Renormalization

Chapter 5

Wednesday 8 May

Before we start:

This week and next week:

	L1	L2
	Quiz: Tuesday May 21 at 11h + lecture	Guest Lecture

Greg Stephens, Theoretical Biophysics

	Monday 27 May: Preparation	Exam:

Issue about schedule of the exam?
Except for *Machine learning for physicists*

Questions: Information about the last quiz?

Scale Invariance and Universality

Chapter 5

Plan: **Lecture 1: Emergent behaviors in the simple example of random walks: scale-invariance and universality**

Lecture 2: Renormalization (next Thursday)

Tutorial: Random walks + (optional) renormalization

Renormalization

Chapter 5. L2

Plan: 1) Introduction: Scale-invariance and idea of real-space renormalisation

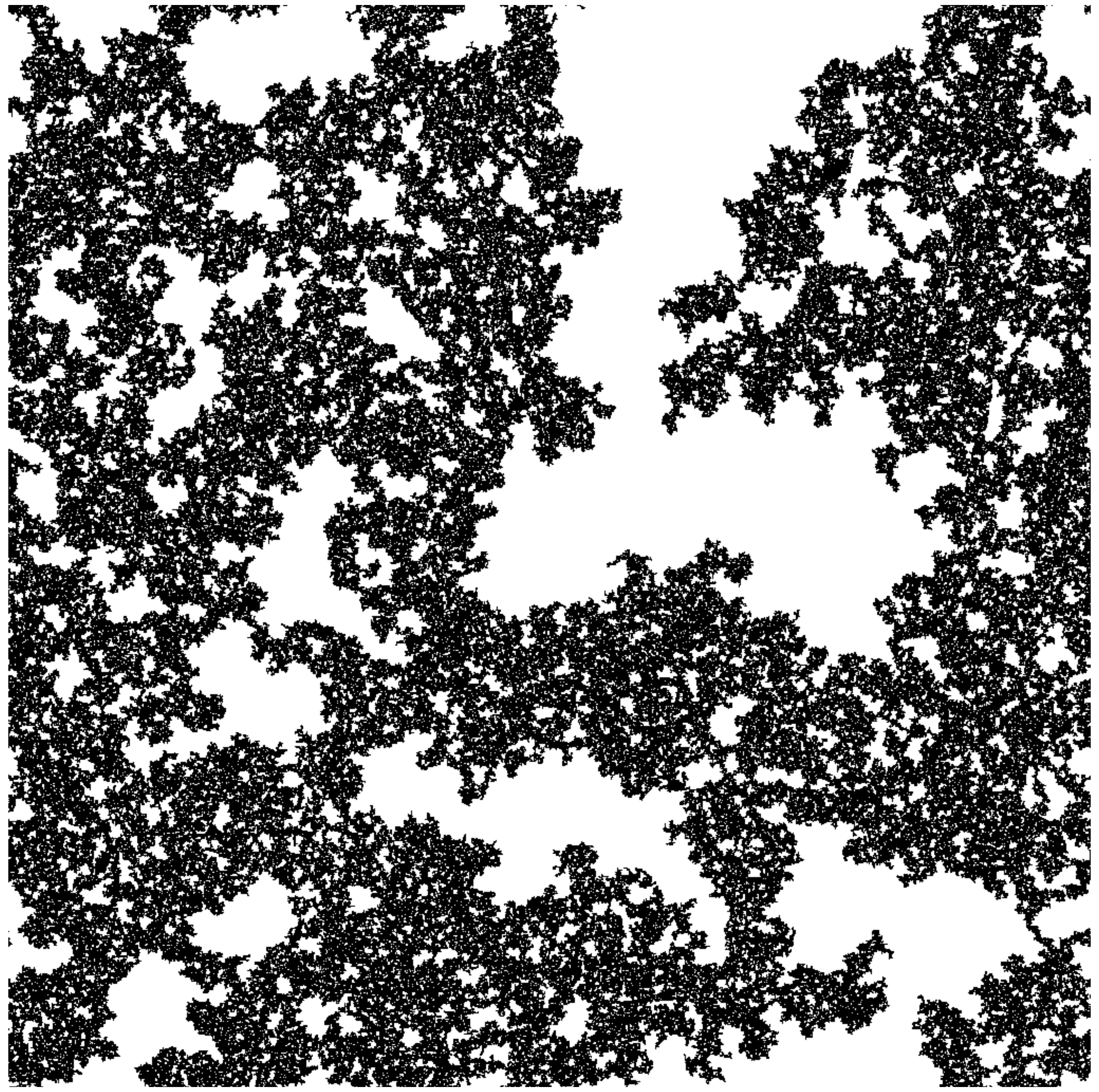
2) Renormalisation in the 1D Ising model

3) Renormalisation in data

Introduction

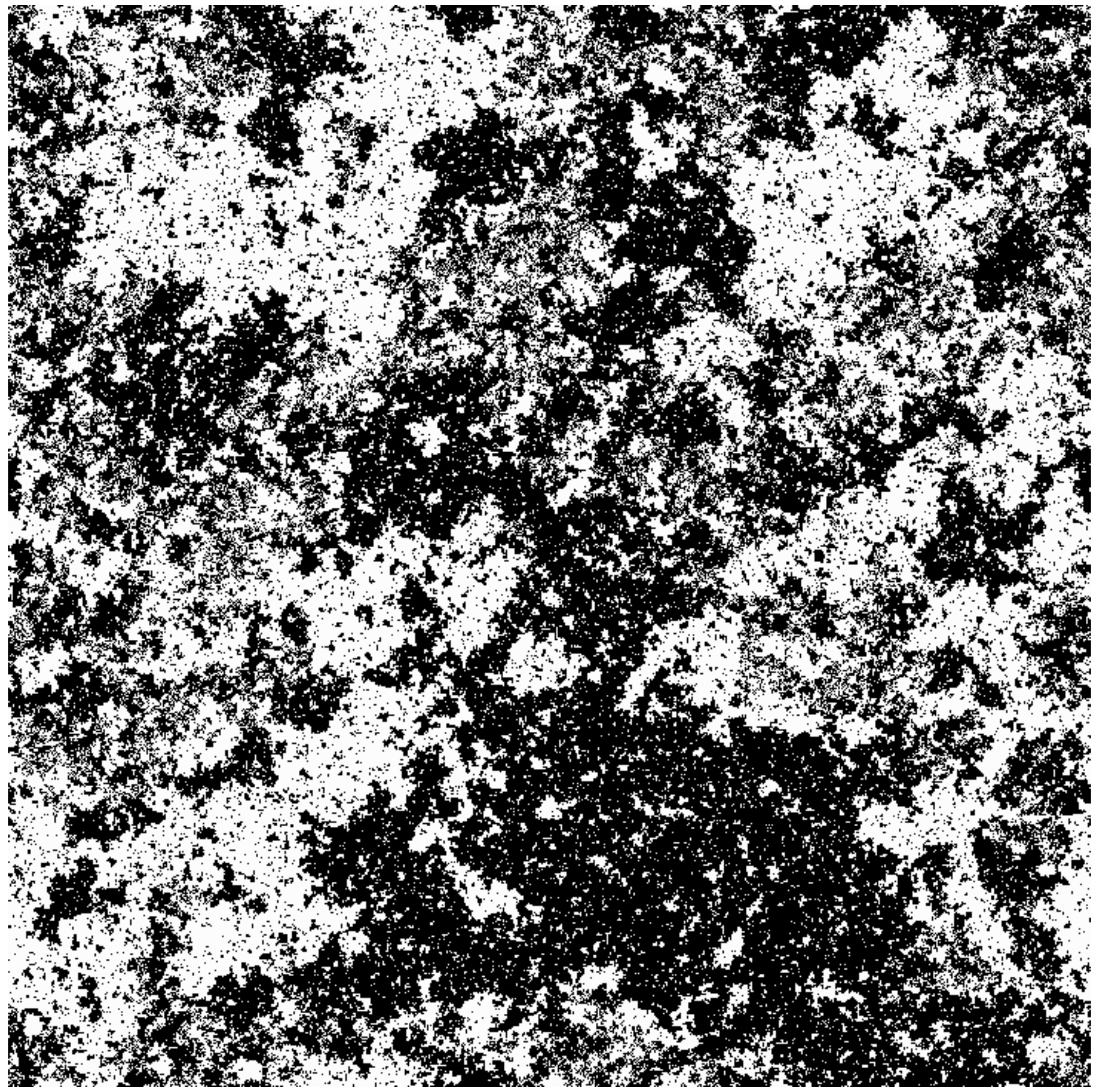
Can you recall what is the meaning of **scale invariance?**

Properties of Critical phenomena: Scale Invariance



Bond percolation

at p = pc = 0.5



Ising model

at T = Tc ~ 2.269

Close to criticality:

Diverging correlation length —> percolating cluster

Cluster of all sizes

Diverging correlation length

Fluctuation of the magnetization at all scales

Scale invariance: **the system "looks the same" at all scales (i.e. when zooming in)**

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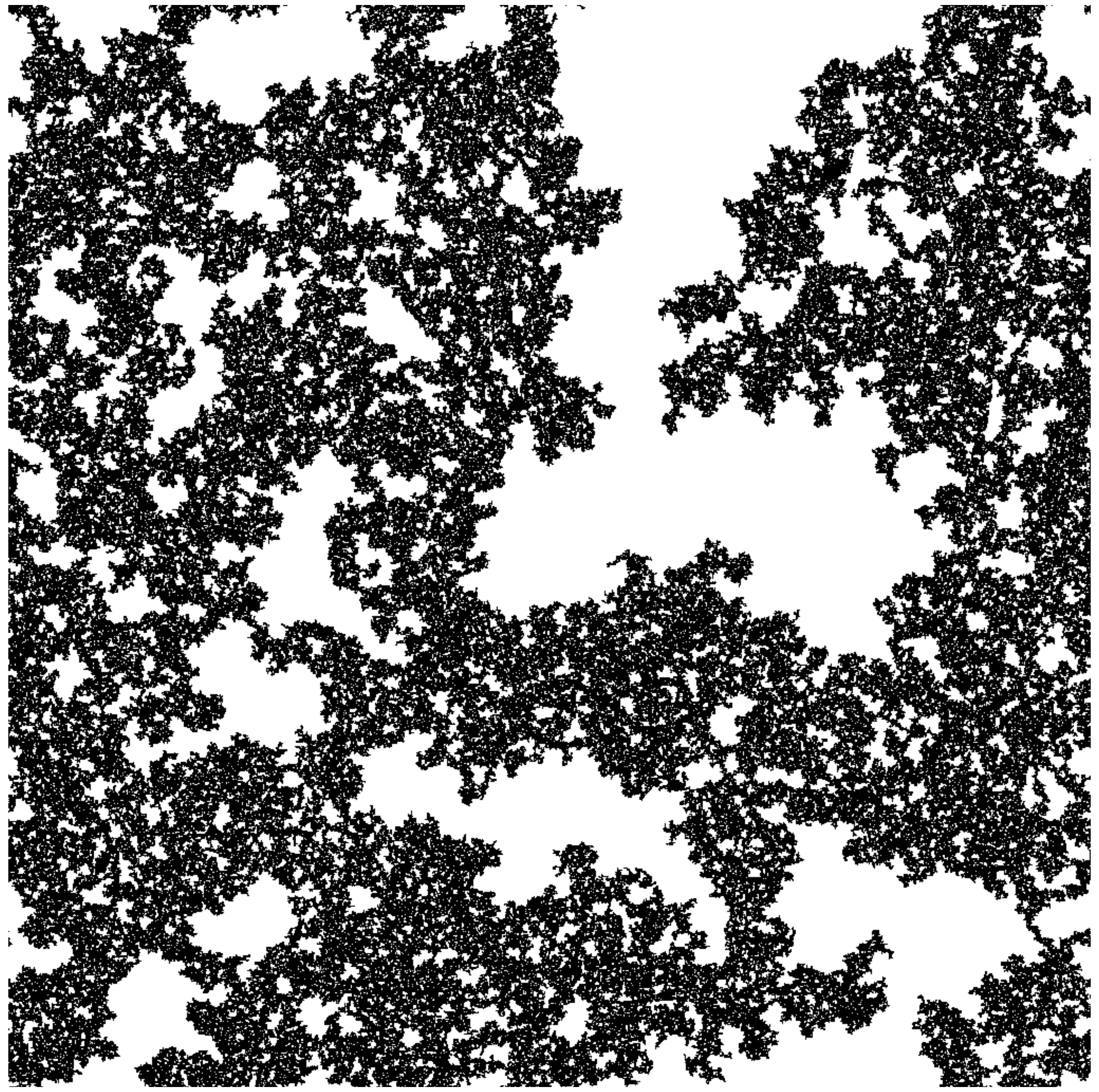

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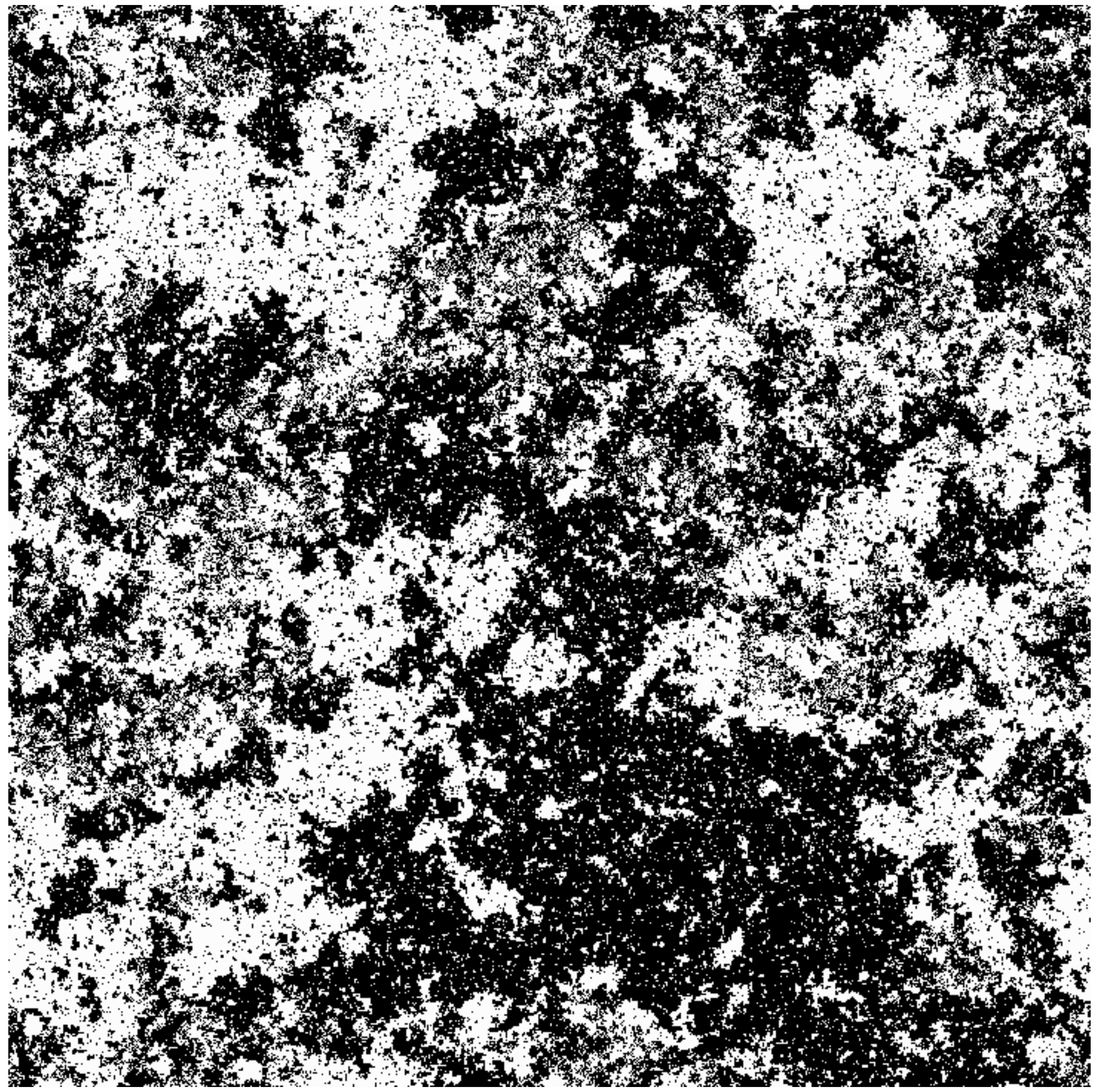

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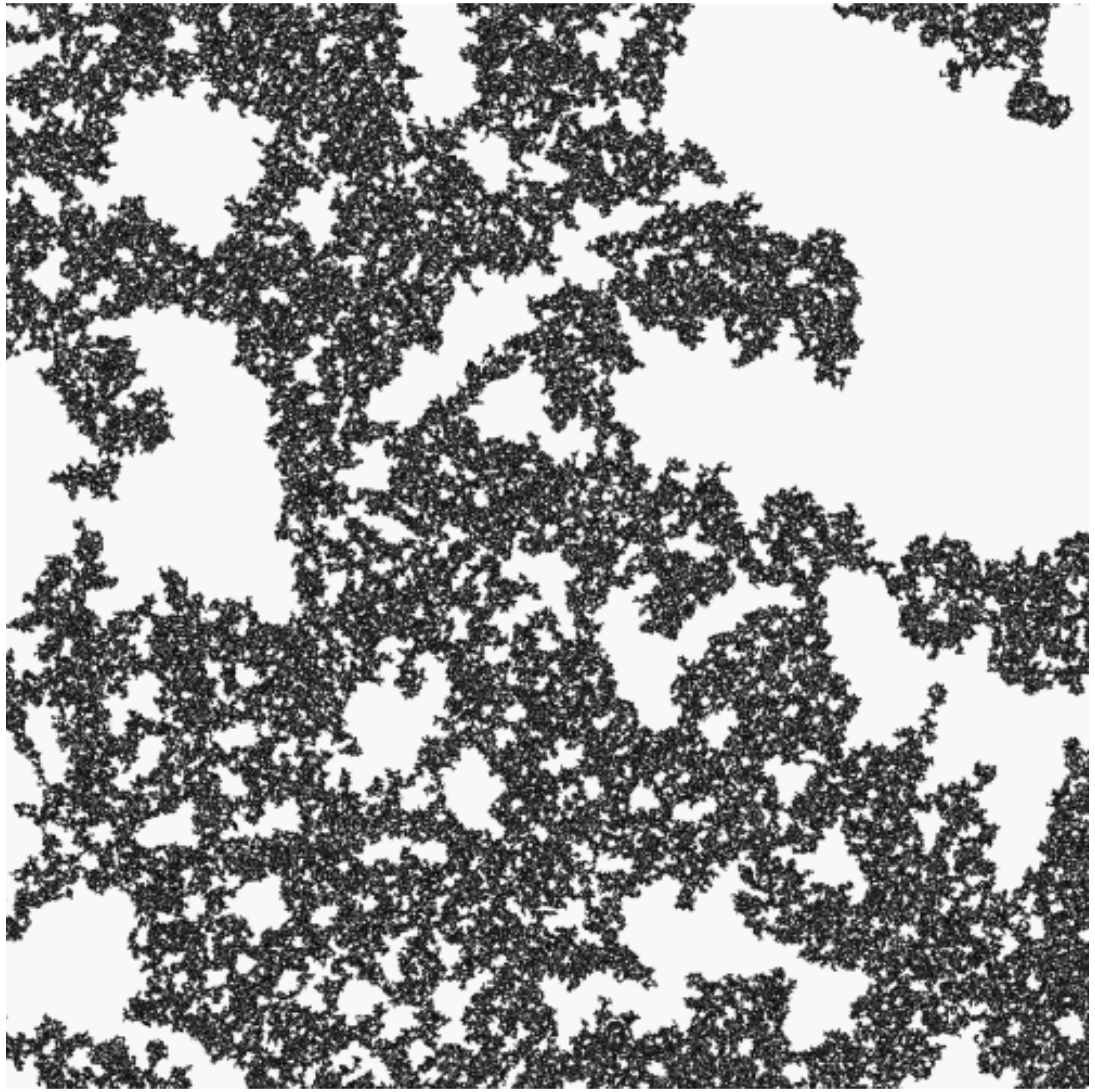
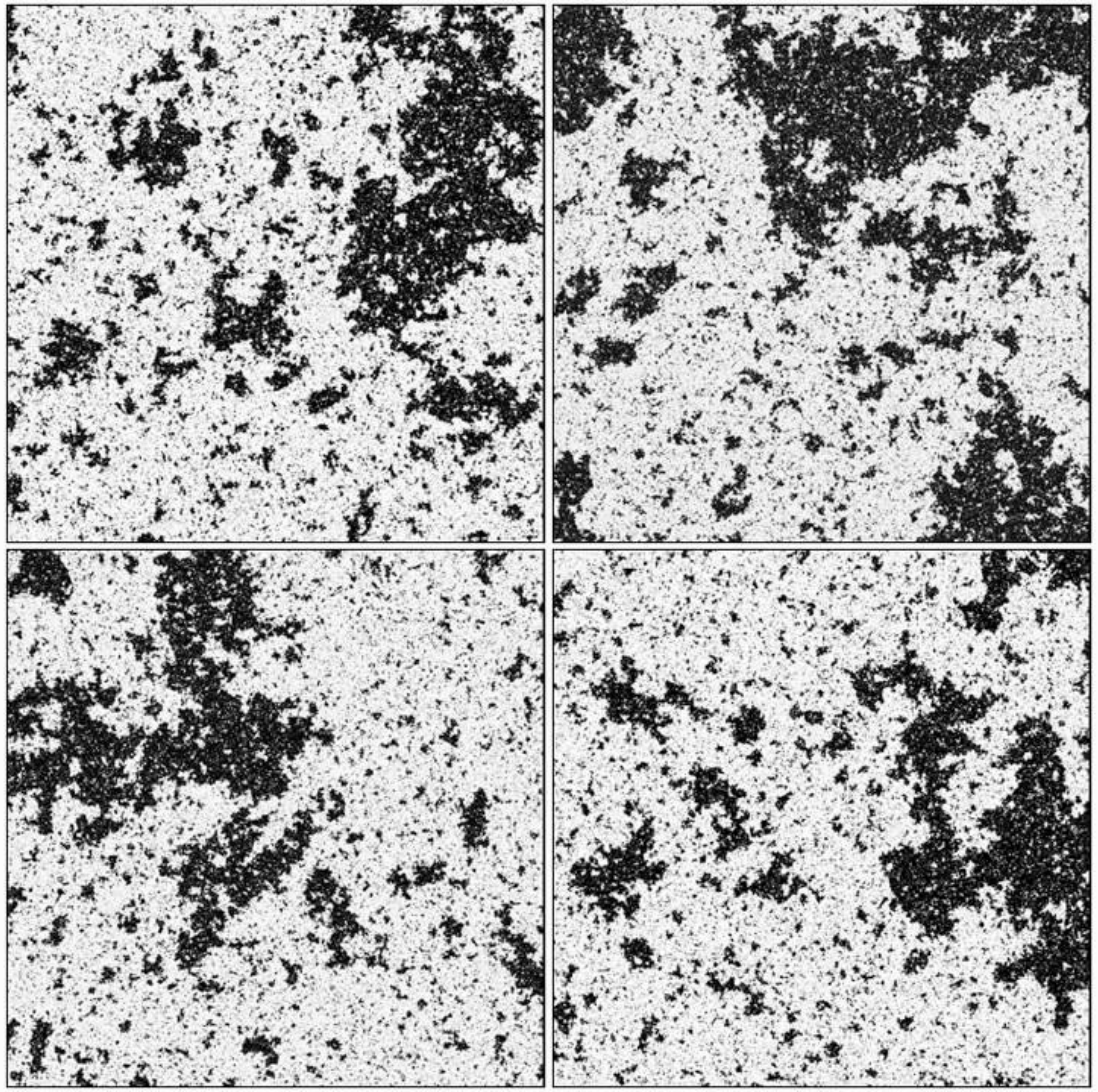
Scale invariance: **the system “looks the same” at all scales (i.e. when zooming in)**

Scale invariance: **the system remains unchanged (in a statistical sense) by coarse-graining operation.**



Scale invariance of critical percolating cluster



Scale invariance in the critical Ising Model

Renormalization Group

Theory of Complex Systems

Niki Stratikopoulou

University of Amsterdam

22 May 2023

Idea of RG

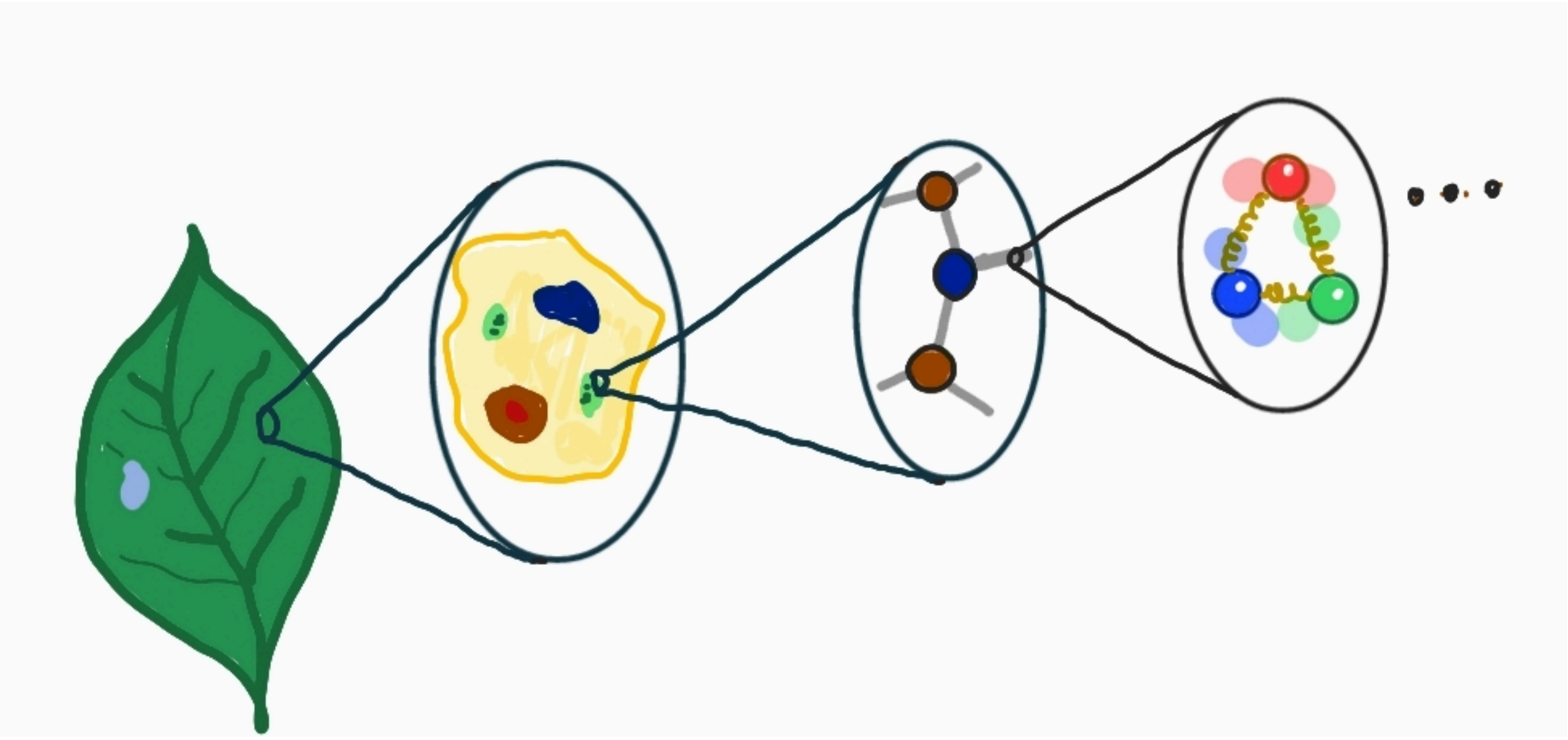
Main obstacle to understanding critical phenomena:

Fluctuations over **all scales** $a < x < \xi = \infty$ must be taken into account.

Can we simplify, by coarse-graining/integrating some of them?

Can we zoom out until smallest fluctuations disappear?

How does a system behave under **change of scales** ?



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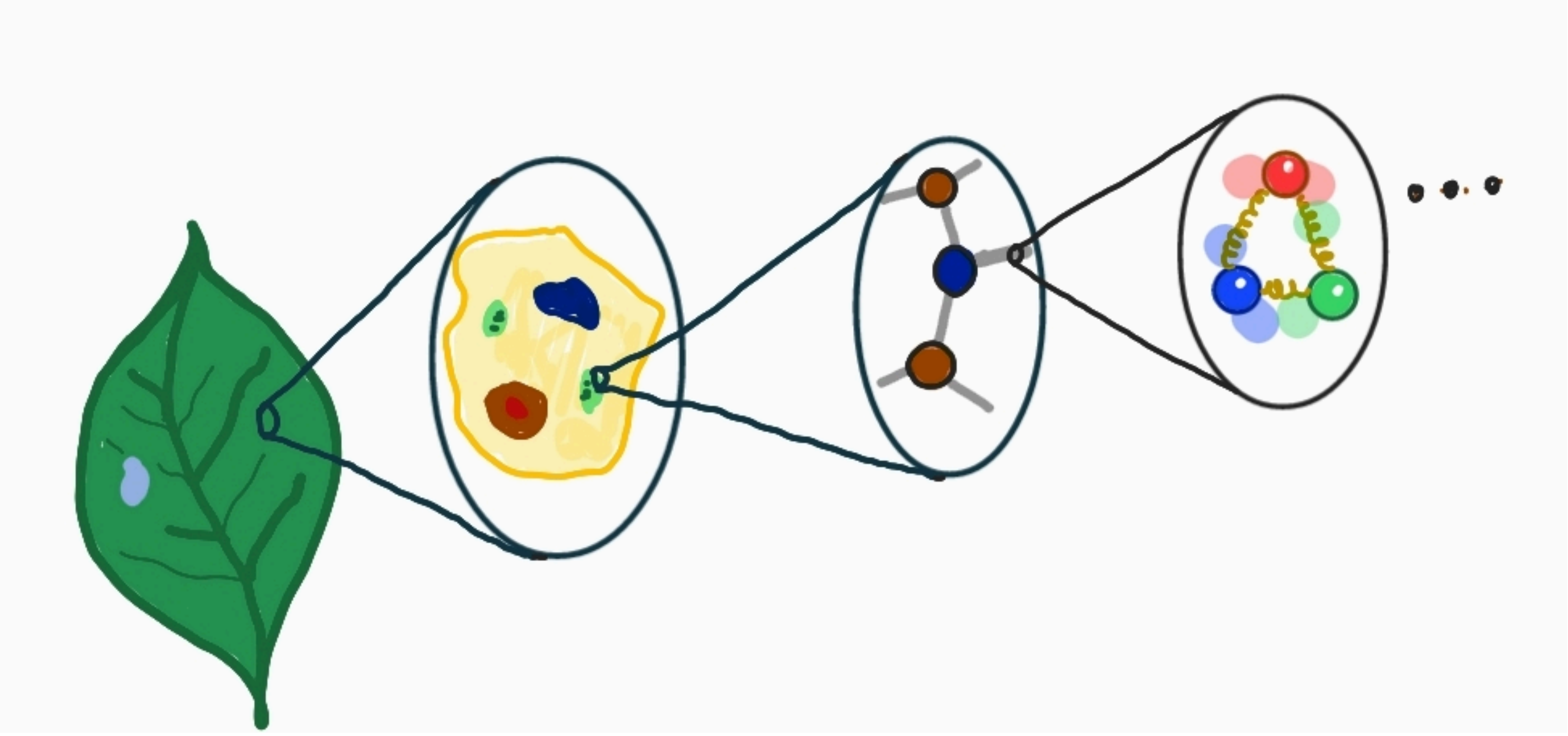


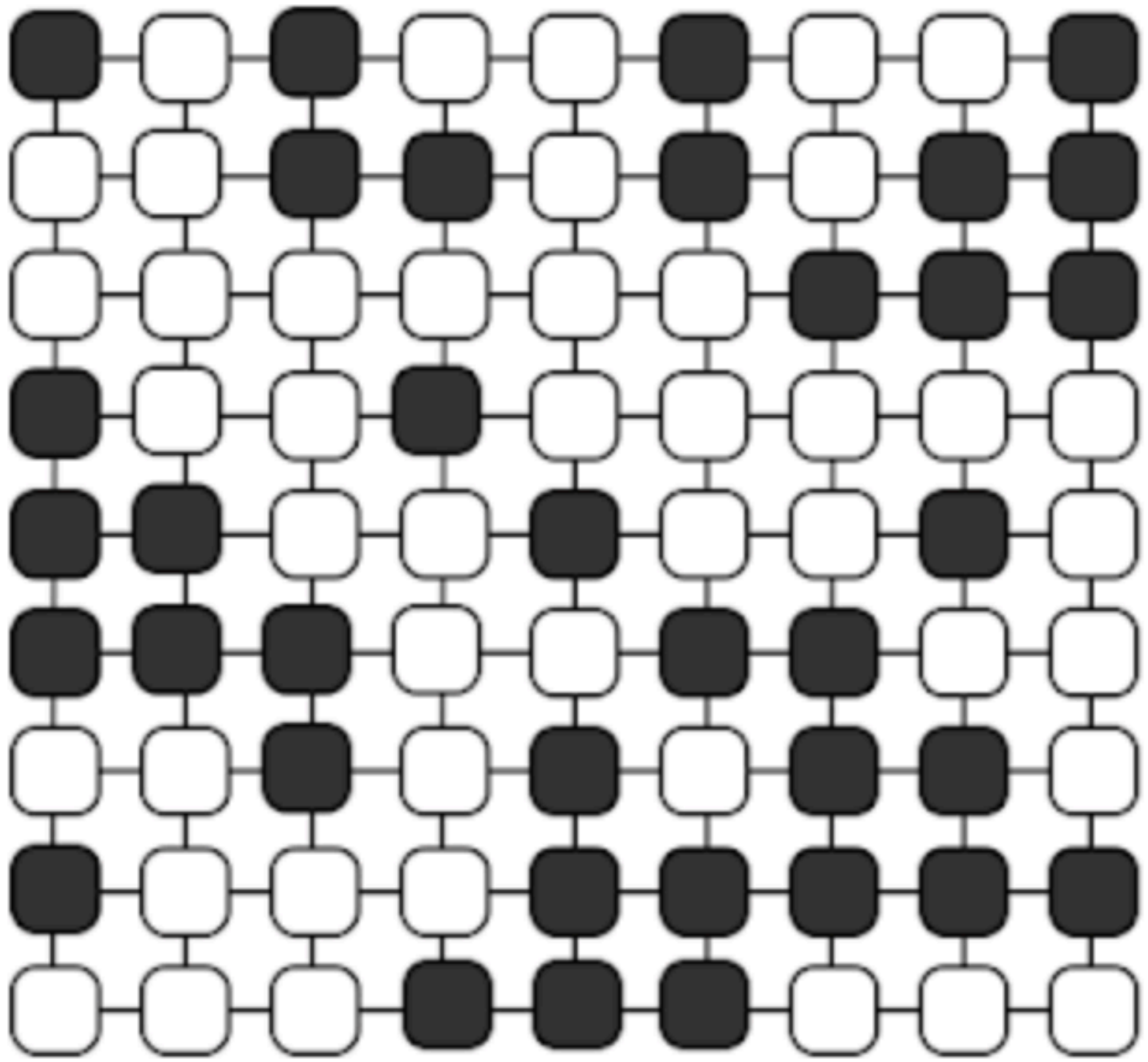
The interactions change…

As we zoom out,
not all the details of fluctuations are important
we could remove them by integrating over them

If the physics is the same on all length scales, then we should be able to able to rescale the system,
to cast it on a different length scale, and get back the original system.

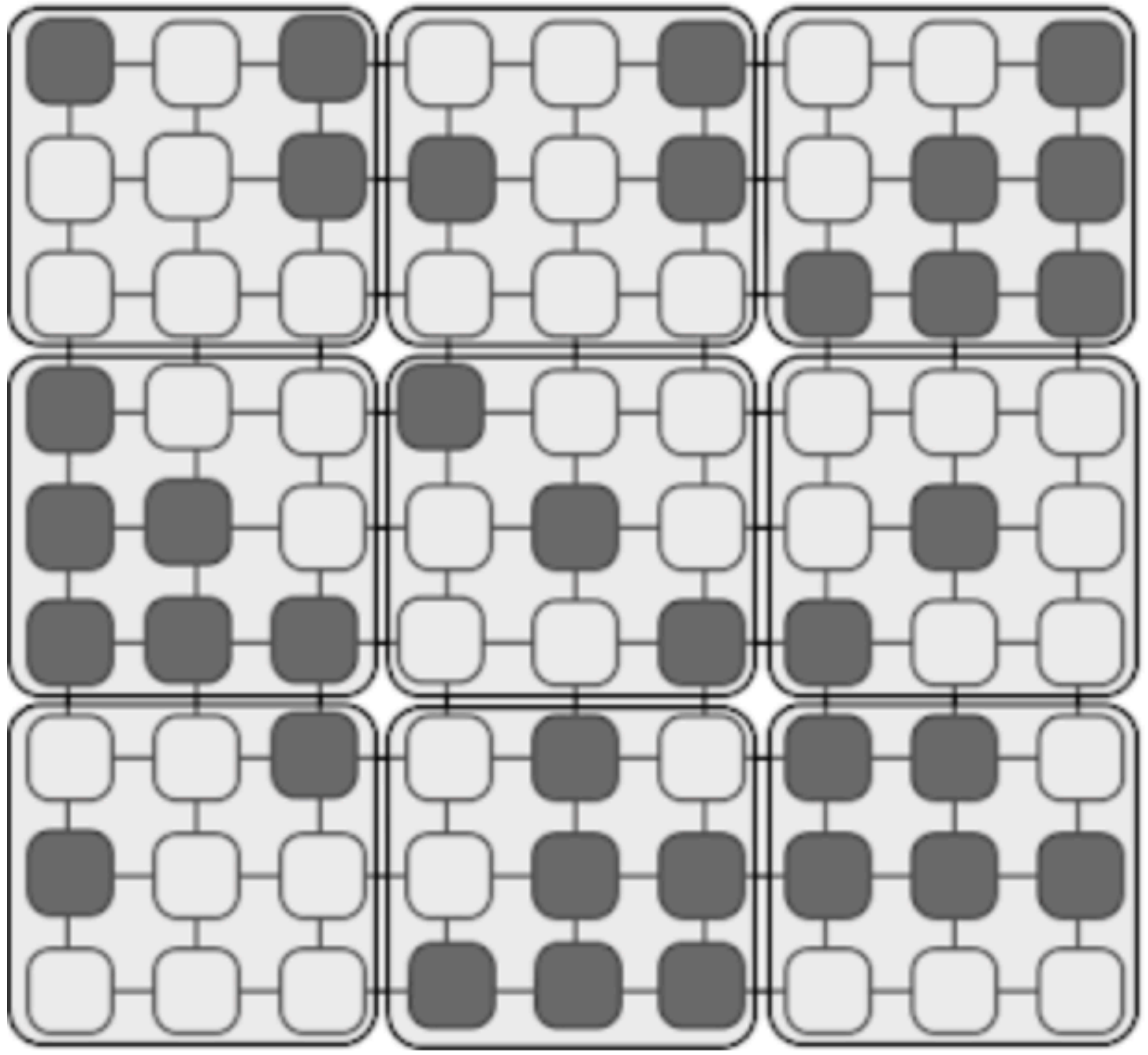
Idea of RG

Example: Take an Ising model configuration



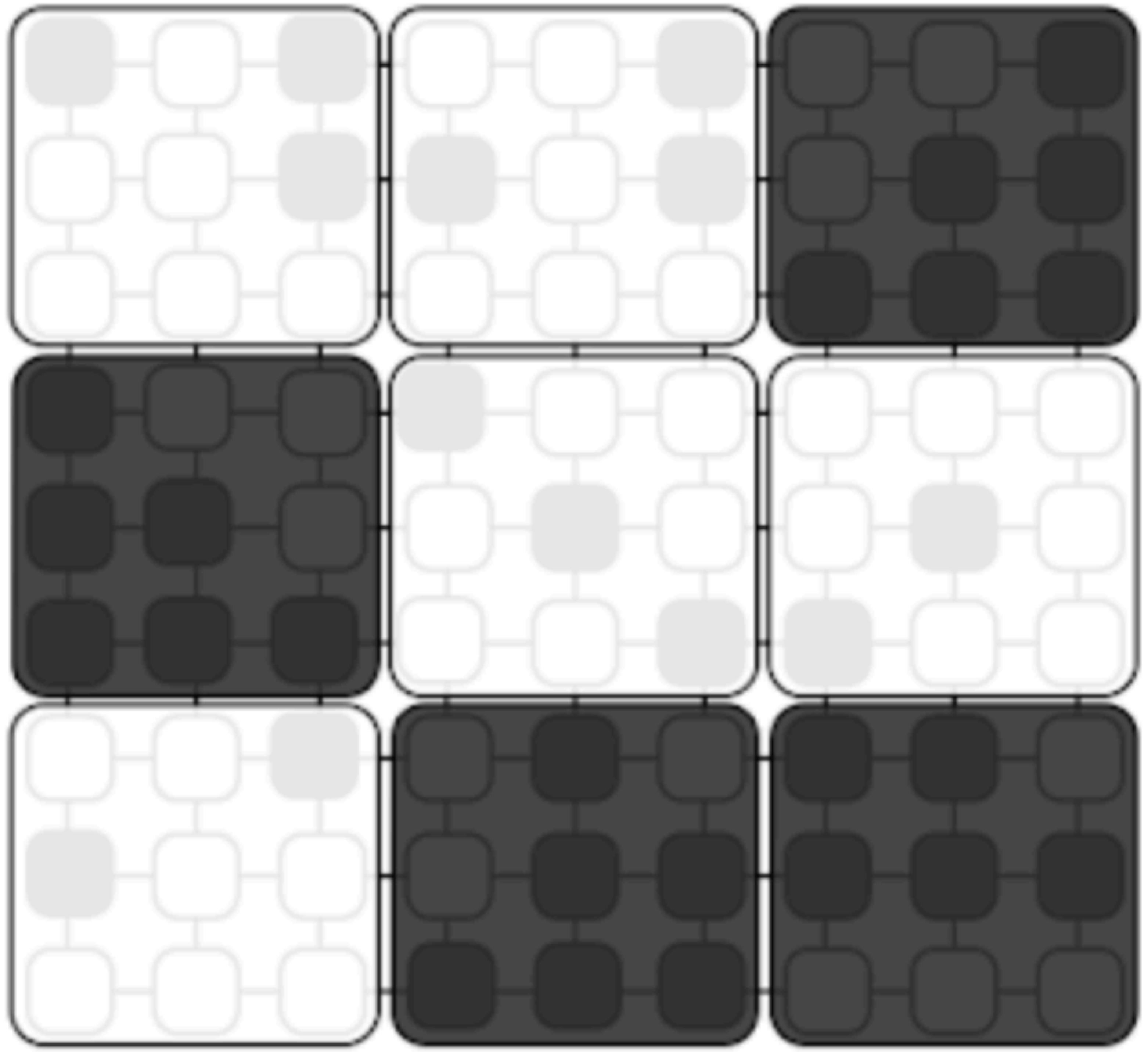
Idea of RG

Example: group neighboring variable by block



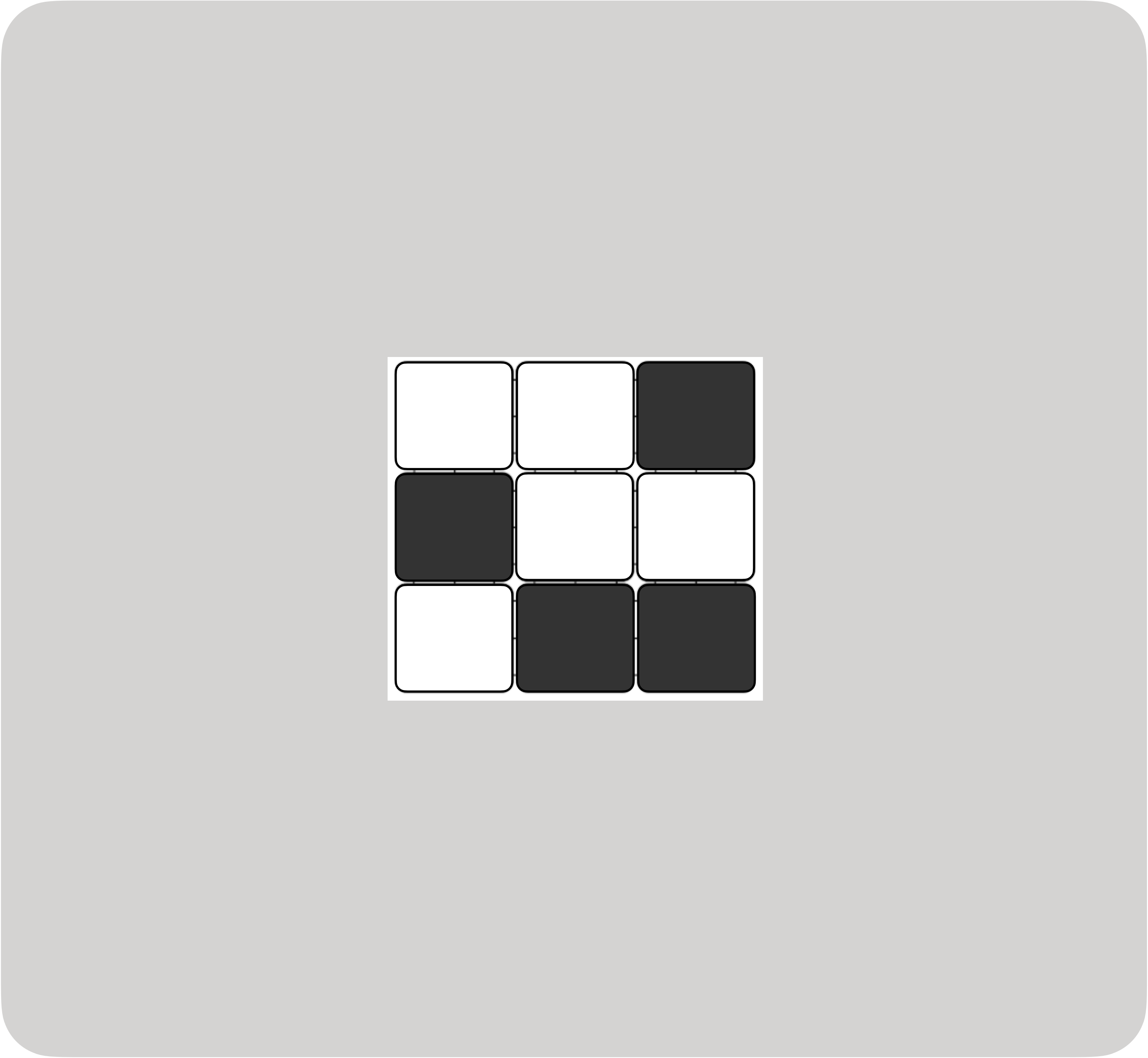
Idea of RG

Example: apply a majority rule



Idea of RG

Example: rescale (zoom out)



Idea of RG

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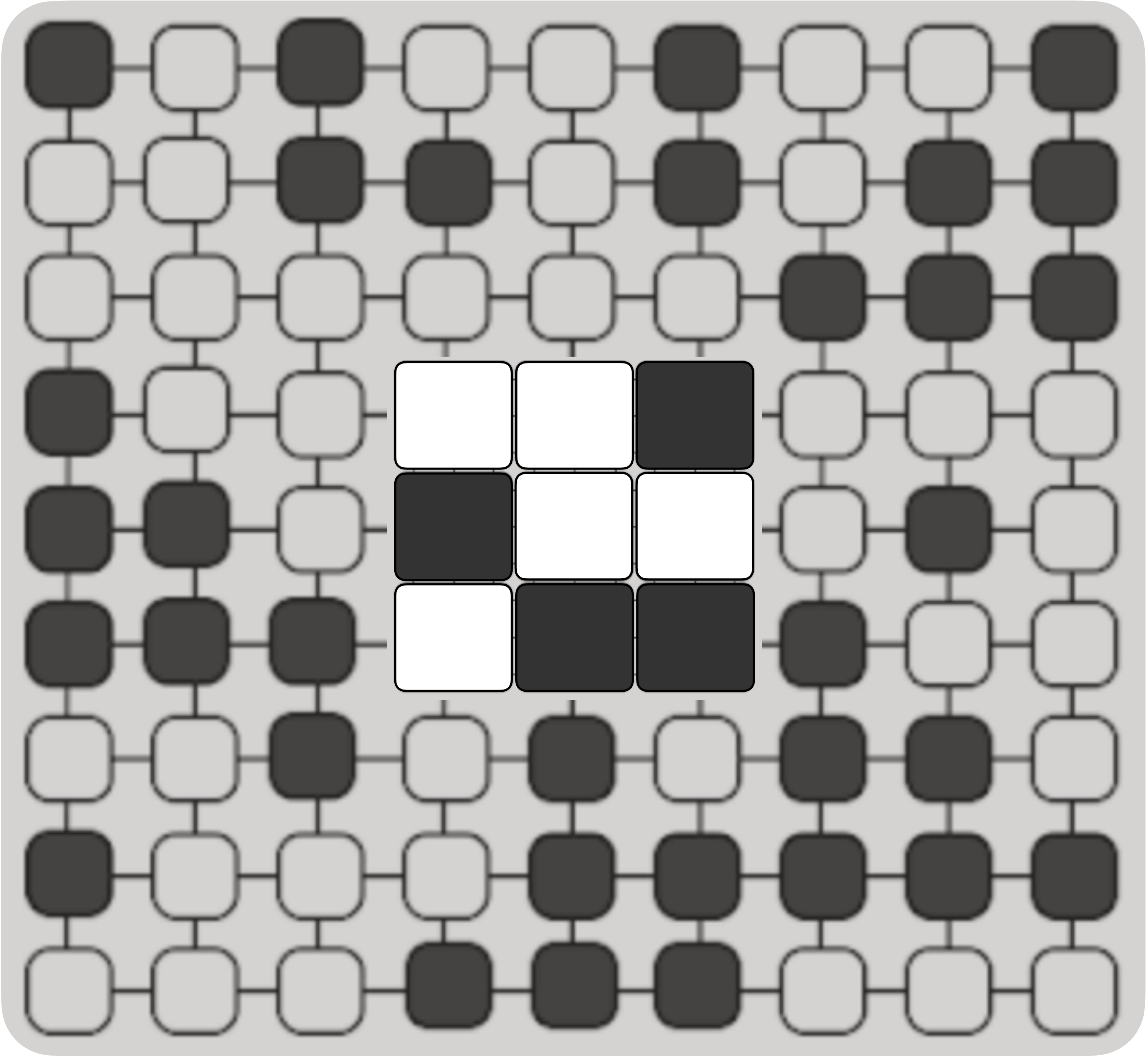

Could this still be modeled by the same Ising model? (Assume we do this procedure for many states for instance, to have a new ensemble distribution)

Idea of RG

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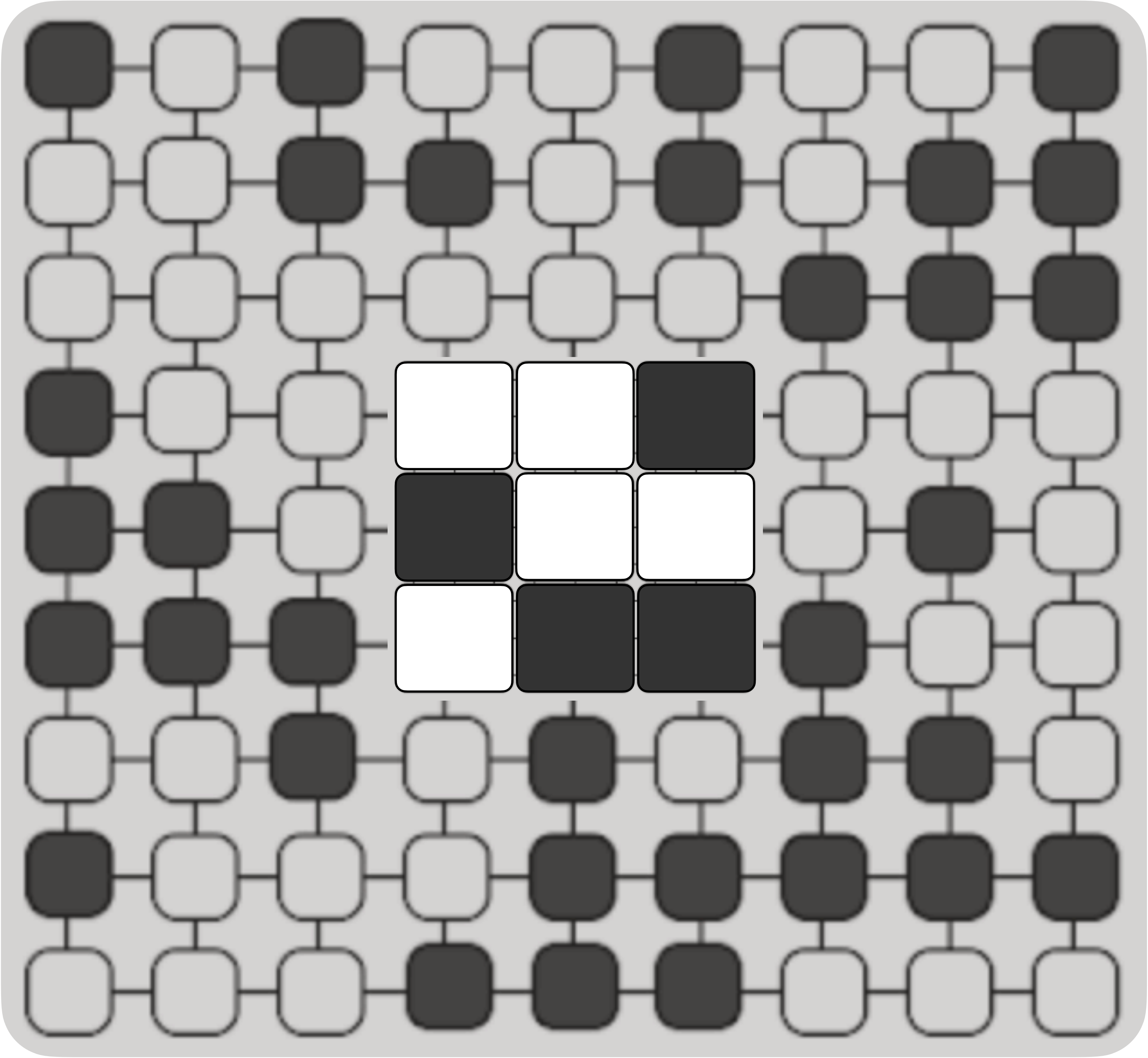

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If so, how would the coupling change compared to the original model?

Idea of RG

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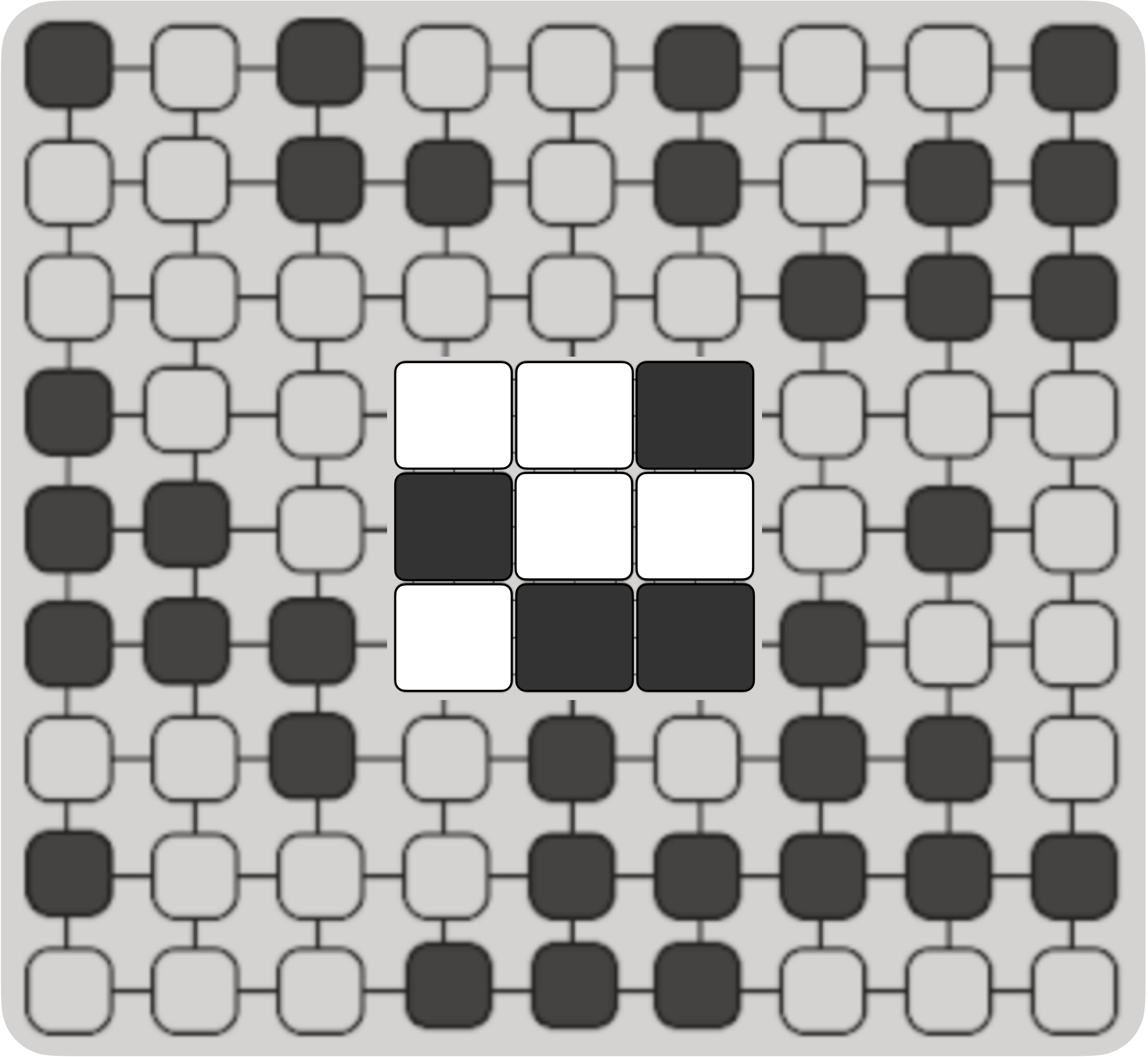

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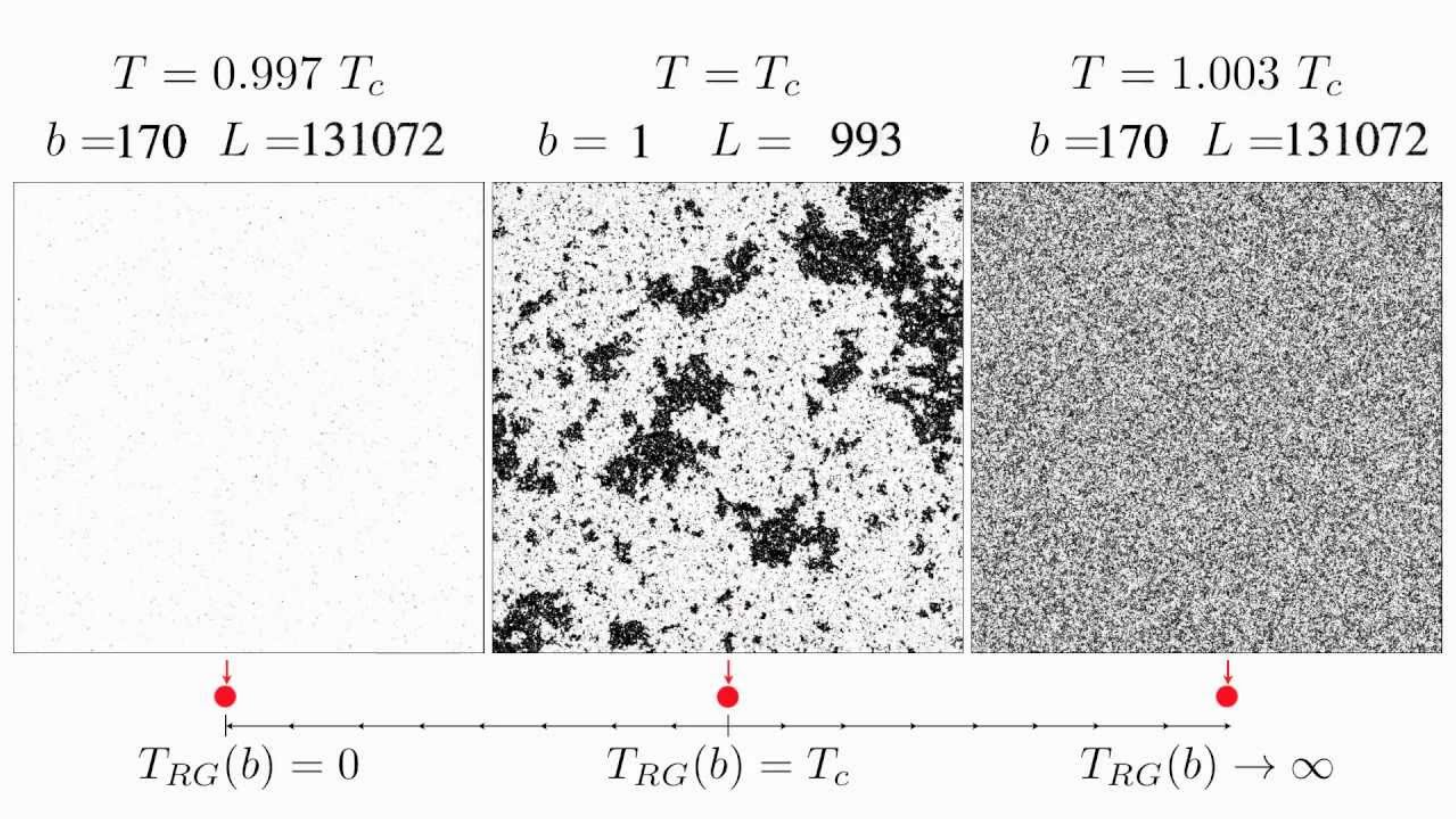
If so, how would the coupling change compared to the original model?

At which temperature would this new system be?

Idea of RG



Renormalization Group (RG): a formal method that allows systematic investigation of the changes of a physical system as viewed at different scales

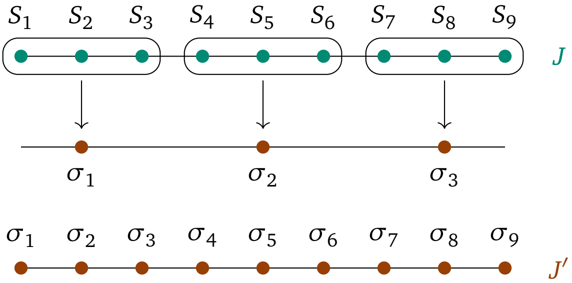
Idea of RG

Renormalization Group (RG): **a formal method that allows systematic investigation of the changes of a physical system as viewed at different scales**

Renormalization 1D Ising

Kadanoff's block spin RG

1. Divide lattice into blocks of size b (b spins) .
2. Replace each block of sites by a single site (1 Big spin).
3. Rescale all lengths by the factor b, to restore the original lattice spacing.
4. Repeat.



There are many choices of RG transformations → different coarse graining procedures / RG schemes.
E.g.

1. Kadanoff's Block spin:

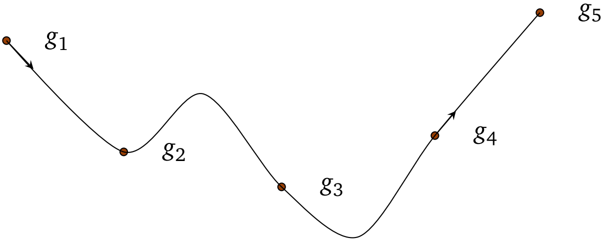
$$\sigma_k^{(n+1)} = \sigma_j^{(n)}, \quad j \in \mathcal{B}_k$$

2. Majority rule

$$\sigma_k^{(n+1)} = A^{(n)} \sum_{i \in \mathcal{B}_k} \sigma_i^{(n)}$$

Fixing one, in each step we generate an *effective Hamiltonian* $\mathrm{H}^{(n+1)}\left(\{\sigma_i^{(n+1)}\}, g_k^{(n+1)}\right)$ describing the system of the block spins, with new effective couplings g_k.
No inverse transformation...

Flow of couplings



Fixed points can occur.

At **criticality** $\xi = \infty$

$$\xi \rightarrow \frac{\xi}{b} \rightarrow \frac{\xi}{b^2} \rightarrow \cdots \rightarrow \frac{\xi}{b^n} = \infty\,.$$

System will still look critical!

RG for the 1d Ising model

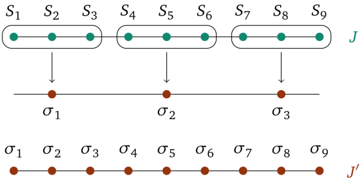
$$\mathrm{H}(s_i;J) = J\sum_i s_i s_{i+1} \quad \text{where} \quad J \leftarrow J/(k_B T)$$

Each pair of spins has the Boltzmann weight

$$W(s_i, s_{i+1}; t) = e^{J s_i s_{i+1}} = \cosh J(1 + t s_i s_{i+1}) \qquad \text{[cf. tutorial]},$$

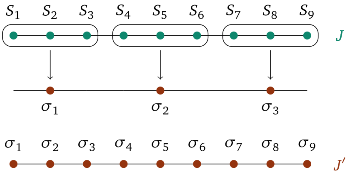
where $t := \tanh(J)$

Divide into blocks of 3 spins & keep spin at center:



1st RG step

Consider two neighboring blocks; fix spins at center $\sigma_1 \equiv S_2$, $\sigma_2 \equiv S_5$ and sum over intermediate spins S_3, S_4. Boltzmann factor:



$$\sum_{S_3,S_4} e^{J\sigma_1 S_3} e^{JS_3S_4} e^{JS_4\sigma_2} = (\cosh J)^3 (1 + t\sigma_1 S_3)(1 + tS_3S_4)(1 + tS_4\sigma_2)$$

$$= 2^2 (\cosh J)^3 (1 + t^3 \sigma_1 \sigma_2)$$

Reflect:
Initial Boltzmann factor: $W(s_i, s_{i+1}; t) = \cosh J(1 + t s_i s_{i+1})$

After 1st RG step: $W(\sigma_i, \sigma_j; t) \sim \cosh^3(J)(1 + t^3 \sigma_i \sigma_j)$
Equivalent if we define

$$t' \equiv \tanh J' = t^3 \equiv (\tanh J)^3$$
$$\Leftrightarrow J' = \tanh^{-1}[(\tanh J)^3]$$

New *effective* Hamiltonian:

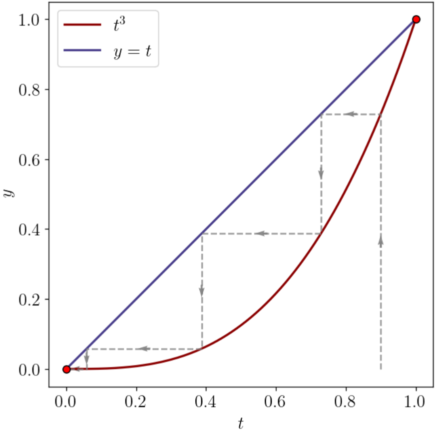
$$H(\sigma_i; J') = \underbrace{A(J)}_{\text{from mult. factors}} - J' \sum_i \sigma_i \sigma_{i+1}$$

+ a recursion relation for couplings.

Plot recursion relation

$$t' = t^3\,.$$

At fixed points: $t' = t$.



RG for the 1d Ising model

Fixed points

- $t = 1$ unstable f.p.
- $t = 0$ stable f.p.

Approaching $t \equiv \tanh J = 0 \Leftrightarrow J = 0$,
corresponds to approaching the high temperature phase ($J \leftarrow J/(k_B T)$).

1d Ising model is always in its **disordered phase**
(non-interacting spins).

$\downarrow$

self-similar with $\xi = 0$ (trivial RG f.p.)

$\nexists$ phase transition in the 1d case.

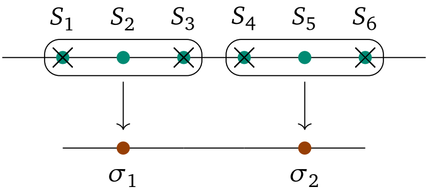
Sum over (integrate out) $k-1$ consecutive spins.

$$\sum_{S_1,\ldots,S_{k-1}} e^{J(S_0S_1+\cdots+S_{k-1}S_k)} \sim \cosh^k(J)\left(1+t^k S_0 S_k\right).$$

Middle terms, with spins that we sum over, cancel out, like in the case of the two intermediate spins $\Longrightarrow$ only boundary spins survive (and identity), with coefficient $t^{\text{len(summed spins)}+1}$

$$t' \equiv \tanh\left(J'\right) = t^k = \tanh^k(J).$$

For the 1d case:



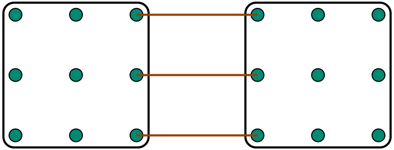
at $T \to 0 \Leftrightarrow J \to \infty$, spins almost aligned in same state

$$J' \simeq J \underbrace{\langle s_3 \rangle_{\sigma_1 = 1} \langle s_4 \rangle_{\sigma_2 = 1}}_{\text{boundary spins we integrate over}} .$$

The interaction between adjacent blocks is mediated by their boundary spins.

$|\langle s_i \rangle| \leq 1$ (max at $T = 0$) $\Rightarrow J' < J$.

In $d > 1$,



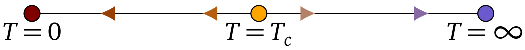
$$J' \simeq b^{d-1} J, \quad \text{for} \quad J \to \infty, \; b \text{ the block size} \; .$$

So for $d > 1$, $J' > J \Rightarrow$ the low-T f.p. is now attractive.
At high-T the system has to be in a paramagnetic phase $\Rightarrow$ high-T f.p. must also be stable.

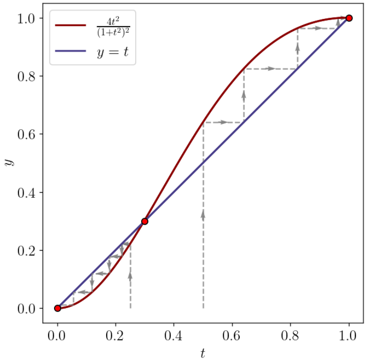
$$\Downarrow$$

There must be a **phase transition**.

RG for Ising in d>1



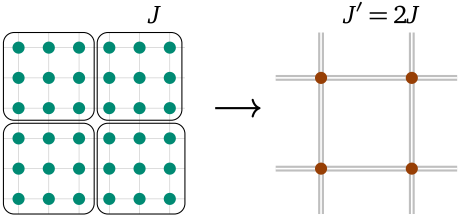
Cobweb plot from analytic recursion relation:



link to video

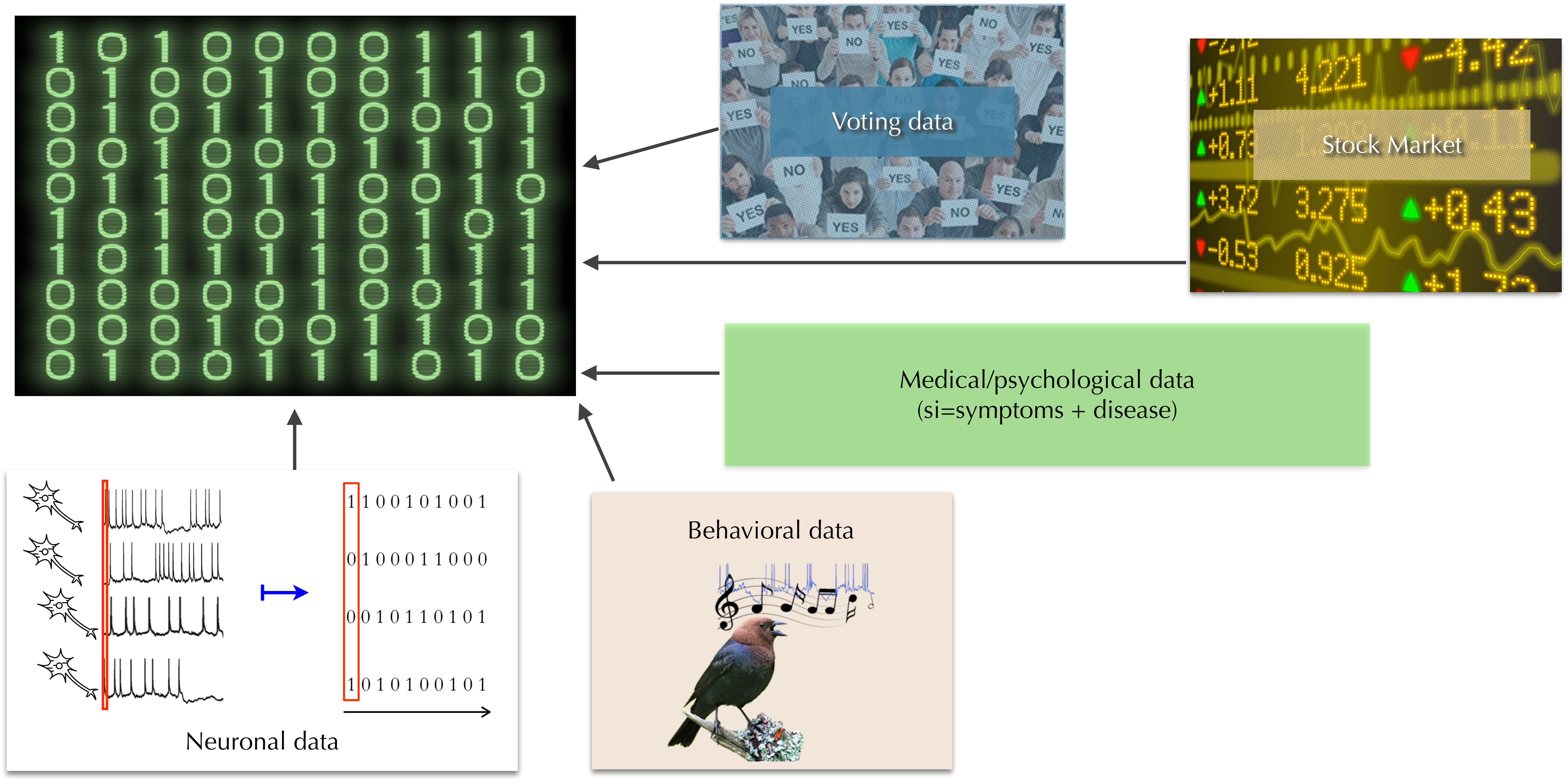
RG for 2d Ising model

One possible RG scheme



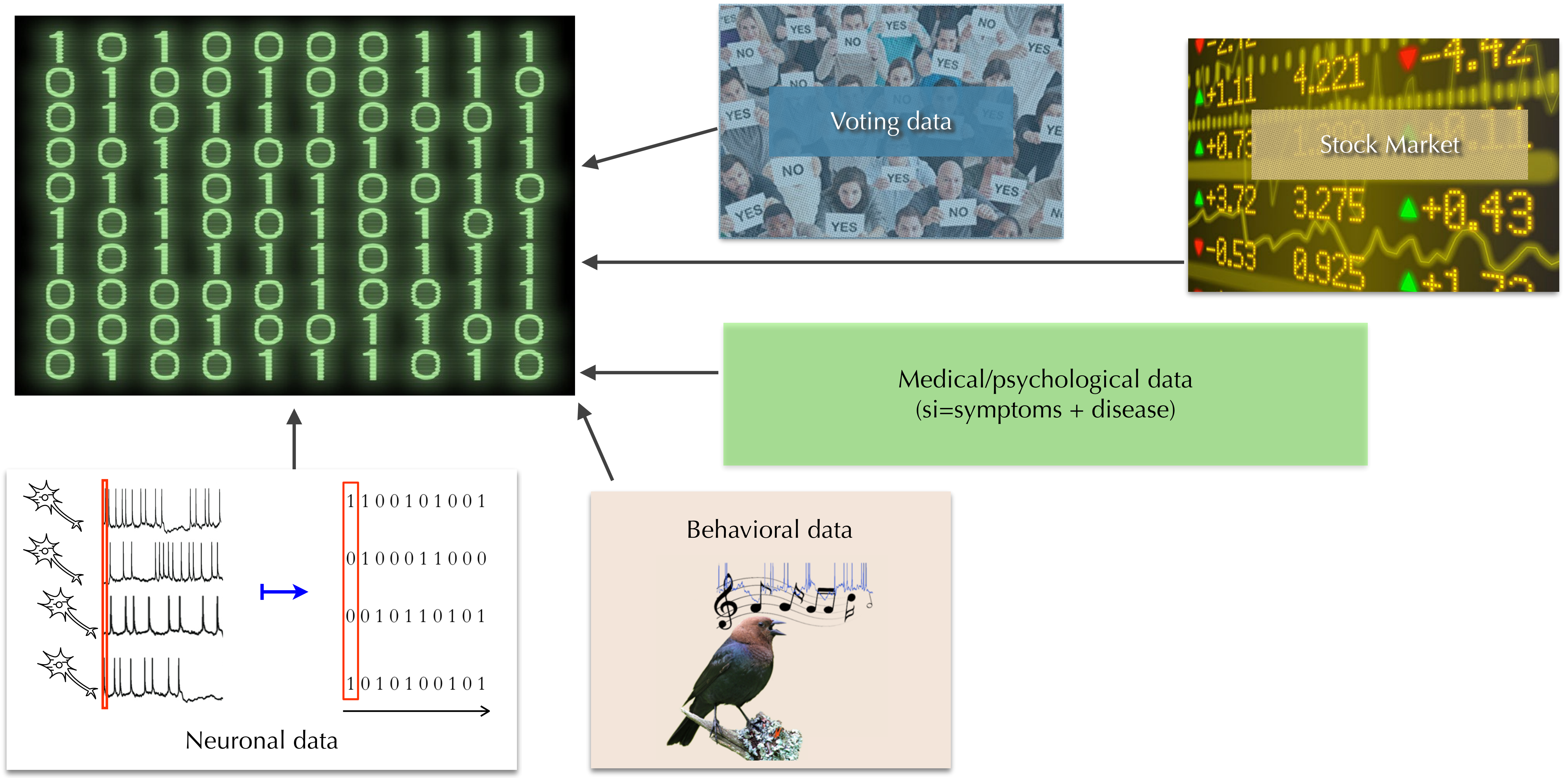
$$t' \equiv \tanh\left(J'\right) = \tanh^2(2J).$$

Use $\tanh(2x) = \frac{2\tanh(x)}{1+\tanh^2(x)}$ → recursion relation → Cobweb plot.



Application to real data?

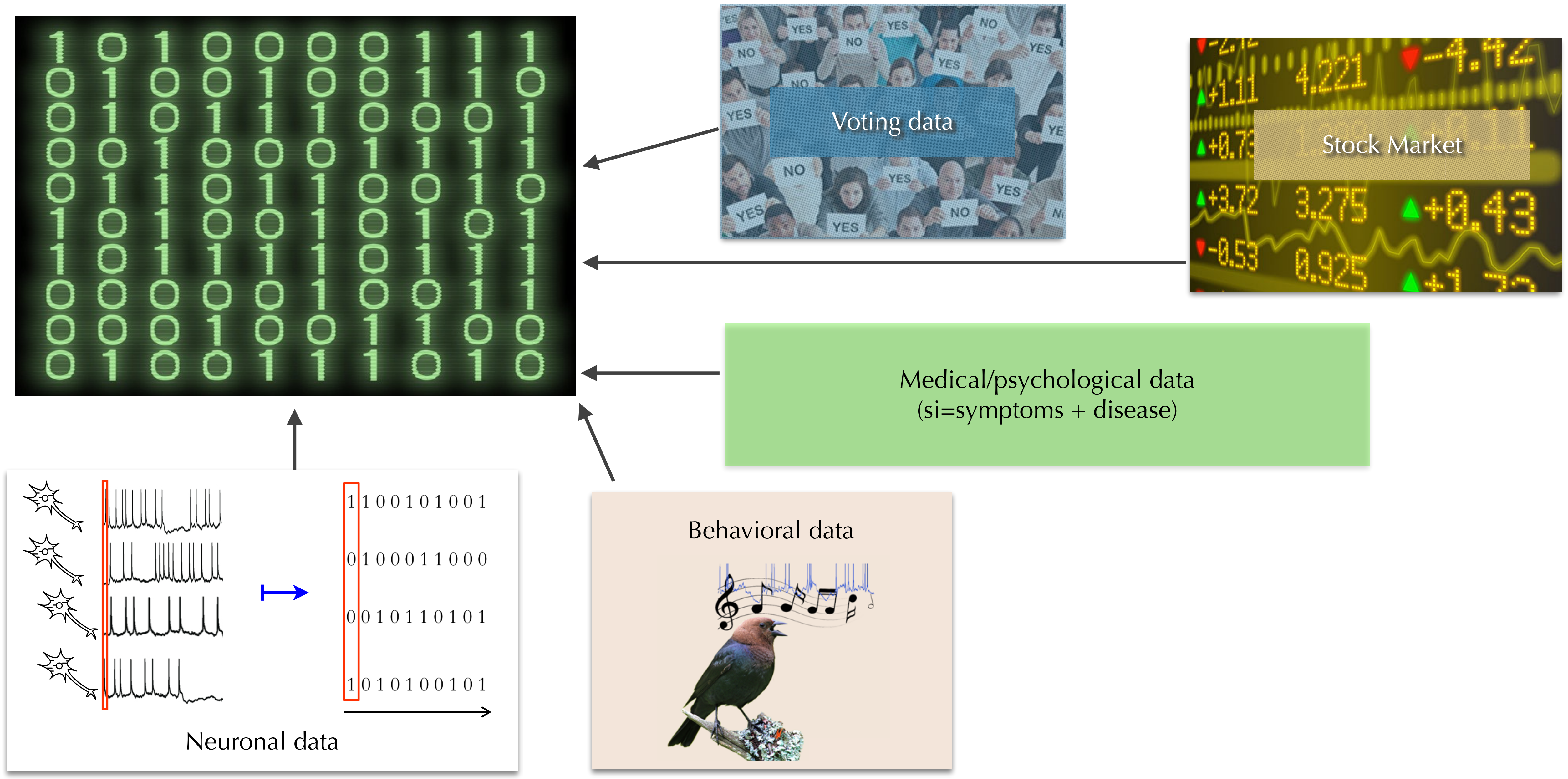
Any issue?



Application to real data?

Issue: There's no notion of locality; we don't have any notion of locality!

we need a way to define locality….



Application to real data?

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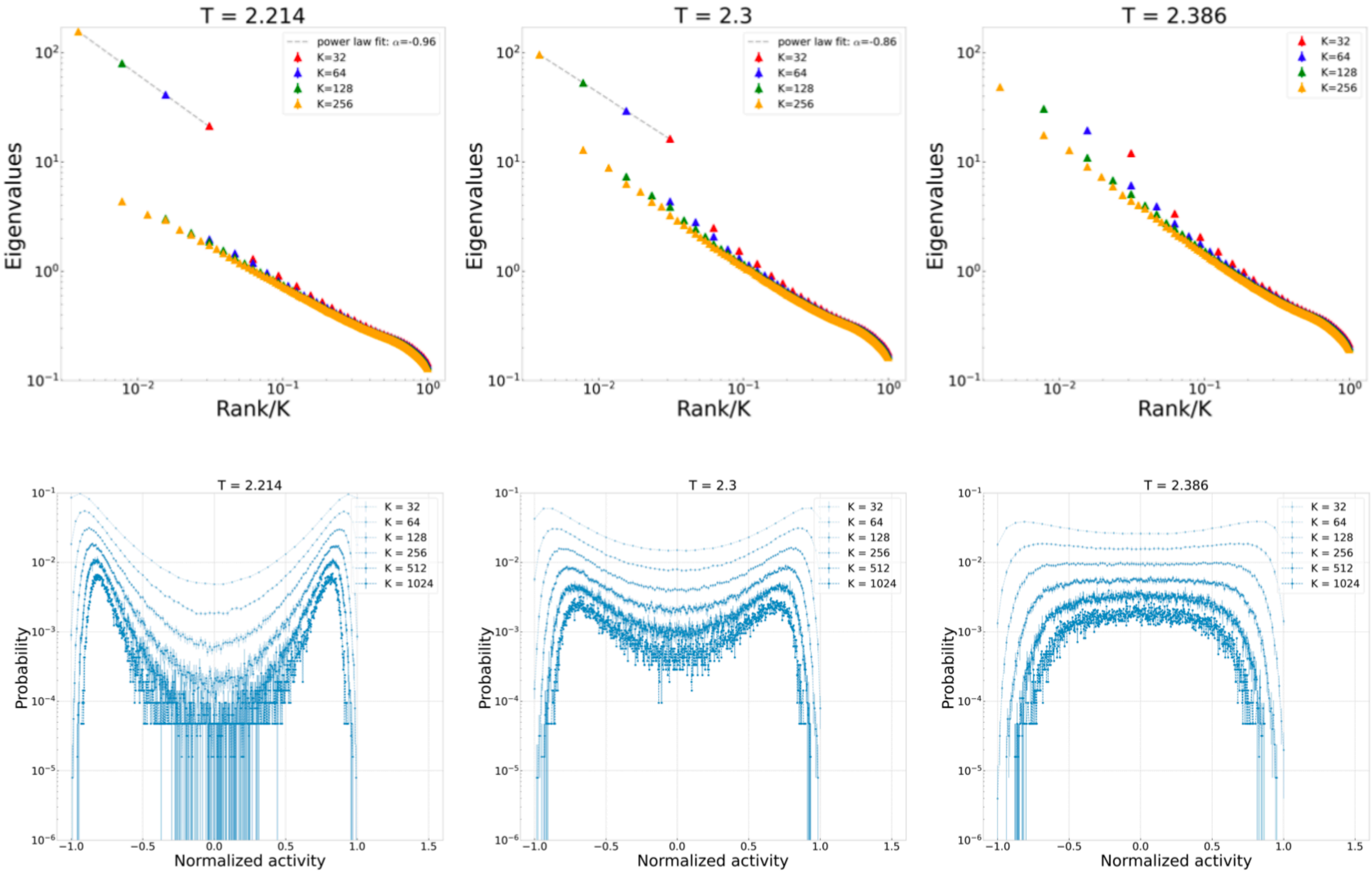
Idea: We will coarse-grain variables that are the most correlated.



T = 2.214
T = 2.3
T = 2.386
power law fit: α=-0.96
power law fit: α=-0.86
K=32
K=64
K=128
K=256
Eigenvalues
Rank/K
Probability
Normalized activity
K = 32
K = 64
K = 128
K = 256
K = 512
K = 1024

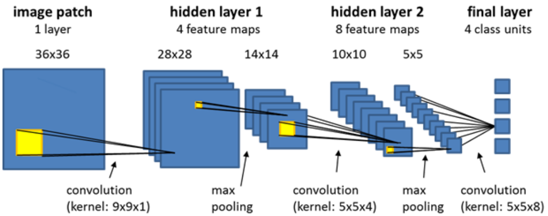
T < Tc | T = Tc | T > Tc

Application to real data?

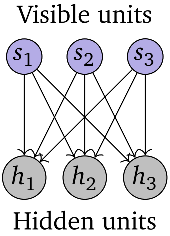
Any issue?

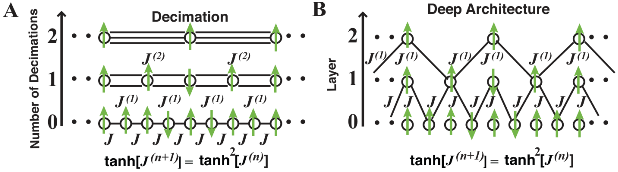
Convolutions

- How CNNs learn to extract the important features from data?
- Iterative **coarse-graining** scheme; each new layer of NN learns increasingly abstract higher-level features.



Restricted Boltzmann Machines (RBM) [Mehta, Schwab, 2014]



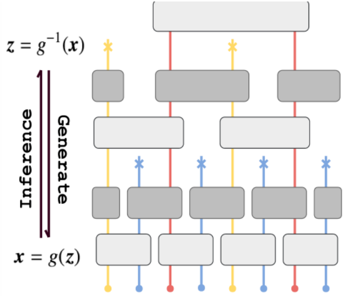


A: 1d RG procedure

B: DNN architecture with weights J^n between $(n+1)$ and n-th hidden layer

RG inspired architectures

Stacking bijections to form a reversible transformation of a probability density.



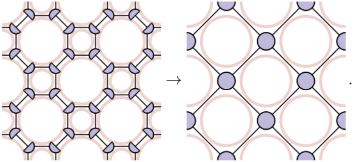
[Li,Wang, 2018]

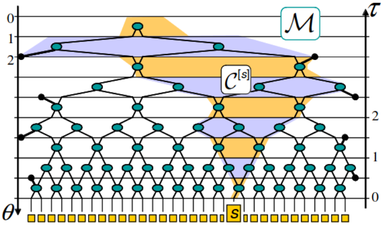
Alternation between disentagler blocks and decimator blocks.

disentagler blocks : Reduce correlaiton between inputs.
decimator blocks : Pass only subset of outpus to next layer.
Crosses →irrelevant latent variables.

Tensor Networks

(transform quantum states instead of probability densities)





Multi-Scale Entanglement Renormalization Ansatz (MERA)

[Vidal, 2006]