Poisson processes

Tutorial 1

Monday 7 February

Poisson process

General Idea

“ Counting process ”

Count events, that happen:

— **randomly** in time (or space);

— completely **independently** from each other;

— with a **constant rate**.

Poisson process

Examples

Supermarket (your local AH) at low hours

On average, **1 customer arrives every 2 minutes**

The arrival times of the costumers are:

— random
— independent (from times of previous customers)
— with constant rate of 1 customer every 2 minutes

Number of customers that have arrived during a certain time?

Poisson process

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Rem: The **rate** at which customers arrive **can** also **change** in time. $\longmapsto$ " Non-homogeneous Poisson process "

Poisson process

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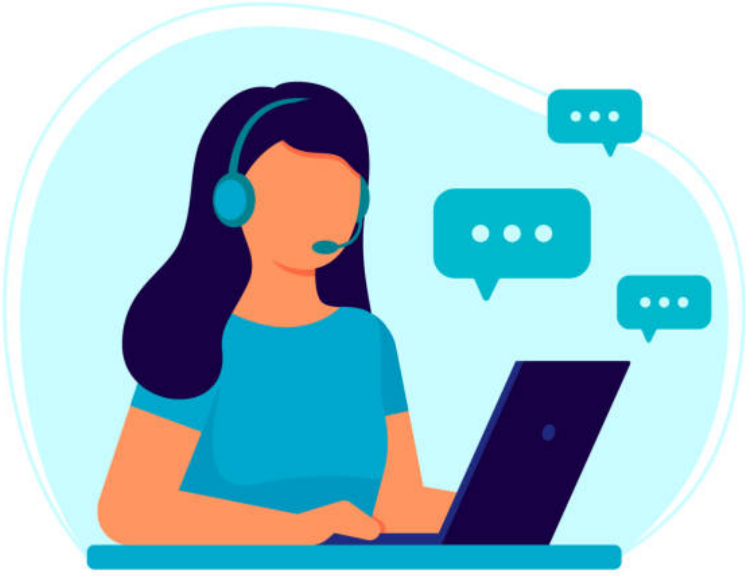
Rem: The **rate** at which customers arrive **can** also **change** in time. $\longmapsto$ " Non-homogeneous Poisson process "

Constant rate $\longmapsto$ " Homogeneous Poisson process "

Poisson process

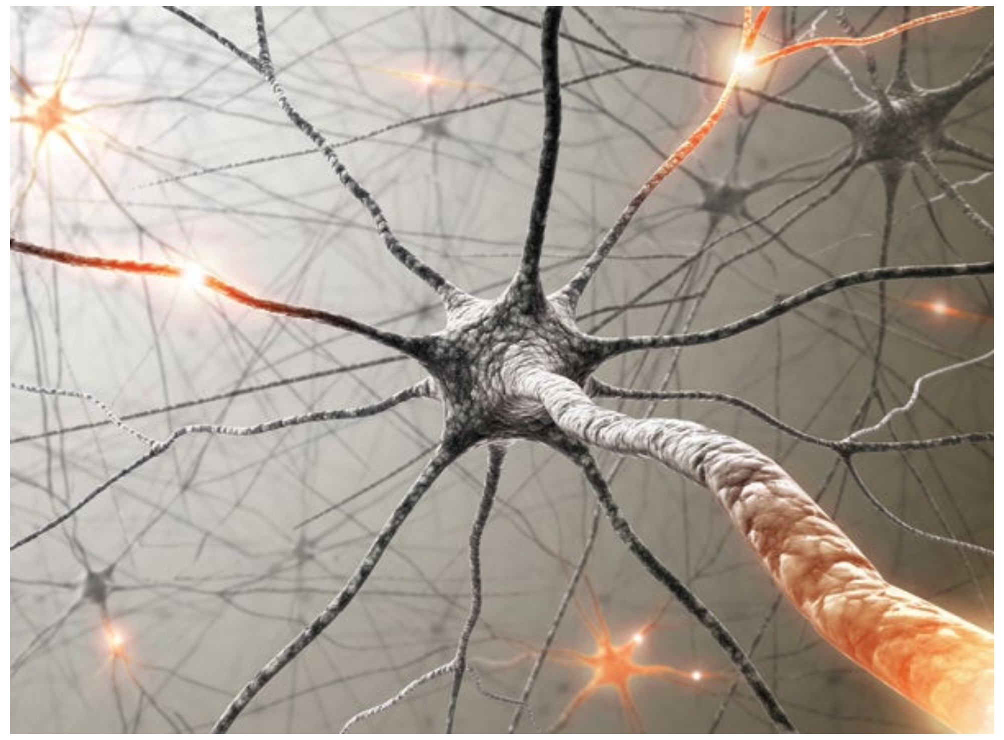
Used to model many other phenomena:

Arrival of customers at a supermarket



Timing of
calls at a help center

Radioactive decay in atoms



Activity of a neuron

Occurrences of earthquakes

Timing of patients
arriving in emergency room

Occurrences of meteors hitting Earth

Occurrences of
neutron-nucleus collisions
in a nuclear reactors

Location of trees in a forest

Movements in a stock price

Etc. Etc. ...

Not a Poisson process!

INDEPENDENCE between events is very important

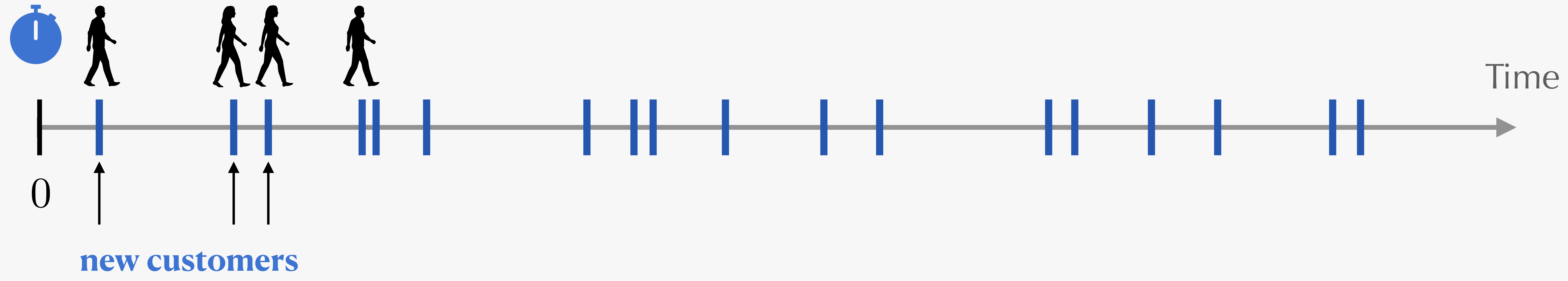
Ex. Occurrences of earthquakes of large magnitudes

Occurrence of earthquakes of large magnitude (>5) in a country may not follow a Poisson process, if one large earthquake increases the probability of aftershocks of similar magnitude.

If events are random,
what can we say?

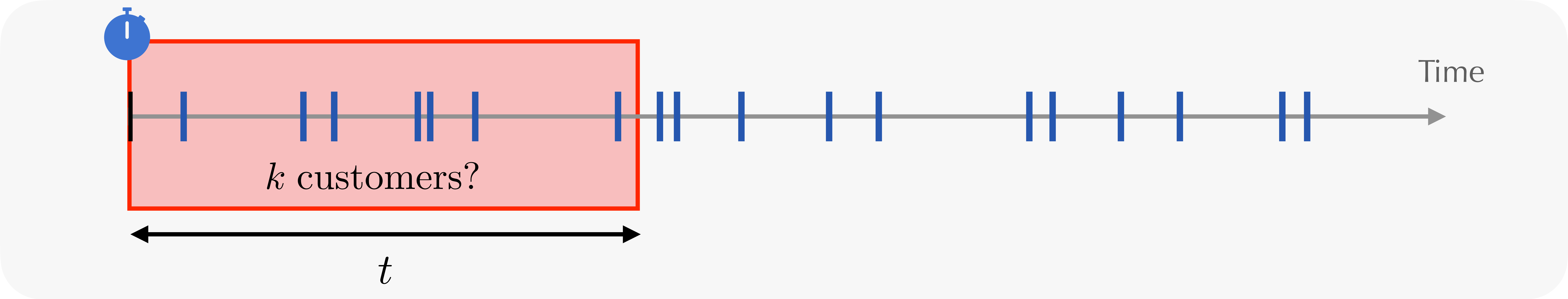
Back at our local AH

— with constant rate of 1 customer every 2 minutes



Poisson distribution

What is the probability that 10 customers have arrived in 5 min?



What is the **probability** that **10 customers** have arrived **in 5 min?**

French mathematician Siméon Denis Poisson,

The counting process $N(t)$ is a **Poisson process** with rate $\lambda > 0$, if:

– $N(0) = 0$

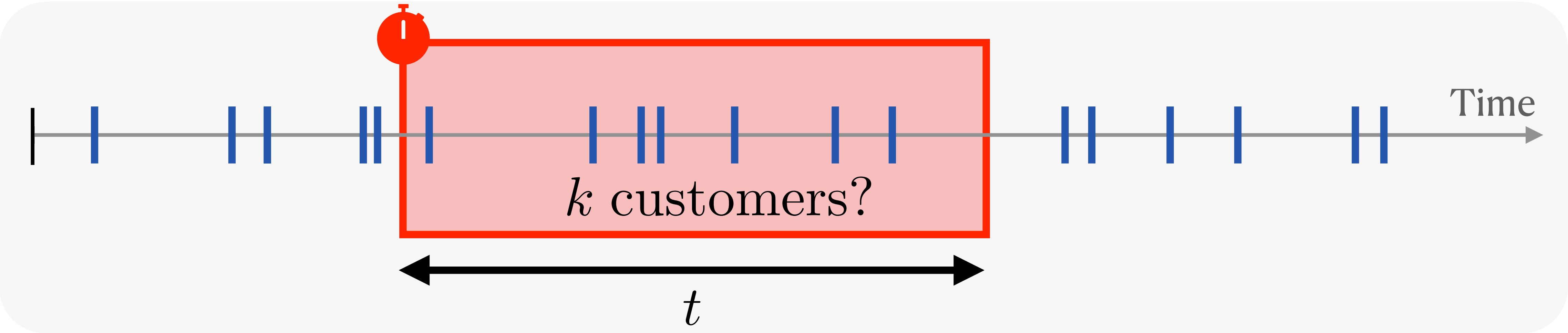
– $N(t)$ has independent increments;

– The probability to observe k events during a time interval t is given by

the **Poisson distribution** with parameter $\mu = \lambda t$

$$P[N(t) = k] = \frac{\mu^k \exp(-\mu)}{k!}$$

What is the probability that 10 customers have arrived in 5 min?



Rate of 1 customer per minute:

$$\lambda = \frac{1 \text{ customer}}{2 \text{ min}} = 0.5 \text{ customer per minute}$$

On average, there are 2.5 customers arriving in t=5 mins:

$$\mu = \lambda\, t = \frac{1 \text{ customer}}{2 \text{ min}} \times 5\,\text{min} = 2.5\,\text{customers}$$

The probability that 10 customers arrive in 5 mins is given by:

$$P[N(t=5)=10] = \frac{2.5^{10}\ \exp(-2.5)}{10!} \simeq 0.0002$$

(Very unlikely to happen)

Poisson distribution with parameter $\mu = \lambda\, t$

$$\boxed{P[N(t)=k] = \frac{\mu^k\ \exp(-\mu)}{k!}}$$

What is the probability that 10 customers have arrived in 5 min?

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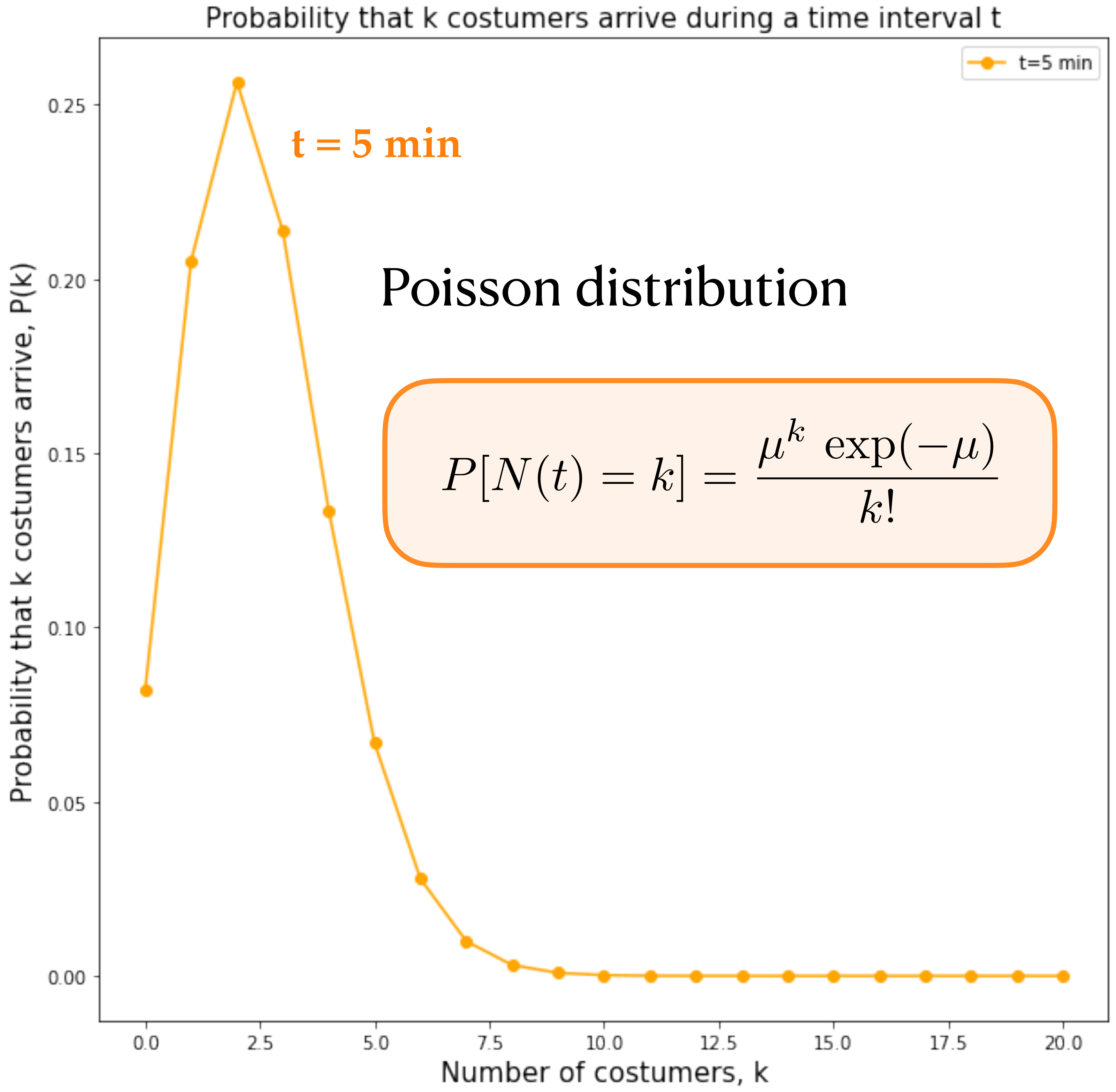

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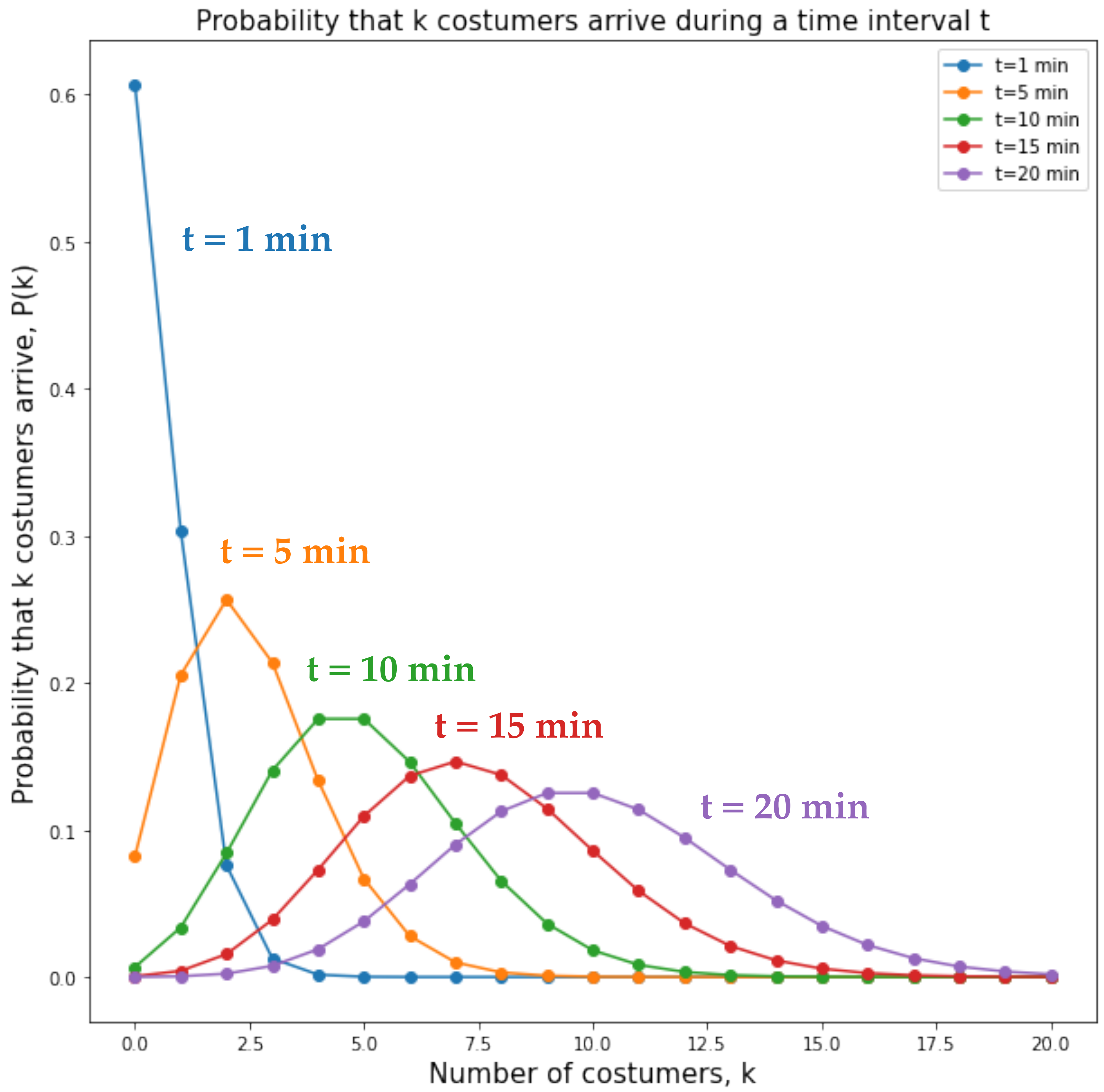
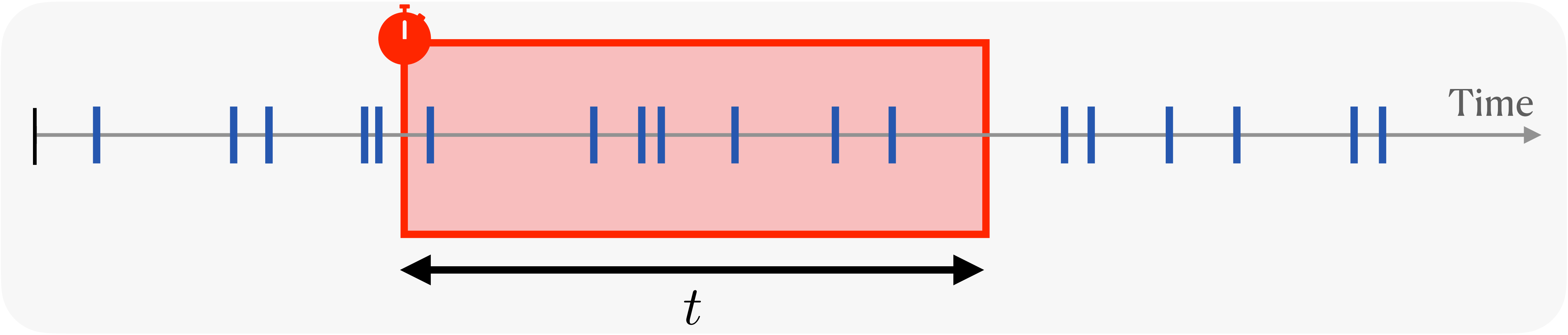
Poisson distribution





Poisson distribution with parameter $\mu = \lambda t$

$$P[N(t) = k] = \frac{\mu^k \exp(-\mu)}{k!}$$

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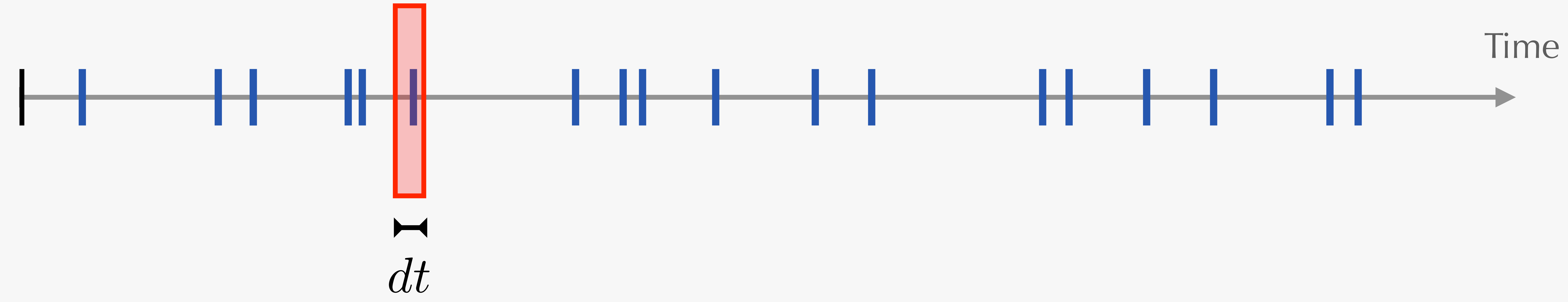
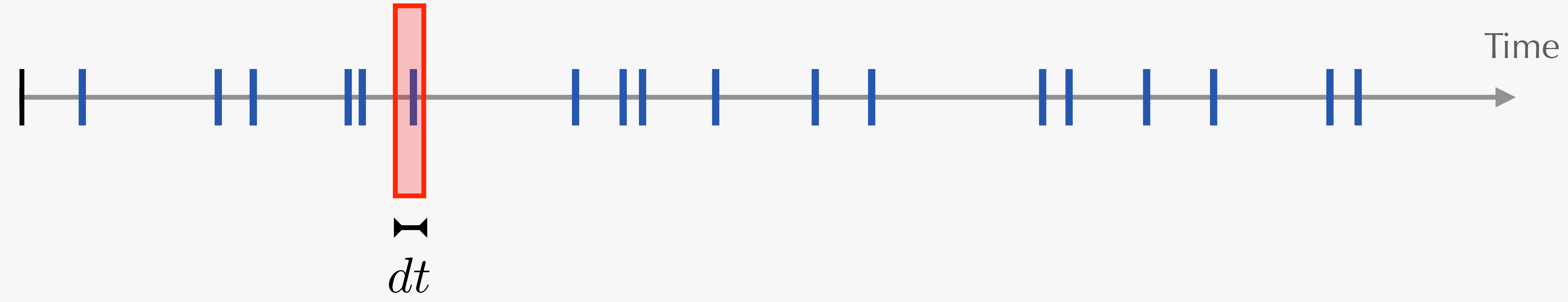
Expansion for
very short time interval dt

Very short time interval dt



Very short time interval dt



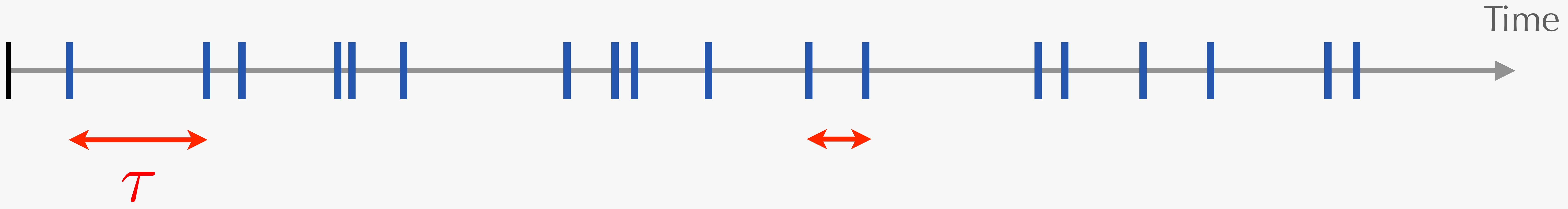
The counting process $N(t)$ is a **Poisson process** with rate $\lambda > 0$, if:

- $N(0) = 0$
- $N(t)$ has independent increments;
- during a very short time interval dt :
  - the probability that 1 customer arrives is $\lambda\, dt + o(dt)$
  - the probability that no customer arrives is $(1 - \lambda\, dt) + o(dt)$
  - the probability that 2 or more customers arrive is $0 + o(dt)$

Distribution of the waiting time

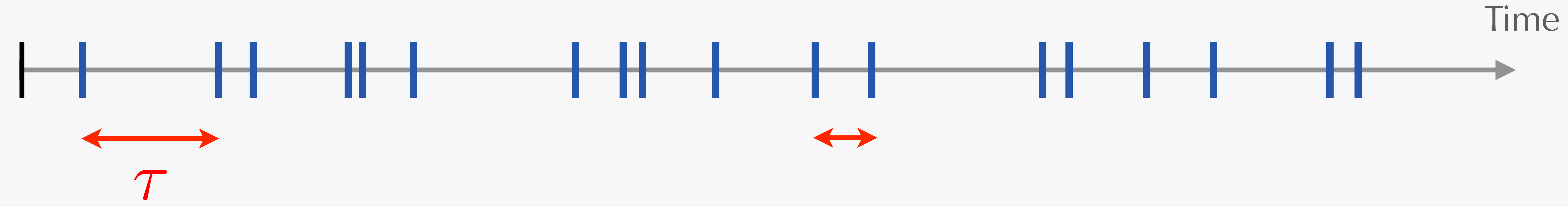
What is the probability that the next customer arrives in the next 1 min?

Distribution of the waiting time



τ = waiting time between the arrivals of two successive customers

Distribution of the waiting time



τ = waiting time between the arrivals of two successive customers

$P(\tau)$ = probability distribution of the waiting times

$P(\tau)\, d\tau$ = probability that a customer arrives between during a very short time interval $[\tau,\ \tau + d\tau)$

What is the probability that the next customer arrives in the next 1 min?

$\tau = 0$ $\tau = T = 1\,\mathrm{min}$

$$U(T) = \int_{\tau=0}^{\tau=T} P(\tau)\, d\tau$$

= probability that the customer arrives at any time between the time $\tau = 0$ ("now") and the time $\tau = T$

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$$\boxed{U(T) = 1 - \exp(-\lambda T)}$$

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Rate of 1 customer per minute:

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Probability that the next customer arrives **within T=1 minutes**: $U(1 \text{ min}) = 1 - \exp(-0.5 \times 1) \simeq 0.39$

within T=5 minutes:

What is the probability that the next customer arrives in the next 1 min?

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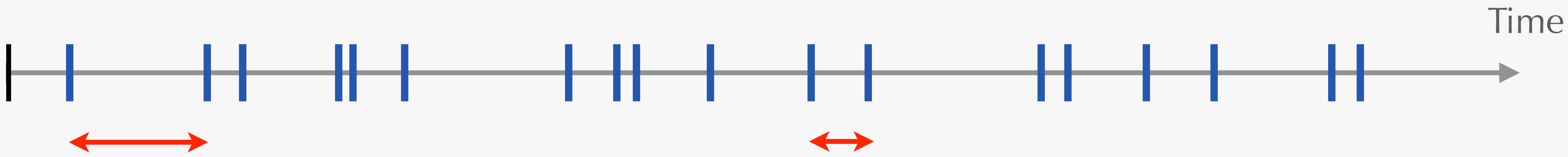
Probability that the next customer arrives **within T=1 minutes**: $U(1 \text{ min}) = 1 - \exp(-0.5 \times 1) \simeq 0.39$

within T=5 minutes: $U(5 \text{ min}) = 1 - \exp(-0.5 \times 5) \simeq 0.92$

Distribution of the waiting time



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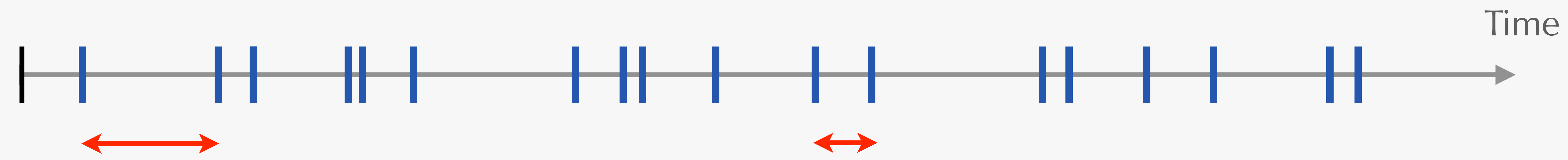
$$U(T) = \int_0^T P(\tau)\, d\tau \quad \Longrightarrow$$

Distribution of the waiting time



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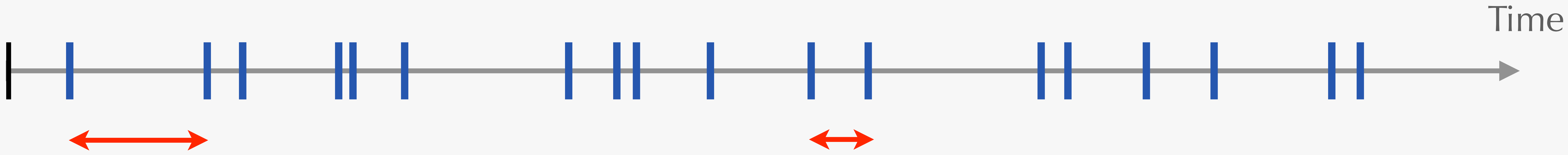

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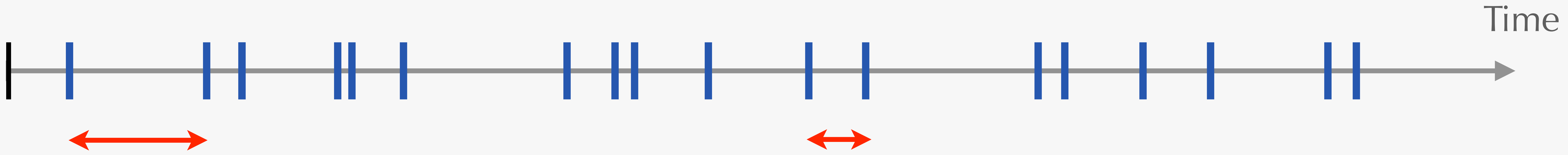
Exponential distribution

$$\boxed{P(\tau) = \lambda \exp(-\lambda\, \tau)}$$

Distribution of the waiting time



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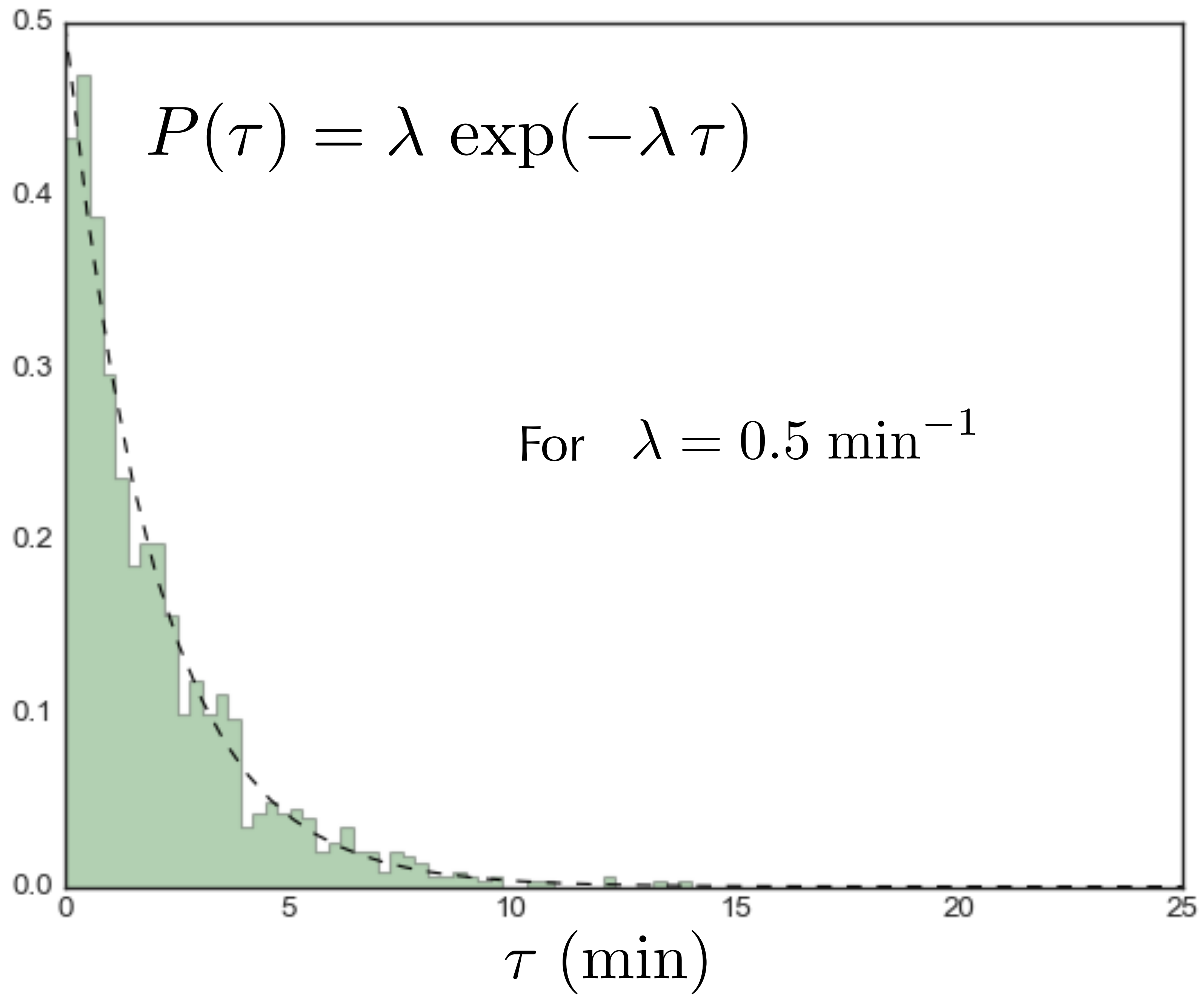

Poisson Processes

Summary

Poisson Process — Summary

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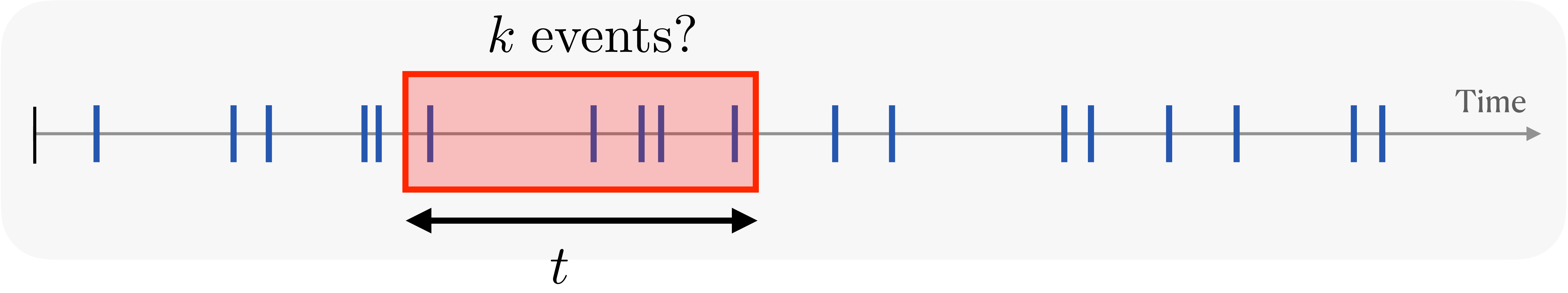
Occurrence of events:

- Random
- Independent
- With constant rate λ

Poisson Process — Summary



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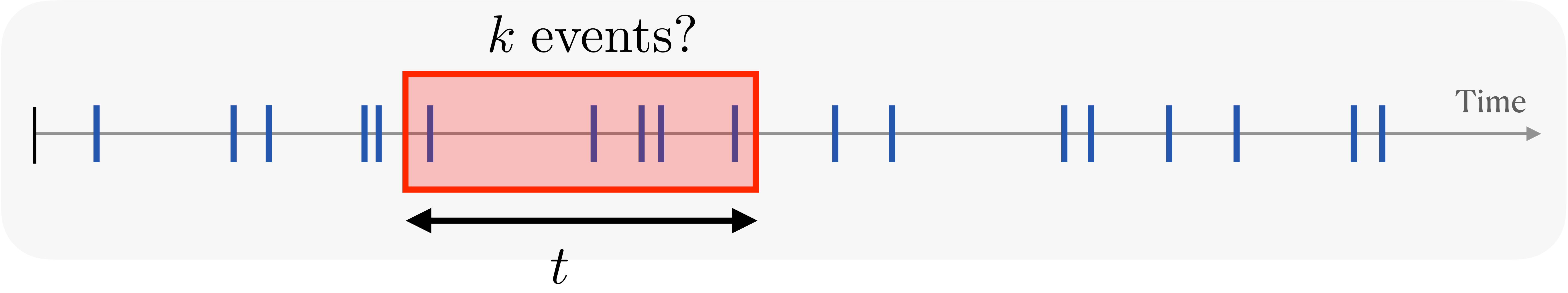
Probability that **k events occur during t** given by:

Poisson Process — Summary



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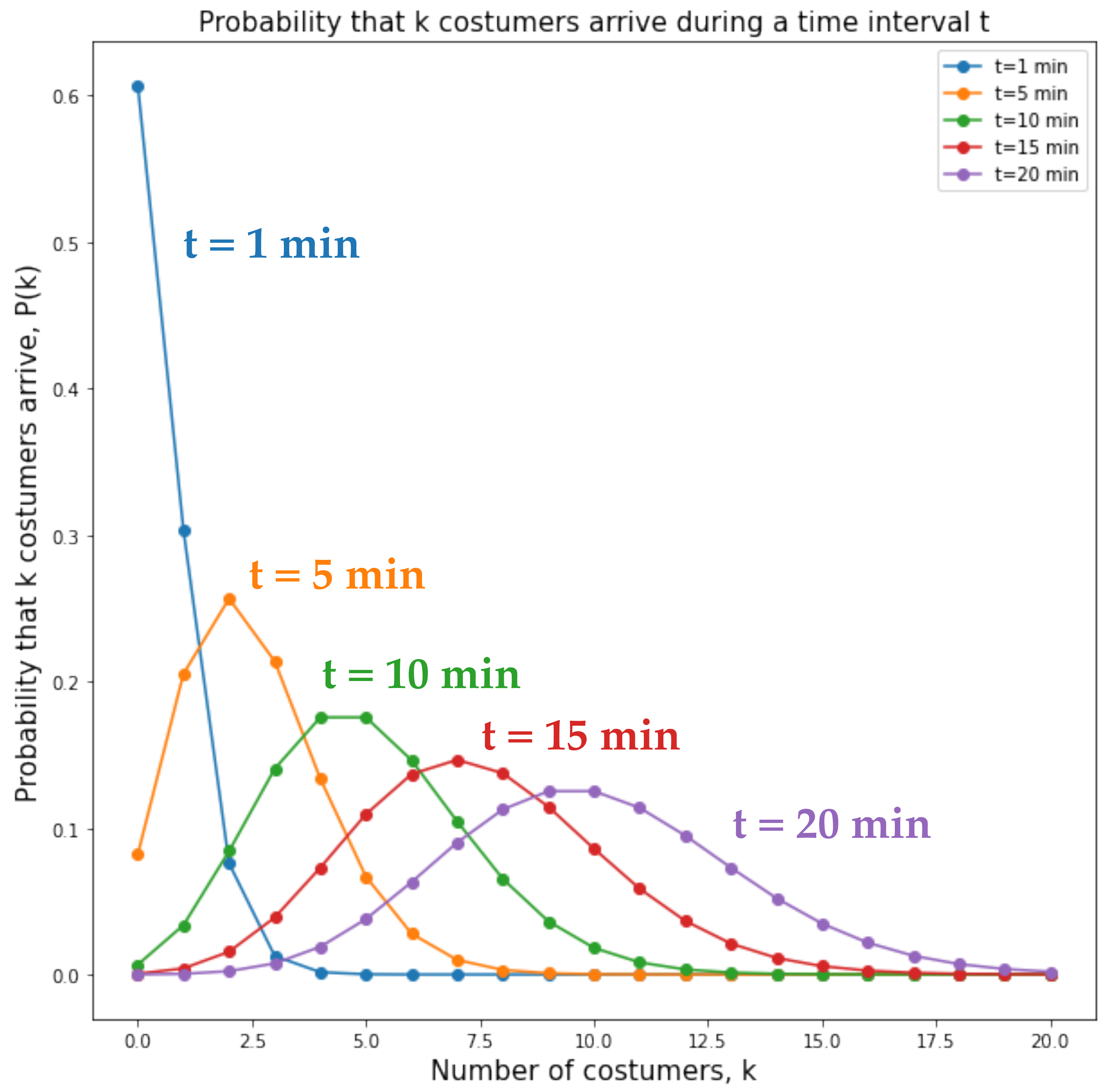


Ex. Answer the question:

What is the probability that exactly 2 customers arrive within 5 min?

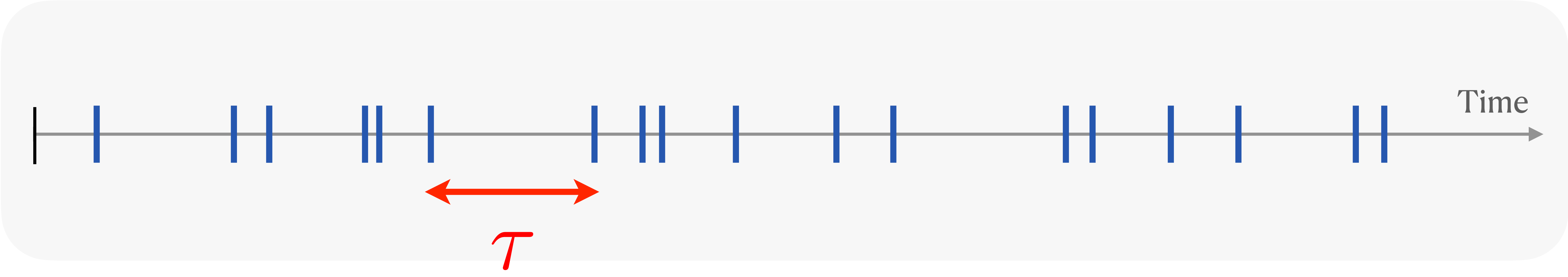
Poisson Process — Summary

Occurrence of events:

— Random
— Independent
— With constant rate λ

Distribution of waiting times is:

Poisson Process — Summary



Occurrence of events:

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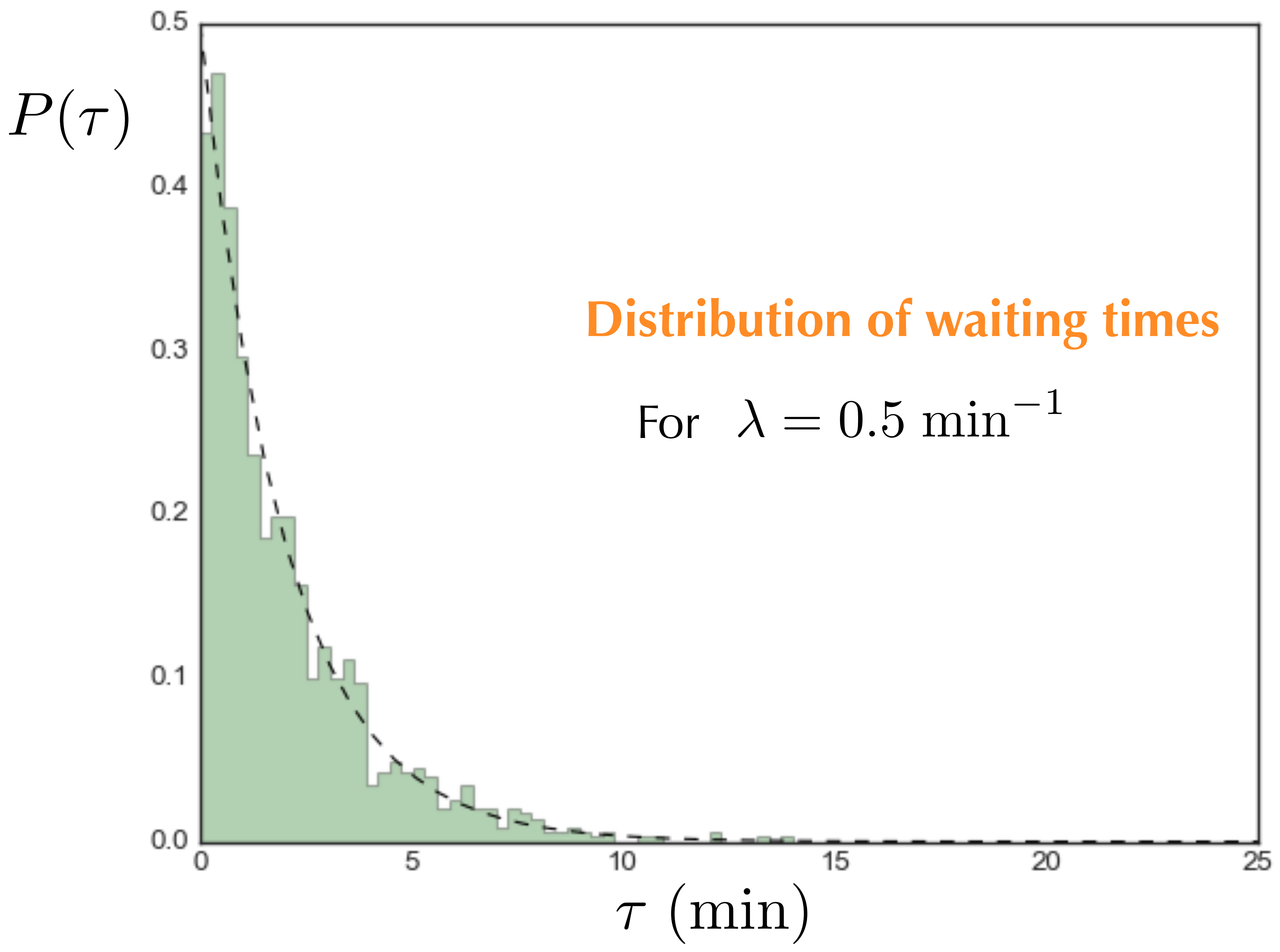

Distribution of waiting times is:

Exponential distribution

$$P(\tau) = \lambda\, \exp(-\lambda\, \tau)$$

Ex. Answer the question: $$U(T) = \int_0^T P(\tau)\, d\tau$$

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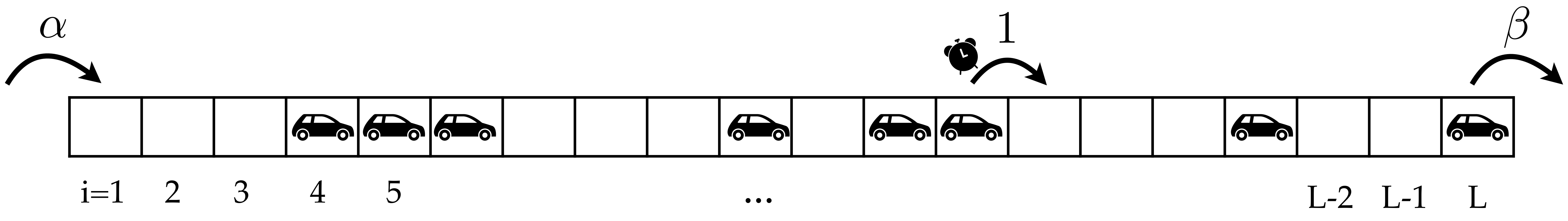
What is the probability that the next customer arrive within 1 min?

Applying what we have just seen to another model…

TASEP

A Simple Traffic Model

Totally Asymmetric Simple Exclusion Process (TASEP)



"**Totally Asymmetric**": cars can only move towards the right

"**Simple**": move only by one site at a time

"**Exclusion**": there can only be one car per site

α = **rate** at which cars **enter from the left** of the lattice

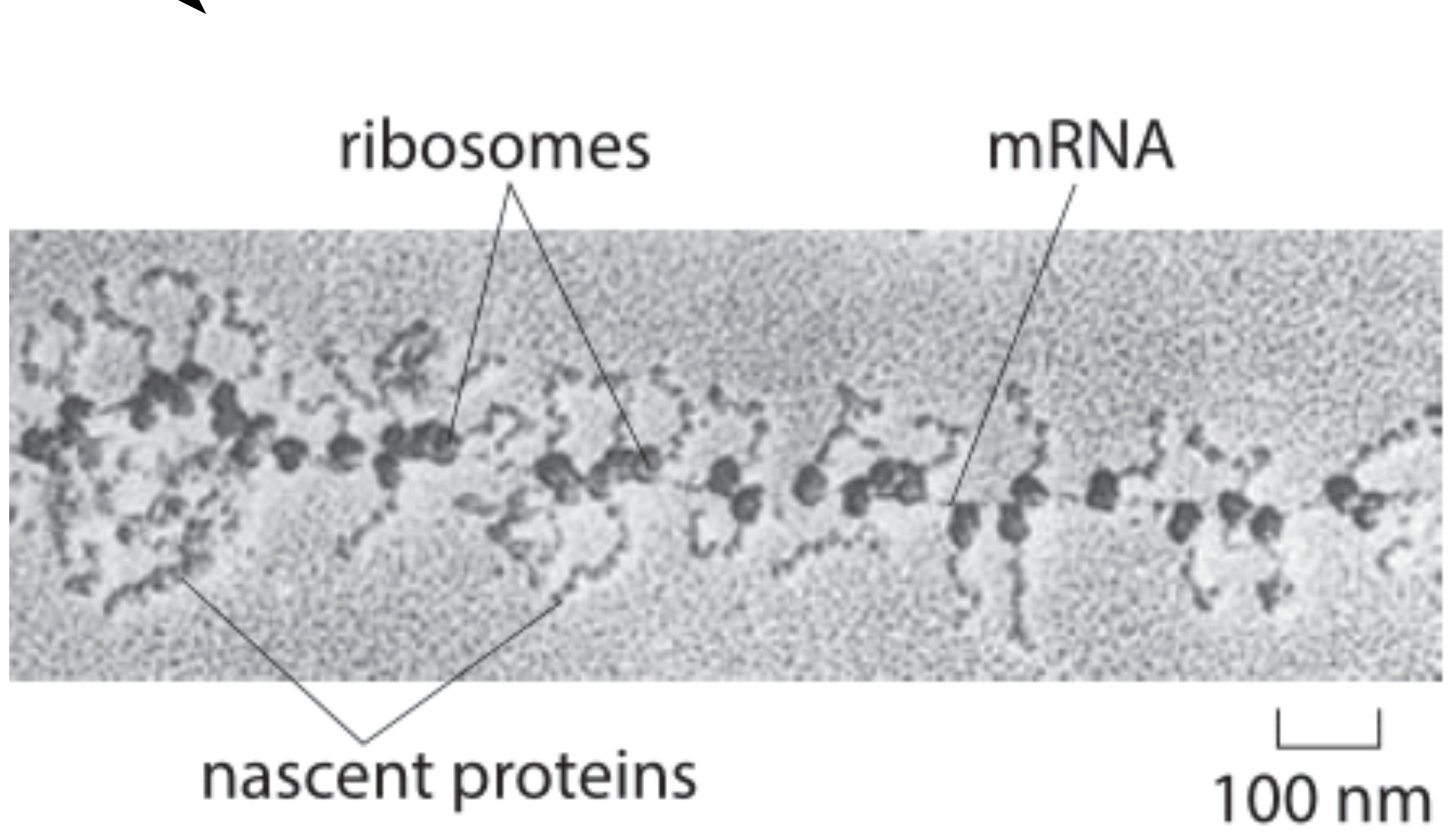
β = **rate** at which cars **exit from the right** of the lattice

1 = **rate** at which cars **move to the right next site**

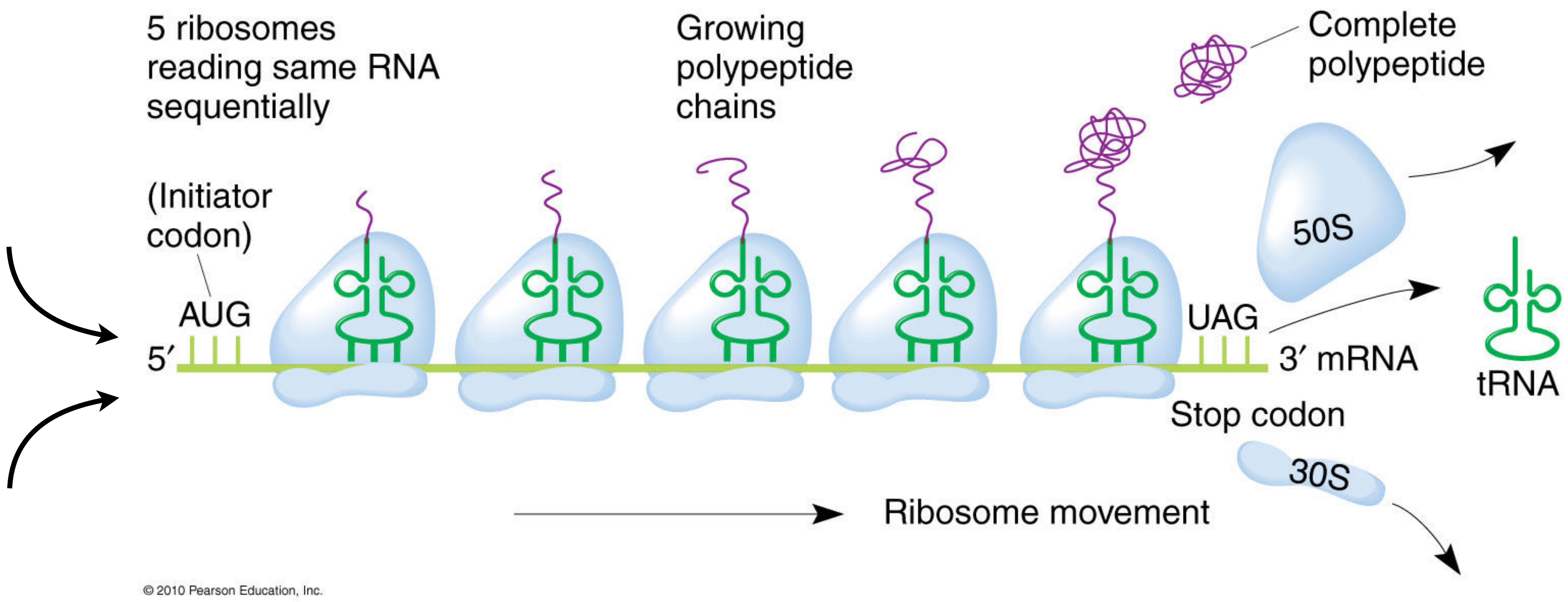
This only happens if the next site is free!

Transcription of mRNA

Ribosome movement



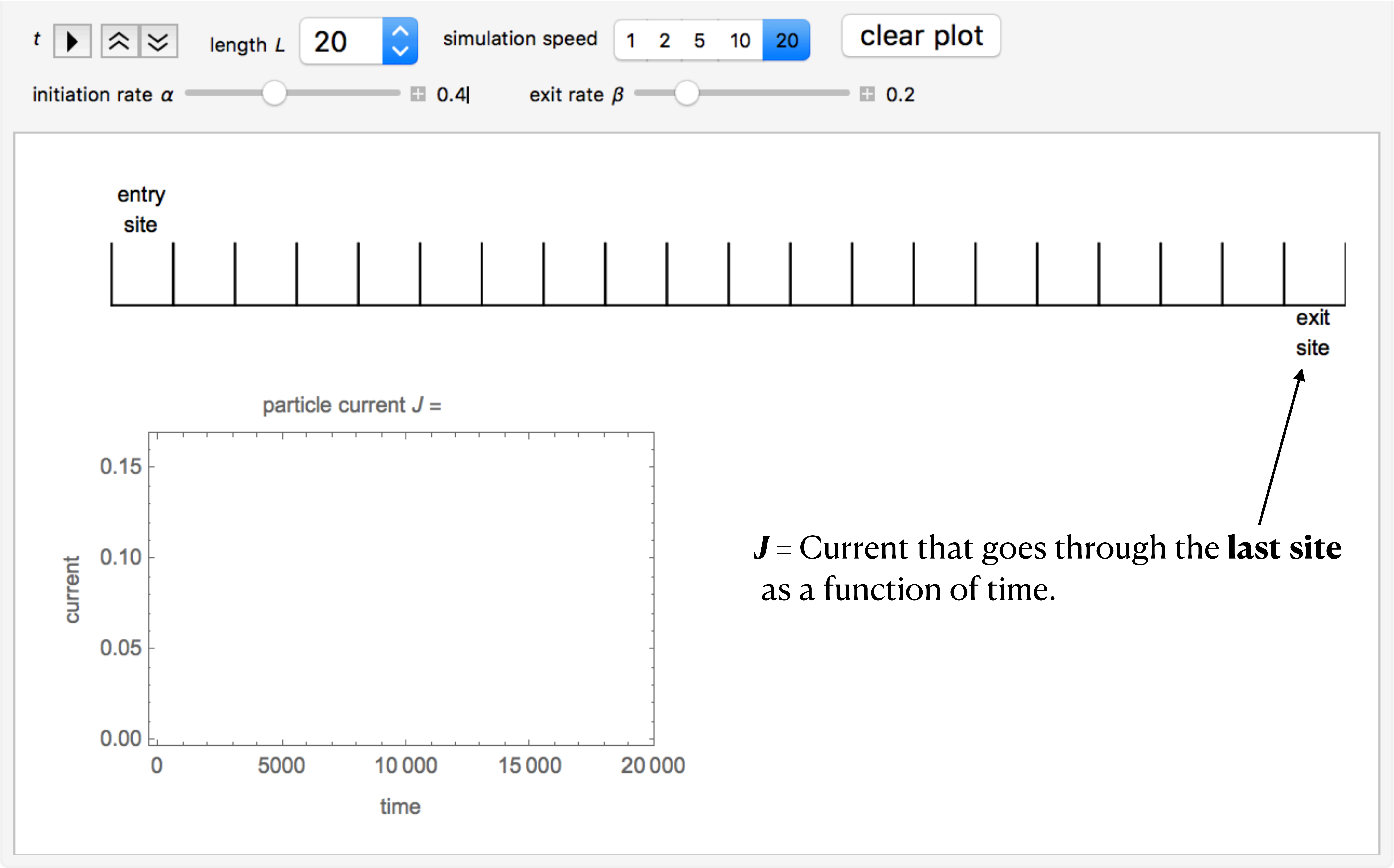
(polypeptide chain gets longer
as ribosome moves to the exit)



Schematic drawing of a polyribosome (polysome)

Simulation

Simulation written in Mathematica: you can find how to get a UvA student licence and install it here



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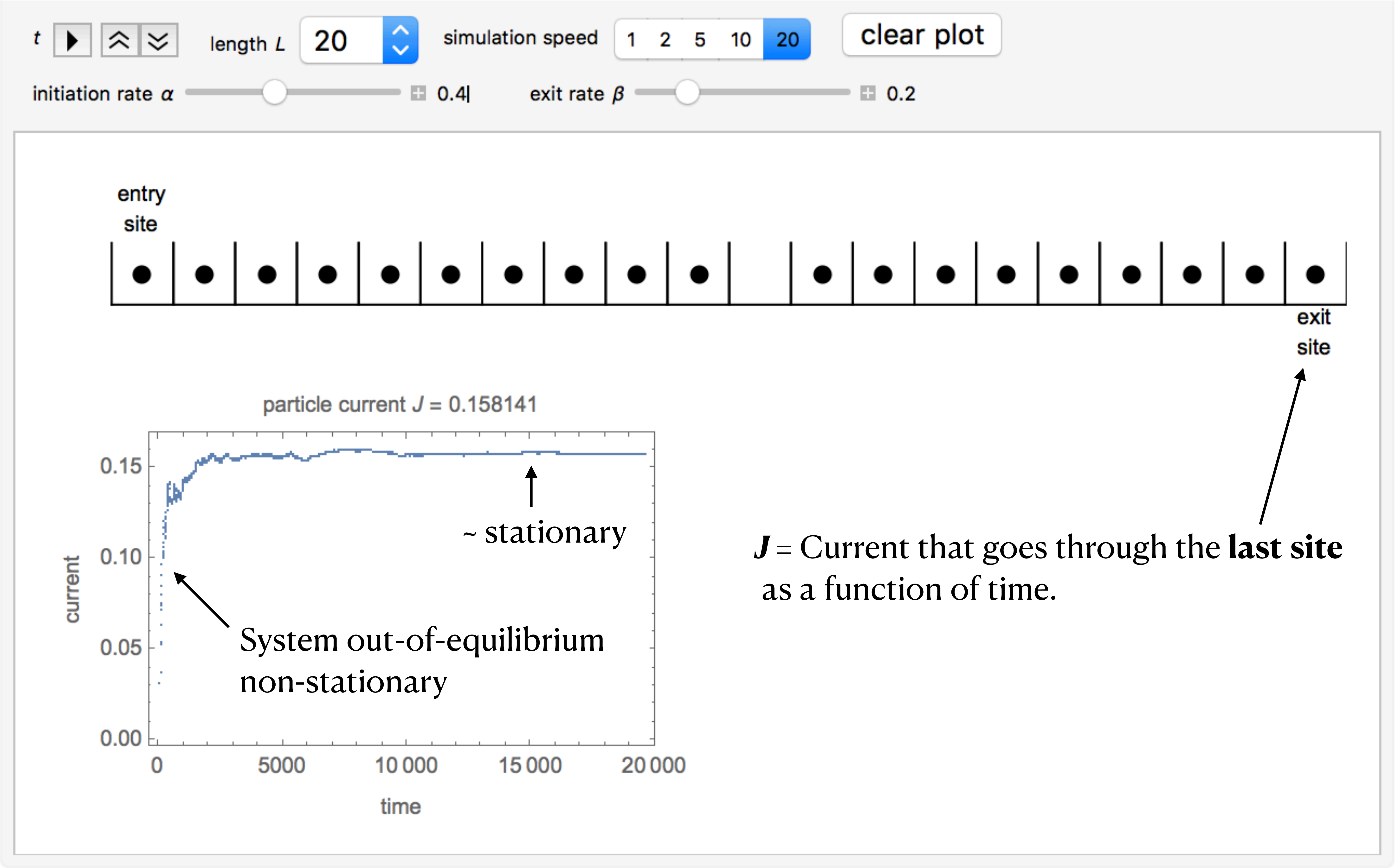


To do before lecture on Wednesday:

Play with the simulation:

— initially out-of-equilibrium

— then stationary

Behaviour of the system for

$(\alpha,\,\beta) = (1,\,0.1)$? $(\alpha,\,\beta) = (0.1,\,1)$?

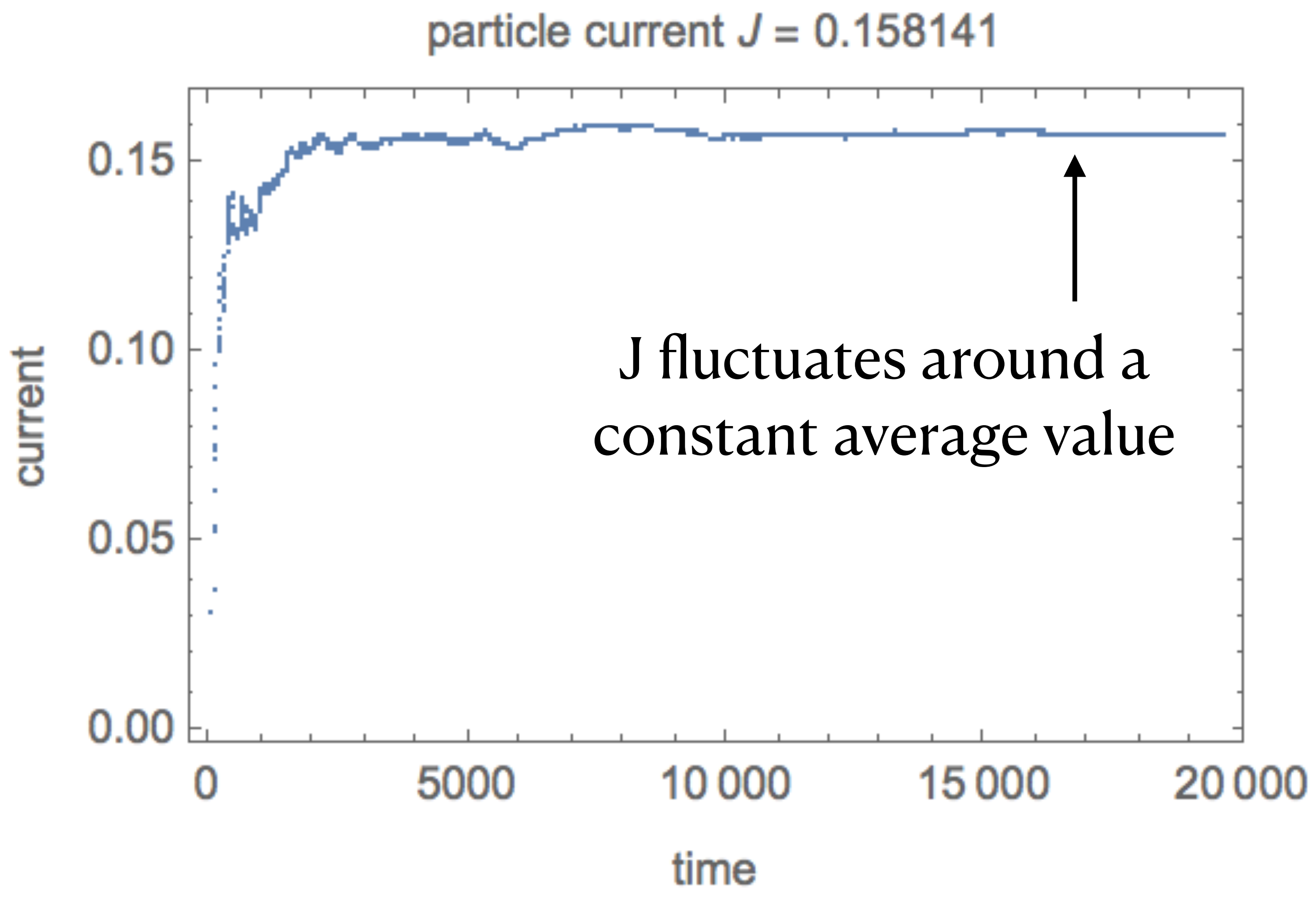
$(\alpha,\,\beta) = (0.2,\,0.2)$? $(\alpha,\,\beta) = (0.6,\,0.6)$?

Which case produce a traffic jam pattern similar to what can be seen on this video?
(link here)

For Wednesday

— **Finish the exercise** —> will be corrected during the tutorial session

— To do **before the lecture on Wednesday:** Fill in this table using the simulation on Mathematica **~2 values per person!**



beta \ alpha	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
1,0										
0,9									0,27	
0,8										
0,7										
0,6	0,09									
0,5				0,24						
0,4										
0,3										
0,2										0,16
0,1										

Advise for the course:

—> work in groups 2-3 people max.
with mix background (theory, computational, CS, …)

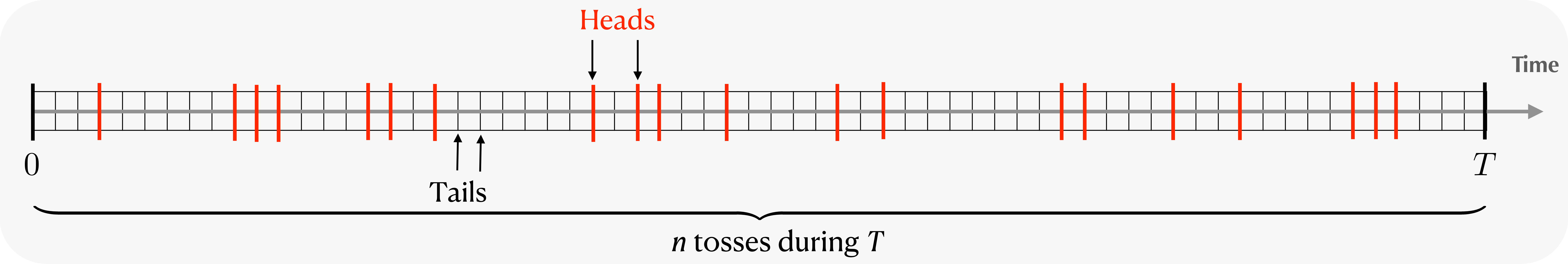
Please contact us if you have any difficulty joining a group!

Fun!

For more practice

— **Make a simulation of the TASEP model** —> check that our simulation behave similarly to the one provided

— **"Super fast" coin flips:** the Poisson distribution can be seen as a limit of the Binomial distribution



During a time interval T, flip de coin n times.

During a small time interval $dt = \frac{T}{n}$:

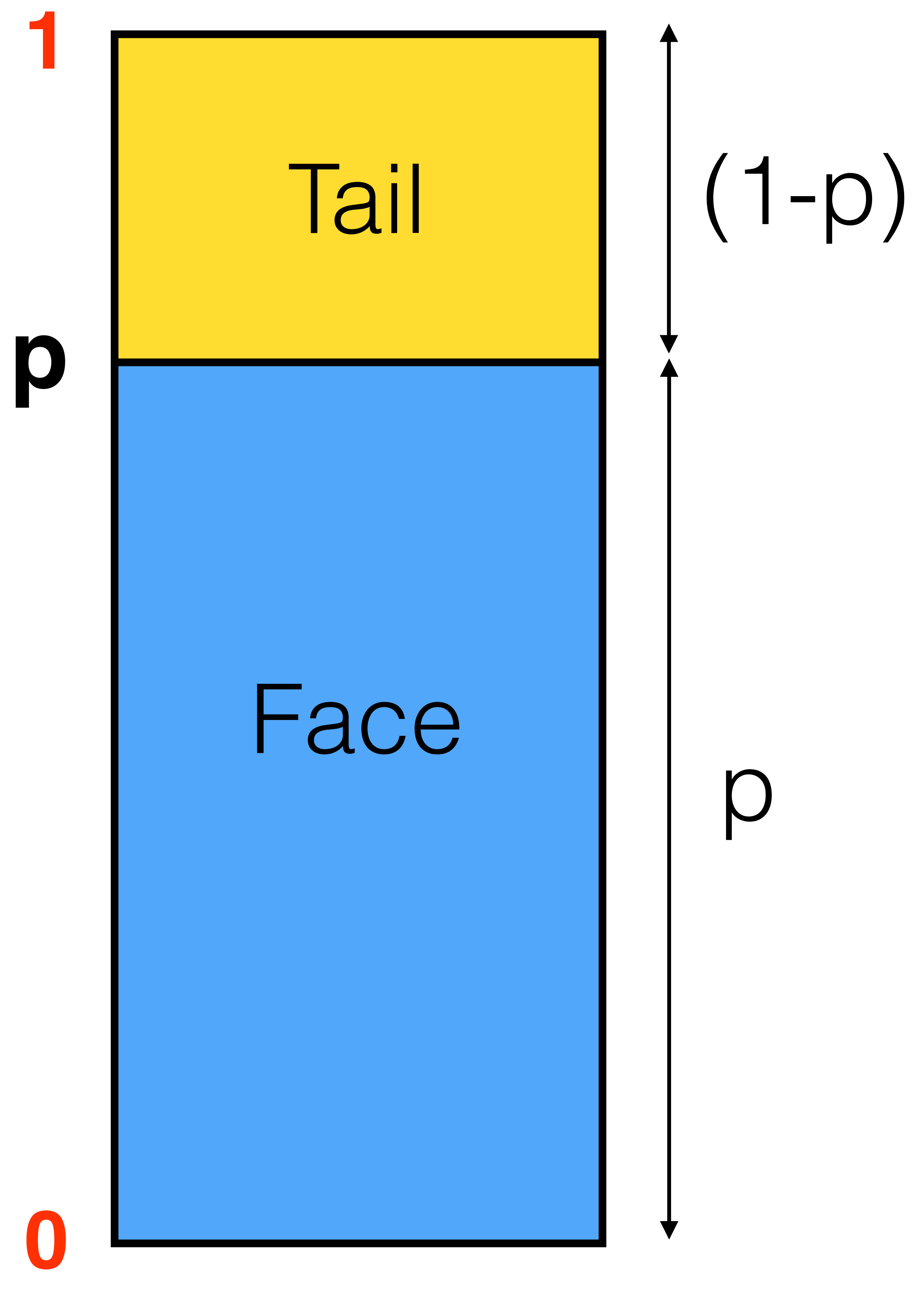
— probability to get Heads: $p = \lambda\, dt$

— probability to get Tails: $(1-p) = (1 - \lambda\, dt)$

Q. What is the **probability to have k heads during T?** What is the limit when ***n* becomes very large**, while ***k* stays finite?**

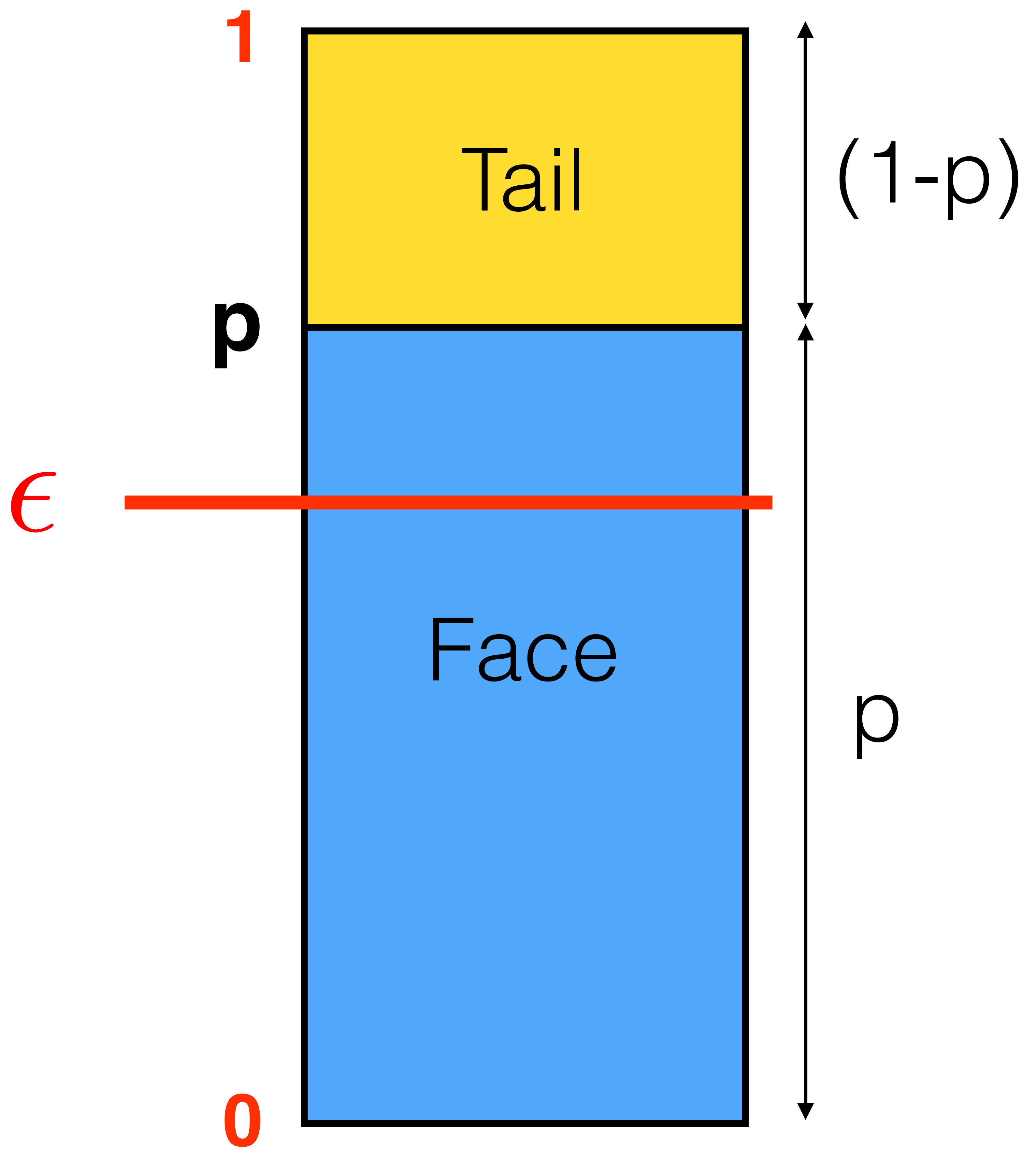
Bonus

Sampling a Binary Discrete Random Variable



Ex. Biased coin:

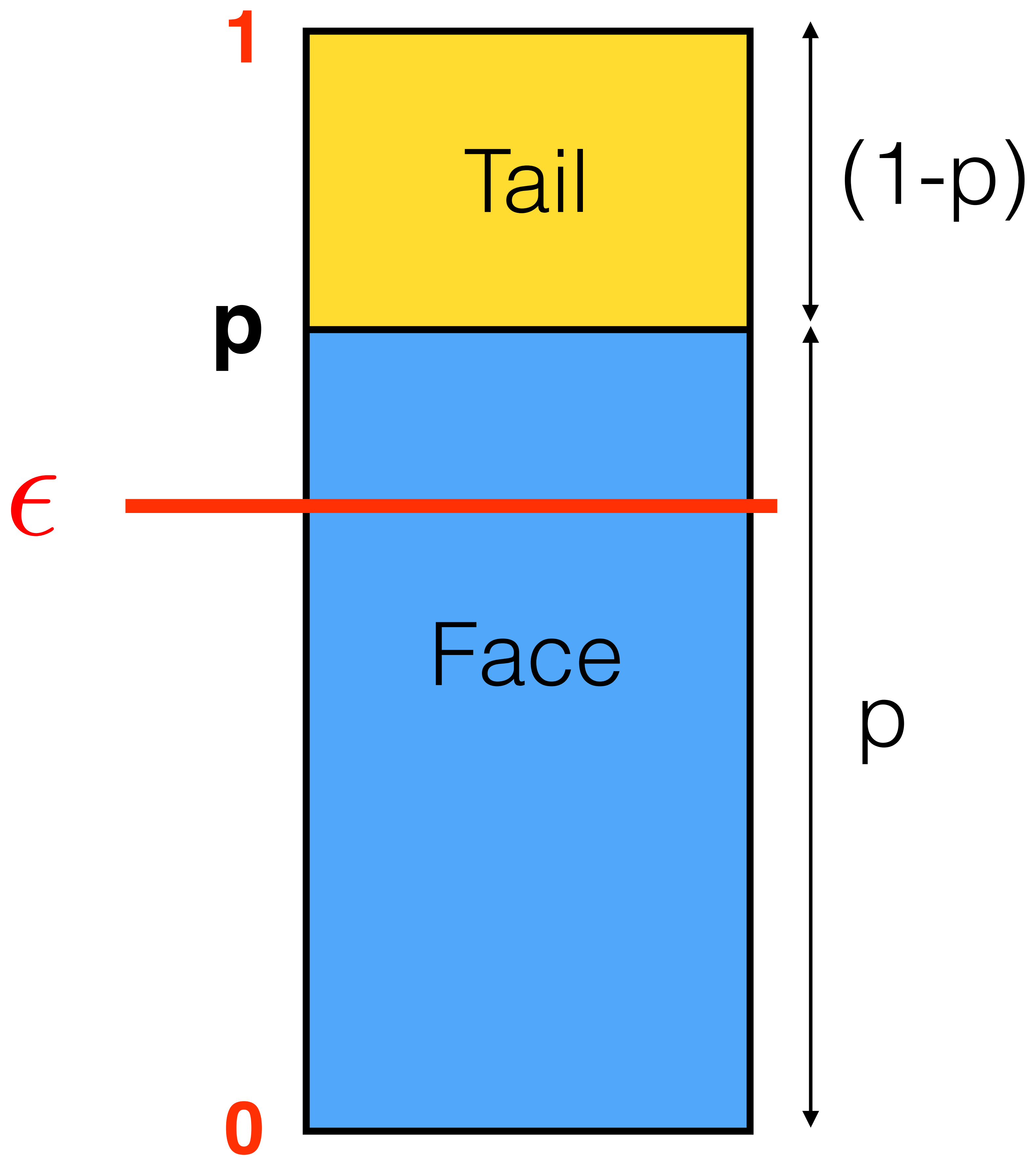
Sampling a Binary Discrete Random Variable



Ex. Biased coin:

$\epsilon \in [0, 1)$ uniform

Sampling a Binary Discrete Random Variable



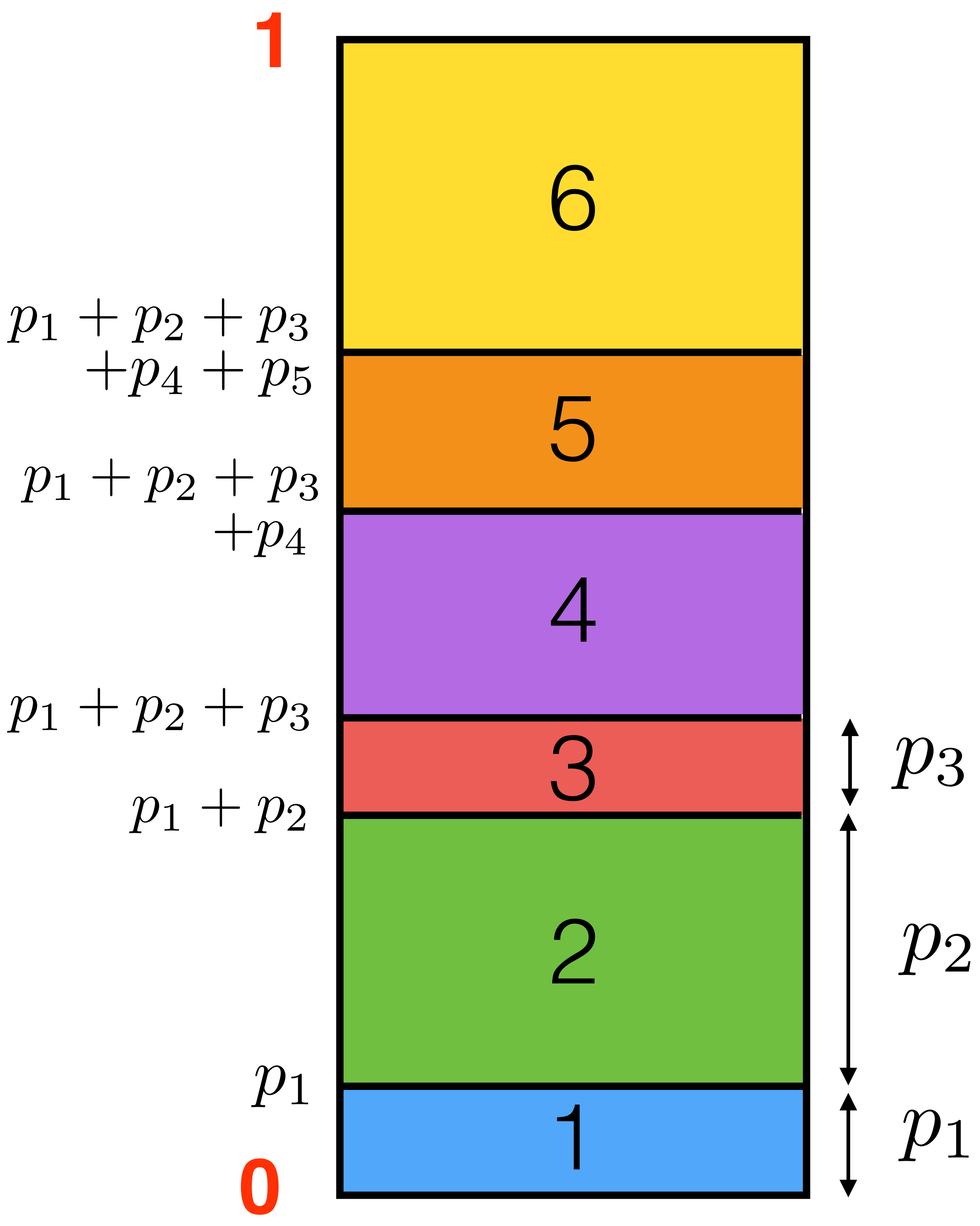
Ex. Biased coin:

$\epsilon \in [0, 1)$ uniform

if $\epsilon < p$, then Face

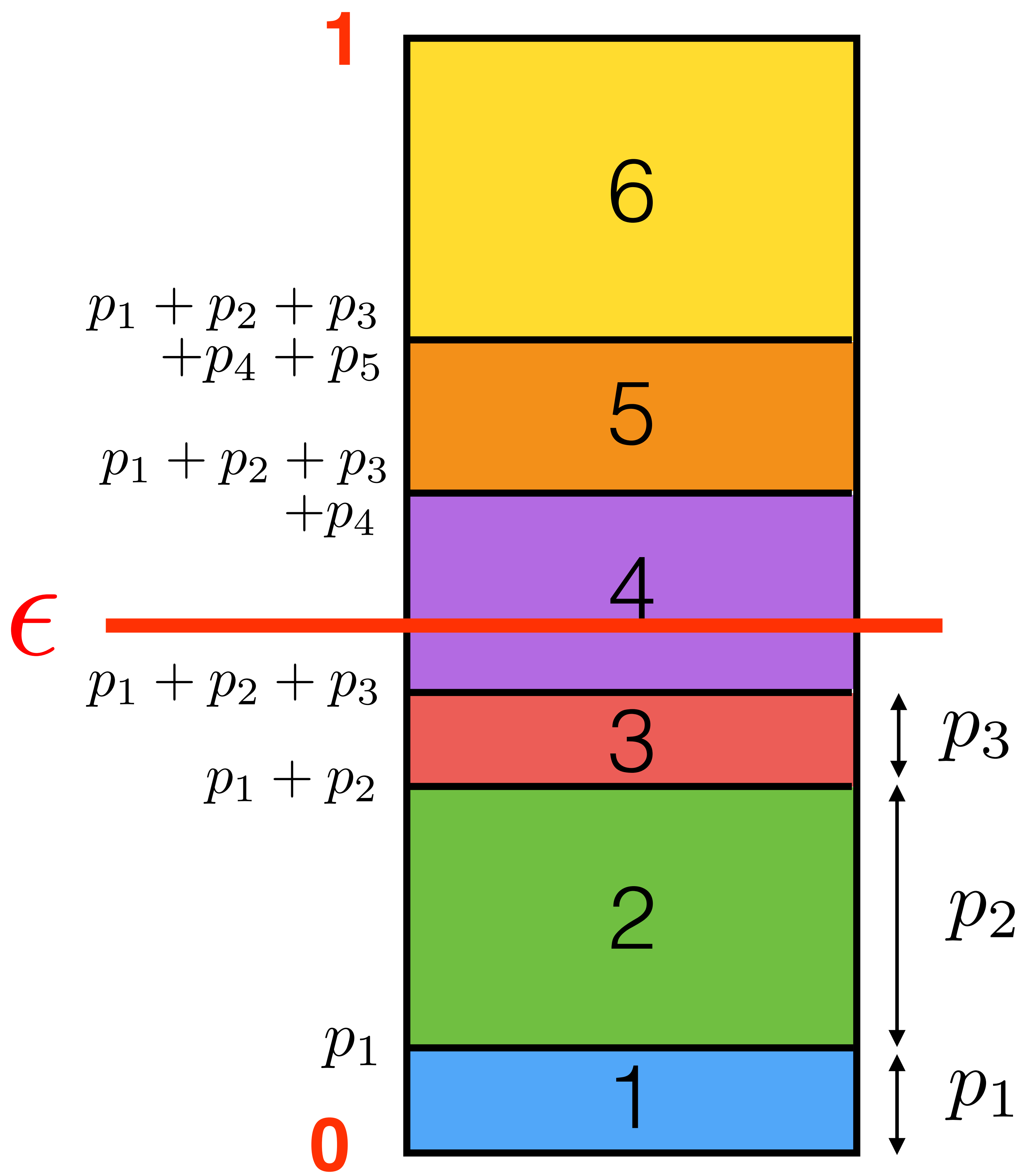
if $\epsilon > p$, then Tail

Sampling a Discrete Random Variable



Ex. Biased dice:

Sampling a Discrete Random Variable



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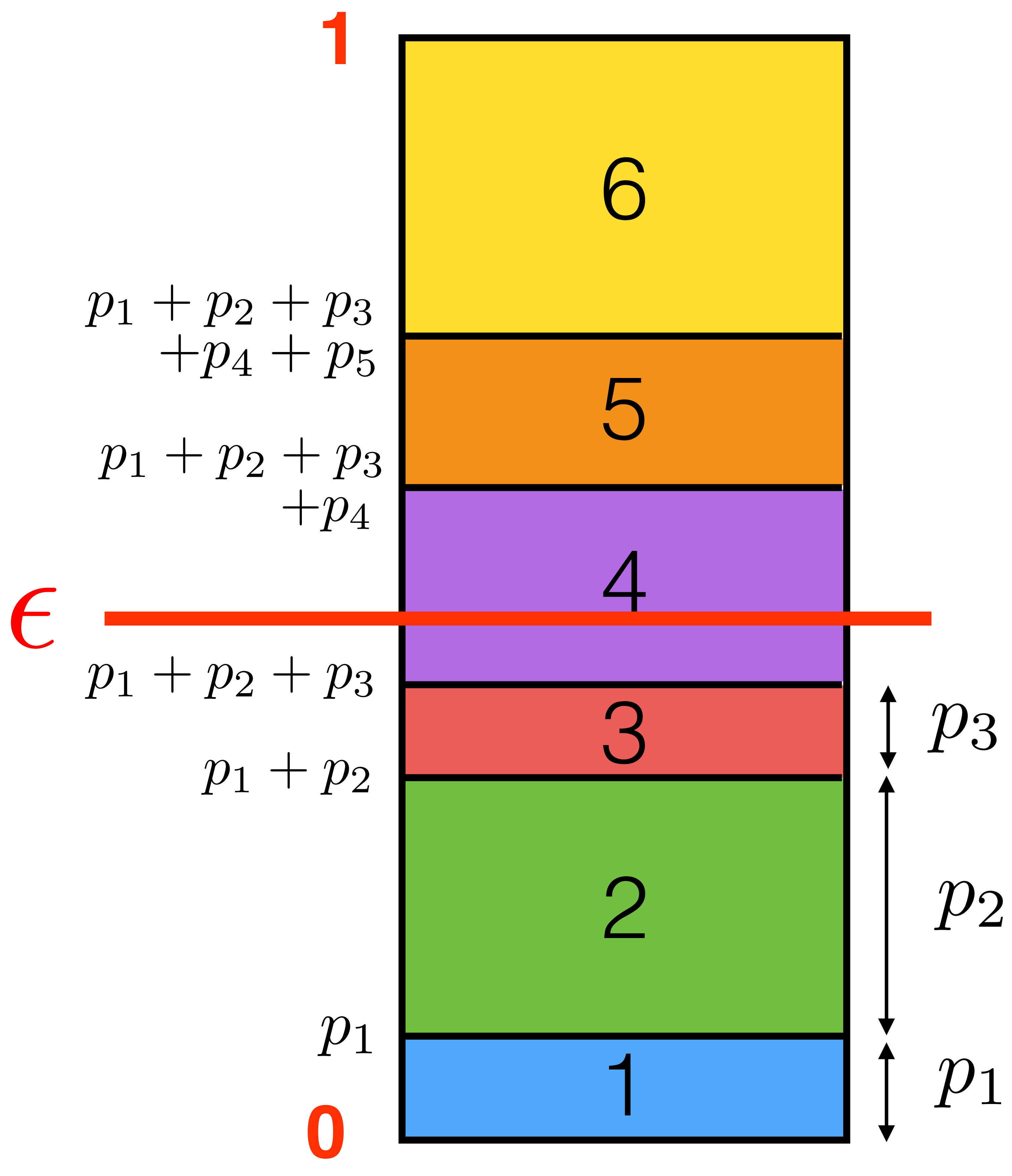


Ex. Biased dice:

$\epsilon \in [0, 1)$ uniform

if $\epsilon < p_1,$ then 1

else if $\epsilon < p_1 + p_2,$ then 2

else if $\epsilon < p_1 + p_2 + p_3,$ then 3

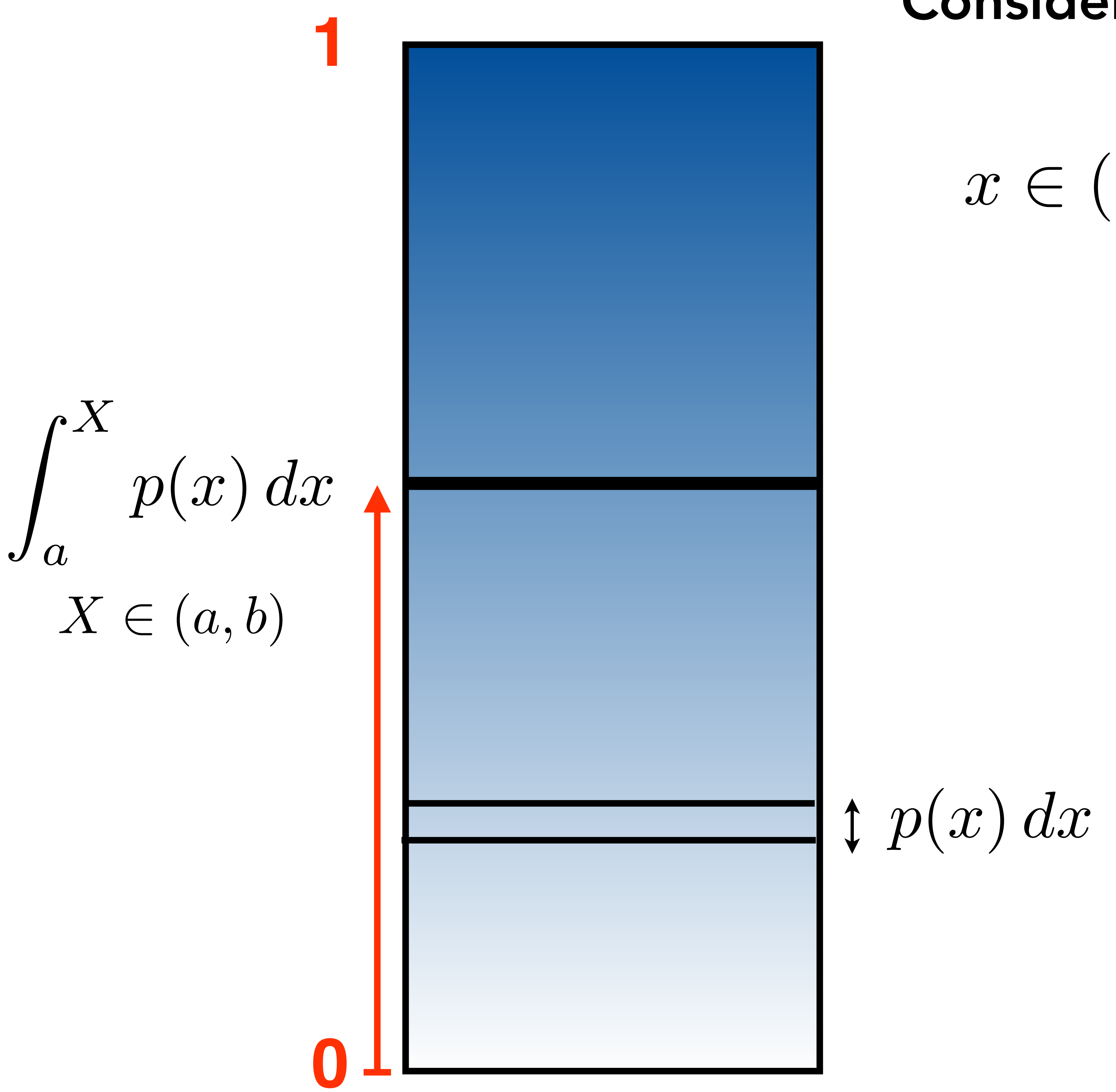
Etc.

Sampling a Continuous Random Variable
Using the inverse of a cumulative

Consider a continuous probability density function:

$$x \in (a,b), \qquad p(x) \qquad \text{with} \quad \int_a^b p(x) = 1$$

Sampling a Continuous Random Variable
Using the inverse of a cumulative



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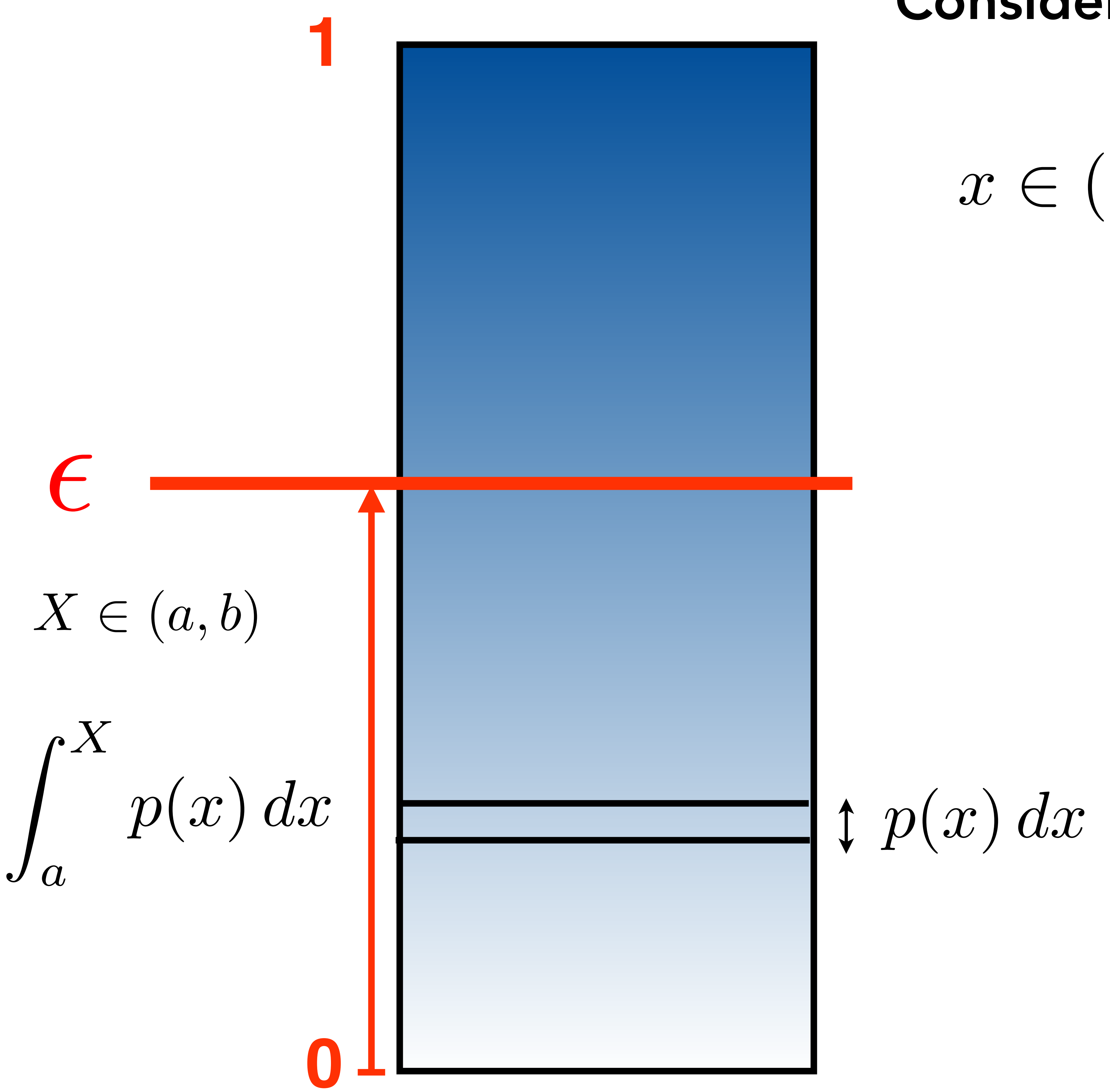

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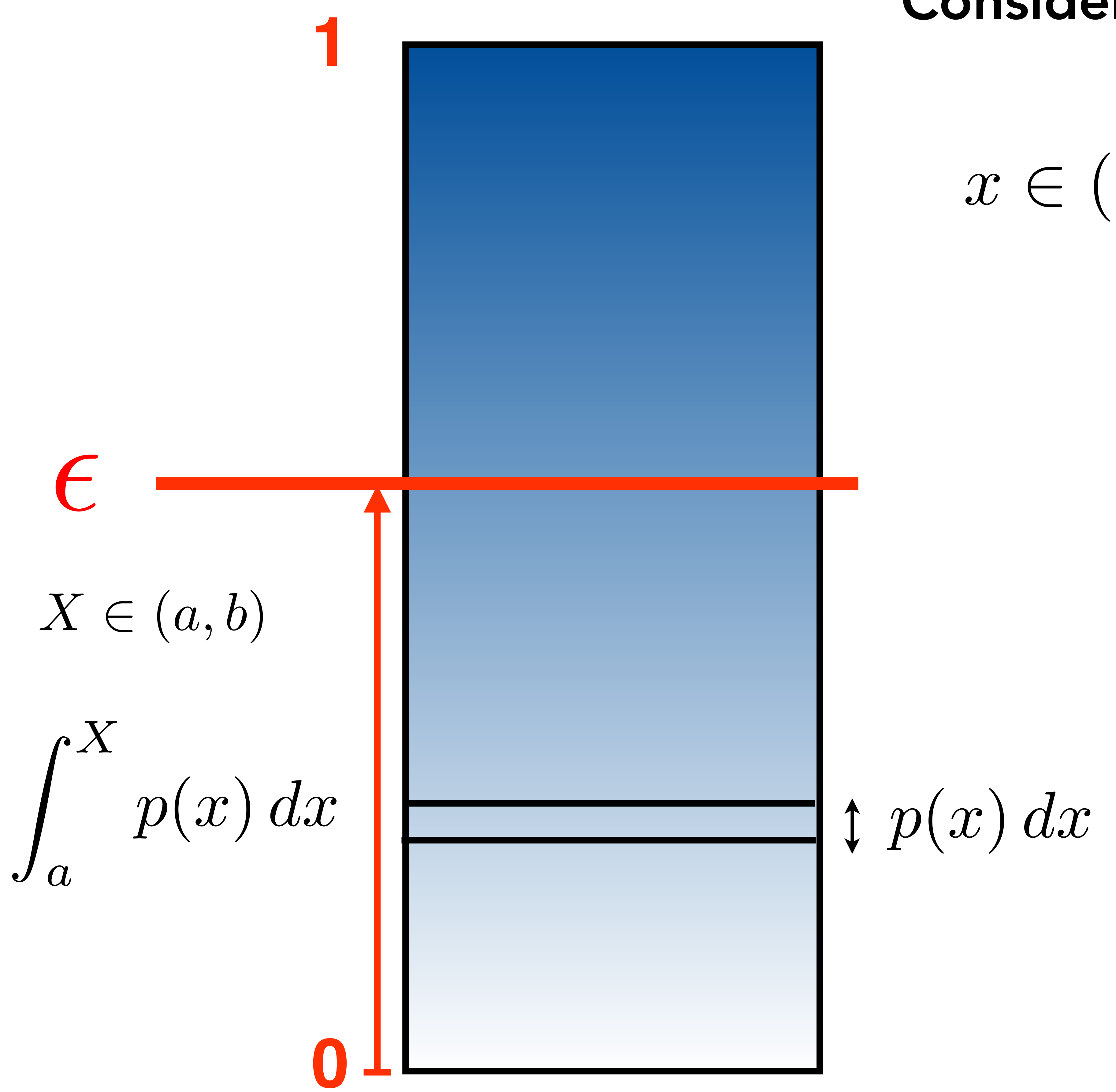
$$x \in (a, b), \qquad p(x) \qquad \text{with} \quad \int_a^b p(x) = 1$$

$$\epsilon \in [0, 1) \qquad \text{uniform}$$

Find $X \in (a, b)$ **such that:**

$$F(X) = \boxed{\int_a^X p(x)\,dx = \epsilon}$$

Sampling a Continuous Random Variable
Using the inverse of a cumulative



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Find $X \in (a, b)$ **such that:**

$$\boxed{F(X) = \int_a^X p(x)\,dx} = \epsilon$$

$$\boxed{X = F^{-1}(\epsilon)}$$

Sampling a Continuous Random Variable Using the inverse of a cumulative

Ex. Exponential distribution:

Consider a series of **independent events** that happen with a **constant rate** λ

Ex. light scattering in a diffusive medium

At any time t_0, the probability that the next event happens at time $t_0 + t$ is independent of t_0 and is given by the **Exponential distribution:**

$$P(t) = \lambda \exp(-\lambda\, t)$$

- $\epsilon = \mathrm{uniform}(0, 1)$
- Find T such that: $$\int_0^T \lambda \exp(-\lambda\, t)\, dt = \epsilon$$

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$$\Rightarrow \quad T = -\frac{1}{\lambda} \log(1 - \epsilon)$$

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$$\Rightarrow \quad T = -\frac{1}{\lambda}\log(1-\epsilon) \quad \Rightarrow \quad \boxed{T = -\frac{1}{\lambda}\log(\eta) \quad \text{where } \eta = \text{uniform}(0,1)}$$