Chapter 4

Examples of non-equilibrium critical phenomena

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In the first part of the course, phase transitions were studied in the equilibrium framework, and the goal was to characterize the order arising below oor above a critical value of the control parameters. Here we would like to investigate how this order is formed dynamically, i.e. when the system is evolving out-of-equilibrium.

Symbol "(⋆)": Questions and exercises indicated with a (⋆) are optional. No worries if you don't have time to try to solve them, or if you don't manage to solve them on your own.

The symbol "</>": Indicates optional questions with numerical simulation.

4.1 Epidemic spreading

In the compartmental SIS model, the fraction of infected in a population is changing in time as:

$$\frac{d\,\mathrm{i}(t)}{dt} = (\beta - \mu)\,\mathrm{i}(t) - \beta\,\mathrm{i}(t)^2\,, \tag{4.1}$$

where μ is the recovery rate and β is the transmission rate.

Q1. What are the steady solutions to (4.1)? Analyze the stability of the steady state solutions by linearizing the equation (4.1) around the steady states i_0 as $i(t) = i_0 + \epsilon(t)$. Depending on the value of $R_0 = \beta/\mu$, which steady states are stable under which circumstances?

> The **SIR model** has individuals recover into a recovered compartment. The rate equations are:
>
> $$\frac{ds(t)}{dt} = -\beta\,\mathrm{i}(t)s(t)\,, \quad \frac{\mathrm{d\,i}(t)}{\mathrm{d}t} = \beta\,\mathrm{i}(t)\,\mathrm{s}(t) - \mu\,\mathrm{i}(t)\,, \quad \frac{\mathrm{d\,r}(t)}{\mathrm{d}t} = \mu\,\mathrm{i}(t)\,, \tag{4.2}$$
>
> where s, i, r are the fractions of susceptible, infected and recovered respectively, such that $1 = \mathrm{s}(t) + \mathrm{i}(t) + \mathrm{r}(t)$ at all times. At late times ($t \longrightarrow \infty$), the total fraction of recovered individuals r_∞ is a control parameter for the epidemic outbreak. Here we will derive a self-consistency relation for r_∞ using the above equations.

Q2. Show that:

$$C(t) = \mathrm{s}(t)\exp\left(\frac{\beta}{\mu}\,\mathrm{r}(t)\right), \tag{4.3}$$

is a **constant of motion**, i.e. show that $\frac{\mathrm{d}}{\mathrm{d}t}C(t) = 0$ by using (4.2).

Q3. Using (4.3), derive an expression for r_∞ by equating $C(t=0) = C(t=\infty)$. Choose as initial conditions a completely susceptible population and use the fact that at late times $i_\infty = 0$.

In the SIS model on a network, before taking a mean-field approximation, the probability of any node i being infected ($X_i(t) = 1$) depends on the probability of it forming an SI-pair with a neighboring node in the network

$$\frac{d\mathbb{P}[X_i(t)=1]}{dt} = -\mu\,\mathbb{P}[X_i(t)=1] + \beta\sum_j A^{ij}\mathbb{P}[X_i(t)=0; X_j(t)=1]\,. \tag{4.4}$$

Q4. Derive the dynamical equation for the joint probability $\mathbb{P}[X_i(t)=0; X_j(t)=1]$ of node i being susceptible and node j being infected. To do so, think about all the different ways that $\mathbb{P}[X_i(t)=0; X_j(t)=1]$ can change, when the dynamical rules of the system are given by recovery of single nodes from infected to susceptible with rate μ and infection by neighboring nodes with rate β.

In the heterogeneous mean-field models on networks, we bin nodes of the same degree k (i.e. nodes with the same number of neighbors). The fraction of infected nodes of degree k is $i_k(t)$ and follows (for the SIS model):

$$\frac{di_k(t)}{dt} = -\mu i_k(t) + \beta k\,(1 - i_k(t))\Theta_k(t) \tag{4.5}$$

where $\Theta_k(t)$ is the fraction of infected neighbors of nodes of degree k. Assuming the absence of degree correlations, this quantity is independent of k and given by:

$$\Theta_k(t) = \Theta(t) = \sum_{k'}\frac{k'-1}{\langle k\rangle}P(k')i_{k'}(t) \tag{4.6}$$

where $\langle k\rangle = \sum_k kP(k)$ is the mean degree and $P(k)$ the degree distribution.

Q5. Assuming that initially $i_k(t) \ll 1$ for all k, neglect quadratic terms in $i(t)$ and derive a linearized equation from (4.5). Use this to derive the rate equation for $\Theta(t)$. How will the number of infected grow with time? What does this mean for the epidemic threshold?

4.2 The (linear) voter model

This exercise is inspired from the Chapter on "Spin dynamics" (Chapter 8) of the book "A Kinetic View of Statistical Physics" by P. L. Krapivsky, S. Redner, and E. Ben Naim [1].

> The **linear voter model** is the simplest kinetic spin system. It has the advantage that, on a lattice structure, it is exactly soluble in any dimensions. The model describes how consensus emerges in a population of individuals that have no firmly fixed opinion. Individuals randomly take the opinion of one of their neighbors. A finite population of such voters eventually achieves consensus in a time that depends on the system size, the spatial dimension, and the number of opinions. Note that a consensus (i.e. everyone has the same opinion) will eventually be reached as it is the only absorbing state of the system.
>
> In the voter model, social interactions between voters are represented by a graph, in which each node represents an individual and links between two nodes indicate which individuals interact. This graph could be a regular lattice in d dimension, or it could be any other type of graph (an Erdös-Rényi random graph, or a graph with broad degree distribution, for example). Let us label the nodes of the graph with an index $i \in \{1, 2, \ldots, N\}$. The opinion of each individual i is indicated by its state s_i , which can take discrete values. For simplicity, we restricted the dynamics to just two opinions (*for* or *against* some issue), such that the s_i are binary variables: $s_i = \pm 1$. For instance, individuals in the state $s_i = +1$ could be "Democrats", and individuals in the state $s_i = -1$ could be "Republicans" (such as in the homework exercise of H1). A state of the system of the N voters is denoted by $\boldsymbol{s} = (s_1, s_2, \ldots, s_N)$.

The dynamics of the voter model is very simple: each voter has no confidence and looks randomly to one of its neighbors to decide what to do. At each time step:

1. Pick a random voter i;
2. Select a random neighbor j of the voter i, and align the opinion of i to the opinion of j, i.e set $s_i = s_j$;
3. Repeat steps 1 & 2 *ad infinitum*, or until a consensus is reached.

Note that in step 2, the voter i changes opinion only when its neighbor j has opposite opinion. Figure 4.2 shows the evolution of the voter model on a square lattice with two different types of initial conditions. In the bottom row, starting from a random initial condition with equal densities of up and down states, one can observe how the system tends to organize in domains with single opinion as time increases.

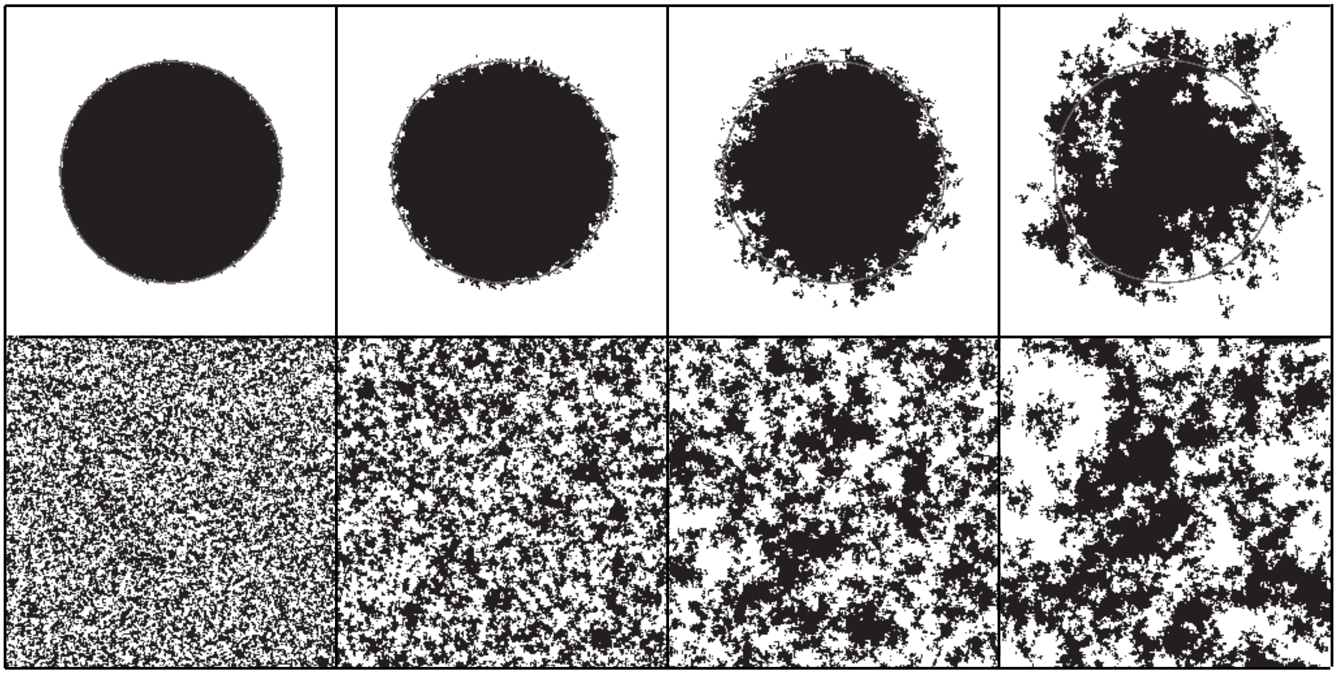

Figure 4.1: Figure is from Dornic et al. [2]: Evolution of the voter model on a 256 × 256 square lattice. Black and white pixels indicate the different opinion states. The top row shows snapshots at times $t = 4$, 16, 64, and 256 starting with an initial circular bubble of one opinion of radius 180 in a sea of the opposite opinion. The bottom row shows the same evolution starting with a random initial condition with equal densities of the two opinion states.

Q0. (bonus) (</>) Starting from one of the two initial conditions of Fig. 4.2, can you perform a Monte Carlo simulation of this simple voter model on a 2D lattice? Do you recover a behavior similar to that of Fig. 4.2?

Q1. Can you show that, at any time step n of the simulation described above, the probability that the voter i changes its opinion from s_i to $-s_i$ is equal to:

$$\mathbb{P}[(s_i, n) \longrightarrow (-s_i, n+1)] = \frac{1}{N} \frac{q_i[\boldsymbol{s}(n)]}{k_i}, \tag{4.7}$$

where N is the total number of nodes on the graph, k_i is the total number of neighbors of node i (i.e. the multiplicity of node i in the graph), and $q_i[\boldsymbol{s}(n)]$ is the number of these neighbors that disagree with voter i (i.e. that hold the opinion opposite to s_i) at the n-th step of the simulation. Note that q_i depends on the state $\boldsymbol{s}(n)$ of the system at the n-th step.

Q2. Let us define a small time interval $dt = 1/N$. The simulation starts at time $t = 0$, and each step of the simulation last a small time interval dt. After n steps, the time is $t = n\,dt$. Can you re-write Eq. (4.7) as a function of t instead of n? Deduce that the rate at which a voter i switches opinion is equal to the fraction of its neighbors that hold the opposite opinion, $w_i[\boldsymbol{s}] = q_i[\boldsymbol{s}]/k_i$, and changes with time. Observe that the rate $w_i(\boldsymbol{s})$ depends on the state of the system, which evolves in time. Each voter switch its opinion following an **inhomogeneous Poisson process**.

Q3. What is the value of the product $s_i s_j$ if the two voters i and j have the same opinion? if they disagree? Using this observation, can you show that $w_i(\boldsymbol{s})$ can be re-written as:

$$w_i(\boldsymbol{s}) = \frac{1}{2}\left[1 - \frac{s_i}{k_i} \sum_{j\in\langle i\rangle} s_j\right], \tag{4.8}$$

where $j \in \langle i\rangle$ denotes a sum over all the neighbors j of i? Tips: you can start by computing the number $q_i[\boldsymbol{s}]$ of neighbors of i that disagree with i when the system is in a state $\boldsymbol{s}$.

Q4. Master equation. In principle, to solve the voter model and describe the dynamical evolution of the system, we would need to compute the probability $P(\boldsymbol{s}, t)$ of the system to be in the state $\boldsymbol{s}$ at any time t. Can you show that $P(\boldsymbol{s}, t)$ satisfies the master equation:

$$\frac{dP(\boldsymbol{s},t)}{dt} = \sum_{i=1}^{N} P(\boldsymbol{s}_{-i}, t)\, w_i(\boldsymbol{s}_{-i}) - P(\boldsymbol{s}, t) \sum_{i=1}^{N} w_i(\boldsymbol{s}), \tag{4.9}$$

in which $\boldsymbol{s}_{-i}$ denotes the state of the voter identical to the state $\boldsymbol{s}$ but in which spin i has been flipped?

Q5. In principle, the master equation (4.9) can be used to derive equations for all the moments of the probability distribution, namely all multi-spin correlation functions of the form $\langle s_i s_j \cdots s_k\rangle$ where the brackets denote the ensemble average $\langle A(\boldsymbol{s})\rangle = \sum_{\boldsymbol{s}} A(\boldsymbol{s})P(\boldsymbol{s})$. Let us consider the simplest of such moments: the mean $\langle s_i\rangle$. The equation for $\langle s_i\rangle$ could be obtained from the master equation, however it is quite cumbersome. Let us take a more direct approach. Assuming that the system is in a state $\boldsymbol{s}$ at time t, depending on the value of $s_i(t)$, which values can be taken by $s_i(t + dt)$ at time $t + dt$, and with which probability? Can you then show that the mean opinion of s_i evolves as:

$$\frac{d\langle s_i\rangle}{dt} = -2\,\langle s_i\, w_i(\boldsymbol{s})\rangle \ ? \tag{4.10}$$

Q6. Replacing $w_i(\boldsymbol{s})$ by its value, can you show that:

$$\frac{d\langle s_i\rangle}{dt} = -\langle s_i\rangle + \frac{1}{k_i} \sum_{j\in<i>} \langle s_j\rangle \ ? \tag{4.11}$$

Q7. As for the Ising model, we can define the total magnetization (per spin) of the system as: $\sum_{i=1}^{N} s_i/N$. We denote by m the mean magnetization: $m = \sum_{i=1}^{N} \langle s_i\rangle /N$. This represents the average orientation of the system. For instance, if the state $s_i = +1$ represents a “Democrat”, and the state $s_i = -1$ represents a “Republican”, then the sign of m would represent the average political orientation of the whole population.

For simplicity, let us consider graphs in which all the nodes have the same multiplicity k, i.e. for all node i, $k_i = k$. Using Eq. (4.11), can you show that the mean magnetization is conserved over time, i.e. that:

$$\frac{dm}{dt} = 0 \ ? \tag{4.12}$$

Notice that while the magnetization of a specific system does change in a single update event, the average over all sites and over all trajectories of the dynamics is conserved. The consequences of this conserved mean magnetization are profound.

Q8. Consider a finite system with an **initial** fraction ρ_0^+ of voters in the +1 state and a fraction $1 - \rho_0^+$ in the −1 state, so that the initial magnetization is $m_0 = 2\,\rho_0^+ - 1$. After a very long time, the system will reach a consensus, i.e. the final total magnetization will be +1 (all voters will be +1), or the final total magnetization will be −1 (all voters will be −1). Let us define $E(\rho)$ as the probability that the system reach the “+1” consensus. Using the conservation of the mean magnetization, can you compute the value of $E(\rho)$ as a function of the original fraction ρ_0^+ of +1 voters in the initial state of the system?

Q9 (bonus) (</>). Write an algorithm that implements the dynamics of the voter model and verify the properties you just derived using many realizations of the dynamics.

References

[1] P. L. Krapivsky, S. Redner, and E. Ben-Naim, *A kinetic view of statistical physics*. Cambridge University Press, 2010.

[2] I. Dornic, H. Chaté, J. Chave, and H. Hinrichsen, "Critical coarsening without surface tension: The universality class of the voter model," *Physical Review Letters*, vol. 87, no. 4, p. 045 701, 2001.

4.3 Stability analysis

Bifurcation theory is a mathematical framework closely related to the study of phase transitions. The difference is that in bifurcation theory the object of study are dynamical systems , while for phase transitions we look at the behaviour of the free energy. For one-dimensional systems, it is possible to define a bifurcation potential $U(x)$ such that

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\mathrm{d}U(x)}{\mathrm{d}x} \,. \tag{4.13}$$

The local minima of such a potential are equivalent to the steady states of the corresponding dynamical system. As an example, consider

$$\dot{x} = \mu x^2 + x \,. \tag{4.14}$$

In bifurcation analysis, μ is considered to be parameter we can change . As a function of μ interesting things can happen to the nature of the solutions. The steady state solution is given by setting $\dot{x} = 0$, which gives $\mu x^2 + x = 0$. The steady-state solutions are therefore $x_0 = 0$ and $x_0 = -\frac{1}{\mu}$. To find the stability of this solution, one typically expands around the steady-state solution by writing $x(t) = x_0 + \varepsilon(t)$ and plugging this back into the equation. For this system we find

$$\frac{d}{dt}(x + \varepsilon) = \dot{x} + \dot{\varepsilon} = \mu(x_0 + \varepsilon)^2 + (x_0 + \varepsilon) = \mu x_0^2 + x_0 + \varepsilon(2\mu x_0 + 1) + \mu\varepsilon^2 \,. \tag{4.15}$$

Since $\dot{x} = \mu x_0^2 + x_0$ is a solution to $\dot{x} = 0$, the only remaining terms are those involving ε. Since ε is assumed to be small, we only keep the terms proportional to ε. This is called linearizing the equation . We now have an equation for the evolution of small perturbations to the steady-state solution in the form of

$$\dot{\varepsilon} = (2\mu x_0 + 1)\varepsilon \,. \tag{4.16}$$

Solving for $\varepsilon(t)$ we find

$$\varepsilon(t) = \varepsilon(0)\exp([2\mu x_0 + 1]t) \,. \tag{4.17}$$

Depending on the sign of $2\mu x_0 + 1$, the perturbation either decays to zero or blows up to infinity. Plugging in $x_0 = 0$, we have

$$\varepsilon(t) = \varepsilon(0)e^t \,. \tag{4.18}$$

Thus regardless of the value of μ the origin is an unstable solution. For $x_0 = -\frac{1}{\mu}$ we get

$$\varepsilon(t) = \varepsilon(0)e^{-t} \,. \tag{4.19}$$

Again, regardless of the value of μ, the solution $x_0 = -\frac{1}{\mu}$ is a stable steady-state solution. This example illustrates the procedure of stability analysis. Using it, we found that no bifurcation occurs. For the following systems we will see that more interesting behaviour is possible.

Q1. For each of the following dynamical systems

$$a)\ \ \dot{x} = \mu - x^2 \qquad b)\ \ \dot{x} = \mu x - x^2 \qquad c)\ \ \dot{x} = \mu x - x^3 \tag{4.20}$$

a. Find the steady-state solutions.

b. Find the value of μ at which a bifurcation occurs.

c. Make a sketch or plot of the steady-state solution branches as a function of μ.

d. Write down the stability of all of the solution branches on either side of the bifurcation point.

e. Find a suitable bifurcation potential $U(x)$ that reproduces the dynamical system. Is the potential a suitable candidate for a physical free energy?

Q2. Now consider the dynamical system

$$\dot{x} = \mu x^2 - x^4 \tag{4.21}$$

Clearly, $x_0 = 0$ is a fixed point of this system.

a. Do a linear stability analysis of the fixed point $x_0 = 0$ and comment on the result. How is this possible?

b. Think of a way to solve the problem you found in the previous question and solve the differential equation that you obtain.

c. Find the stability of the resulting solution. In particular, what happens at $\mu = 0$?

d. For several positive and negative values of μ, sketch or plot $\dot{x}$ as a function of x. What can this plot tell you about the stability of the steady-state solutions?

4.4 Solutions

4.4.1 Epidemic spreading

A1. The steady states are solutions to $\frac{d}{dt}i(t) = 0$. There are two:

$$i_0 = 0\,, \qquad i_1 = 1 - \frac{1}{R_0}\,, \qquad \text{with: } R_0 = \frac{\beta}{\mu}\,. \tag{4.22}$$

- If we expand the equation around i_0 as $i(t) = i_0 + \epsilon(t)$, we see that, for small positive $\epsilon(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{i}(t) = \frac{\mathrm{d}\epsilon(t)}{\mathrm{d}t} \sim (\beta - \mu)\epsilon(t)\,. \tag{4.23}$$

  This means that the fluctuations $\epsilon(t)$ will grow in size if $(\beta - \mu) > 0$ and decay in size when $\beta - \mu < 0$, or phrased in terms of R_0: when $R_0 < 1$ the steady state $i_0 = 0$ is stable, and when $R_0 > 1$ it is unstable.

- Likewise, we can expand the equation around the endemic steady state $i_1 = 1 - \frac{1}{R_0}$ as

$$\frac{\mathrm{d}\epsilon(t)}{\mathrm{d}t} = \mu(R_0 - 1)\left(1 - \frac{1}{R_0} + \epsilon(t)\right) - \beta\left(1 - \frac{1}{R_0} + \epsilon(t)\right)^2 \sim -\beta\left(1 - \frac{1}{R_0}\right)\epsilon(t)\,, \tag{4.24}$$

  where we use that $\frac{\mathrm{d}i_1}{\mathrm{d}t} = 0$.

  When $R_0 > 1$, we have $(1 - \frac{1}{R_0}) > 0$ and fluctuation around the endemic steady state will decay in size. When $R_0 < 1$ the fluctuation around the endemic steady state will grow in size and the system will flow to the other stable steady state at $i = i_0 = 0$.

A2. The solution is straight forward by application of the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}C(t) = \frac{ds(t)}{dt}\exp\left(\frac{\beta}{\mu}\,\mathrm{r}(t)\right) + \mathrm{s}(t)\frac{\beta}{\mu}\frac{\mathrm{d}\,\mathrm{r}(t)}{\mathrm{d}t}\exp\left(\frac{\beta}{\mu}\,\mathrm{r}(t)\right) \tag{4.25}$$

$$= -\beta\,\mathrm{i}(t)\,\mathrm{s}(t)\exp\left(\frac{\beta}{\mu}\,\mathrm{r}(t)\right) + \beta\,\mathrm{i}(t)\,\mathrm{s}(t)\exp\left(\frac{\beta}{\mu}\,\mathrm{r}(t)\right) = 0\,. \tag{4.26}$$

A3. Because $C(t)$ is actually constant in time, we can equate $C(t=0) = C(t=\infty)$ to obtain

$$s(0)\exp\left(\frac{\beta}{\mu}r(0)\right) = s_\infty\exp\left(\frac{\beta}{\mu}r_\infty\right)\,. \tag{4.27}$$

At $t = 0$ we choose a completely susceptible population $s(0) = 1$, such that $r(0) = 0$. At late times, the infection has died out, so $i_\infty = 0$ and by conservation of total probabilities $s_\infty = 1 - i_\infty - r_\infty = 1 - r_\infty$. We hence obtain an expression solely in terms of r_∞ and R_0

$$1 = (1 - r_\infty)e^{R_0 r_\infty}\,, \qquad \text{or: } e^{-R_0 r_\infty} = 1 - r_\infty\,. \tag{4.28}$$

A4. We have to analyze all possible ways in which SI pairs of nodes i and j can be created or destroyed. There are five possibilities, two contribute positively and three contribute negatively:

Positive contributions:

- The node pair i and j were both infected and node i recovers to form the SI pair. This increases the probability of being i and j forming an SI pair and happens with probability $\mu\mathbb{P}[X_i(t) = 1; X_j(t) = 1]$
- The node pair i and j were both susceptible, then node j became infected by one of its neighbours. This also contributes positively and happens with probability $\beta\sum_k A^{jk}\mathbb{P}[X_i(t) = 1; X_j(t) = 0; X_k(t) = 1]$.

Negative contributions:

- The node pair i and j are already in the SI configuration, and node j infects node i. This happens with probability $\beta A^{ij}\mathbb{P}[X_i(t) = 0; X_j(t) = 1]$.

- The node pair i and j are already in the SI configuration, and node j recovers. This happens with probability $\mu\mathbb{P}[X_i(t) = 0; X_j(t) = 1]$.
- The node pair i and j are already in the SI configuration, and node i is infected by another neighbor k. This happens with probability $\beta\sum_{k\neq j} A^{ik}\mathbb{P}[X_i(t) = 0; X_j(t) = 1; X_k(t) = 1]$.

Putting it all together we get the equation

$$\begin{aligned}\frac{d}{dt}\mathbb{P}[X_i(t) = 0; X_j(t) = 1] &= \mu\mathbb{P}[X_i(t) = 1; X_j(t) = 1] - \beta A^{ij}\mathbb{P}[X_i(t) = 0; X_j(t) = 1] \\ &\quad - \mu\mathbb{P}[X_i(t) = 0; X_j(t) = 1] + \beta\sum_k A^{jk}\mathbb{P}[X_i(t) = 1; X_j(t) = 0; X_k(t) = 1] \\ &- \beta\sum_{k\neq j} A^{ik}\mathbb{P}[X_i(t) = 0; X_j(t) = 1; X_k(t) = 1]\,.\end{aligned} \tag{4.29}$$

A5. The linear approximation for small fractions of infected gives

$$\frac{d}{dt}i_k(t) = -\mu i_k(t) + \beta k\Theta(t)\,, \tag{4.30}$$

with

$$\Theta(t) = \sum_{k'}\frac{k'-1}{\langle k\rangle}P(k')i_{k'}(t)\,. \tag{4.31}$$

Taking the time derivative of $\Theta(t)$ and using (4.30) gives:

$$\frac{d}{dt}\Theta(t) = \frac{1}{\langle k\rangle}\sum_{k'}(k'-1)P(k')\left(-\mu i_{k'}(t) + \beta k'\Theta(t)\right) \tag{4.32}$$

$$= -\mu\Theta(t) + \frac{\beta}{\langle k\rangle}\sum_{k'}(k'^2 - k')P(k')\Theta(t) \tag{4.33}$$

$$= -\mu\Theta(t) + \frac{\beta}{\langle k\rangle}\left(\langle k^2\rangle - \langle k'\rangle\right)\Theta(t)\,. \tag{4.34}$$

$$\tag{4.35}$$

This implies that

$$\Theta(t) \sim e^{t/\tau}\,, \qquad \text{with: } \tau^{-1} = \mu\left(\frac{\beta}{\mu}\frac{\langle k^2\rangle - \langle k\rangle}{\langle k\rangle} - 1\right)\,. \tag{4.36}$$

And so the basic reproductive number R_0 is defined in terms of transmission and recovery rates and the network degree fluctuations as

$$R_0 = \frac{\beta}{\mu}\frac{\langle k^2\rangle - \langle k\rangle}{\langle k\rangle}\,. \tag{4.37}$$

For large $\langle k^2\rangle$, the reproduction number is increased, meaning that the epidemic threshold at $R_0 = 1$ is lowered!

4.4.2 Voter model

A1. At each time step:

- the probability that the voter i is chosen is $1/N$, where N is the number of voters (= the number of nodes);
- if selected, the probability that the voter i changes its opinion is then equal to the probability that the randomly chosen neighbor j has the opposite opinion; this probability is equal to the fraction of neighbors of i that have opposite opinion, which is the ratio $q_i(n)/k_i$ of the number of neighbors with opposite opinion $q_i(n)$ and the total number of neighbors k_i.

This finally gives that the probability that the voter i changes its opinion between the time steps n and $(n+1)$ in Eq. (4.7).

A2. Re-writing Eq. (4.7) as a function of the time t, we get:

$$\mathbb{P}[s_i(t) \longrightarrow -s_i(t+dt)] = \frac{q_i[\boldsymbol{s}(t)]}{k_i}\, dt = w_i[\boldsymbol{s}(t)]\, dt\,. \tag{4.38}$$

The voter i therefore changes its opinion with a rate $w_i[\boldsymbol{s}(t)] = \frac{q_i[\boldsymbol{s}(t)]}{k_i}$, which may evolve in time, while the opinions of the neighbors of i are still changing.

> Observe that the rate $w_i(\boldsymbol{s})$ depends on the state of the system, which evolves in time. Each voter switch its opinion following an **inhomogeneous Poisson process**.

A3. $s_i s_j = -1$ if the two voters i and j have opposite opinions, and $s_i s_j = +1$ if the two voters have the same opinion. Using this, we can define the following marker function :

$$\delta(s_i, s_j) = \frac{1}{2}(1 - s_i s_j)\,, \tag{4.39}$$

which is equal to 1 if i and j have opposite opinions, and 0 if they have the same opinion. The number of neighbors of i that have an opinion opposite to s_i is therefore given by:

$$q_i(\boldsymbol{s}) = \sum_{j\in<i>} \delta(s_i, s_j) \tag{4.40}$$

$$= \frac{1}{2}\sum_{j\in<i>}(1 - s_i s_j) = \frac{1}{2}\left[k_i - s_i \sum_{j\in<i>} s_j\right], \tag{4.41}$$

where we used that $\sum_{j\in<i>} 1 = k_i$ is equal to the total number of numbers of i. Using that $w_i(\boldsymbol{s}) = q_i(\boldsymbol{s})/k_i$ finally gives Eq. (4.8).

A4. In the master equation, the loss term accounts for all possible transitions out of the state $\boldsymbol{s}$, while the gain term accounts for transitions to the state $\boldsymbol{s}$ from states in which a single spin differ. In more details: the probability that the system is in state $\boldsymbol{s}$ at time $t + dt$ can be connected to the states of the system at time t in the following way:

$$P(\boldsymbol{s}, t+dt) = \underbrace{\sum_{i=1}^{N} P(\boldsymbol{s}_{-i}, t)\; w_i(\boldsymbol{s}_{-i})\, dt}_{(1)} + \underbrace{P(\boldsymbol{s}, t)\left[1 - \sum_{i=1}^{N} w_i(\boldsymbol{s})\, dt\right]}_{(2)}, \tag{4.42}$$

where

(1) is the probability that, at time t, the system was in a state $\boldsymbol{s}_{-i}$ that differs from the state $\boldsymbol{s}$ by only a single spin s_i, and that this spin flipped from $-s_i$ to s_i during dt;

(2) is the probability that the system was already in state $\boldsymbol{s}$ at time t and that no spin were flipped during dt (i.e., no voter changed its opinion during dt).

Re-organizing the terms in Eq. (4.42), we get:

$$\frac{P(\boldsymbol{s}, t+dt) - P(\boldsymbol{s}, t)}{dt} = \underbrace{\sum_{i=1}^{N} P(\boldsymbol{s}_{-i}, t)\; w_i(\boldsymbol{s}_{-i})}_{(1)} + \underbrace{P(\boldsymbol{s}, t)\left[\sum_{i=1}^{N} w_i(\boldsymbol{s})\right]}_{(2)}, \tag{4.43}$$

which finally leads to the master equation (4.9).

A5. Assuming that the system is in a state $\boldsymbol{s}$ at time t, and knowing $s_i(t)$, the value of $s_i(t+dt)$ is given by:

$$s_i(t+dt) = \begin{cases} s_i(t) & \text{with probability} \;\; 1 - w_i(\boldsymbol{s}, t)\, dt \\ -s_i(t) & \text{with probability} \;\; w_i(\boldsymbol{s}, t)\, dt \end{cases} \tag{4.44}$$

i.e., the spin i is flipped with probability $w_i(\boldsymbol{s}, t)\, dt$ and remains unchanged with probability $(1 - w_i(\boldsymbol{s}, t)\, dt)$. Note that in the case where the spin i remains unchanged, other spins could have been flipped. Using this equation, the average value of $s_i(t+dt)$ at time $t+dt$ can be computed by summing over all possible state $\boldsymbol{s}$ in which the system could have been at time t, which gives:

$$\langle s_i(t+dt)\rangle = \sum_{\boldsymbol{s}} P(\boldsymbol{s}, t)\, s_i(t)[1 - w_i(\boldsymbol{s}, t)\, dt] + \sum_{\boldsymbol{s}} P(\boldsymbol{s}, t)\, (-s_i(t))\, w_i(\boldsymbol{s}, t)\, dt\,. \tag{4.45}$$

Cleaning up a bit:

$$\langle s_i(t+dt)\rangle = \sum_{\boldsymbol{s}} P(\boldsymbol{s}, t)\, s_i(t) - 2 \sum_{\boldsymbol{s}} P(\boldsymbol{s}, t)\, s_i(t)\, w_i(\boldsymbol{s}, t)\, dt\,, \tag{4.46}$$

$$= \langle s_i(t)\rangle - 2\,\langle s_i(t)\, w_i(\boldsymbol{s}, t)\rangle\, dt \tag{4.47}$$

Re-organizing the terms finally leads to:

$$\frac{\langle s_i(t+dt)\rangle - \langle s_i(t)\rangle}{dt} = -2\,\langle s_i(t)\, w_i(\boldsymbol{s}, t)\rangle \tag{4.48}$$

which corresponds to equation (4.10).

A6. Replacing $w_i(\boldsymbol{s})$ by its value in Eq. (4.10), one gets:

$$\frac{d\,\langle s_i\rangle}{dt} = -\left\langle s_i \left[1 - \frac{s_i}{k_i} \sum_{j\in<i>} s_j\right]\right\rangle \tag{4.49}$$

$$= -\,\langle s_i\rangle + \frac{1}{k_i}\left\langle s_i^2 \sum_{j\in<i>} s_j\right\rangle \tag{4.50}$$

$$= -\,\langle s_i\rangle + \frac{1}{k_i} \sum_{j\in<i>} \langle s_j\rangle\;\;, \tag{4.51}$$

where we used that $s_i^2 = 1$.

A7. To obtain the evolution of the average total magnetization in time, we must sum Eq. (4.11) over all the nodes:

$$\frac{dm(t)}{dt} = \sum_{i=1}^{N} \frac{d\,\langle s_i(t)\rangle}{dt} \tag{4.52}$$

$$= -\sum_{i=1}^{N} \langle s_i(t)\rangle + \frac{1}{k} \sum_{i=1}^{N} \sum_{j\in<i>} \langle s_j(t)\rangle \tag{4.53}$$

$$= -m(t) + \frac{1}{k} \sum_{i=1}^{N} \sum_{j\in<i>} \langle s_j(t)\rangle\;\;. \tag{4.54}$$

As each node has exactly k neighbors, each of the term $\langle s_i\rangle$ is counted k times in the double sum:

$$\sum_{i=1}^{N} \sum_{j\in<i>} \langle s_j(t)\rangle = k \sum_{i=1}^{N} \langle s_i(t)\rangle = k\, m(t)\,. \tag{4.55}$$

Replacing this result in Eq. (4.54), we finally get:

$$\frac{dm(t)}{dt} = -m(t) + \frac{1}{k}\,k\,m(t) = 0\,. \tag{4.56}$$

A8. As a consequence of Eq. (4.12), we have that $m(t)$ is a constant. The average magnetization over all sites and over all trajectories of the dynamics is constant throughout the evolution of the system, and therefore at all time $m(t) = m_0$. In particular, the final magnetization is equal to the initial magnetization: $m_\infty = m_0$.

The initial magnetization is given by:

$$m_0 = (+1) \times \rho_0^+ + (-1) \times (1 - \rho_0^+) = 2\rho_0^+ - 1\,. \tag{4.57}$$

At infinite time, the system will always reach a consensus, which is either everyone votes +1, or everyone votes −1. These two states are the absorbing states of the system, so as soon as the system is in one of them, it will stay there. Averaging over all possible realizations of the dynamics, the final magnetization can be computed as:

$$m_\infty = (+1) \times E(\rho) + (-1) \times (1 - E(\rho)) = 2\,E(\rho) - 1\,, \tag{4.58}$$

where $E(\rho)$ is the probability that the system reaches the "+1" consensus. Using that $m_\infty = m_0$, we finally obtain that the probability that the system reach the consensus "+1" is equal to the original fraction of +1 voters in the initial state, i.e.: $E(\rho) = \rho_0^+$.

4.4.3 Stability analysis

A1. a. Saddle-node bifurcation. For the first system $\dot{x} = \mu - x^2$, the steady-state solutions are given by

$$\dot{x} = \mu - x^2 = 0 \implies x_0 = \pm\sqrt{\mu}\,. \tag{4.59}$$

To find the bifurcation point, we expand around the steady-state solution. We let $x(t) = x_0 + \varepsilon(t)$ and plug this into the governing equation

$$\dot{x} = \cancel{\dot{x}_0} + \dot{\varepsilon} = \dot{\varepsilon} = \mu - (x_0 + \varepsilon)^2 = \mu - x_0^2 - 2x_0\varepsilon - \varepsilon^2 \approx -2x_0\varepsilon\,. \tag{4.60}$$

Note that we neglect terms quadratic in ε. The above computation results in an evolution equation for the perturbation ε,

$$\dot{\varepsilon} = -2x_0\varepsilon\,, \tag{4.61}$$

which is solved by

$$\varepsilon(t) = \varepsilon(0)\exp(-2x_0 t)\,. \tag{4.62}$$

It is clear that the behavior of the perturbation depends on the sign of x_0. If $x_0 < 0$, the perturbation grows exponentially and if $x_0 > 0$, the perturbation decays exponentially.
To find the point where the bifurcation occurs, we look at $x_0 = \pm\sqrt{\mu}$. We see that the steady-state solutions are created at $\mu = 0$. For $\mu < 0$ there are no (real) steady-state solutions.
The stability of the solutions is found by plugging the different steady-state solutions into the governing equation for the perturbation ε. For $x_0 = +\sqrt{\mu}$, perturbations decay to zero and so this solution is stable. For $x_0 = -\sqrt{\mu}$, perturbations grow rapidly and therefore this solution is unstable.

b. Transcritical bifurcation. For the second system $\dot{x} = \mu x - x^2$, the steady-state solutions are given by

$$\dot{x} = \mu x - x^2 = x(\mu - x) \implies x_0 = 0\,,\,,\mu\,. \tag{4.63}$$

We again expand around the steady-solution to find their stability and the bifurcation point.

$$\dot{\varepsilon} = \mu(x_0 + \varepsilon) - (x_0 + \varepsilon)^2 = \mu x_0 + \mu\varepsilon - x_0^2 - 2x_0\varepsilon - \varepsilon^2 \approx (\mu - 2x_0)\varepsilon\,. \tag{4.64}$$

In this case, the equation governing the perturbations is

$$\dot{\varepsilon} = (\mu - 2x_0)\varepsilon \tag{4.65}$$

and is solved by

$$\varepsilon(t) = \varepsilon(0)\exp([\mu - 2x_0]t)\,. \tag{4.66}$$

Note that now the stability of the solution is determined by the sign of the term $\mu - 2x_0$. For the solution $x_0 = 0$, the stability depends on the sign of μ. If $\mu < 0$, this solution is stable. For the solution $x_0 = \mu$, we have $\mu - 2x_0 = -\mu$ and so the stability depends on the sign of μ but opposite to the $x_0 = 0$ solution. In this case, if $\mu < 0$, this solution is unstable. Clearly, the stability of the solutions switches at $\mu = 0$, which is the location of the bifurcation point.

c. Pitchfork bifurcation. For the third system $\dot{x} = \mu x - x^2$, the steady-state solutions are given by

$$\dot{x} = \mu x - x^3 = x(\mu - x^2) \implies x_0 = 0, \pm\sqrt{\mu}\,. \tag{4.67}$$

As usual, we expand around the steady-state solution.

$$\dot{\varepsilon} = \mu(x_0+\varepsilon) - (x_0+\varepsilon)^3 = \mu x_0 + \mu\varepsilon - (x_0^3 + 3x_0^2\varepsilon + 3x_0\varepsilon^2 + \varepsilon^3) \approx (\mu - 3x_0^2)\varepsilon \tag{4.68}$$

and so the perturbations are governed by

$$\varepsilon(t) = \varepsilon(0)\exp\big([\mu - 3x_0^2]t\big) \tag{4.69}$$

The stability now depends on the sign of $\mu - 3x_0^2$. We plug in the three different steady-state solutions

$$x_0 = 0, \quad \mu - 3x_0^2 = \mu, \quad x_0 = \pm\sqrt{\mu}, \quad \mu - 3x_0^2 = -2\mu\,. \tag{4.70}$$

Thus, the solutions $x_0 = \pm\sqrt{\mu}$ are stable if $\mu > 0$ and the solution $x_0 = 0$ is stable for $\mu < 0$. As with the other two systems, the bifurcation point is $\mu = 0$.

A2. **a.** Doing a linear stability analysis around $x_0 = 0$ we find

$$\dot{\varepsilon} = \mu x_0^2 - x_0^4 + \varepsilon\big(2x_0(\mu - 2x_0^2)\big) + \varepsilon^2\big(\mu - 6x_0^2\big) + O\big(\varepsilon^3\big) \approx 2x_0\varepsilon\big(\mu - 2x_0^2\big)\,. \tag{4.71}$$

If we solve this for ε we find

$$\varepsilon(t) = \varepsilon(0)\exp\big(2x_0(\mu - 2x_0^2)\big)\,. \tag{4.72}$$

Plugging in $x_0 = 0$ gives

$$\varepsilon(t) = \varepsilon(0)\,, \tag{4.73}$$

implying that the perturbation is a constant and neither grows or decays. This is due to the fact that the term in front of ε in our expansion is zero at $x_0 = 0$. In order to find the behavior of small perturbations, we therefore have include terms of order ε^2.

b. c. Including terms of order ε^2 and setting $x_0 = 0$ gives us the equation

$$\dot{\varepsilon} = \mu\varepsilon^2\,. \tag{4.74}$$

Solving this equation gives

$$\varepsilon(t) = \frac{\varepsilon(0)}{1 - \mu\varepsilon(0)t}\,. \tag{4.75}$$

If μ and $\varepsilon(0)$ have the same sign, perturbations grow. If μ and $\varepsilon(0)$ have different signs, then perturbations decay. This means that the solution $x_0 = 0$ is semi-stable. On one side of $x_0 = 0$ perturbations decay, on the other side perturbations grow, depending on the sign of μ. If $\mu = 0$, the original equation becomes $\dot{x} = -x^4$ and therefore terms of order ε^2 give no information about the stability of solutions.