Chapter 6

Collective Behavior

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6.1 Noisy Kuramoto Model

For a popular science introduction to the Kuramoto model you can watch the following video. Here a couple of applications of the Kuramoto model are mentioned, such as for the synchronization of clocks, humans walking, fireflies and planets or moons. Another application is in the study of synchrony between neurons in the brain. The suprachiasmatic nucleus, also known as the body-clock, is a cluster of neurons in the brain responsible for dictating the rhythm of bodily functions. For a short introduction into the application of the Kuramoto model see the following article. The article also contains an interactive animation that you can use to develop intuition regarding this problem set.

The noisy Kuramoto model is defined by the following system of coupled differential equations

$$\frac{\mathrm{d}\theta_i(t)}{\mathrm{d}t} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin\left[\theta_j(t) - \theta_i(t)\right] + \xi_i(t), \tag{6.1}$$

with $\xi_i(t)$ independent noise with expected values $\langle \xi_i(t) \rangle = 0$ and $\langle \xi_i(t)\xi_j(t') \rangle = 2D\delta(t-t')\delta_{i,j}$. Here we have

- N – number of oscillators
- $\omega_i \sim \mu(\omega)$ – natural frequency of i^{th} oscillator drawn from distribution $\mu(\cdot)$
- $\theta_i(t)$ – phase of i^{th} oscillator
- $K \in (0,\infty)$ – interaction strength
- $D \in (0,\infty)$ – noise strength (contained in $\xi_i(t)$).

Q1. Consider a system of two oscillators. What is the effect of the interaction term $\frac{K}{N}\sum_{j=1}^{N} \sin\left[\theta_j(t) - \theta_i(t)\right]$? Note that K is a positive constant. Try this situation in the simulation here and check that your intuition is correct.

We define the order parameter as

$$r_N(t)\mathrm{e}^{i\psi_N(t)} = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{i\theta_j(t)}. \tag{6.2}$$

Q2. Show that (6.1) can be written as

$$\frac{\mathrm{d}\theta_i(t)}{\mathrm{d}t} = \omega_i + Kr_N(t)\sin\left[\psi_N(t) - \theta_i(t)\right] + \xi_i(t). \tag{6.3}$$

Q3. What is the advantage of (6.3)?

Let $p_t(\theta, \omega)$ represent the density of phase oscillator with angle θ, natural frequency ω at time t. In the continuum limit ($N \to \infty$) this density evolves according to:

$$\frac{\partial p_t(\theta, \omega)}{\partial t} = -\frac{\partial}{\partial \theta}\Big[p_t(\theta, \omega) v_t(\theta, \omega)\Big] + D\frac{\partial^2 p_t(\theta, \omega)}{\partial \theta^2} \tag{6.4}$$

with

$$v_t(\theta, \omega) = \omega + Kr(t)\sin(\psi(t) - \theta) \tag{6.5}$$

and satisfies

$$p_t(\theta + 2\pi, \omega) = p_t(\theta, \omega) \quad \text{and} \quad \int_0^{2\pi} p_t(\theta, \omega)\mathrm{d}\theta = 1. \tag{6.6}$$

Q4. Solve for the stationary density of (6.4) in the case that $\mu(\omega) = \delta_0$.

Q5. In the large N limit the expression for the order parameter becomes:

$$r(t)\mathrm{e}^{i\psi(t)} = \int_0^{2\pi} \int_{\mathbb{R}} \mathrm{e}^{i\theta} p_t(\theta, \omega)\mathrm{d}\omega\mathrm{d}\theta. \tag{6.7}$$

1. Use the solution you calculated in **Q4.** to find an implicit equation for the stationary synchronization level r in the case that $\mu(\omega) = \delta_0$.
2. Taylor expand (around $r = 0$) of the equation you found in a) to determine the critical threshold for K above which nonzero solutions for r exists. *Hint: Define the right hand side of the equation from a) to be a function* $V(2Kr/D)$. *Use that* $V(x)$ *is monotonically increasing, concave and that* $\lim_{x\to\infty} V(x) = 1$.
3. Reflect on whether or not the critical threshold makes sense.

6.2 Swarm behavior and Vicsek Model

6.2.1 Bird flocks

Watch this 20min talk (starting at 1h 11 min of the video), which discusses this paper on bird flocks [1].

Q1. In the Vicsek model, how is the evolution of the positions of the birds described? Can you recall the expression of the position and the direction of the birds at time $t + 1$ as a function of its position and direction at time t? Can you comment on the meaning of each term?

Q2. How would you generalize this equation of evolution in the case of birds that move at a constant speed v_0?

Q3. In the Vicsek model, what is the meaning of the metric range? The authors of Ref. [1] argue that animal collective behavior depends on topological distance rather than metric distance. What is the difference between a metric range and a topological range? How do the authors of Ref. [1] suggest to modify the Vicsek model to capture better the behavior of bird flocks?

Q4. The author of the paper give the following relation between the distance r_1 to the first neighbor and the distance r_N to the N-th neighbor:

$$r_N \sim r_1 \, N^{1/3} \,, \tag{6.8}$$

where the $\sim$ indicates the scaling of r_N with N to some constant factor (to be more specific, this equation means that we expect r_N to behave like: $r_N = \alpha \, r_1 N^{1/3} + \beta$, where α and β don't depend on N). Assuming that a flock has a constant density of birds, can you re-derive this result?

Q5. Considering the following two hypotheses:

(a) the birds interact with a metric interaction: they align to their neighbors that are within a fixed distance r_c from them;

(b) the birds interact with a topological interaction: they align with their N_c first neighbors, where N_c is fixed.

Assume that you have access to the recordings of many bird flocks, how would you use (6.8) to check which of these two hypotheses is the good one? Comment on Fig. 3.c and d of Ref. [1].

We are interested in understanding how the collective behavior (flocking) emerges in the Vicsek model as a dynamical phase transition. In section 2.2, we remove the movement of the "animals" by fixing them on a lattice, but we still allow them to have different directions. In this case, the system can only exhibit a collective behavior (long-range order in the alignment of the orientation of the "animals") if the dimension d is strictly larger than 2. In section 2.3, we see that the Vicsek model can already exhibit such collective behavior in dimension $d = 2$ thanks to the spatial movements of the "animals" (which allows information to propagate faster and further in the system). **Note: the following exercises are optional.**

6.2.2 (Bonus) Scaling arguments for the XY models – "Birds on a lattice"

Consider "bird-spins" on a d-dimensional (rigid) lattice that have the ability to align themselves to their neighbouring spins. Instead of taking binary values, these spins can take on continuous values in the range $\theta_i \in (-\pi, \pi)$. Suppose we prepare the lattice in a configuration where all spins point in the same direction, $\langle \boldsymbol{\theta} \rangle = \theta_0$. We now introduce a single mis-aligned spin that has alignment $\theta_i = \theta_0 + \delta\theta_0$. Due to the interaction between adjacent lattice sites, the error will propagate throughout the lattice.

Q1. Assuming that the propagation on the lattice follows a diffusion law (i.e. behaves like a random walker), derive (using scaling arguments) that the spin error decays as

$$\delta\theta \sim \delta\theta_0 \tau^{-d/2} \tag{6.9}$$

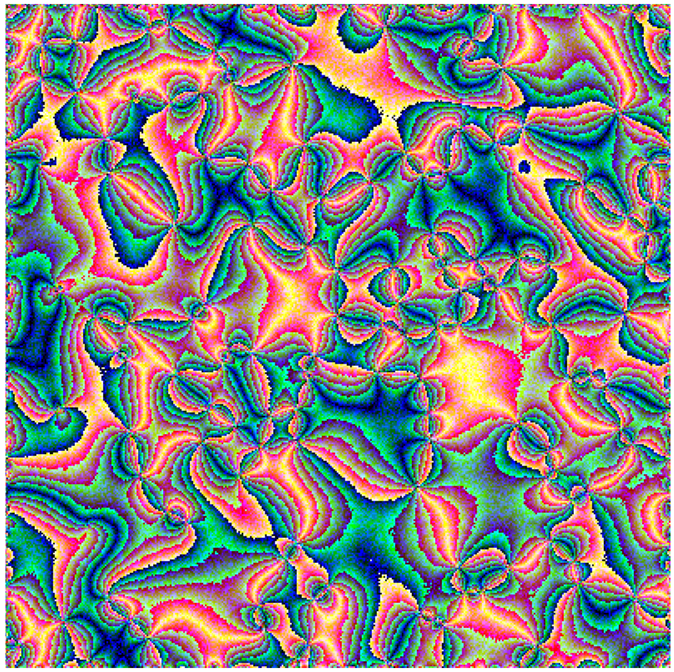

Figure 6.1: Realization of the "birds on a lattice" model (also known as the xy-model) on a 400×400 grid. Colors indicate the angle of each of the spins.

where τ is the time elapsed since introducing the mis-aligned spin and d is the lattice dimension.

Q2. Having seen what happens to a single defect, we now wish to find the influence of noise on all the spins on the lattice. Assuming that noise fluctuations produce a certain number of errors within some volume and that this number scales with the time τ, derive that the number of errors is proportional to

$$n_e \sim \tau^{1+d/2} \tag{6.10}$$

Q3. Show that the total expected error Ω_e in the system scales with n_e as

$$\Omega_e \sim \sqrt{n_e} \tag{6.11}$$

Hint: since the errors can point in any direction, how would you measure the total error?

Q4. a. Combining these results, show that the error amplitude per spin is given by

$$\Delta\theta \sim r_e^{1-d/2} \tag{6.12}$$

where r_e is the radius of the volume within which the errors have propagated.

b. Comment on the behaviour of the systems in different dimensions. In which dimensions is it possible to have long-range order?

c. For the physicists; do you know of a theorem that could have predicted the absence of long-range order in some dimensions? How does it apply to this system?

6.2.3 (Bonus) Birds not on a lattice: the Vicsek model

In the Vicsek model, the fluctuations are coupled to the motion of the "bird-spins". Assume that the mean motion of the spins is in the θ_0 direction. We align our frame of reference to this direction, such that $\theta_0 = 0$ in this frame.

Q1. Argue or draw a diagram to show that we can write the seperations induced by angular fluctuations around the mean direction as

$$\delta x_\perp \sim v_0 \tau \sin(\delta\theta), \quad \delta x_\parallel \sim v_0 \tau (1 - \cos(\delta\theta)) \tag{6.13}$$

Q2. Use the fact that $\delta\theta$ is small to derive the leading order terms in $\delta\theta$ for the expressions of $\delta x_\perp$ and $\delta x_\parallel$. What does this tell you about the nature of the transport of orientation information?

Q3. Decomposing the error propagation volume as $w_\perp^{d-1} w_\parallel$ and introducing two diffusion constants $D_\perp$ and $D_\parallel$, which encode the loss of orientation information due to diffusion. The growth of the error propagation volume is given by

$$w_\perp \sim \delta x_\perp + D_\perp \tau^{1/2}, \quad w_\parallel \sim \delta x_\parallel + D_\parallel \tau^{1/2} \tag{6.14}$$

Using the same reasoning as you did in the question on the lattice spins, show that

$$\Delta\theta \sim \frac{\tau^{1/2}}{\sqrt{w_\perp^{d-1} w_\parallel}} \tag{6.15}$$

Q4. We introduce three timescales which govern the propagation of orientation errors:

$$w_\perp \sim \tau^\alpha, \quad w_\parallel \sim \tau^\beta, \quad \Delta\theta \sim \tau^\gamma \tag{6.16}$$

Show that for large τ these yield a system of three equations given by

$$2\gamma = 1 - \beta - (d-1)\alpha, \quad \alpha = \max\left(1 + \gamma, \frac{1}{2}\right), \quad \beta = \max\left(1 + 2\gamma, \frac{1}{2}\right) \tag{6.17}$$

Q5. a. Solve (by hand or by using Mathematica) this system of equations. Determine the range of validity of each of the solutions you find by taking the constraints on α and β into account. You can use that

$$\max(x, y) = \frac{1}{2}(x + y + |x - y|) \tag{6.18}$$

Comment of the nature of the orientation information diffusion in the different regimes. You should find three distinct regimes.

b. What is the minimal dimension for which long-range order is possible?

c. For the physicists; why doesn't this contradict the theorem of question **2.1,Q4c**?

References

[1] M. Ballerini, N. Cabibbo, R. Candelier, *et al.*, "Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study," *Proceedings of the national academy of sciences*, vol. 105, no. 4, pp. 1232–1237, 2008.

6.3 Solutions

6.3.1 Kuramoto model

A1. The interaction is attractive.

A2. Multiplying both sides of (6.2) by $e^{-i\theta_i}$ gives

$$r_N(t)e^{i(\psi_N(t)-\theta_i(t))} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j(t)-\theta_i(t))}. \tag{6.19}$$

Now expressing both complex exponentials using Euler we have

$$\begin{aligned} r_N(t)\cos(\psi_N(t)-\theta_i(t)) &+ i r_N(t)\sin(\psi_N(t)-\theta_i(t)) \\ &= \frac{1}{N}\sum_{j=1}^{N}\Big[\cos\big(\theta_j(t)-\theta_i(t)\big) + i\sin\big(\theta_j(t)-\theta_i(t)\big)\Big]. \end{aligned} \tag{6.20}$$

Here we just equate the terms without an i on the left with the terms without an i on the right and do the same for terms with an i so that

$$r_N(t)\cos(\psi_N(t)-\theta_i(t)) = \frac{1}{N}\sum_{j=1}^{N}\cos\big(\theta_j(t)-\theta_i(t)\big)$$
$$r_N(t)\sin(\psi_N(t)-\theta_i(t)) = \frac{1}{N}\sum_{j=1}^{N}\sin\big(\theta_j(t)-\theta_i(t)\big).$$

The right hand side of the second equation matches the sum that appears in our interaction term so we can replace it with the left hand side, giving us:

$$K r_N(t)\sin(\psi_N(t)-\theta_i(t)).$$

A3. The equation basically says that you only interact with the average angle of the oscillators and that your interaction is modulated by the amount of synchronization there is. This makes it much easier to analyze and simulate the model.

A4. Since $\omega = 0$ for all oscillators and we are considering the stationary density (i.e. time independent density) we can write $p(\theta)$ instead of $p_t(\theta,\omega)$ as well as r and ψ instead of $r(t)$ and $\psi(t)$. This simplifies (6.4) to

$$0 = -\frac{\partial}{\partial\theta}\Big[p(\theta)Kr\sin(\psi-\theta)\Big] + D\frac{\partial^2 p(\theta)}{\partial\theta^2}. \tag{6.21}$$

We can rewrite this in a more suggestive form:

$$\frac{\partial^2 p(\theta)}{\partial\theta^2} = \frac{\partial}{\partial\theta}\Big[p(\theta)\frac{Kr}{D}\sin(\psi-\theta)\Big]. \tag{6.22}$$

This implies that

$$\frac{\partial p(\theta)}{\partial\theta} = p(\theta)\frac{Kr}{D}\sin(\psi-\theta) \tag{6.23}$$

The solution has the form

$$p(\theta) = \frac{1}{Z}\exp\left[\frac{Kr}{D}\cos(\psi-\theta)\right] + C. \tag{6.24}$$

Using the boundary conditions we have

$$p(\theta) = \frac{1}{Z}\exp\left[\frac{Kr}{D}\cos(\psi-\theta)\right] + C = \frac{1}{Z}\exp\left[\frac{Kr}{D}\cos(\psi-\theta+2\pi)\right] + C = p(\theta+2\pi), \tag{6.25}$$

which implies that $C = 0$. For normalization

$$\int_0^{2\pi} p(\theta)\mathrm{d}\theta = 1 \tag{6.26}$$

so that

$$Z = \int_0^{2\pi} \mathrm{e}^{\frac{Kr}{D}\cos(\psi-\theta)}\mathrm{d}\theta, \tag{6.27}$$

which is a special function called a *modified Bessel function of the first kind.*

A5.

a. Plugging the solution of the previous question into (6.7) we have

$$r\mathrm{e}^{i\psi} = \int_0^{2\pi} \mathrm{e}^{i\theta}\frac{1}{Z}\exp\left[\frac{Kr}{D}\cos(\psi-\theta)\right]\mathrm{d}\theta. \tag{6.28}$$

Multiplying both sides by $\mathrm{e}^{-i\psi}$ and collecting real and imaginary parts gives

$$r = \frac{\int_0^{2\pi}\cos(\psi-\theta)\exp\left[\frac{Kr}{D}\cos(\psi-\theta)\right]\mathrm{d}\theta}{\int_0^{2\pi}\mathrm{e}^{\frac{Kr}{D}\cos(\psi-\theta)}\mathrm{d}\theta}. \tag{6.29}$$

b. We can consider the equation for the synchronization level to be of the form $r = V(\frac{Kr}{D})$ where.

$$V(x) := \frac{\int_0^{2\pi}\cos(\psi-\theta)\exp\left[x\cos(\psi-\theta)\right]\mathrm{d}\theta}{\int_0^{2\pi}\mathrm{e}^{x\cos(\psi-\theta)}\mathrm{d}\theta}. \tag{6.30}$$

From this expression we can see that $V(0) = 0$. The derivative of $V(x)$ is

$$V'(x) = \frac{\int_0^{2\pi}\cos^2(\psi-\theta)\exp\left[x\cos(\psi-\theta)\right]\mathrm{d}\theta\int_0^{2\pi}\mathrm{e}^{x\cos(\psi-\theta)}\mathrm{d}\theta}{\left(\int_0^{2\pi}\mathrm{e}^{x\cos(\psi-\theta)}\mathrm{d}\theta\right)^2} \tag{6.31}$$

$$-\frac{\left(\int_0^{2\pi}\cos(\psi-\theta)\exp\left[x\cos(\psi-\theta)\right]\mathrm{d}\theta\right)^2}{\left(\int_0^{2\pi}\mathrm{e}^{x\cos(\psi-\theta)}\mathrm{d}\theta\right)^2} \tag{6.32}$$

$$= \frac{\int_0^{2\pi}\cos^2(\psi-\theta)\exp\left[x\cos(\psi-\theta)\right]\mathrm{d}\theta\int_0^{2\pi}\mathrm{e}^{x\cos(\psi-\theta)}\mathrm{d}\theta}{\left(\int_0^{2\pi}\mathrm{e}^{x\cos(\psi-\theta)}\mathrm{d}\theta\right)^2} - V(x). \tag{6.33}$$

From this we see that

$$V'(0) = \frac{1}{2\pi}\int_0^{2\pi}\cos^2(\psi-\theta)\mathrm{d}\theta = \frac{1}{2}. \tag{6.34}$$

the equation for r around small r is thus

$$r = \frac{Kr}{2D}. \tag{6.35}$$

By plotting both left and right hand side of the full equation as functions of r, we see that there is a non-zero solution for r when $K > K_c$ with

$$K_c := 2D. \tag{6.36}$$

c. For synchronization to occur the interaction strength must overcome the tendency of the noise term to distribute the oscillators evenly on the circle. It thus makes sense that the critical value increases as the noise strength increases.

6.3.2 Swarm behavior and Vicsek Model

A1. At each time step, a randomly chosen bird moves forward by a small step of length 1 in the direction given by θ_t (equation given in the case of a 2-dimensional model):

$$\boldsymbol{x}_{t+1} = \boldsymbol{x}_t + \cos\theta_t \boldsymbol{e}_x + \sin\theta_t \boldsymbol{e}_y \,. \tag{6.37}$$

It then updates its direction θ_{t+1} to be equal to the average value of the directions of the other birds in its neighbourhood plus some random noise:

$$\theta_{t+1} = \langle\theta_t\rangle_{r_c} + \eta_t \,, \tag{6.38}$$

The term η_t represents a Gaussian noise, i.e. $\langle\eta_t\rangle = 0$ and $\langle\eta_t\, \eta_{t'}\rangle = \alpha\, \delta_{t,t'}$, where α is a constant controlling the amplitude of the noise. The constant r_c is the radius of the disk around the bird that defines its neighbourhood. The considered bird interacts with other birds in its neighbourhood by updating its new direction to be the average direction of these birds.

A2. If the birds move at a constant speed v_0, then, at each time steps of the simulation, the time is increased by a small quantity dt and the positions of the birds are updated using:

$$\boldsymbol{x}_{t+dt} = \boldsymbol{x}_t + v_0\, dt\, (\cos\theta_t \boldsymbol{e}_x + \sin\theta_t \boldsymbol{e}_y) \,. \tag{6.39}$$

The update of the directions of the birds remain given by Eq. (6.38).

A3. Metric range: In the Vicsek model, the metric range is the constant denoted r_c in the previous equations. It represents the radius of the area around each bird that is considered to be the neighbourhood of the bird. Birds aligned their direction with other birds in that neighbourhood.

Topological range: The authors of Ref. [1] suggest that birds don't use a metric range to align their direction, as described above, but instead they align their direction with their N_c-th nearest neighbours. The constant N_c, which represents the number of birds to which they align to, is the topological range.

A4. The quantity r_1 is the average distance of the closest neighbor of any bird in the flock. The quantity r_N is the average distance of the N-th closest neighbor of any bird in the flock. Let us denote ρ the density of birds in the flock; we assume that this density is constant over the flock. We then obtained that:

$$\rho \sim \frac{1}{r_1^3} \qquad \text{and} \qquad \rho \sim \frac{N}{r_N^3} \,, \tag{6.40}$$

as we expect to find on average 1 bird in a ball of radius r_1 and N birds in a ball of radius r_N. Assuming that the density is constant in the flock, this leads to:

$$\frac{1}{r_1^3} \sim \frac{N}{r_N^3} \,, \tag{6.41}$$

which thus gives Eq. (6.8): $r_N \sim r_1\, N^{1/3}$. This relation applies to $N = n_c$, for which $r_N = r_c$; one gets that:

$$r_c \sim r_1\, n_c^{1/3} \,, \tag{6.42}$$

A5. Considering the following two hypotheses:

(a) **metric scenario:** birds align to their neighbors that are within a fixed distance r_c from them, where the distance r_c is a constant over all flocks: r_c = constant. Equation (6.8) then gives a dependence of n_c in r_1:

$$r_c = \text{constant} \qquad \text{and} \qquad n_c^{1/3} \sim r_1^{-1} \,. \tag{6.43}$$

(b) **topological scenario:** the birds align with their n_c first neighbors, where the number n_c is fixed and constant over all flocks (of these type of birds): n_c = constant. From Eq. (6.8) we then obtain a dependence of r_c in r_1:

$$n_c = \text{constant} \qquad \text{and} \qquad r_c \sim r_1\,. \tag{6.44}$$

The authors of Ref. [1] recorded multiple flocks for the same species of birds and computed the values of n_c (using a measure of anisotropy in the positions N-th birds), r_1, and finally r_c (using the relation (6.42)). Fig. 3.c and d shows the behavior of n_c and r_c as a function of r_1 for different flocks. We observe that the value of n_c doesn't seem to depend on the flock, while r_c depends linearly in r_1. The topological distance seems to be the appropriate hypothesis for this problem.

Birds on a lattice

A1. Since the propagation of errors was assumed to be diffusive, we know that in a time τ the error will spread out over a distance $r \sim \sqrt{\tau}$. In d dimensions, the volume over which the error propagates is given by $V_e \sim r^d \sim \tau^{d/2}$. Because the error is conserved, the spin error decays as

$$\delta\theta \sim \delta\theta_0/V_e = \delta\theta_0\tau^{-d/2} \tag{6.45}$$

A2. If we assume that errors are produced within a volume V_e at a rate proportional to τ, then the total number of errors should scale with the volume. We also know that errors propagate over the volume V_e as $V_e \sim \tau^{d/2}$. Combining both assumptions we have

$$n_e \sim \tau V_e \sim \tau^{1+d/2} \tag{6.46}$$

A3. The total error is given by

$$\Omega_e = \sqrt{\sum_{i=1}^{n_e}(\delta\theta_i)^2} \approx \sqrt{n_e\,\langle(\delta\theta)^2\rangle} \sim \sqrt{n_e} \tag{6.47}$$

A4. a. The error amplitude per spin is the total expected error divided by the volume over which the errors have propagated. Therefore

$$\Delta\theta \sim \Omega_e/V_e \sim \frac{\sqrt{n_e}}{V_e} \sim \frac{\sqrt{\tau V_e}}{V_e} \sim \sqrt{\frac{\tau}{V_e}} \sim \sqrt{\frac{r^2}{r^d}} \sim r^{1-d/2} \tag{6.48}$$

b. If $d < 2$, then $1 - d/2 > 0$ and therefore errors propagate throughout the entire system and long-range order is not possible. If $d > 2$, then $1 - d/2 < 0$ and errors will decay. This means that order is resistant to fluctuations, enabling the possibility of long-range order. In $d = 2$, fluctuations can still propagate throughout the entire system, but order will be disturbed on a very slow timescale.

c. The absence of long-range order is predicted by the Mermin-Wagner theorem, which states that continuous symmetries cannot be spontaneously broken in systems with sufficiently short-range interactions in $d \leq 2$. For this model, that means that in $d \leq 2$, there is no spontaneous magnetization, i.e. $\langle s_i\rangle = 0$.

Birds not on a lattice: the Vicsek model

A1. If two birds start from the same location $\boldsymbol{x}_0(\tau = 0) = (0, 0)$ with angles $\theta_1 = 0$ and $\theta_2 = \delta\theta$, then after a time τ we have

$$\boldsymbol{x}_1(\tau) = (v_0\tau, 0), \quad \boldsymbol{x}_2(\tau) = (v_0\tau\cos\delta\theta, v_0\tau\sin\delta\theta) \tag{6.49}$$

The difference between these locations is

$$\delta\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1 = (\delta x_\parallel, \delta x_\perp) = (v_0\tau(1-\cos(\delta\theta)), v_0\tau\sin(\delta\theta_0)) \tag{6.50}$$

A2. Expanding the sine and cosine we get

$$\delta x_\perp = v_0\tau\sin(\delta\theta) \approx v_0\tau\delta\theta \sim \tau\delta\theta \tag{6.51}$$

$$\delta x_\parallel = v_\tau(1-\cos(\delta\theta)) \approx v_0\tau\left(1 - 1 + \frac{1}{2}(\delta\theta)^2\right) \sim \tau(\delta\theta)^2 \tag{6.52}$$

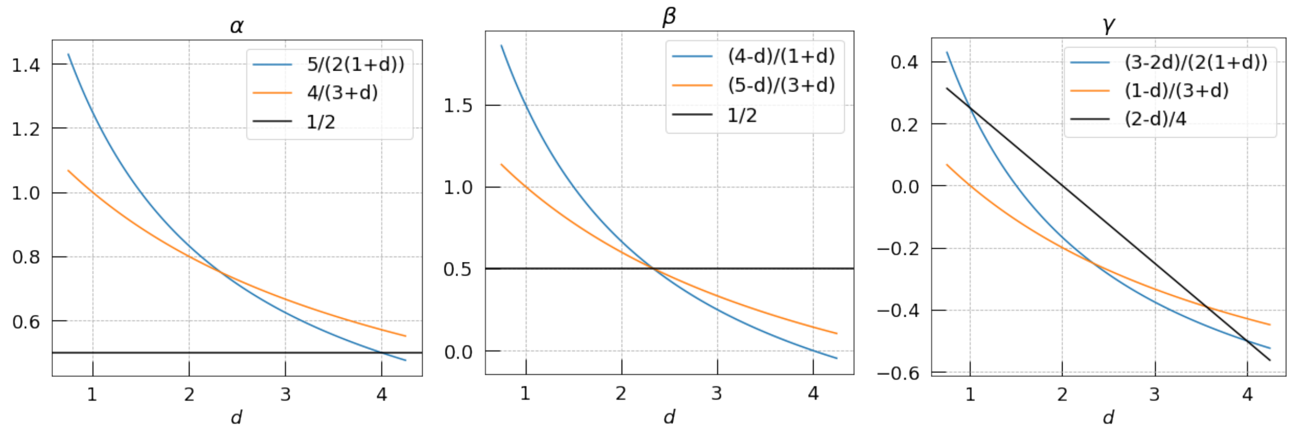

Figure 6.2: Solution regimes of the exponents α, β and γ.

A3. Using the same reasoning as for the lattice spins, we have

$$\delta\theta \sim \frac{\Omega_e}{V_e} \sim \sqrt{\frac{\tau}{V_e}} \sim \frac{\tau^{1/2}}{\sqrt{w_\perp^{d-1} w_\parallel}} \tag{6.53}$$

A4. We start with the equation for $\delta\theta$ and write

$$\delta\theta \sim \frac{\tau^{1/2}}{\sqrt{\tau^{\alpha(d-1)}\tau^{\beta}}} = \tau^{\gamma} \tag{6.54}$$

Squaring both sides we have

$$\frac{\tau}{\tau^{\alpha(d-1)}\tau^{\beta}} = \tau^{2\gamma} \tag{6.55}$$

and therefore

$$\tau = \tau^{2\gamma+\alpha(d-1)+\beta} \implies 1 = 2\gamma + \alpha(d-1) + \beta \tag{6.56}$$

Secondly, we have

$$\tau^{\alpha} = \tau(\delta\theta) + D_\perp \tau^{1/2} = \tau^{\gamma+1} + D_\perp \tau^{1/2} \tag{6.57}$$

and

$$\tau^{\beta} = \tau(\delta\theta)^2 + D_\parallel \tau^{1/2} = \tau^{2\gamma+1} + D_\parallel \tau^{1/2} \tag{6.58}$$

At large values of τ, the behavior of the exponent will tend to the largest exponent in the sum terms, thus

$$\alpha = \max\left(1+\gamma, \frac{1}{2}\right), \quad \beta = \max\left(1+2\gamma, \frac{1}{2}\right) \tag{6.59}$$

A5. a. Plugging the system of equations into Mathematica and using the analytic expression for the maximum of two numbers, we obtain four solutions. The trick here is to look at the value of γ as a function of d and check the consistency relations $\alpha = \max\left(1+\gamma, \frac{1}{2}\right)$ and $\beta = \max\left(1+2\gamma, \frac{1}{2}\right)$. An easier way to find the different regimes is to plot the different solution curves and find the only consistent way to match the three regimes. Looking at Fig. 6.2, we see that in order to continuously move between the different solution regimes, we should move from the black curve to the blue curve at $d = 4$ and move from the blue curve to the orange curve at $d = 7/3$.
For $d \geq 4$ the solution is consistent if

$$\alpha = \beta = \frac{1}{2}, \quad 2\gamma - 1 + \frac{1}{2}(d-1) + \frac{1}{2} = 0 \implies \gamma = \frac{1}{2} - \frac{d}{4} \tag{6.60}$$

In this case, we have a diffusive system (characterized by the exponent $1/2$) and $\gamma < 0$ indicates the existence of long-range order. For $7/3 \leq d < 4$, the solution is given by

$$\alpha = \frac{5}{2(d+1)}, \quad \beta = \frac{1}{2}, \quad \gamma = \frac{3-2d}{2(d+1)} \tag{6.61}$$

For d in this range, $\gamma < 0$. Note that in this case we have super-diffusive propagation (due to $\alpha > \frac{1}{2}$) in the direction transverse to the motion. For $d < 7/3$, the valid solution is given by

$$\alpha = \frac{4}{d+3}, \quad \beta = \frac{5-d}{d+3}, \quad \gamma = \frac{1-d}{d+3} \tag{6.62}$$

giving $\gamma < 0$ for any $d > 1$ and super-diffusive propagation in both the longitudinal and transverse direction.

b. From the above, we see that the minimal dimension in which long-range order is possible is $d > 1$.

c. The Mermin-Wagner theorem holds for equilibrium systems. The Vicsek model - due to the active nature of the particles - is out of equilibrium. The ability to move allows for faster propagation of orientation information through the flock, thereby allowing for long-range order in dimensions lower than 2.