Scale Invariance and Universality

Chapter 5

Wednesday 8 May

Before we start:

This week and next week:

	L1	L2
	Guest lecture Fernando	**Chap 5. L1:** RW
	Chap4. L2: Epidemic modeling	**Chap 5. L2:** renormalization + guest lecture
	Wout Merbis, from the Dutch Institute for Emergent Phenomena	**Vítor Vasconcelos,** from the Computational Science Lab
	Quiz: Tuesday May 21 at 11h	

Questions: **Comment:** Bonus homework?

Information about the last quiz.

Issue aboout schedule of the exam? Except for *Machine learning for physicists*

Scale Invariance and Universality

Chapter 5

Plan: **Lecture 1: Emergent behaviors in the simple example of random walks: scale-invariance and universality**

Lecture 2: Renormalization (next Thursday)

Tutorial: Random walks + (optional) renormalization

Emergent behaviors in the simple example of random walks: scale-invariance and universality

Chapter 5. L1

Plan: **1) Introduction: properties of critical phenomena**

2) Fractals: notion of self-similarity and fractal dimension

3) Random walk and Scale invariance

Expectations: Participate in the discussions, take notes, try to do the analytical derivations

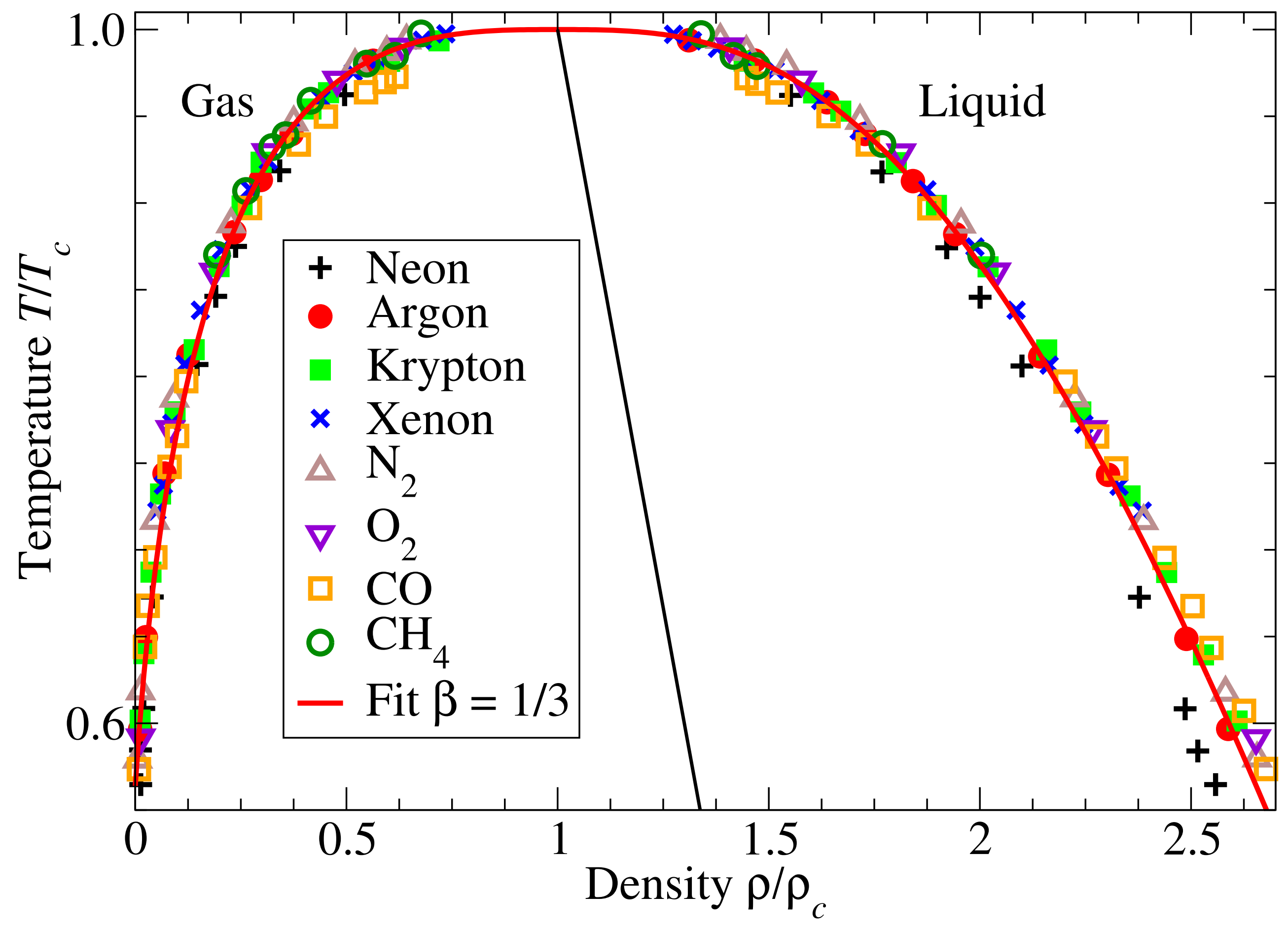
References: "Entropy, Order Parameters, and Complexity", by J. P. Sethna, Chap. 2 and 12.2

Introduction

Do you remember main **properties** of **continuous phase transitions at criticality?**

Properties of Critical phenomena: ________

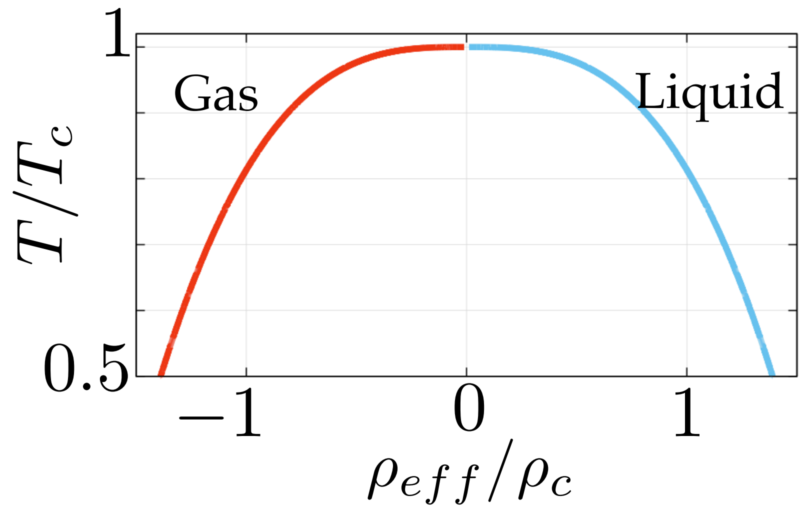
?? ________ ?? : two systems, **microscopically** completely **different**, can exhibit precisely the **same critical behavior near their critical point**.



Ex. ________ at the liquid-gas critical point:

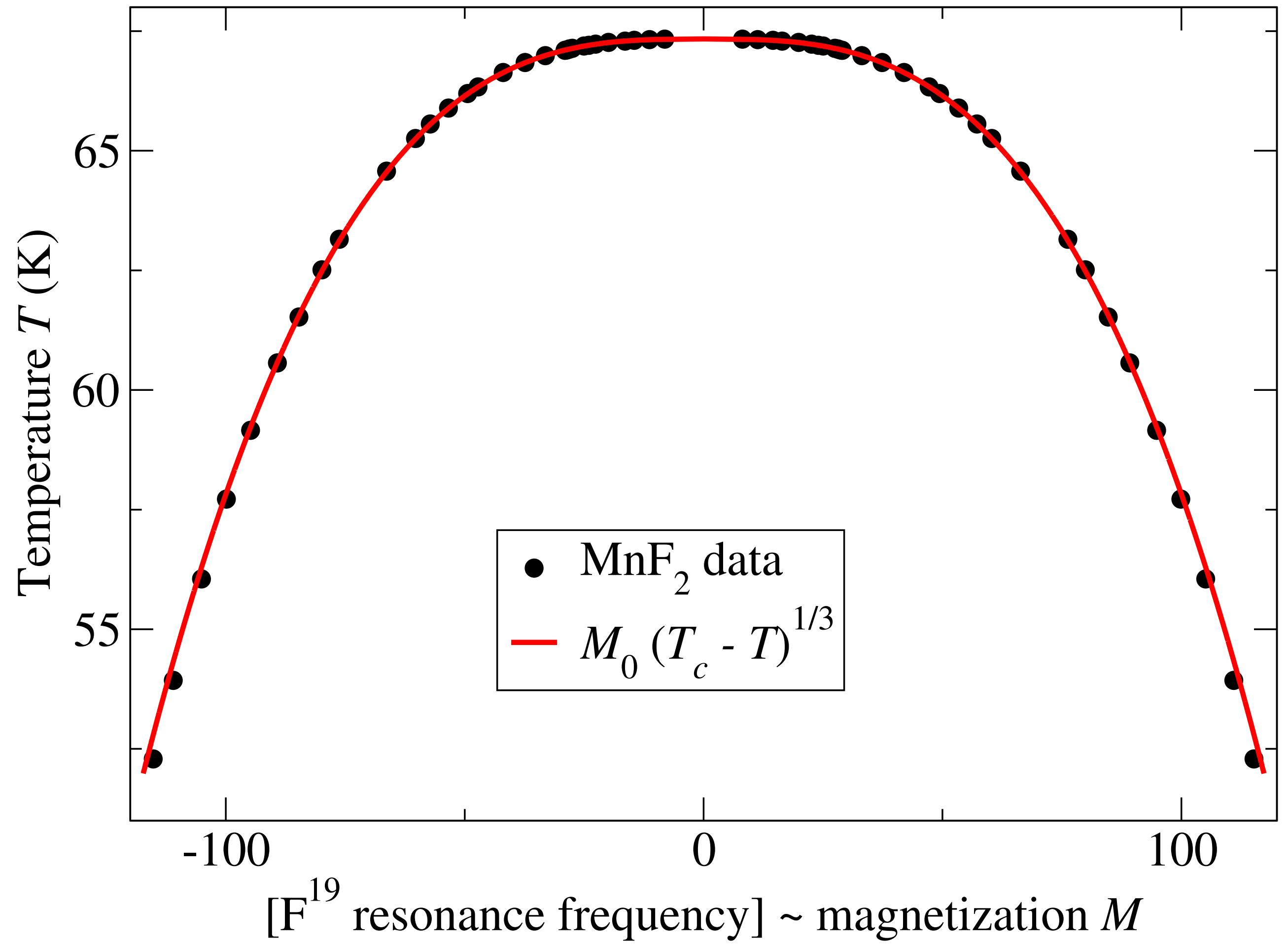
Liquid-gas coexistence lines for a variety of atoms and small molecules near their critical point.

Tilted



$$\frac{\rho_{eff}}{\rho_c} = \pm\rho_0 \left(1 - \frac{T}{T_c}\right)^{\beta}$$

$$\begin{cases} \rho_0 = 1.75 \\ \beta = 1/3 \end{cases}$$



Ex. ________ : 3D ferromagnetic–paramagnetic critical point

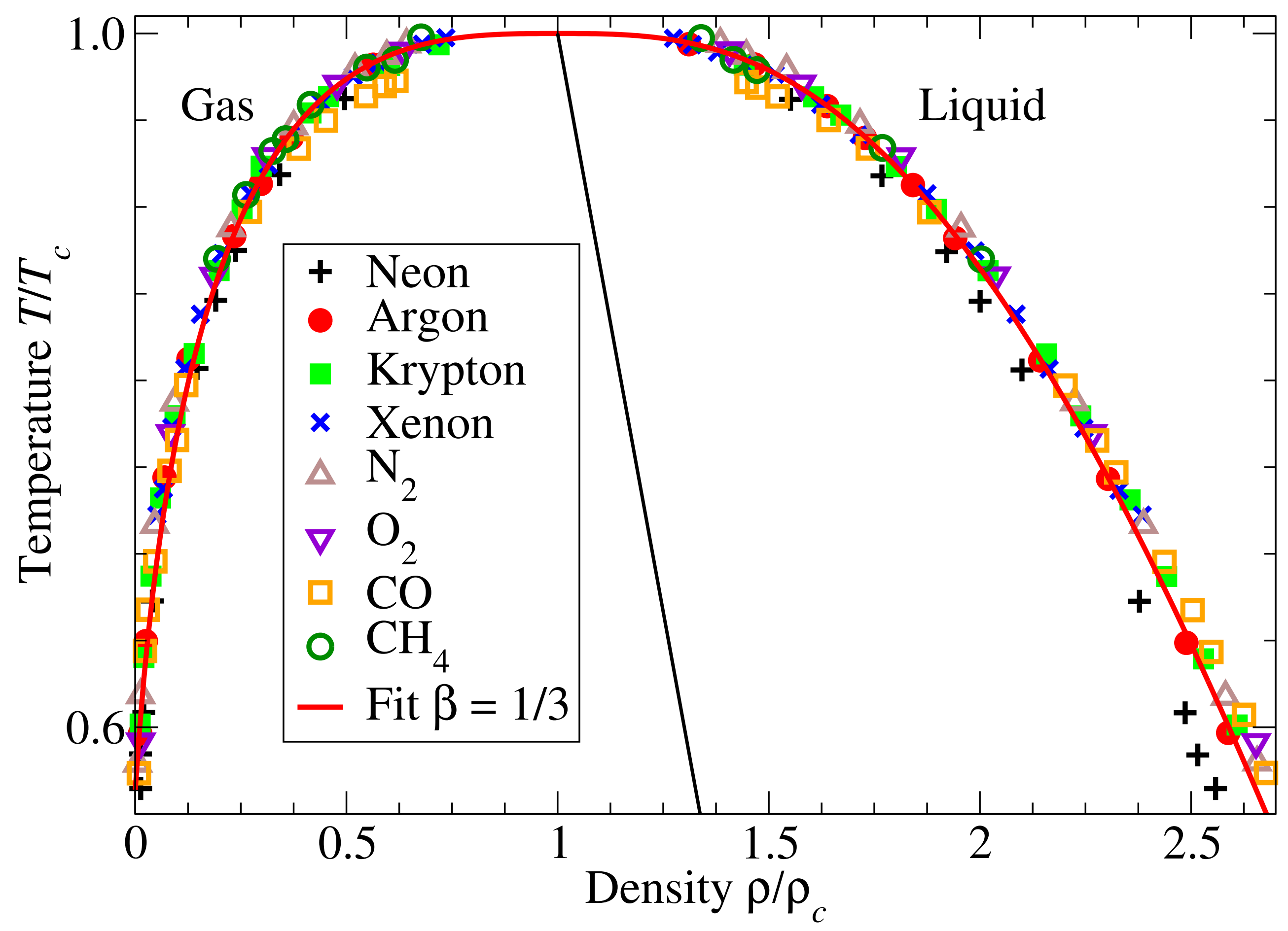
Uniaxial antiferromagnet MnF_2 $\quad \langle M \rangle = M_0\, T_c \left(1 - \frac{T}{T_c}\right)^{\beta}$ $\quad \beta = 1/3$

Current estimate: $\beta = 0.3264...$

Critical exponents all identical: at the critical point, correlation length, susceptibility, specific heat have power-law singularities with the same exponents.

Properties of Critical phenomena: Universality!

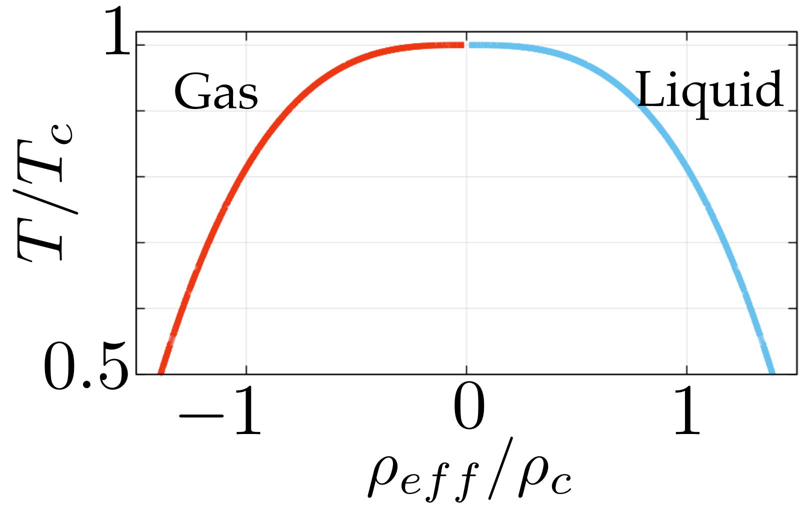
Universality: two systems, **microscopically** completely **different**, can exhibit precisely the **same critical behavior near their critical point**.



Ex. Universality at the liquid-gas critical point:

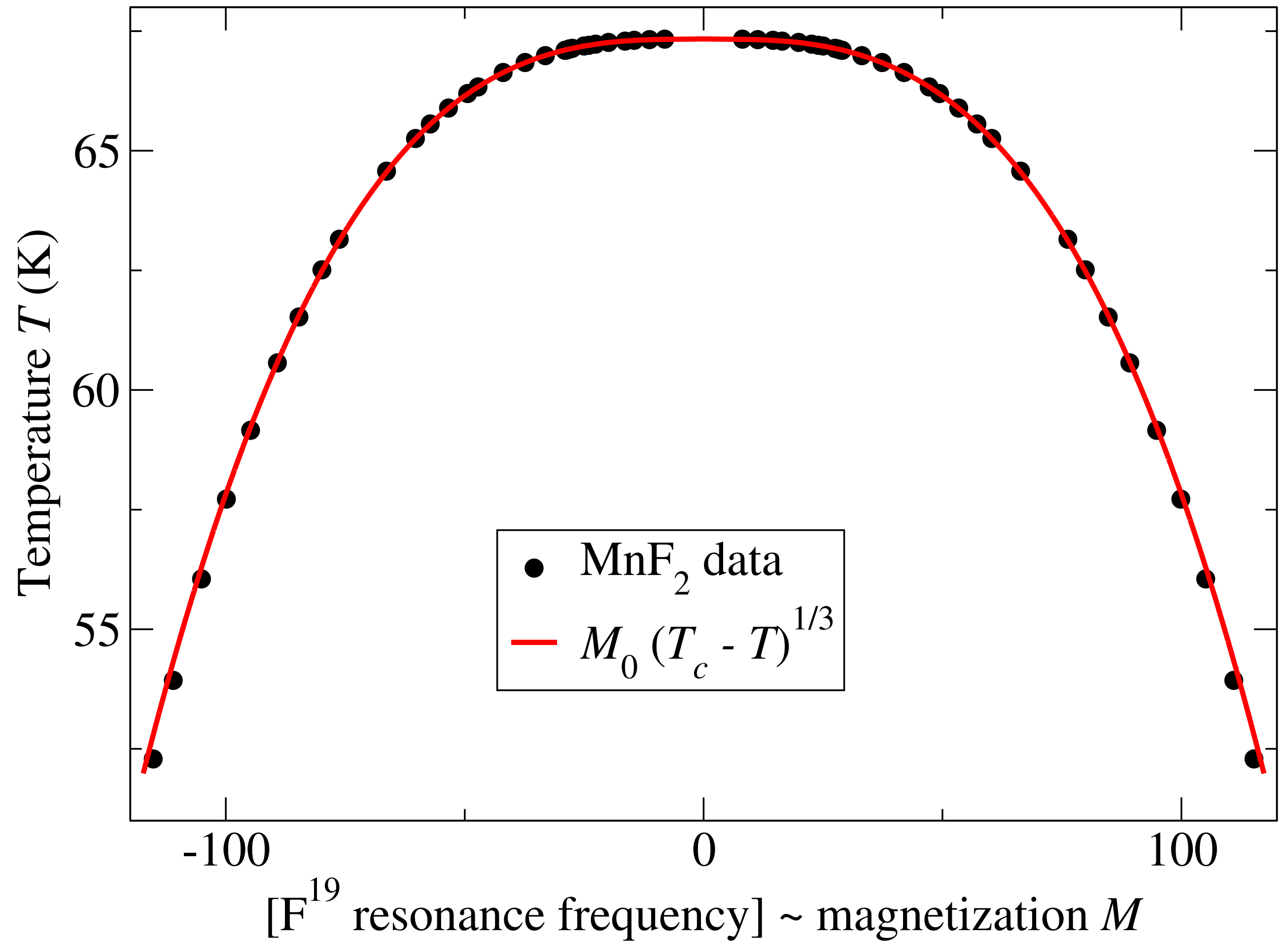
Liquid-gas coexistence lines for a variety of atoms and small molecules near their critical point.

Tilted



$$\frac{\rho_{eff}}{\rho_c} = \pm\rho_0 \left(1 - \frac{T}{T_c}\right)^{\beta}$$

$$\begin{cases} \rho_0 = 1.75 \\ \beta = 1/3 \end{cases}$$



Ex. Universality: 3D ferromagnetic–paramagnetic critical point

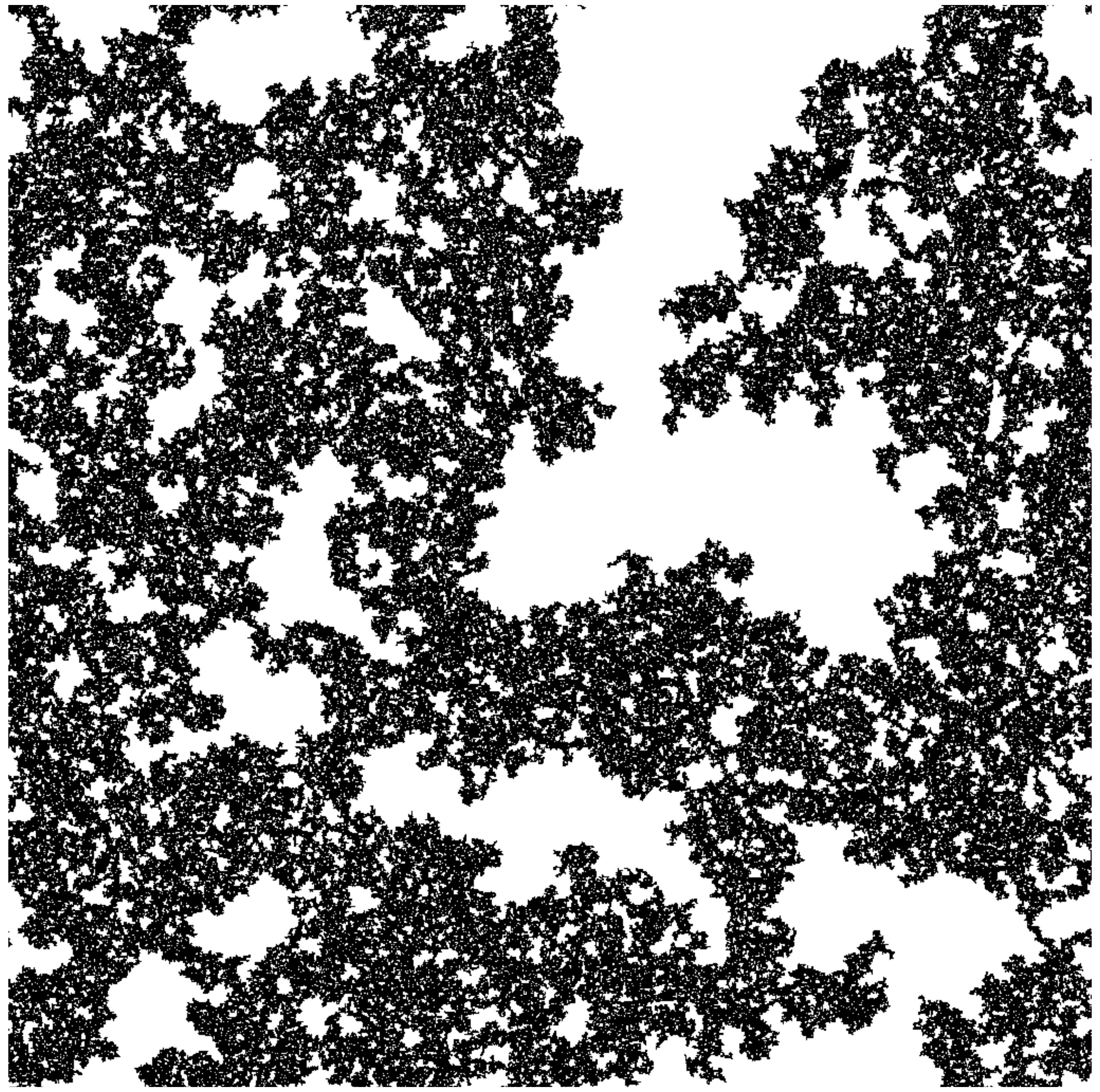
Uniaxial antiferromagnet MnF_2 $\quad \langle M \rangle = M_0\, T_c \left(1 - \frac{T}{T_c}\right)^{\beta}$ $\quad \beta = 1/3$

Current estimate: $\beta = 0.3264...$

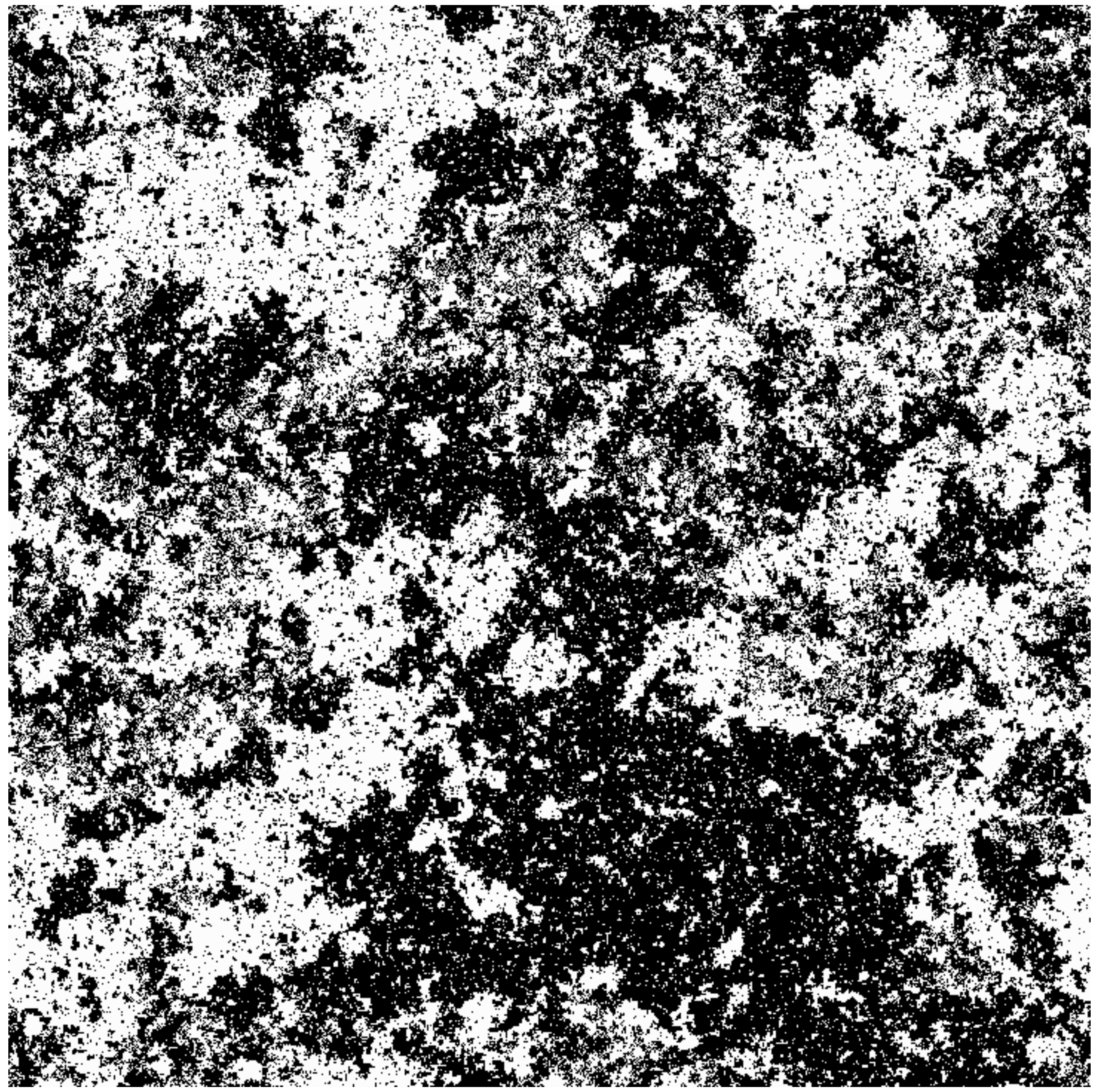
Critical exponents all identical: at the critical point, correlation length, susceptibility, specific heat have power-law singularities with the same exponents.

They are in the **same universality class**, along with the **3D Ising model**
(despite drastic simplifications!)

Properties of Critical phenomena: ____________?



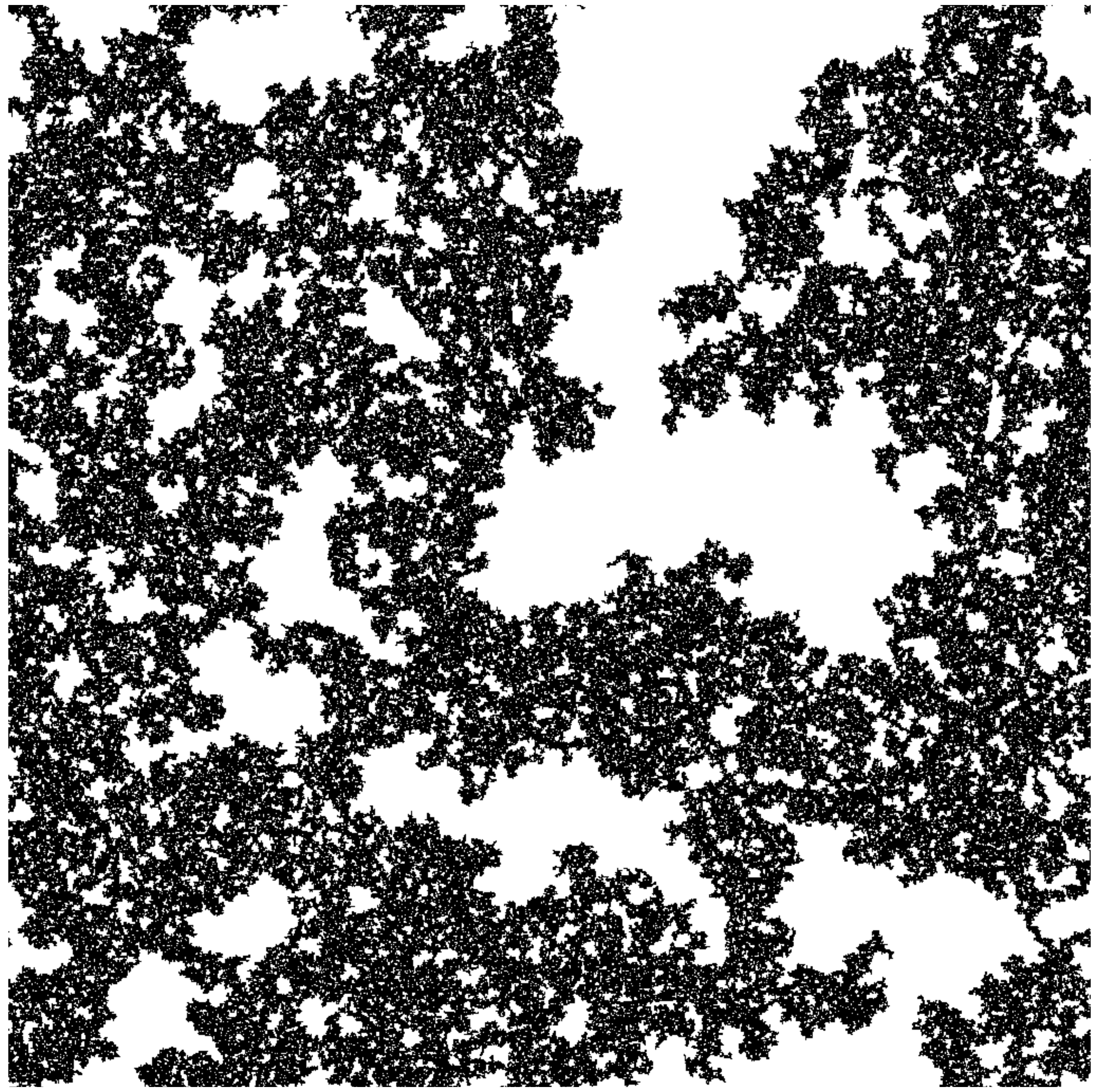
Bond percolation
at $p = p_c = 0.5$



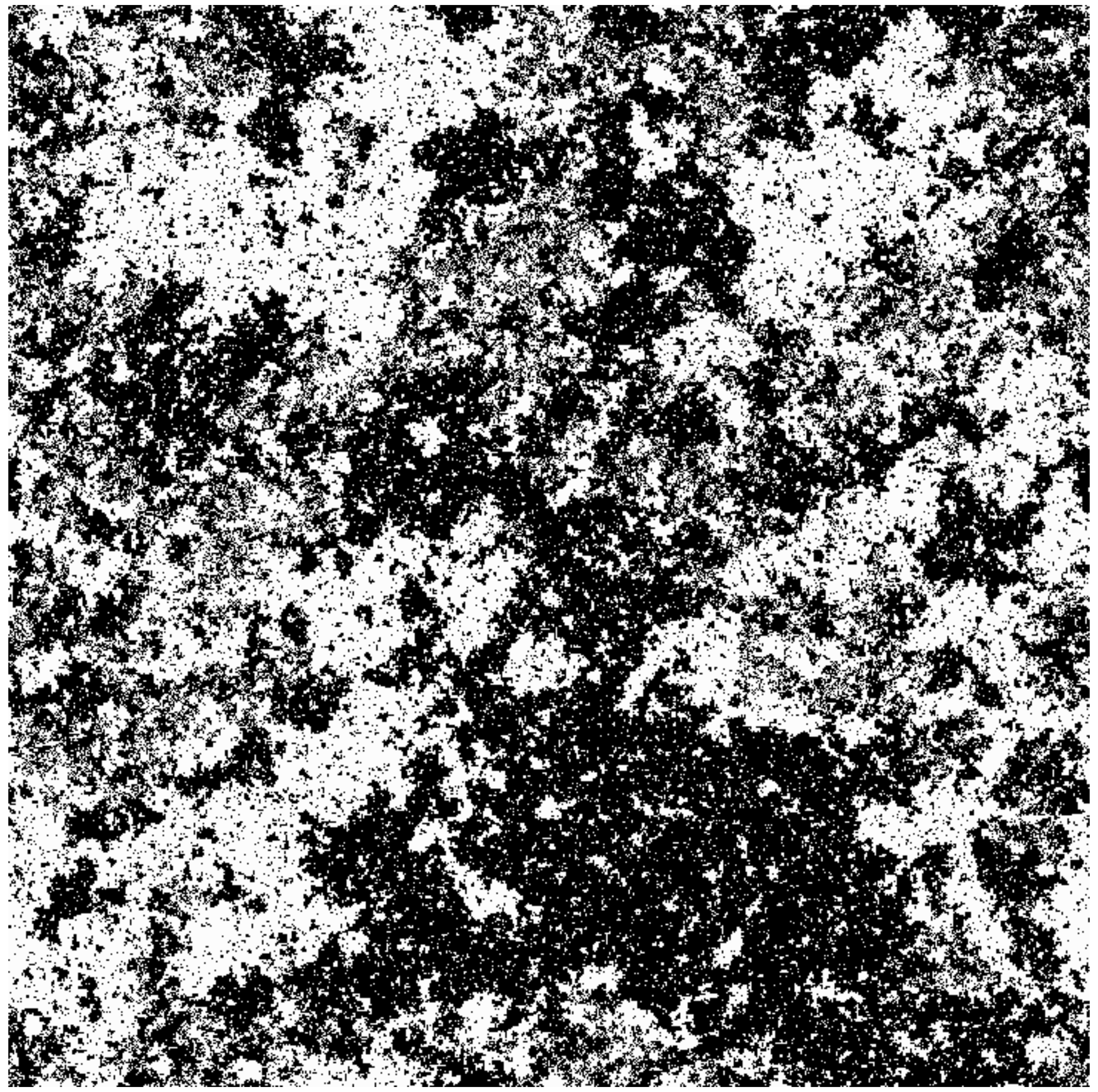
Ising model
at $T = T_c \sim 2.269$

Anything interesting on these pictures?

Properties of Critical phenomena: ______________?



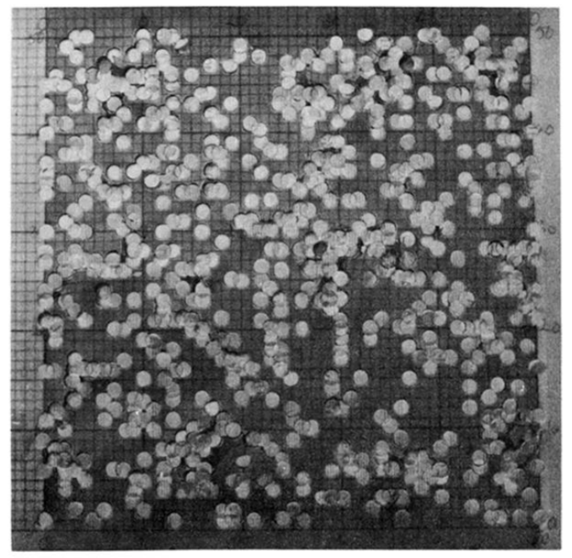
Bond percolation
at p = pc = 0.5



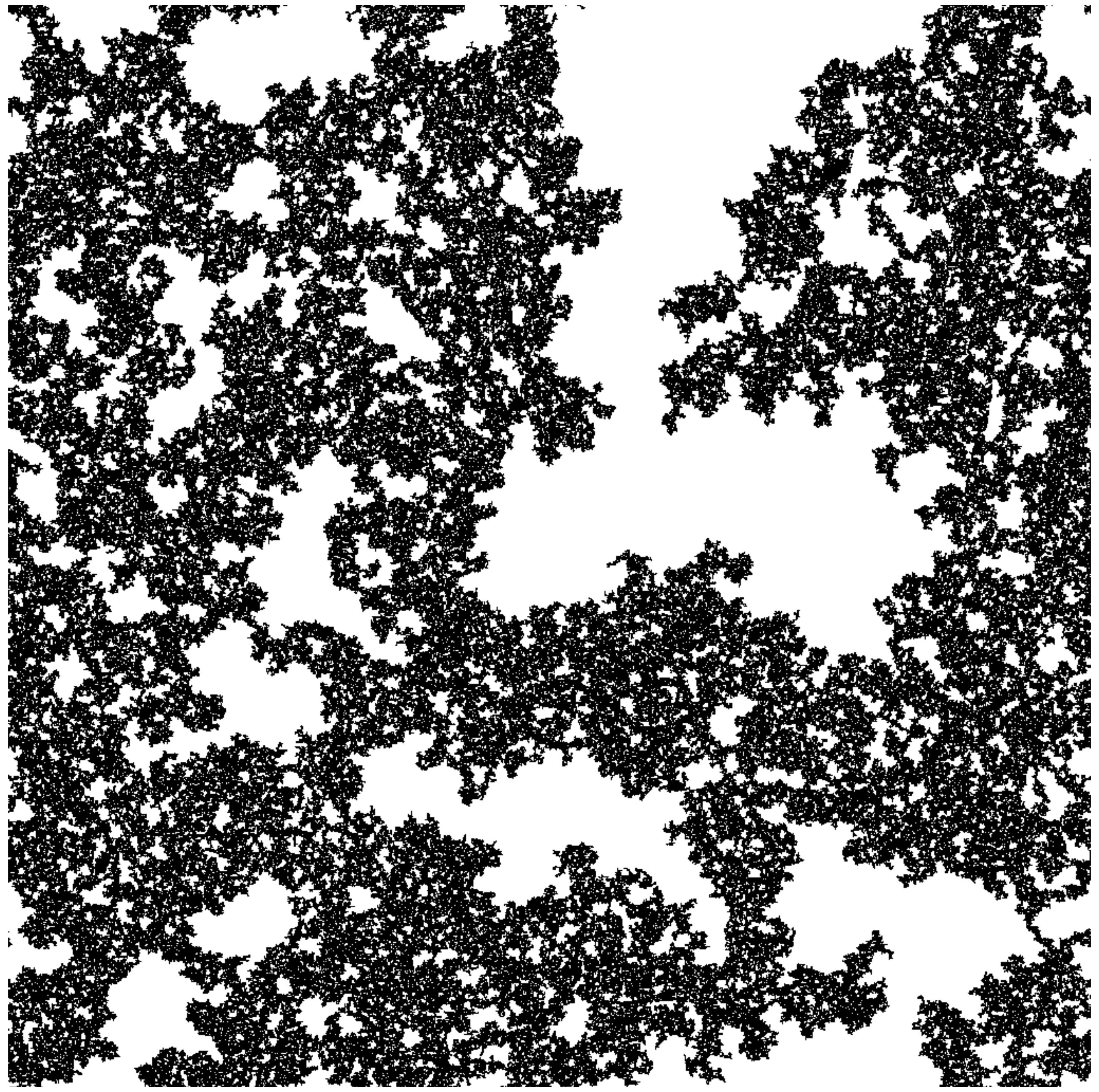
Ising model
at T = Tc ~ 2.269

"Fractal-like" structure: **very "tortuous" paths**

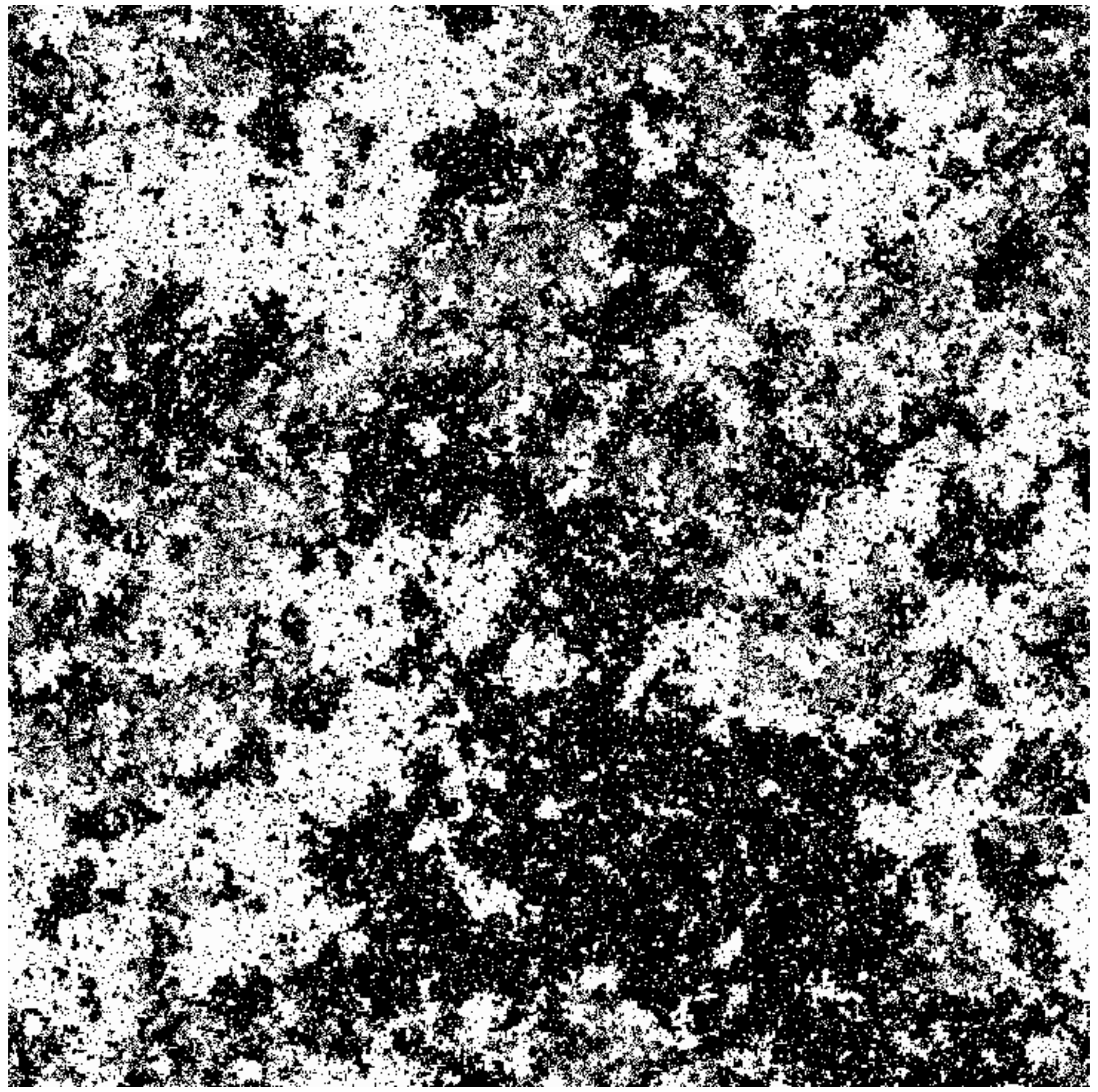
cf. recall:
conductivity fall to small value close to critical concentration of wholes,
paths become very few and tortuous just before the sheet fall apart.



Properties of Critical phenomena: Scale Invariance



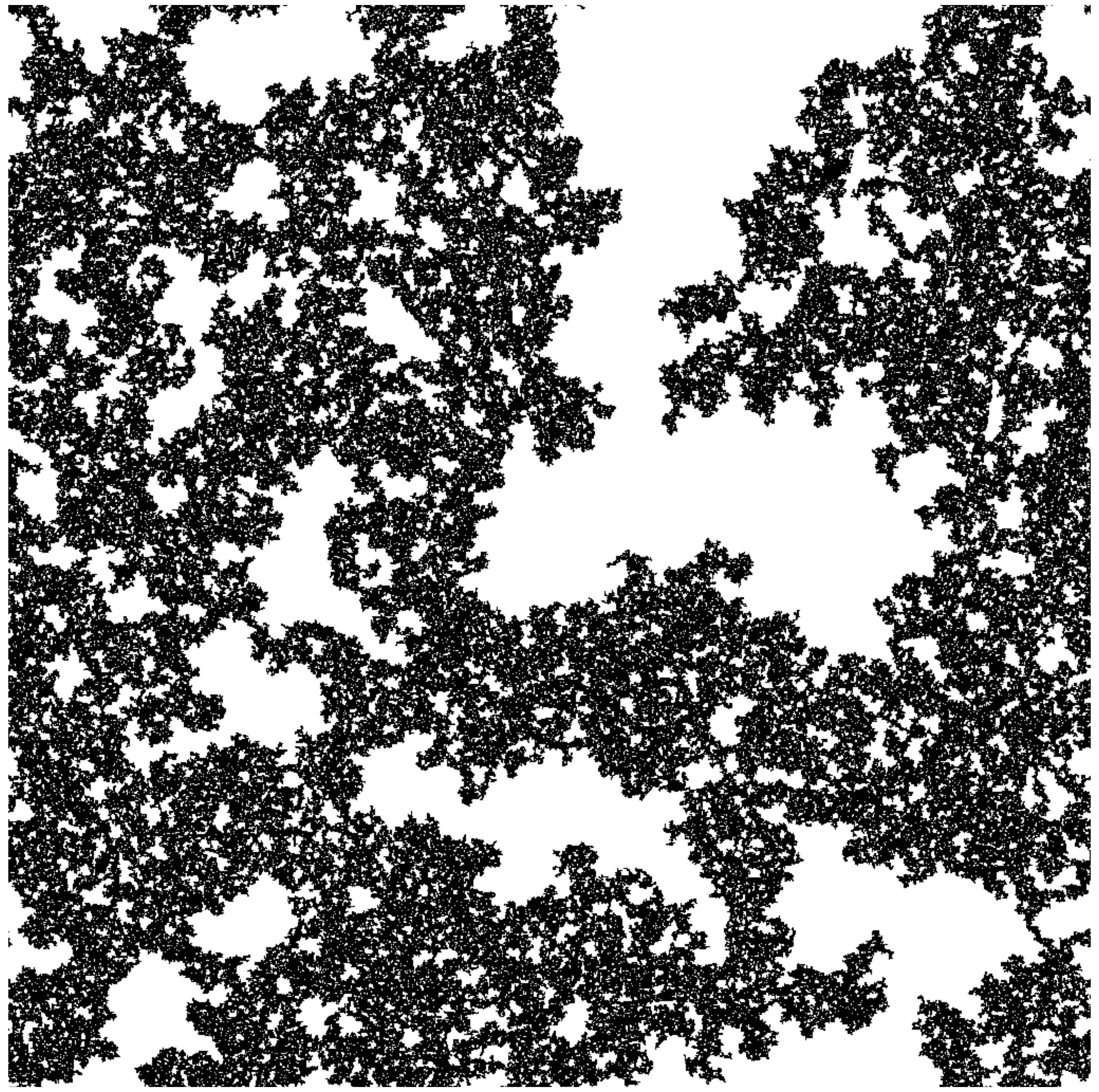
Bond percolation
at p = pc = 0.5



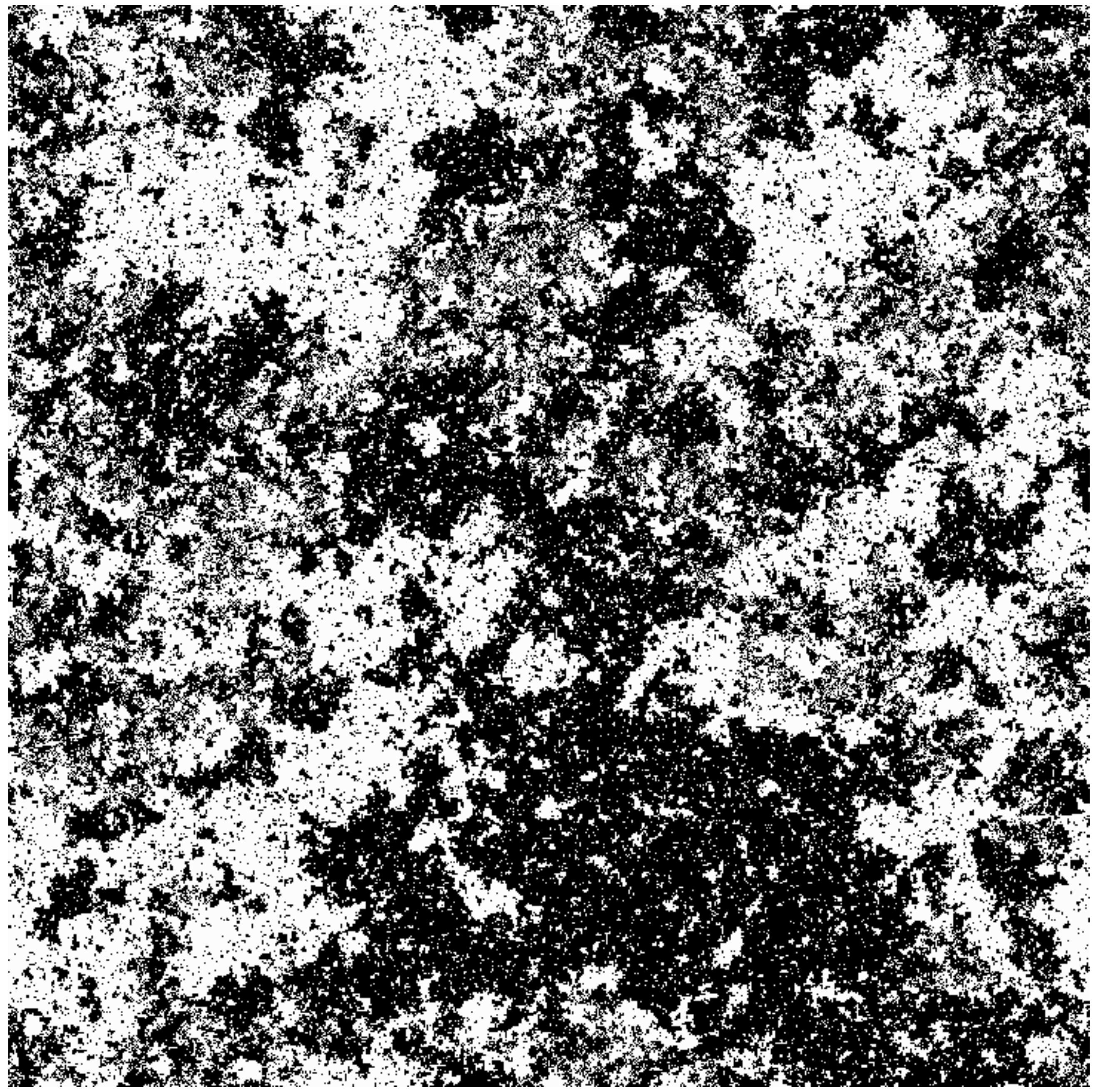
Ising model
at T = Tc ~ 2.269

Explanation for the **"Fractal-like" structure?**

Properties of Critical phenomena: Scale Invariance



Bond percolation
at p = pc = 0.5



Ising model
at T = Tc ~ 2.269

Explanation for the **"Fractal-like" structure?**

Close to criticality:

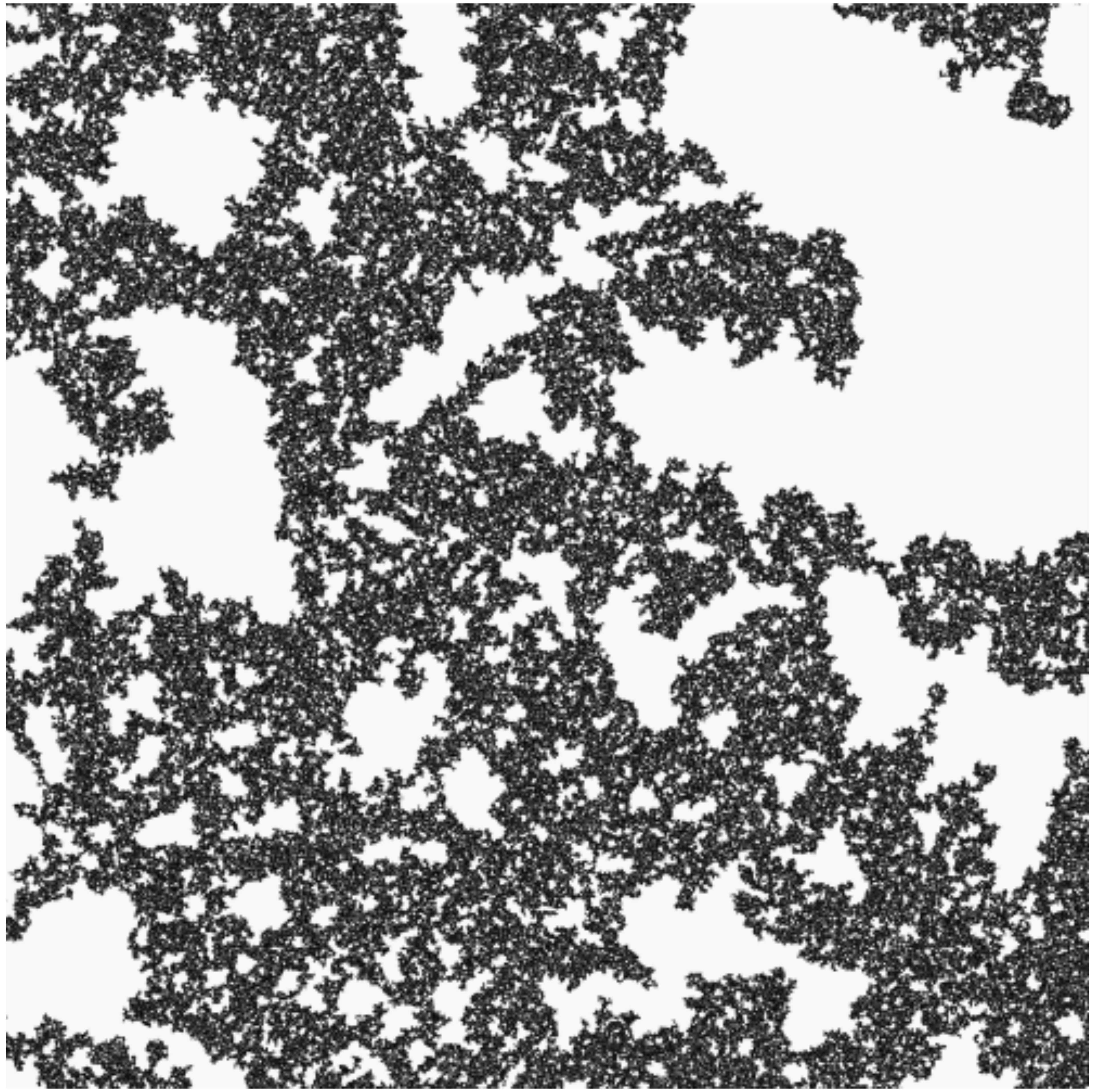
Diverging correlation length —> percolating cluster

Cluster of all sizes

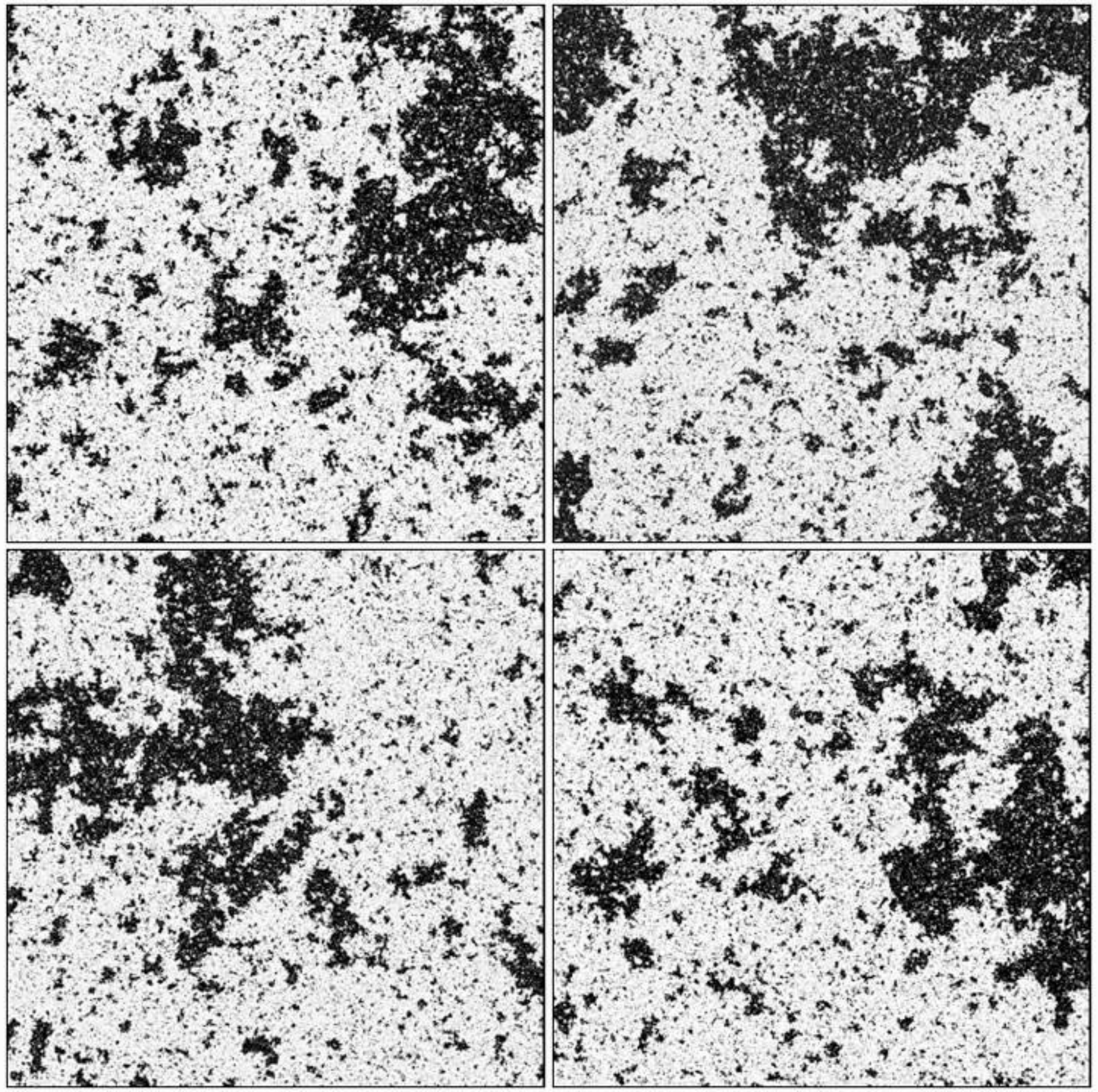
Diverging correlation length

Fluctuation of the magnetization at all scales

Scale invariance: **the system looks the same at all scales! (i.e. when zooming in)**



Scale invariance of critical percolating cluster



Scale invariance in the critical Ising Model

Fractals

Self-similar object

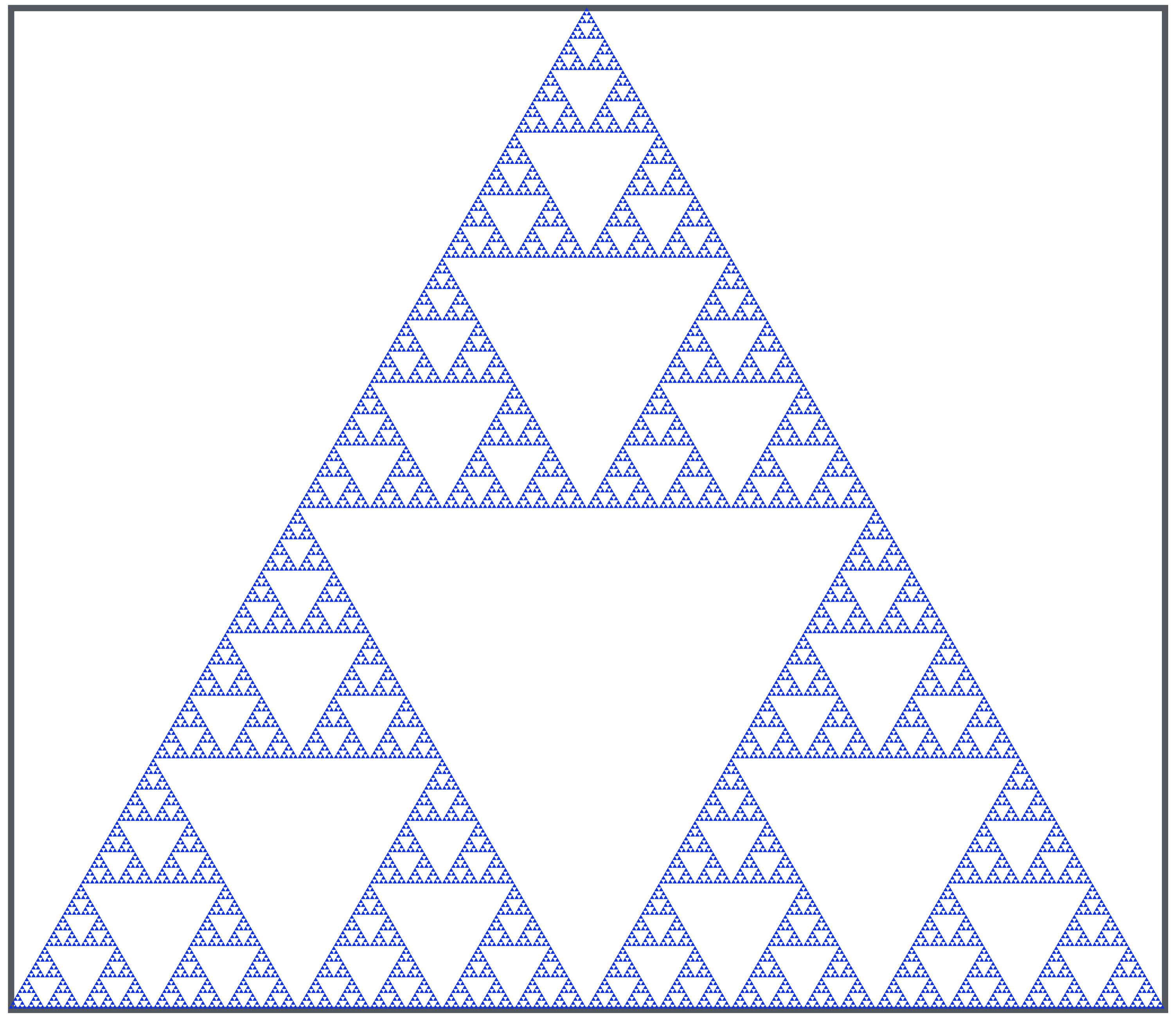
Self-similar object: is exactly or approximately similar to a part of itself

Self-similar object

Self-similar object: is exactly or approximately similar to a part of itself

It is a typical properties of **fractals**

Example: the Sierpiński triangle

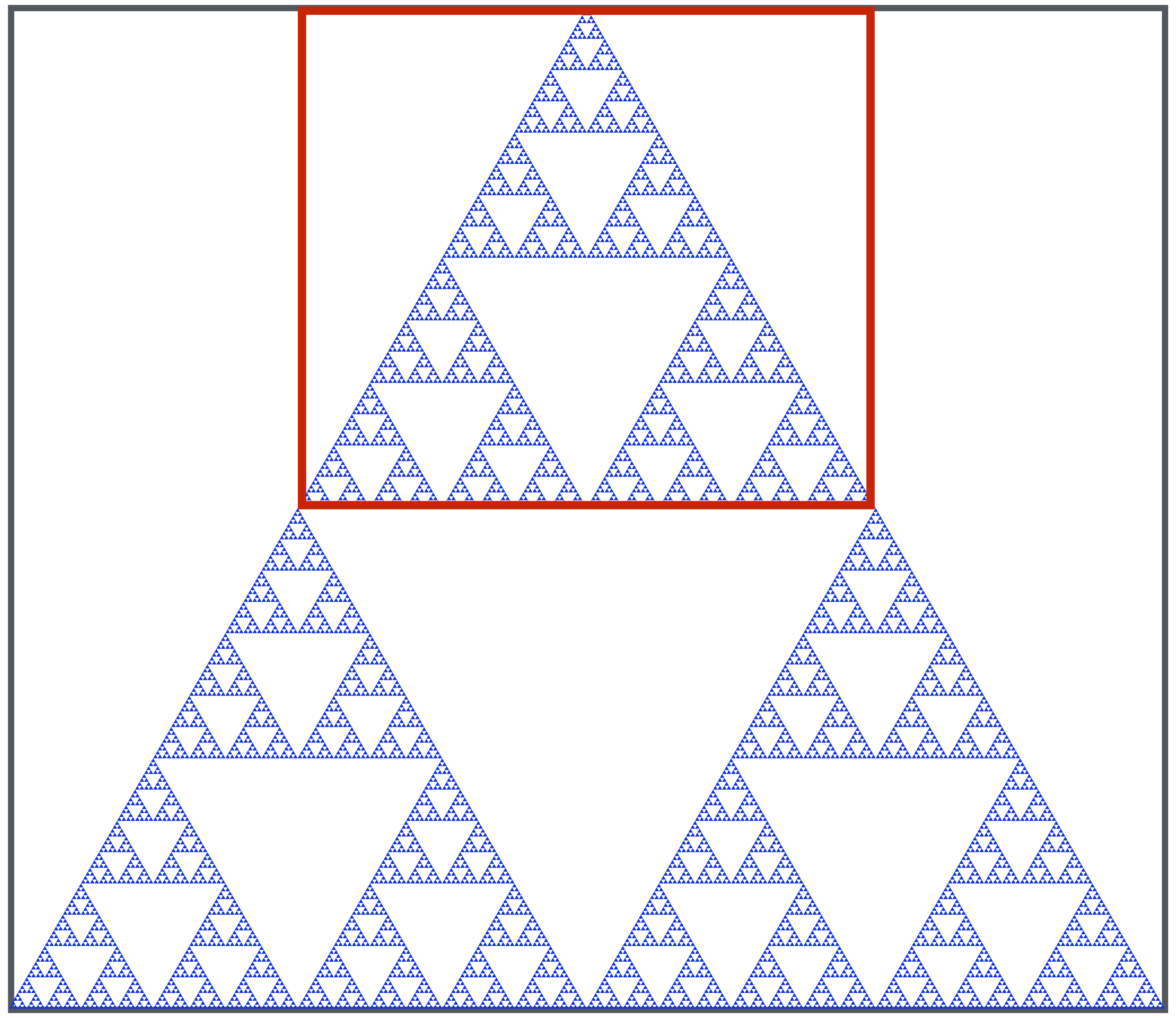


Self-similar object

Self-similar object: is exactly or approximately similar to a part of itself

It is a typical properties of **fractals**

Example: the Sierpiński triangle

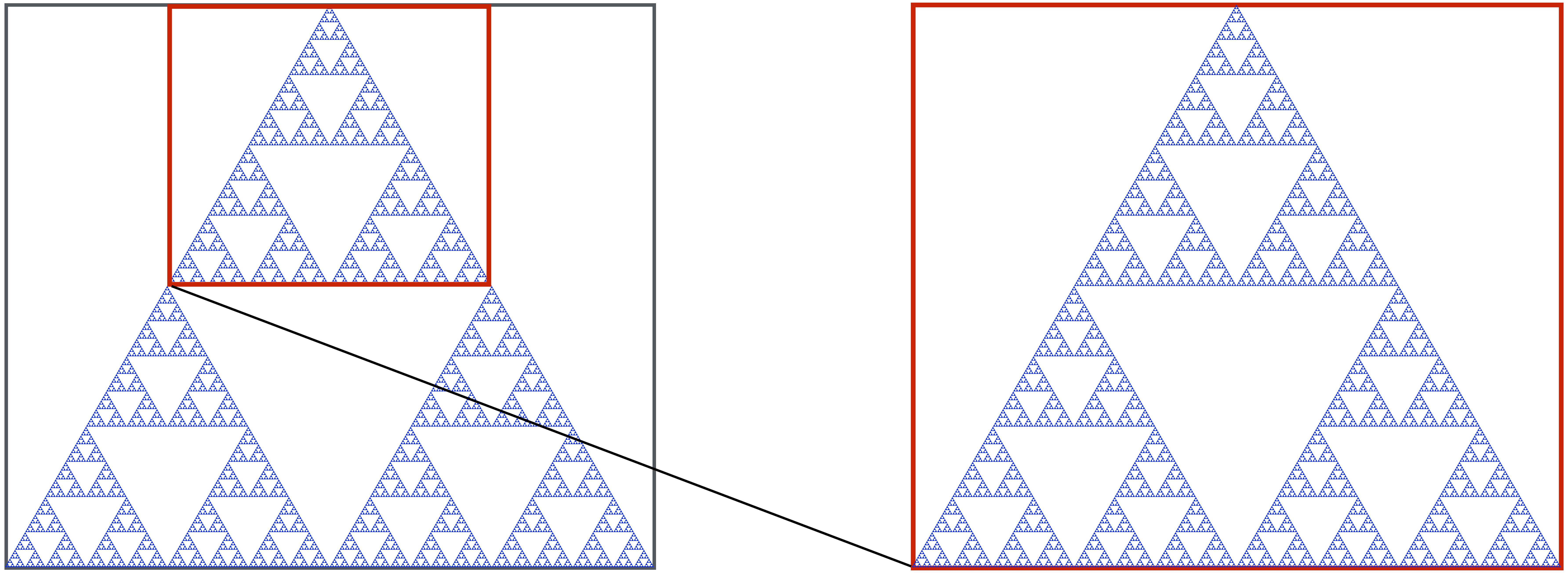


Self-similar object

Self-similar object: is exactly or approximately similar to a part of itself

It is a typical properties of **fractals**

Example: the Sierpiński triangle



Ex. of Fractals/Self-similarity in nature?

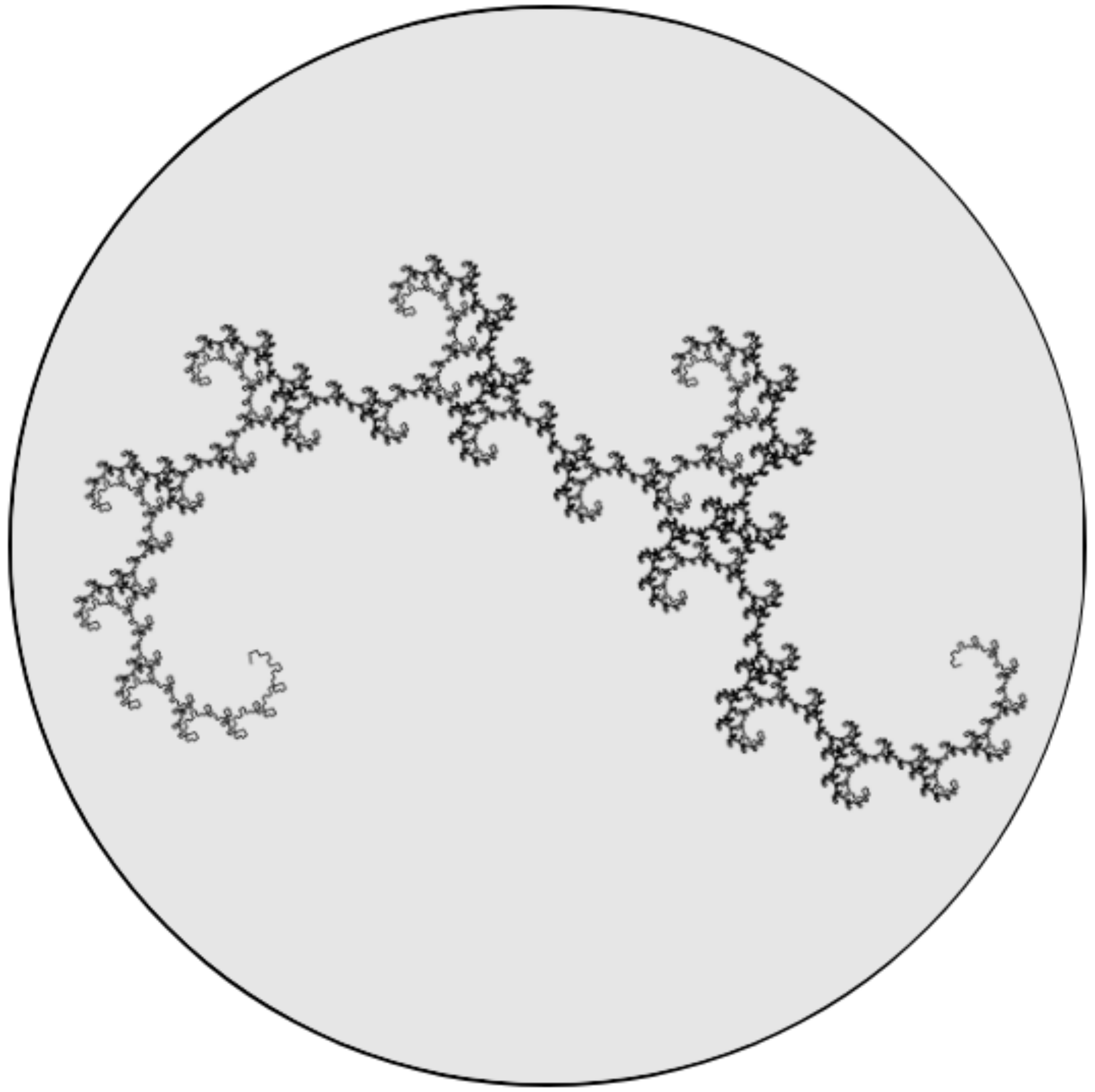
Many real world elements are **statistically self-similar**:

Romanesco broccoli

Frost crystals

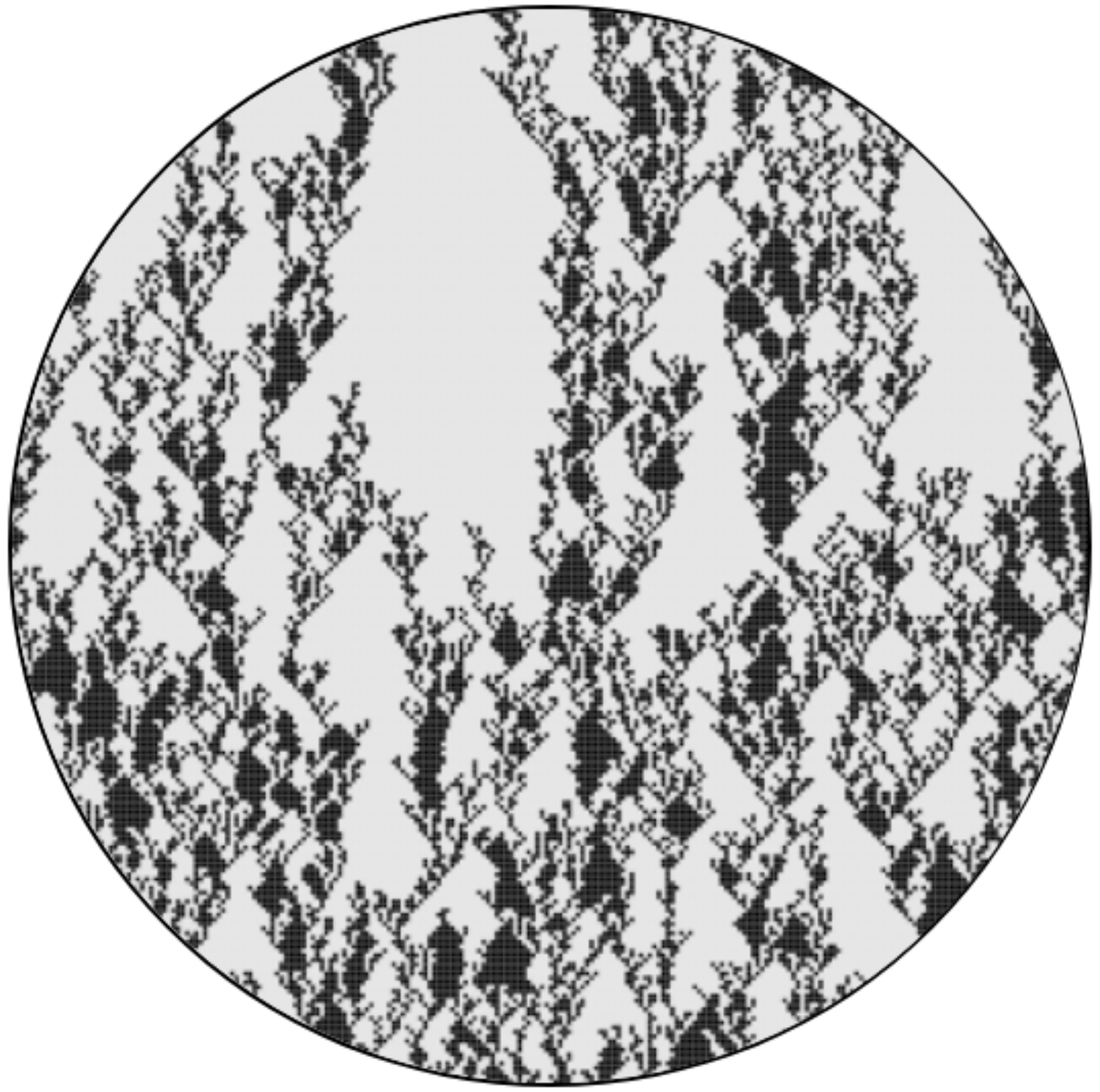
High-voltage breakdown within a block of acrylic glass

Examples of Models



Iteration process
in which elements of a structure are replaced by a smaller version of the whole structure

Lindenmayer Systems
Statistical iterative system:
how fractal patterns observed in natural systems, particularly structural properties of some plants

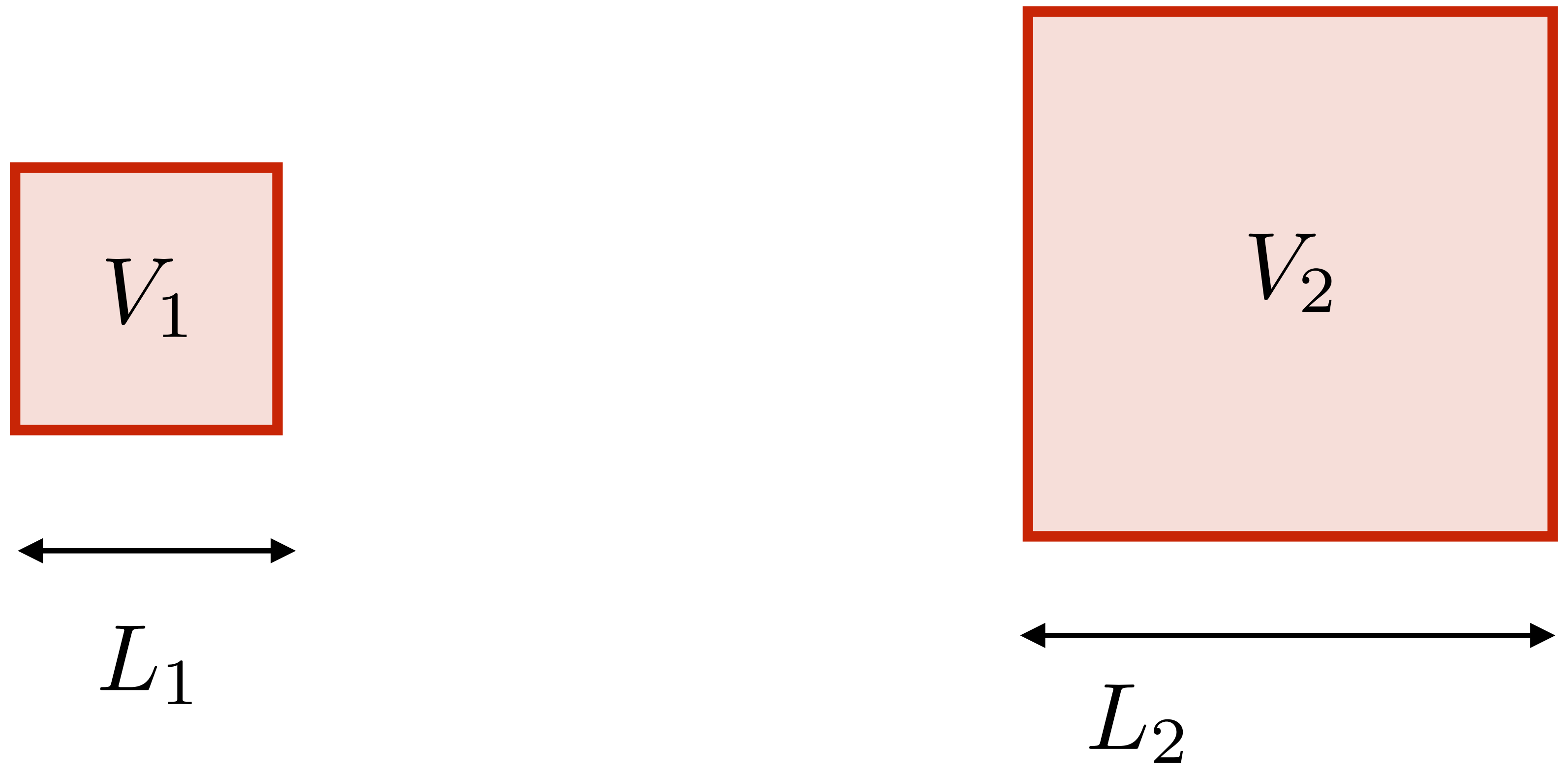


Stochastic cellular automaton
generating fractal growth patterns

Fractal dimension

Usual objects: expect volume (or mass) scales as $V \propto L^d$ where d = dimension of the object

= is an integer

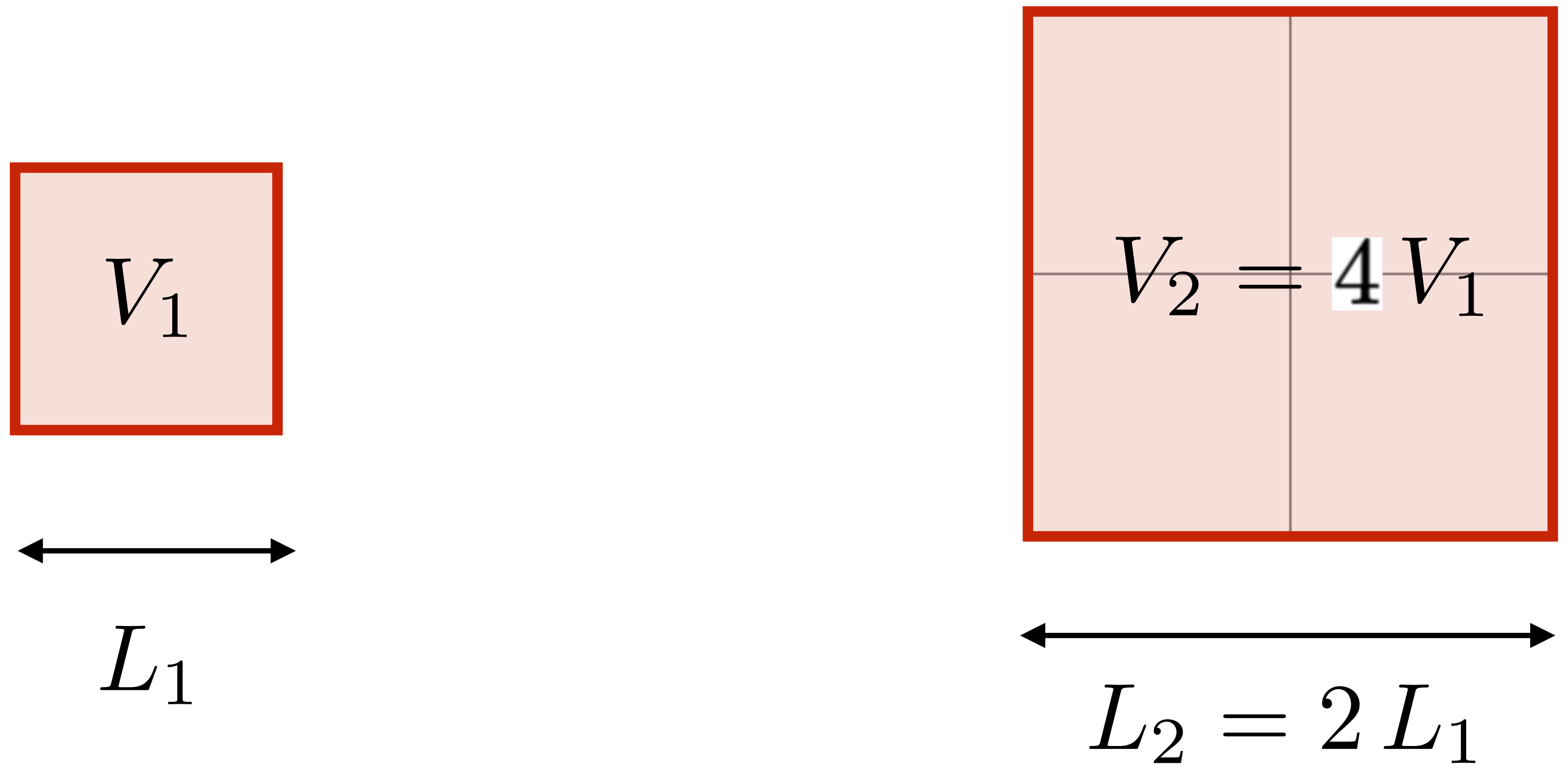


$$\rho = \frac{V_1}{L_1^d} = \text{constant}$$

Fractal dimension

Usual objects: expect volume (or mass) scales as $V \propto L^d$ where d = dimension of the object

= is an integer



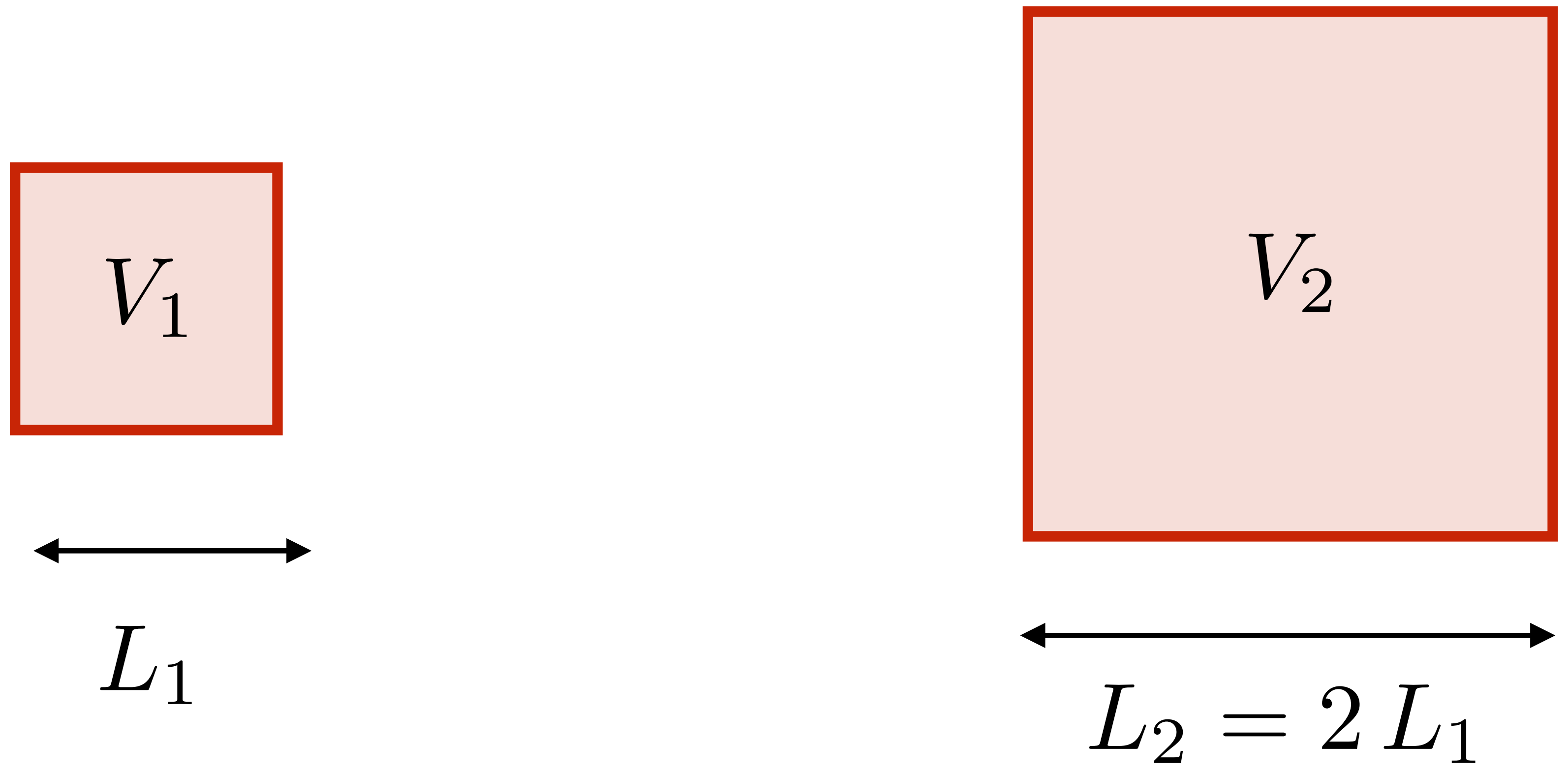
$$L_2 = \lambda\, L_1 \qquad\qquad V_2 = \lambda^d\, V_1$$

Density is independent of the length scale at which it is measured

$$\rho = \frac{V_1}{L_1^d} = \frac{V_2}{L_2^d} = \text{constant}$$

Fractal dimension

Usual objects: usually the density is independent of the length scale at which it is measured: $\rho = \frac{V_1}{L_1^d} = \frac{V_2}{L_2^d}$



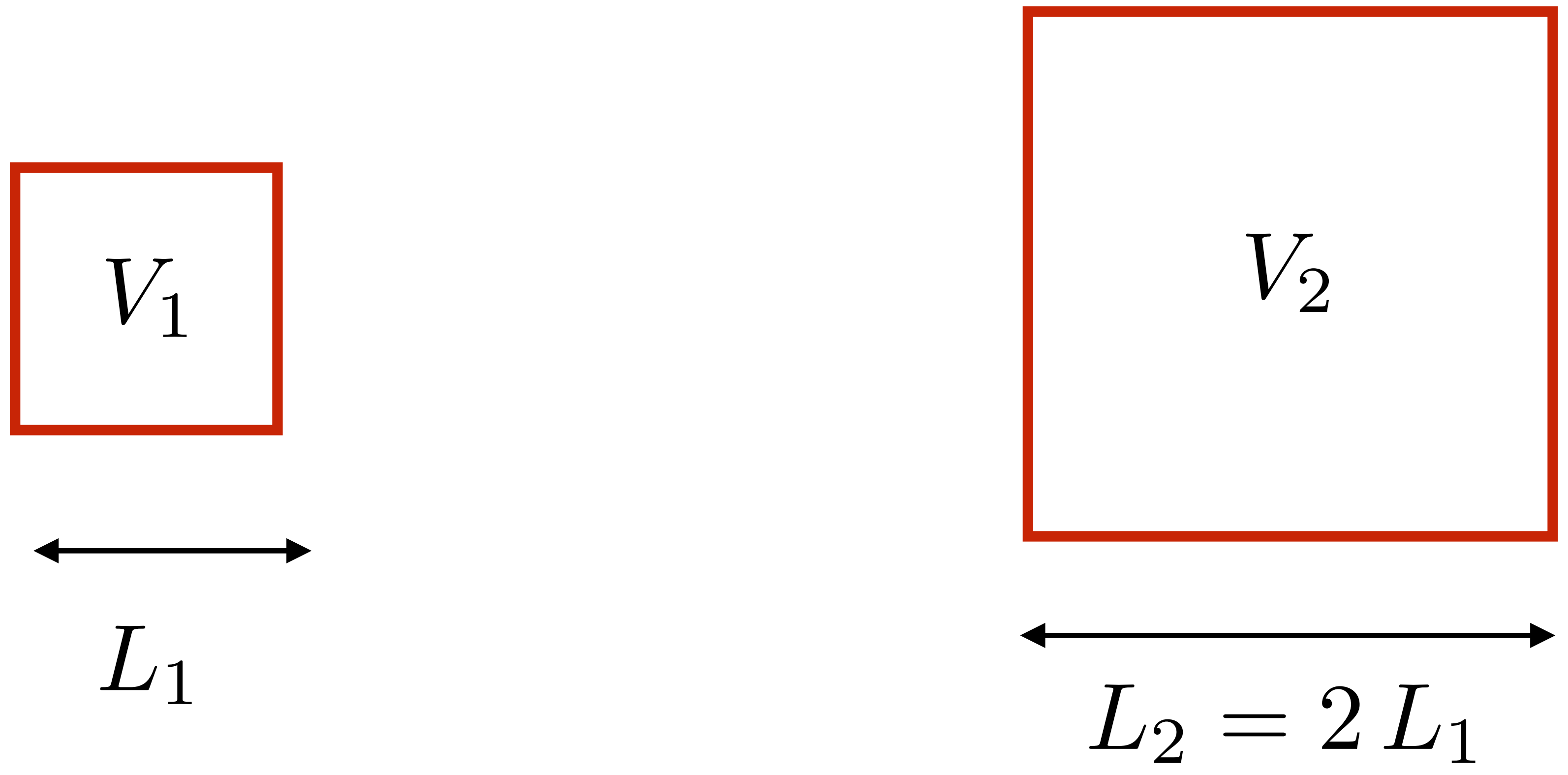
L_1 $\qquad$ $L_2 = \lambda\, L_1$

V_1 $\qquad$ $V_2 = \lambda^d\, V_1$

Fractal objects: this idea $[V_2 = \lambda^d\, V_1]$ is extended to fractal objects —> defines the notion of **fractal dimension**

Fractal dimension

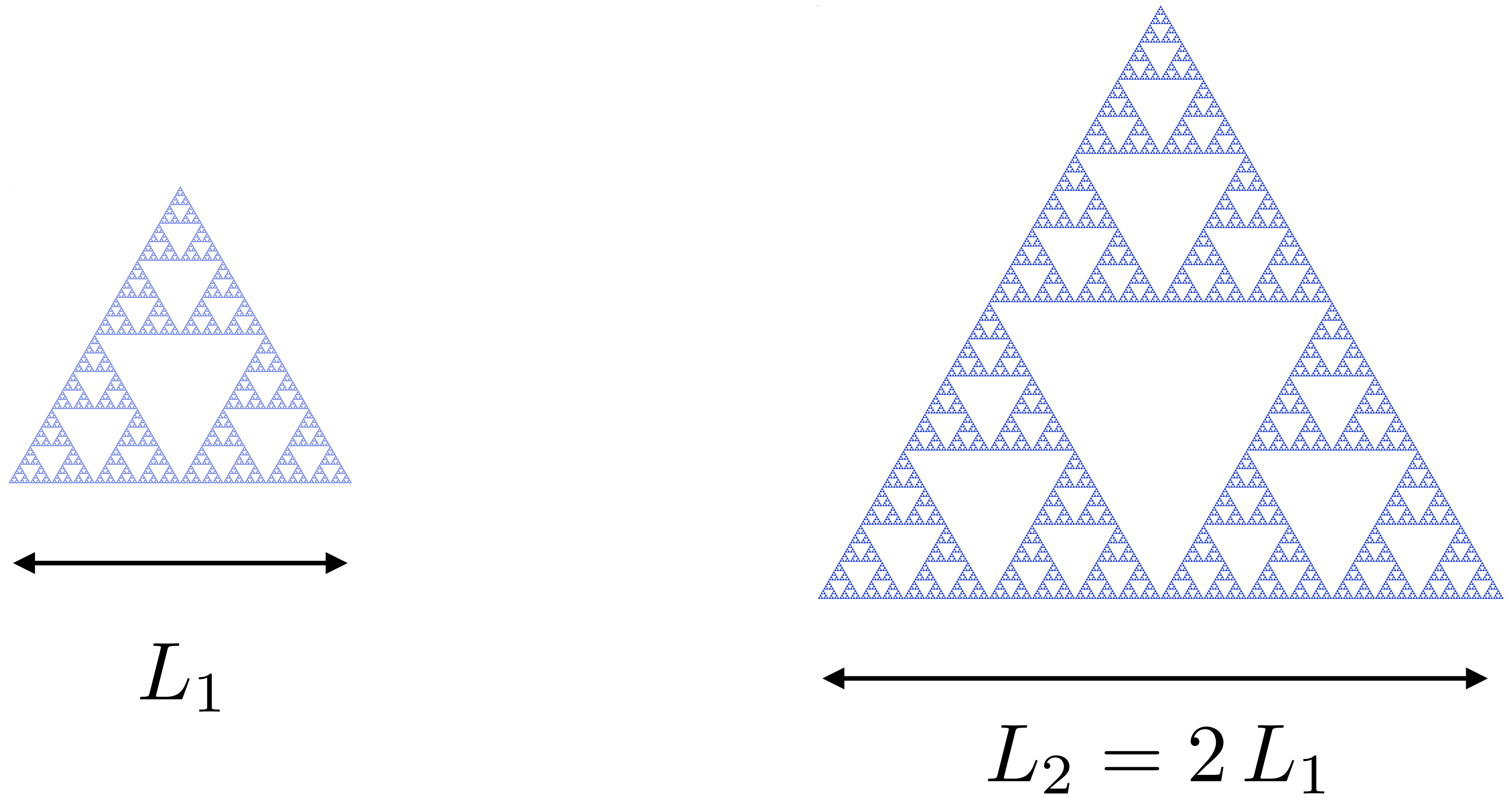
Usual objects: usually the density is independent of the length scale at which it is measured: $\rho = \frac{V_1}{L_1^d} = \frac{V_2}{L_2^d}$



L_1 $\qquad L_2 = \lambda\, L_1$

V_1 $\qquad V_2 = \lambda^d\, V_1$

Fractal objects: this idea $[V_2 = \lambda^d\, V_1\,]$ is extended to fractal objects —> defines the notion of **fractal dimension**

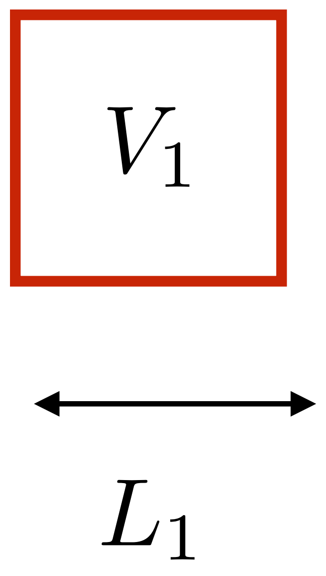


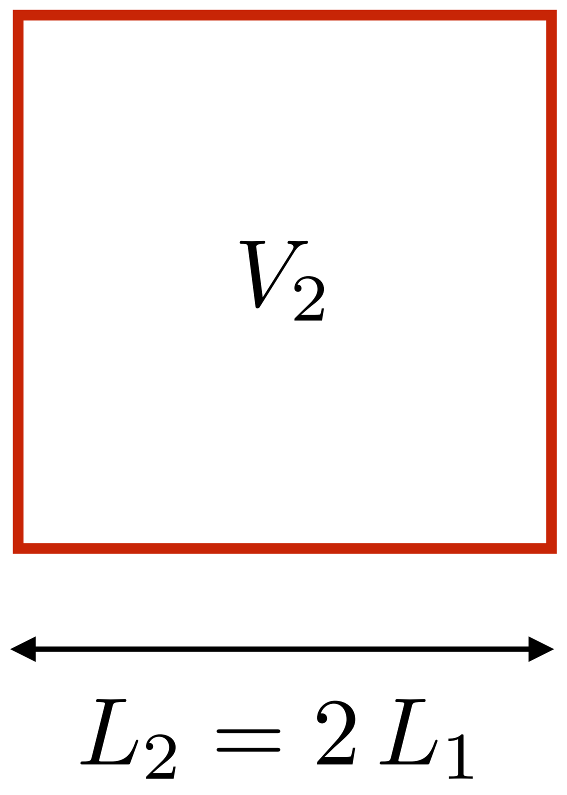
L_1 $\qquad L_2 = 2\,L_1$

V_1

Fractal dimension

Usual objects: usually the density is independent of the length scale at which it is measured: $\rho = \frac{V_1}{L_1^d} = \frac{V_2}{L_2^d}$

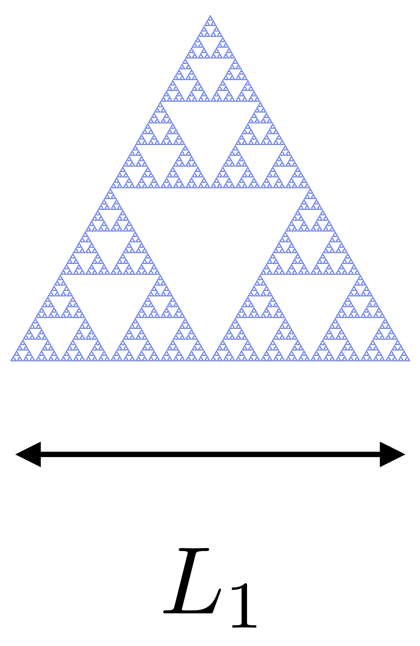


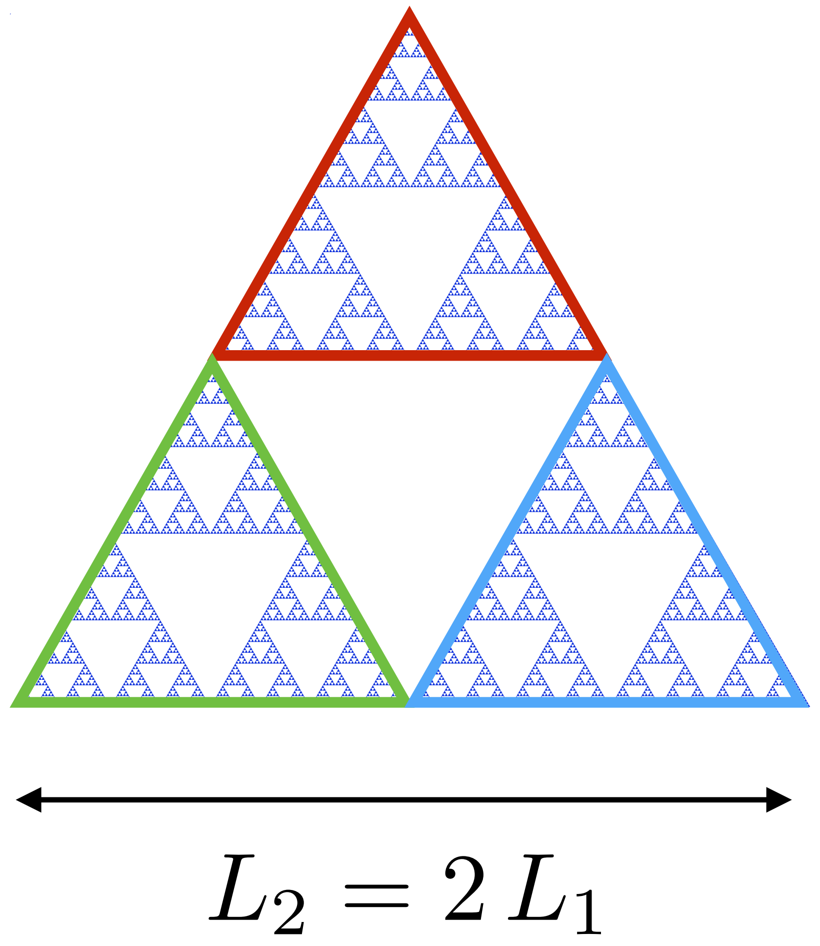


L_1 $\qquad$ $L_2 = \lambda\, L_1$

V_1 $\qquad$ $V_2 = \lambda^d\, V_1$

Fractal objects: this idea $[V_2 = \lambda^d\, V_1\,]$ is extended to fractal objects —> defines the notion of **fractal dimension**



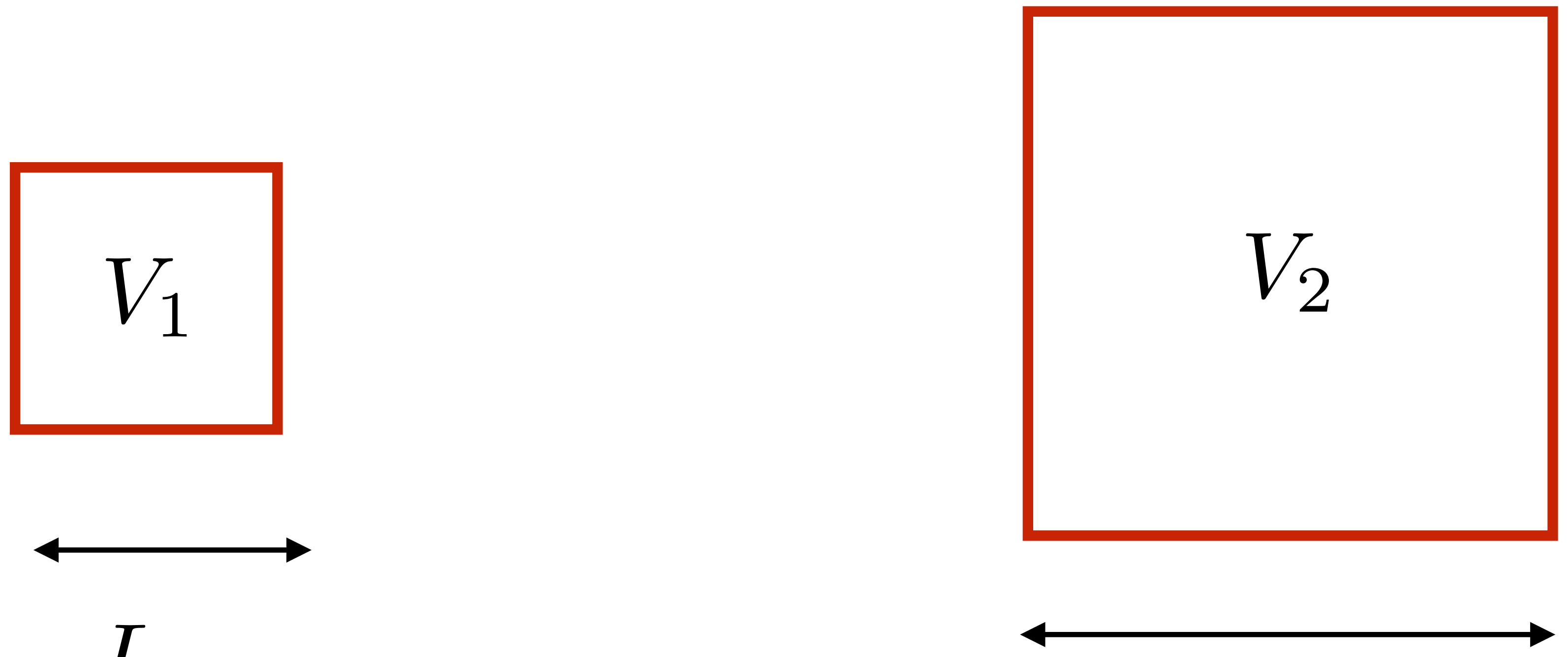


L_1 $\qquad$ $L_2 = 2\,L_1$

V_1 $\qquad$ $V_2 = 3\,V_1 \;= 2^{d_F}\,V_1$

Fractal dimension

Usual objects: usually the density is independent of the length scale at which it is measured: $\rho = \dfrac{V_1}{L_1^d} = \dfrac{V_2}{L_2^d}$

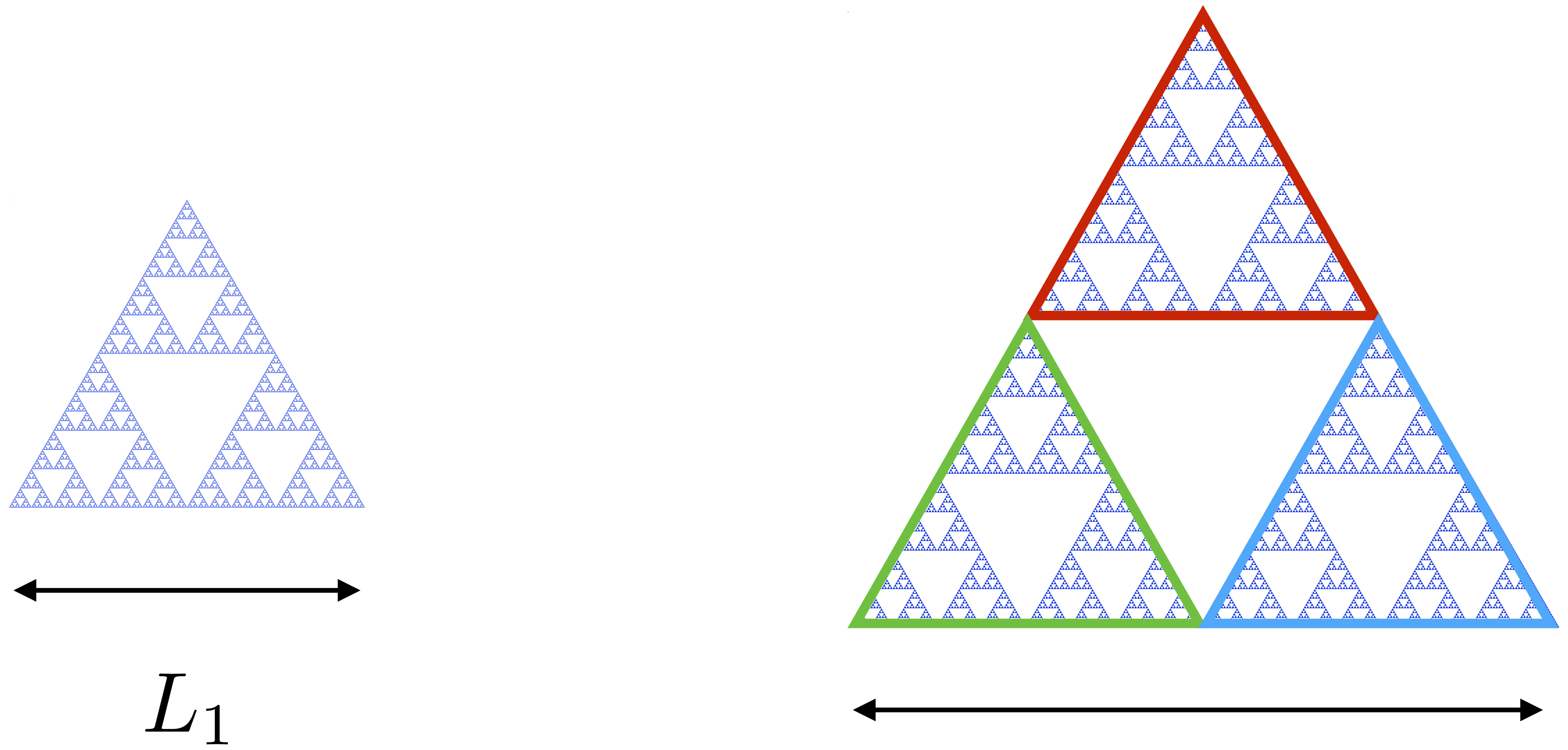


L_1 $\quad$ $L_2 = 2\,L_1$

$L_1 \qquad L_2 = \lambda\, L_1$

$V_1 \qquad V_2 = \lambda^d\, V_1$

Fractal objects: this idea $[V_2 = \lambda^d\, V_1\,]$ is extended to fractal objects —> defines the notion of **fractal dimension**



L_1 $\quad$ $L_2 = 2\,L_1$

$L_1 \qquad L_2 = 2\,L_1$

$V_1 \qquad V_2 = 3\,V_1 \; = 2^{d_F}\, V_1$

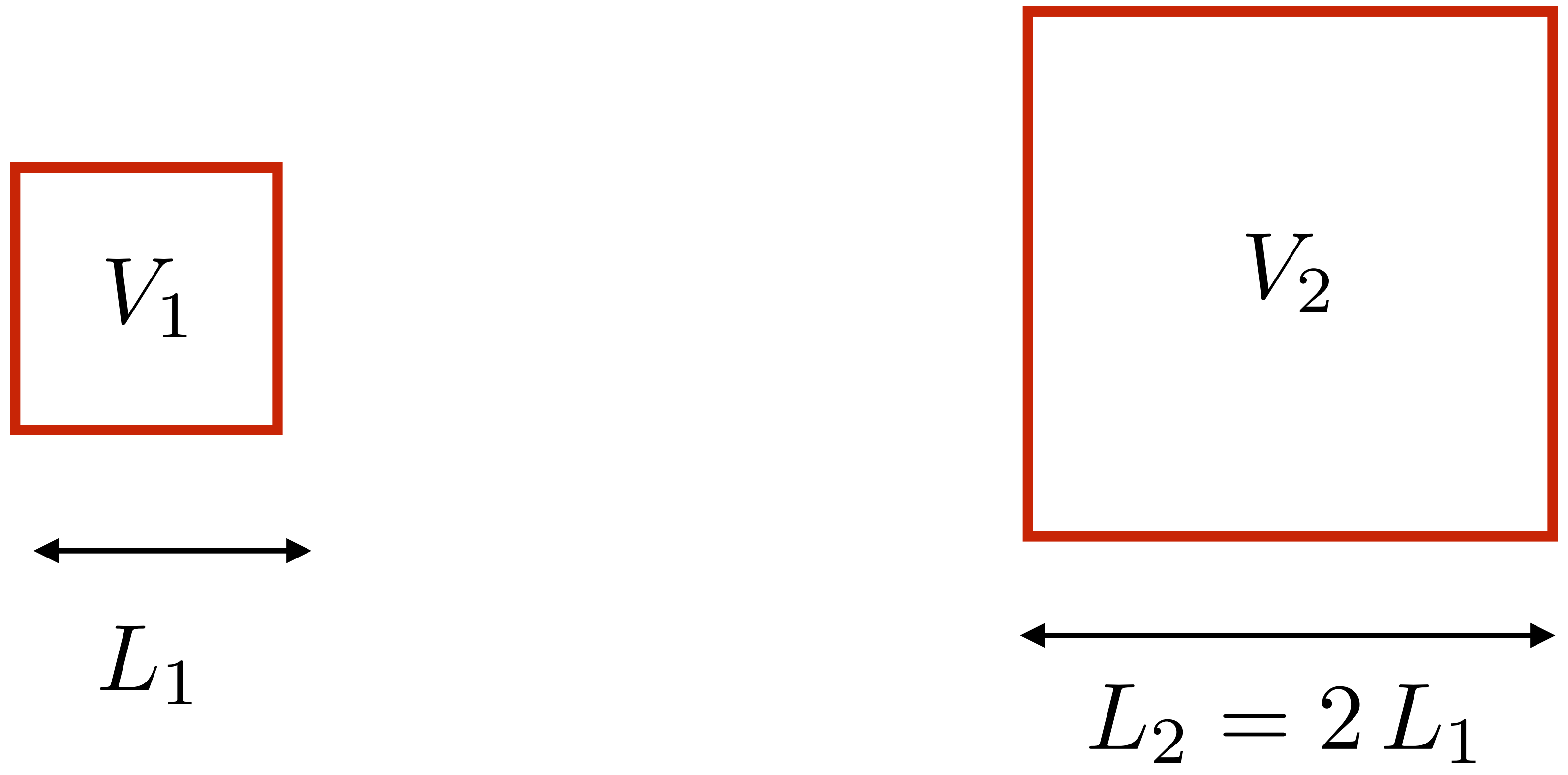
$$d_F = \frac{\ln 3}{\ln 2} \sim 1.58 \quad \text{is not an integer!}$$

$1 < d_F < 2$

—> defines notion of **fractal dimension**

Fractal dimension

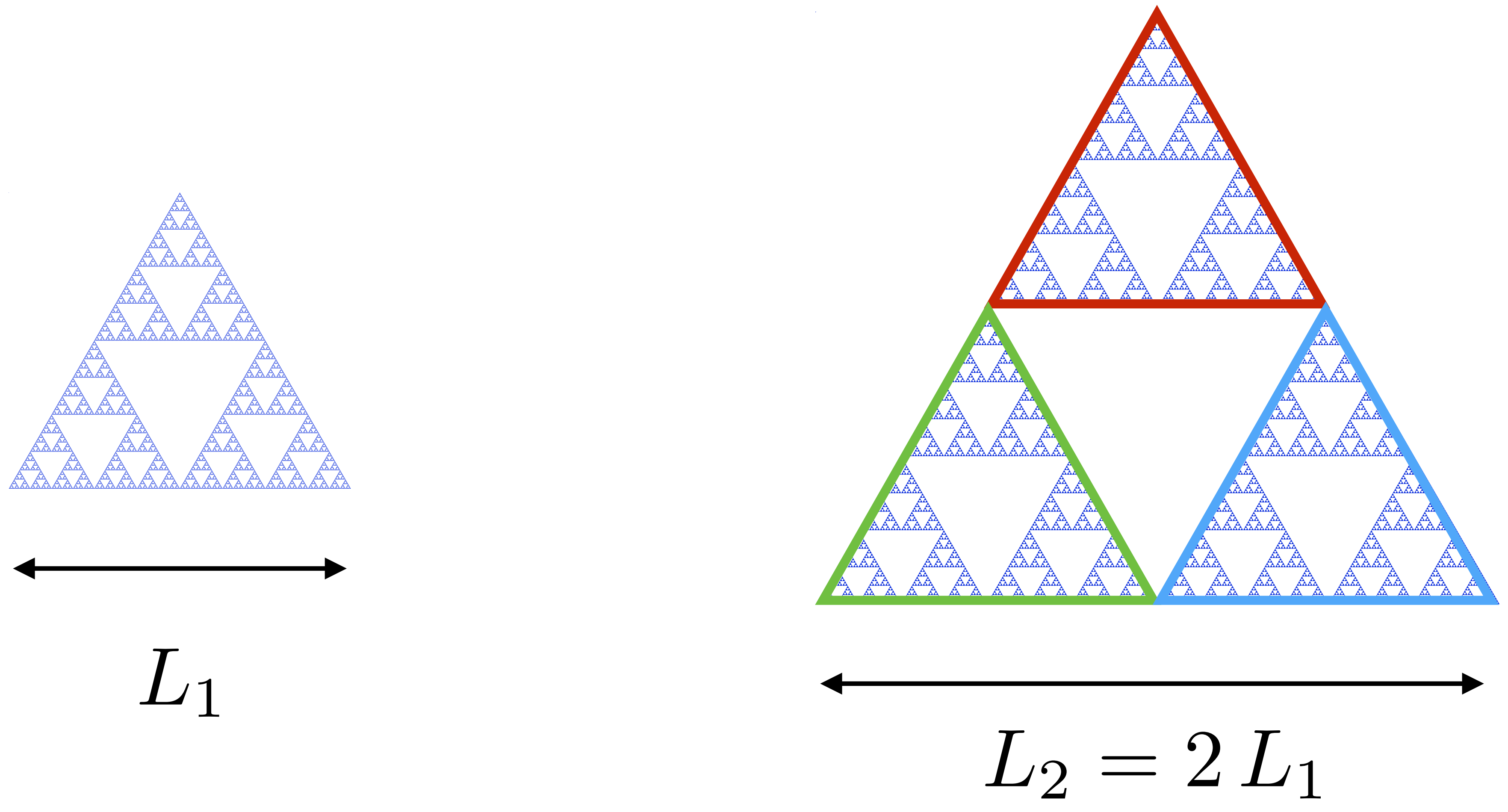
Usual objects: usually the density is independent of the length scale at which it is measured: $\rho = \dfrac{V_1}{L_1^d} = \dfrac{V_2}{L_2^d}$



L_1 $\qquad$ $L_2 = \lambda\, L_1$

V_1 $\qquad$ $V_2 = \lambda^d\, V_1$

Fractal objects: this idea $[V_2 = \lambda^d\, V_1\,]$ is extended to fractal objects —> defines the notion of **fractal dimension**



L_1 $\qquad$ $L_2 = 2\,L_1$

V_1 $\qquad$ $V_2 = 3\,V_1 = 2^{d_F}\,V_1$

$$\rho_2 = \frac{V_2}{L_2^d} = \frac{\lambda^{d_F}}{\lambda^d}\,\frac{V_1}{L_1^d} = \frac{1}{\lambda^{d-d_F}}\,\rho_1 \qquad 1 < d_F < 2 = d$$

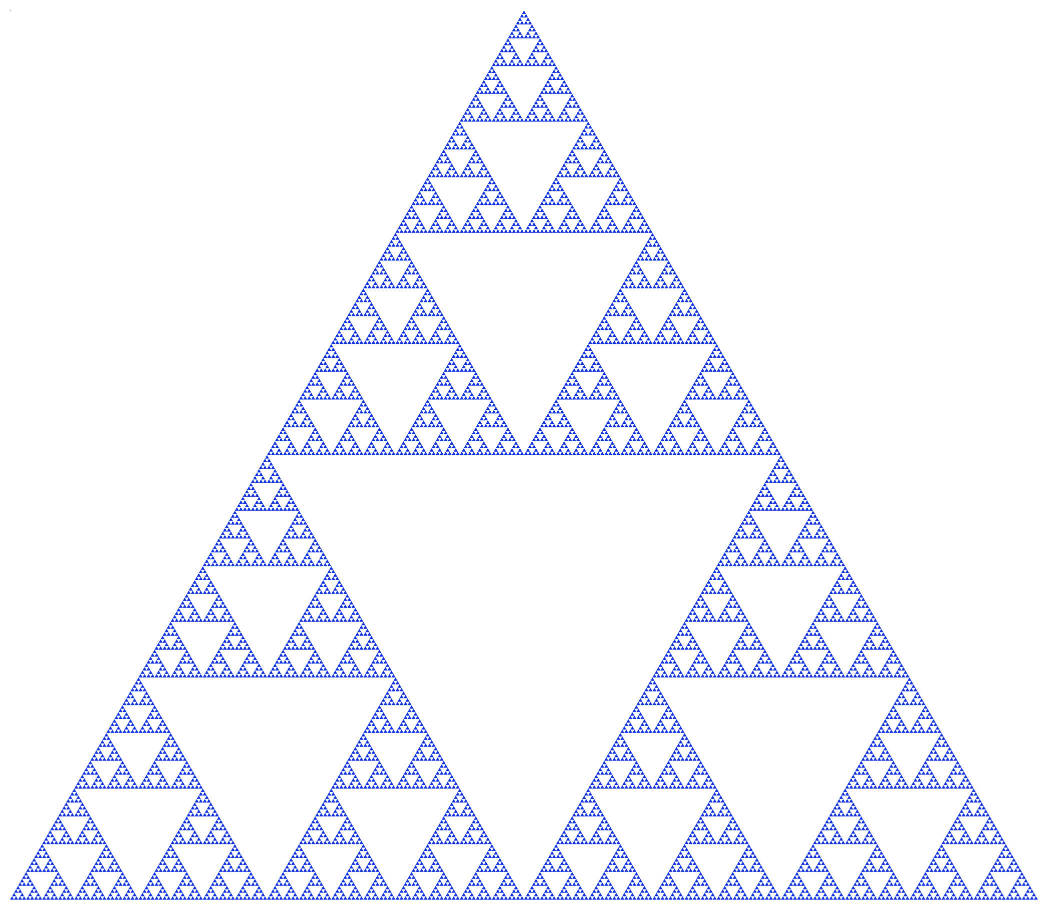
Density depends on the scale! $\quad \rho_2 = \dfrac{1}{\lambda^{d-d_F}}\,\rho_1$

Decrease for larger lambda

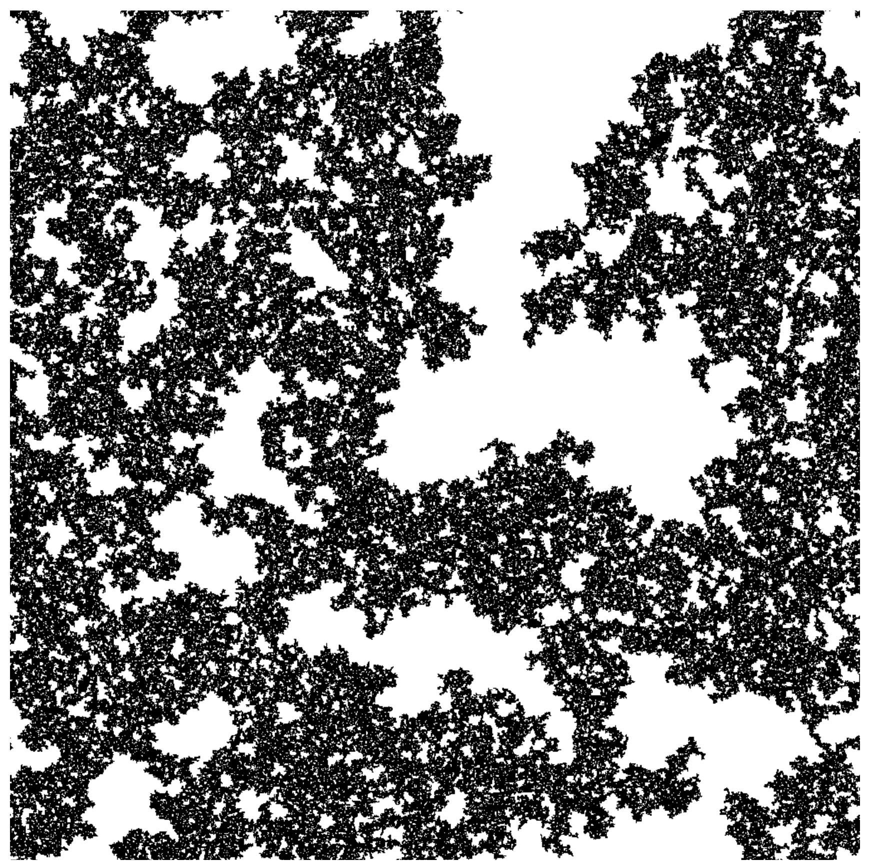
At criticality:

Statistical Self-similarity and Scale Invariance

Exact self-similarity

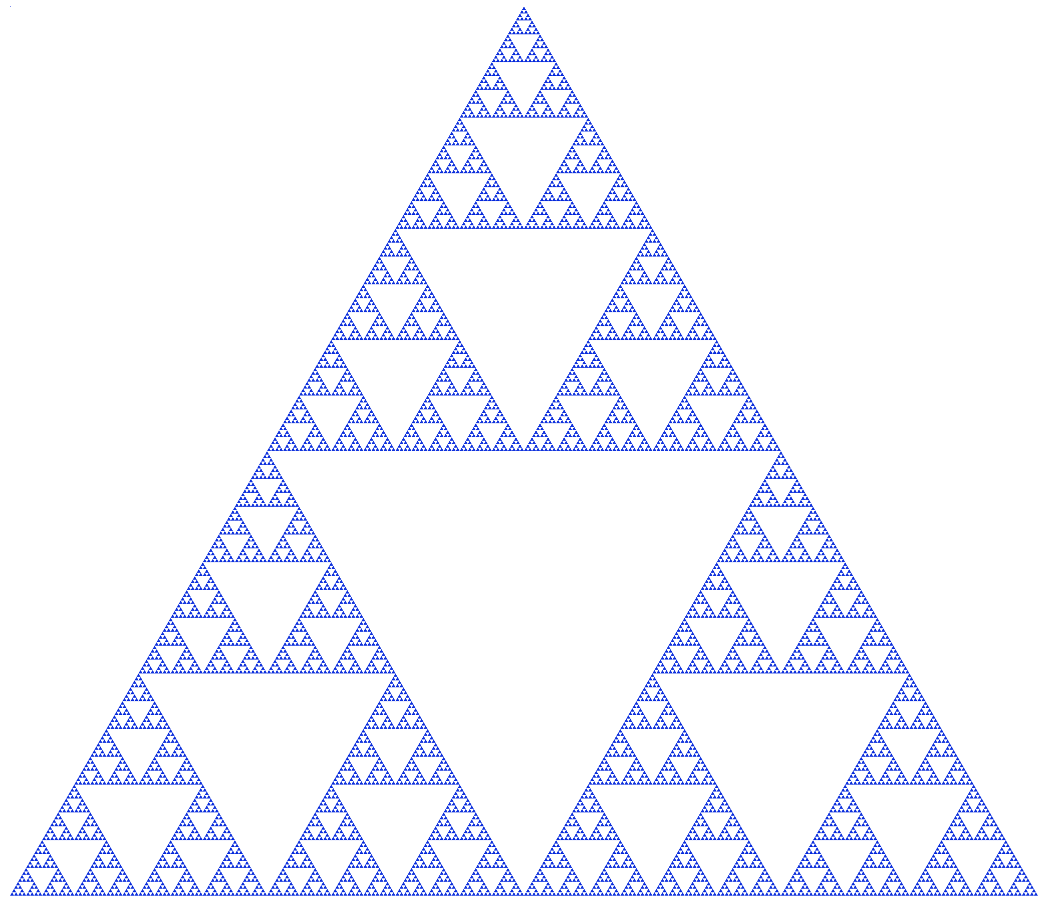


Scale invariance



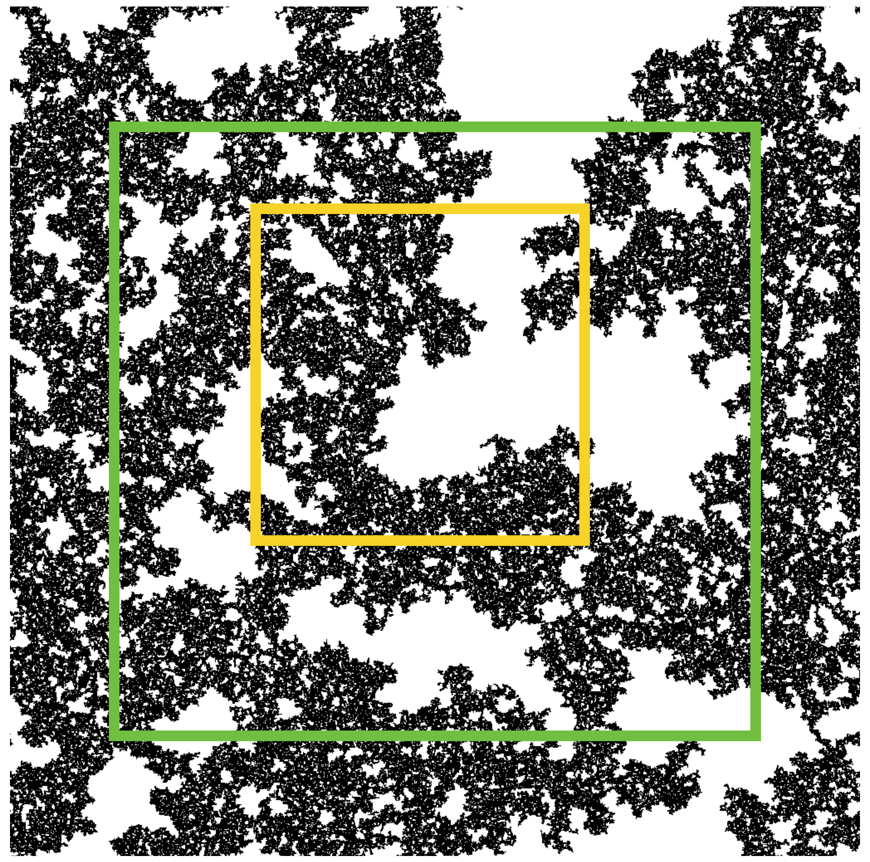
At criticality:
Statistical Self-similarity and Scale Invariance

Exact self-similarity



Self-similar only for specific values of scaling factor λ

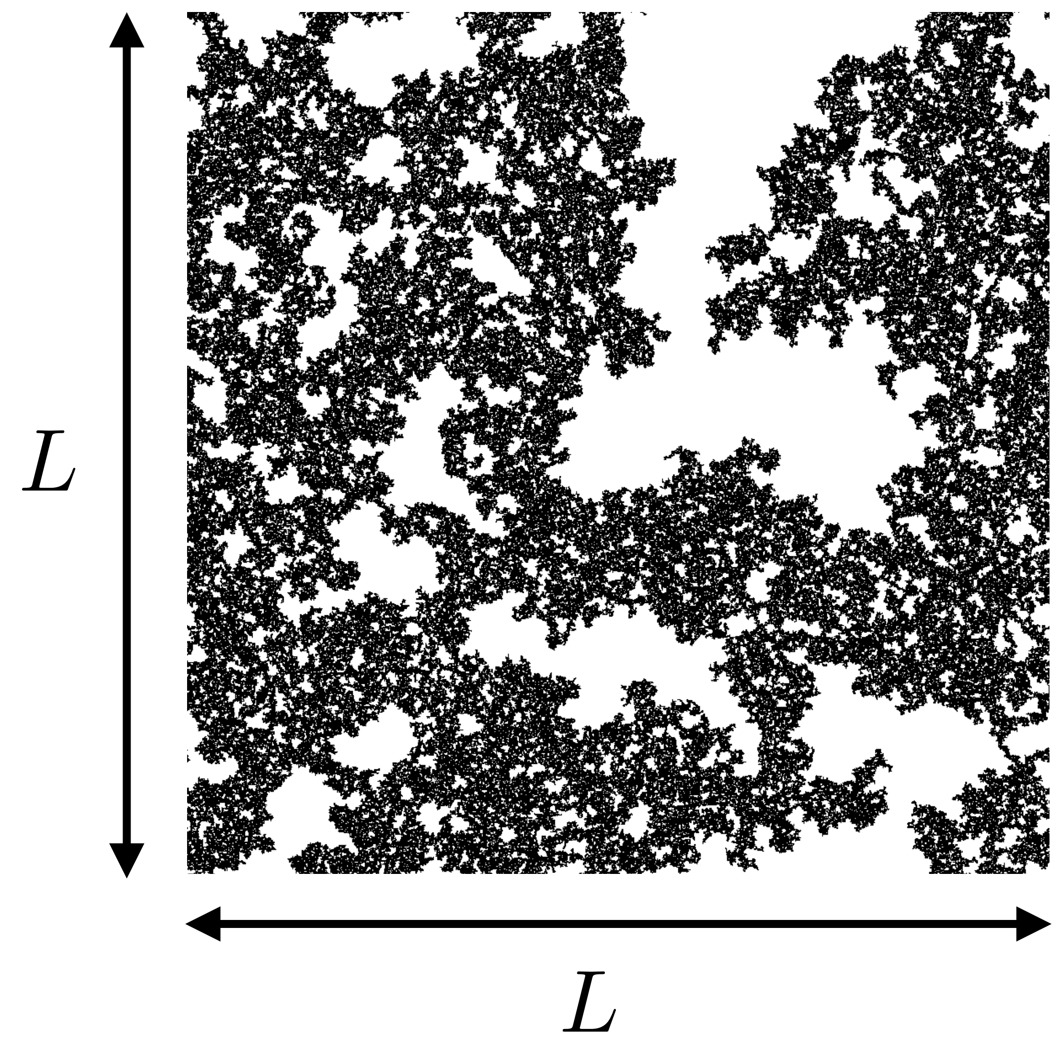
Scale invariance



There is a small piece of object that is similar to the whole, at any choice of the scaling factor λ

is self-similar for all choices of scaling factor λ

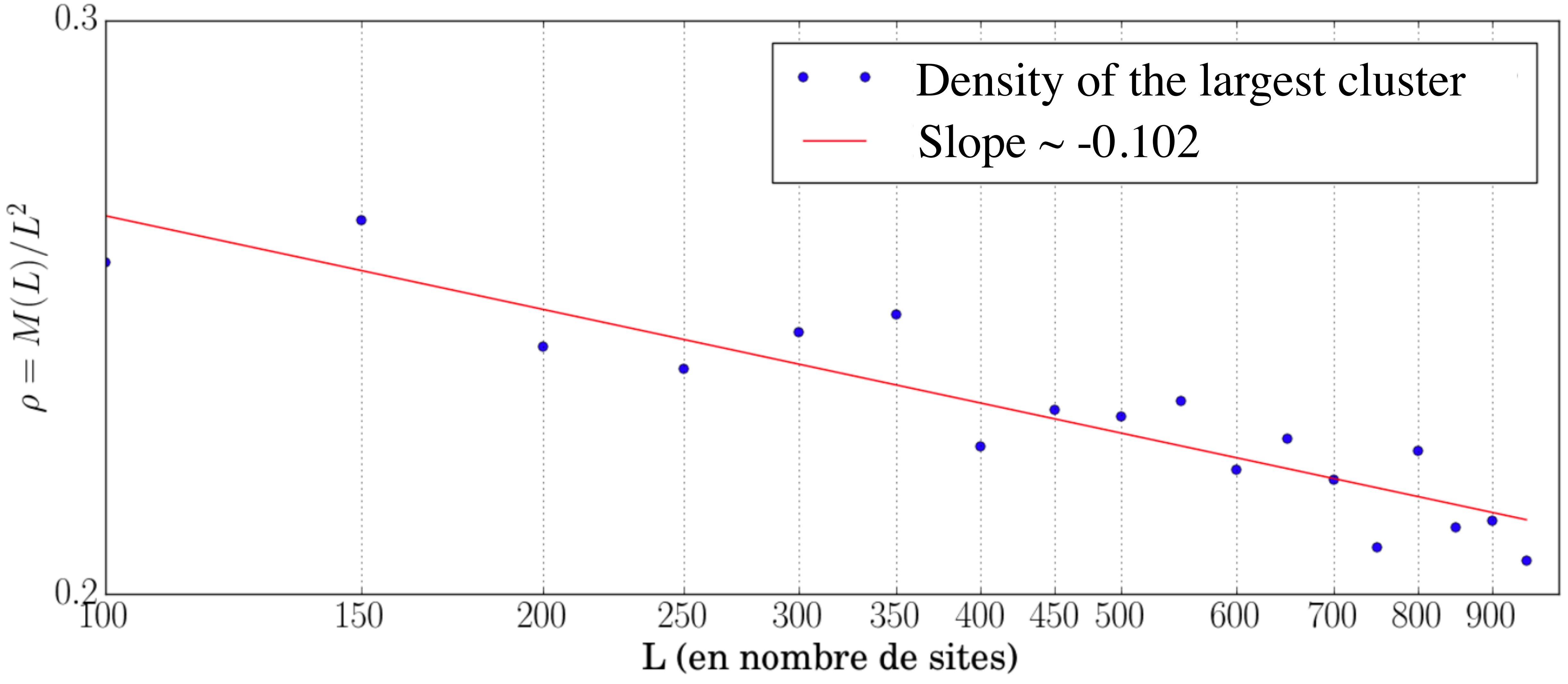
At criticality: Scale Invariance



Site percolation
on a square lattice

at p = $p_c \sim 0.59$

ρ decreases for larger L

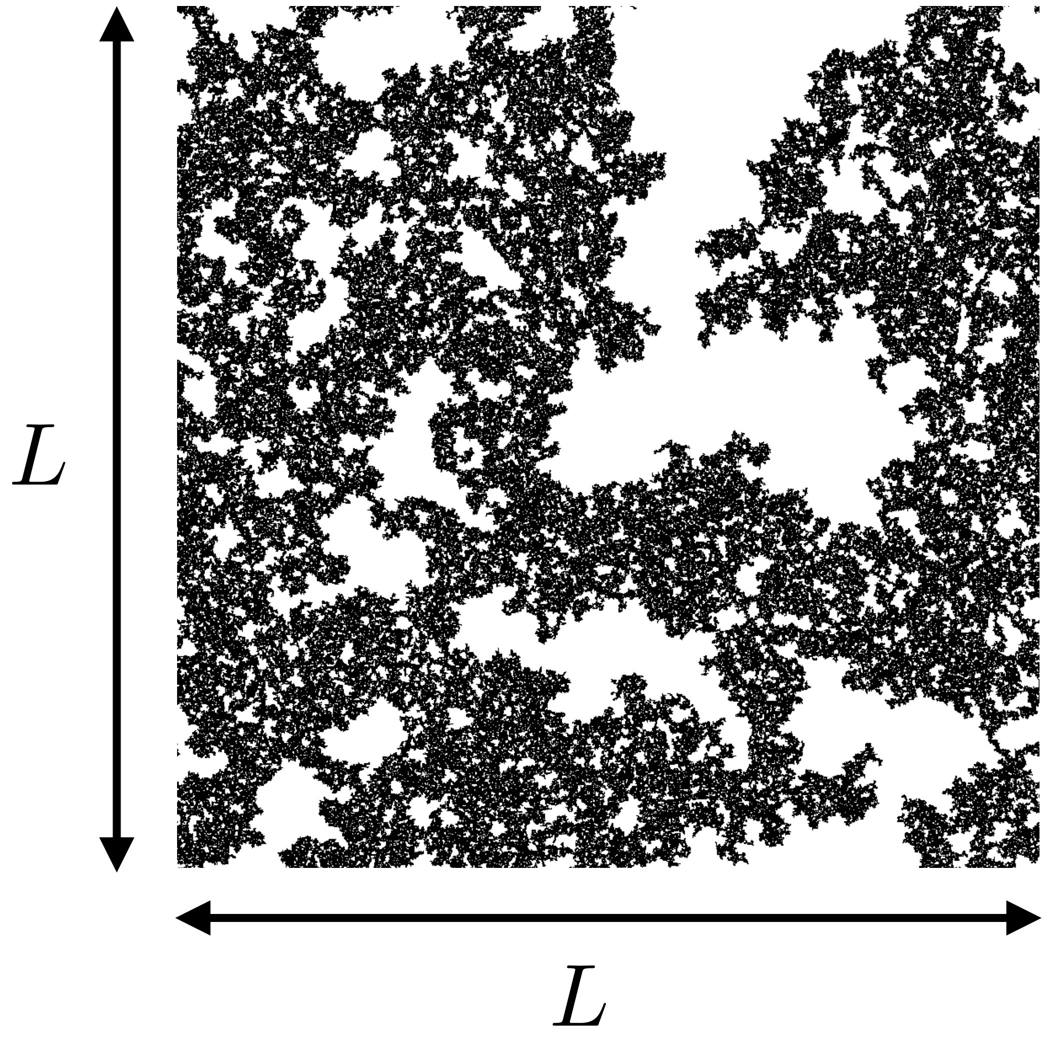


$$\frac{\rho_2}{\rho_1} = \left(\frac{L_2}{L_1}\right)^{-(d-d_F)}$$

$$\log \rho = -(d - d_F)\ \log(L) + \text{const.}$$

$$\boxed{d_F \simeq -0.102 + d \simeq 1.898}$$

At criticality: Scale Invariance

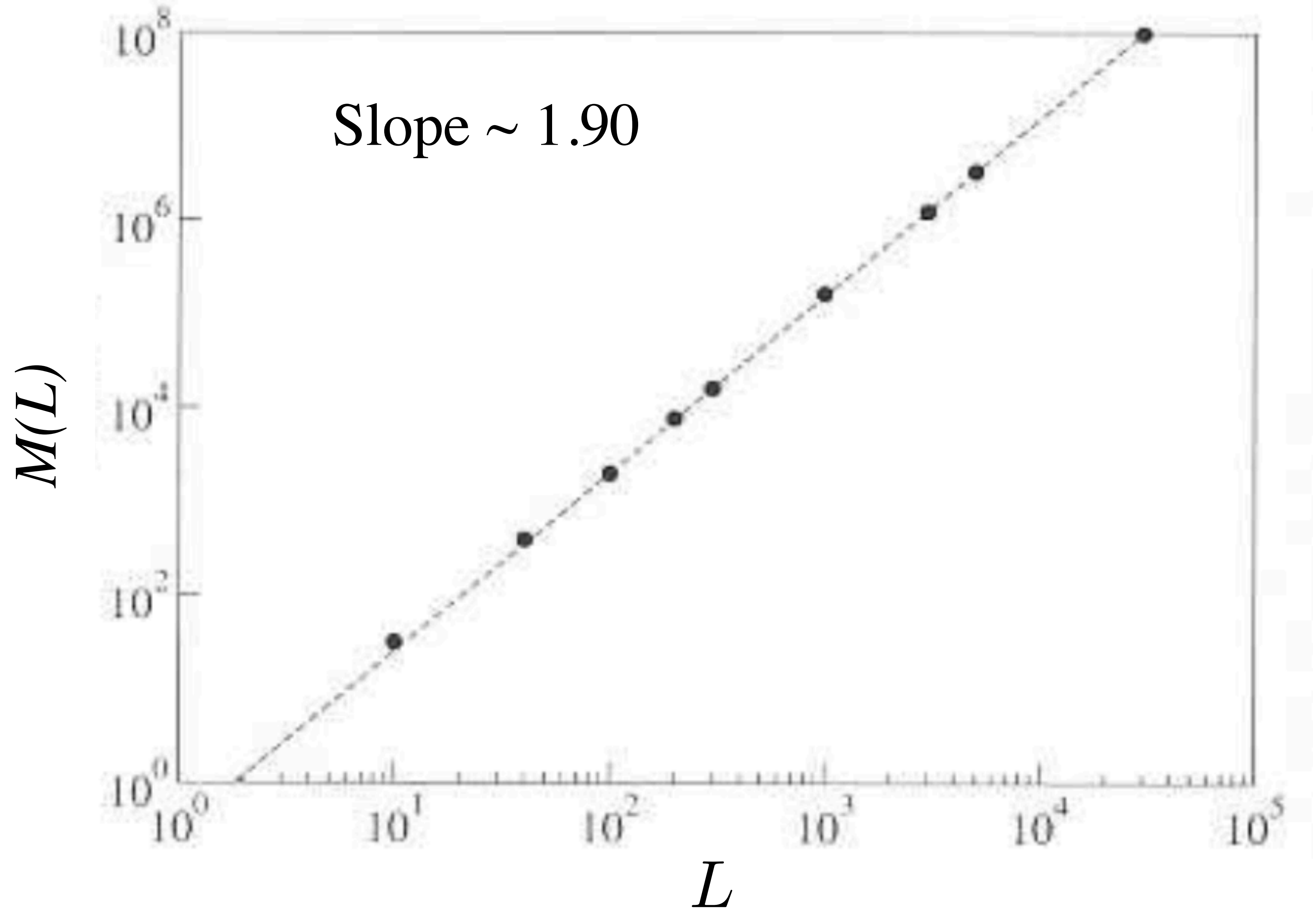


Site percolation
on a square lattice

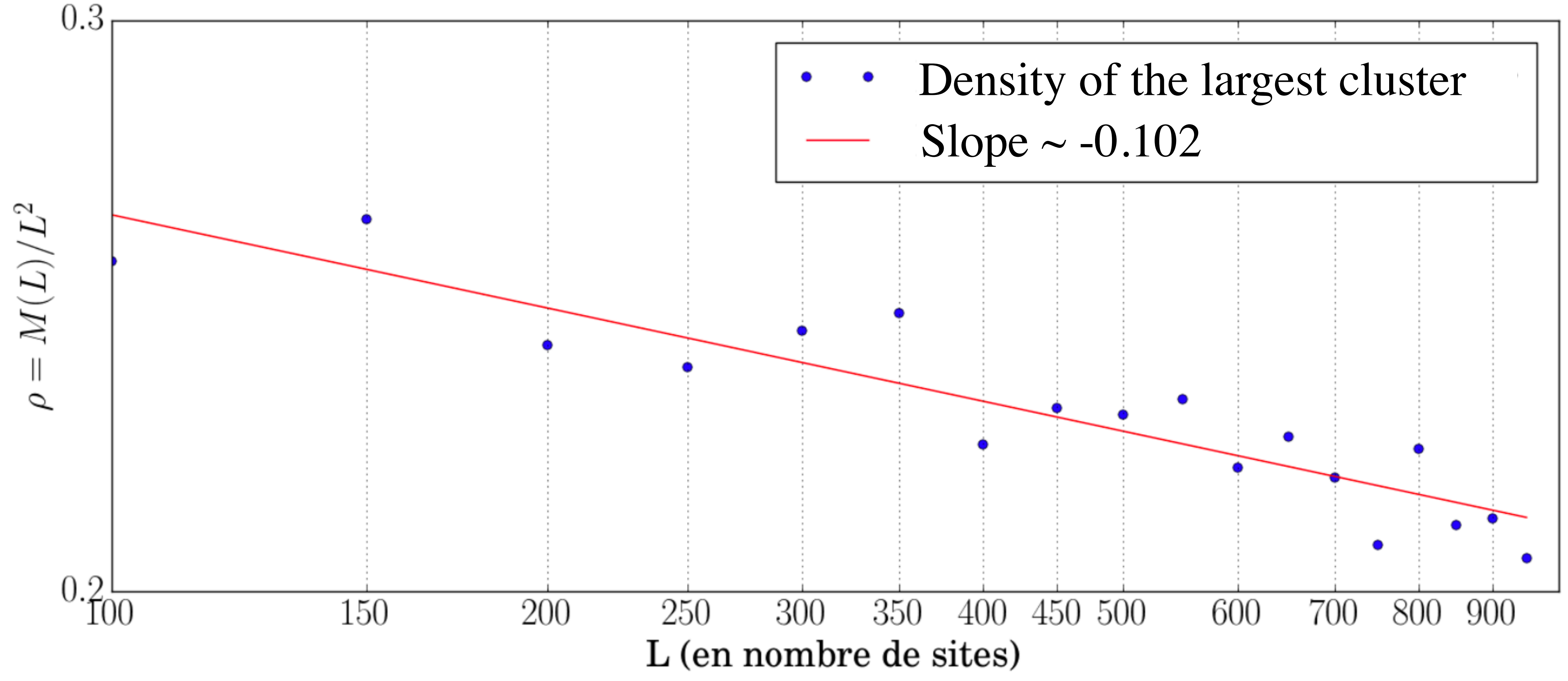
at p = $p_c \sim 0.59$

$$M(L) \propto L^{d_F}$$

$$\boxed{d_F \simeq 1.90}$$



ρ decreases for larger L



$$\frac{\rho_2}{\rho_1} = \left(\frac{L_2}{L_1}\right)^{-(d-d_F)}$$

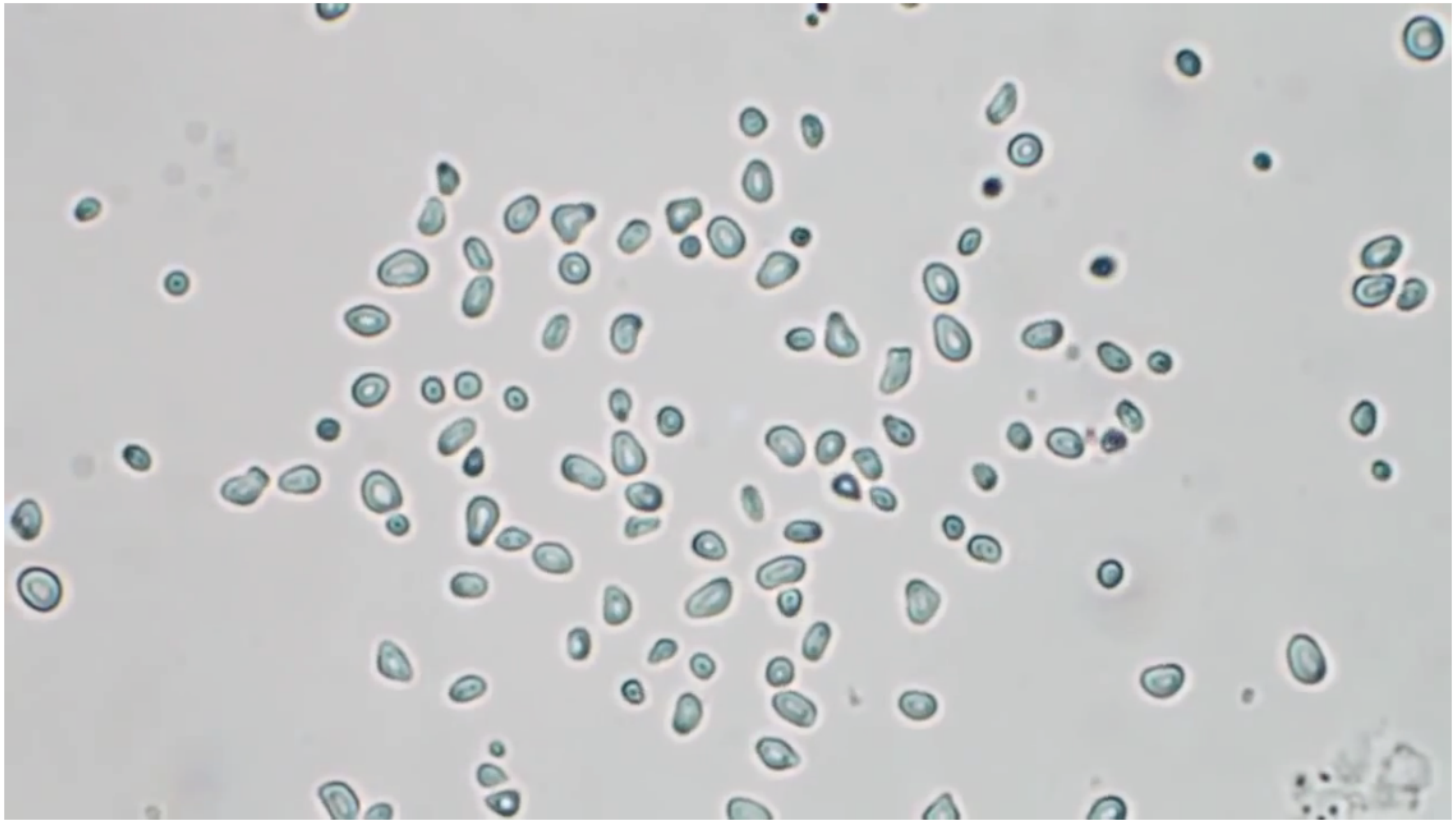
$$\log \rho = -(d - d_F)\,\log(L) + \text{const.}$$

$$\boxed{d_F \simeq -0.102 + d \simeq 1.898}$$

Random walk,
Self-Similarity, Universality,
Fractal dimension

Brownian motion

1827: Robert Brown observed irregular motion of small pollen grains suspended in water

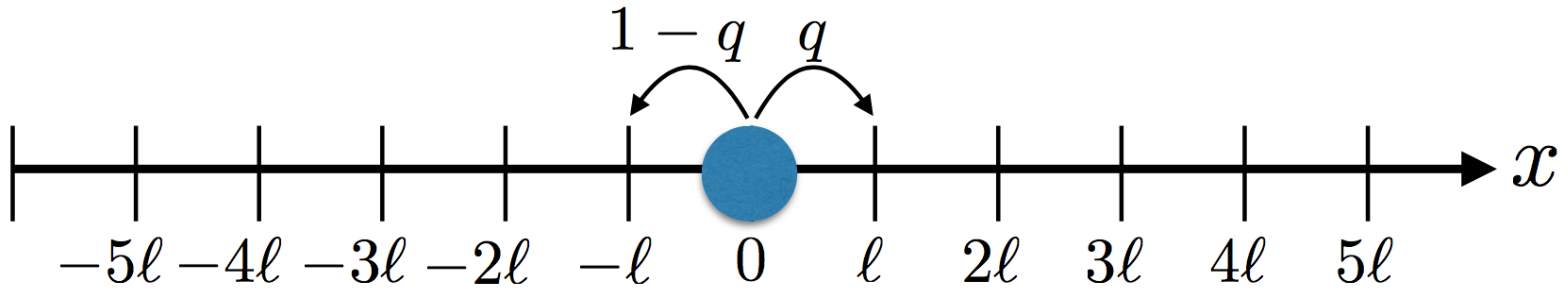


1905-06 Albert Einstein, Marian Smoluchowski: microscopic description of Brownian motion and relation to diffusion equation.

Random walks

Random walks: paths that take successive steps in random directions.

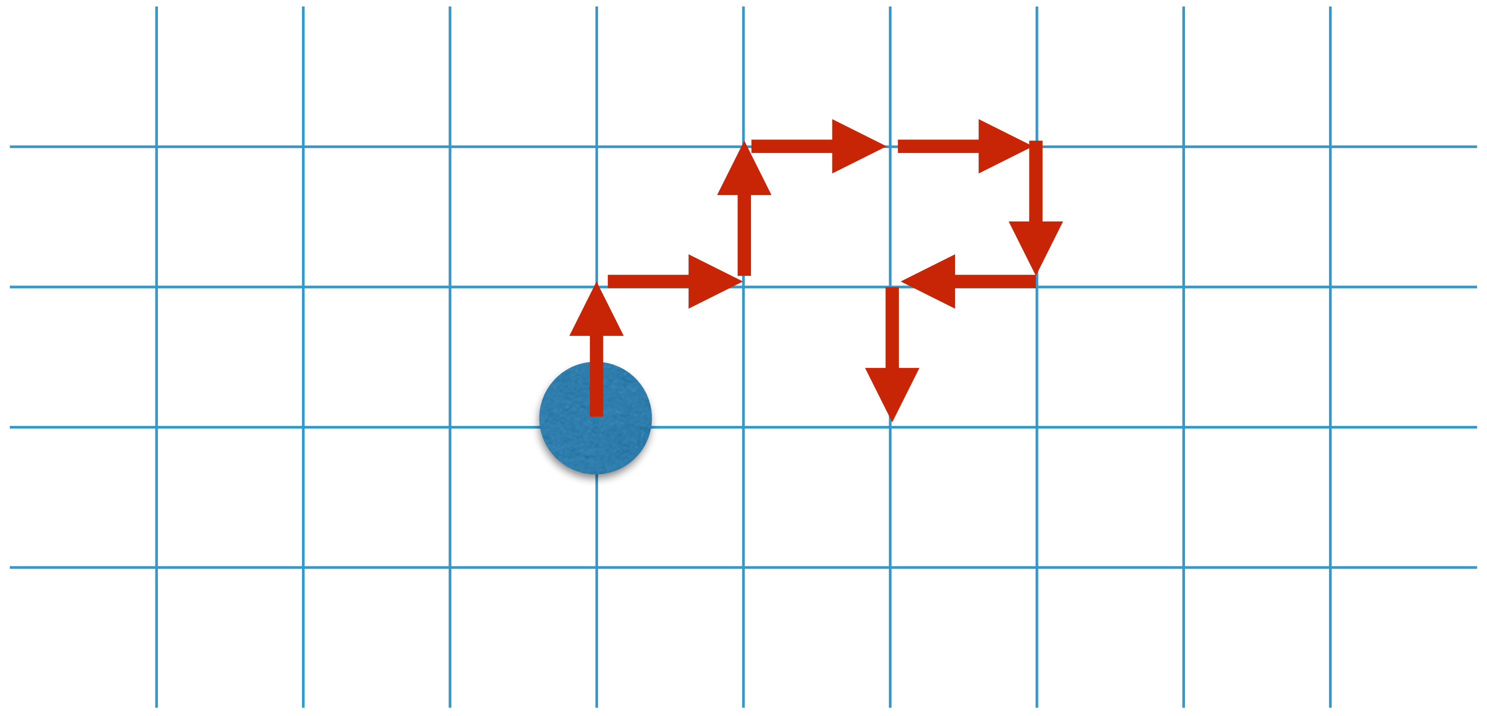
Ex. 1D



Ex. 2D

Ex. Random walk on a 2D square lattice

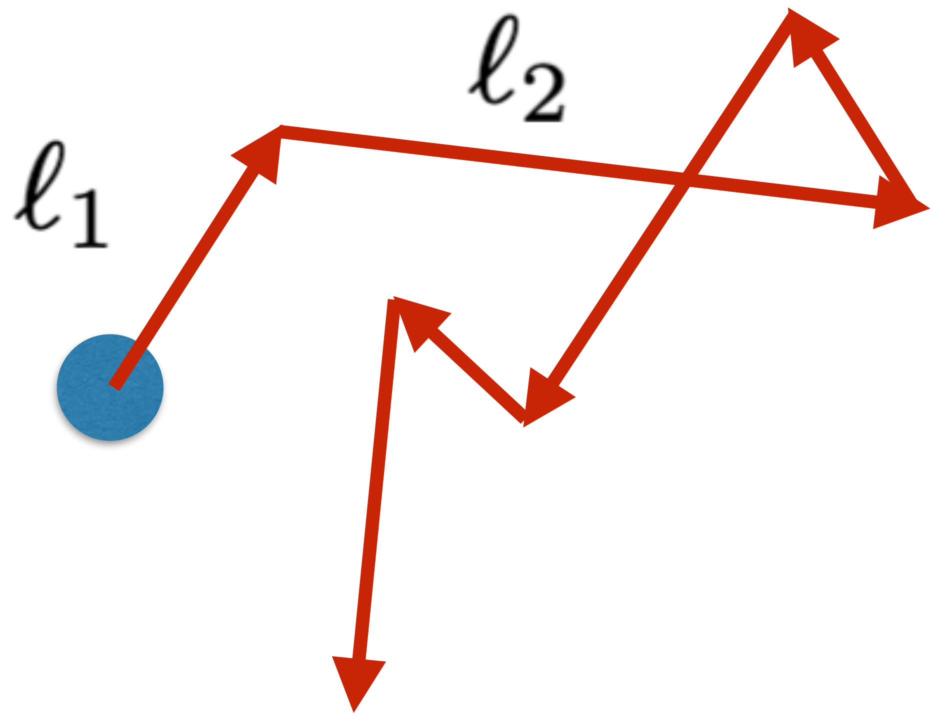
1/4 chance to go in any of the 4 directions



Ex. Random walk in the continuous 2D plane

Choose a random direction uniformly

Jump of length ℓ sampled from a continuous distribution $P(\ell)$

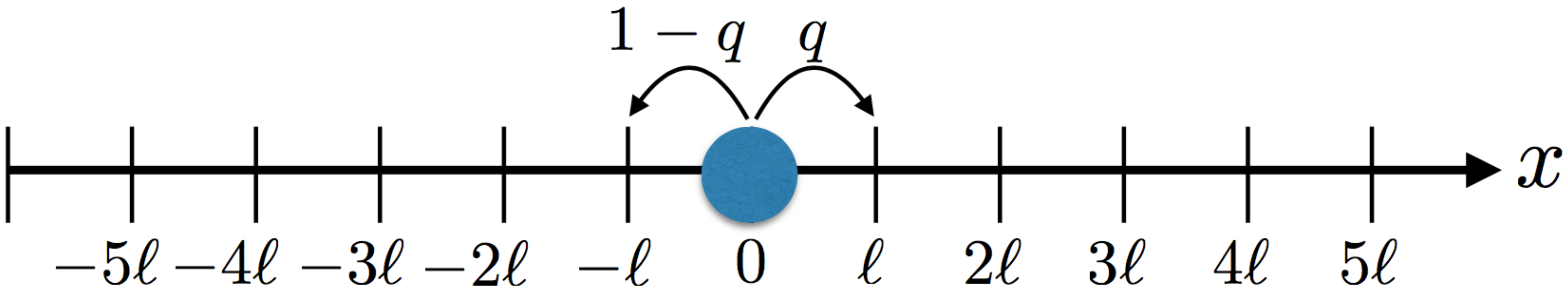


Random walks

Random walks: paths that take successive steps in random directions.

What can we say about this system?

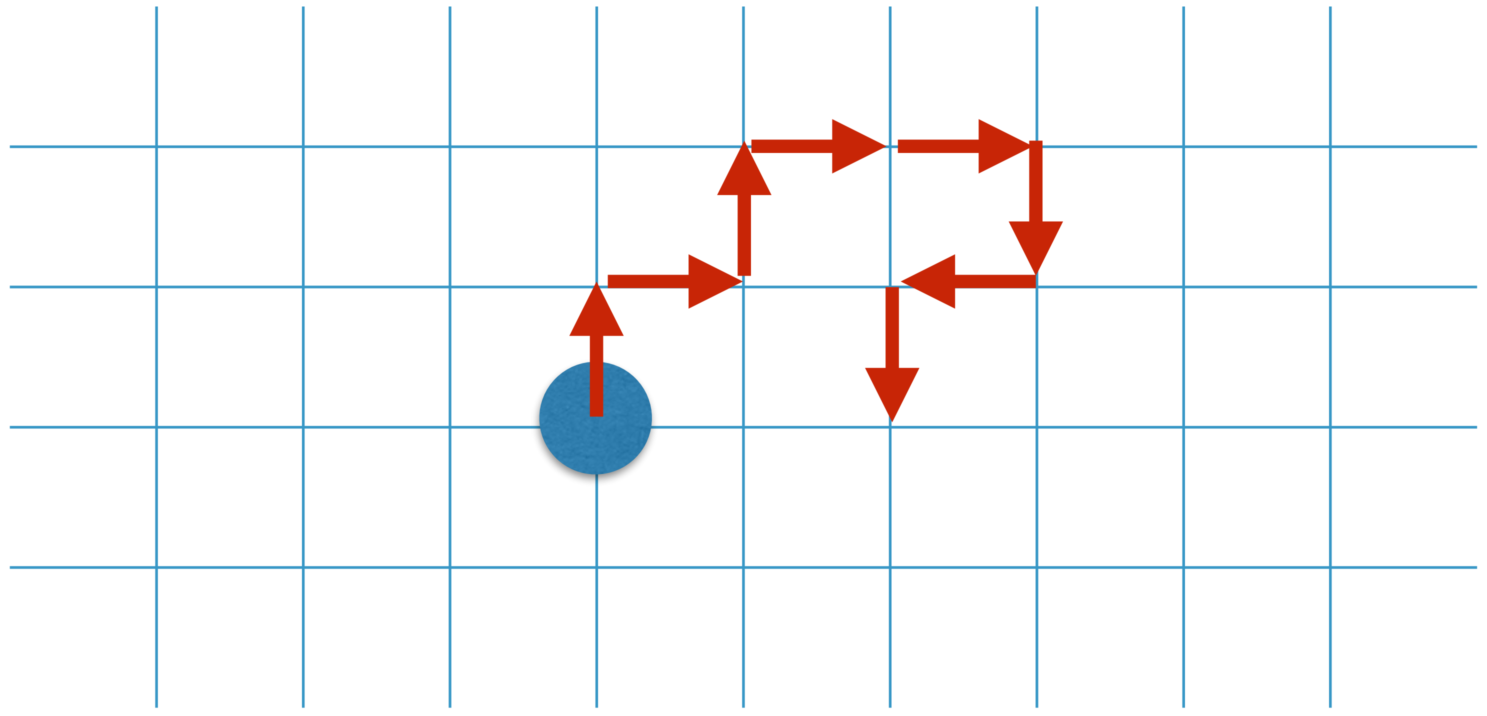
Ex. 1D



Ex. 2D

Ex. Random walk on a 2D square lattice

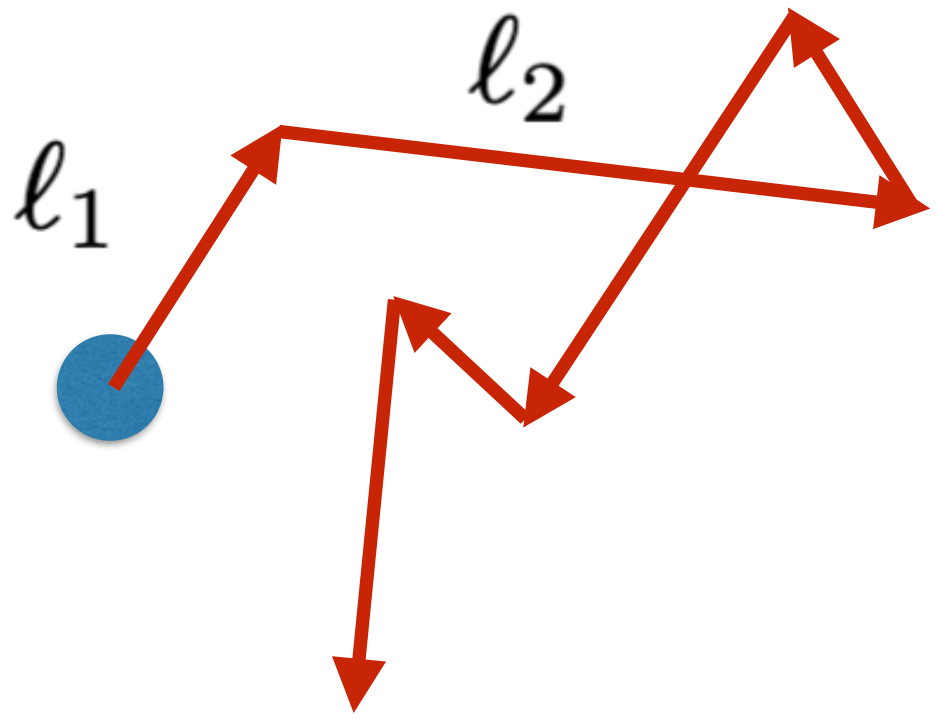
1/4 chance to go in any of the 4 directions



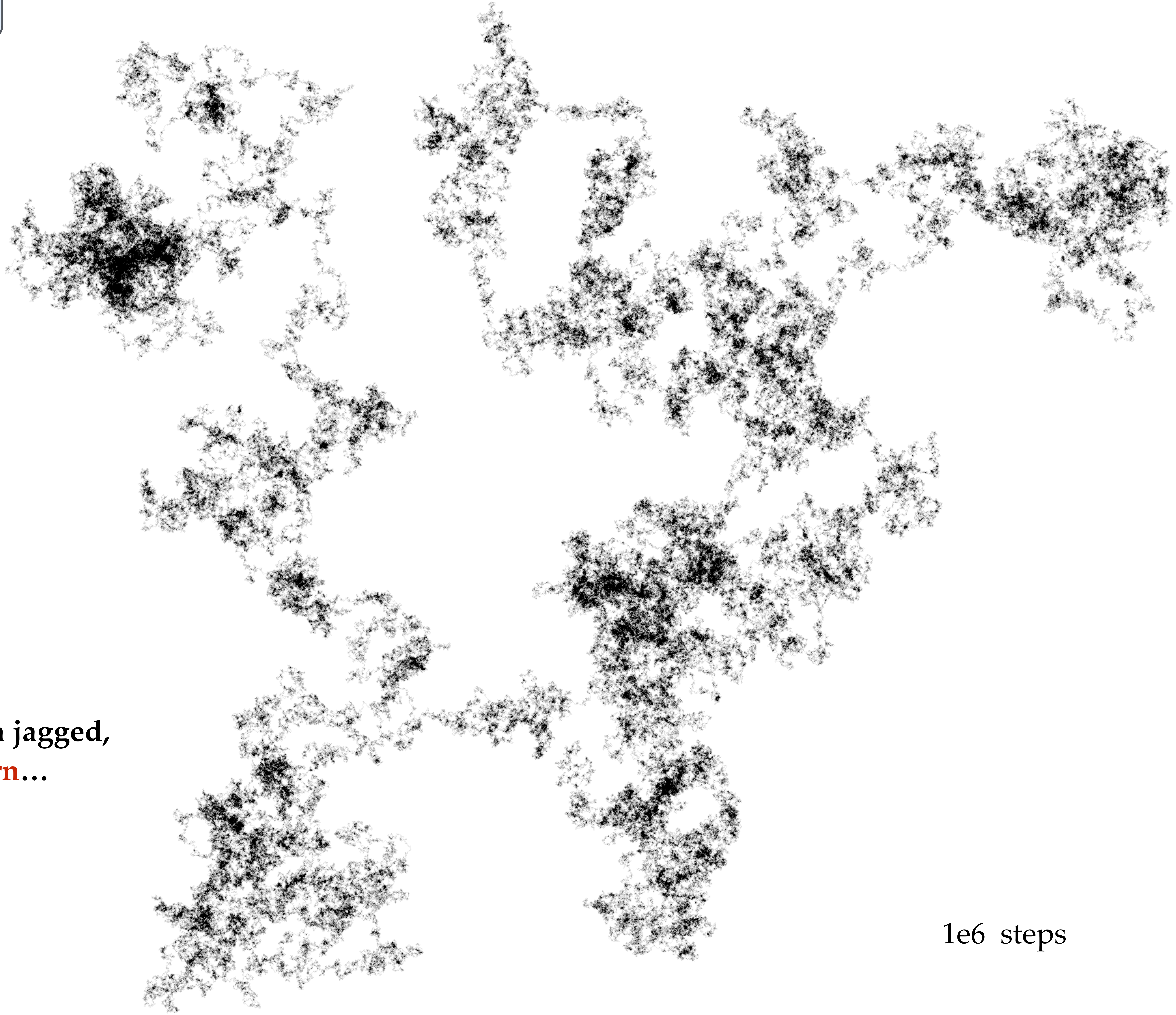
Ex. Random walk in the continuous 2D plane

Choose a random direction uniformly

Jump of length ℓ sampled from a continuous distribution $P(\ell)$

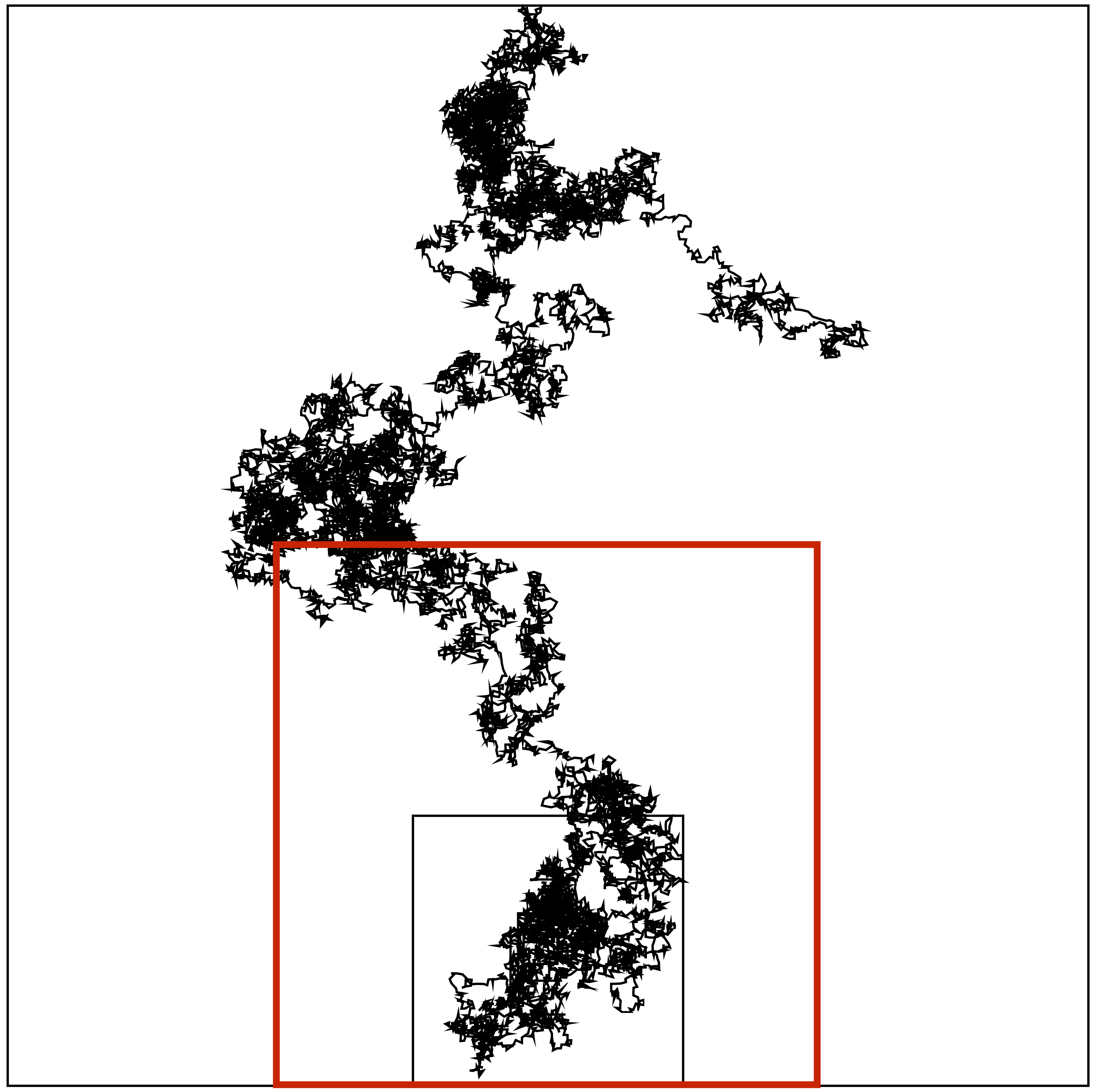


Random walks

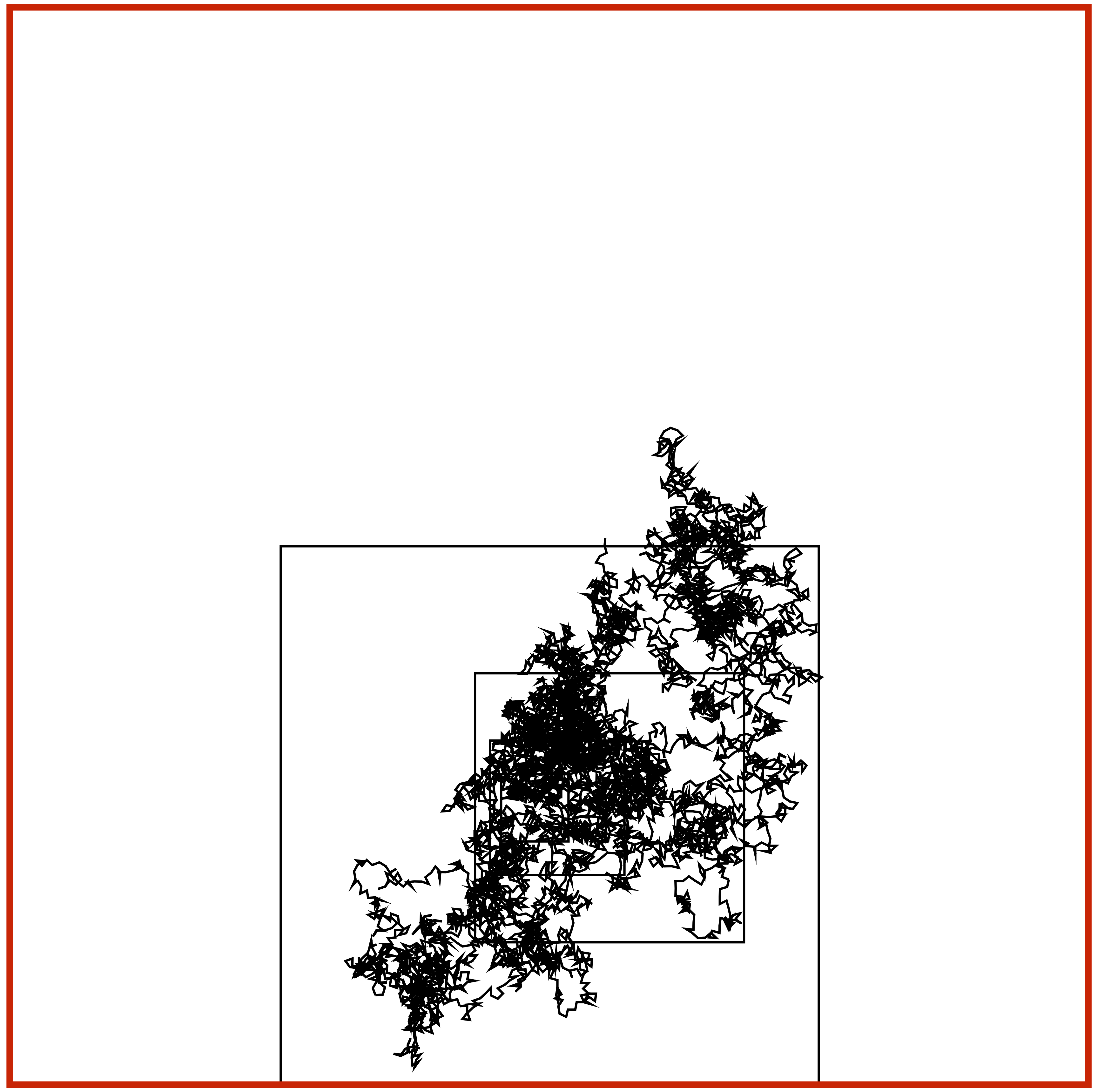


Random walks form a jagged, fractal-like pattern…

Scale invariance of random walks

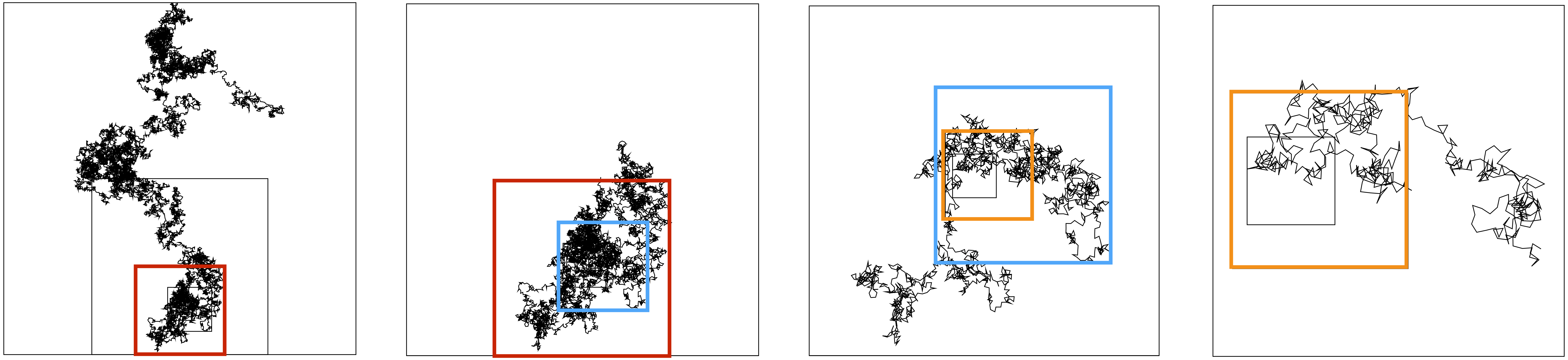


32 000 steps



First 8 000 steps

Scale invariance of random walks

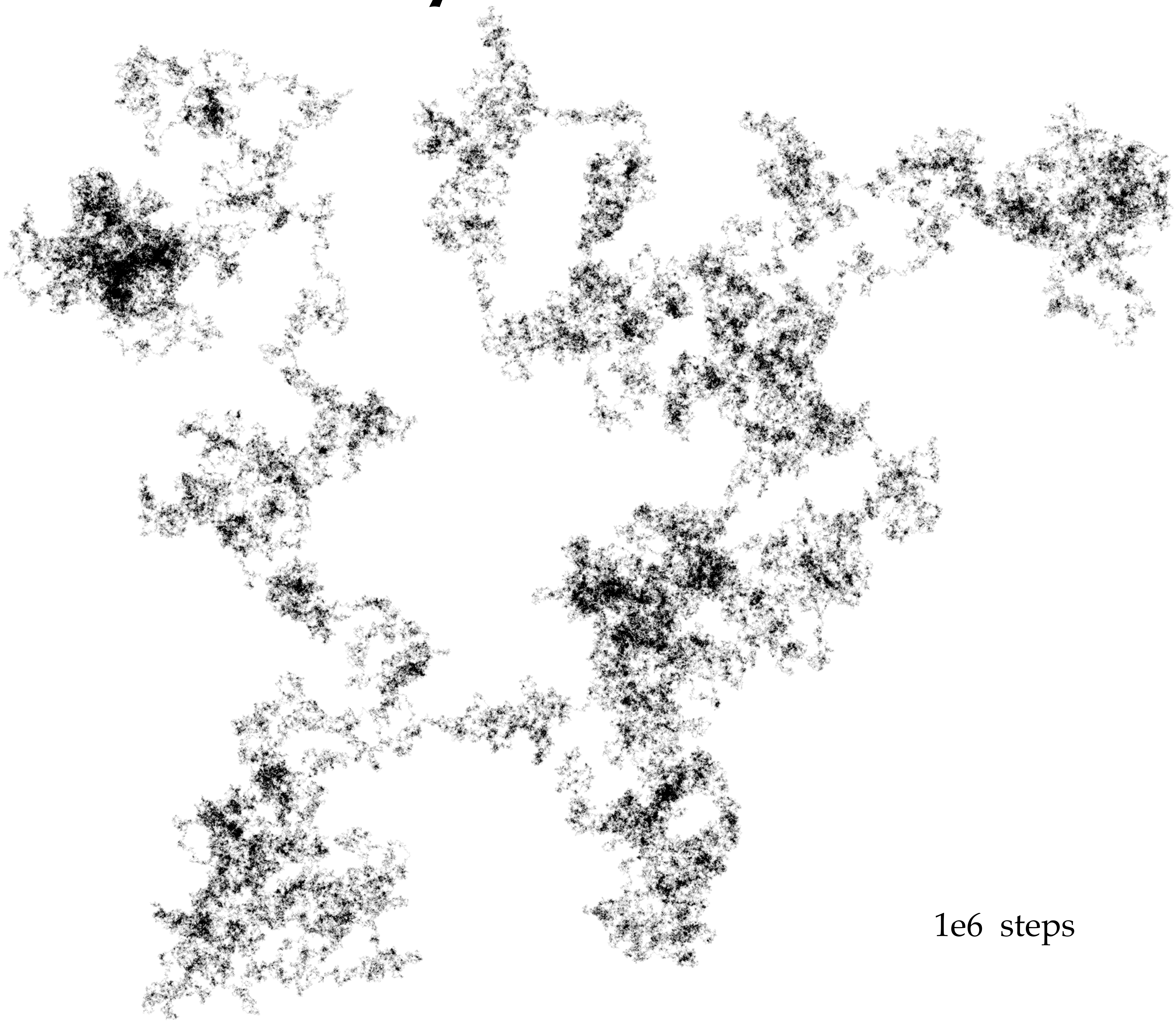


32 000 steps | First 8 000 steps | First 2 000 steps | First 500 steps

Random walks form a jagged,
fractal-like pattern…
which looks the same when rescaled. → *scale invariant*

Universality

Play with the simulations <u>here</u>



1e6 steps

Universality

Play with the simulations here

On scales where the individual steps are not distinguishable,
all these random walks look the same:

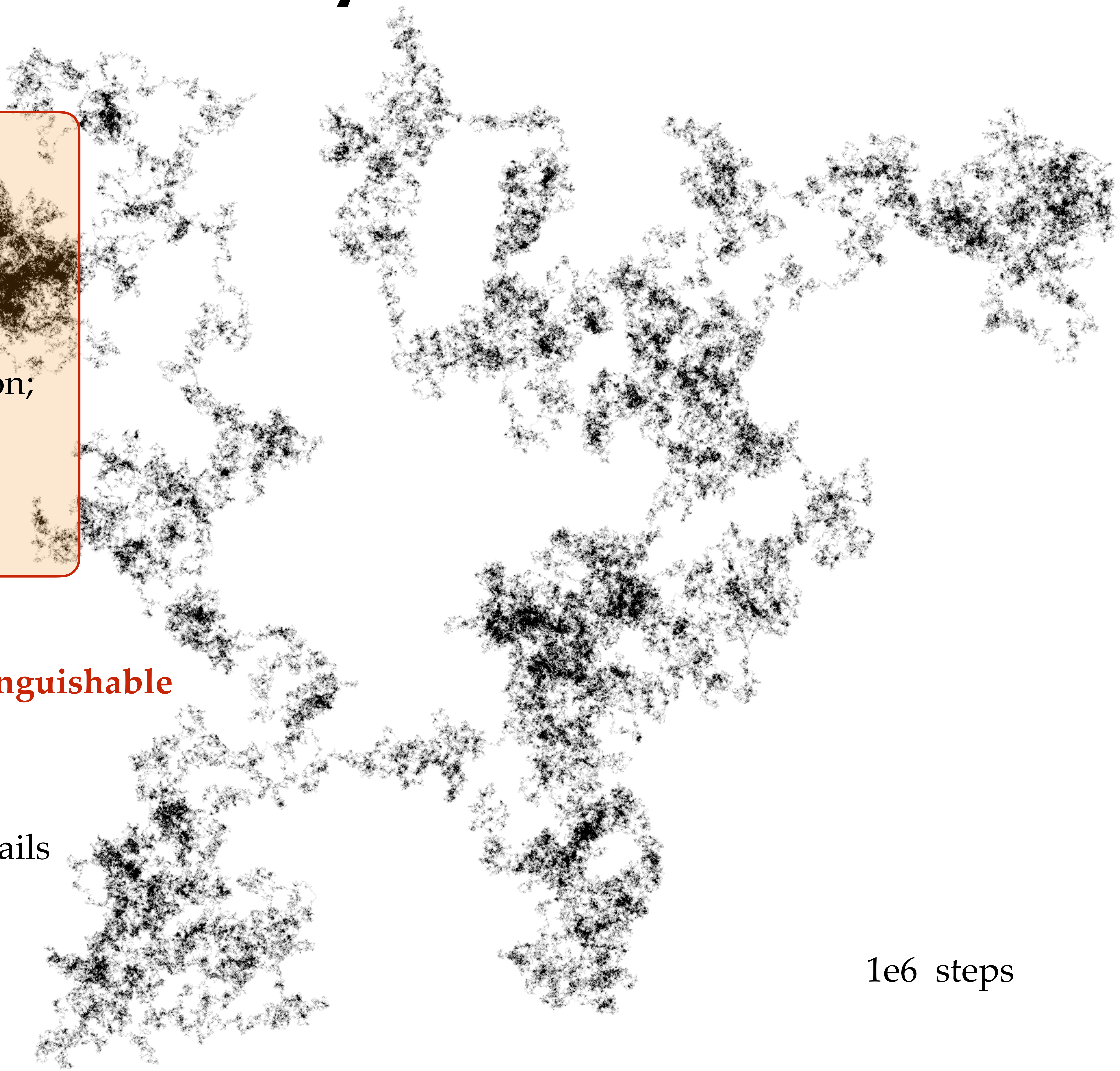
— RW on a 2D grid;

— 2D RW with fixed length, but uniformly distributed direction;

— 2D RW uniformly distributed direction,
and exponentially jump lengths;

After a few steps, all of these random walks are **statistically indistinguishable**

Universality: independence of the behavior on the microscopic details



1e6 steps

Universality

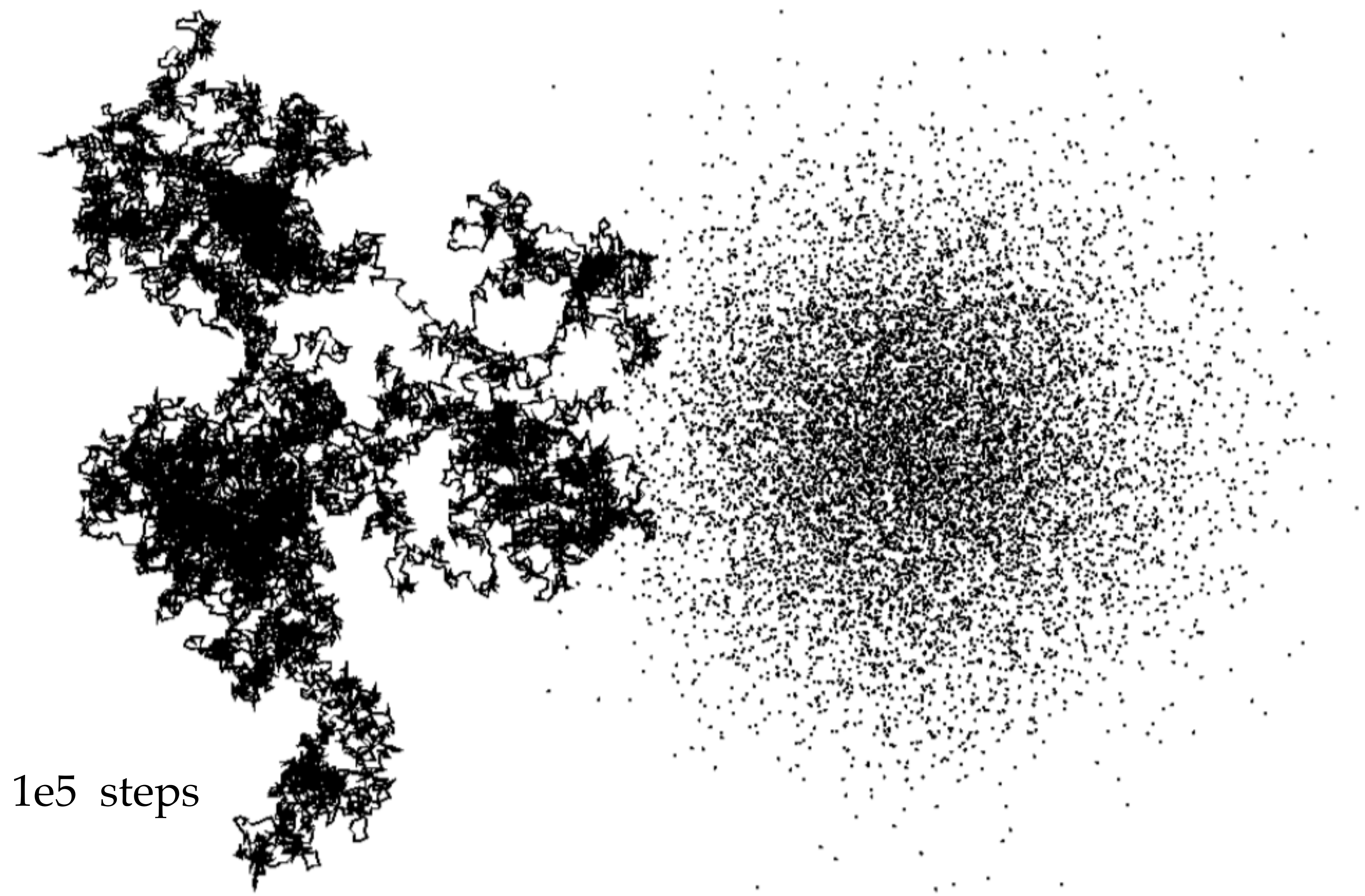
Emergent behavior

Simple laws/behaviors that emerge from underlying complexity

Observe on the simulations here

Two types of emergent behavior:

An **individual random walk**, after a large number of steps, becomes *fractal* or ***scale invariant***



The **endpoint** of the random walk has a **probability distribution** that obeys a **simple continuum law**, the diffusion equation

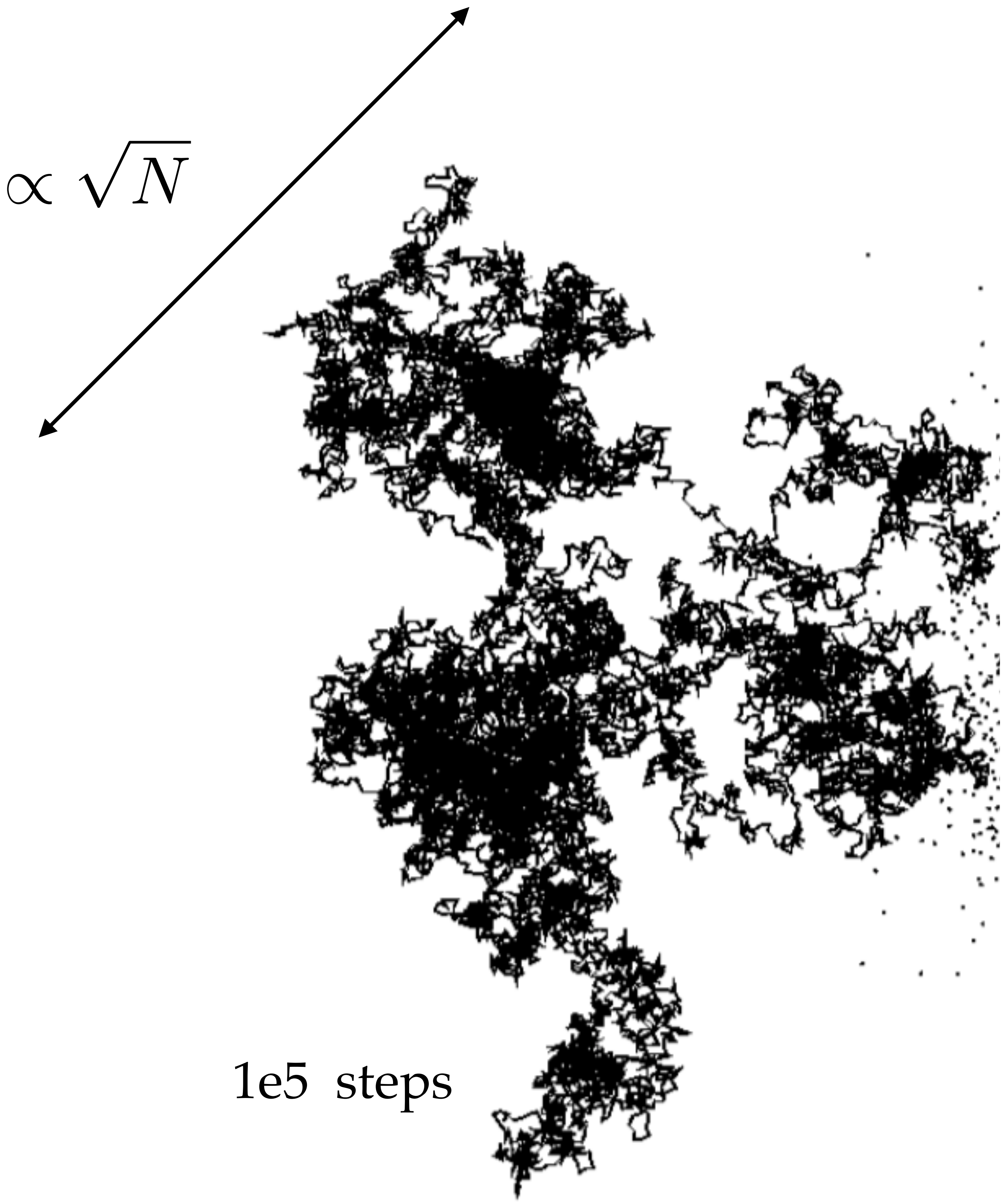
Both of these behaviors are **independent of the microscopic details** of the walk: they are **universal.**

Extension of a random walk

Simple laws/behaviors that emerge from underlying complexity

"Spatial extension" of the walk:

$L \propto \sqrt{N}$

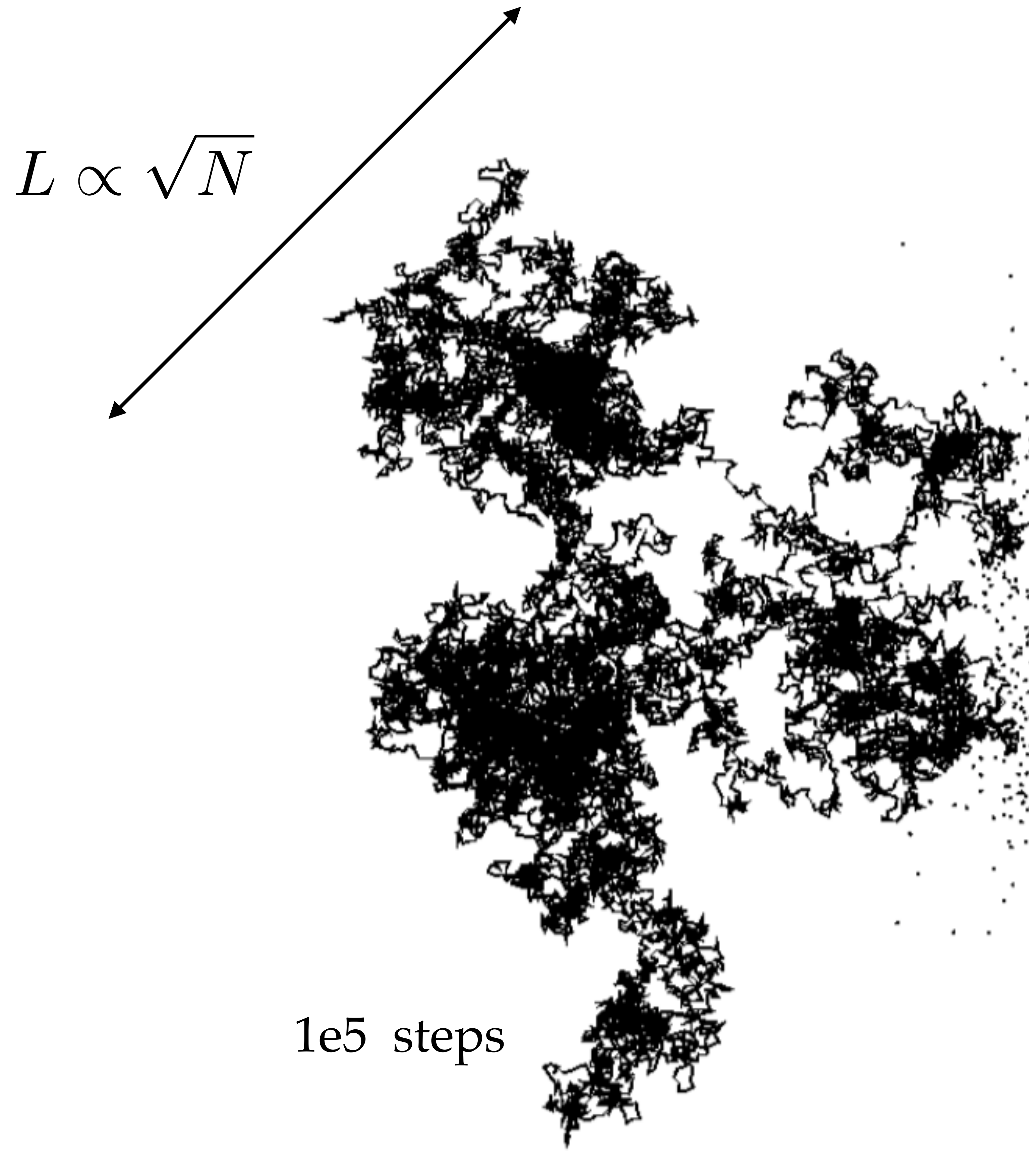


1e5 steps

Extension of a random walk

Simple laws/behaviors that emerge from underlying complexity

"Spatial extension" of the walk:



Independent and Identically Distributed random variables (or, i.i.d. random variables):

For any variable X_N defines as the sum of N independently and identically distributed (i.i.d.) random variables r_i for which σ_{r_i} is finite, we have that:

- the mean of X_N is equal to N times the mean of r_i:

$$\boxed{\langle X_N \rangle = N \langle r_i \rangle \,;} \tag{5.34}$$

- the variance of X_N is equal to N times the variance of r_i:

$$\boxed{\sigma^2_{X_N} = N\,\sigma^2_{r_i} = N\left(\langle r_i^2 \rangle - \langle r_i \rangle^2\right).} \tag{5.35}$$

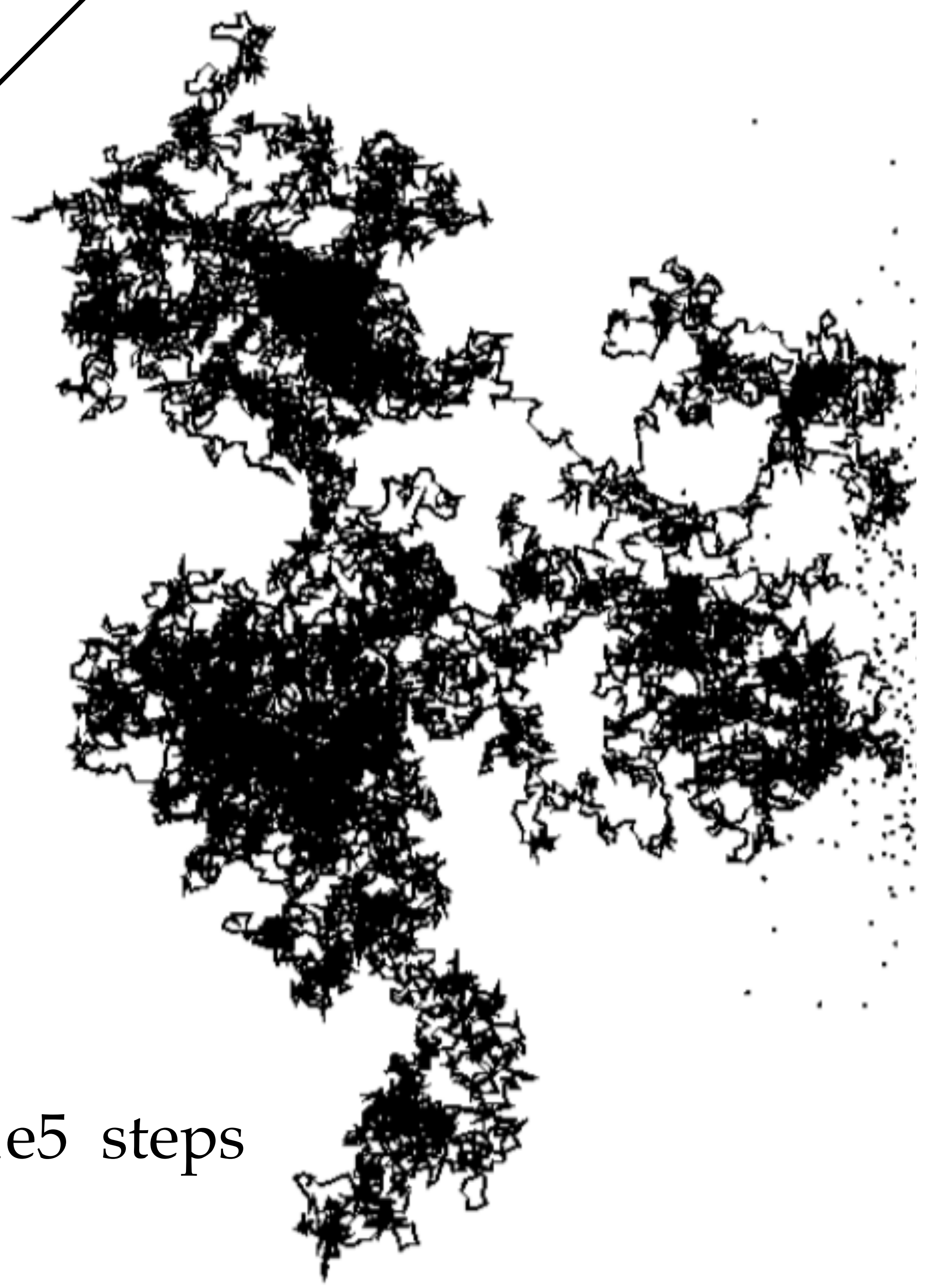
As a consequence, the standard deviation of the final positions of a 1-dimensional random walk defined by X_N always grows as $\sqrt{N}$, as long as σ_{r_i} is finite (no matter the detailed shape of the distribution of r_i).

Fractal dimension

Simple laws/behaviors that emerge from underlying complexity

"Spatial extension" of the walk:

$L \propto \sqrt{N}$



1e5 steps

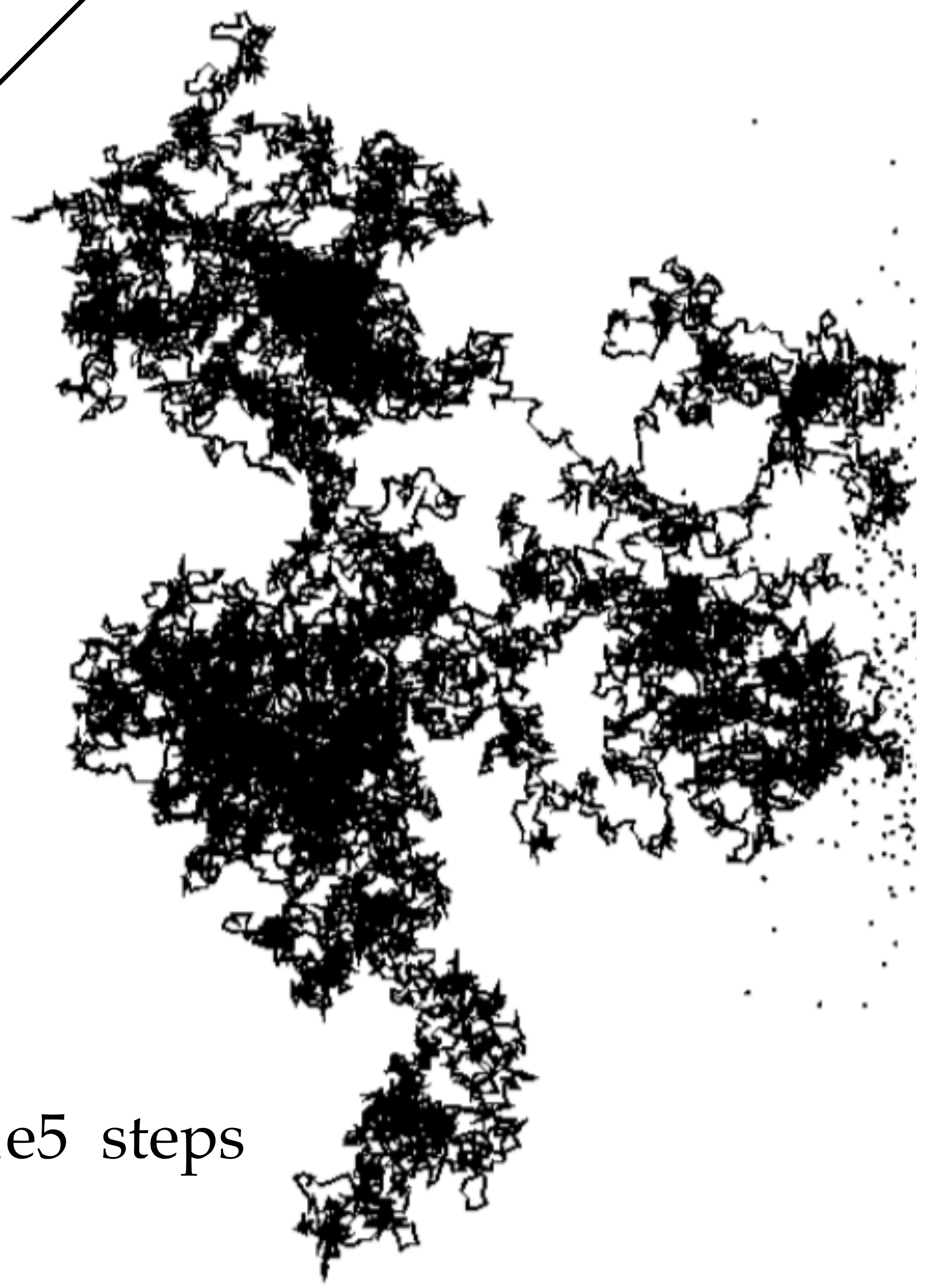
Mass of the walk: $V \propto N$

Fractal dimension

Simple laws/behaviors that emerge from underlying complexity

"Spatial extension" of the walk:

$L \propto \sqrt{N}$


1e5 steps

Mass of the walk: $M \propto N$

$L \propto \sqrt{N} \implies L \propto \sqrt{M}$

$L_1 \longrightarrow L_2 = 2\,L_1$

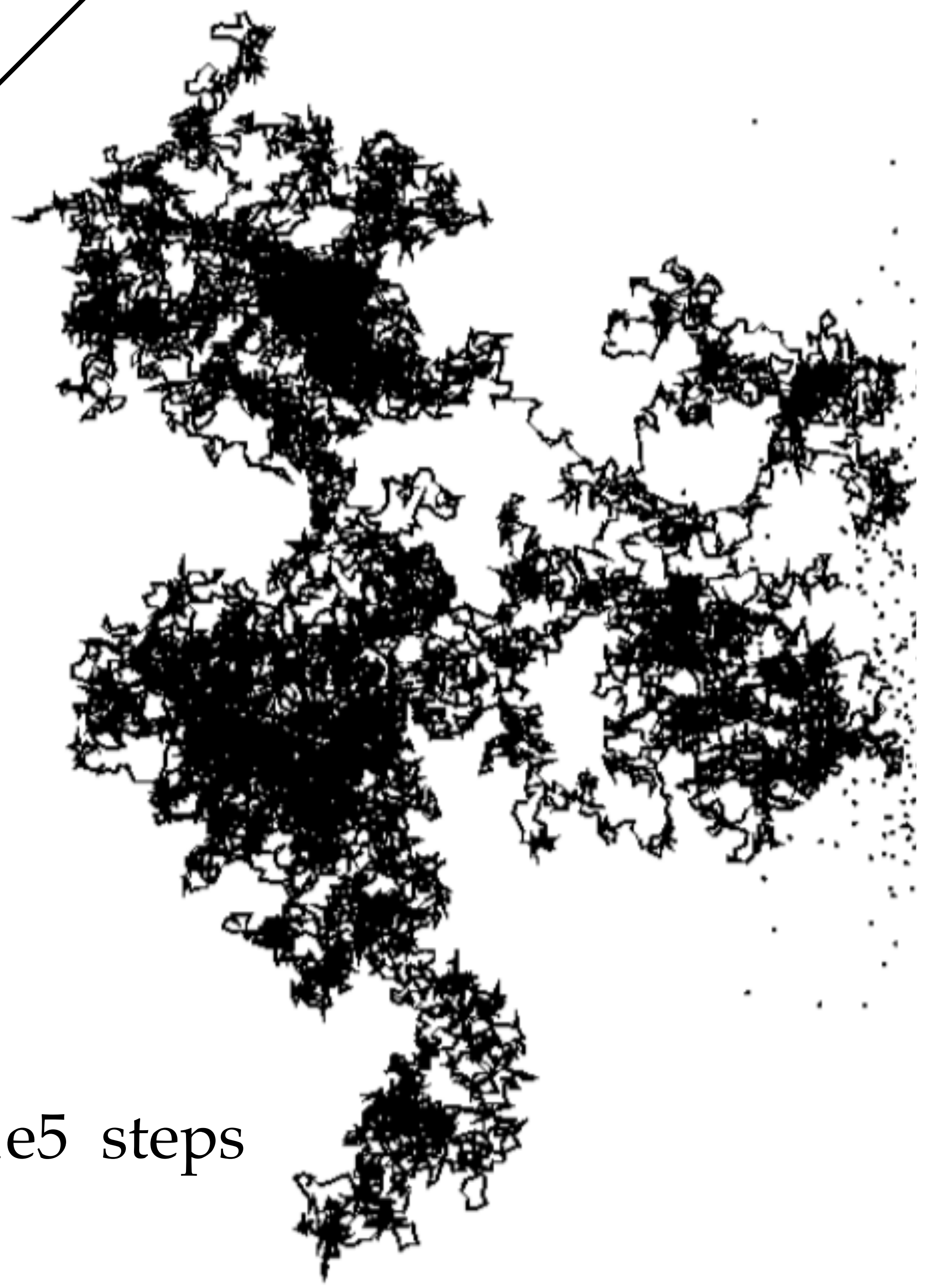
$M_1 \longrightarrow M_2 = 2^{\,d_F}\,M_1$

Fractal dimension

Simple laws/behaviors that emerge from underlying complexity

"Spatial extension" of the walk:

$L \propto \sqrt{N}$



1e5 steps

Mass of the walk: $M \propto N$

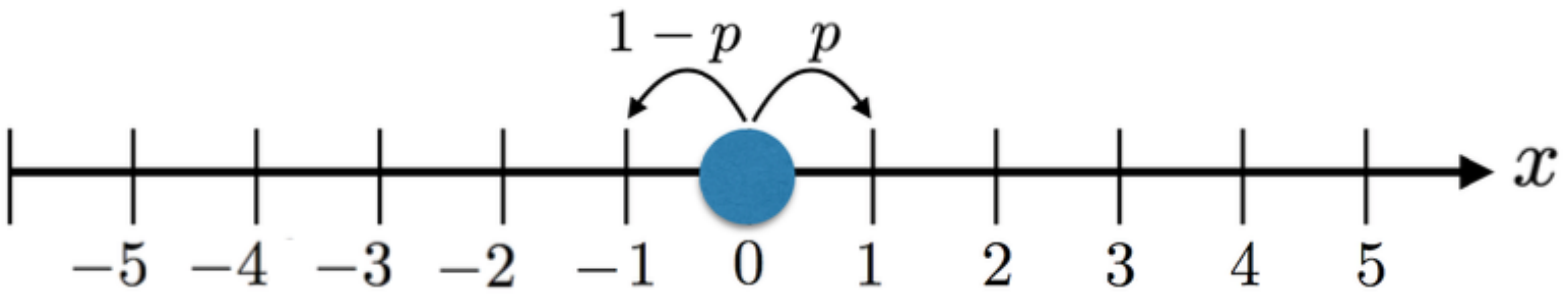
$$L \propto \sqrt{N} \implies L \propto \sqrt{M}$$

$$L_1 \longrightarrow L_2 = 2\,L_1$$

$$M_1 \longrightarrow M_2 = 2^{d_F}\,M_1$$

$$d_F = 2$$

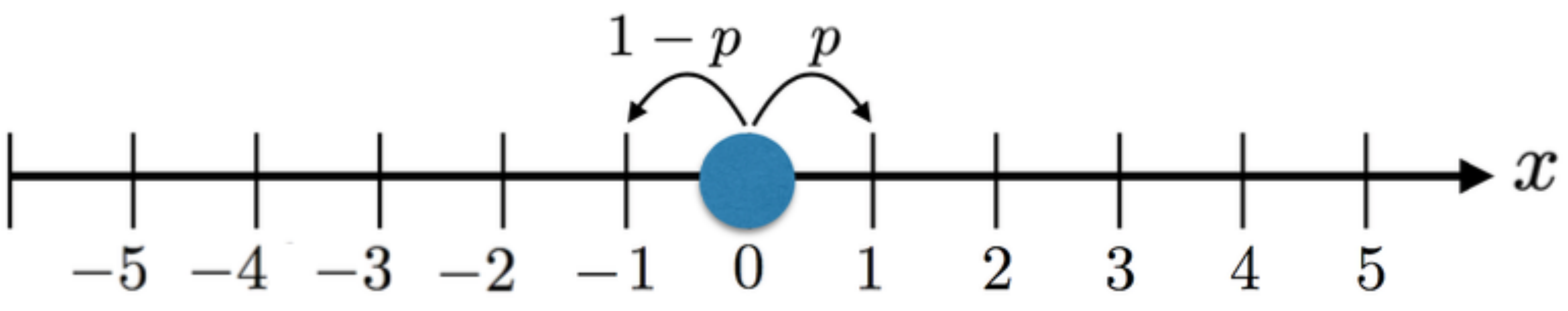
Master equation and Fokker-Planck equation



$P(x, N)$ = probability to find the walker in position x after N jumps.

Discrete Master equation:

Master equation and Fokker-Planck equation



$P(x, N)$ = probability to find the walker in position x after N jumps.

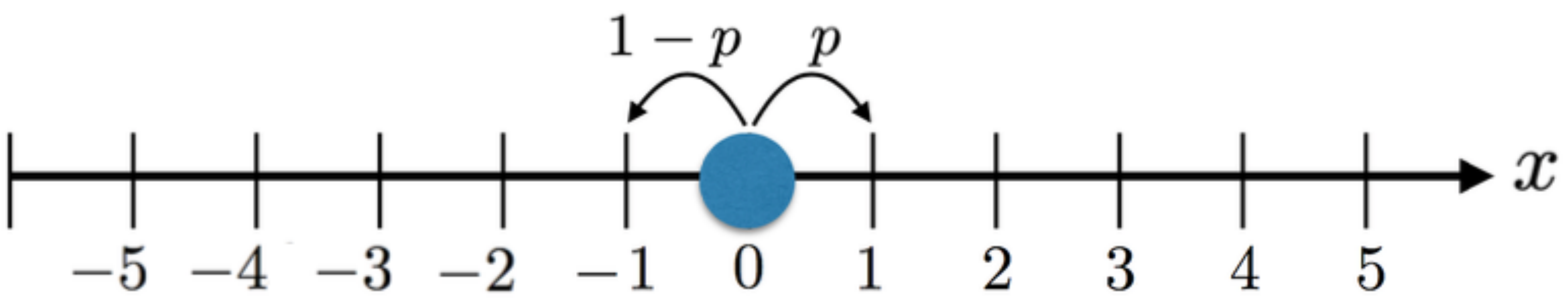
Discrete Master equation:

$$P(x,\ N+1) = \underbrace{P(x-1,\ N)\times p}_{(1)} + \underbrace{P(x+1,\ N)\times(1-p)}_{(2)},$$

Small jumps ℓ **in small time intervals** dt **:** $\quad t = N\,dt$

$$P(x,\ t+dt) = \underbrace{P(x-\ell,\ t)\times p}_{(1)} + \underbrace{P(x+\ell,\ t)\times(1-p)}_{(2)}$$

Master equation and Fokker-Planck equation



$P(x, N)$ = probability to find the walker in position x after N jumps.

Discrete Master equation:

$$P(x,\ N+1) = \underbrace{P(x-1,\ N)\times p}_{(1)} + \underbrace{P(x+1,\ N)\times(1-p)}_{(2)},$$

Small jumps ℓ **in small time intervals** dt **:** $t = N\,dt$

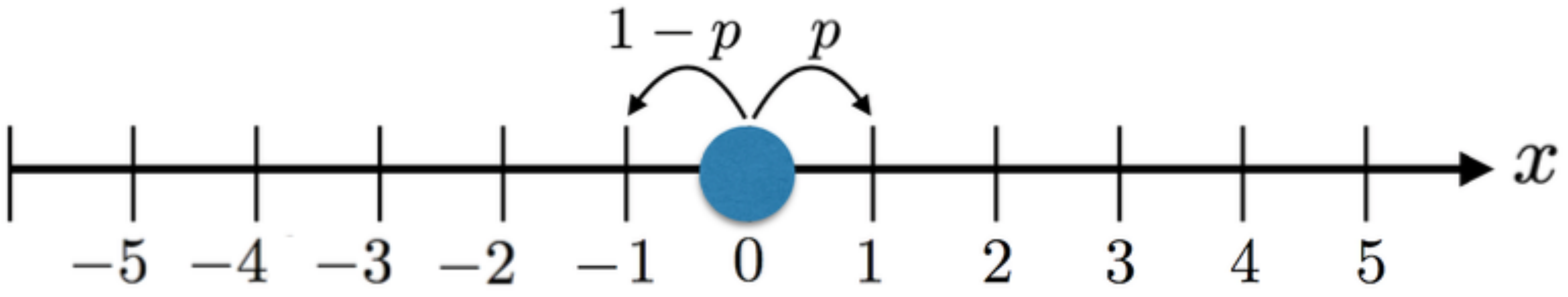
$$P(x,\ t+dt) = \underbrace{P(x-\ell,\ t)\times p}_{(1)} + \underbrace{P(x+\ell,\ t)\times(1-p)}_{(2)}$$

> We recall that the Taylor expansion of a function $f(x)$ (of a single variable x) that is infinitely differentiable at a point x_0 is given by the infinite sum:
>
> $$f(x_0+h) = f(x_0) + h\,f'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \cdots. \tag{5.42}$$
>
> For very small values of h, one may then truncate this sum at a chosen order.

Taylor expansions: — 1rst order in dt

— 2nd order in ℓ

Master equation and Fokker-Planck equation



$P(x, N)$ = probability to find the walker in position x after N jumps.

Discrete Master equation:

$$P(x,\ N+1) = \underbrace{P(x-1,\ N) \times p}_{(1)} + \underbrace{P(x+1,\ N) \times (1-p)}_{(2)},$$

Small jumps ℓ in small time intervals dt : $t = N\,dt$

$$P(x,\ t+dt) = \underbrace{P(x-\ell,\ t) \times p}_{(1)} + \underbrace{P(x+\ell,\ t) \times (1-p)}_{(2)}$$

Taylor expansions:
— 1rst order in dt
— 2nd order in ℓ

We recall that the Taylor expansion of a function $f(x)$ (of a single variable x) that is infinitely differentiable at a point x_0 is given by the infinite sum:

$$f(x_0+h) = f(x_0) + h\,f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \cdots \,. \qquad (5.42)$$

For very small values of h, one may then truncate this sum at a chosen order.

$$\frac{\partial P}{\partial t}(x,t)\,dt + o(dt) = (1-2p)\,\ell\,\frac{\partial P}{\partial x}(x,t) + \frac{\ell^2}{2}\,\frac{\partial^2 P}{\partial x^2}(x,t) + o(\ell^2)\,.$$

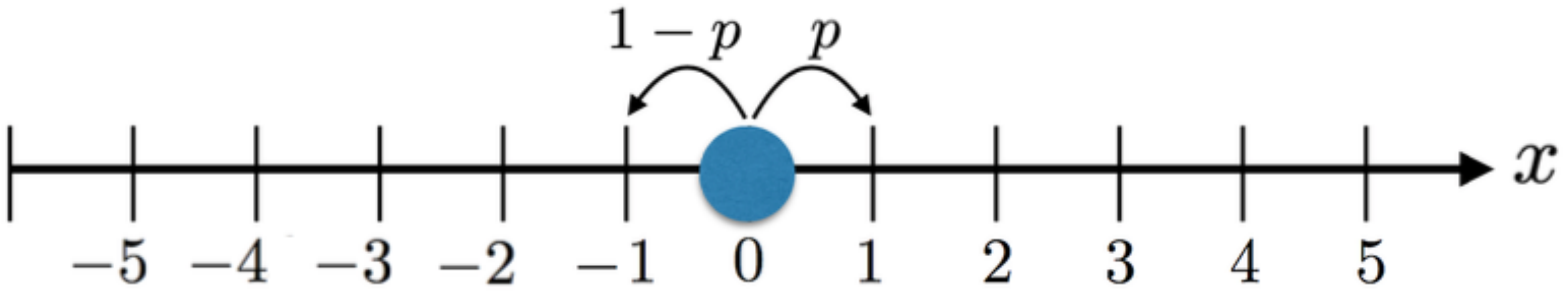
Fokker-Planck equation:

$$\boxed{\frac{\partial P}{\partial t}(x,t) = -v\,\frac{\partial P}{\partial x}(x,t) + D\,\frac{\partial^2 P}{\partial x^2}(x,t)\,,}$$

diffusion coefficient $D = \ell^2/(2\,dt)$

drift $v = (2p-1)\,\ell/dt$

Master equation and Fokker-Planck equation



$P(x,N)$ = probability to find the walker in position x after N jumps.

Discrete Master equation:

$$P(x,\ N+1) = \underbrace{P(x-1,\ N)\times p}_{(1)} + \underbrace{P(x+1,\ N)\times(1-p)}_{(2)},$$

Small jumps ℓ in small time intervals dt : $t = N\,dt$

$$P(x,\ t+dt) = \underbrace{P(x-\ell,\ t)\times p}_{(1)} + \underbrace{P(x+\ell,\ t)\times(1-p)}_{(2)}$$

Taylor expansions: — 1rst order in dt
— 2nd order in ℓ

We recall that the Taylor expansion of a function $f(x)$ (of a single variable x) that is infinitely differentiable at a point x_0 is given by the infinite sum:

$$f(x_0+h) = f(x_0) + h\,f'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \cdots . \tag{5.42}$$

For very small values of h, one may then truncate this sum at a chosen order.

$$\frac{\partial P}{\partial t}(x,t)\,dt + o(dt) = (1-2p)\,\ell\,\frac{\partial P}{\partial x}(x,t) + \frac{\ell^2}{2}\,\frac{\partial^2 P}{\partial x^2}(x,t) + o(\ell^2)\,.$$

Diffusion equation:

$p = 1/2$

$$\frac{\partial P}{\partial t}(x,t) = D\,\frac{\partial^2 P}{\partial x^2}(x,t)$$

diffusion coefficient $D = \ell^2/(2\,dt)$

Fokker-Planck equation:

$$\frac{\partial P}{\partial t}(x,t) = -v\,\frac{\partial P}{\partial x}(x,t) + D\,\frac{\partial^2 P}{\partial x^2}(x,t)\,,$$

diffusion coefficient $D = \ell^2/(2\,dt)$

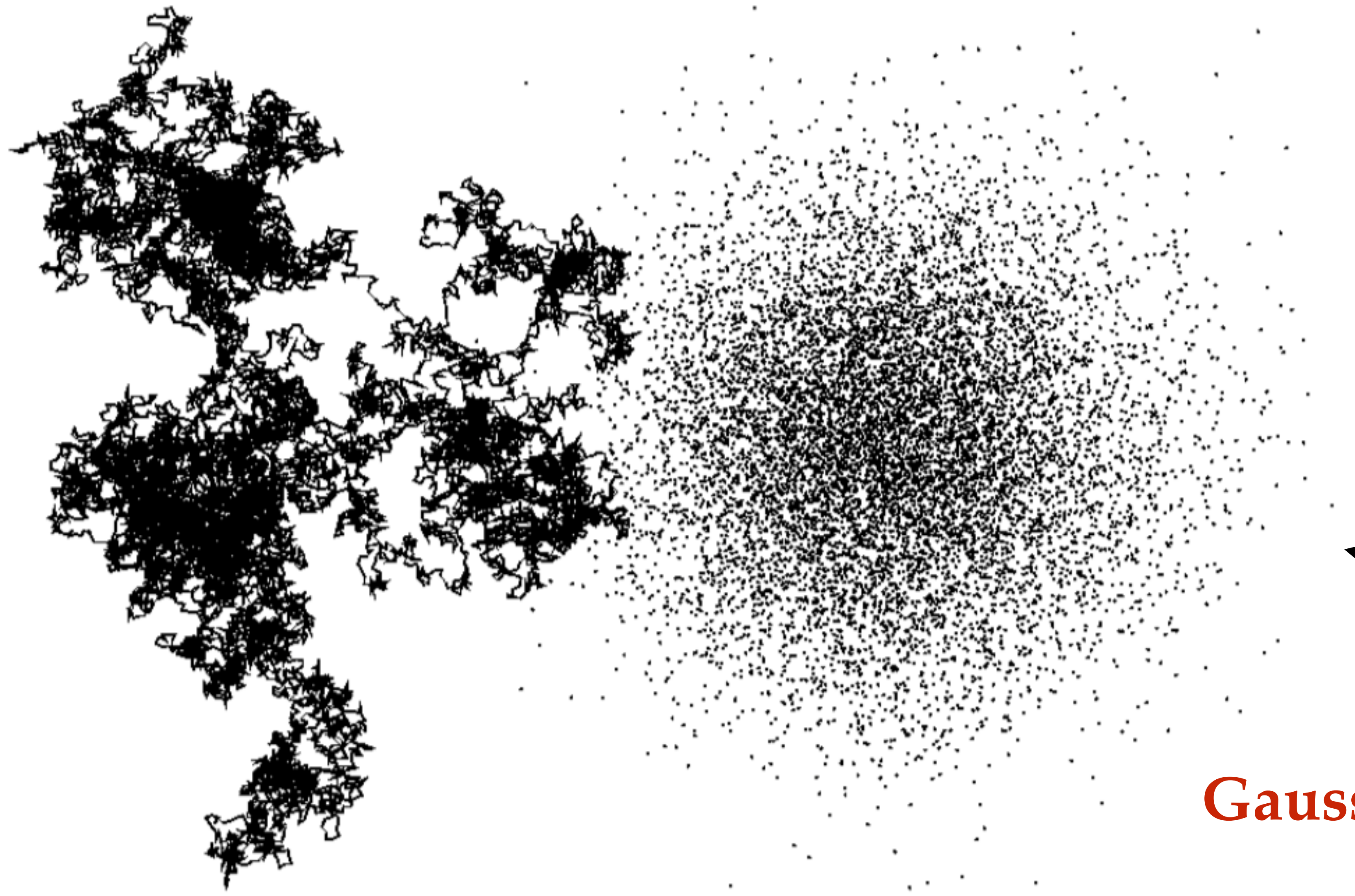
drift $v = (2p-1)\,\ell/dt$

Solution:

$$p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = \text{Gaussian}\,(\mu,\sigma^2)$$

where $\mu = v\,t$

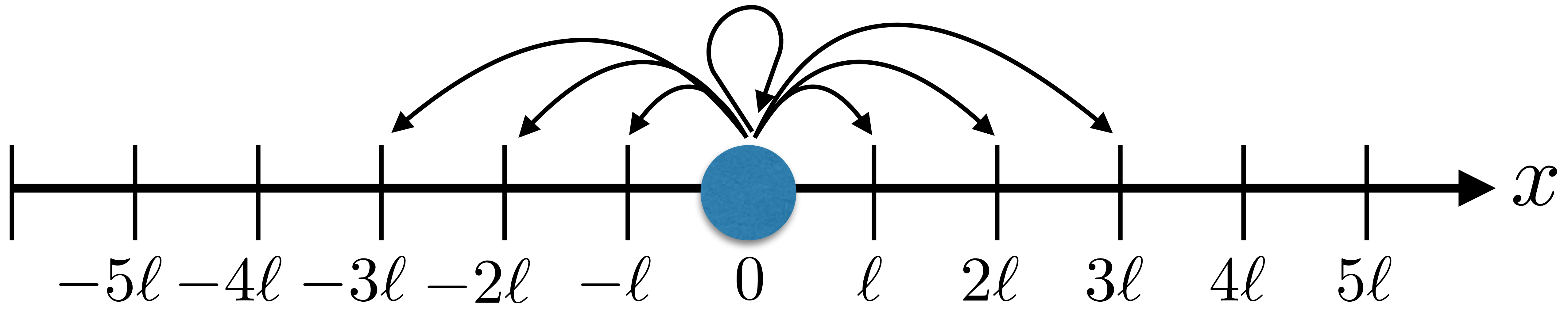
$\sigma^2 = 2Dt$



The **endpoint** of the random walk has a **probability distribution** that obeys a **simple continuum law**, the diffusion equation

Gaussian distribution of the endpoints

Fokker-Planck equation



In general the probability distribution Π of jump lengths *s* can depend on the particle position *x*

$$\Pi(s|x)$$

Generalized master equation:

$$p(x, t+\Delta t) = \sum_s \Pi(s|x-s)p(x-s, t)$$

Again Taylor expand the master equation above to derive the Fokker-Planck equation:

$$\boxed{\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\Big[v(x)p(x,t)\Big] + \frac{\partial^2}{\partial x^2}\Big[D(x)p(x,t)\Big]}$$

drift velocity
(external fluid flow, external potential)

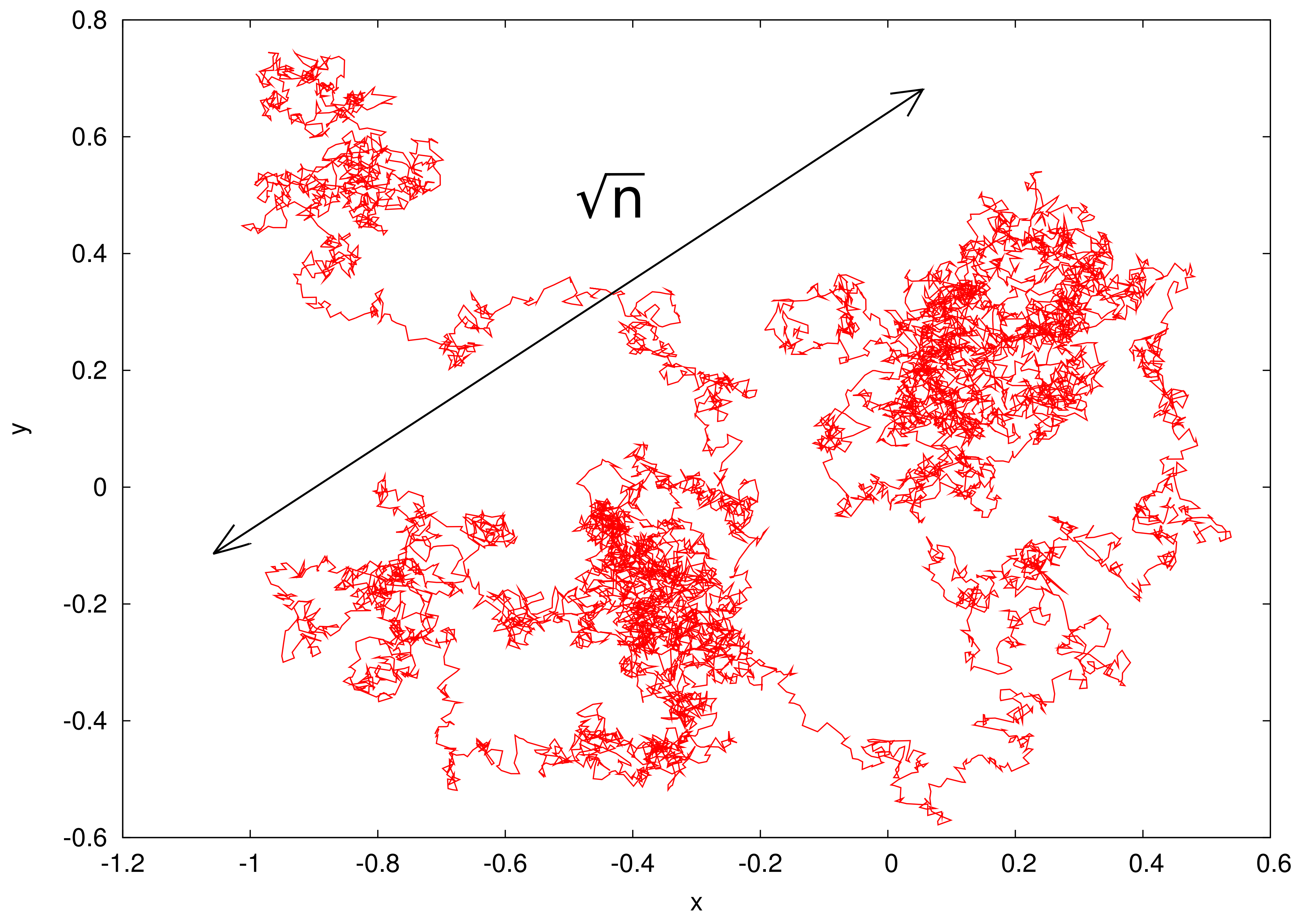
$$v(x) = \sum_s \frac{s}{\Delta t}\Pi(s|x) = \frac{\langle s(x)\rangle}{\Delta t}$$

diffusion coefficient
(e.g. position dependent temperature)

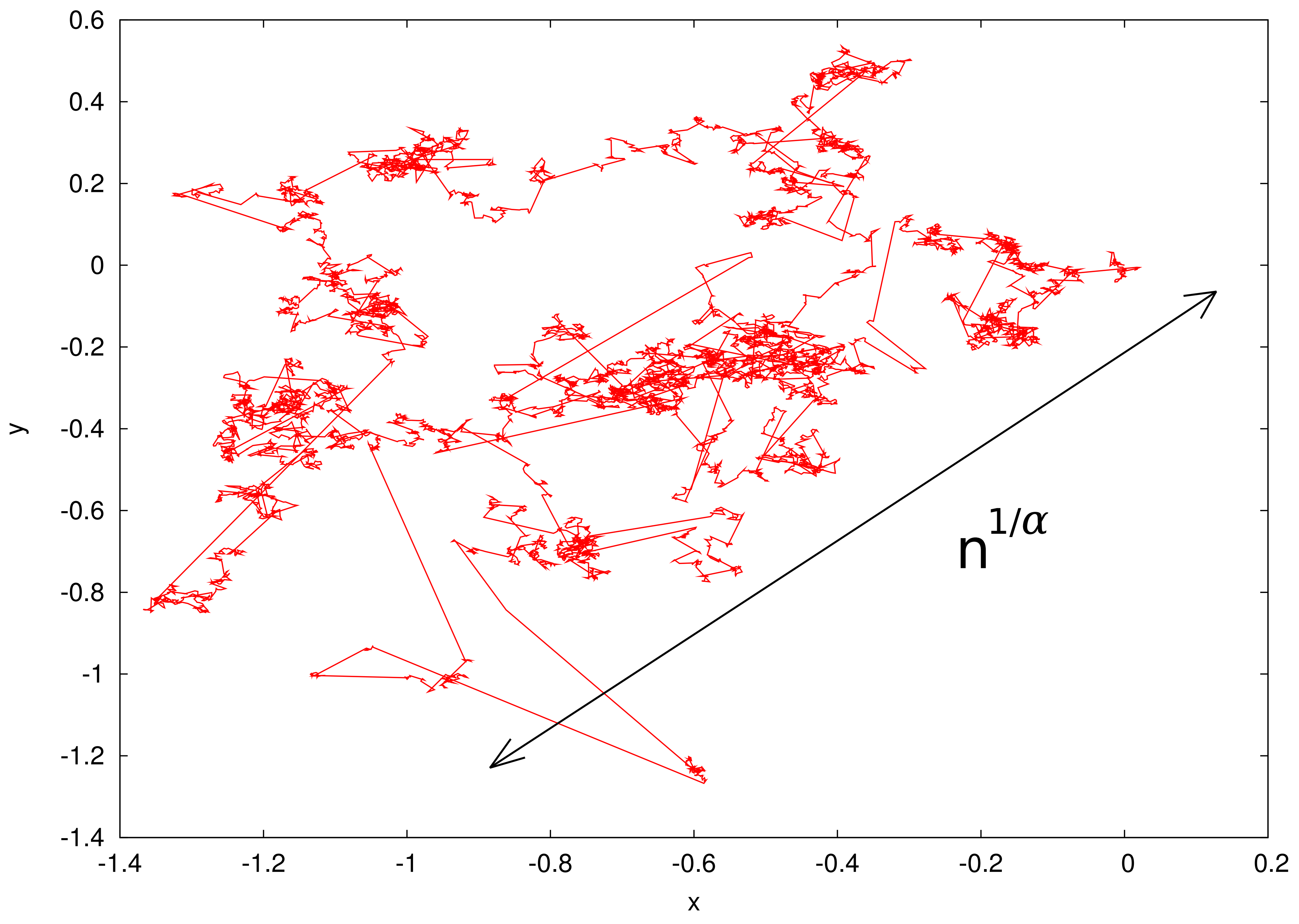
$$D(x) = \sum_s \frac{s^2}{2\Delta t}\Pi(s|x) = \frac{\langle s^2(x)\rangle}{2\Delta t}$$

Lévy flights

Jump length distribution: $\Pi(s) \sim \dfrac{c}{s^{\alpha+1}}$



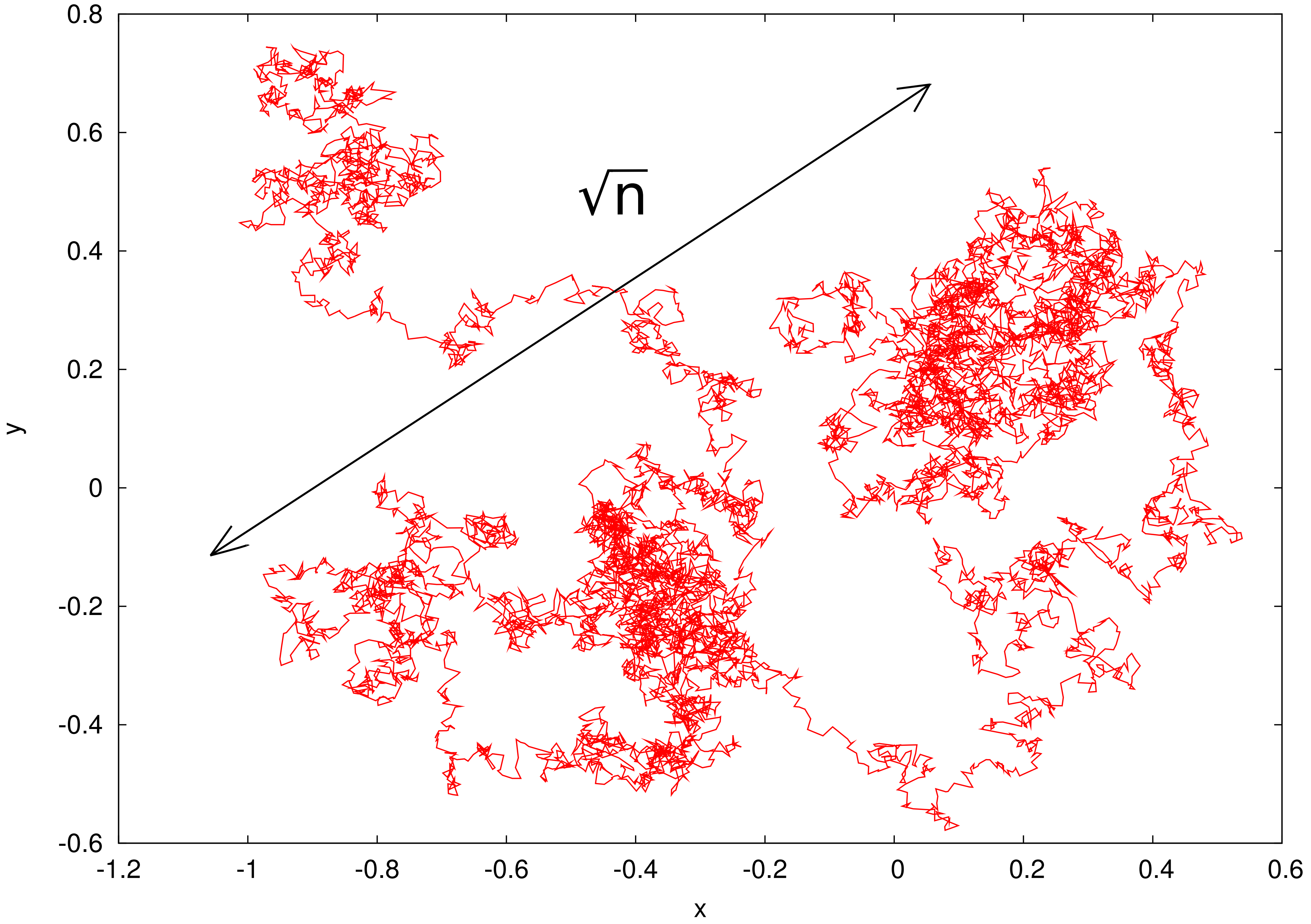
$\alpha > 2$

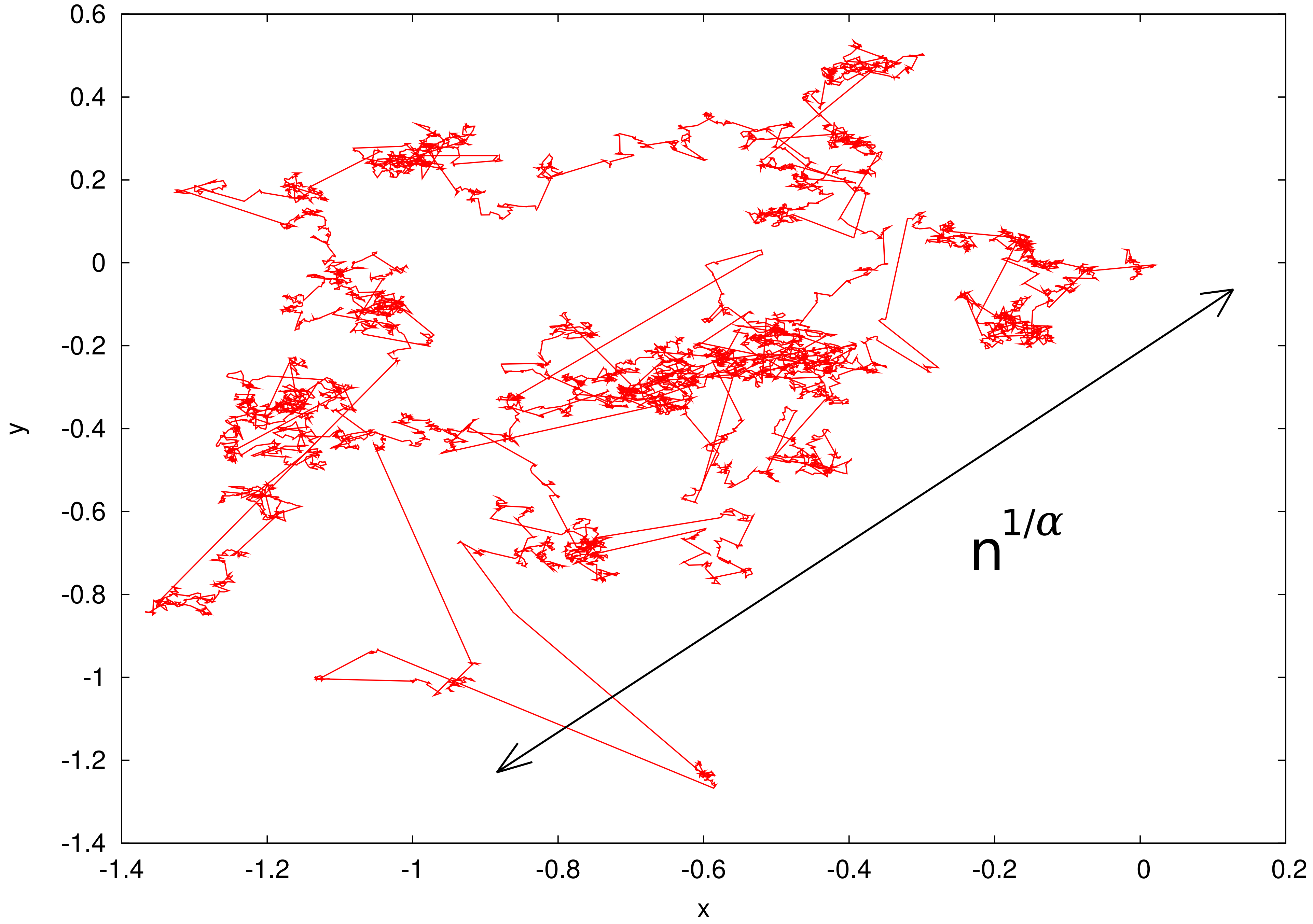


$\alpha \leq 2$

Lévy flights

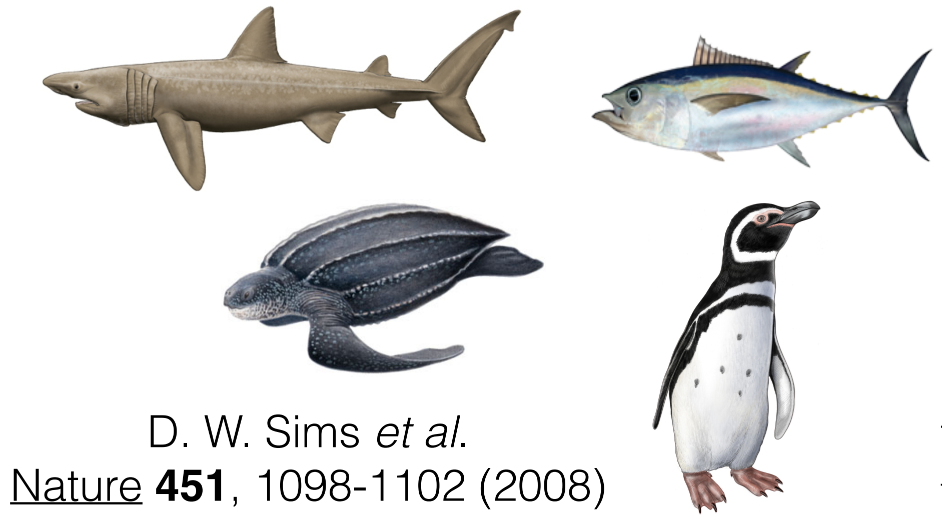
Jump length distribution: $\Pi(s) \sim \dfrac{c}{s^{\alpha+1}}$





$\alpha > 2$

$\alpha \leq 2$



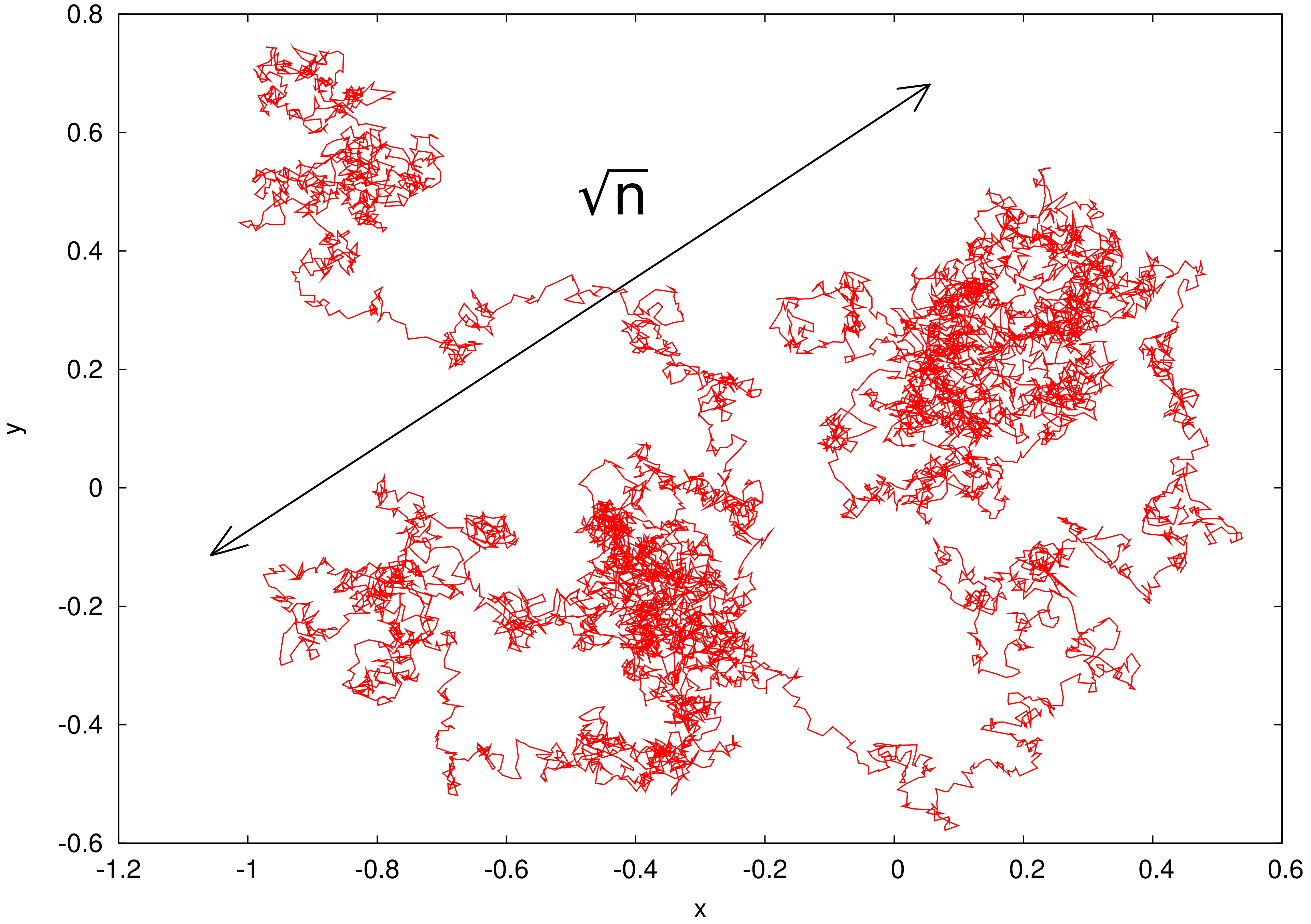
D. W. Sims *et al.*
Nature **451**, 1098-1102 (2008)

Spread faster than diffusion —-> "Super-diffusion"

Argue that Lévy flights are a better search strategy when prey is scarce

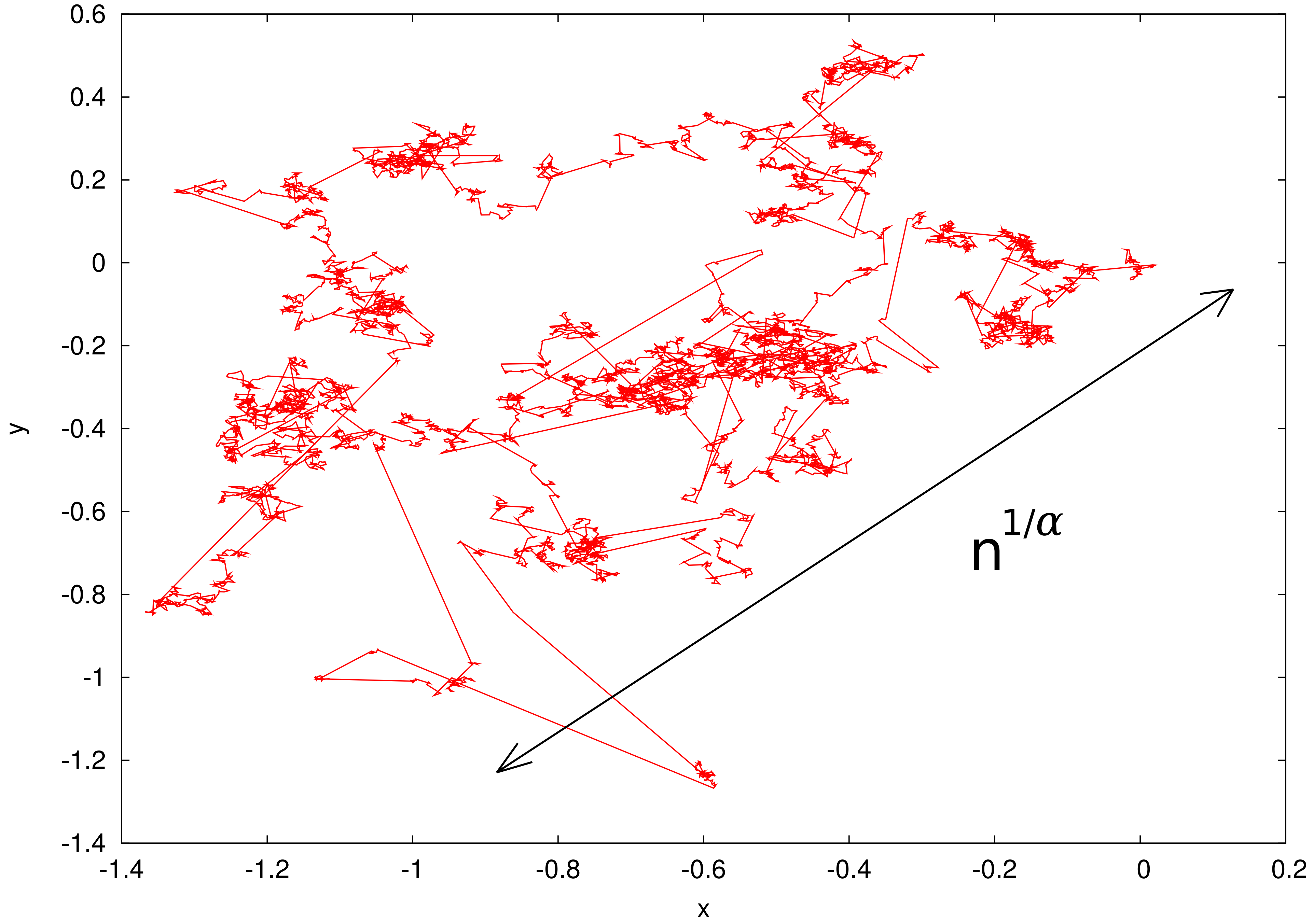
Lévy flights

Jump length distribution: $\Pi(s) \sim \frac{c}{s^{\alpha+1}}$



$\alpha > 2$

Mean and variance defined —> Brownian motion



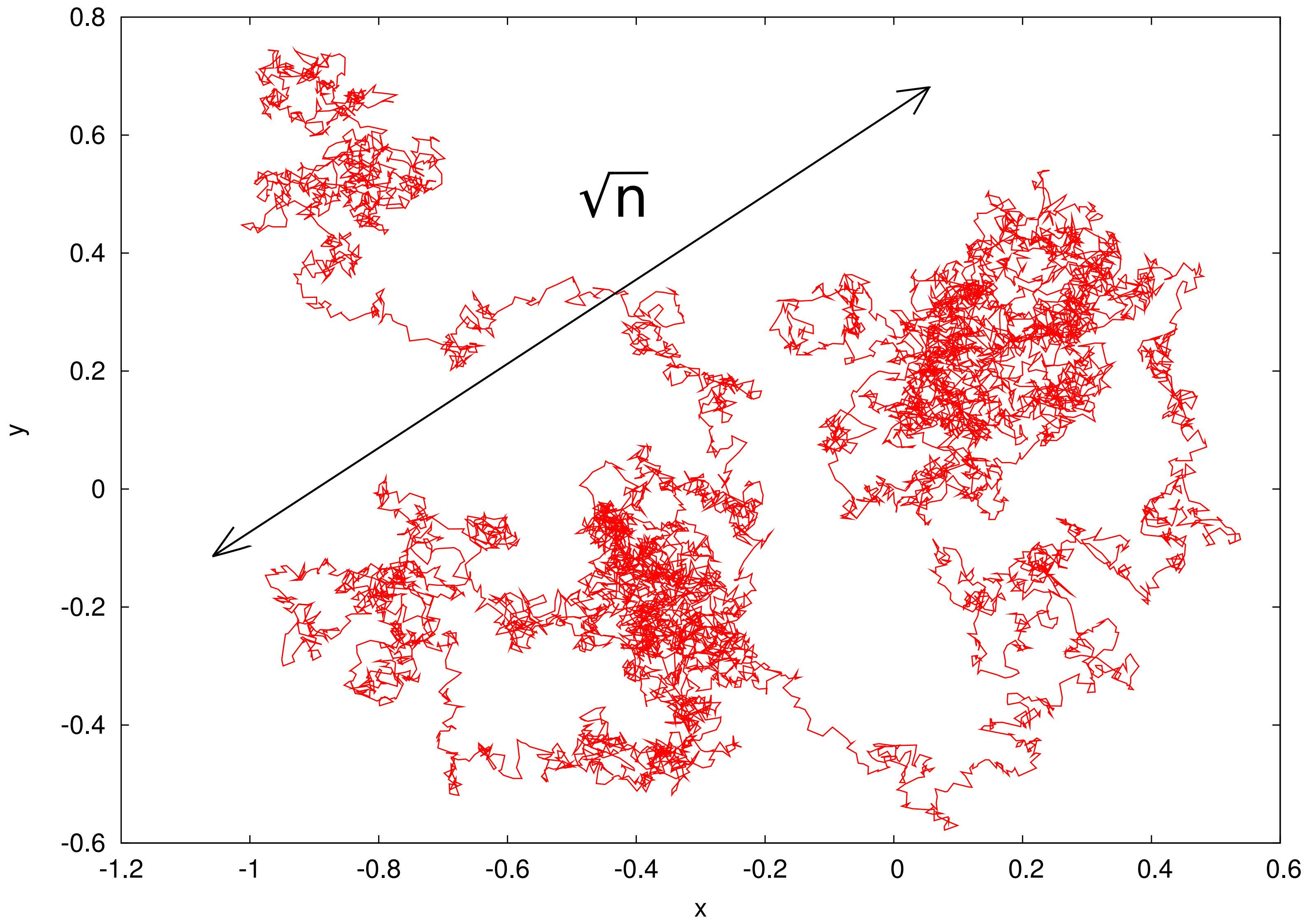
$\alpha \leq 2$

Variance not defined (infinite) —> some rare but very large jumps

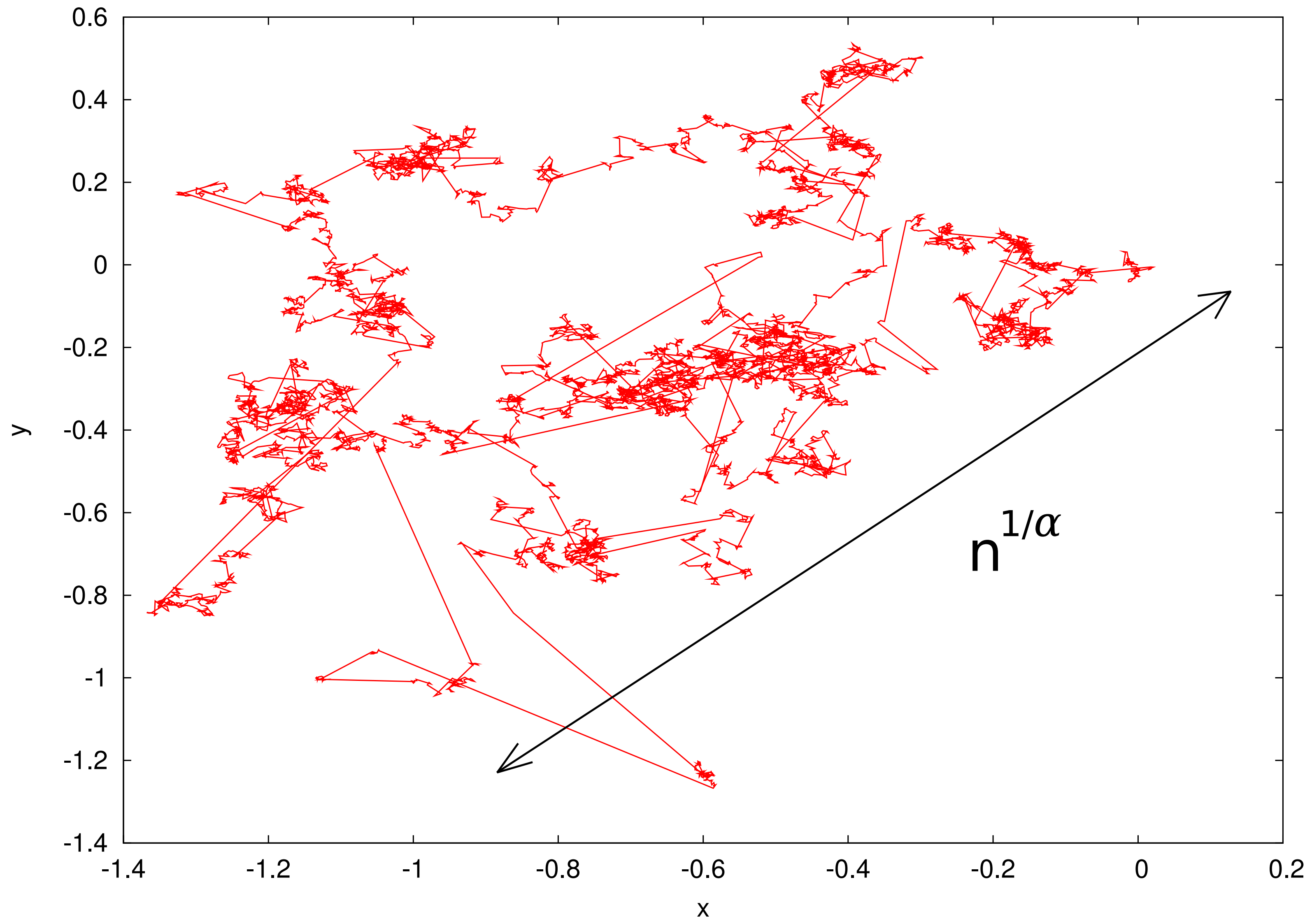
—> Lévy Flights

Lévy flights

Jump length distribution: $\Pi(s) \sim \frac{c}{s^{\alpha+1}}$



$\alpha > 2$



$\alpha \leq 2$

Mean and variance defined —> Brownian motion

Distribution of the walker position = Gaussian distribution (CLT)

Evolution described by: Diffusion Equation — Fokker-Planck Equation

Variance not defined (infinite) —> Lévy Flights

Distribution of the walker position

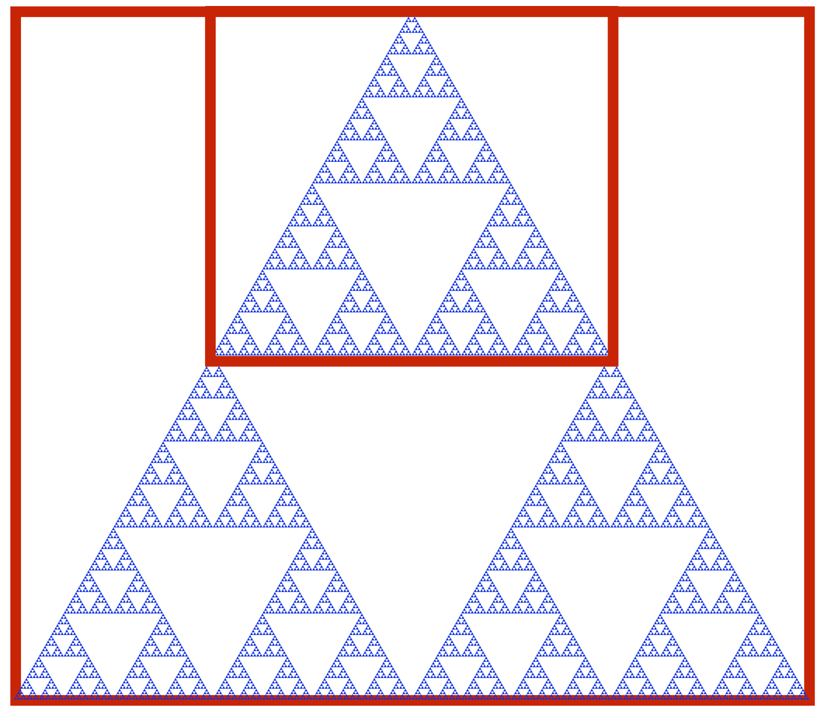
= Lévy stable distribution with same exponent α (generalized CLT)

Evolution described by: fractional Fokker-Planck Equation

Self-Similarity and Scale Invariance

Self-similar object: is exactly or approximately similar to a part of itself

It is a typical properties of **fractals:**



Many real world elements are **statistically self-similar**:

Scale invariance: **At any scale**, there is a small piece of object that is similar to the whole.