Landau Theory

Lecture 4 — Part 2

Monday Feb 28

Landau Theory

Lecture 4 part II

Plan:

Feedback on the quiz

Part I — Mean-Field Approximation

Part II — Landau Theory

Tutorial: more on Mean-Field approximation

During this lecture, we will use the mean-field approximation to solve various problem.
we will come back on the approximation in more details in the tutorial.

Questions: Any question about the homework?

Quiz 2 next week: Wednesday afternoon?

Expectations: Participate in the discussions, take notes, try to do the analytical derivations

References: Book "Complexity and Criticality", K. Christensen, N. Moloney, **Chapter 1 and 2**

Finding the exponents (cf. exercises)

Self-consistency relation: m is solution of: $$m = \tanh\left[\frac{T_c}{T}m + \frac{H}{k_B T}\right] \quad (1)$$

Free energy: $$f = \frac{k_B T_c}{2}m^2 - k_B T \log\left[2\cosh\left(\frac{T_c}{T}m + \frac{H}{k_B T}\right)\right] \quad (2)$$

Exposant	Champ moyen
α	discont.
β	0.5
γ	1
δ	3

- $m_0 \sim (T_c - T)^\beta \quad (T < T_c)$ **for *H*=0 ?**
  - Start from Eq. (1) with H=0
  - Expand for small m (as m is continuous and *m*=0 at *Tc*)
- **Susceptibility per spin:** $\chi \sim |T - T_c|^{-\gamma}$ **?** $\quad \chi = \lim_{H\to 0}\left(\frac{\partial m}{\partial H}\right)_T$
  - Start from Eq. (1) with H non 0
  - Apply derivative on both sides of Eq.(1), then take the limit ***H*=0**
  - Expand for small m
- **Heat capacity:** $C \sim |T - T_c|^{-\alpha}$ **?** $\quad C = -T\left(\frac{\partial^2 f}{\partial T^2}\right)_{H=0}$
  - Start from Eq. (2) with H=0
  - Replace **m** by its expression **m(T) for H=0. Treat T>Tc and T<Tc separately**
  - Derive twice by ***T***
  - Expand for small m
- **magnetization** $m \sim H^{1/\delta}$ **at *T*=*Tc* and *H* small ?**
  - Start from Eq. (1) with *T*=*Tc*
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(as m is small, for T=Tc and H small)

Close to T_C, m is very small !

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m is solution of: $\dfrac{\partial f}{\partial m} = 0$

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  - Expand for small m (as m is small, for T=Tc and H small)

Close to T_C, m is very small !

Everything can be obtained from knowing *f (m; T,H)*

Expansion of the Free-energy to 4-th order in m

Free energy: $$f = \frac{k_B T_c}{2} m^2 - k_B T \log\left[2\cosh\left(\frac{T_c}{T} m + \frac{H}{k_B T}\right)\right] \qquad \text{(2)}$$

Behavior at $(T, H) \to (T_c, 0)$ **?**

Useful expansions: $\log(1+x) = x - \frac{x^2}{2} + \frac{x^4}{4} + o(x^4)$ $\quad \cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)$ $\Rightarrow$ $\log[\cosh(x)] = \frac{x^2}{2} - \frac{x^4}{12} + o(x^4)$

Expansion at 4-th order in *m*:

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Expansion at 4-th order in *m*:

$$f = f_0 - Hm + \frac{k_B}{2}(T - T_c) m^2 + \frac{k_B T}{12} m^4 + o(m^4)$$

where $f_0 = -k_B T \log(2)$ pure entropy-part of the free energy

remaining term when T —> + infinity

Rem: we neglected terms in $H^2, Hm^3, H^2m^2, H^3m, H^4$ which are all $o(m^4)$

as $H \sim m^3$

Expansion of the Free-energy to 4-th order in m

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$$f = f_0 - Hm + \frac{k_B}{2}(T - T_c) m^2 + \underbrace{\frac{k_B T}{12} m^4}_{} + o(m^4)$$

Behavior of the function *f*
for *m* —> +/- *infinity*

Expansion of the Free-energy to 4-th order in *m*

Free energy: $$f = \frac{k_B T_c}{2} m^2 - k_B T \log\left[2\cosh\left(\frac{T_c}{T} m + \frac{H}{k_B T}\right)\right] \qquad \textbf{(2)}$$

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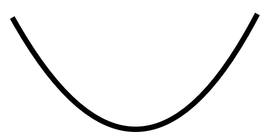
Expansion at 4-th order in *m*:

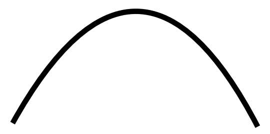
$$f = f_0 - Hm + \boxed{\frac{k_B}{2}(T - T_c)m^2} + \underline{\frac{k_B T}{12}} m^4 + o(m^4)$$

For H=0

Behavior of the function *f* near ***m = 0***



(T - Tc) > 0



(T - Tc) < 0

Behavior of the function *f* for ***m*** —> +/- ***infinity***

Expansion of the Free-energy to 4-th order in *m*

Free energy: $$f = \frac{k_B T_c}{2} m^2 - k_B T \log\left[2\cosh\left(\frac{T_c}{T} m + \frac{H}{k_B T}\right)\right] \qquad (2)$$

Behavior at $(T, H) \to (T_c, 0)$ **?**

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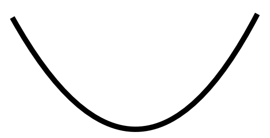

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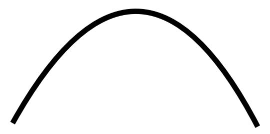


(T - Tc) < 0

Behavior of the function *f* for ***m* —> +/- *infinity***

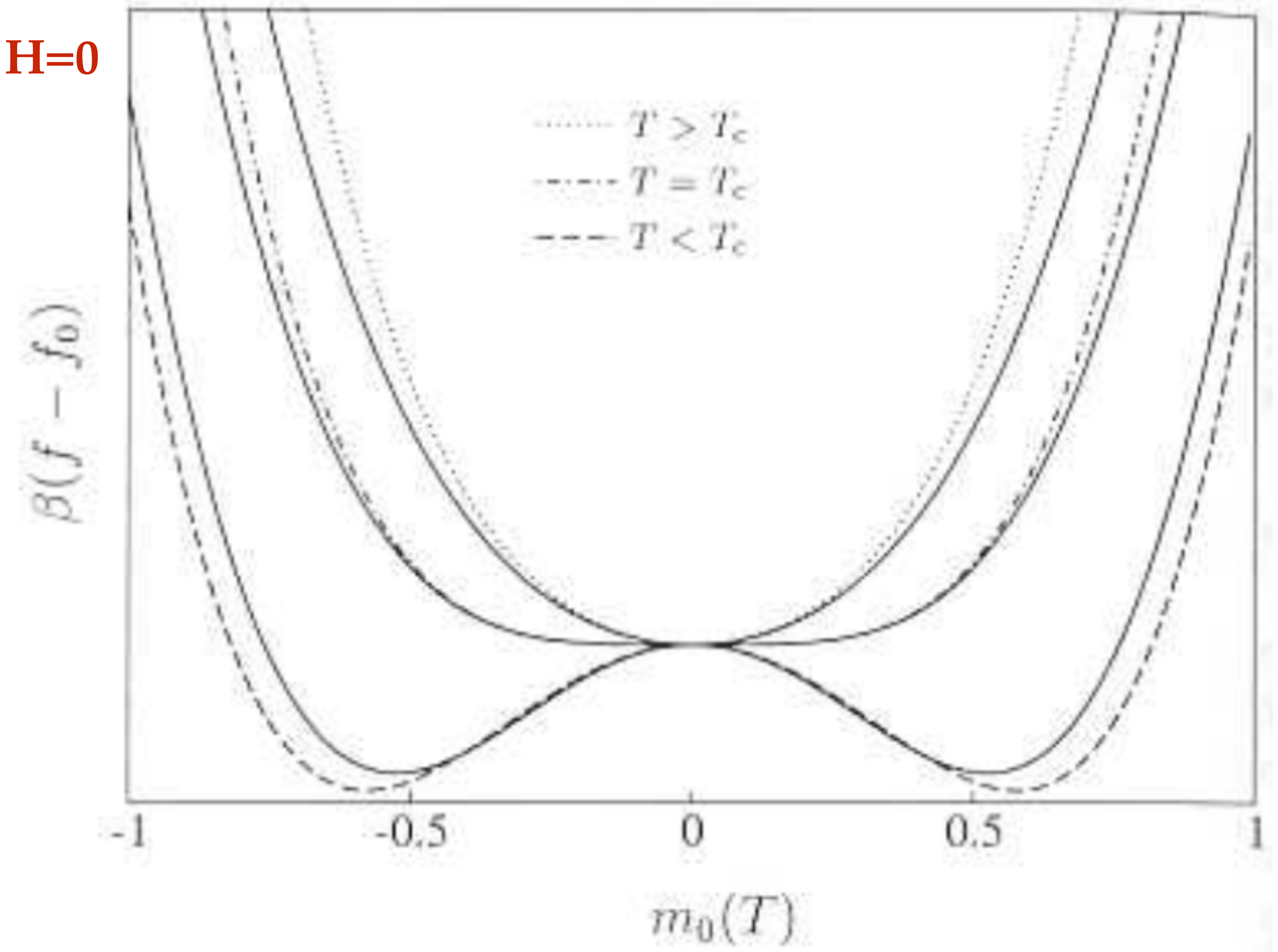
At 4-th order, there is still the transition:

If we had cut at order 2, there will be no transition

+ issues, as for T<Tc, f would be going to -infinity

—— **Mean-field** ------ **After expansion**

For H=0



Expansion of the Free-energy to 4-th order in m

Free energy: $$f = \frac{k_B T_c}{2} m^2 - k_B T \log\left[2\cosh\left(\frac{T_c}{T} m + \frac{H}{k_B T}\right)\right] \qquad (2)$$

Behavior at $(T, H) \to (T_c, 0)$ **?**

Useful expansions: $\log(1+x) = x - \frac{x^2}{2} + \frac{x^4}{4} + o(x^4)$ $\quad \cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)$ $\quad \Rightarrow \quad \log[\cosh(x)] = \frac{x^2}{2} - \frac{x^4}{12} + o(x^4)$

Expansion at 4-th order in *m*:

$$f = f_0 - Hm + \underbrace{\frac{k_B}{2}(T - T_c)m^2}_{} + \underbrace{\frac{k_B T}{12} m^4}_{} + o(m^4)$$

Behavior of the function *f* near *m* = 0

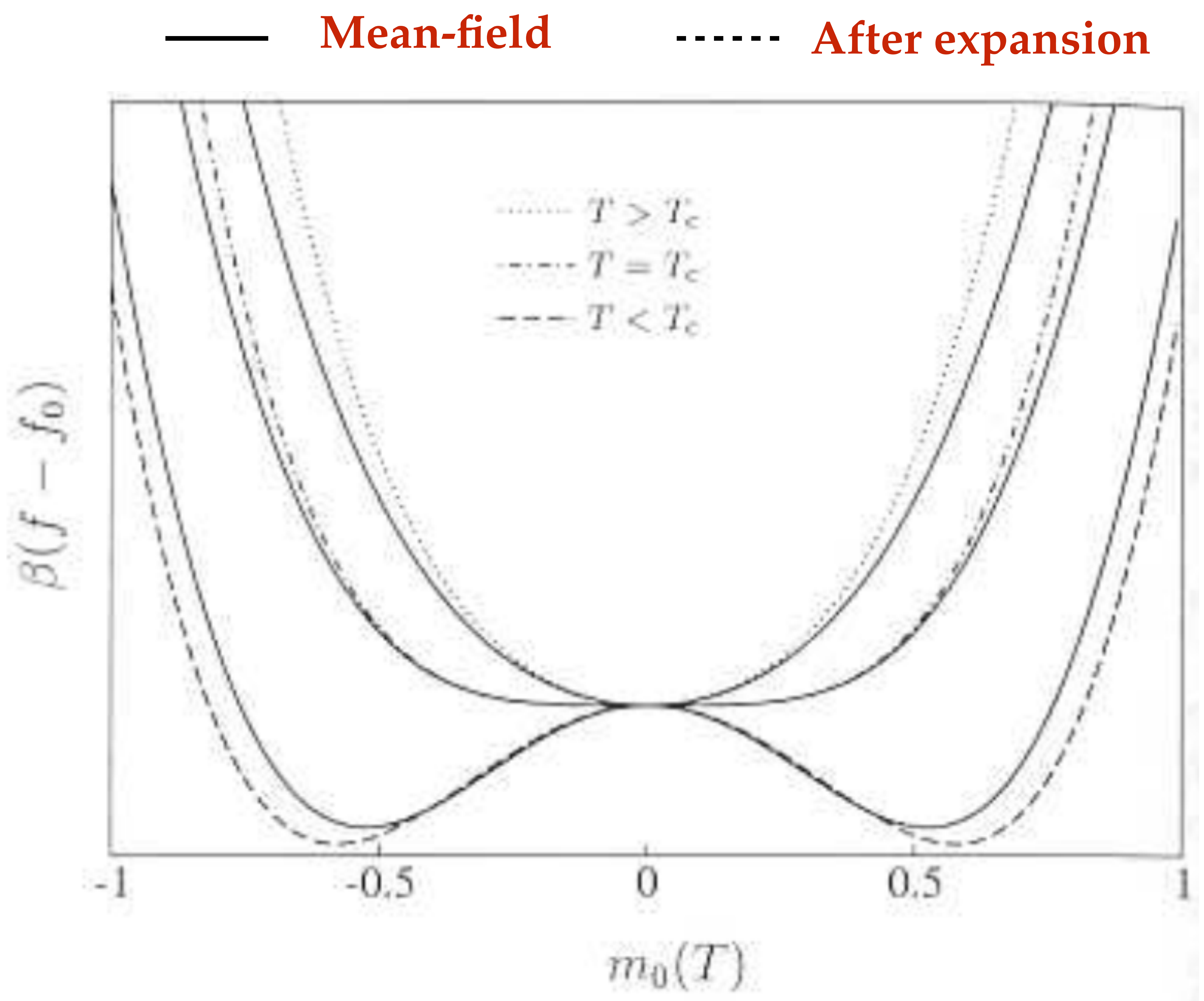
Behavior of the function *f* for *m* —> +/- *infinity*

Landau approximation of the Ising model:

corresponds to a 4-th order expansion of the free energy per spin in the order parameter.

Expansion, but:

preserves **all the information** required to **extract the critical exponents** that determine the behavior of the mean-field Ising model **close to the critical point *(T, H) = (Tc, 0)***



Expansion of the Free-energy to 4-th order in *m*

Expansion at 4-th order in *m*: for $(T, H) \to (T_c, 0)$

$$f = f_0 - Hm + \underline{a_2(T - T_c)}m^2 + \underline{a_4}m^4 + o(m^4)$$

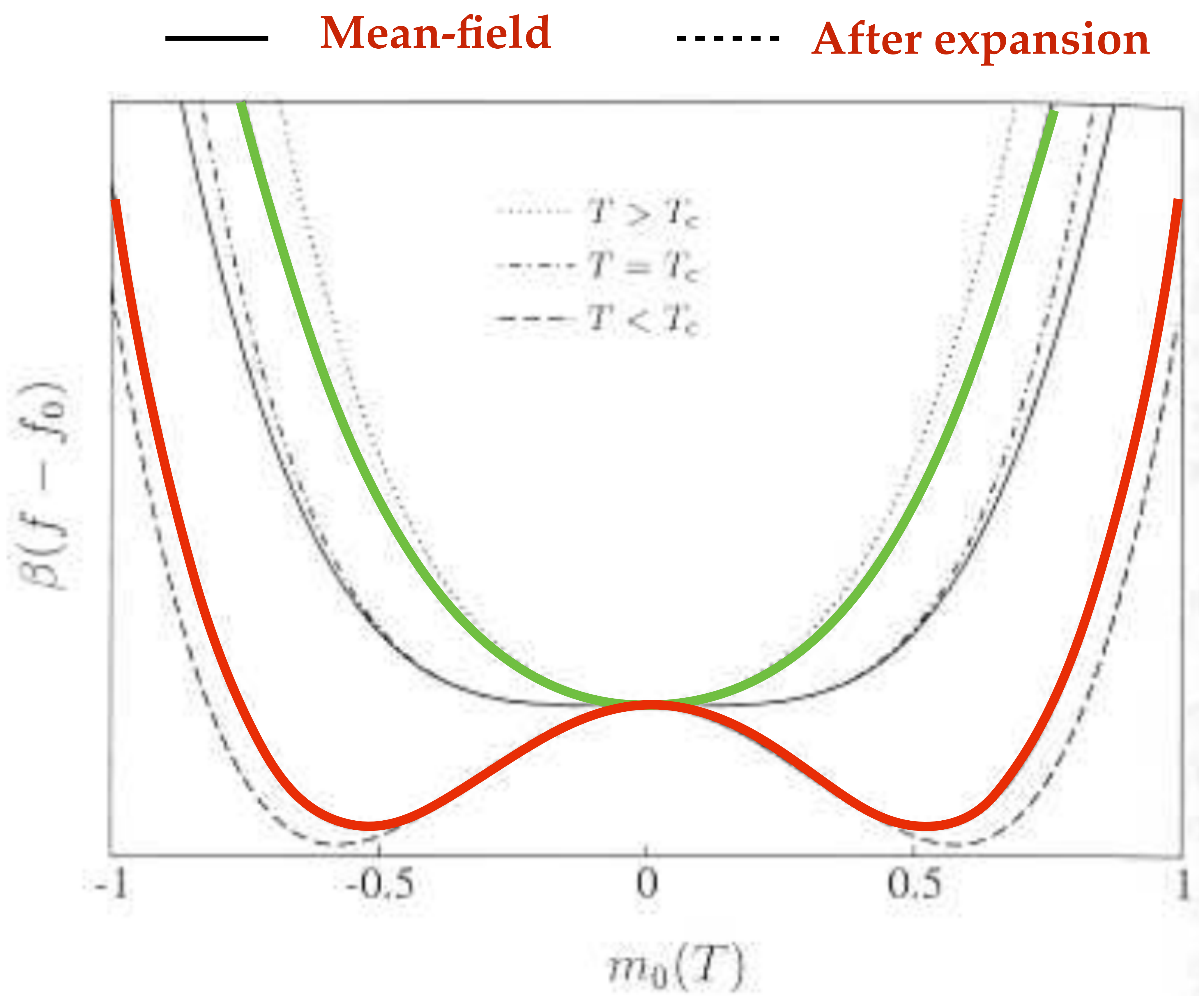
positive if T > Tc
negative if T < Tc

positive

$$f_0 = -k_B T \log(2) \qquad a_2 = \frac{k_B}{2} \qquad a_4 = \frac{k_B T}{12} > 0$$

f is bounded from under: there is a minimum

Play with the little app.



Critical exponents??

Expansion at 4-th order in *m*: for $(T, H) \to (T_c, 0)$

$$f = f_0 - Hm + a_2(T - T_c)m^2 + a_4 m^4 + o(m^4)$$

$$f_0 = -k_B T \log(2) \qquad a_2 = \frac{k_B}{2} \qquad a_4 = \frac{k_B T}{12}$$

Self-consistency relation: m is solution of: $\frac{\partial f}{\partial m} = 0$

$$-H + 2\,a_2 (T - T_c)\,m + 4\,a_4\,m^3 = 0$$

H=0

Check that we recover the exponents:

- $m_0 \sim (T_c - T)^\beta \qquad (T < T_c)$ **for *H=0*** **?**

$$m_0^2 = -\frac{a_2}{2\,a_4}(T - T_c) \qquad \beta = 1/2$$

Critical exponents??

Expansion at 4-th order in *m*: for $(T, H) \to (T_c, 0)$

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Self-consistency relation: m is solution of: $\dfrac{\partial f}{\partial m} = 0$

$$-H + 2\,a_2(T - T_c)\,m + 4\,a_4\,m^3 = 0$$

$$\lim_{H\to 0}\left(\frac{\partial[\cdot]}{\partial H}\right)_T$$

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- $m_0 \sim (T_c - T)^\beta \qquad (T < T_c)$ **for *H*=0** **?**

$$m_0^2 = -\frac{a_2}{2\,a_4}(T - T_c) \qquad \beta = 1/2$$

- **Susceptibility per spin:** $\chi \sim |T - T_c|^{-\gamma}$ **?**

$$\chi = \lim_{H\to 0}\left(\frac{\partial m}{\partial H}\right)_T$$

$$-1 + 2\,a_2(T - T_c)\,\chi + 12\,a_4\,m_0^2\,\chi = 0$$

$$\chi = \frac{1}{2\,a_2(T - T_c) + 12\,a_4\,m_0^2}$$

$$\chi = \begin{cases} \frac{1}{k_B}(T - T_c)^{-1} & \text{for } T \to T_c^+ \\ \frac{1}{2k_B}(T_c - T)^{-1} & \text{for } T \to T_c^- \end{cases}$$

Etc. We obtain the same results, as exponents were previously already obtained by taking the expansion in *m*

Landau Theory
for continuous phase transitions

Description of continuous phase transition (Lev Landau 1937) [Landau, Lifshitz, Pitaevskij. Statistical physics]

Since the order parameter grows continuously from zero at the critical temperature,

Landau suggested that, if the free energy is analytic (as a function of the order parameter) near the critical point,

then the **free energy can be expanded as a Taylor series in terms of the order parameter** which is small near the PT

this expansion would tell us about the behavior near the transition.

$$f(T, H; \eta) = \sum_{k=0}^{\infty} a_k(T, H)\, \eta^k \qquad \text{for } \; T \to T_c, H \to H_c$$

control parameters order parameter

Use symmetry arguments to constrains the $a_k(T, H)$

Ex. in the Ising model, ***at H=0***: $f(T, 0; -m) = f(T, 0; m)$ f is even —>>> $a_k(T, 0) = 0$ for k odd

Close to the PT: m is small —>> **high-order terms** are **negligible**

$$f(T, 0; m) = a_0(T, 0) + a_2(T, 0)\, m^2 + a_4(T, 0)\, m^4$$

Simplest expansion that would still have a PT

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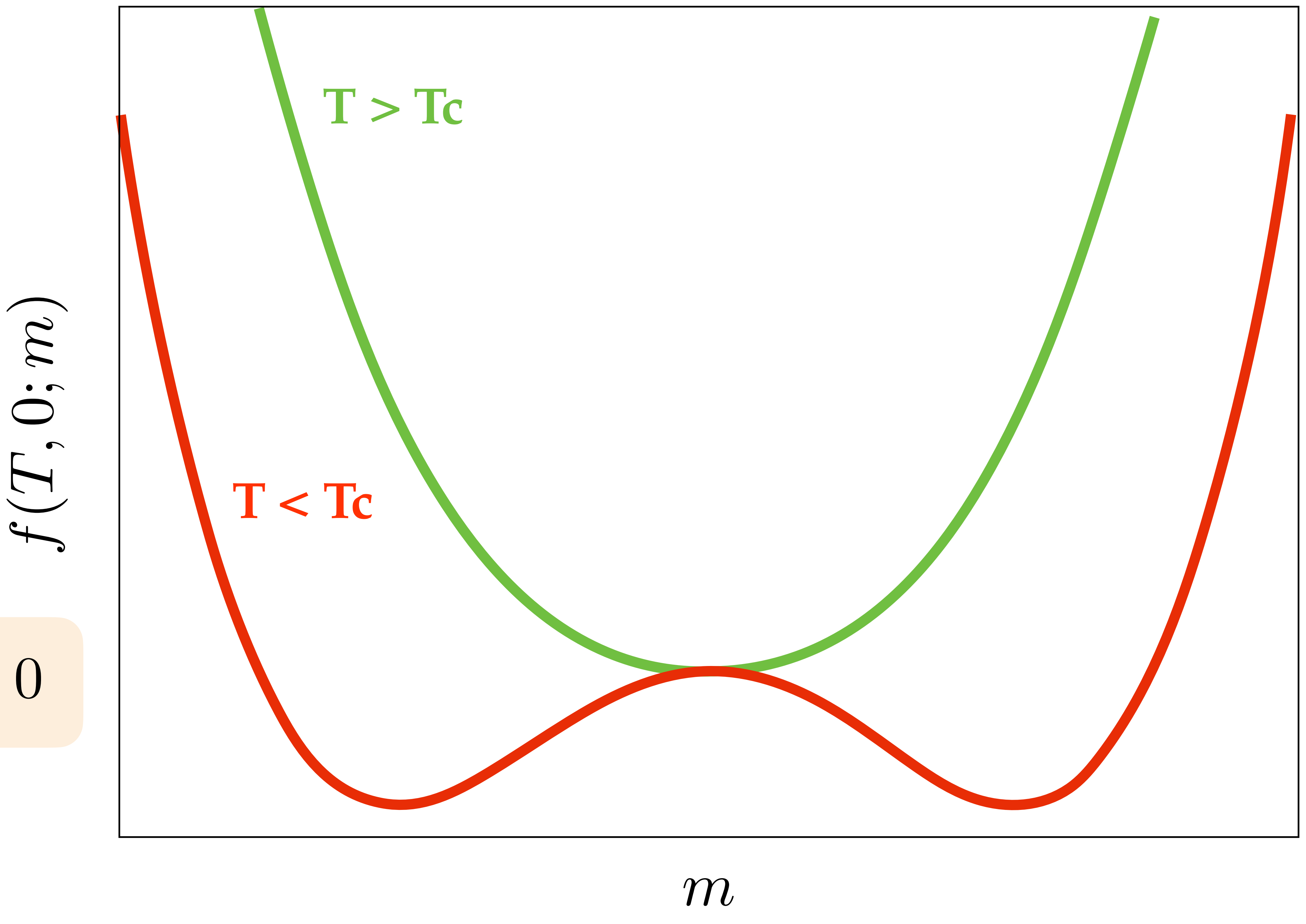
Description of continuous phase transition (Lev Landau 1937)

$$f(T,0;m) = a_0(T,0) + a_2(T,0)\,m^2 \; + a_4(T,0)\,m^4$$

Simplest expansion that would still have a PT

$a_4(T,0) > 0$ Expansion must **stop at even terms** with **positive coefficient** so that ***f(T, 0; m)*** has a minimum



$a_2(T,0) > 0$ for $T > T_c$

—>> at ***T > Tc*** there is only one minimum, at **m=0**

$a_2(T,0) < 0$ for $T < T_c$

—>> at ***T < Tc*** there are two non-zero symmetric minima

—>> $a_2(T,0) = 0$ at $T = T_c$

$a_2(T,0) = \tilde{a}_2(T - T_c)$ with $\tilde{a}_2 > 0$

For T > Tc, the only terms that remains is $a_0(T,0) = f_0(T)$

Entropic part of the free energy

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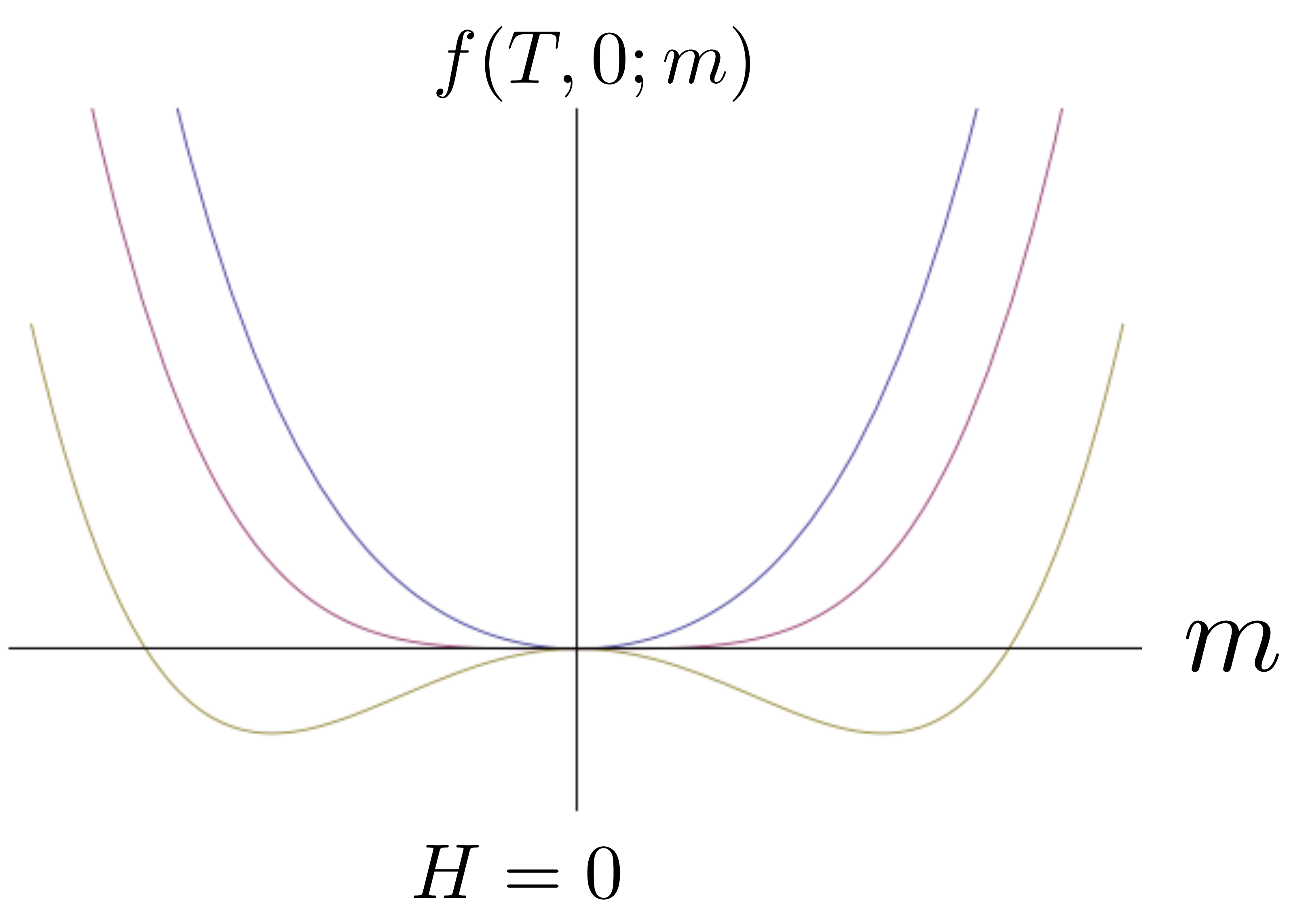
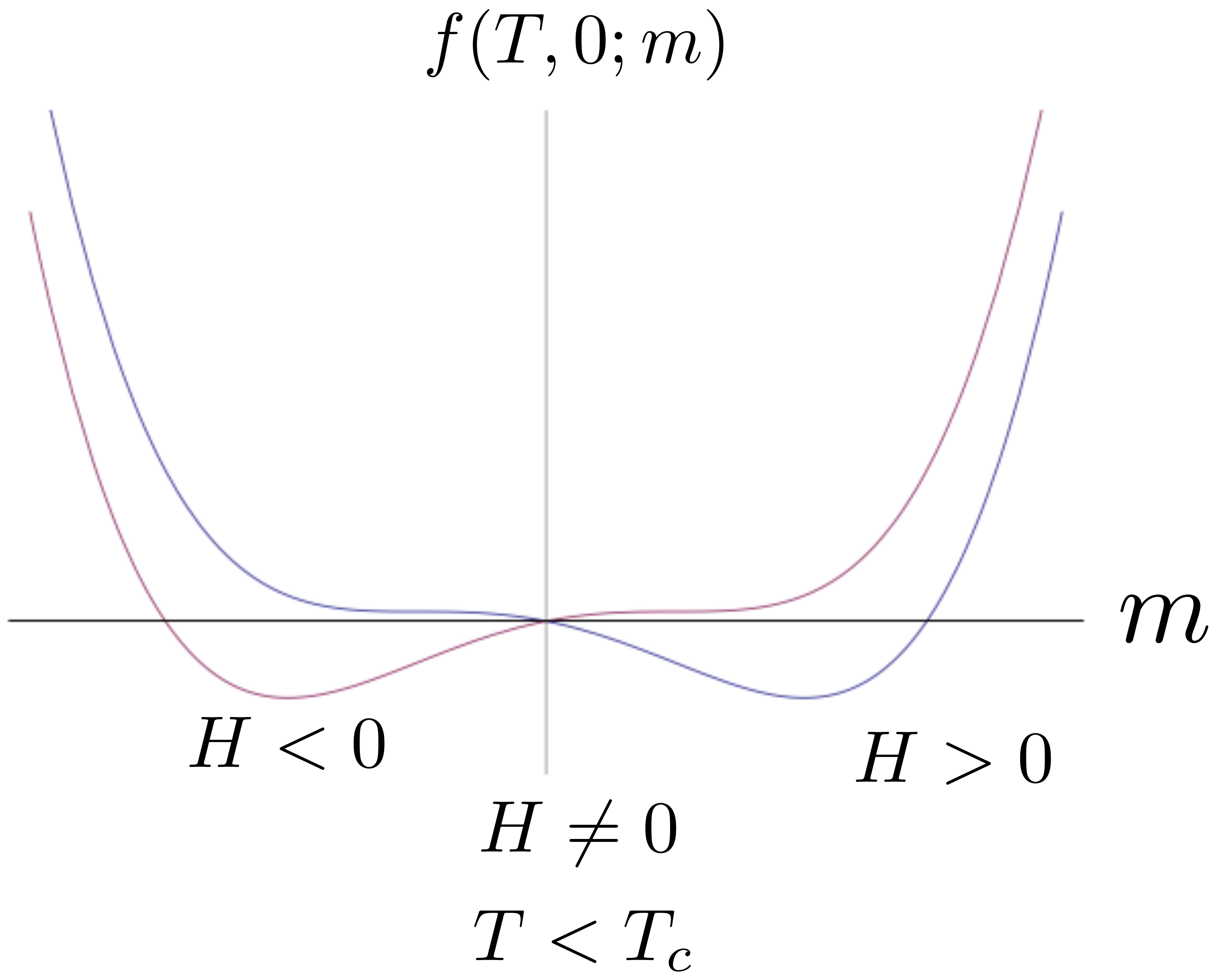
for continuous phase transitions

Description of continuous phase transition (Lev Landau 1937):

When *H=0*: $f(T,0;m) = f_0(T) + \tilde{a}_2(T - T_c)\, m^2 \; + a_4(T,0)\, m^4$

With an external field *H* : $f(T,0;m) = f_0(T) - Hm + \tilde{a}_2(T - T_c)\, m^2 \; + a_4(T,0)\, m^4$





Landau Theory
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Description of continuous phase transition (Lev Landau 1937):

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With an external field *H* : $$f(T,0;m) = f_0(T) - Hm + \tilde{a}_2(T - T_c)\, m^2 \ + a_4(T,0)\, m^4$$

This is the same expansion as the one we obtained previously from Expansion of the Mean-field Ising:

$$f = f_0 - Hm + \underline{a_2(T - T_c)}m^2 + \underline{a_4}m^4 + o(m^4)$$

$$f_0 = -k_B T \log(2)$$

positive if T > Tc

negative if T < Tc

positive

We can recover all the critical exponents

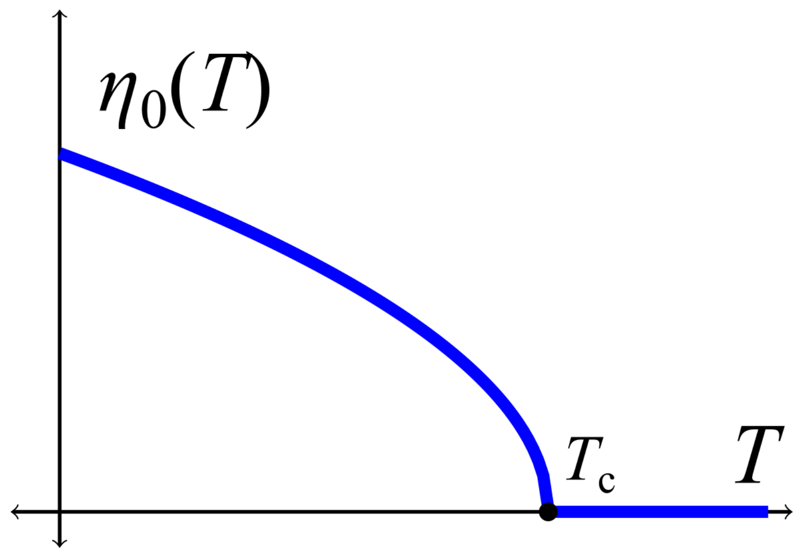
Description of continuous phase transition (Lev Landau 1937):

With an external field H : $f(T,0;m) = f_0(T) - Hm + \tilde{a}_2(T-T_c)\,m^2 + a_4(T,0)\,m^4$

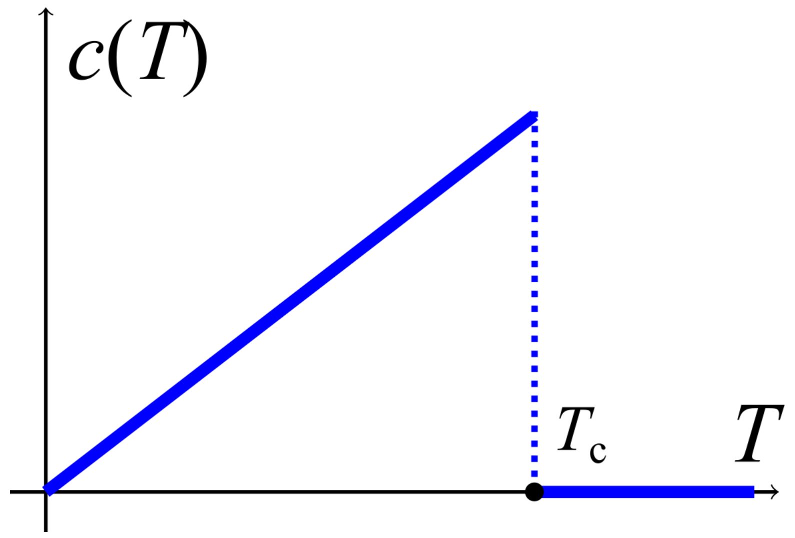
$$F - F_0 = \begin{cases} -\dfrac{a_0^2}{2b_0}(T-T_c)^2, & T < T_c \\ 0, & T > T_c \end{cases}$$

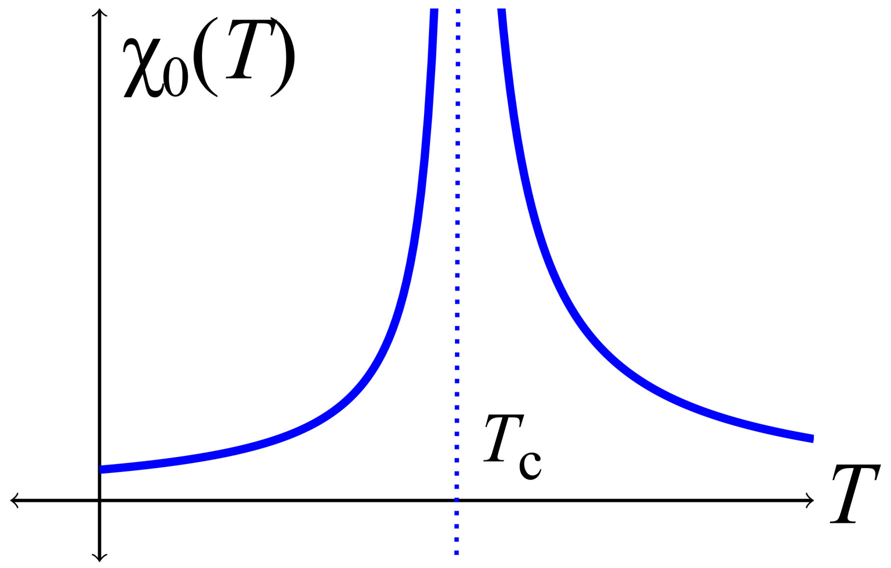
$$\eta_0^2 = -\frac{a}{b} = -\frac{a_0}{b_0}(T-T_c)$$

$$\eta(T) \propto |T-T_c|^{1/2}$$



$$c_p = -T\frac{\partial^2 F}{\partial T^2} = \begin{cases} \dfrac{a_0^2}{b_0}T, & T < T_c \\ 0, & T > T_c \end{cases}$$





$$\chi(T, h \to 0) = \begin{cases} \frac{1}{2a_0(T-T_c)}, & T > T_c \\ \frac{1}{-4a_0(T-T_c)}, & T < T_c \end{cases} \propto |T-T_c|^{-\gamma}$$

Validity of the theory: Ginzburg criterion

The mean-field approximation assumes **fluctuations** of the energy can be **neglected**

$$E(\vec{s}) = E_0(\vec{s}) + \Delta E(\vec{s})$$

where

$$\Delta E(\vec{s}) = -\frac{Jq}{2}\sum_{i=1}^{N}(s_i - \langle s_i\rangle)(s_j - \langle s_j\rangle)$$

$$|\langle \Delta E(\vec{s})\rangle| = \frac{Jq}{2}\sum_{i=1}^{N} g(s_i, s_j) = \frac{Jq}{2}k_B T\chi$$

——>>>> Fluctuations are negligible if: $|\langle \Delta E(\vec{s})\rangle| \ll |\langle E_0(\vec{s})\rangle|$

—>> Ginzburg criterion: $\sqrt{\langle M^2\rangle - \langle M\rangle^2|} \ll \langle M\rangle$

—>> In the case of the mean-field Ising: **Gives d > 4**

Exponent agreement between **mean-field** and **d > 4** is a result of the **"unimportance" of fluctuations in higher dimensions.**

Universality

Critical Temperatures are Non-Universal:

Lattice	z	$k_B T_c / J$
$d = 1$ line	2	0
$d = 2$ hexagonal	3	$2/\ln(2+\sqrt{3})$ [a]
square	4	$2/\ln(1+\sqrt{2})$ [b] ≈ 2.269185
triangular	6	$4/\ln 3$ [a]
$d = 3$ diamond	4	2.70 [c]
simple cubic	6	4.51152 [d]
body-centred cubic	8	6.40 [e]
face-centred cubic	12	9.79 [e]
Mean-field	z	z

Critical Exponents are Universal:

$$C \sim |T - T_c|^{-\alpha} \qquad \chi \sim |T - T_c|^{-\gamma}$$

$$m_0 \sim (T_c - T)^{\beta} \quad (T < T_c) \qquad m \sim H^{1/\delta} \quad (T = T_c)$$

Exponents	$d = 2$	$d = 3$	$d >= 4$	Mean-field
α	$\ln\|T - T_c\|$	0.01 ± 0.01	0	0 (discont.)
β	0.125	0.312 ± 0.003	0.5	0.5
γ	1.75	1.250 ± 0.002	1	1
δ	15 (*)	5.0 ± 0.05	3	3

Upper critical dimension = 4

Critical exponents for ***d* >= 4** remains unchanged

Exponents of Mean-field Ising are the same than for d >= 4

Summary

Mean-field approximation: the theory assume that we **neglect the fluctuations**

Landau's theory: — is a Mean-field theory

— gives a **good qualitative description** of the phase transitions

— quantitatively it was inconsistent with experiments

Mean-field approximation: fluctuations of the order parameters are neglected **—>>** not a good idea when close to critical point!

Valid above a critical dimension ***dc* = *4*** **("unimportance" of fluctuations in higher dimensions)**

Below *dc*, fluctuations can't be neglected: **statistical field theory** **—>>** **Ginzburg-Landau theory** Ginzburg-Landau ϕ^4 model

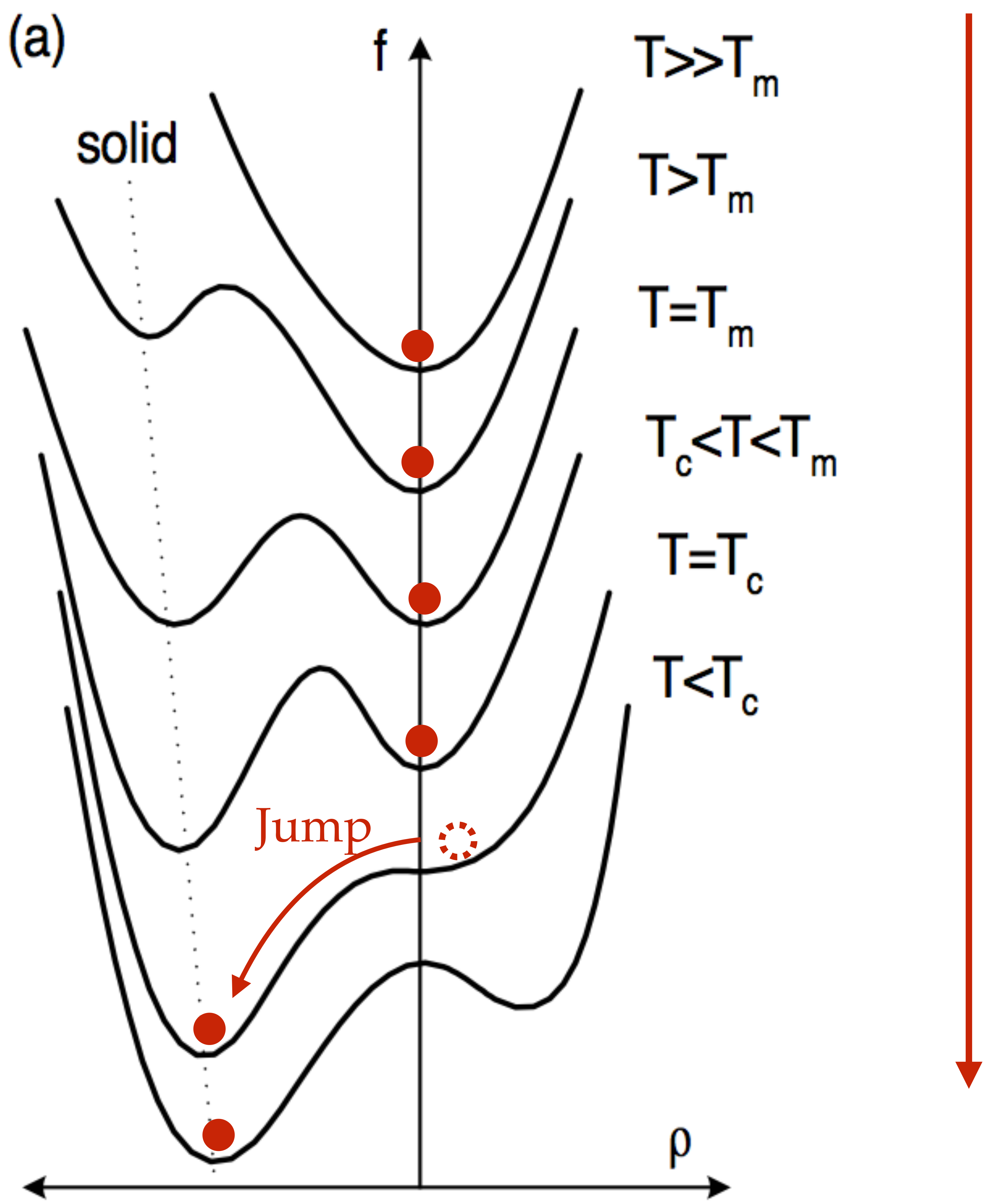
[Play with the little app.]

Landau Theory
for discontinuous phase transitions

Liquid to Solid water:

$$f - f_0 = a(T - T_c)\rho^2 + c\rho^3 + \frac{1}{2}b\rho^4$$

All coefficients are **positive**



Play with the little app.

Landau Theory for discontinuous phase transitions

Other example: symmetric function

$$f(T) = f_0(T) + \alpha_0 (T - T_c) m^2 + \frac{1}{2}\beta m^4 + \frac{1}{3}\gamma m^6$$

$$\alpha_0 > 0 \qquad \beta < 0 \qquad \gamma > 0$$

(b)