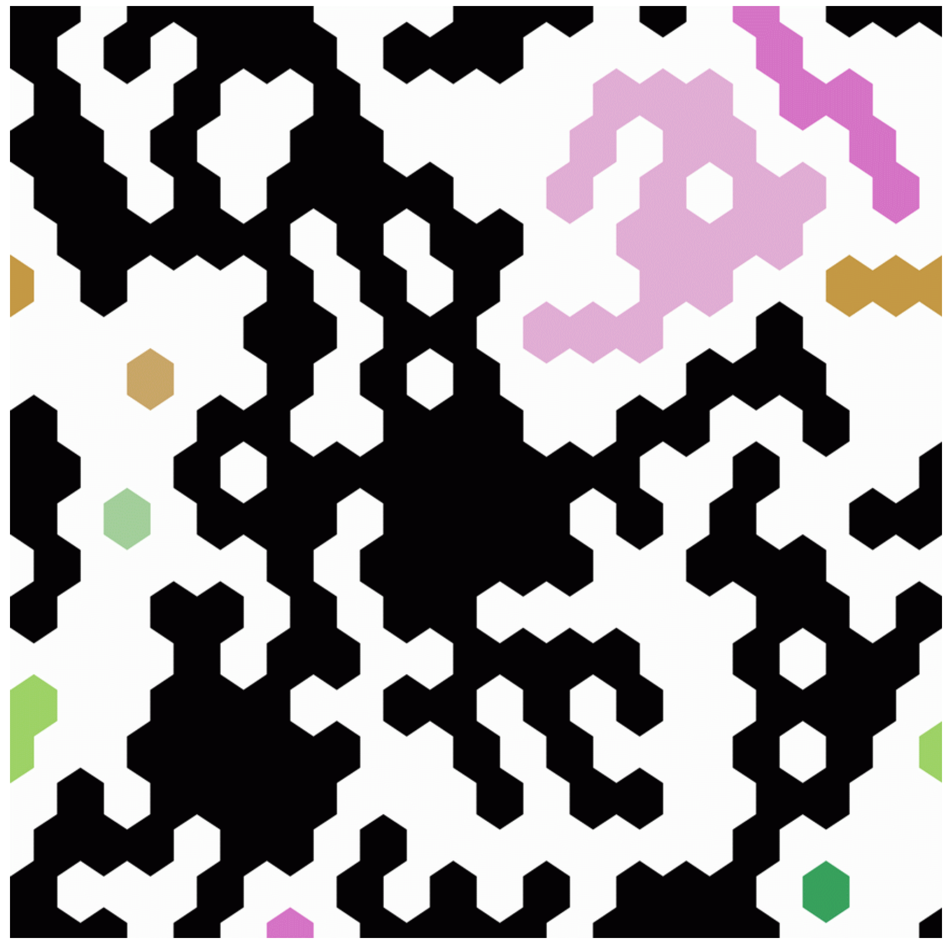
Bond percolation on a lattice:

bonds are randomly occupied with probability p



Site percolation on a triangular lattice:

hexagonal sites that are occupied with probability p



Percolation and Universality

Bond percolation on a lattice:

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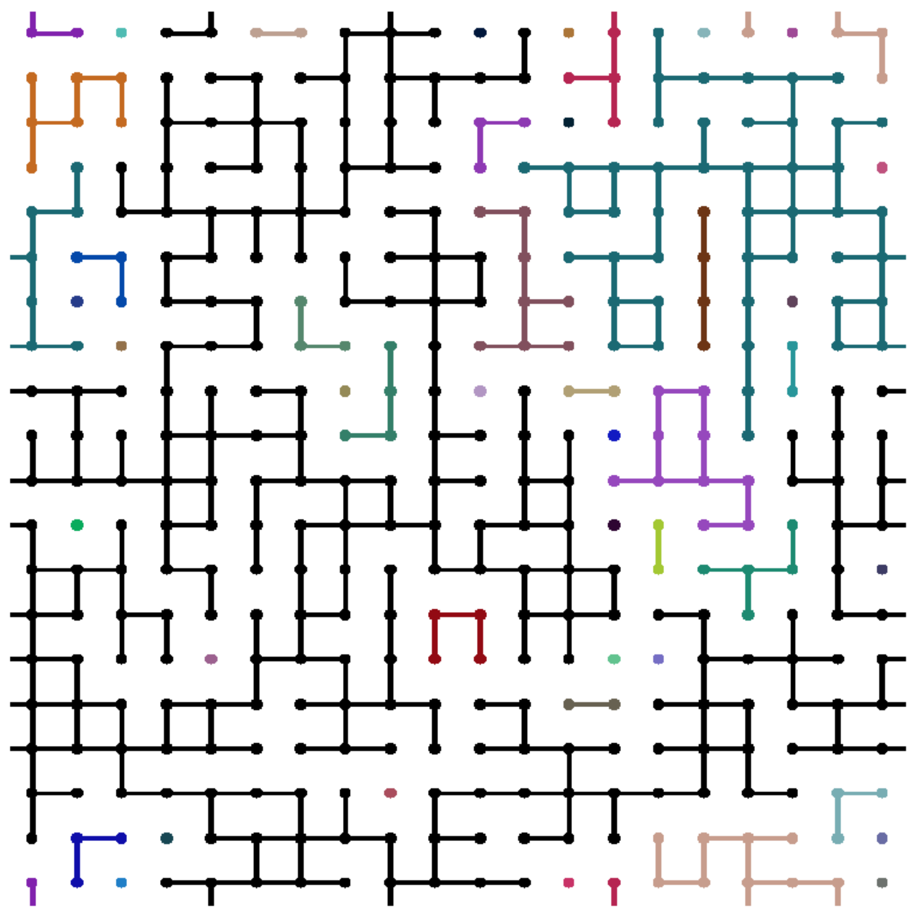
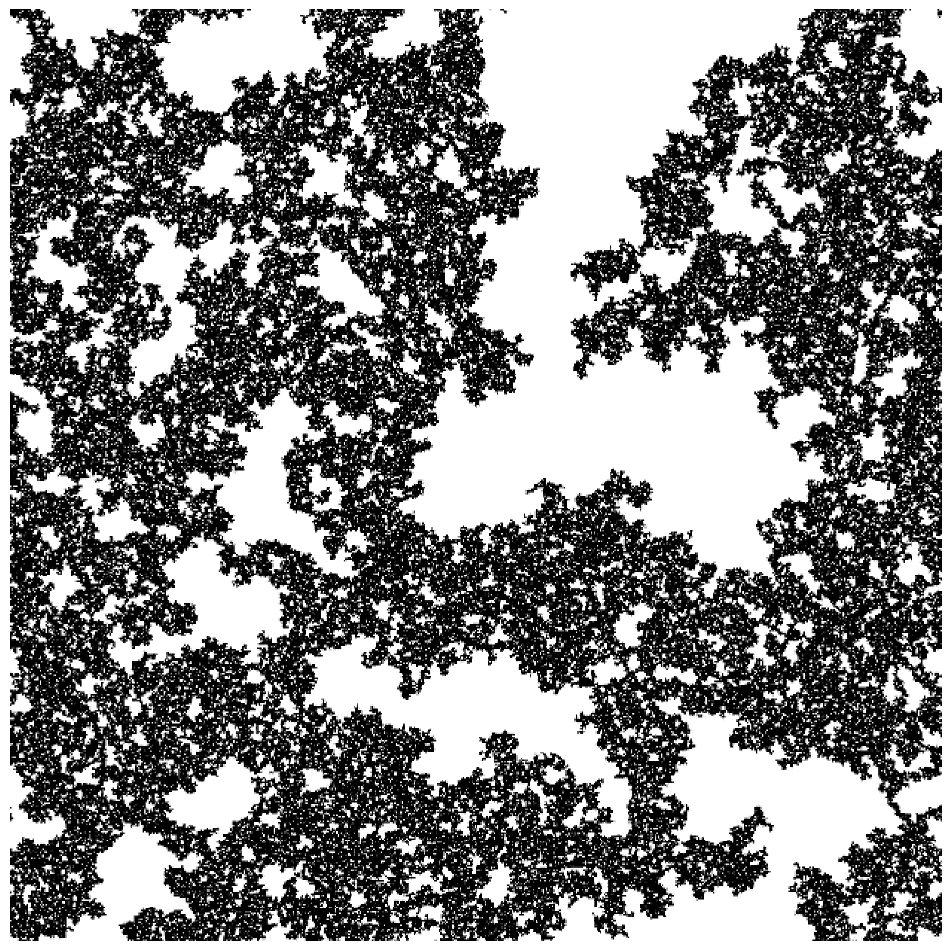
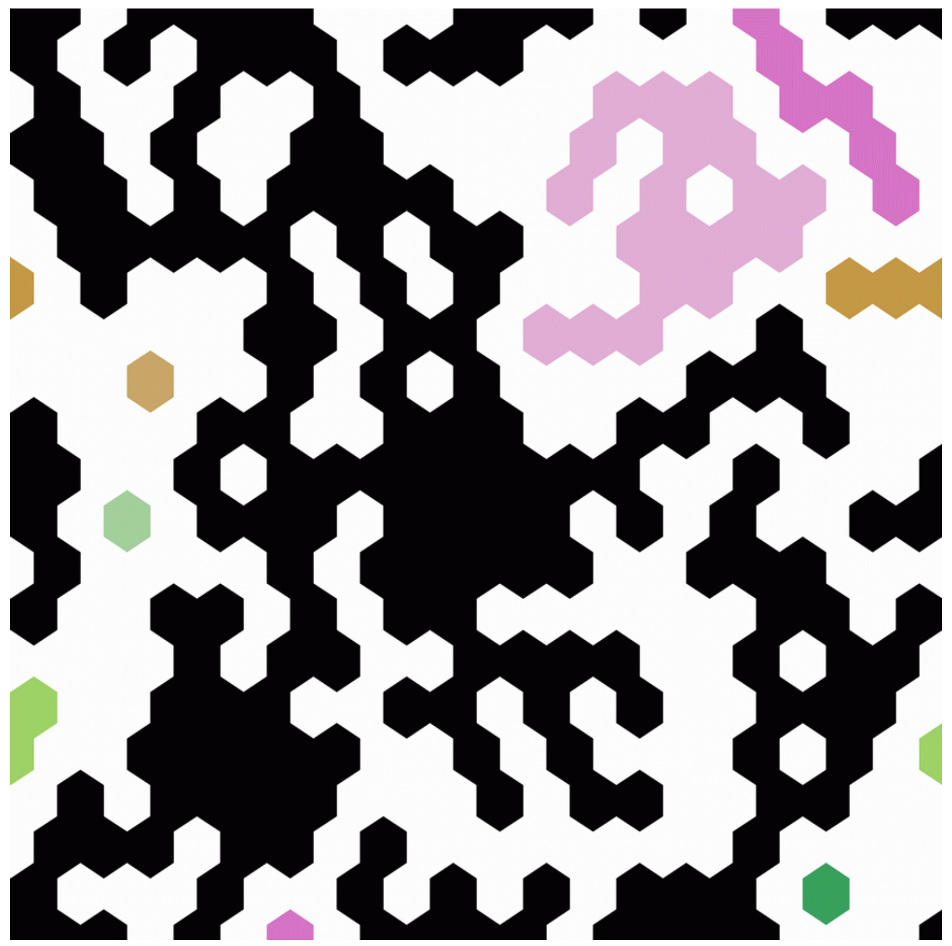




Infinite cluster
at $p_c = 0.5$

Site percolation on a triangular lattice:

hexagonal sites that are **occupied** with probability **p**



Percolation and Universality

Bond percolation on a lattice:

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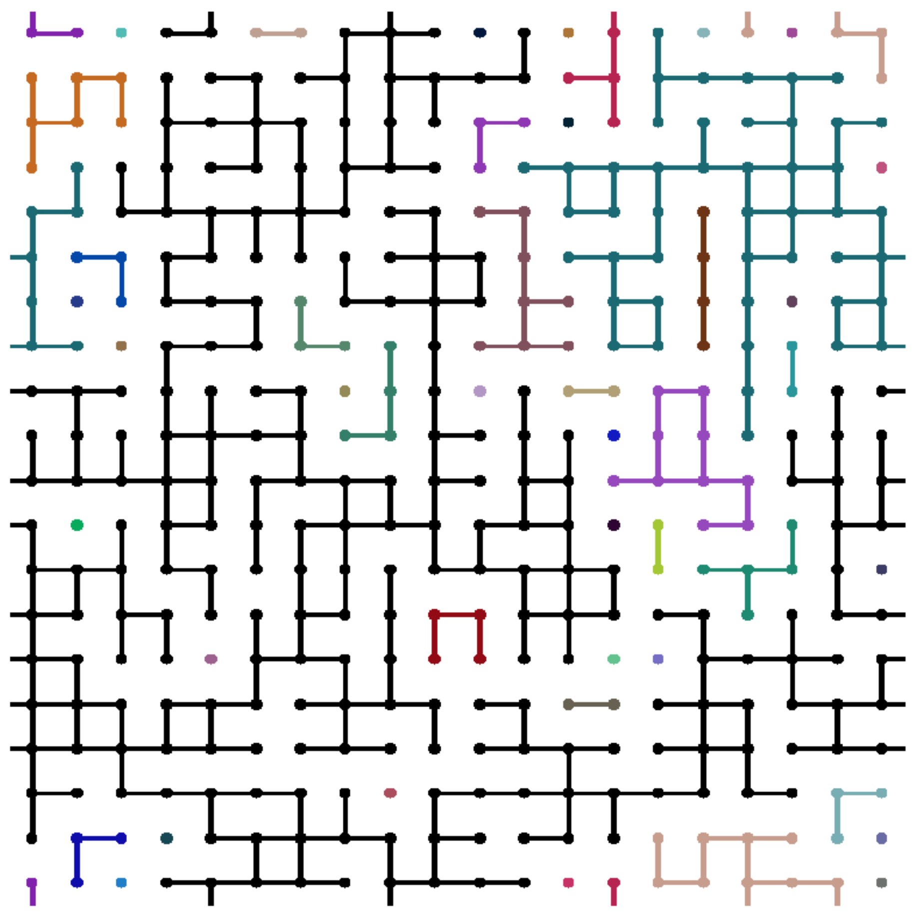
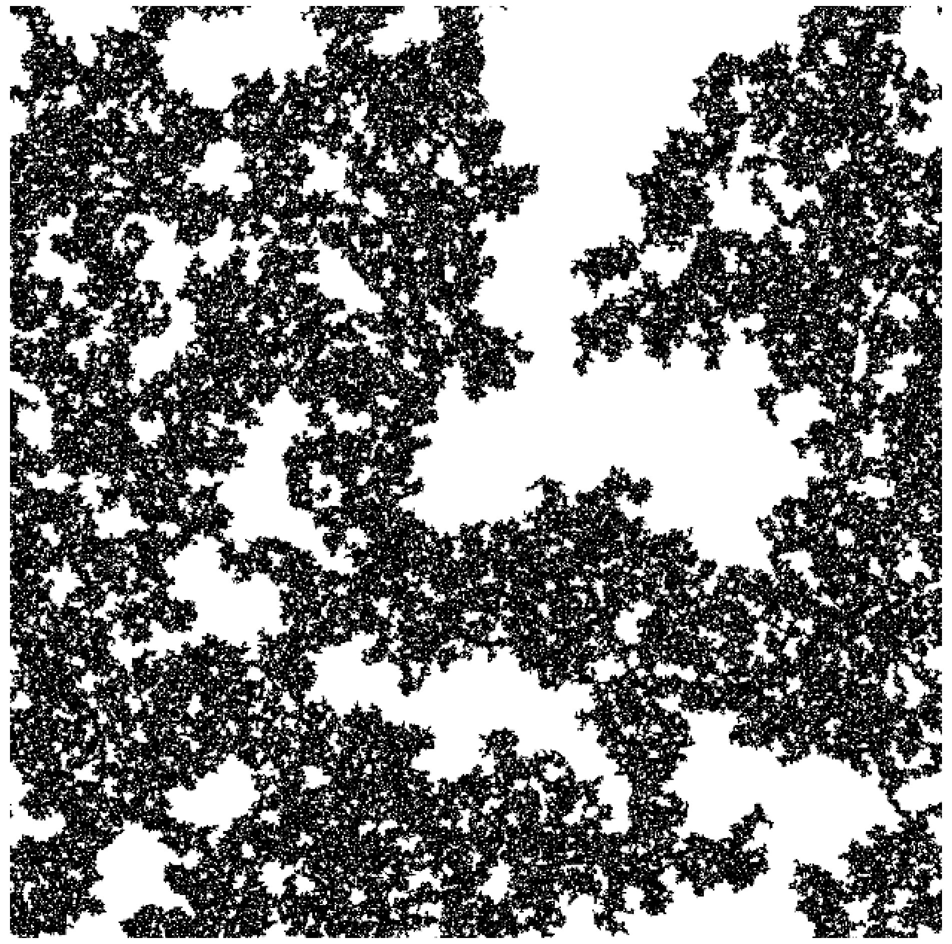
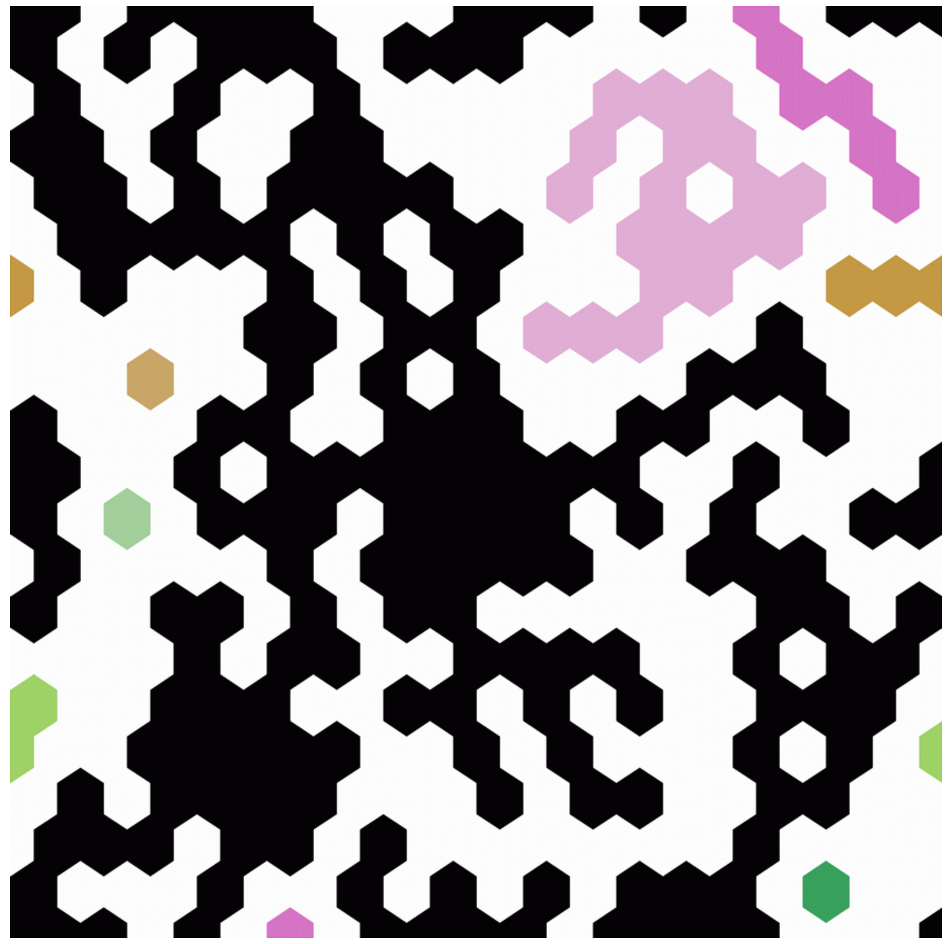


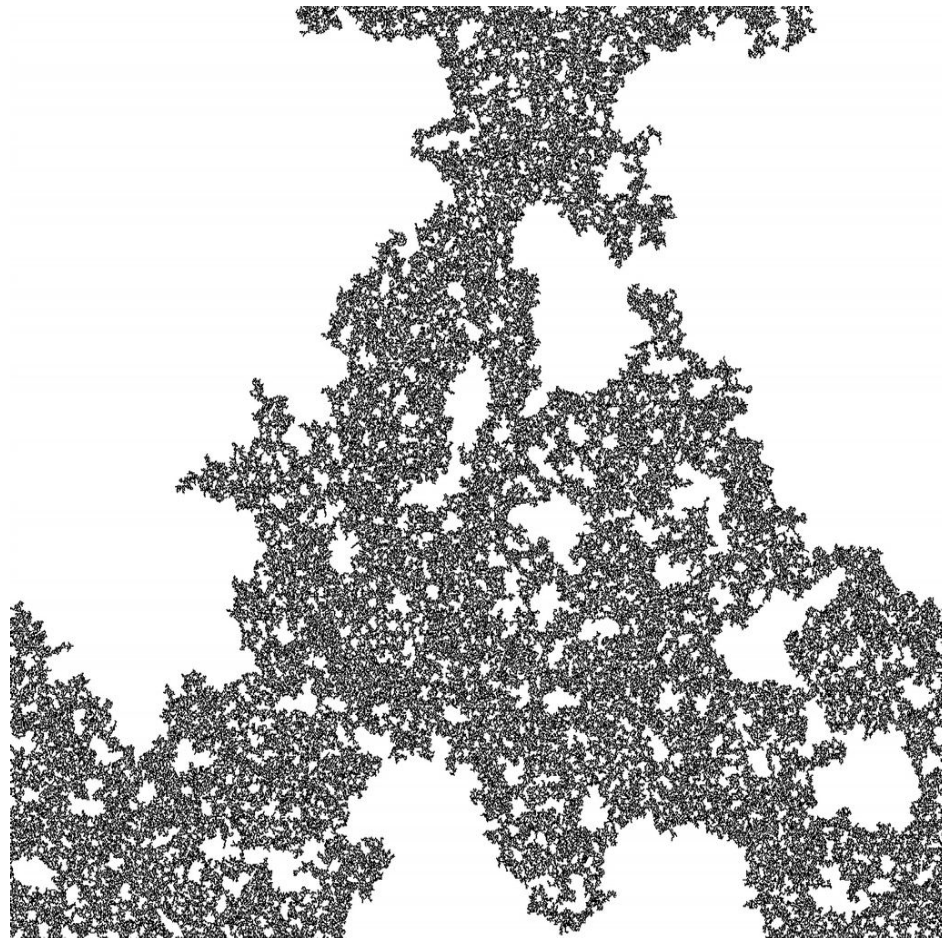


Infinite cluster at p_c

Site percolation on a triangular lattice:

hexagonal sites that are **occupied** with probability **p**



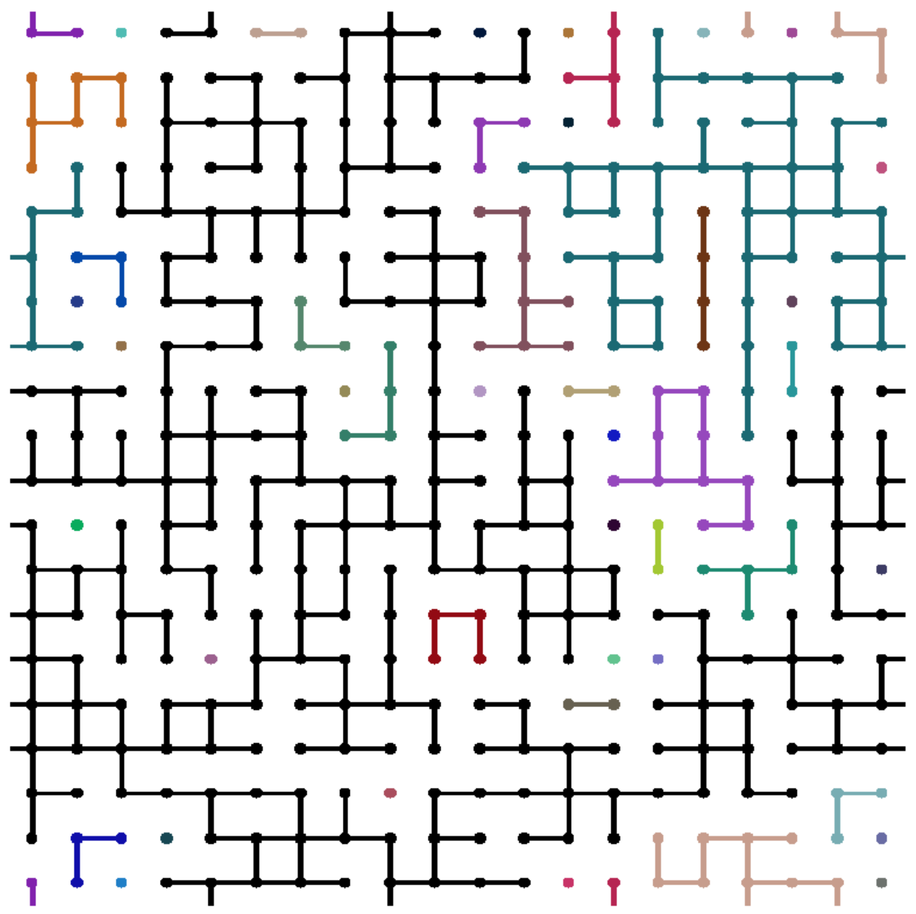


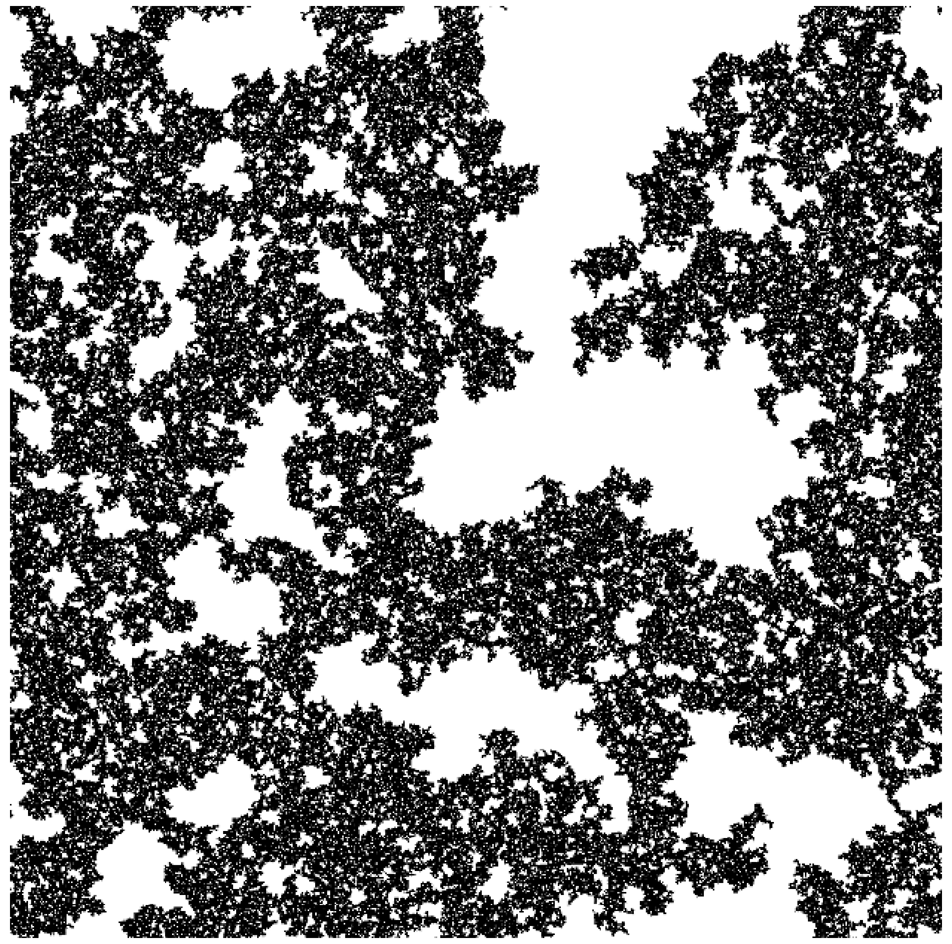
Infinite cluster at p_c

Percolation and Universality

Bond percolation on a lattice:

bonds are randomly **occupied** with probability **p**





Infinite cluster at $\mathbf{p_c}$

Site percolation on a triangular lattice:

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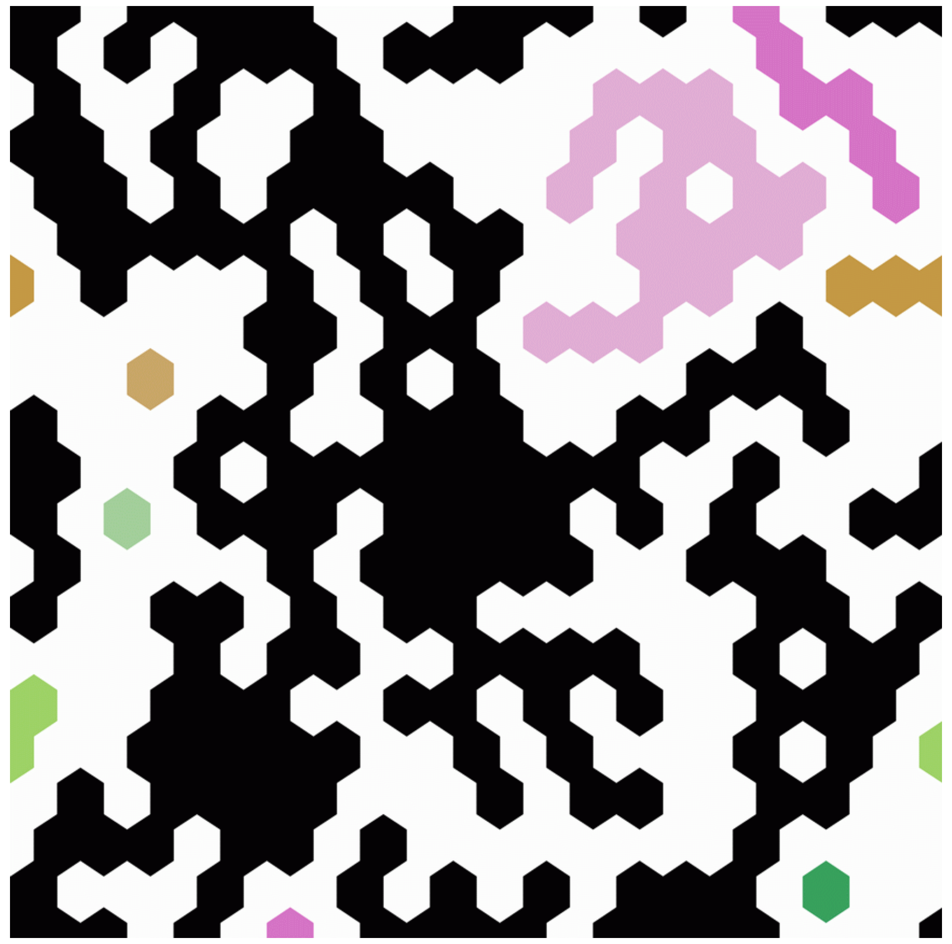
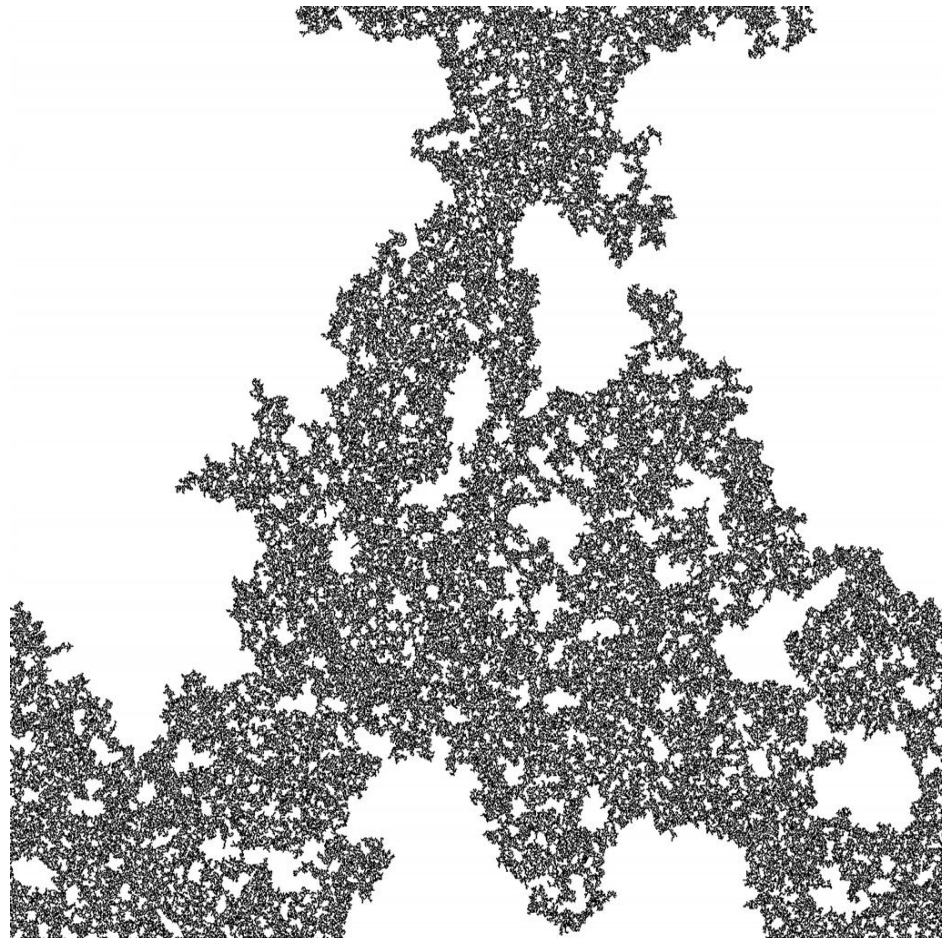




Infinite cluster at $\mathbf{p_c}$

Even though the microscopic **lattices** and **occupation rules** are **different**, the resulting **clusters at p_c look identical**.

Universality: the morphology of the percolation cluster at $\mathbf{p_c}$ is **independent of microscopic details**.

Non-universal critical occupation probability

p_c is not universal

Lattice	z	Site percolation	Bond percolation
$d = 1$ Line	2	1	1
$d = 2$ Hexagonal	3	0.6971	1 - 2 sin(pi/18)
Square	4	0.5927	1/2
Triangular	6	1/2	2 sin(pi/18)
$d = 3$ Diamond	4	0.4301	0.3893
Simple cubique	6	0.3116	0.2488
$d = 4$ Hypercubic	8	0.1968	0.1601
$d = 5$ Hypercubic	10	0.1408	0.1181
$d = 6$ Hypercubic	12	0.1090	0.0942
$d = 7$ Hypercubic	14	0.0889	0.0786
Bethe	z	1 / (z - 1)	1 / (z - 1)

z = number of nearest neighbors

values of p_c

Depend on the lattice structure and on the occupation rule

Universal critical exponents

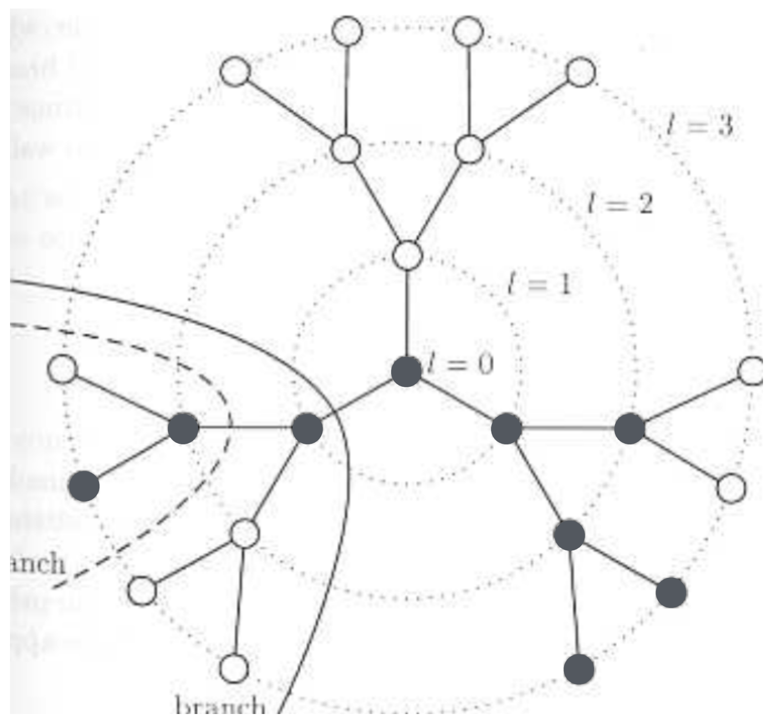
Critical exponents are universal

They are **insensitive to the details** of the underlying lattice (such as bound or site, value of **z**)

They only depend on dimensionality

	Exponent	Quantity	d=1	d=2	d=3	d=4	d=5	$d \geq 6$	Bethe
Pba site belongs to percolating cluster:	β	$P_\infty(p) \propto (p-p_c)^\beta$	0	5/36	0.4181(8)	0.657(9)	0.830(10)	1	1
Mean Cluster Size:	γ	$\chi(p) \propto \|p-p_c\|^{-\gamma}$	1	43/18	1.793(3)	1.442(16)	1.185(5)	1	1
Correlation length:	ν	$\xi(p) \propto \|p-p_c\|^{-\nu}$	1	4/3	0.8765(16)	0.689(10)	0.569(5)	1/2	1/2
Cutoff cluster size:	σ	$s_\xi(p) \propto \|p-p_c\|^{-1/\sigma}$	1	36/91	0.4522(8)	0.476(5)	0.496(4)	1/2	1/2
has exponent:	D	$\sigma_\xi \propto \xi^D$	1	91/48	2.523(6)	3.05(5)	3.54(4)	4	4

Bethe lattice:



Values are independent of the geometry of the lattice

Upper critical dimension $d_u = 6$

above d_u critical exponent remain unchanged

Bethe lattice: correspond to an infinite-dimensional lattice, as there are no loops of occupied sites

The **behavior near the critical threshold**, p_c, is characterized by **universal critical exponents**

Introduction

Critical exponents and Universality

"Entropy, Order Parameters, and Complexity", by J. P. Sethna, Section 12.1

Last week: 2D Ising model

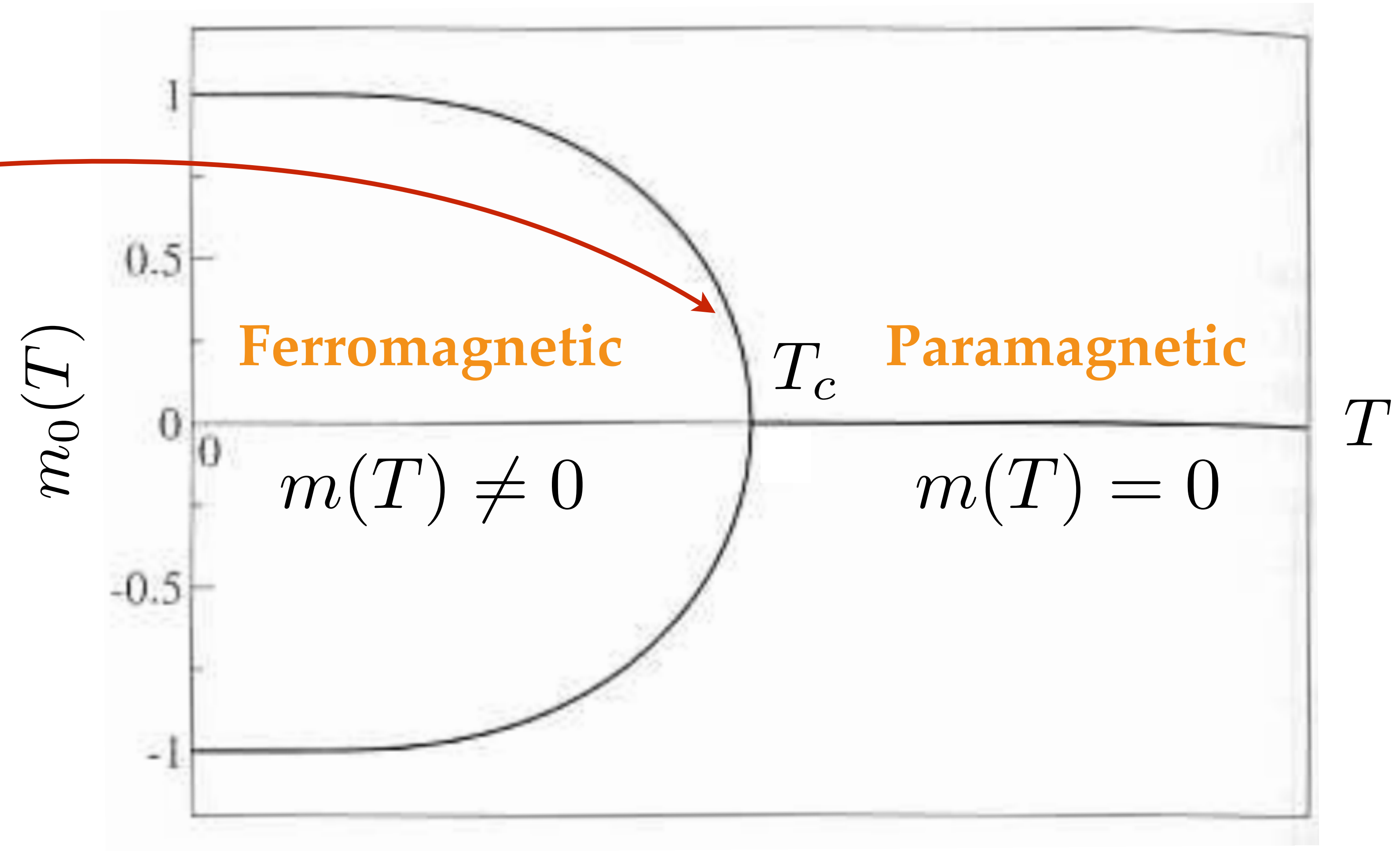
Exact solution for H=0 by Onsager (1944):

proves the existence of a critical phase transition at $$k_B T_c = \frac{2J}{\log(1+\sqrt{2})} \simeq 2.269\, J$$

with a magnetization: $$m(T) = \left(1 - \sinh^{-4}\left(\frac{2\,J}{k_B T}\right)\right)^{1/8}$$

Critical exponent

Universality: $$m(T) \underset{T_c}{\sim} (T - T_c)^{1/8}$$



Difficult to derive exact results
with field and in larger dimensions

Universality

Universality: two systems, **microscopically** very **different**, can exhibit the **same behavior at criticality**

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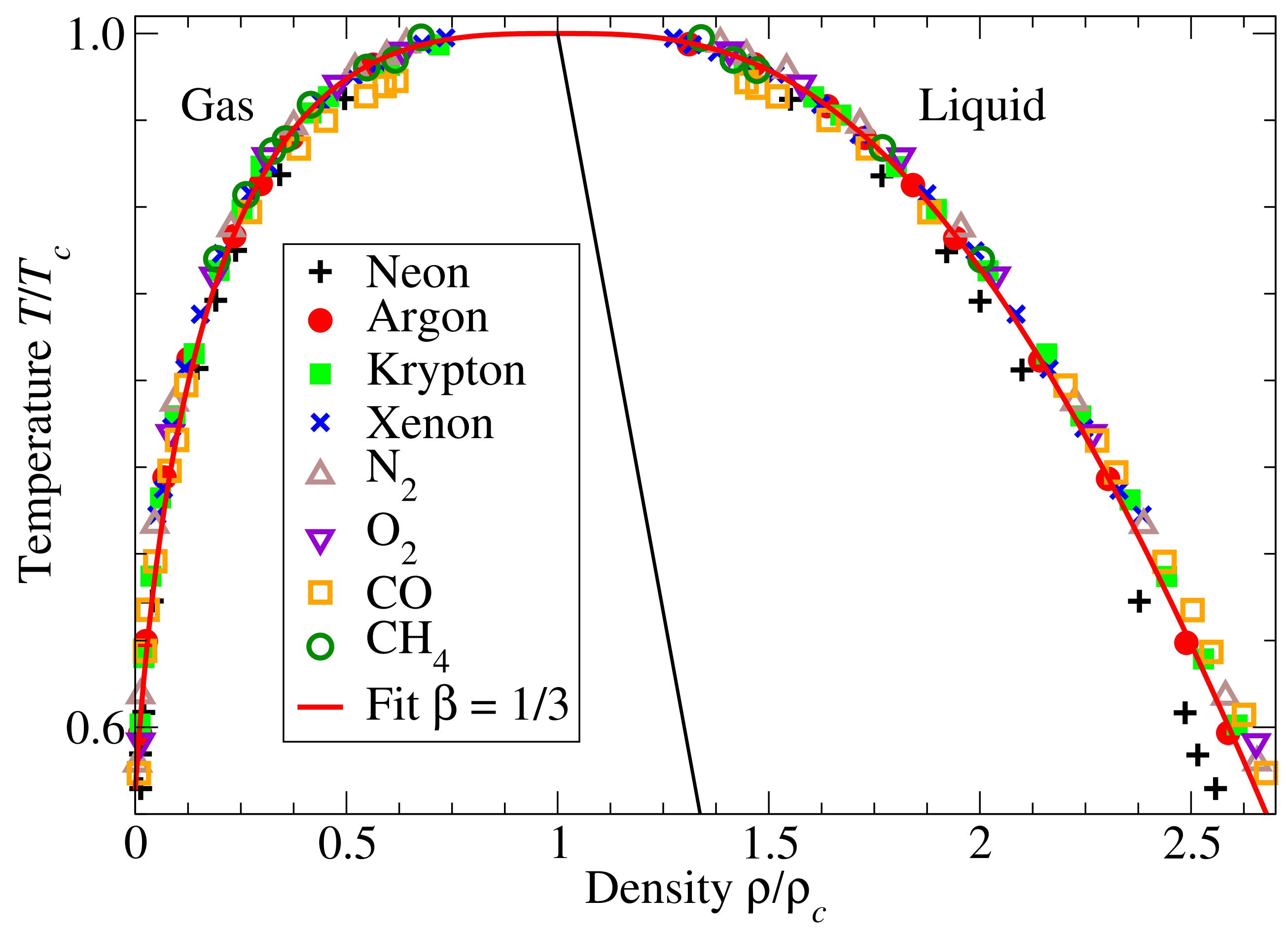

rescaled by their critical density

rescaled by their critical temperature

Ex. **Universality at the liquid-gas critical point:**

Liquid-gas coexistence lines for a variety of atoms and small molecules near their critical point.

Fit to Argon data:
$$\frac{\rho}{\rho_c} = 1 + s\left(1 - \frac{T}{T_c}\right) \pm \rho_0 \left(1 - \frac{T}{T_c}\right)^{\beta} \qquad \begin{cases} s = 0.75 \\ \rho_0 = 1.75 \\ \beta = 1/3 \end{cases}$$

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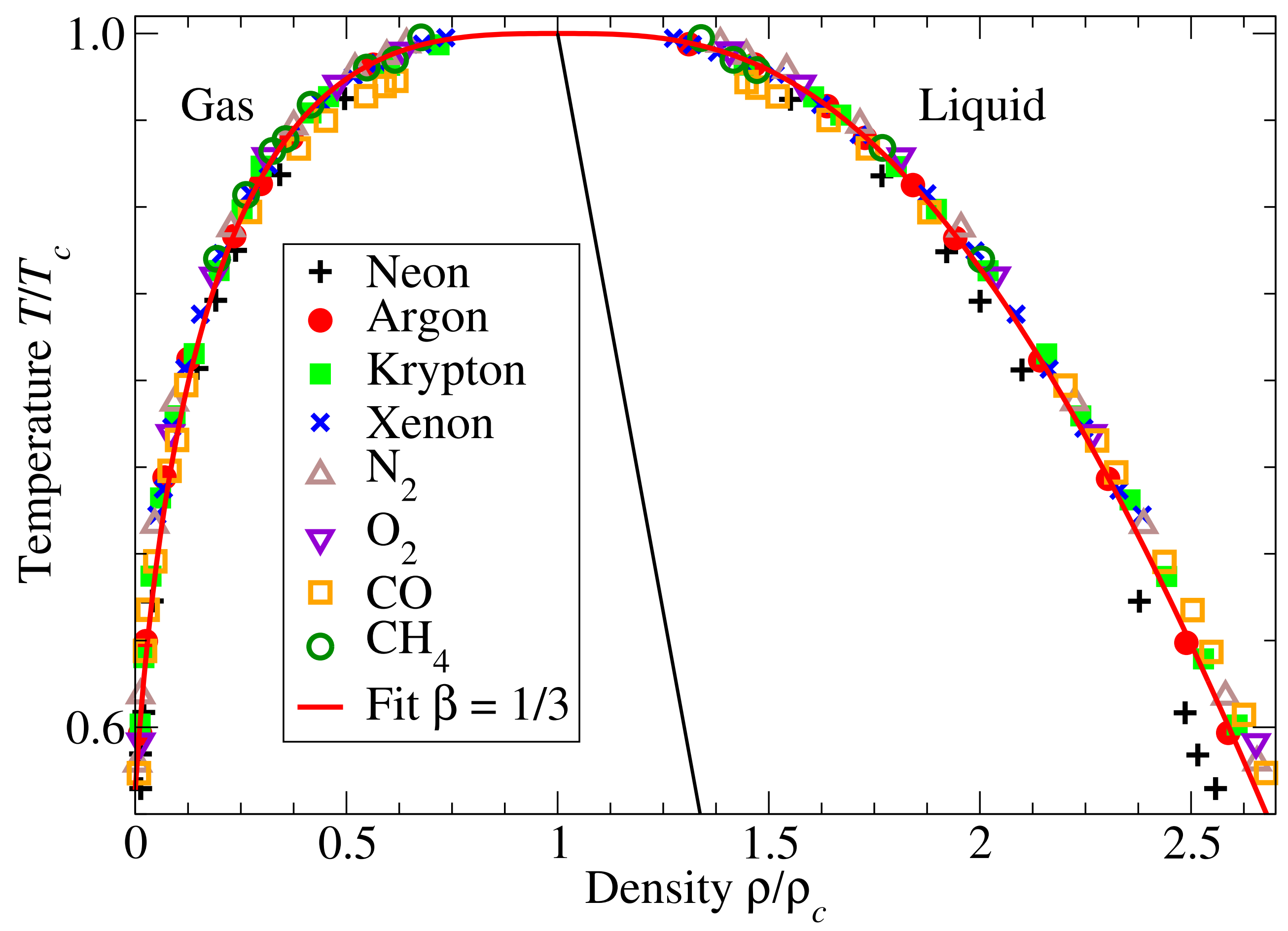

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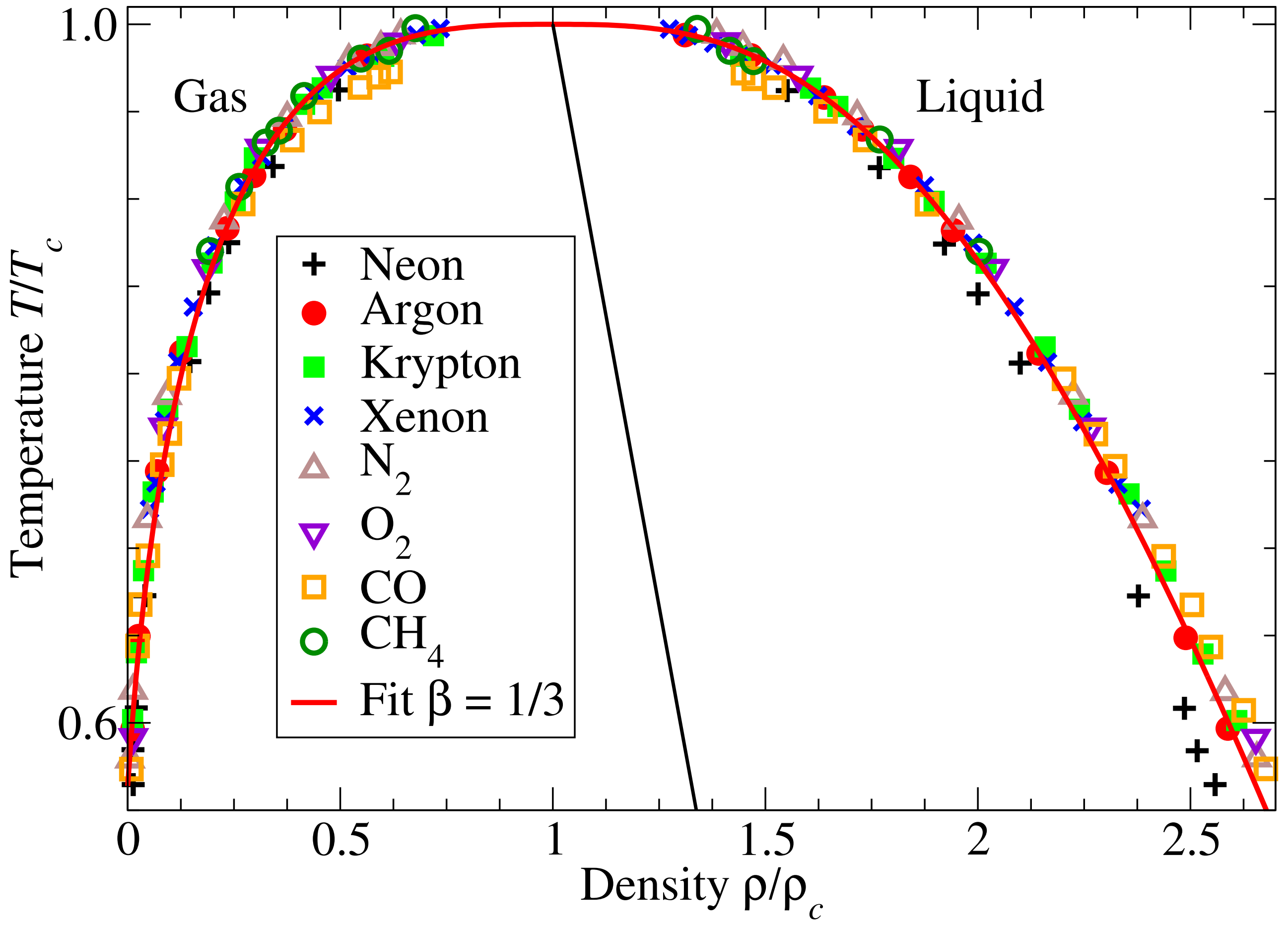

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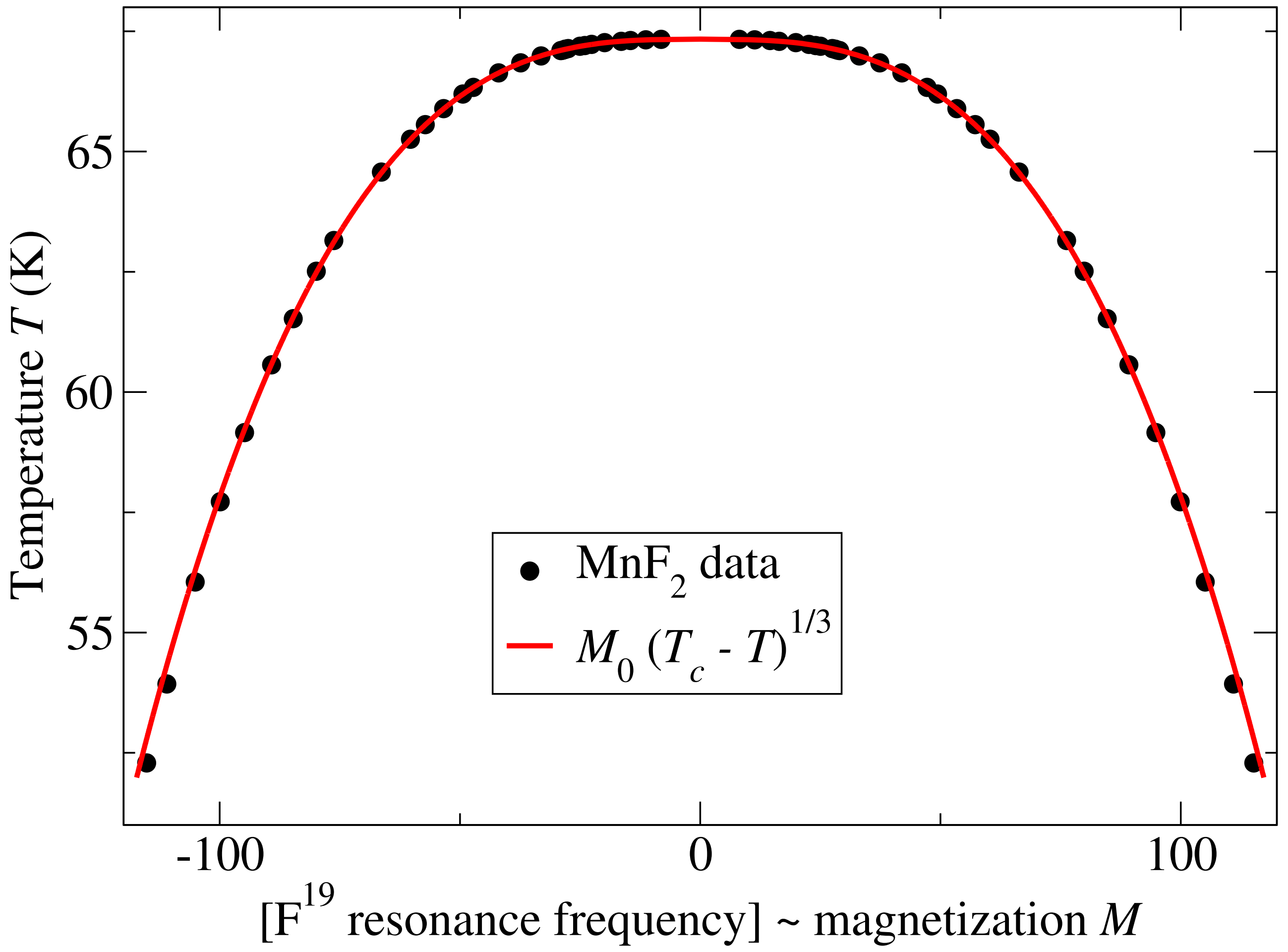
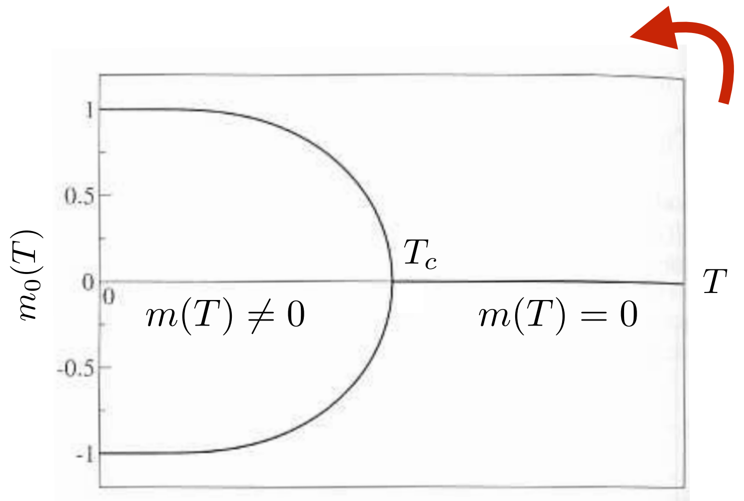


Ex. Universality: ferromagnetic–paramagnetic critical point

Uniaxial antiferromagnet MnF_2
$$\langle M \rangle = M_0\, T_c \left(1 - \frac{T}{T_c}\right)^{\beta} \qquad \beta = 1/3$$



Critical exponents universality

Universality

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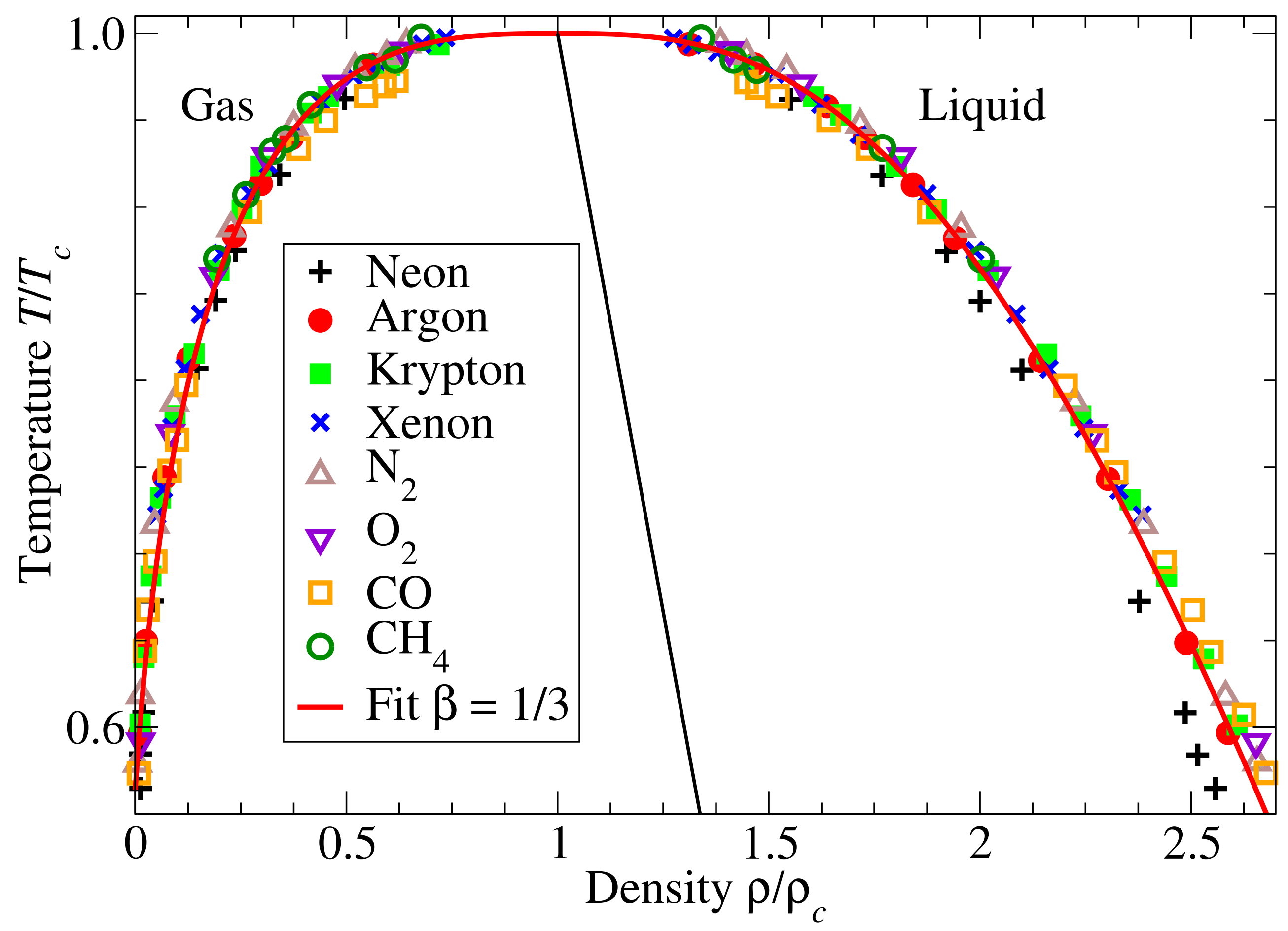

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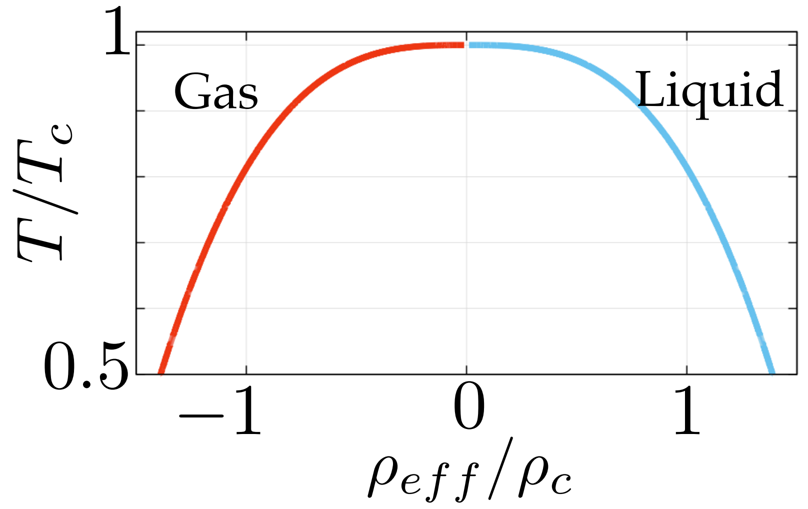
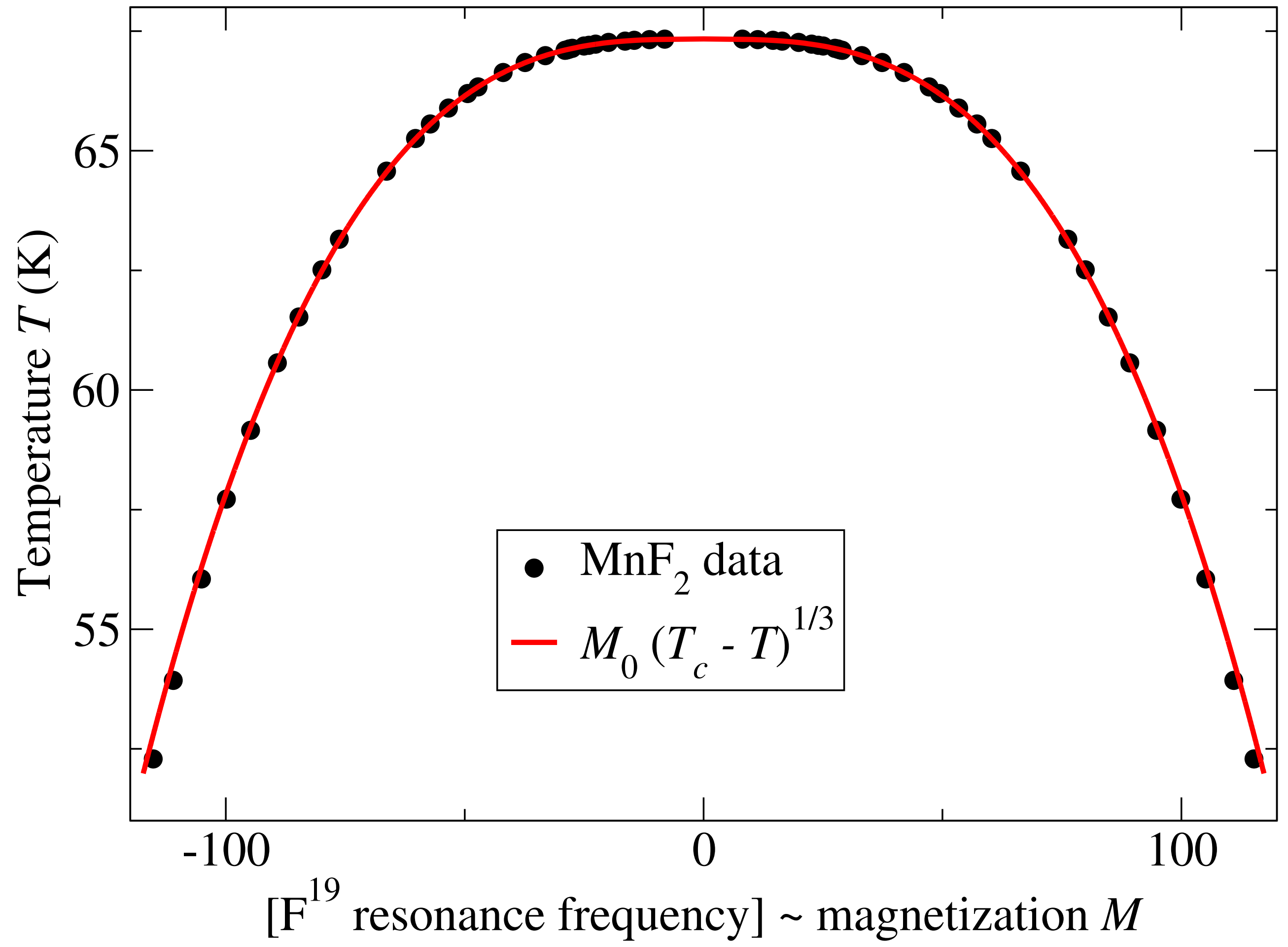
Tilted



$$\frac{\rho_{eff}}{\rho_c} = \pm\rho_0 \left(1 - \frac{T}{T_c}\right)^{\beta} \qquad \begin{cases} \rho_0 = 1.75 \\ \beta = 1/3 \end{cases}$$



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Current estimate: $\beta = 0.3264\ldots$

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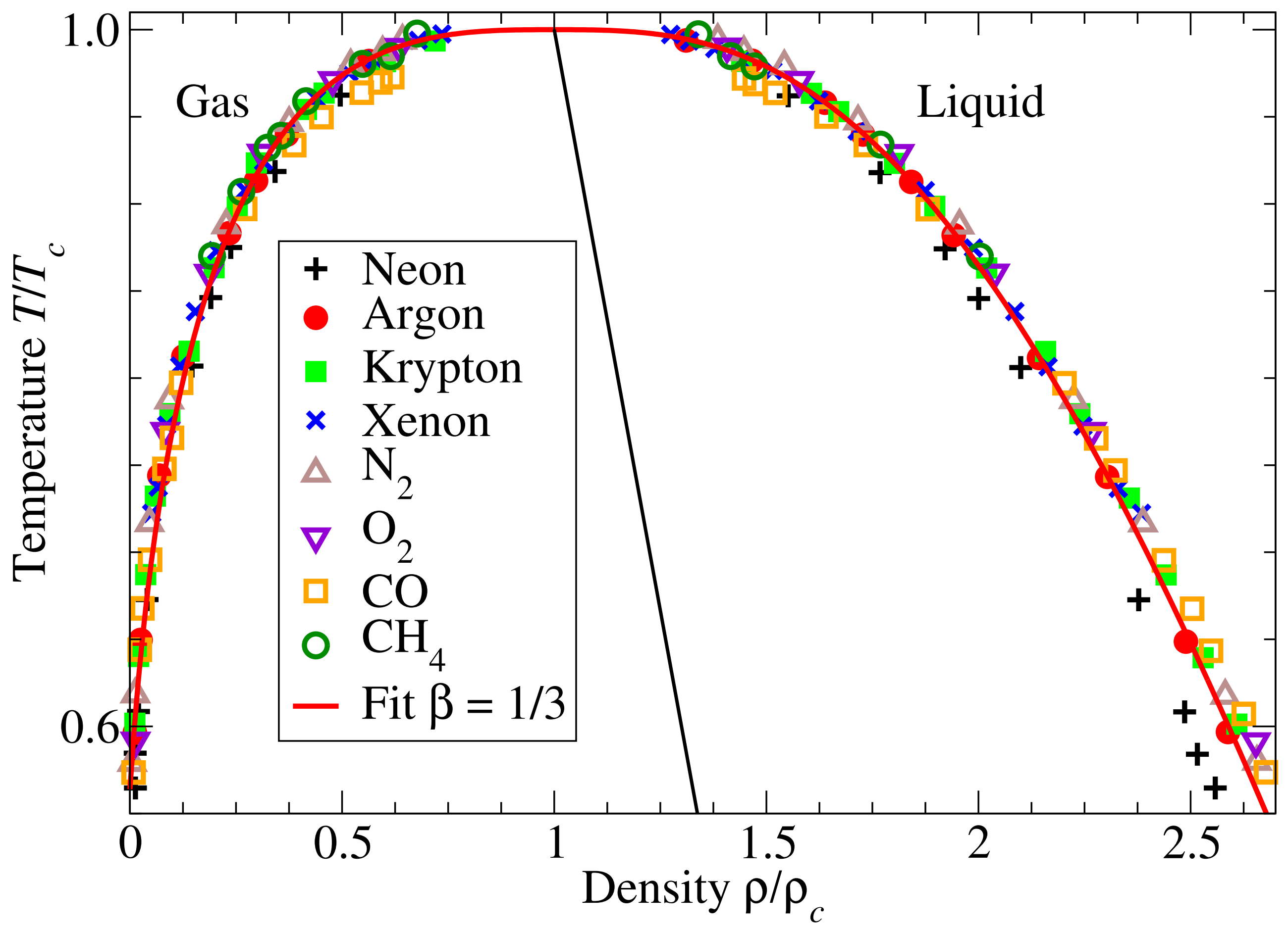

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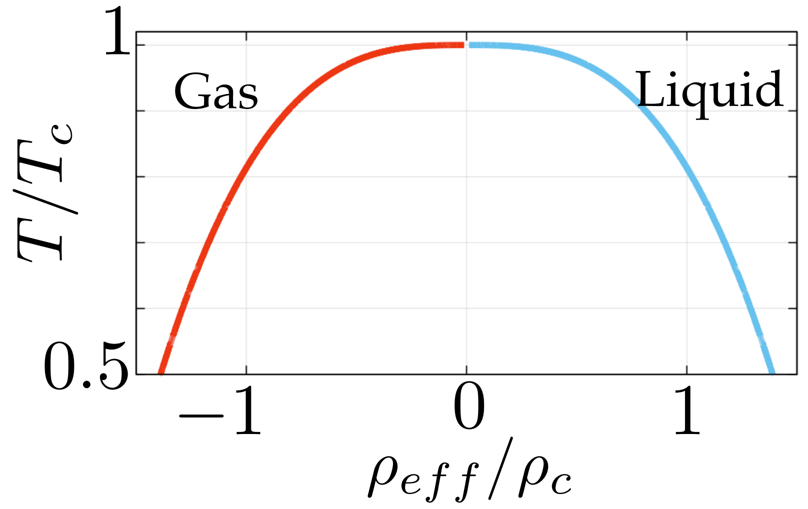

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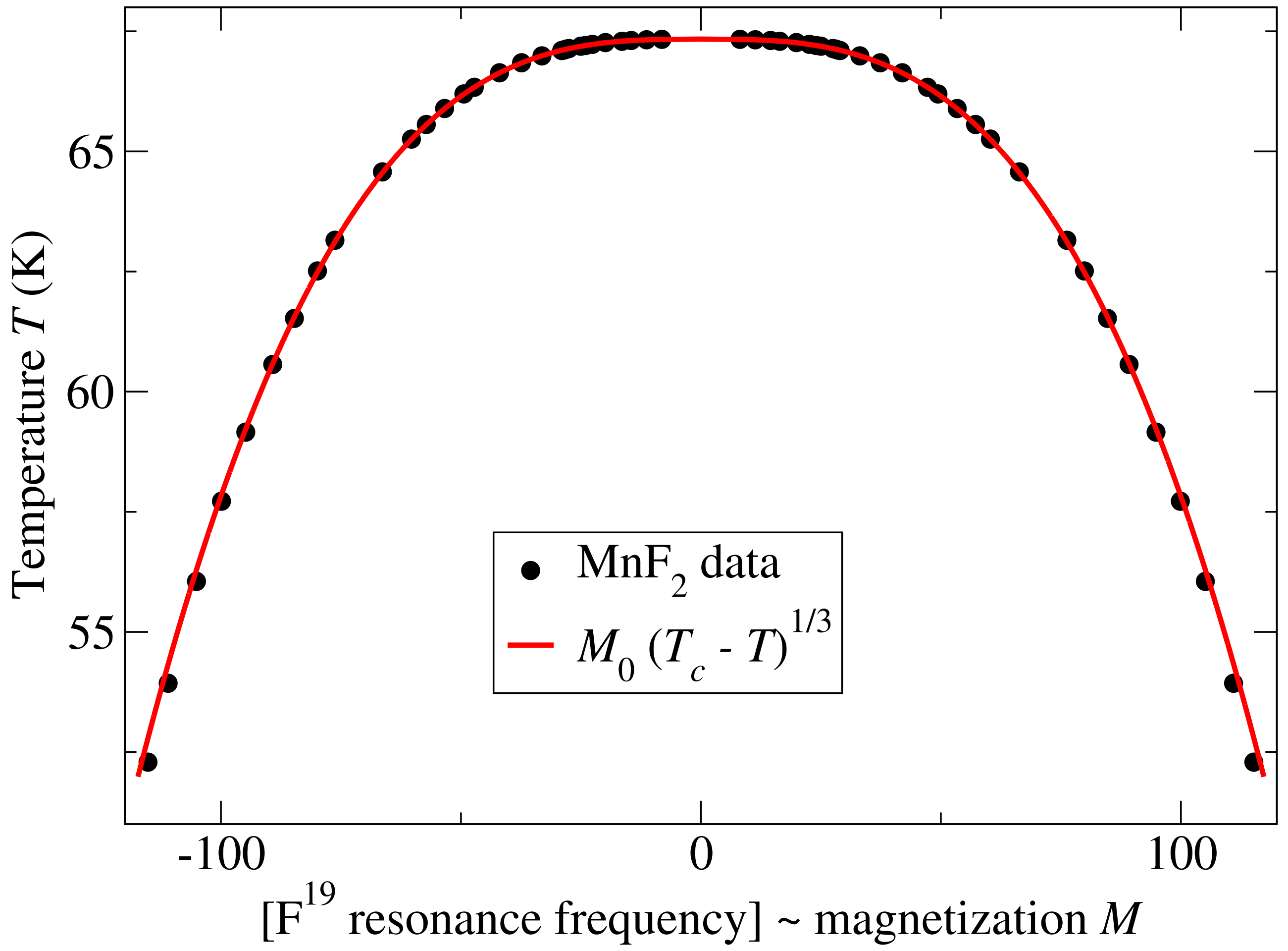

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Critical exponents all identical: at the critical point, correlation length, susceptibility, specific heat have power-law singularities with the same exponents.

Critical exponents universality

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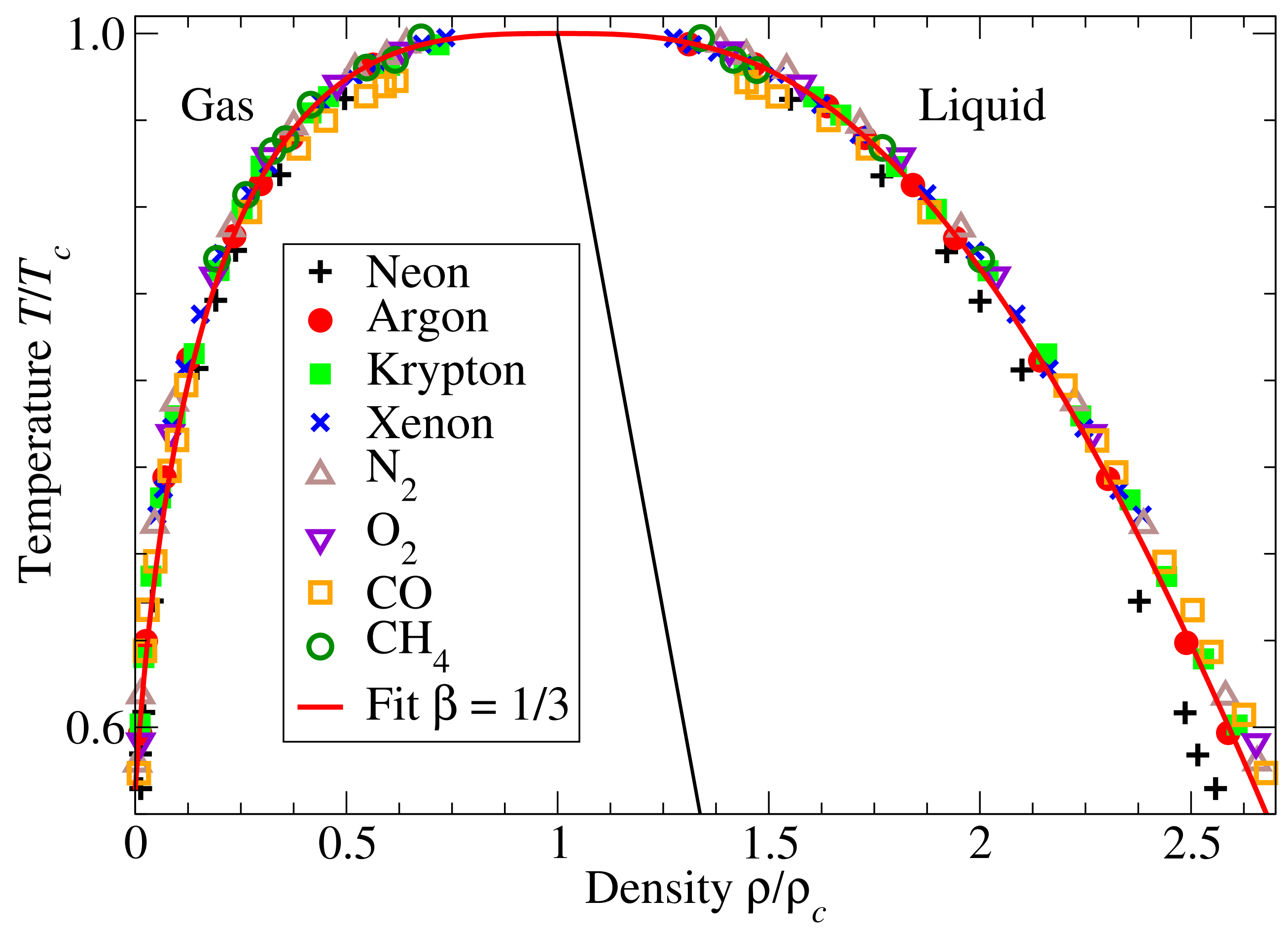

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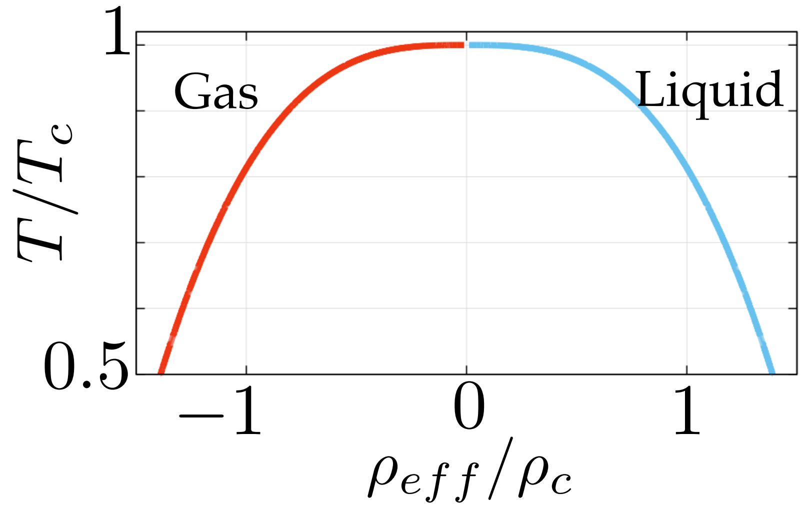

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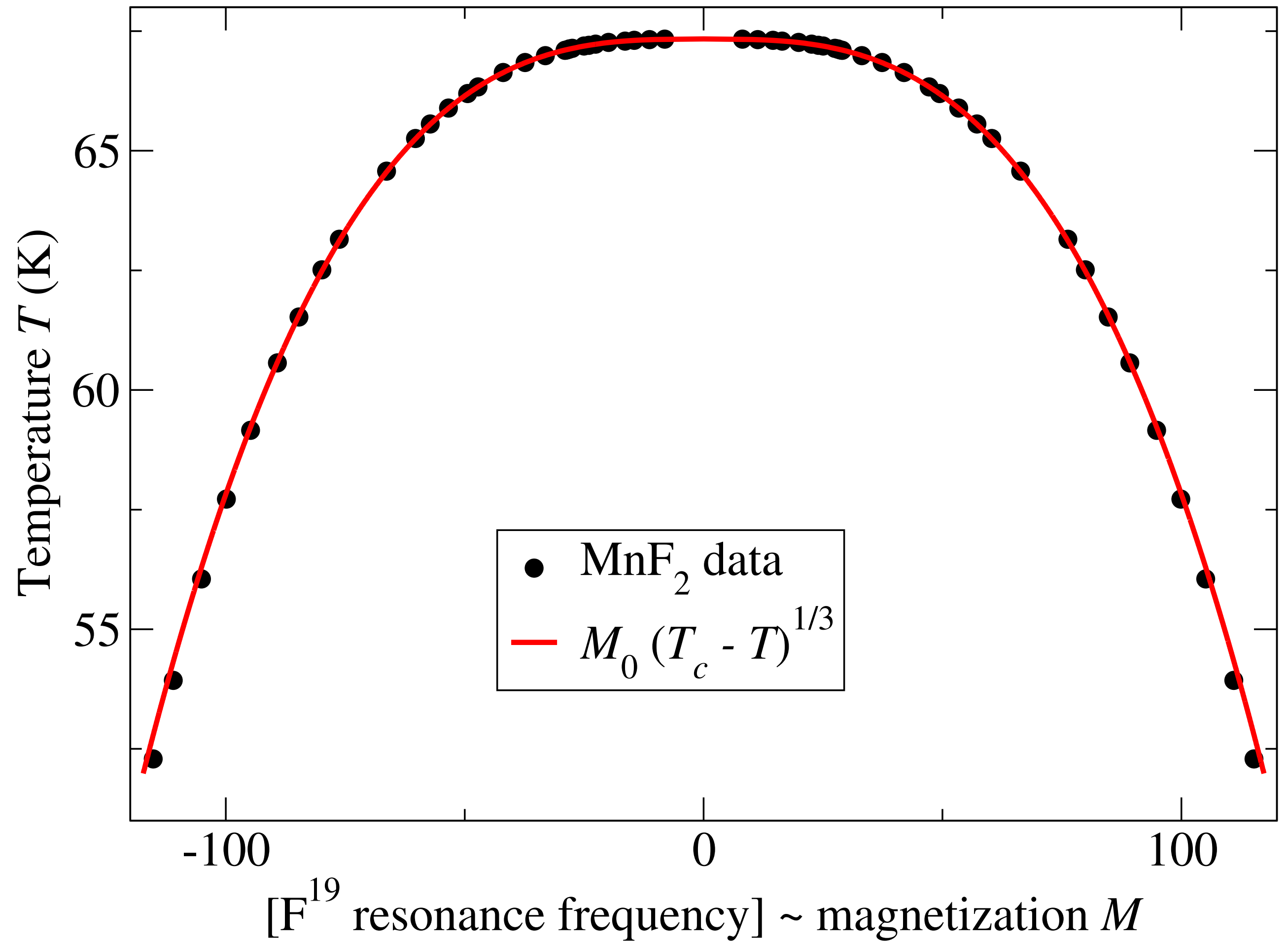

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Current estimate: $\beta = 0.3264...$

Critical exponents all identical: at the critical point, correlation length, susceptibility, specific heat have power-law singularities with the same exponents.

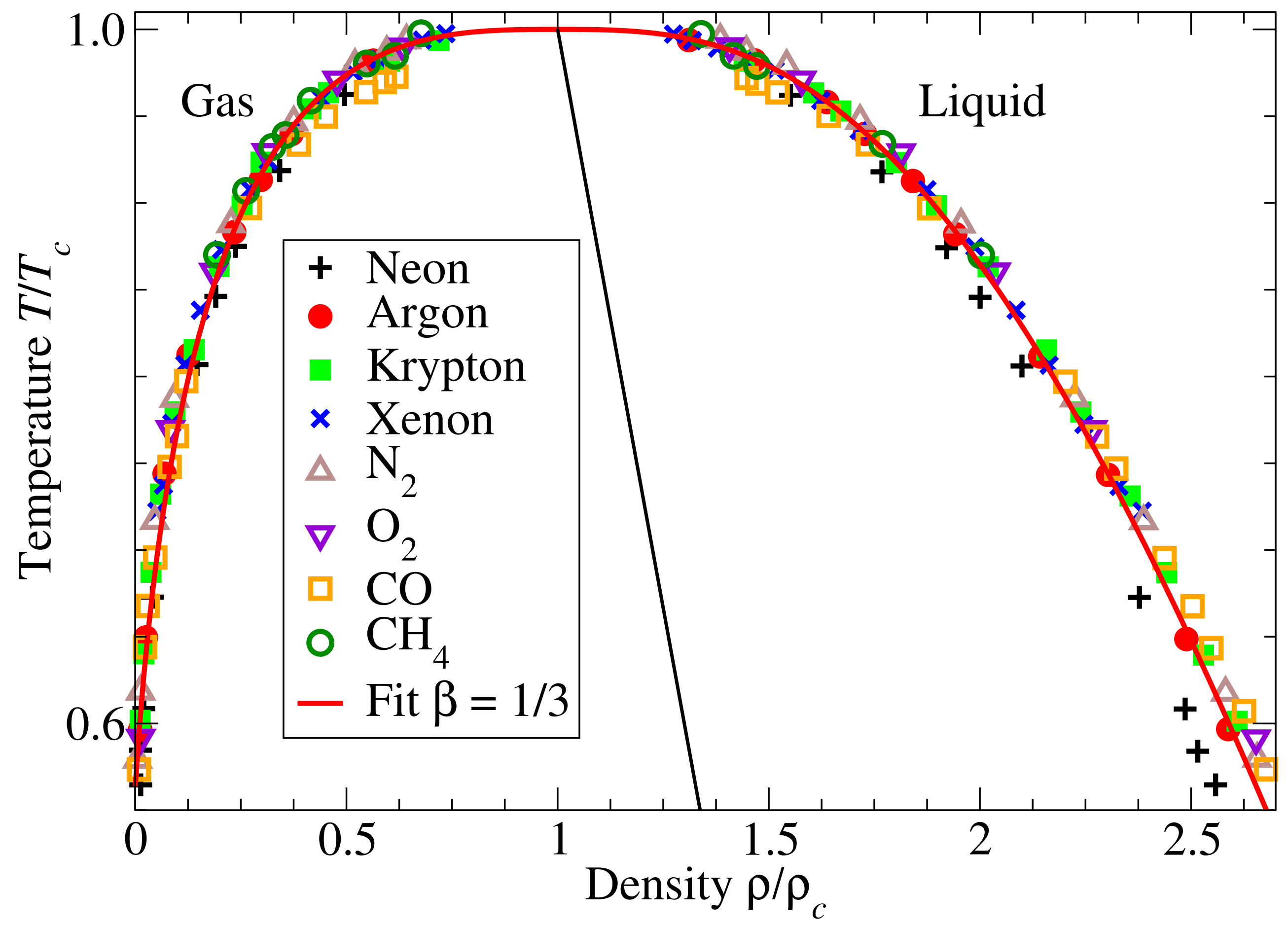
They are in the **same universality class**, along with the **3D Ising model**!

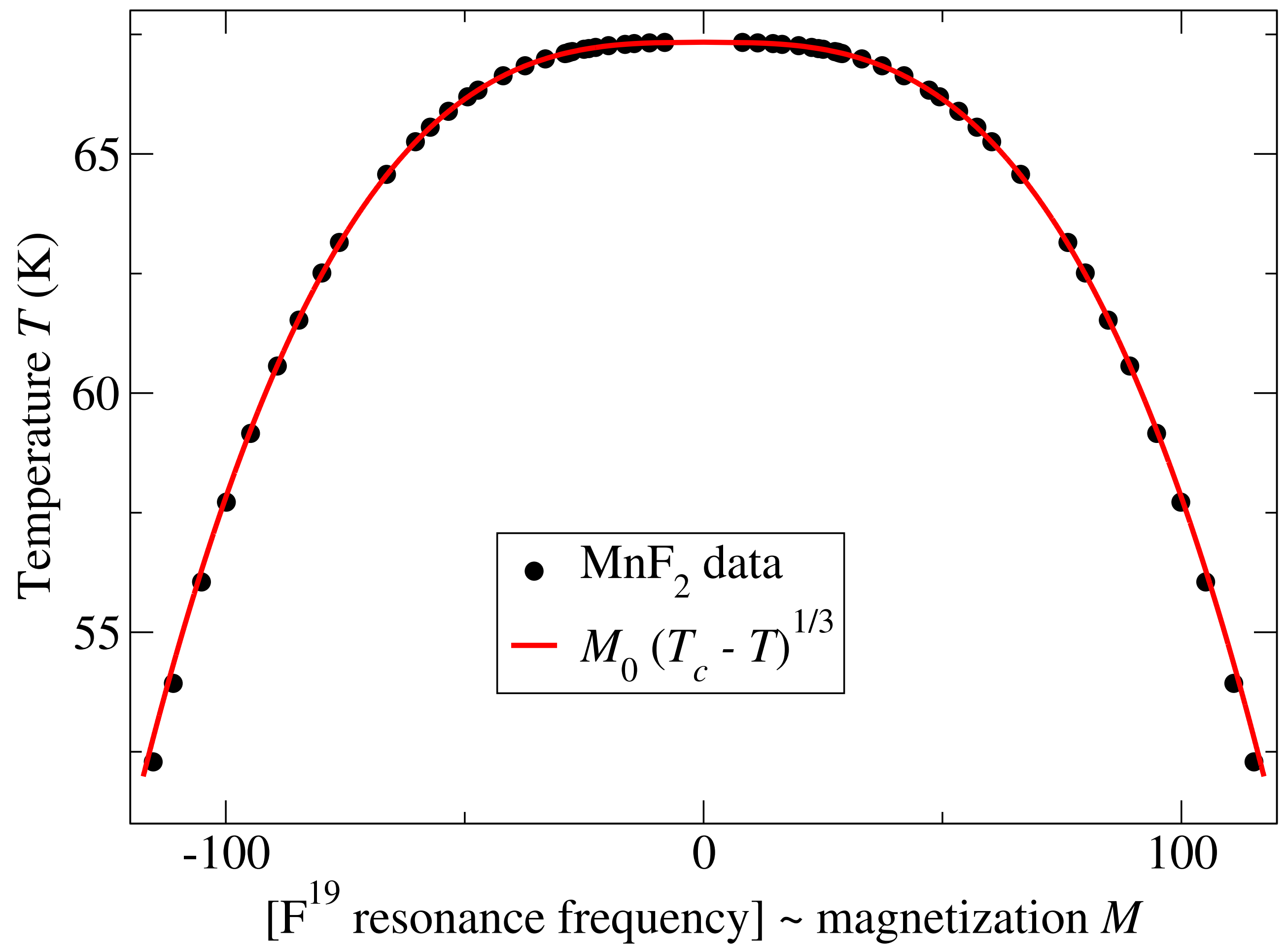
(despite drastic simplifications!)

Critical exponents universality

Coming up

Universality: two systems, **microscopically** very **different**, can exhibit the **same behavior at criticality**



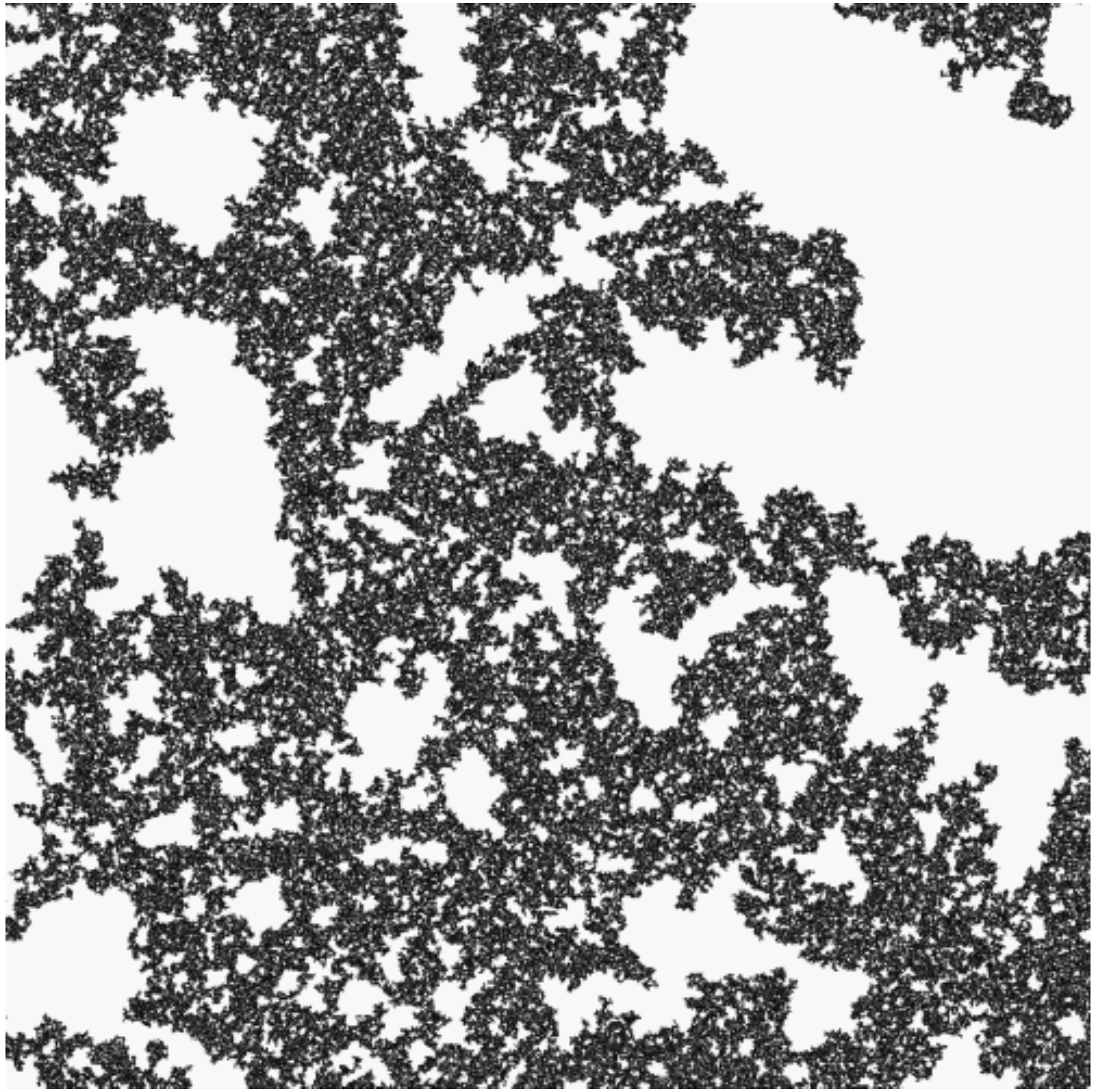


In the coming chapters: We will discuss

— How to obtain critical exponent for $d > d_u$ using a **mean-field approximation**

— systems belonging to **universality class** of percolation problems: epidemic models

Coming up



CPCS (and others interested) please stay a few more minutes: **Question date for master project bazaar**