Chapter 5

Scale-invariance and Universality: Random Walks, Renormalization

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Symbol "(⋆)": Questions and exercises indicated with a (⋆) are optional. No worries if you don't have time to try to solve them, or if you don't manage to solve them on your own.

The symbol "</>": indicates optional questions with numerical simulation.

5.1 Discrete Space Random walks

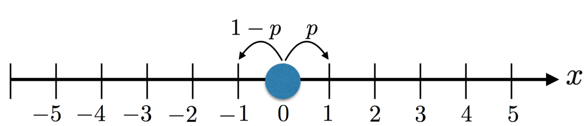


Fig 1: 1D random walk on a lattice.

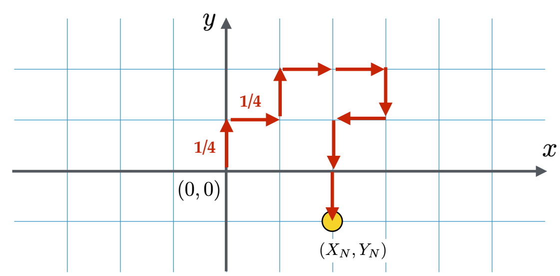


Fig 2: 2D random walk on a lattice.

5.1.1 1D random walker

> Take a walker hopping on the sites of a 1D lattice (see Fig. 1). The walker starts from the position $X_0 = 0$. It then randomly hops:
>
> - to the next site on its right with probability p;
> - or, to the next site on its left with probability $(1-p)$.
>
> We denote by X_N the position of the walker after N steps. This position is a random variable, as its value can change for each new realization of the walk. We would like to characterize better this random position.

For example, imagine playing a "Heads or Tails?" game with a (unfair) coin that gives "Heads" with probability p. You have chosen "Heads", and will win \$1 each times a toss returns "Heads" and lose \$1 each time its returns "Tails". The evolution of the amount of the money that you earn or lose during the course of the game is represented by the simple 1d random walk model described above. The position X_i of the walker after i steps models the amount of money you have earned ($X_i > 0$) or lost ($X_i < 0$) after i tosses of the coin.

> **Symmetric walk:** $p = 0.5$**.**
> Let us start with the simple case $p = (1 - p) = 0.5$: the walker jumps to right or to the left with equal probability.

We denote r_i the jumps performed by the walker at the i-th step (or toss): $r_i = +1$ if the walker moves to the next site on its right and $r_i = -1$ if it moves to the left, such that the position after i steps is given by: $X_i = X_{i-1} + r_i$.

Q1. Can you show that $X_N = \sum_{i=1}^N r_i$? Can you deduce that for $p = 0.5$, $\langle X_N \rangle = 0$?

Q2. Can you show that $X_N^2 = \sum_{i=1}^N \sum_{j=1}^N r_i r_j = \sum_{i=1}^N \sum_{j=1, j\neq i}^N r_i r_j + \sum_{i=1}^N r_i^2$? Can you deduce the value of $\langle X_N^2 \rangle$?

> **Asymmetric walk:** $p \neq 0.5$**.**
> The walker now jumps to the right with probability p and to the left with probability $(1 - p)$.

Q3. Using a numerical simulation, sample 10^4 realizations of the walk until 100 steps. What seems to be the distribution of final positions X_{100} of the walker?

Q4. Using the same reasoning as in Q1, can you deduce the value of $\langle X_N \rangle$ for any value of p? Check that you recover $\langle X_N \rangle = 0$ for $p = 0.5$.

Q5. Can you show that the variance of X_N is equal to N times the variance of r_i:

$$\sigma_{X_N}^2 = \langle X_N^2 \rangle - \langle X_N \rangle^2 = N(\langle r_i^2 \rangle - \langle r_i \rangle^2) = N\,\sigma_{r_i}^2\,. \tag{5.1}$$

Can you deduce the value of $\sigma_{X_N}^2$ for any value of p? Check that you recover the value obtained in Q2 for $p = 0.5$.

Q6. Using a numerical simulation, sample 10^4 realizations of the walk with $p = 2/3$ until 100 steps. What seems to be the distribution of final positions X_{100} of the walker?

Q7. Let denote X_N^+, resp. X_N^-, the total number of jumps to the right, resp. to the left, performed by the walker until the N-th steps. Both X_N^+ and X_N^- are random variables. What is the distribution of X_N^+? What is its mean value? its standard deviation? For large values of N to which distribution is the distribution of X_N^+ converging to (see de Moivre–Laplace theorem)?

Q8. Can you express X_N as a function of X_N^+? Using this result, can you check that you re-obtain the values of the $\langle X_N \rangle$ and $\sigma_{X_N}^2$ derived in the previous question? For large values of N, which distribution is then expected for X_N? Can you compare this distribution to the distribution of the 10^4 final positions obtained in Q6?

5.1.2 Random walk on a 2D lattice

> **2D random walk on a lattice.** We consider the following 2D generalization of the previous walker (see Fig. 2). The walker is now randomly evolving on a 2D square lattice: it jumps in each direction (up, down, left or right) with the same probability $1/4$. The position of the walker after N steps on the 2D lattice is indicated by its coordinate (X_N, Y_N). The walker initially starts from the origin $(X_0, Y_0) = (0, 0)$.

Q9. Can you generate and plot one realization of such 2D random walk with 100 000 steps? How does the plot of this trajectory looks like if you cut it to its first 10 000 steps? Observe the scale invariance structure of the random trajectory (looks the same at all scale when N is sufficiently large).

Q10. We are interested in characterizing better the statistics of the position of the walker after N steps. Can you generate 10^4 trajectories of 100 steps and plot their final positions? What do you expect regarding the distribution of the walker positions? Can you plot this distribution?

Q11. We denote by $\vec{R}_N = (X_N,\ Y_N)$ the vector indicating the position of the walker after N steps in the 2D plane. Can you show that the average final position of the walker is $\langle \vec{R}_N \rangle = (0, 0)$?

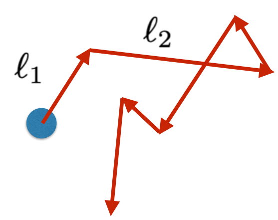
Q12. We observe that $\vec{R}_N = \sum_{i=1}^{N} \vec{r}_i$, where $\vec{r}_i = (x_i, y_i)$ is the vector of coordinates (x_i, y_i) corresponding to the i-th jump. Can you show that $\langle \vec{R}_N^2 \rangle = N\ell^2$, where $\ell = ||r_i|| = \sqrt{2}$?

5.1.3 Random walk on a lattice of any dimension d.

Q13. Can you show that the result of Q12 holds for any symmetric random walks on a square lattice in any dimension?

Q14. Can you show that the fractal dimension associated with the trajectory of the random walker is $d_f = 2$?

5.2 Continuous Space Random walks



5.2.1 Universality

Let's go hunt for mushroom in the forest... We are looking for a specific type of mushrooms that grows on trees. Our strategy is the following: we start from a tree, we then take a random direction and go straight in that direction until we reach another tree, we check if there are mushroom on that tree, and re-iterate.

Q15. Let's assume that trees are uniformly distributed in the forest. Starting from a random tree and a random direction, the average distance to the next tree is of the order of 5 meters. What do you expect for the distribution of the distance to travel before reaching the next tree?
A similar model (extended to 3 dimensions) is used to simulate the diffusion of light in a diffusive material or the diffusive transport of neutrons in a nuclear reactor.

Q16. Can you generate a trajectory of 10 000 steps of this walker? Can you compare the trajectory to the one sampled in question Q9?

Q17. Can you generate 10^4 realizations of 100 steps of this walkers with an average distance between trees of $\ell = \sqrt{2}$ and plot the distribution of their final positions? Compare to the distribution found in Q10.

5.2.2 Diffusion equation – Fokker-Planck equation

Q18. Going back to the discrete 1D case described in Sec. 5.1.1, let $P(x, N)$ be the probability to find the walker in position x after N jumps. Can you give the master equation describing the discrete evolution of the walker on the lattice? The equation gives the probability to find the walker in x after $N+1$ steps, $P(x, N+1)$, as a function of the probabilities to find the walker on other places on the lattice after N steps.

Q19. Let's assume that the walker starts from x_0 at time $t = 0$ and that jumps occur at regular time intervals dt. After N steps, the corresponding time is $t = N\,dt$, and $P(x, N) = P(x, t)$. We also take the lattice step size to be equal to a small value ℓ. Assuming that dt is very small, can you write a Taylor expansion of $P(x, t + dt)$ to first order in dt? Assuming that ℓ is very small, can you write a Taylor expansion of $p(x + \ell, N)$ in ℓ to order 2?

Q20. Let us take $p = 1/2$. Combining the Taylor expansions in the master equation, can you recover the diffusion equation? Take the limit of very small jump length ℓ and very small duration dt, such that the ratio ℓ^2/dt goes to a constant.

Q21. Let us take $p \neq 1/2$. How is this equation modified? Do you recognize this equation?

5.3 (⋆) Real Space renormalization

5.3.1 Renormalization in $d = 1$ Ising model

We consider the usual Ising Hamiltonian (Energy) with nearest-neighbor interactions

$$E(S_i; J) = -J \sum_i S_i S_{i+1}\,, \quad \text{where} \quad S_i = \pm 1\,. \tag{5.2}$$

The Boltzmann weight for each pair of spins is

$$W(S_i, S_{i+1}; v) = e^{K S_i \cdot S_{i+1}}\,, \qquad \text{where } K = \frac{J}{k_B T}\,. \tag{5.3}$$

Q1. Write the Boltzmann weight in the form

$$e^{K S_i S_{i+1}} = \cosh(K)\,\left(1 + \tanh(K)\, S_i S_{i+1}\right)\,. \tag{5.4}$$

Q2. Warm up: Consider three consecutive spins S_i, S_{i+1} and S_{i+2}, with S_i and S_{i+2} being fixed. The corresponding Boltzmann factor where they appear is

$$\exp\left(K\, S_i S_{i+1} + K\, S_{i+1} S_{i+2}\right)\,. \tag{5.5}$$

Using eq. (5.4) we can "integrate out" the middle spin S_{i+1}, by taking the sum over all its possible values. Can you prove that

$$\sum_{S_{i+1}=\pm 1} e^{K S_i S_{i+1} + S_{i+1} S_{i+2}} = 2\cosh^2(K)(1 + t^2\, S_i S_{i+2})\,, \tag{5.6}$$

where for convenience $t \equiv \tanh(K)$.

Note: this procedure of "integrating out" a random variable from a probability distribution is often called "*marginalization*": i.e., by computing the probability distribution:

$$P(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n) = \sum_{s_i} P(s_1, \cdots, s_{i-1}, s_i, s_{i+1}, \cdots, s_n)\,, \tag{5.7}$$

we are *marginalizing out* the variable s_i.

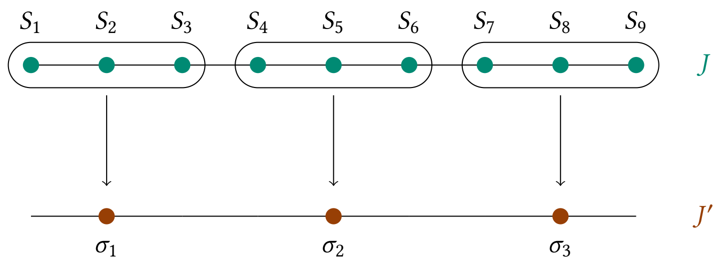


Figure 5.1: RG transformation: block spins and decimate.

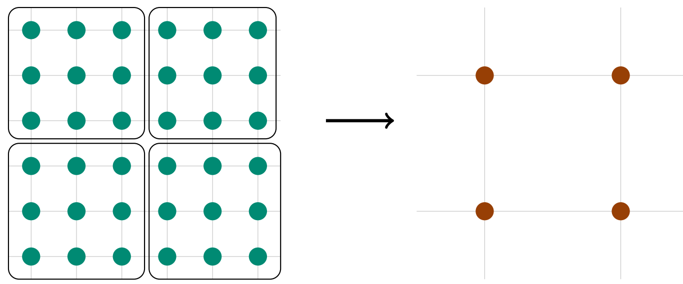


Figure 5.2: One possible decimation procedure.

RG transformation. One possible RG transformation is to divide the system into blocks, each including three spins, and apply the *decimation* rule: for each block we choose as a spin of the new system, the one that is at the center (see examples in Fig. 5.1 and 5.2). Applying this procedure to the probability original probability distribution would mean marginalizing over all the spins that are not kept at the next step.

Q3. For the first RG step, we can consider having two neighboring blocks and summing over the intermediate spins (S_3, S_4), while keeping fixed the values of the spins at the center of the blocks $\sigma_1 \equiv S_2$ and $\sigma_2 \equiv S_5$. Write the new (*effective*) Boltzmann factor and show that the new couplings K' can be rewritten from K, through a recursion relation.

Q4. Plot the recursion relation, and graphically show that there are two fixed points. Remembering that we have absorbed a factor $1/k_B T$ in K, which fixed points correspond to the high and low temperature phases of the system? In which phase does the 1d Ising model live? *Hint:* Use Cobweb plots.

Q5 Now we would like to generalize eq. (5.6), by *marginalizing out* ($\Leftrightarrow$ summing over) $(k-1)$ spins consecutive spins. Given eq. (5.5) can you induce the equivalent, generalized, form of the summing procedure and the recursion relation for the coupling constants?

5.3.2 Renormalization in $d = 2$ Ising model

Moving to the 2d lattice, soon we figure out that the decimation procedure we applied in $1d$ cannot be done exactly in $2d$, as depicted in Fig. 5.2, since new spin interaction terms are generated at the end of each step. Nevertheless, it is possible to construct another, more sophisticated, decimation scheme that allows us to have the same recursion relation that we had in the $1d$ case. Specifically the scheme relies on *moving* the bonds that are not connected, to the spins that are kept. These spins are then connected by bonds of strength $2J$ instead of J.

Q 2.1 Argue that the new recursion relation after decimating 2 spins is

$$t' \equiv \tanh(K') = \tanh^2(2K) \tag{5.8}$$

Q 2.2 Using the recursion relation (5.8), find the fixed points J_c of this decimation scheme. Give a qualitative analysis of the fixed points.
Hint: Use the identity $\tanh(2x) = \frac{2\tanh(x)}{1+\tanh^2(x)}$ and set $t = \tanh(K)$ for convenience.

5.4 Solutions

5.4.1 Discrete Space Random walks

1D random walker

A1. The walker initially starts from the position $X_0 = 0$. Then, at each step (indexed by i), it performs a random jump $r_i \in \{-1, +1\}$. Therefore its position is:

$$\text{after } N = 1 \text{ step}: \quad X_1 = X_0 + r_1 = r_1\,, \tag{5.9}$$
$$\text{after } N = 2 \text{ steps}: \quad X_2 = X_1 + r_2 = r_1 + r_2\,, \tag{5.10}$$
$$\text{after } N = 3 \text{ steps}: \quad X_3 = X_2 + r_3 = r_1 + r_2 + r_3\,, \tag{5.11}$$
$$\text{after } N \text{ steps}: \quad X_N = X_{N-1} + r_N = (r_1 + \cdots + r_{i-1}) + r_N = \sum_{i=1}^{N} r_i\,. \tag{5.12}$$

Taking the ensemble average on both sides gives:

$$\langle X_N \rangle = \left\langle \sum_{i=1}^{N} r_i \right\rangle = \sum_{i=1}^{N} \langle r_i \rangle \tag{5.13}$$

as the average of a sum of terms is equal to the sum of the averages of each terms. Here we observe that the random variables r_i are independent random variables, all sampled from the same distribution, which is $P(r_i) = [\delta(r_i + 1) + \delta(r_i - 1)]/2$, i.e. that r_i is equal to $+1$ with probability $1/2$ and is equal to -1 with probability $1/2$. Therefore all the variables r_i have the same mean which is given by:

$$\langle r_i \rangle = \frac{1}{2} \times (+1) + \frac{1}{2} \times (-1) = 0\,. \tag{5.14}$$

Replacing this result in Eq. (5.13) finally gives that $\langle X_N \rangle = N \times \langle r_i \rangle = 0$.

A2. Erratum: In the text of question Q2, the correct expression is $X_N^2 = \sum_{i=1}^N \sum_{j=1}^N r_i r_j$ (the expression previously appearing in the tutorial text had an extra term that was not correct). This has been adjusted in the present text. The last result was correct though, i.e. $X_N^2 = \sum_{i=1}^N \sum_{j=1, j\neq i}^N r_i r_j + \sum_{i=1}^N r_i^2$ is correct.

Starting from $X_N = \sum_{i=1}^N r_i$, we take the square:

$$X_N^2 = \left(\sum_{i=1}^{N} r_i \right) \times \left(\sum_{i=1}^{N} r_i \right)\,, \tag{5.15}$$
$$= (r_1 + r_2 + \cdots + r_N) \times (r_1 + r_2 + \cdots + r_N)\,, \tag{5.16}$$
$$= (r_1 \times r_1 + r_1 \times r_2 + \cdots + r_1 \times r_N) + (r_2 \times r_1 + r_2 \times r_2 + \cdots + r_2 \times r_N) + \cdots \tag{5.17}$$
$$= \sum_{i=1}^{N} \sum_{j=1}^{N} r_i r_j \tag{5.18}$$

which gives all possible products of the type $r_i r_j$ for all indexes i and j (up to N). Separating the terms for which $i = j$ from the terms with $i \neq j$, one finally gets:

$$X_N^2 = \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} r_i r_j + \sum_{i=1}^{N} r_i^2\,. \tag{5.19}$$

Note: you certainly recognize here the identity $(r_1 + r_2)^2 = r_1^2 + 2\, r_1 r_2 + r_2^2$, which can be re-written under the form: $(r_1 + r_2)^2 = r_1^2 + r_1 r_2 + r_2 r_1 + r_2^2$. The result above is just a generalization of this identity when there are more terms in the sum.

Taking the ensemble average on both sides of Eq. (5.19):

$$\langle X_N^2 \rangle = \left\langle \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} r_i r_j + \sum_{i=1}^{N} r_i^2 \right\rangle, \tag{5.20}$$

$$= \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \langle r_i r_j \rangle + \sum_{i=1}^{N} \langle r_i^2 \rangle, \tag{5.21}$$

where we used again that the average of a sum of terms is the sum of the averages of the terms. Besides, for $i \neq j$, the two random variables r_i and r_j are independently sampled. This implies that they are not correlated, i.e. that[1] $\langle r_i r_j \rangle = \langle r_i \rangle \langle r_j \rangle$. Given that, $\langle r_i \rangle = 0$ at any step i, we get that:

$$\langle r_i r_j \rangle = \langle r_i \rangle \langle r_j \rangle = 0\,, \qquad \text{for all } i \neq j\,. \tag{5.22}$$

Finally, here again we observe that the random variables r_i are independent random variables, all sampled from the same distribution, $P(r_i) = [\delta(r_i+1) + \delta(r_i-1)]/2$, i.e. that r_i is equal to $+1$ with probability $1/2$ and is equal to -1 with probability $1/2$. Therefore all the variables r_i have the same second order moment, which is given by:

$$\langle r_i^2 \rangle = \frac{1}{2} \times (+1)^2 + \frac{1}{2} \times (-1)^2 = 1\,. \tag{5.23}$$

Replacing the results (5.22) and (5.23) in Eq. (5.21) finally gives that $\langle X_N^2 \rangle = N \times \langle r_i^2 \rangle = N$.

We observe that the variance of the position of the walker after N steps grows linearly with N ($\sigma_N^2 = \langle X_N^2 \rangle - \langle X_N \rangle^2 = N$), and therefore the standard deviation of the position of the walker grows as $\sqrt{N}$.

A3. We sample 10^4 realizations of the 1D random walk described in this section (see Fig. 1). Each realization of the walk is stopped at $N = 100$ steps. The distribution of the final positions of the 10^4 walkers is expected to be a Gaussian distribution with mean $\langle X_{100} \rangle = 0$ and variance $\sigma_{100}^2 = \langle X_{100}^2 \rangle - \langle X_{100} \rangle^2 = N = 100$ (i.e. a standard deviation of $\sigma_{100} = \sqrt{100} = 10$).

A4. Starting from Eq. (5.13), and using that the r_i are all sampled from the same distribution, we obtain:

$$\langle X_N \rangle = \sum_{i=1}^{N} \langle r_i \rangle = N \, \langle r_i \rangle\,. \tag{5.24}$$

The random variables r_i can take two values: either $r_i = +1$ with the probability p, or $r_i = -1$ with the probability $(1-p)$. Therefore the mean of r_i is simply given by:

$$\langle r_i \rangle = (+1) \times p + (-1) \times (1-p) = 2p - 1\,, \tag{5.25}$$

which finally gives:

$$\langle X_N \rangle = N\,(2p-1)\,. \tag{5.26}$$

Taking $p = 0.5$ gives back $\langle X_N \rangle = 0$ for the symmetric random walk. Besides, if $p > 0.5$ (resp. $p < 0.5$), we have that $\langle X_N \rangle > 0$ (resp. $\langle X_N \rangle < 0$): the walker is drifting towards the right (resp. left) side of the x-axis.

A5. Starting from Eq. (5.21), and using that the r_i are all independently sampled from the same distribution, we obtain:

$$\langle X_N^2 \rangle = \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \langle r_i r_j \rangle + \sum_{i=1}^{N} \langle r_i^2 \rangle, \tag{5.27}$$

$$= \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \langle r_i \rangle \langle r_j \rangle + \sum_{i=1}^{N} \langle r_i^2 \rangle, \qquad \text{using the independence between } r_i \text{ and } r_j \tag{5.28}$$

$$\langle X_N^2 \rangle = N(N-1)\,\langle r_i \rangle^2 + N\,\langle r_i^2 \rangle, \qquad \text{using that all the } r_i \text{ are sampled from the same distribution.} \tag{5.29}$$

[1] Note that this is not true in general: the average of a product of terms is not equal to the product of the averages of the terms. This is only correct when the random terms in the product are statistically independent. For instance, this doesn't work for the product $r_i \times r_i$: $\langle r_i^2 \rangle \neq \langle r_i \rangle^2$.

There are $N(N-1)$ terms in the first sum: $\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} 1 = N(N-1)$ (think for instance of the number of elements of an $N \times N$-matrix, excluding the diagonal elements).

The variance of X_N is defined as $\sigma_{X_N}^2 = \langle X_N^2\rangle - \langle X_N\rangle^2$. Replacing that $\langle X_N\rangle = N\,\langle r_i\rangle$ (see Eq. (5.24)) and the previous results Eq. (5.29) gives:

$$\sigma_{X_N}^2 = \Big(N(N-1)\,\langle r_i\rangle^2 + N\,\langle r_i^2\rangle\Big) - \Big(N\,\langle r_i\rangle\Big)^2\,, \tag{5.30}$$

$$= N^2\,\langle r_i\rangle^2 - N\langle r_i\rangle^2 + N\,\langle r_i^2\rangle - \Big(N\,\langle r_i\rangle\Big)^2\,, \tag{5.31}$$

$$= N\,\Big(\langle r_i^2\rangle - \langle r_i\rangle^2\Big) \tag{5.32}$$

$$\sigma_{X_N}^2 = N\,\sigma_{r_i}^2\,. \tag{5.33}$$

The variance of X_N is equal to N times the variance of r_i. This is true for any variable X_N defines as the sum of N independently and identically distributed (i.i.d.) random variables r_i for which σ_{r_i} is finite.

Independent and Identically Distributed random variables (or, i.i.d. random variables):

For any variable X_N defines as the sum of N independently and identically distributed (i.i.d.) random variables r_i for which σ_{r_i} is finite, we have that:

- the mean of X_N is equal to N times the mean of r_i:

$$\langle X_N\rangle = N\,\langle r_i\rangle\,; \tag{5.34}$$

- the variance of X_N is equal to N times the variance of r_i:

$$\sigma_{X_N}^2 = N\,\sigma_{r_i}^2 = N\,\Big(\langle r_i^2\rangle - \langle r_i\rangle^2\Big)\,. \tag{5.35}$$

As a consequence, the standard deviation of the final positions of a 1-dimensional random walk defined by X_N always grows as $\sqrt{N}$, as long as σ_{r_i} is finite (no matter the detailed shape of the distribution of r_i).

The random variables r_i can take two values: either $r_i = +1$ with the probability p, or $r_i = -1$ with the probability $(1-p)$. Therefore the second moment of r_i is given by:

$$\langle r_i^2\rangle = (+1)^2 \times p + (-1)^2 \times (1-p) = 1\,. \tag{5.36}$$

The mean of r_i was already computed in Eq. (5.25), which gives the variance:

$$\sigma_{r_i}^2 = \langle r_i^2\rangle - \langle r_i\rangle^2 = 1 - (2p-1)^2 \tag{5.37}$$

$$= 4\,p\,(1-p)\,. \tag{5.38}$$

Finally, the variance of the position of the walker after N steps, computed using $\sigma_{X_N}^2 = N\,\sigma_{r_i}^2$, is:

$$\sigma_{X_N}^2 = 4\,p\,(1-p)\,N\,. \tag{5.39}$$

For a symmetric random walk, $p = 1/2$, we recover that $\sigma_{X_N}^2 = N$.

For $p = 0$ and for $p = 1$, $\sigma_{X_N}^2 = 0$. This is normal, as for both values the walk becomes deterministic (there is no randomness/stochasticity anymore), and therefore we can expect that there will be zero variance in the final position of the walker: for $p = 1$ (resp. $p = 0$) the position of the walker after N steps is always exactly $X_N = N$ (resp. $X_N = -N$), i.e. the walker has done N steps to the right (resp. to the left).

A6. Taking $p = 2/3$, we expect the final positions of the walker to be Gaussian distributed with a mean $X_{100} = N\,(2p-1) \simeq 33.33$ and a variance of $\sigma_{100}^2 = 4\,p\,(1-p)\,N \simeq 88.88$.

Random walk on a 2D lattice

Random walk on a lattice of any dimension d

5.4.2 Continuous Space Random walks

Universality

Diffusion equation – Fokker-Planck equation

A18. Because the walker can only jump to one of the two nearest site at each step, for the walker to be in position x after $(N+1)$ steps, it must have been in position $(x+1)$ or in position $(x-1)$ just before, i.e. at the N-th step. Therefore, the probability to find the walker in position x after $(N+1)$ steps is given by:

$$P(x,\ N+1) = \underbrace{P(x-1,\ N)\times p}_{(1)} + \underbrace{P(x+1,\ N)\times(1-p)}_{(2)}\,, \tag{5.40}$$

where

(1) is the probability that the walker was at the position $(x-1)$ at the previous step N and then jumped to the right from the position $(x-1)$ to the position x,

(2) is the probability that the walker was at the position $(x+1)$ at the previous step N and then jumped from to the left from the position $(x+1)$ to the position x.

A19. Note that taking the lattice step size equal to ℓ instead of just "1", and replacing jump numbers N by the time $t = N\,dt$, the previous equation is modified as:

$$P(x,\ t+dt) = \underbrace{P(x-\ell,\ t)\times p}_{(1)} + \underbrace{P(x+\ell,\ t)\times(1-p)}_{(2)}\ . \tag{5.41}$$

> We recall that the Taylor expansion of a function $f(x)$ (of a single variable x) that is infinitely differentiable at a point x_0 is given by the infinite sum:
>
> $$f(x_0+h) = f(x_0) + h\,f'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \cdots\ . \tag{5.42}$$
>
> For very small values of h, one may then truncate this sum at a chosen order.

Assuming dt small, a Taylor expansion of $P(x,\ t+dt)$ to first order in dt gives:

$$P(x,\ t+dt) = P(x,t) + dt\,\frac{\partial P}{\partial t}(x,t) + o(dt)\,. \tag{5.43}$$

Note that this expansion is done by considering $P(x,t)$ as a function of the variable t, while x is kept constant.

Assuming ℓ small, a Taylor expansion of $P(x+\ell,\ t)$ and of $P(x-\ell,\ t)$ to second order in ℓ gives:

$$P(x+\ell,\ t) = P(x,t) + \ell\,\frac{\partial P}{\partial x}(x,t) + \frac{\ell^2}{2}\,\frac{\partial^2 P}{\partial x^2}(x,t) + o(\ell^2)\,. \tag{5.44}$$

and

$$P(x-\ell,\ t) = P(x,t) - \ell\,\frac{\partial P}{\partial x}(x,t) + \frac{\ell^2}{2}\,\frac{\partial^2 P}{\partial x^2}(x,t) + o(\ell^2)\,. \tag{5.45}$$

Note that these two expansions are done by considering $P(x,t)$ as a function of the variable x, while t is kept constant. Also note that the sign in front of the first order term is the only thing that changes between Eq. (5.44) and Eq. (5.45).

A20. Taking $p = 1/2$, and replacing the three Taylor expansions above into the master equation (5.41) gives:

$$P(x,t) + \frac{\partial P}{\partial t}(x,t)\,dt + o(dt) = \left[P(x,t) - \ell\,\frac{\partial P}{\partial x}(x,t) + \frac{\ell^2}{2}\,\frac{\partial^2 P}{\partial x^2}(x,t) + o(\ell^2)\right] \times p$$
$$+ \left[P(x,t) + \ell\,\frac{\partial P}{\partial x}(x,t) + \frac{\ell^2}{2}\,\frac{\partial^2 P}{\partial x^2}(x,t) + o(\ell^2)\right] \times (1-p)\,. \tag{5.46}$$

After replacing $p = 1/2$, simplifying and reorganizing of the terms in Eq. (5.68), we get:

$$\frac{\partial P}{\partial t}(x,t)\,dt + o(dt) = \frac{\ell^2}{2}\,\frac{\partial^2 P}{\partial x^2}(x,t) + o(\ell^2)\,. \tag{5.47}$$

Note that because the first order terms in ℓ cancel each other, one needs to go to the next order in ℓ (second order) to get a non-zero term in the previous equation. Besides, the text of the question assumes that ℓ^2/dt goes to a constant, which implies that dt will be of the same order as ℓ^2. It then makes sense to take the expansion in Eq. (5.68) simultaneously to second order in ℓ and to first order in dt.

Diffusion equation: Taking the limit dt and ℓ very small as ℓ^2/dt goes to a constant finally gives the one-dimensional diffusion equation:

$$\frac{\partial P}{\partial t}(x,t) = D\,\frac{\partial^2 P}{\partial x^2}(x,t)\,. \tag{5.48}$$

in which $D = \ell^2/(2\,dt)$ can be identified as the diffusion constant.

A21. More generally, for any values of p, we can restart from Eq. (5.68), and after simplying and reorganizing the terms, we get:

$$\frac{\partial P}{\partial t}(x,t)\,dt + o(dt) = (1-2p)\,\ell\,\frac{\partial P}{\partial x}(x,t) + \frac{\ell^2}{2}\,\frac{\partial^2 P}{\partial x^2}(x,t) + o(\ell^2)\,. \tag{5.49}$$

Fokker-Planck equation: For small values of ℓ and dt, we therefore obtain the equation:

$$\frac{\partial P}{\partial t}(x,t) = -v\,\frac{\partial P}{\partial x}(x,t) + D\,\frac{\partial^2 P}{\partial x^2}(x,t)\,, \tag{5.50}$$

which is a one-dimensional Fokker-Planck equation, with a constant diffusion coefficient $D = \ell^2/(2\,dt)$ and a constant drift $v = (2p-1)\,\ell/dt$.

Comments:

- The diffusion constant D is the same as in the diffusion equation (5.48). In fact if we take $p = 1/2$ we obtain $v = 0$ and recover Eq. (5.48).
- For $p > 1/2$, the drift is positive $v > 0$, which means that the walker would tend to drift towards larger values of x (towards the right of the x-axis). This is compatible with a probability to jump to the right that is larger than one half, $p > 1/2$. Similarly, if $p < 1/2$, then the walker would tend to drift towards the left.

5.4.3 (⋆) Real space renormalization

Renormalization in d=1 Ising model

A1.

$$\begin{aligned} e^{KS_iS_{i+1}} &= \frac{1}{2}\left(e^K + e^{-K} + (e^K - e^{-K})S_iS_{i+1}\right) \quad \text{since} \quad S_iS_{i+1} = \pm 1 \\ &= \cosh(K) + \sinh(K)\,S_iS_{i+1} \\ &= \cosh(K)\,(1 + \tanh(K)\,S_iS_{i+1})\,. \end{aligned} \tag{5.51}$$

Another way is to directly use the relation $\exp(x) = \cosh(x) + \sinh(x)$ and the facts that:
– $\cosh(s_i\, x) = \cosh(x)$, because $s_i \in \{-1, +1\}$ and cosh is an even function;
– $\sinh(s_i\, x) = s_i\, \sinh(x)$, because $s_i \in \{-1, +1\}$ and sinh is an odd function.

A2.

$$\begin{aligned}\sum_{S_{i+1}=\pm 1} e^{KS_iS_{i+1}+S_{i+1}S_{i+2}} &= \cosh^2(K)(1+tS_iS_{i+1})(1+tS_{i+1}S_{i+2})\\ &= \sum_{S_{i+1}=\pm 1} \cosh^2(K)\left(1+tS_iS_{i+1}+tS_{i+1}S_{i+2}+t^2S_iS_{i+2}\right)\\ &= 2\cosh^2(K)(1+t^2S_iS_{i+2}).\end{aligned}$$

A3.

$$\begin{aligned}\sum_{S_3,S_4} e^{K\sigma_1S_3}e^{KS_3S_4}e^{KS_4\sigma_2} &= (\cosh K)^3(1+t\sigma_1S_3)(1+tS_3S_4)(1+tS_4\sigma_2)\\ &= 2^2(\cosh K)^3(1+t^3\sigma_1\sigma_2).\end{aligned} \tag{5.52}$$

Besides the multiplicative normalization constant (independent of the spins), (5.52) is the same as (5.51) and so we can define a new Boltzmann weight $W(\sigma_1, \sigma_2; K')$ of the block spins σ_1 and σ_2 with

$$t' \equiv \tanh K' = t^3 \equiv (\tanh K)^3 \tag{5.53}$$

$$\Leftrightarrow K' = \tanh^{-1}[(\tanh K)^3] \tag{5.54}$$

such that the new Hamiltonian of the system is given by

$$H(\sigma_i; K') = A(K) - K' \sum_i \sigma_i\sigma_{i+1} \tag{5.55}$$

where $A(K)$ comes from the multiplicative factors in (5.52) and contributes only to the Free energy of the system.

A4. At the fixed points, the coupling constants will no longer change, which translates to

$$t' = t \tag{5.56}$$

The intersections of the above line with the recursion relation (5.53), will give the location of these fixed points, and using the Cobweb plots we can deduce their stability, as shown in Fig. 5.3.

Note: We have defined $K = \frac{J}{k_B T}$. This means approaching $t = 0$, or equivalently $K = 0$, corresponds to approaching the high temperature phase, while approaching $t \longrightarrow 1$ (i.e., $K \longrightarrow \infty$), corresponds to $T \longrightarrow 0$.

Since the stable fixed point is at $t \equiv \tanh(K) = 0 \Leftrightarrow K = 0$, when starting from any point $t_0 \neq 1$, we conclude that the **1d Ising model is always in its disordered phase**, with non interacting spins ($J = 0$). A system with non-interacting spins is self-similar with correlation length $\xi = 0$ and qualitatively it looks the same on all length scales. Quantitatively, the probability distribution of microstates is invariant under the renormalization transformation.

Overall, the renormalization transformation correctly predicts that there is no phase transition in the $d = 1$ Ising model.

A5. The middle terms, containing the spins to be summed over, will vanish like in the case of one, or two intermediate spins, and the only term that will survive, besides the identity, will be the last one containing the 'boundary spins', with a coefficient $t^{\text{len(summed spins)+1}}$. Hence, we get the following recursion relation:

$$\sum_{S_1,\ldots,S_{k-1}} e^{K(S_0S_1+\cdots+S_{k-1}S_k)} = 2^{(k-1)}\cosh^k(K)\left(1+t^k\, S_0S_k\right). \tag{5.57}$$

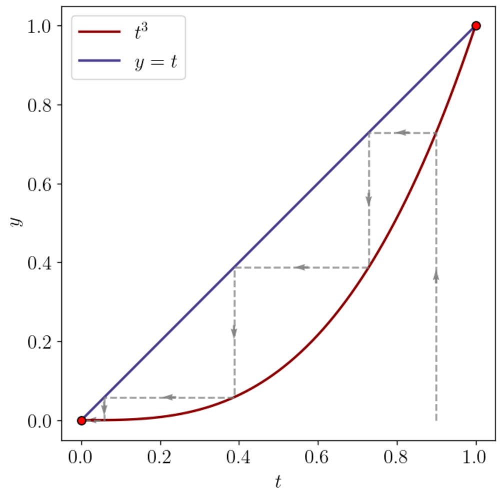


Figure 5.3: Cobweb plot, showing the fixed point $(0, 0)$ of the recursion relation when starting from $t_0 = 0.9$. $(1, 1)$ is an unstable fixed point.

Again, Besides the multiplicative normalization constant (independent of the spins), (5.57) is the same as (5.51) and so we can define a new Boltzmann weight $W(S_0, S_k; J')$ with

$$t' \equiv \tanh(K') = t^k = \tanh^k(K)\,. \tag{5.58}$$

We see that the mapping has two fixed points at $t_1^* = 0$ and $t_2^* = 1$. The first is an attractive fixed point and the second repulsive; Unless t is exaclty $t = 1$, each iteration moves the values of t towards the origin.

Renormalization in $d = 2$ Ising model

A 2.1. Set $K \longrightarrow 2K$ and $k = 2$ in eq. (5.58)

A 2.2. At the fixed points, the couplings will no longer change, so $t' = t$. We should therefore look for the intersections of the curves

$$y = \frac{4t^2}{(1+t^2)} \quad \text{and} \quad y = t\,. \tag{5.59}$$

This time there are three intersections $\Longrightarrow$ three fixed points. If we use the diagram as a Cobweb plot, we see that the points $t = 0, 1$ are stable fixed points, that correspond to $K = 0$ (high T) and $K \longrightarrow \infty$ (low T) respectively. This implies that there must be also a critical point at a non-trivial t_c (K_c), for which the 2d Ising model undergoes a **phase transition**.

With the current choice of decimation scheme, $t_c \simeq 0.2956$, which corresponds to $K_c = \frac{J}{k_B T_C} \simeq 0.30469*$. The exact theoretical value obtained by Onsager (1944) is $K_c \simeq 0.44$ (Onsager: $k_B\,T_c \simeq 2.26\,J$).

Note: Different decimation schemes will lead to slightly different values for K_c. However, the qualitative behavior of the system predicted by any RG transformation, will remain the same.

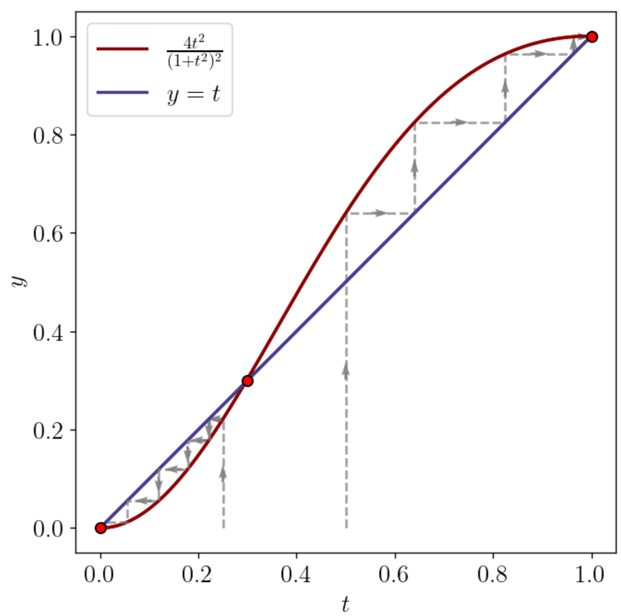


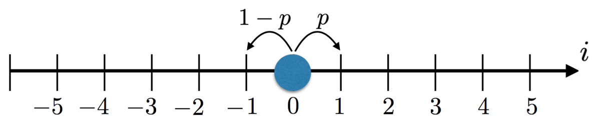
Figure 5.4: Graphical solution of the recursion relation and Cobweb plots showing the stability of the three fixed points.

5.5 Example of Exam Exercise

5.5.1 Random walks and control theory (5/20 pts) – estimated 0h45

General information. This is an example from the Exam of 2023. The topics assessed here are: Random walks, summing of i.i.d random variables, Markov chains and master equation, Diffusion equation, simulation of random walks, critical assessment of the results. The exam has exercises of various difficulty levels, this one is considered to go from easy to medium to high difficulty level, depending on the question (see ⋆ indications).

This exercise is inspired from the paper **S. Majumdar**, **C. Godreche**.



Q1. 1D discrete Random Walk. Consider a random walker evolving on a one dimensional lattice (see Figure above): at each time step the walker jumps to the next site to the right or to the left with the same probability $p = 1/2 = (1-p)$. Imagine performing many simulations of this random walk up to a fixed number N of steps. Describe briefly in words: How do you expect the final positions of the walker to be distributed if N is very large? What would be the mean and the variance of this distribution? Justify very briefly why (no need to prove it).

Q2. Master equation for the discrete process. Can you write down the master equation that gives the evolution of the probability $P(i, N)$ to find the walker in site i after N steps? (relate $P(i, N)$ with the probabilities to find the walker in any position on the lattice at the previous step $(N-1)$).

Q3. Continuous limit and Diffusion equation. We consider that the walker takes a small time dt to jump from site to site. After n jumps the elapsed time is thus $t = n\,dt$. Let us also describe the position of the walker by the variable $x = i\,\ell$, where ℓ is the length of the lattice steps.
Taking the jump length ℓ and the duration dt to be very small such that the ratio ℓ^2/dt goes to a constant, can you show that the probability density to find the walker near the position x at time t is given by the diffusion equation:

$$\frac{\partial P}{\partial t}(x,t) = D\frac{\partial^2 P}{\partial^2 x}(x,t)\,, \tag{5.60}$$

where D is a constant that must be specified (see Eq. (**??**) for a recall of Taylor expansion).
Considering that the walker is free to evolve along the x-axis and that it starts from $x = 0$ at time $t = 0$, we recall that the solution of this equation is:

$$P(x,t) = \frac{1}{\sqrt{2\pi\sigma^2}}\ \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad \text{where}\ \ \mu = 0\ \ \text{and}\ \ \sigma^2 = 2Dt\,. \tag{5.61}$$

How does this result compare to the distribution you suggested in question Q1?

Q4. Langevin equation and simulation of the continuous random walk. To simulate random trajectories of a diffusion process described by the Fokker-Planck equation:

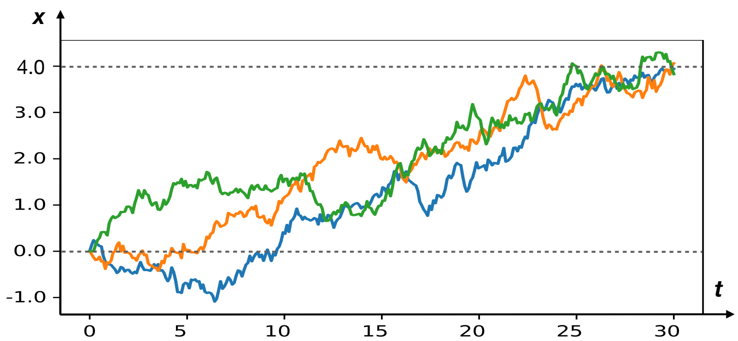
$$\frac{\partial P}{\partial t}(x,t) = D\frac{\partial^2 P}{\partial^2 x}(x,t) - \frac{\partial}{\partial x}\left[u(x,t)\,P(x,t)\right]\,, \tag{5.62}$$

where $u(x,t)$ represents a local drift, one can use the discrete Langevin equation:

$$x(t+dt) = x(t) + u(x,t)\,dt + \sqrt{2D\,dt}\,\eta_t\,, \tag{5.63}$$

where η_t is a random value sampled from a Gaussian distribution with mean 0 and variance 1. This equation gives the position of the walker at time $(t+dt)$ given its position at time t. Using this equation, how would you simulate random trajectories of the diffusion process described by Eq. (5.60)? Consider that the walker starts from the position $x(0) = 0$ at time $t = 0$.

Q5. ($\star$) **Brownian bridges and Rare events.** We would like to simulate trajectories of this random walk that start from $x_0 = 0$ at time $t = 0$, but that also arrives in a chosen position $x_F = 4$ at the chosen time $t_F = 30$, as shown in the figure below. Is it possible to sample such trajectories with the algorithm just described? What may be the difficulty?



Such random walk is called a *Brownian bridge*, as all the trajectories form bridges between the two chosen points. Simulating Brownian bridges can be useful in many contexts. In finance for instance, they can model the random evolution of the price of an asset when you happen to know the exact price at two points in time.

Q6. ($\star\star$) **Backward process.** We would like to sample this process more efficiently. To force the walker to arrive at the position x_F at the final time t_F, we imagine reversing the time: i.e., we start the process at the final time t_F and observe the diffusion of the random walk back in time. We introduce the time $\tau = t - t_F$, such that:
– at the final time $t = t_F$, we have $\tau = 0$;
– and, as τ increases, we go back in time.
Looking at the process backward, we are interested in computing the probability distribution $Q(x, t)$ to find the walker in the position x at time t given that it will be in x_F at time $t = t_F$ (or $\tau = 0$). We assume that the backward process follows the same diffusion process as the forward process. Can you show that $Q(x, t)$ satisfies the (backward) diffusion equation:

$$\frac{\partial Q}{\partial t}(x,t) = -D\frac{\partial^2 Q}{\partial^2 x}(x,t)\,? \tag{5.64}$$

What is the solution of this equation?

Q7. ($\star\star\star$) **Control process.** At any intermediate time t between $t = 0$ and t_F, the process is now described by two probability distributions: $P(x, t)$ if one consider the forward diffusion process started in $x = 0$ at $t = 0$; and $Q(x, t)$ if one consider the backward process starting in x_F at time t_F and moving back in time. The position of the walker constrained to be both in $x(0) = 0$ and $x(t_F) = x_F$ is thus described by a combination of these two distributions:

$$\mathcal{P}(x,t) = \frac{P(x,t)\,Q(x,t)}{A}\text{, where } A \text{ is a normalisation constant.} \tag{5.65}$$

Can you show that $\mathcal{P}(x, t)$ satisfies the Fokker-Planck equation (5.62)? What is the value of the drift $u(x, t)$? What does this implies for the simulation of Brownian bridges? Can one use this result to more easily sample Brownian bridges?

5.5.2 Random walks, Brownian Bridge, and Control theory – Answers

A1. 1D discrete Random Walk. We expect the final positions of the walker to be Gaussian distributed, with a mean N times the means of the single jump, i.e. a mean of 0, and a variance that is N times the variance of a single jumps, i.e. $N \times 1$. This result is a consequence of the fact that the position of the walker after N steps is the sum of N i.i.d. random variables r_i that are the successive jumps.

Keywords: Gaussian distributed; mean = N x mean; variance = N x variance; iid R.V.

A2. Master equation for the discrete process. The probability to find the walker in i after N steps is given by:

$$P(i, N) = \frac{1}{2}\, P(i-1, N-1) + \frac{1}{2}\, P(i+1, N-1)\,. \tag{5.66}$$

This is because to be in i at step N the walker can have been either in $(i-1)$ or in $(i+1)$ with the same probability $1/2$ at the previous step.

A3. Continuous limit and Diffusion equation. We shift the previous equation by 1 steps, and re-write it in terms of x and t, instead of i and N:

$$P(x, t+dt) = \frac{1}{2}\, P(x-\ell, t) + \frac{1}{2}\, P(x+\ell, t)\,. \tag{5.67}$$

We take a Taylor expansion of $P(x+\ell, t)$ and of $P(x-\ell, t)$ in second order in ℓ, and of $P(x, t+dt)$ in first order in dt:

$$\begin{aligned} P(x,t) + \frac{\partial P}{\partial t}(x,t)\, dt + o(dt) = &\left[P(x,t) - \ell\, \frac{\partial P}{\partial x}(x,t) + \frac{\ell^2}{2}\, \frac{\partial^2 P}{\partial x^2}(x,t) + o(\ell^2)\right] \times \frac{1}{2} \\ &+ \left[P(x,t) + \ell\, \frac{\partial P}{\partial x}(x,t) + \frac{\ell^2}{2}\, \frac{\partial^2 P}{\partial x^2}(x,t) + o(\ell^2)\right] \times \frac{1}{2}\,. \end{aligned} \tag{5.68}$$

After simplification, we get:

$$\frac{\partial P}{\partial t}(x,t)\, dt + o(dt) = \frac{\ell^2}{2}\, \frac{\partial^2 P}{\partial x^2}(x,t) + o(\ell^2)\,. \tag{5.69}$$

which gives the diffusion equation, with $D = \ell^2/(2\,dt)$. This result is the same as the one expected in Q1: we have a Gaussian distribution with mean $\mu = 0$ and variance:

$$\sigma^2 = 2Dt = 2\, \frac{\ell^2}{2\, dt}\, (N\, dt) = \ell^2\, N\,, \tag{5.70}$$

in which we previously had $\ell = 1$.

A4. Langevin equation and simulation of the continuous random walk. To simulate a Brownian trajectory, I will first start the walker from position $x_0 = 0$ at time $t = 0$. Then, at each time step:
– time is incremented by dt; i.e. the new time is $t + dt$;
– sample a new random number η_t from a Gaussian distribution with mean 0 and variance 1;
– the new position $x(t+dt)$ of the walker is given by Eq. (5.63), in which $u(x,t) = 0$.

A5. (★) **Brownian bridges and Rare events.** To simulate trajectories that start in 0 and arrives in $x_F = 4$ at time $t = 30$, one can generate a huge number of the Brownian trajectories with the previous algorithm and only select the one happen to pass by x_F at time t_F. The issue is that the occurrence of such trajectories may be very rare, and it will be difficult to get a good statistics from this way of simulating.

A6. (★★) **Backward process.** Assuming that the backward trajectories follows the same diffusion process as the forward one, we have that $Q(x, \tau)$ satisfies the same diffusion equation than $P(x,t)$, i.e. Eq. (5.60),

$$\frac{\partial Q}{\partial \tau}(x, \tau) = D \frac{\partial^2 Q}{\partial^2 x}(x, \tau)\,, \tag{5.71}$$

for which now the initial position at $\tau = 0$ is $x(\tau = 0) = x_F$ and the time given by τ is evolving backward. Using that $t = t_F - \tau$,