# Chapter 2

# Examples of Equilibrium Critical Phenomena

Questions and exercises indicated with a ($\star$) are optional. No worries if you don't have time to try to solve them, or if you don't manage to solve them on your own.

Most of the exercises below are mostly based on the lectures and are here to help you re-derive some of the results we have seen in class. For equilibrium systems, all the information about the properties of the systems are contained in the partition function (denoted $Z$ below). The exercises on the Ising and Potts models thus focus on deriving the partition function.

## 2.1 Ising Model

### 2.1.1 Macroscopique quantities of interest ("Observables")

> Consider a system of $N$ spin interacting placed in a uniform external field $H$. The total energy of the system is:
>
> $$E(\vec{s}) = -J \sum_{\langle i,j \rangle} s_i s_j - H \sum_{i=1}^{N} s_i \,, \tag{2.1}$$
>
> where $J$ is a coupling constant and $H$ is the external field. The notation $\langle i, j \rangle$ means that the summation is over pairs of spins $i$ and $j$ that are nearest neighbors. Note that which pairs of spins are considered "neighbors" will depend on the underlying lattice structure of the system, which we haven't yet specified for the moment

**Q1.** The temperature $T$ of the system is set by a thermostat, such that the probability distribution of the microstates of the system is given by the Boltzmann distribution. Can you recall the form of this distribution? What is the partition function $Z(T, H)$?

**Q2.** The free energy $F$ is a good potential thermodynamic for this system. Can you recall it's thermodynamic definition? How can one compute the free energy of this system from its microscopic description?
Because the free energy is an extensive quantities (i.e. it scales linearly with the system size), we introduce the free energy per spin $f = F/N$ (which is an intensive quantity, i.e. it doesn't scale with $N$ anymore).

**Q3.** Interesting quantities for the system are the average total magnetization $\langle M \rangle$ and the average total energy $\langle E \rangle$ of the system. Can you recall their definitions? Observe that the average magnetization and the average energy are both extensive quantities (i.e. they are proportional to the number of spin $N$). We introduce the average total magnetization per spin $m = \langle M \rangle / N$ and the average total energy per spin $\epsilon = \langle E \rangle / N$.

**Q4.** The average magnetization and the average energy can be obtained from the free energy (or the log-partition function) by derivation:

$$m(T,H) = -\left(\frac{\partial f}{\partial H}\right)_T = \frac{1}{N}\frac{1}{\beta}\left(\frac{\partial \log Z}{\partial H}\right)_T \qquad \text{and} \qquad \epsilon(T,H) = -\frac{1}{N}\left(\frac{\partial \log Z}{\partial \beta}\right)_H . \tag{2.2}$$

Can you prove these relations?

We recall that $f$ is the free energy per spin, and that $\beta = \frac{1}{k_B T}$ is the inverse temperature.

**Partial derivative:** for a multivariate function $f(x,y)$, one can derive $f$ with respect to any of the two variables $x$ or $y$. To indicate precisely according to which variable we derive $f$, one introduces partial derivatives.
We denote respectively:

$$\left(\frac{\partial f}{\partial x}\right)_y (x,y) \qquad \text{and} \qquad \left(\frac{\partial f}{\partial y}\right)_x (x,y)\,, \tag{2.3}$$

the partial derivative of $f$ with respect to the variable $x$ and the partial derivative of $f$ with respect to the variable $y$. The first partial derivative indicates that we derive $f(x,y)$ with respect to $x$ while keeping the variable $y$ constant; while the second indicates that we derive $f(x,y)$ with respect to $y$ while keeping the variable $x$ constant.

**Q5.** ($\star$)

**Fluctuation-dissipation theorem.** The susceptibility per spin $\chi(T,H)$ characterizes how sensitive is the magnetisation of the system to small modifications of the external field $H$ at any given temperature. Thus, for any given temperature, the susceptibility corresponds to the slope of the magnetisation $m(T,H)$ as a function of $H$:

$$\chi(T,H) = \left(\frac{\partial m}{\partial H}\right)_T . \tag{2.4}$$

Similarly, the specific heat $c(T,H)$ characterizes how much the average energy per spin $\epsilon(T,H)$ changes under small modification of the temperature $T$. For any given external field, the susceptibility corresponds to the slope of the energy $\epsilon(T,H)$ as a function of $T$:

$$c(T,H) = \left(\frac{\partial \epsilon}{\partial T}\right)_H . \tag{2.5}$$

According to the fluctuation-dissipation theorem, the variance of the total energy is related to the specific heat, and the variance of the magnetisation is related to the susceptibility per spin through:

$$\frac{\langle E^2\rangle - \langle E\rangle^2}{N} = k_B T^2\, c \qquad \text{and} \qquad \frac{\langle M^2\rangle - \langle M\rangle^2}{N} = k_B T\, \chi\,. \tag{2.6}$$

Can you prove these relations?

### 2.1.2 System of non-interacting Ising spins

Consider a system of $N$ spin non-interacting ($J = 0$) placed in a uniformed external field $H$. Since the spin are non-interacting, the problem is independent of the underlying lattice and of the dimensionality. The result can serve as a cross reference for the behaviour of the Ising model in the weak-coupling limit $\beta J \longrightarrow 0$ (i.e. $T \longrightarrow +\infty$). The total energy of the system is:

$$E(\vec{s}) = -H\sum_{i=1}^{N} s_i\,. \tag{2.7}$$

**Q1.** The behavior of the system is determined by the relative strength between the external field $H$ and the thermal energy $k_B T$. Can you comment on the behavior that you expect for the system in the limit $H/k_B T \to 0$ and in the limit $H/k_B T \to \pm\infty$? Which values do you expect for the average magnetisation $\langle M \rangle$? For the average energy $\langle E \rangle$?
Observe that the average magnetisation and the average energy are both extensive quantities (i.e. they are proportional to the number of spin $N$). We introduce the average total magnetisation per spin $m = \langle M \rangle / N$ and the average total energy per spin $\epsilon = \langle E \rangle / N$.

**Q2.** Can you compute the partition function $Z(T, H)$ for this system?

**Q3.** From your result for the partition function, can you compute the free energy for this system, using the statistical formulation of the free energy? Deduce that the expression for the free energy per spin is: $f(T, H) = F(T, H)/N = -k_B T \log(2\cosh(\beta H))$.

**Q4.** Using the relation in Eq. (2.2), can you compute $m(T, H)$ and $e(T, H)$ for this system? Observe that when there is no external field ($H = 0$), there is no spontaneous magnetization of the system ($m = 0$) at any temperature.

**Q5.** Using the relation in Eq. (2.6), can you show that, in presence of an external field ($H \neq 0$), the relative fluctuations of the energy and the relative fluctuations of the magnetization both behave as:

$$\frac{\sqrt{\langle E^2 \rangle - \langle E \rangle^2}}{\langle E \rangle} \propto \frac{1}{\sqrt{N}} \qquad \text{and} \qquad \frac{\sqrt{\langle M^2 \rangle - \langle M \rangle^2}}{\langle M \rangle} \propto \frac{1}{\sqrt{N}} \tag{2.8}$$

As the system size is increased (larger and larger $N$), the distribution (over many realization of the system) of a quantity of interests (energy and magnetization, here) becomes more and more sharply peaked around its average value. This is a standard behavior for a system with non-interacting components.

### 2.1.3 (⋆) 1-dimensional system with no external field

> Consider a system of $N$ interacting spin placed on a 1-dimensional line. The total energy of the system can be written as:
>
> $$E(\vec{s}) = -J \sum_{i=1}^{N-1} s_i s_{i+1} \,. \tag{2.9}$$
>
> where $J$ is a coupling constant.

**Q1.** Let $\phi$ be is a binary variable that can take values $\pm 1$ and $x$ a real number. Can you show that:

$$\exp(x\phi) = \cosh(x\phi)(1 + \phi\, \tanh x) \tag{2.10}$$

**Q2.** Using this relation, can you show that the partition function can be re-written under the form:

$$Z = (\cosh \beta J)^{N-1} \sum_{\vec{s}} \prod_{i=1}^{N-1} [1 + s_i s_{i+1} \tanh(\beta J)]\ ? \tag{2.11}$$

**Q3.** Can you show that for any spin $s_i$, its (non-weighted) average value over all the states of the system is zero:

$$\sum_{\vec{s}} s_i = 0\ ? \tag{2.12}$$

Can you show that this is also true for any sub-product of spins? i.e., let $\phi(\vec{s})$ be a product of a subset of spins:

$$\sum_{\vec{s}} \phi(\vec{s}) = 0\ ? \tag{2.13}$$

**Q4.** Expand the product over $i$ in Eq. (2.11). Using the results of question Q3, can you show that the only term in this expansion that doesn't cancel after summing over the states $\vec{s}$ is the term "1"?

**Q5.** Finally, deduce that, for a one-dimensional Ising system with no external field:

$$Z = 2^N (\cosh \beta J)^{N-1} . \tag{2.14}$$

**Q6.** How is this result modified if we were to consider periodic boundary conditions? i.e.

$$E(\vec{s}) = -J \sum_{i=1}^{N} s_i s_{i+1} \qquad \text{with } s_{N+1} = s_1 . \tag{2.15}$$

Can you show that the two results are identical in the thermodynamic limit?

## 2.2 (★) Potts model

Consider a Potts model with $q$ states ($s_i \in \{1, 2, \cdots, q\}$) on a 1-d lattice with periodic boundary conditions:

$$E(\vec{s}) = -J \sum_{i=1}^{N-1} \delta(s_i, s_{i+1}) - J\delta(s_1, s_N) \tag{2.16}$$

where $\delta(x, y) = 1$ if $x = y$ and $\delta(x, y) = 0$ if $x \neq y$.

**Q1.** Compute the partition function $Z(T)$ of this system using the transfer matrix method.

**Q2.** We place $N$ dots on a circle and we have $q$ colors. From the result of the last question, can you obtain the number of ways to color the $N$ dots such that there is no two consecutive dots with the same color.

## 2.3 Percolation

Try to re-derive the results seen during the lectures.

## 2.4 Short reminder on conditional probabilities

> **Independent events.** Two events $A$ and $B$ are independent if the probability of their simultaneous occurrence is the product of the probabilities that each of them occurs: $P(A \cap B) = P(A)\, P(B)$.

> **Conditional probability.** The probability of an event $A$ conditional on the occurrence of an event $B$, is defined
>
> $$P(A\,|\,B) \doteq \frac{P(A \cap B)}{P(B)} .$$
>
> This probability can also be read as the probability that an event $A$ occurs, given that another event $B$ has already occurred.
> Equivalently, $P(A \cap B) = P(A\,|\,B)\, P(B)$, i.e. that the probability that both $A$ and $B$ occur (*joint probability of $A$ and $B$*) is the probability that $B$ occurs, times the probability that $A$ occurs given $B$. If $A$ and $B$ are independent then $P(A\,|\,B) = P(A)$.

**Q1.** Consider families with two children $\Omega = \{bb, bg, gb, gg\}$ where the first (second) character stands for the sex (boy or girl) for the elder (younger) child. Imagine that all four possibilities occur with the same probability $P(\omega) = 1/4$.

(a) Given that a family has a boy, what is the probability that the other child is also a boy?

(b) Given that the older child of a family is a boy, what is the probability that the younger is also a boy?

## 2.5 Solutions

### 2.5.1 Ising model: Macroscopic quantities of interest

**Q1.** The probability to find the system in a particular state $\mathbf{s}$ is weighted by the energy of that stated in the for of a Boltzmann factor $\left(e^{-\beta E(\mathbf{s})}\right)$. The corresponding Boltzmann distribution is given by

$$P(\mathbf{s}) = \frac{1}{Z}\exp\left(-\beta E(\mathbf{s})\right) , \tag{2.17}$$

where $\beta = 1/(k_B T)$. The partition function ensures that this distribution is normalized and is consequently given by the sum of the Boltzmann factors over all the system configurations.

$$Z = \sum_{\mathbf{s}} \exp\left(-\beta E(\mathbf{s})\right) . \tag{2.18}$$

**Q2.** The free energy is defined as

$$F = E - TS , \tag{2.19}$$

where $E$ is the internal energy of the system and $S$ is the entropy. An equivalent expression is given by

$$F = -k_B T \log Z = -\frac{1}{\beta}\log Z . \tag{2.20}$$

Since $Z$ contains the sum over all specific configurations, this is how to compute the free energy given the knowledge of its microscopic description in terms of the partition function.

**Q3.** We can find averages of any relevant observable by computing its thermal average

$$\langle A\rangle = \sum_{\mathbf{s}} A(\mathbf{s})P(\mathbf{s}) = \frac{1}{Z}\sum_{\mathbf{s}} A(\mathbf{s})\exp\left(-\beta E(\mathbf{s})\right) . \tag{2.21}$$

The total magnetization is given by the sum over all the values of the spins, i.e.

$$M = \sum_{i=1}^{N} s_i . \tag{2.22}$$

Hence,

$$\begin{aligned}
\langle M\rangle &= \sum_{\mathbf{s}} M(\mathbf{s})\exp\left(-\beta E(\mathbf{s})\right) = \sum_{\mathbf{s}}\sum_{i=1}^{N} s_i \exp\left(-\beta E(\mathbf{s})\right) \\
\langle E\rangle &= \sum_{\mathbf{s}} E(\mathbf{s})\exp\left(-\beta E(\mathbf{s})\right) .
\end{aligned}$$

**Q4.** We can find many of the thermal averages by taking appropriate derivatives of the partition function (2.18). For instance, to find the average total energy we take a derivative with respect to $\beta$,

$$\frac{\partial Z}{\partial \beta} = \sum_{\mathbf{s}}\left(-E(\mathbf{s})\right)\exp\left(-\beta E(\mathbf{s})\right) = -Z\langle E\rangle . \tag{2.23}$$

This is almost in the form of a thermal average. We can complete the expression by writing

$$\langle E\rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \equiv -\frac{\partial \log Z}{\partial \beta} \tag{2.24}$$

For the magnetization we can do something similar. Note that the expression for the magnetization occurs in the expression for the system energy (2.1) with the prefactor $H$. Therefore, if we take a derivative of the partition function with respect to $H$ we find

$$\frac{\partial Z}{\partial H} = \sum_{\mathbf{s}} \beta\left[\sum_{i=1}^{N} s_i\right]\exp\left(-\beta E(\mathbf{s})\right) = \beta Z\langle M\rangle \tag{2.25}$$

and thus we can also write that the average total magnetization is given by

$$\langle M\rangle = \frac{1}{\beta Z}\frac{\partial Z}{\partial H} = \frac{1}{Z}\frac{\partial}{\partial H}\frac{Z}{\beta} = \frac{\partial}{\partial H}\left(\frac{\log Z}{\beta}\right) \equiv -\frac{\partial}{\partial H}F \tag{2.26}$$

Having defined the magnetization and energy per spin,

$$m \equiv \langle M\rangle/N\,, \quad \epsilon \equiv \langle E\rangle/N\,, \tag{2.27}$$

We can divide equations (2.24), (2.5.1) with N and derive the desired expressions:

$$m = -\frac{\partial f}{\partial H}\,, \quad \epsilon = -\frac{1}{N}\left(\frac{\partial \log Z}{\partial \beta}\right) \tag{2.28}$$

**Q5.**

- For the specific heat: We showed that

$$\epsilon = -\frac{1}{N}\left(\frac{\partial \log Z}{\partial \beta}\right)\,, \tag{2.29}$$

  but we now want to link it with $c$ which involves a derivative with respect to T. It is useful to use the chain rule to switch to a derivative with respect to $\beta$

$$\beta \equiv \frac{1}{k_B T} \implies \mathrm{d}\beta = -\frac{1}{k_B T^2}\,\mathrm{d}T \implies \frac{\partial}{\partial \beta} = -k_B T^2\frac{\partial}{\partial T} \implies \frac{\partial}{\partial T} = -k_B\beta^2\frac{\partial}{\partial \beta} \tag{2.30}$$

  Using this relation we find that

$$c \equiv \frac{\partial \epsilon}{\partial T} = \frac{k_B\beta^2}{N}\frac{\partial^2 \log Z}{\partial \beta^2} = \frac{k_B\beta^2}{N}\frac{\partial}{\partial \beta}\left[\frac{1}{Z}\frac{\partial Z}{\partial \beta}\right] = \frac{k_B\beta^2}{N}\left[\frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2} - \left(\frac{1}{Z}\frac{\partial Z}{\partial \beta}\right)^2\right] \tag{2.31}$$

  The last term is equal to the square of $\langle E\rangle$ (see eq. (2.24)). The second derivative of the partition function with respect to $\beta$ is given by

$$\frac{\partial^2 Z}{\partial \beta^2} = \frac{\partial}{\partial \beta}\left[\sum_{\mathbf{s}}(-E(\mathbf{s}))\exp\left(-\beta E(\mathbf{s})\right)\right] = \sum_{\mathbf{s}}\left[-E(\mathbf{s})\right]^2\exp\left(-\beta E(\mathbf{s})\right) = Z\langle E^2\rangle \tag{2.32}$$

  Upon substitution of both these relations we obtain the relation between the specific heat and the variance of the energy

$$c = k_B\beta^2\left[\frac{\langle E^2\rangle - \langle E\rangle^2}{N}\right] = \frac{1}{k_B T^2}\left[\frac{\langle E^2\rangle - \langle E\rangle^2}{N}\right] \tag{2.33}$$

- For the magnetization we do the same trick

$$\chi = \frac{\partial m}{\partial H} = -\frac{\partial^2 f}{\partial H^2} = \frac{1}{\beta N}\frac{\partial^2}{\partial H^2}\log Z\,, \tag{2.34}$$

  and the proof follows in the same way as for the specific heat.

### 2.5.2 System of non-interacting Ising spins

**Q1.** The probability of finding the system in a state **s** is given by

$$P(\mathbf{s}) = \frac{1}{Z}\exp\left(\beta H\sum_{i=1}^{N} s_i\right) = \frac{1}{Z}\exp(\beta H M)\,.$$

- In the limit $H/k_BT = \beta H \to 0$ , the thermal energy "wins" over the external field. Then we have

$$P(\mathbf{s}) = \frac{1}{Z}\,. \tag{2.35}$$

  So each state becomes equally likely, which means that the most probable microstates consist of spins randomly pointing up and down. Therefore we expect a vanishing magnetization $\langle M\rangle = 0$ and zero energy $\langle E\rangle = 0$.

- In the limit $\beta H \to \pm\infty$, the external field "wins" over the thermal energy: The most probable microstates consists of spins that are aligned in the same direction as the external magnetic field $H$, corresponding to a minimum disorder. The behaviour depends on the sign of the external field. If $H > 0$, as $\beta H$ goes to (plus) infinity, the state with all spins pointing up becomes the lowest energy state, to such an extent as to completely dominate the probability distribution. We thus have

$$\langle M\rangle = N, \quad \langle E\rangle = -HN, \quad \beta H \to \infty \tag{2.36}$$

  **NOTE:** The way to derive $\langle E\rangle$ is to notice that in eq. (2.1), when $|H| \to \infty$, the second term dominates, so we can safely neglect the first one.

  The minus infinity (i.e. $H < 0$) case is the same but with all the spins pointing down. Thus

$$\langle M\rangle = -N, \quad \langle E\rangle = -HN, \quad \beta H \to -\infty\,. \tag{2.37}$$

**Q2.** The partition function is sum of the Boltzmann factors over all configurations of the system. So explicitly we have

$$Z = \sum_{\mathbf{s}} \exp\left(\beta H \sum_{i=1}^{N} s_i\right) = \sum_{\mathbf{s}} \prod_{i=1}^{N} \exp(\beta H s_i) = \underbrace{\sum_{s_1} \exp(\beta H s_1)}_{Z_1} \sum_{s_2} \exp(\beta H s_2) \cdots \sum_{s_N} \exp(\beta H s_N) = (Z_1)^N\,. \tag{2.38}$$

Note that because the spins are all independent, the partition function factorizes into contributions from each individual spin. Each spin $s_i$ can take the value $\pm1$, so each factor is just

$$Z_1 = \sum_{s_i=\pm1} \exp(\beta H s_i) = 2\cosh(\beta H)\,. \qquad \text{Reminder: } \cosh x = \frac{e^x + e^{-x}}{2} \tag{2.39}$$

So, for a system with N spins, the partition function is

$$Z = (Z_1)^N = 2^N [\cosh(\beta H)]^N\,. \tag{2.40}$$

**Q3.** Since we know that $F = -k_BT \log Z$ we find that

$$F = -k_BT \log Z = -Nk_BT \log[2\cosh(\beta H)]\,, \tag{2.41}$$

and therefore

$$f = \frac{F}{N} = -k_BT \log[2\cosh(\beta H)]\,. \tag{2.42}$$

**Q4.** The average magnetization per site follows directly from the free energy expression

$$m = -\frac{df}{dH} = k_BT \frac{1}{2\cosh(\beta H)} 2\beta \sinh(\beta H) = \tanh(\beta H)\,. \tag{2.43}$$

Likewise, the average energy is given by

$$\varepsilon = -\frac{1}{N}\left(\frac{\partial \log Z}{\partial \beta}\right) = -\frac{\partial}{\partial \beta} \log[2\cosh(\beta H)] = -\frac{1}{2\cosh(\beta H)} 2H \sinh(\beta H) = -H\tanh(\beta H)\,. \tag{2.44}$$

**Q5.** For the energy, we use the relation

$$\langle E^2 \rangle - \langle E \rangle^2 = Nk_B T^2 c \sim N, \qquad \langle E \rangle = \varepsilon N = -HN\tanh(\beta H) \sim N\ . \tag{2.45}$$

Since $c$ is related to $\varepsilon$ $\left(c \equiv \frac{\partial \varepsilon}{\partial T}\right)$, and $\varepsilon$ does not depend on $N$, then $c$ does not depend on N too. Thus,

$$\frac{\sqrt{\langle E^2 \rangle - \langle E \rangle^2}}{\langle E \rangle} \sim \frac{\sqrt{N}}{N} \sim \frac{1}{\sqrt{N}}. \tag{2.46}$$

The same argument holds for the fluctuations of the magnetization.

### 2.5.3 One-dimensional Ising system with no external field

**Q1.** Note first that

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}. \tag{2.47}$$

From this follows that

$$\exp(x\phi) = \cosh(x\phi) + \sinh(x\phi). \tag{2.48}$$

Now remember that $\cosh(x)$ is an even function and $\sinh(x)$ is an odd function of $x$. This we know that

$$\cosh(x) = \cosh(-x), \quad \sinh(x) = -\sinh(-x). \tag{2.49}$$

Since $\phi$ can only be $\pm 1$ we can use the above relations to write

$$\cosh(\phi x) = \cosh(x), \quad \sinh(\phi x) = \phi \sinh(x). \tag{2.50}$$

With these relations we can write that

$$\exp(x\phi) = \cosh(x) + \phi \sinh(x) = \cosh(x)\left(1 + \phi \frac{\sinh(x)}{\cosh(x)}\right) = \cosh(x)(1 + \phi \tanh(x)). \tag{2.51}$$

**Q2.** Starting from the formal definition of the partition function we have

$$Z = \sum_{\mathbf{s}} \exp\left(\beta J \sum_{i=1}^{N-1} s_i s_{i+1}\right). \tag{2.52}$$

We identify $\phi_i = s_i s_{i+1} = \pm 1$ and write

$$Z = \sum_{\mathbf{s}} \prod_{i=1}^{N-1} \exp(\beta J \phi_i). \tag{2.53}$$

Since $\phi_i$ is a variable that takes binary values $\pm 1$, we can use the identity from before and write

$$Z = \sum_{\mathbf{s}} \prod_{i=1}^{N-1} \cosh(\beta J)\left[1 + \phi_i \tanh(\beta J)\right]. \tag{2.54}$$

Since $\cosh(\beta J)$ does not depend on $s_i$ we can take it out of the product and obtain

$$Z = \cosh(\beta J)^{N-1} \sum_{\mathbf{s}} \prod_{i=1}^{N-1} \left[1 + \phi_i \tanh(\beta J)\right]. \tag{2.55}$$

**Q3.** Consider a state $\mathbf{s} = (s_1, s_2, \ldots, s_i, \ldots, s_N)$.
For all such states, there exists another state of the form $\mathbf{s}' = (s_1, s_2, \ldots, -s_i, \ldots, s_N)$ where $s_i$ is flipped. Since there are $2^N$ states, there are $2^{N-1}$ states with $s_i = 1$ and $2^{N-1}$ states with $s_i = -1$. Thus we can write

$$\sum_{\mathbf{s}} s_i = 2^{N-1} - 2^{N-1} = 0. \tag{2.56}$$

A product oif spins can always be written as $\phi(\mathbf{s}) = s_i \times \phi'(\mathbf{s})$ where $\phi'(\mathbf{s})$ is the remainder of the product without $s_i$, and so

$$\sum_{\mathbf{s}} \phi(\mathbf{s}) = \sum_{\mathbf{s}} s_i \times \phi'(\mathbf{s}) = \sum_{s_i=\pm 1} s_i \sum_{\mathbf{s}'} \phi'(\mathbf{s}) = \sum_{\mathbf{s}'} \phi'(\mathbf{s}) - \sum_{\mathbf{s}'} \phi'(\mathbf{s}) = 0. \tag{2.57}$$

**Q4.** The product has the form

$$\begin{aligned}\sum_{\mathbf{s}} (1+\phi_1)(1+\phi_2)\ldots(1+\phi_N) &= \sum_{\mathbf{s}} (1+\phi_1+\phi_2+\cdots+\phi_1\phi_2+\phi_1\phi_3+\cdots+\phi_1\phi_2\ldots\phi_N) \\ &= \sum_{\mathbf{s}} 1 + \sum_{\mathbf{s}} \phi_1 + \cdots + \sum_{\mathbf{s}} \phi_1\phi_2\ldots\phi_N = \sum_{\mathbf{s}} 1 = 2^N\end{aligned}$$

since we just showed that single-spin and product-operators vanish when summed over the possible configurations of the system. Since there are $2^N$ possible configurations for $N$ binary spins, the remaining sum is just the number of configurations.

**Q5.** We have all the ingedients necessary to find that

$$Z = 2^N \cosh(\beta J)^{N-1}. \tag{2.58}$$

**Q6.** A first guess for the partition function would be $Z_{WRONG} = 2^N \cosh(\beta J)^N$, but this is just the partition function for a line of strings of length $N+1$ instead of $N$ and doesn't take the periodic boundary conditions into account. To include these, consider again the product

$$\sum_{\mathbf{s}} \prod_{i=1}^{N} [1 + s_i s_{i+1} \tanh(\beta J)]. \tag{2.59}$$

Since each of the spins $s_i$ can only take values $\pm 1$, a square $(s_i)^2 = 1$ necessarily. Now note that because of the periodic boundary conditions, there is a term in the product expansion of the form

$$\sum_{\mathbf{s}} \prod_{i=1}^{N} [1 + s_i s_{i+1} \tanh(\beta J)] = 1 + \cdots + (s_1 s_2 \times s_2 s_3 \times \cdots \times s_{N-1} s_N \times s_N s_1) \tanh(\beta J)^N.$$

In the final term each spin $s_i$ occurs as a square and so the product evaluates to 1 and therefore

$$\sum_{\mathbf{s}} \prod_{i=1}^{N} [1 + s_i s_{i+1} \tanh(\beta J)] = (1 + \tanh(\beta J)^N) 2^N, \tag{2.60}$$

and so the partition function for the periodic system is given by

$$Z_{\text{periodic}} = 2^N \cosh(\beta J)^N \left(1 + \tanh(\beta J)^N\right) = 2^N \left(\cosh(\beta J)^N + \sinh(\beta J)^N\right). \tag{2.61}$$

**Q7.** We can use the same expansion to derive the average magnetization $m = \langle s_i \rangle$ and the spin-spin correlation function $\langle s_i s_j \rangle$. We start with the average magnatization which we write formally as

$$\langle s_j \rangle = \frac{1}{Z} \sum_{\mathbf{s}} s_j \exp\left(\beta J \sum_{i=1}^{N-1} s_i s_{i+1}\right). \tag{2.62}$$

Note that we use the label $j$ to distinguish the expectation of the spin from the spin's index in the sum. We use the same tricks as above to write this as

$$\langle s_j \rangle = \frac{1}{Z} \sum_{\mathbf{s}} s_j \prod_{i=1}^{N-1} \cosh(\beta J) \left[1 + s_i s_{i+1} \tanh(\beta J)\right]. \tag{2.63}$$

Since the hyperbolic cosine function does not depend on the state of the system, we can take it out of both the sum and the product and write

$$\langle s_j \rangle = \frac{\cosh(\beta J)^{N-1}}{Z} \sum_{\mathbf{s}} s_j \prod_{i=1}^{N-1} \left[1 + s_i s_{i+1} \tanh(\beta J)\right]. \tag{2.64}$$

Let $\phi_i \equiv s_i s_{i+1}$ and define $t = \tanh(\beta J)$. Then we can write the product as

$$s_j \prod_{i=1}^{N-1} [1 + \phi_i t] = s_j \left[1 + \phi_1 t + \phi_2 t + \cdots + \phi_1 \phi_2 t^2 + \cdots + \phi_1 \cdots \phi_{N-1} t^{N-1}\right]. \tag{2.65}$$

Note that, due to the nearest-neighbour lattice, no spin can occur more than twice in any of the products of $\phi$ operators (e.g. $\phi_1 \phi_2 = s_1 s_2 s_2 s_3 = s_1 s_3$). Thus, all of these products consist of a multiple of two spins, all of which are distinct. Therefore, multiplying by any $s_j$ will result in terms that involve $2n + 1$ different spins for $n = 0, 1, 2, \ldots$. If we sum these over all states we find

$$\langle s_j \rangle = \frac{\cosh(\beta J)^{N-1}}{Z} \sum_{\mathbf{s}} \left[s_j + s_j \phi_1 t + \cdots + s_j \phi_1 \cdots \phi_{N-1} t^{N-1}\right] = 0. \tag{2.66}$$

For the spin-spin correlation function we have to evaluate the product

$$s_j s_k \prod_{i=1}^{N-1} [1 + \phi_i t]. \tag{2.67}$$

Note that we can express $s_j s_k$ as the following product of the $\phi$ ( assuming $j < k$ without loss of generality)

$$\phi_j \phi_{j+1} \ldots \phi_{k-2} \phi_{k-1} = s_j (s_{j+1})^2 (s_{j+2})^2 \ldots (s_{k-2})^2 (s_{k-1})^2 s_k = s_j s_k. \tag{2.68}$$

The number of operators involved in constructing $s_j s_k$ like this is directly related to the distance between the spins $|k - j|$. If we call this distance $r$, then the term in the product expansion comes with a factor of $\tanh(\beta J)^r$, i.e. there is a single term in the product expansion of the form

$$s_j s_{j+r} \prod_{i=1}^{N-1} [1 + s_i s_{i+1} \tanh(\beta J)] = s_j s_{j+r} + \cdots + \tanh(\beta J)^r + \ldots \tag{2.69}$$

Summing over all the spins, the hyperbolic tangent term is the only surviving term and we obtain

$$\langle s_j s_{j+1}\rangle = \frac{\cosh(\beta J)^{N-1}}{Z}\sum_{\mathbf{s}}\Big[s_j s_{j+r} + \cdots + \tanh(\beta J)^r + \dots\Big] = \frac{2^N\cosh(\beta J)^{N-1}}{2^N\cosh(\beta J)^{N-1}}\tanh(\beta J)^r = \tanh(\beta J)^r. \tag{2.70}$$

We now define the correlation length $\xi$ and write

$$e^{-r/\xi} = \tanh(\beta J)^r \Rightarrow \xi = -\frac{1}{\log(\tanh(\beta J))}. \tag{2.71}$$

### 2.5.4 Potts model

In the transfer matrix method we define the transfer matrix as a matrix with the appropriate Boltzmann factor in each of its entries. In this case, the diagonal and off-diagonal elements are given by

$$(T_{s_i,s_{i+1}})_{ii} = \exp(\beta J)\;, \quad (T_{s_i,s_{i+1}})_{ij} = 1\,, \, j\neq i\,. \tag{2.72}$$

Consequently, the transfer matrix is a $q\times q$ matrix of the form

$$T_{s_i,s_{i+1}} = \exp(\beta J)\,\mathbb{I}_q + \left(\mathbf{1}_{q\times q} - \mathbb{I}_q\right) = [\exp(\beta J) - 1]\,\mathbb{I}_q + \mathbf{1}_{q\times q}\,, \tag{2.73}$$

where $\mathbb{I}_q$ is the identity matrix and $\mathbf{1}_{q\times q}$ is a matrix filled with ones.

Let $a \equiv \exp(\beta J)$. To solve for the eigenvalues we write

$$\left[(a-1)\mathbb{I}_q + \mathbf{1}_{q\times q}\right]\mathbf{v} = \lambda\mathbf{v} \tag{2.74}$$

Note that since $\mathbf{1}_{q\times q}$ applied to any vector, results to a vector with the sum of its components as entries, a vector with an equal number of $+1$ and $-1$ returns the zero vector. Denoting these vectors as $\mathbf{u}_\pm$ we find that

$$\left[(a-1)\mathbb{I} + \mathbf{1}_{q\times q}\right]\mathbf{u}_\pm = (a-1)\mathbf{u}_\pm \tag{2.75}$$

And so $a-1$ is an eigenvalue. The eigenvectors of this type are generated by a basis of $q-1$ eigenvectors which all have the same eigenvalue (check!).

Also note that if we have a vector with all ones, which we denote $\mathbf{u}_1 = (1,1,\dots,1)$ then $\mathbf{1}_{q\times q}\mathbf{u}_1 = q\mathbf{u}_1$. So in that case

$$\left[(a-1)\mathbb{I} + \mathbf{1}_{q\times q}\right]\mathbf{u}_1 = (a-1+q)\mathbf{u}_1\,, \tag{2.76}$$

thus $a-1+q$ are also eigenvalues ($q$ of those). For $q=0$ we recover the previous eigenvalue. We have thus found all the eigenvalues of the $q\times q$ transfer matrix.

The eigenvectors of the transfer matrix are the same as those of $\mathbf{1}_{q\times q}$. The partition function is given by the trace of the transfer matrix to the $N-th$ power and therefore

$$Z = \mathrm{Tr}\left[T^N\right] = \sum_{i=1}^{q}(\lambda_i)^N = \left(e^{\beta J} - 1 + q\right)^N + (q-1)\left(e^{\beta J} - 1\right)^N\;. \tag{2.77}$$

**Q2.** This translates to taking the limit $J\to-\infty$. Then, a state with two consecutive dots with the same color, would cost an infinite amount of energy to construct. In that case, only states with no consecutive dots contribute to the partition function (with contribution $e^0 = 1$) and so the number of ways, $n$, of constructing such a state is equal to the partition evaluated at $J\to-\infty$

$$n = \lim_{J\to-\infty} Z = (q-1)^N + (-1)^N(q-1)\,. \tag{2.78}$$