Mean-Field Approximation

Chapter 3

Monday 22 April

Mean-Field Approximation

Chapter 3

Plan: **Lecture 1: Mean-Field Approximation: Ex. of the Ising model**

Lecture 2: Landau Theory

Tutorial: more on Mean-Field approximation

During the coming weeks, we will use the mean-field approximation to solve various problems (TASEP, simple epidemic model, Voter model).

Questions: **homework:** Is there anyone still looking for a group to work with?
Any question?

Information Quiz 2 : will include 2 questions from Quiz 1 (Quiz 1 available as a “training quiz” on Canvas)
When: just after the holidays: Monday 11h, fine?

Mean-Field (MF) approximation
Application to the Ising model

Chapter III — Lecture 1

Plan:

1) Introduction to MF approximation
   a) General idea
   b) Ex. Mean-Field Ising Model
   c) Summary

2) Computing the Order Parameter (m)
   a) Self-Consistency Relation for m
   b) Solving the Self-Consistency Relation
   c) Stability of the solutions
   d) Phase transitions

3) MF Critical exponents
   a) Reminder Critical Exponents
   b) Exercise
   c) Critical exponents and universality

Expectations: Participate in the discussions, take notes, try to do the analytical derivations

References: Book "Complexity and Criticality", K. Christensen, N. Moloney, **Chapter 2: the Ising model (mean-field part)**

Introduction to Mean-Field (MF) approximation

MF approximation: General idea

Mean-field approximation: Weiss (1907): Simplest approximation to try when studying a phase transition

Not restricted to the Ising model

In the context of Ising model: Qualitatively good insight in the phase transition **for $d > 1$**

For $d = 1$: the theory predict a phase transition at a finite temperature **$T_c > 0$** for **$H=0$** **—> not correct, even qualitatively**

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For *d* = *1*: the theory predict a phase transition at a finite temperature **Tc > 0** for **H=0** —> **not correct, even qualitatively**

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$$s_i = \langle s_i \rangle + \epsilon_i \qquad |\epsilon_i| \ll |\langle s_i \rangle|$$

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Application to the Ising model

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$$E_{int}(\vec{s}) =$$

Application to the Ising model

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Application to the Ising model

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$m = \langle s_i \rangle = \langle s_j \rangle$ $\qquad$ q = number of nearest neighbors

Application to the Ising model

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Application to the Ising model

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Total energy: $E(\vec{s}) = E_{int}(\vec{s}) - H \sum_{i=1}^{N} s_i$

Application to the Ising model

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$$E_{int}(\vec{s}) = N \frac{Jm^2 q}{2} - qJm \sum_{i=1}^{N} s_i$$

where $\quad H_{eff} = qJm + H$

Total energy: $$E(\vec{s}) = E_{int}(\vec{s}) - H \sum_{i=1}^{N} s_i = E_0 - H_{eff} \sum_{i=1}^{N} s_i$$

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~ System of non-interacting spins
in the effective field created by the nearest neighbors

Mean-Field Ising model

Mean-field Ising model: Correspond to a system of non-interacting spins, immersed in an effective field of strength $H + Jqm$

where $m = \langle s_i \rangle$ q = number of nearest neighbors

Total energy: $E(\vec{s}) = E_0 - H_{eff} \sum_{i=1}^{N} s_i$ where $H_{eff} = qJm + H$

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$$\boxed{P(\vec{s}) = \prod_{i=1}^{N} P(s_i)}$$

where

$$\boxed{\begin{array}{l} P(s_i) = \dfrac{\mathrm{e}^{-\beta E(s_i)}}{Z_i} \\ E(s_i) = -H_{eff}\, s_i \end{array}}$$

$$Z_i = 2\cosh[\beta H_{eff}]$$

$\Longrightarrow$ **Replace the joint probability distribution by a product of the probability distribution for each spin**

Mean-field Ising model: $\Longrightarrow$ **As if spins are independent**

Each spin is immersed in the average local field created by its nearest neighbors: $H_{loc} = q \times Jm$

Mean-field approximation for Ising Model Summary

Mean-field approximation: Weiss (1907): Simplest approximation to try when studying a phase transition

Not restricted to the Ising model

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Mean-field Ising model: Correspond to a **system of non-interacting spin**, immerse in an effective field of strength $H + Jqm$

where $m = \langle s_i \rangle$ $\quad q$ = number of nearest neighbors $\quad H_{eff} = qJm + H$

$\Longrightarrow$ **As if:**

— **spins are independent**

— each spin is **immersed in** the average **local field created by its nearest neighbors**: $H_{loc} = q \times Jm$

Computing the order parameter (magnetisation per spin)

Self-consistency relation

$$P(\vec{s}) = \prod_{i=1}^{N} P(s_i) \qquad \text{where} \qquad P(s_i) = \frac{\mathrm{e}^{-\beta E(s_i)}}{Z_i} \qquad Z_i = 2\cosh[\beta(qJm + H)]$$

$$E(s_i) = -(qJm + H)s_i$$

Average magnetization per spin: $m = \langle s_i \rangle = \mathbb{P}(s_i = 1) - \mathbb{P}(s_i = -1)$

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Self-consistency relation: Solution

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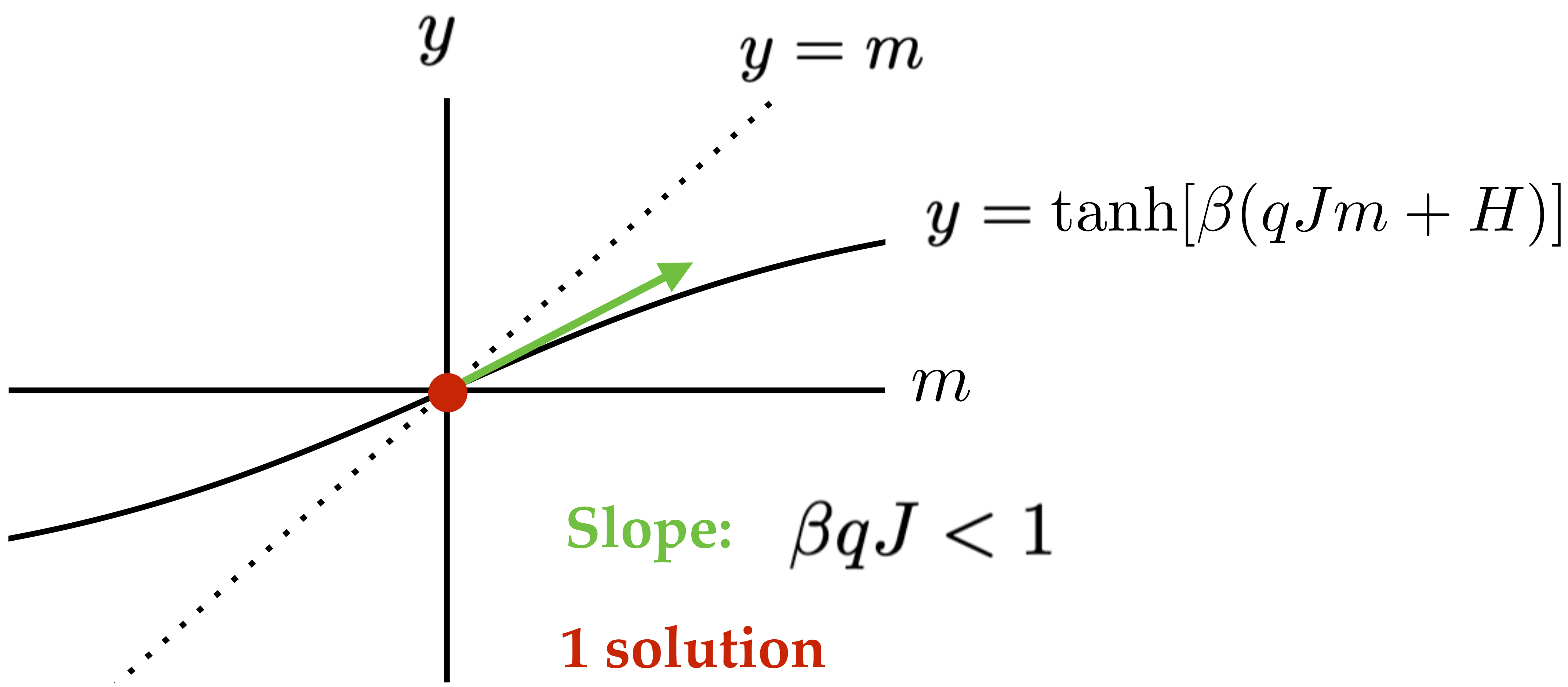
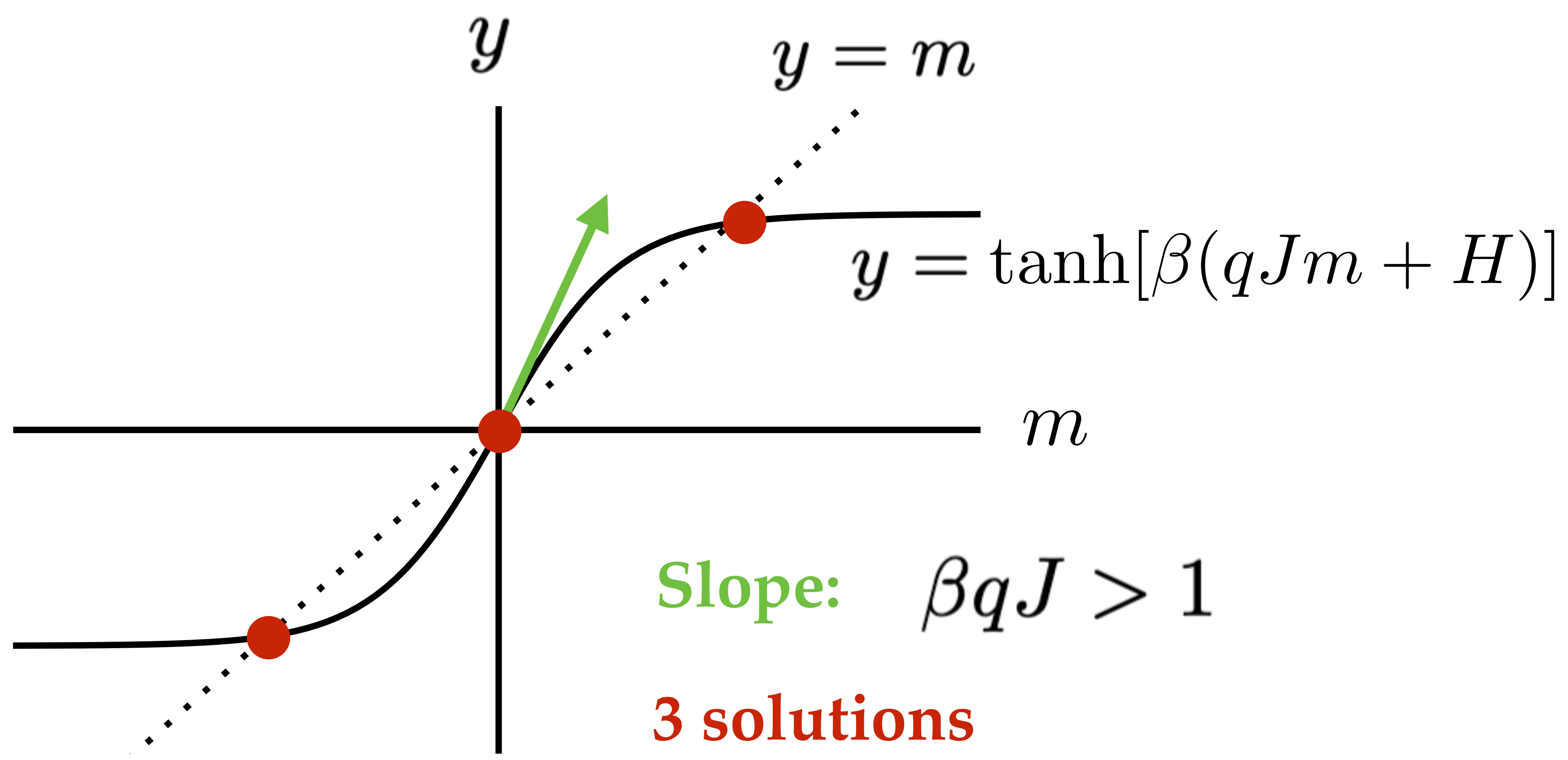
Self-consistency relation: m is solution of: $m = \tanh[\beta(qJm + H)]$

● solutions

$H = 0$





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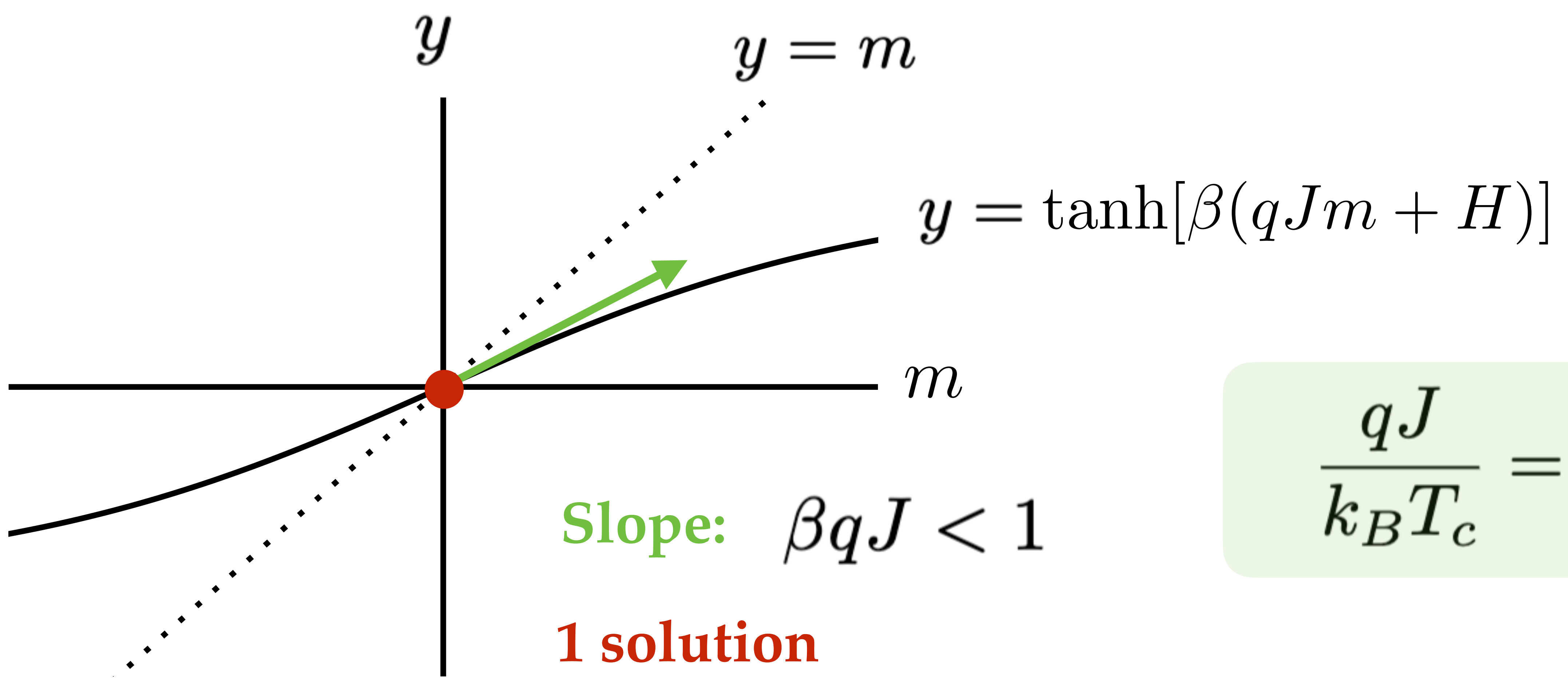
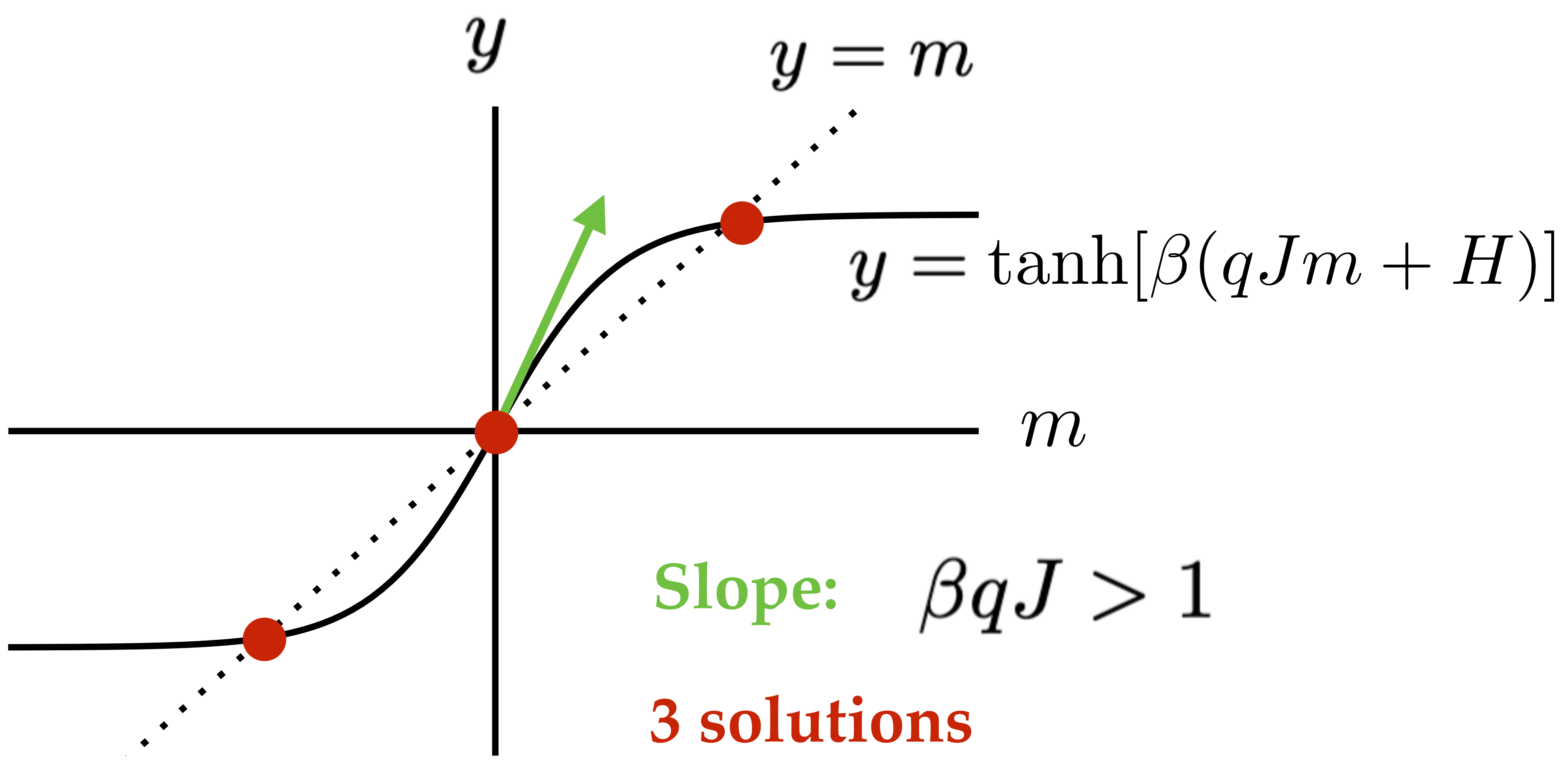


Slope: $\beta qJ < 1$

1 solution

$$\frac{qJ}{k_B T_c} = 1$$



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3 solutions

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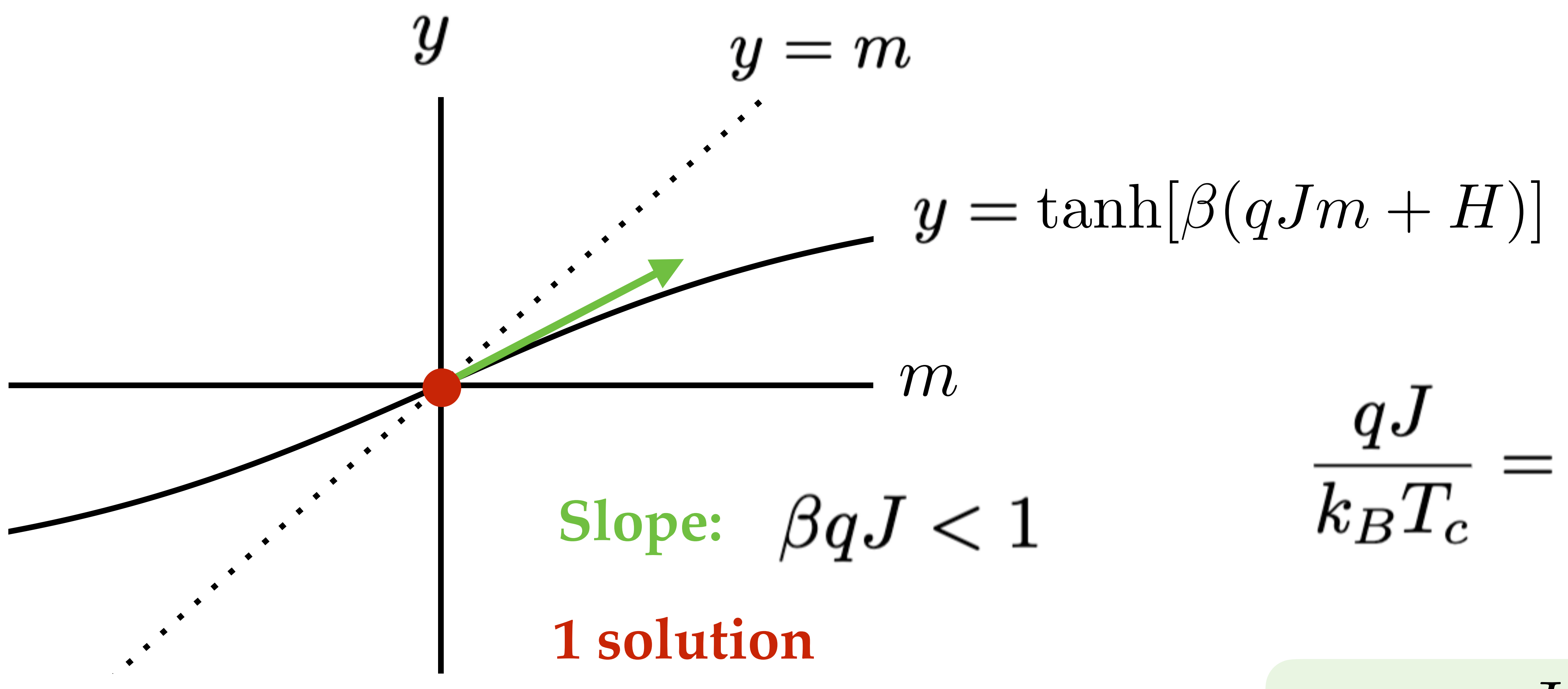
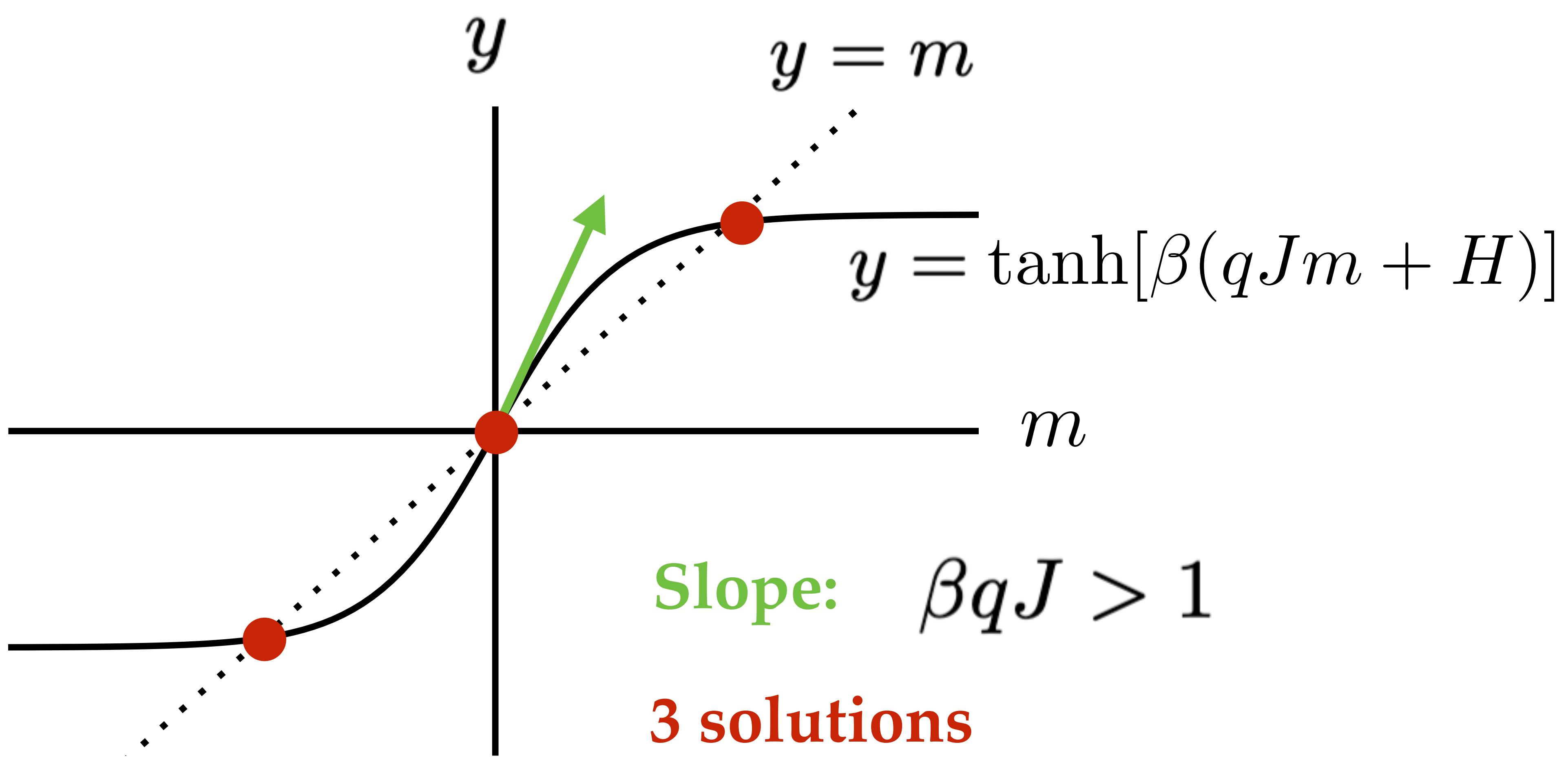
● solutions

$H = 0$



$T > T_c$

$$\frac{qJ}{k_B T_c} = 1$$



$T < T_c$

$$T_c = \frac{qJ}{k_B}$$

Solving the self-consistency equation

Self-consistency relation: Solution

Self-consistency relation: m is solution of: $$m = \tanh[\beta(qJm + H)]$$

● solutions

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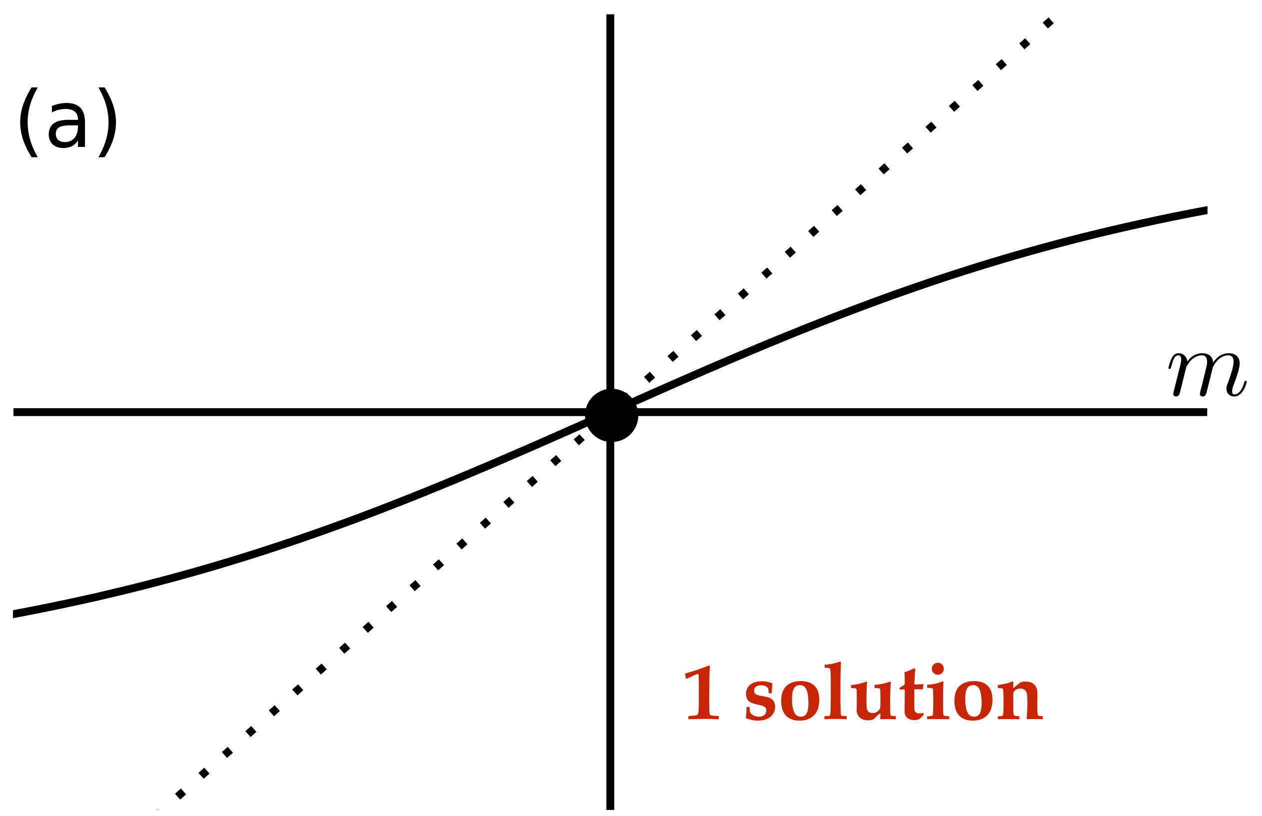
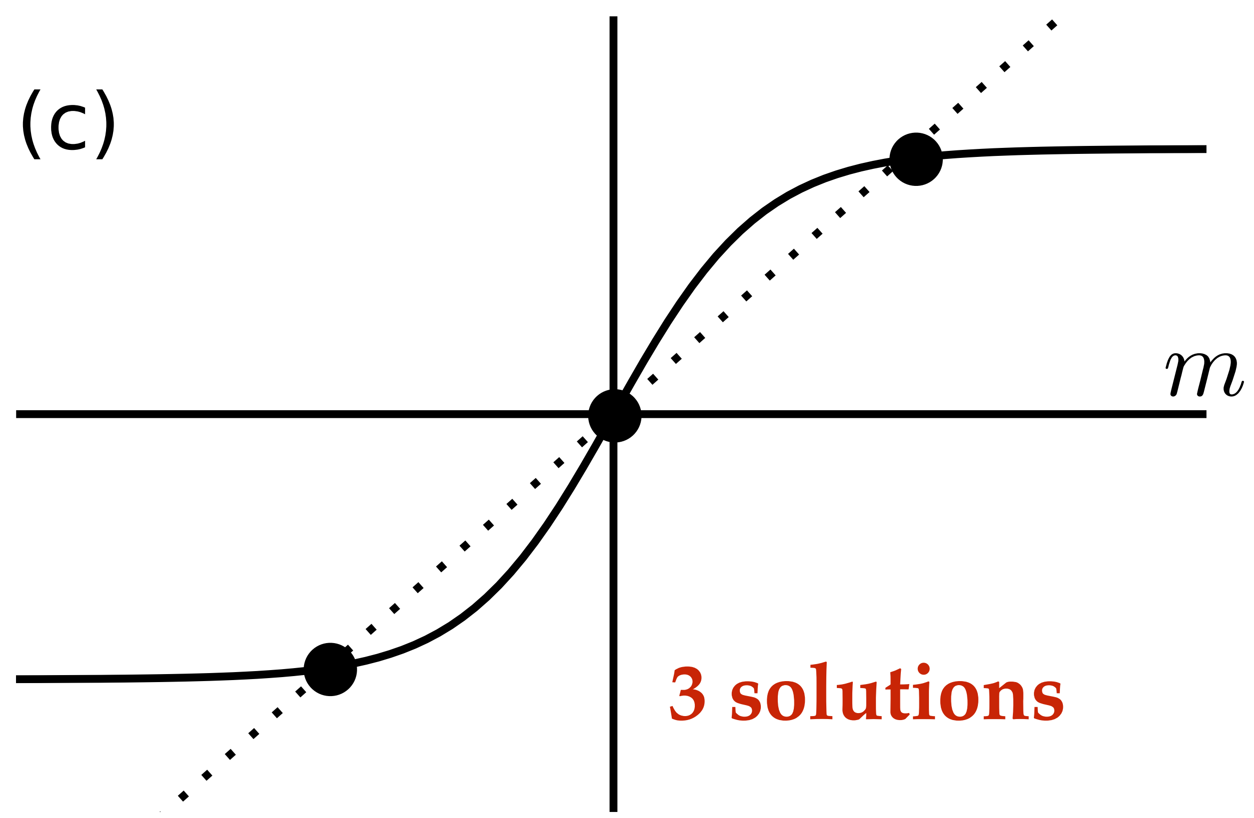


Slope: $\dfrac{qJ}{k_B T}$



$T > T_c$

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Are all three solutions Stables?

Solving the self-consistency equation

Self-consistency relation: Solution

Self-consistency relation: m is solution of: $$m = \tanh[\beta(qJm + H)]$$

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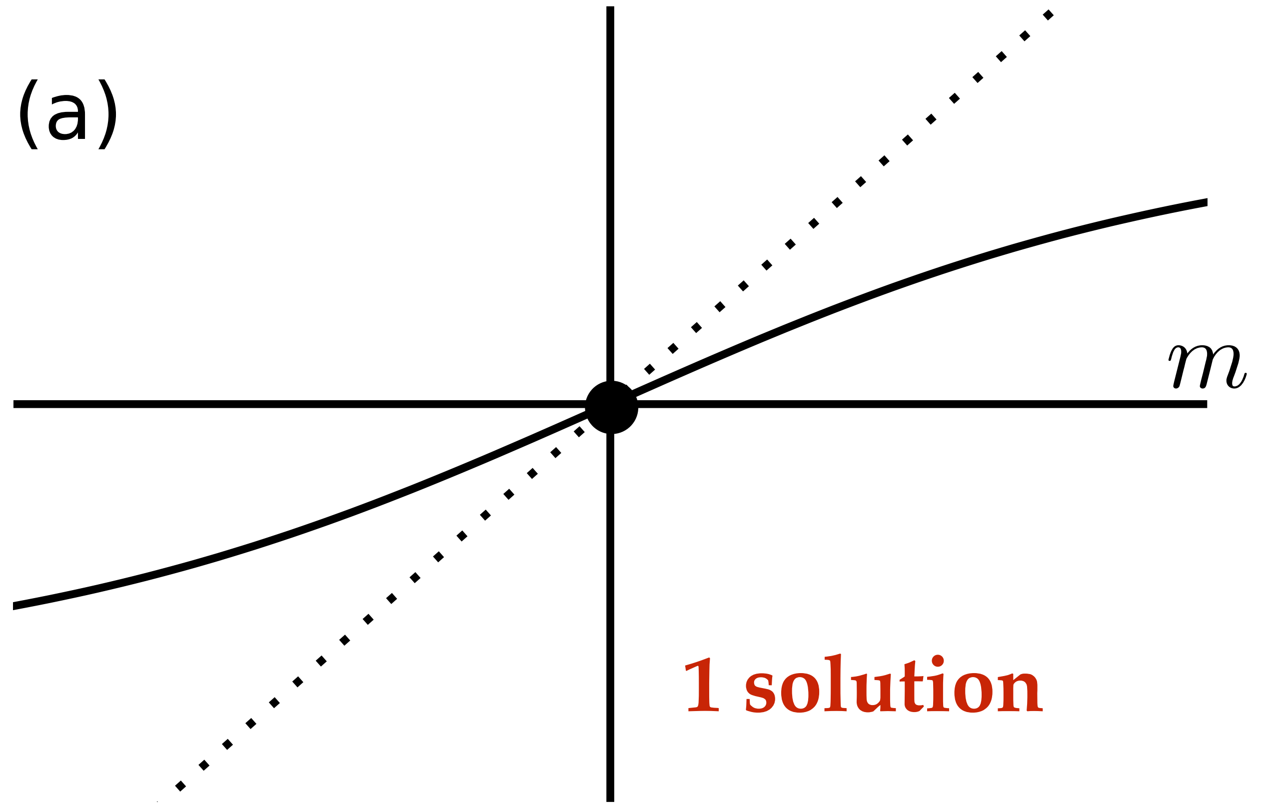

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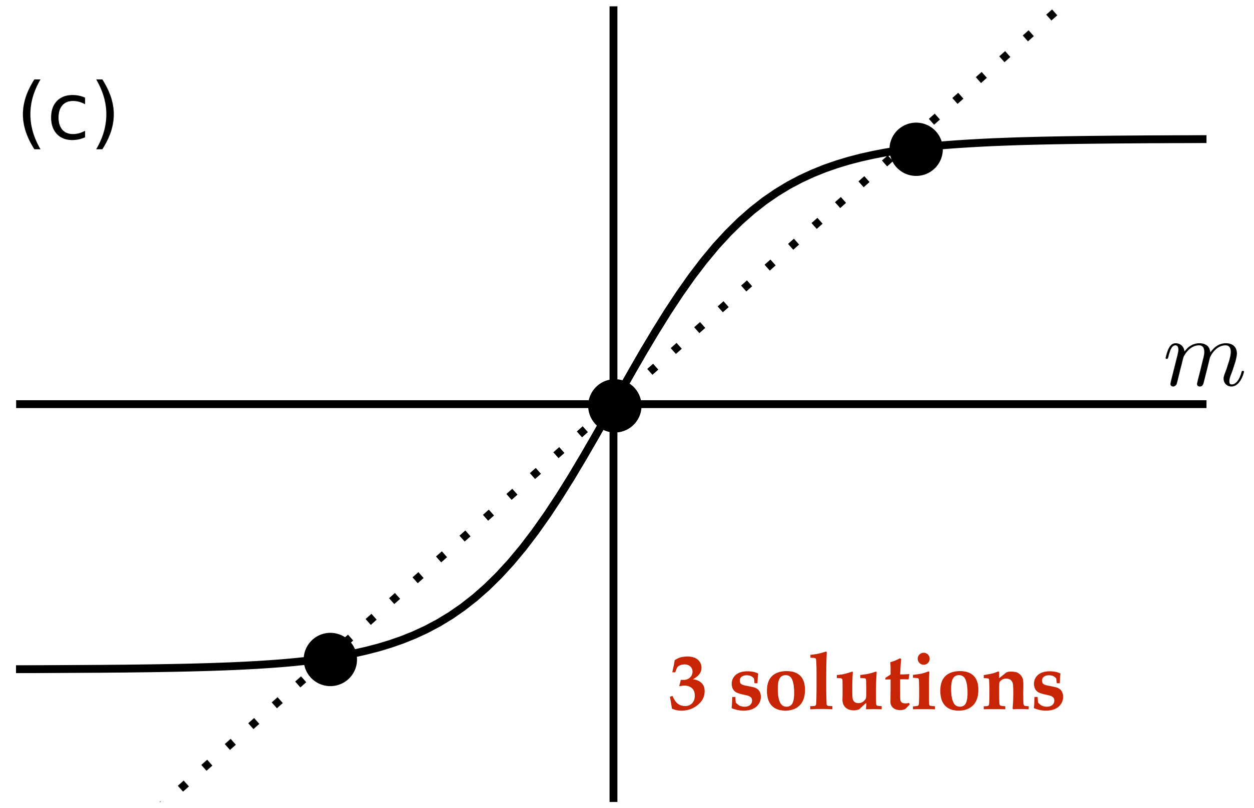

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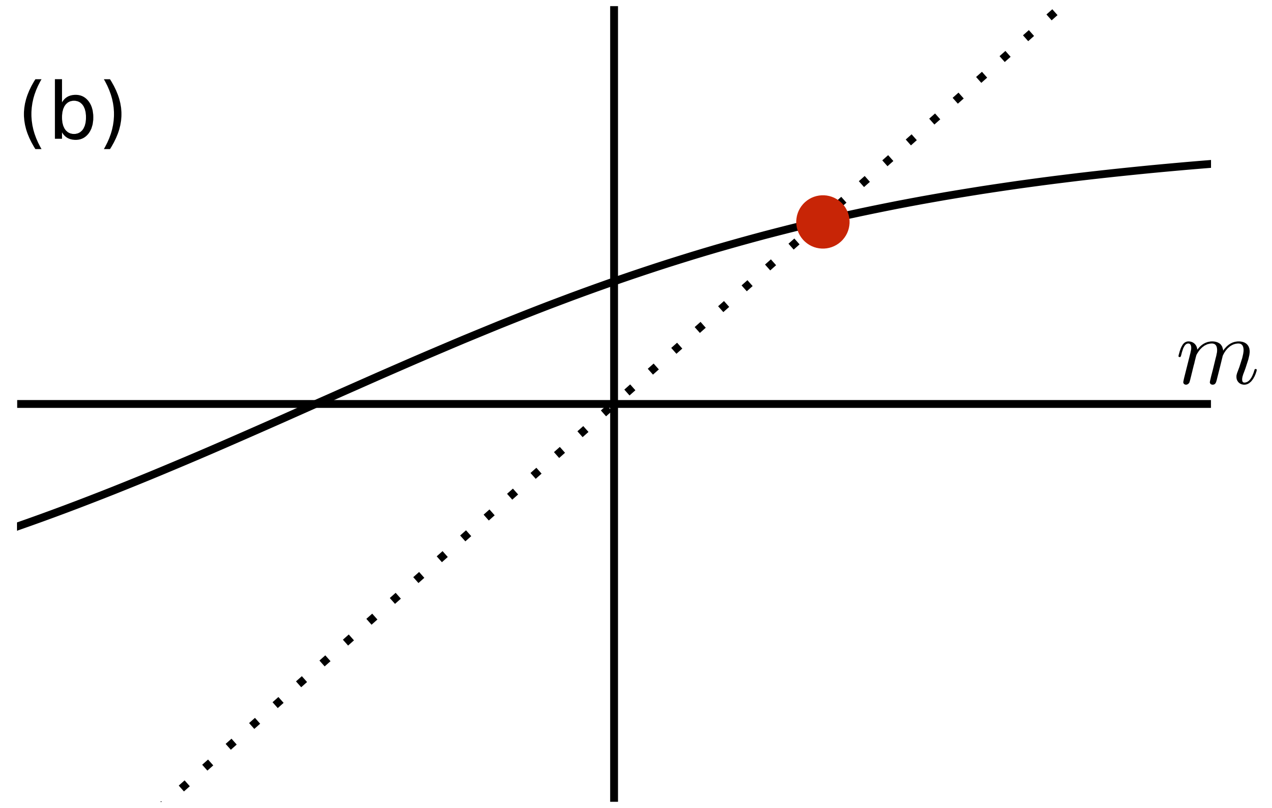

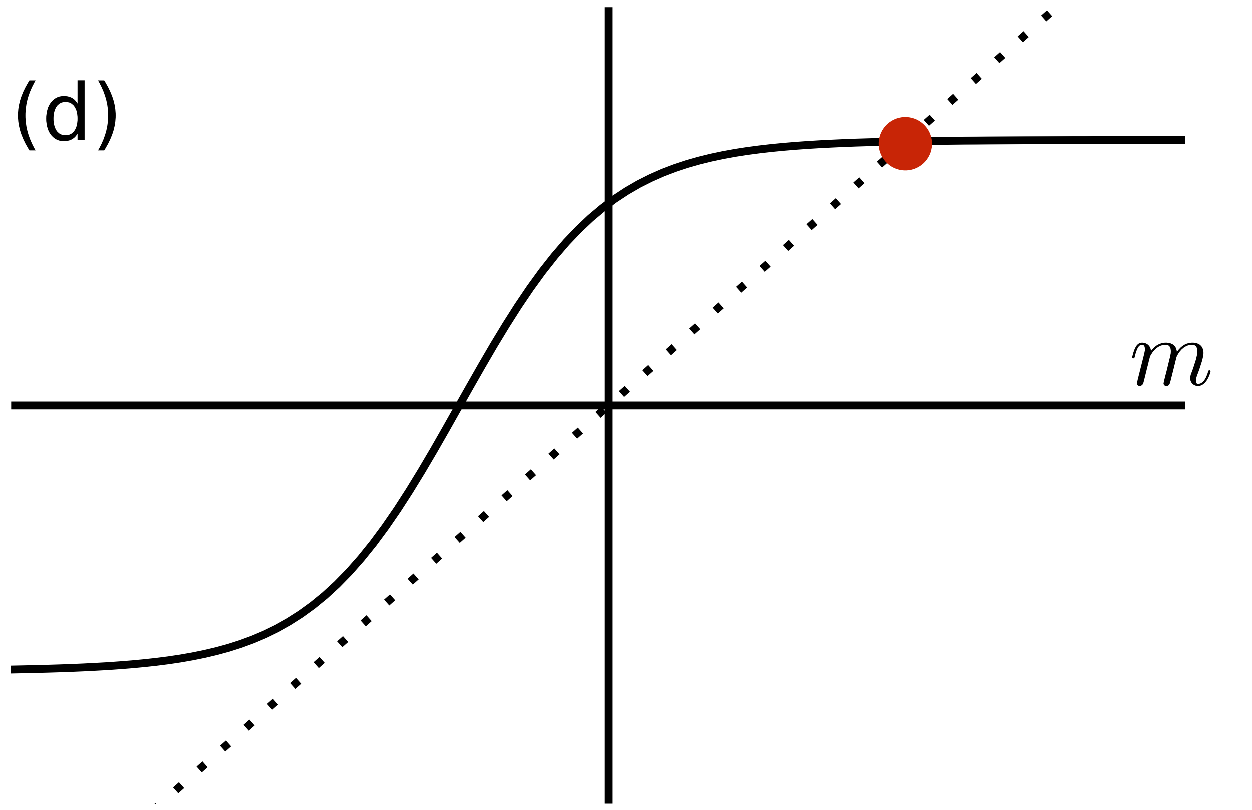

Self-consistency relation: Solution

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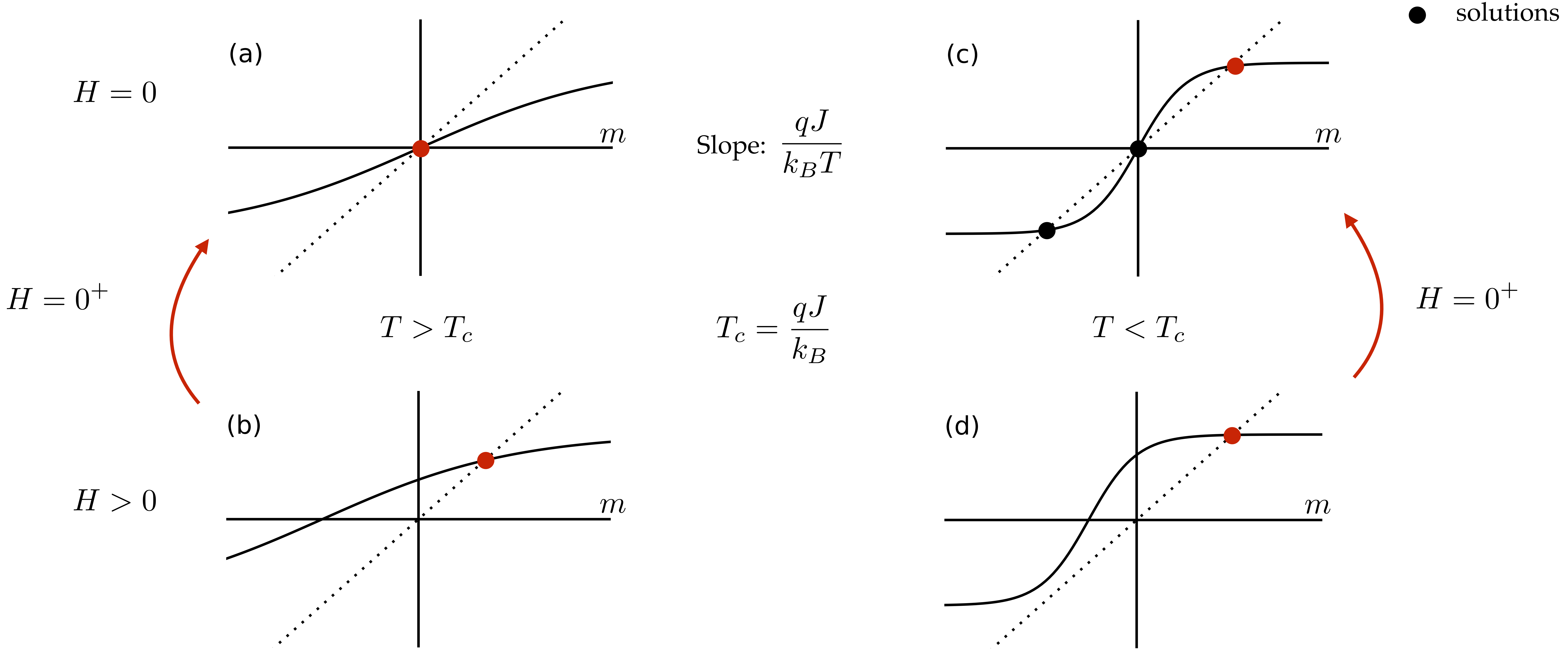

Free energy: analyzing stability

Stability of the solutions

To check which solutions are stable:

$$F = -k_B T \log Z \qquad \text{where} \qquad Z = \exp\left(-\frac{\beta N J m^2 q}{2}\right) \prod_{i=1}^{N} \overbrace{[2\cosh[\beta(qJm + H)]]}^{Z_i}$$

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$$f = \frac{k_B T_c}{2} m^2 - k_B T \log\left[2\cosh\left(\frac{T_c}{T} m + \frac{H}{k_B T}\right)\right]$$

$$T_c = \frac{qJ}{k_B}$$

Stability of the solutions

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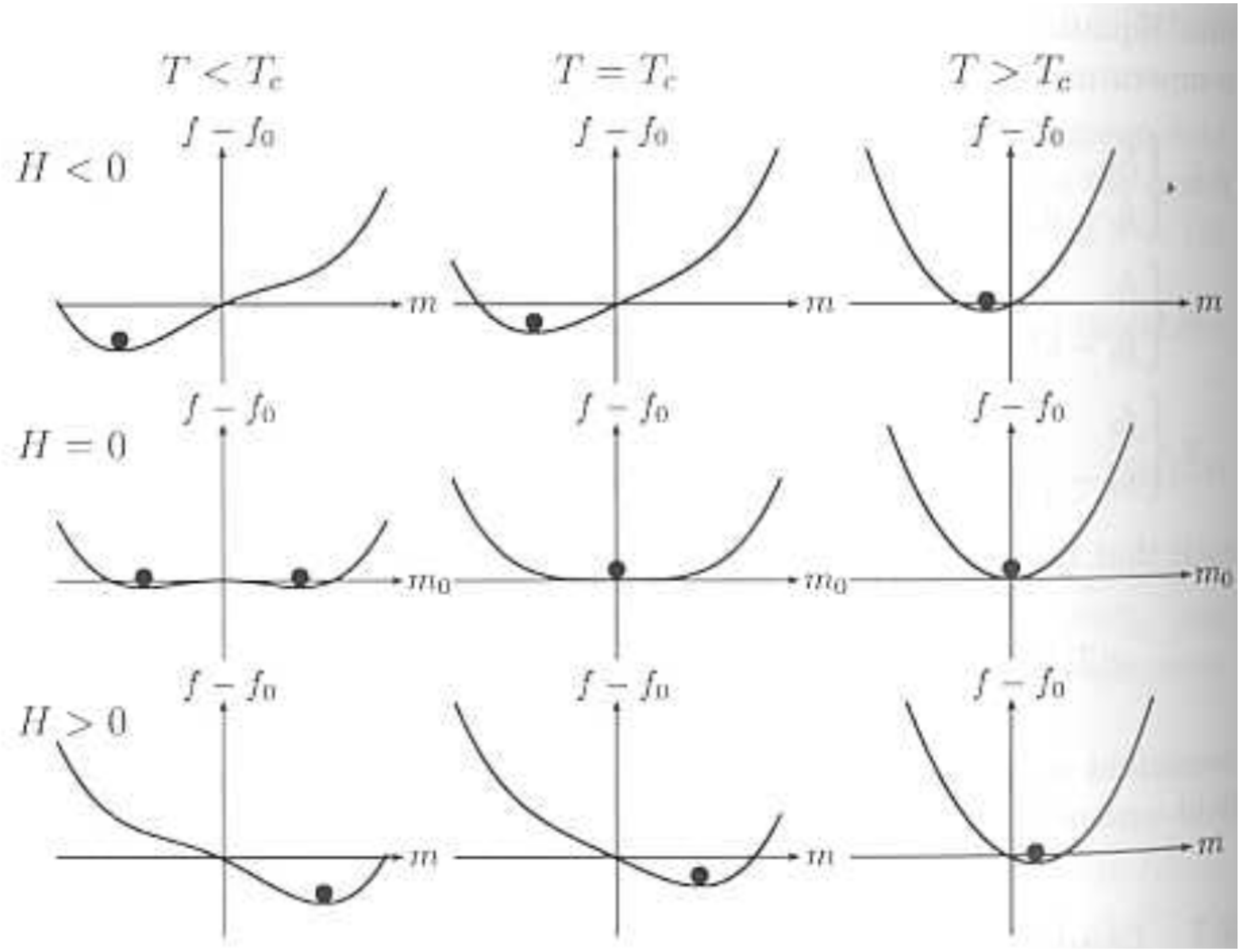

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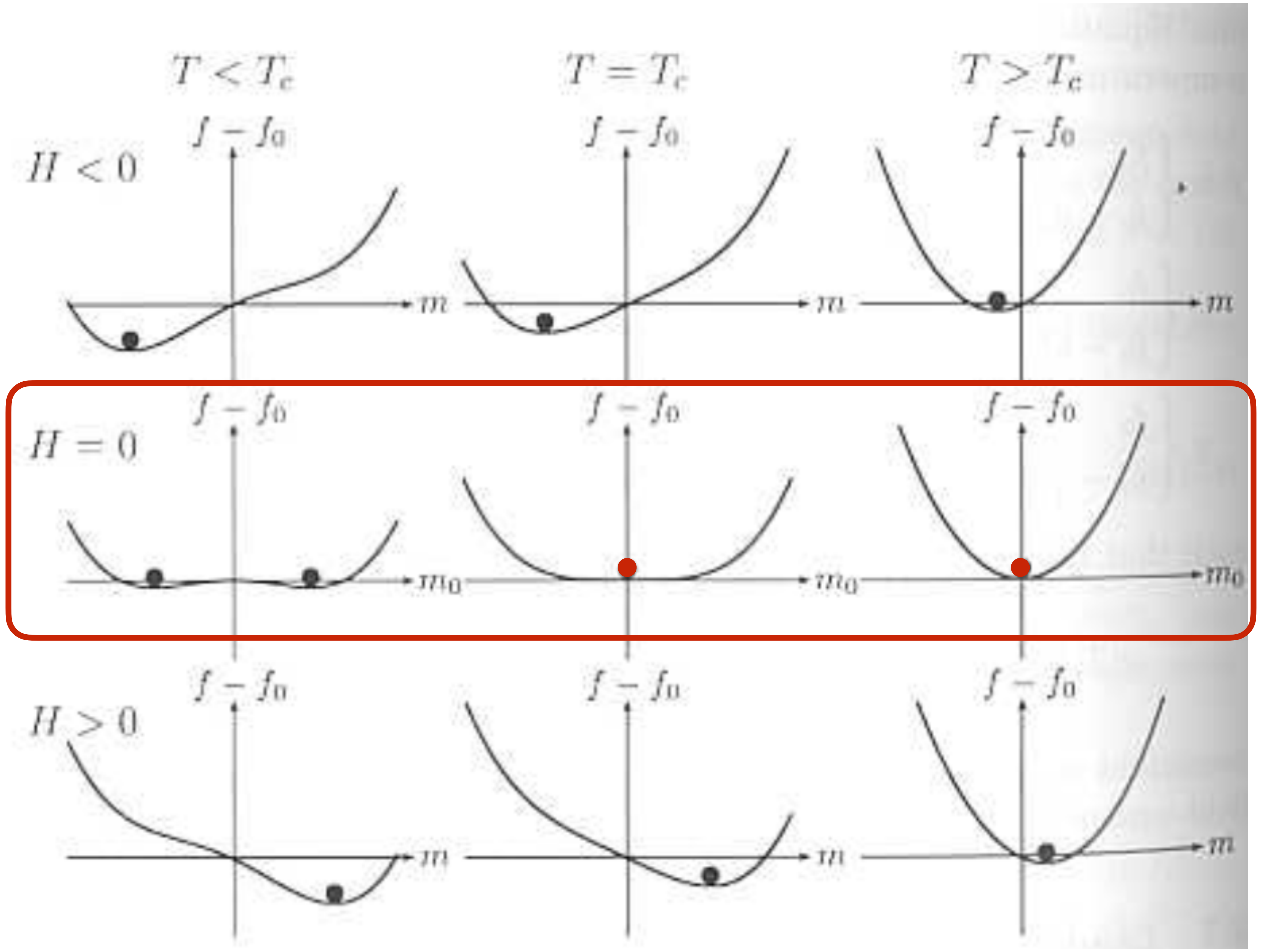
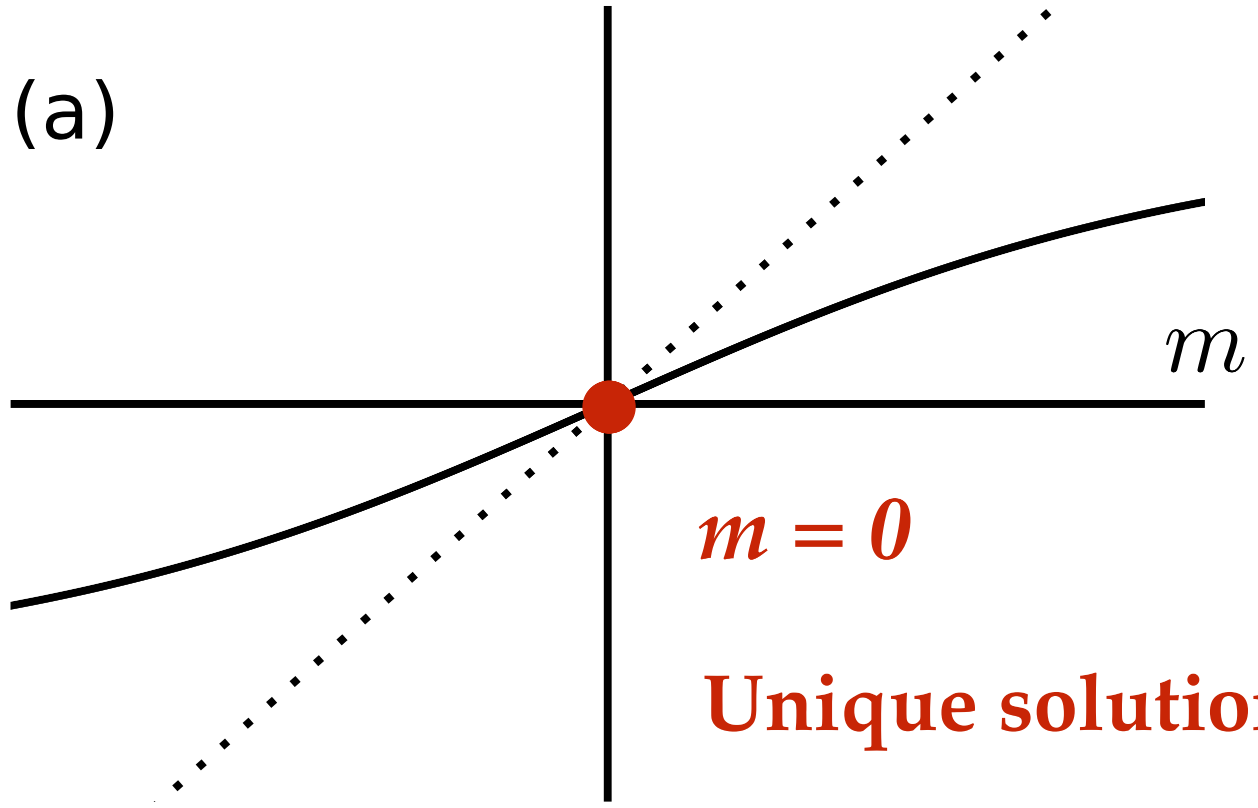


$\boxed{H = 0}$ $\quad T > T_c$



m = 0

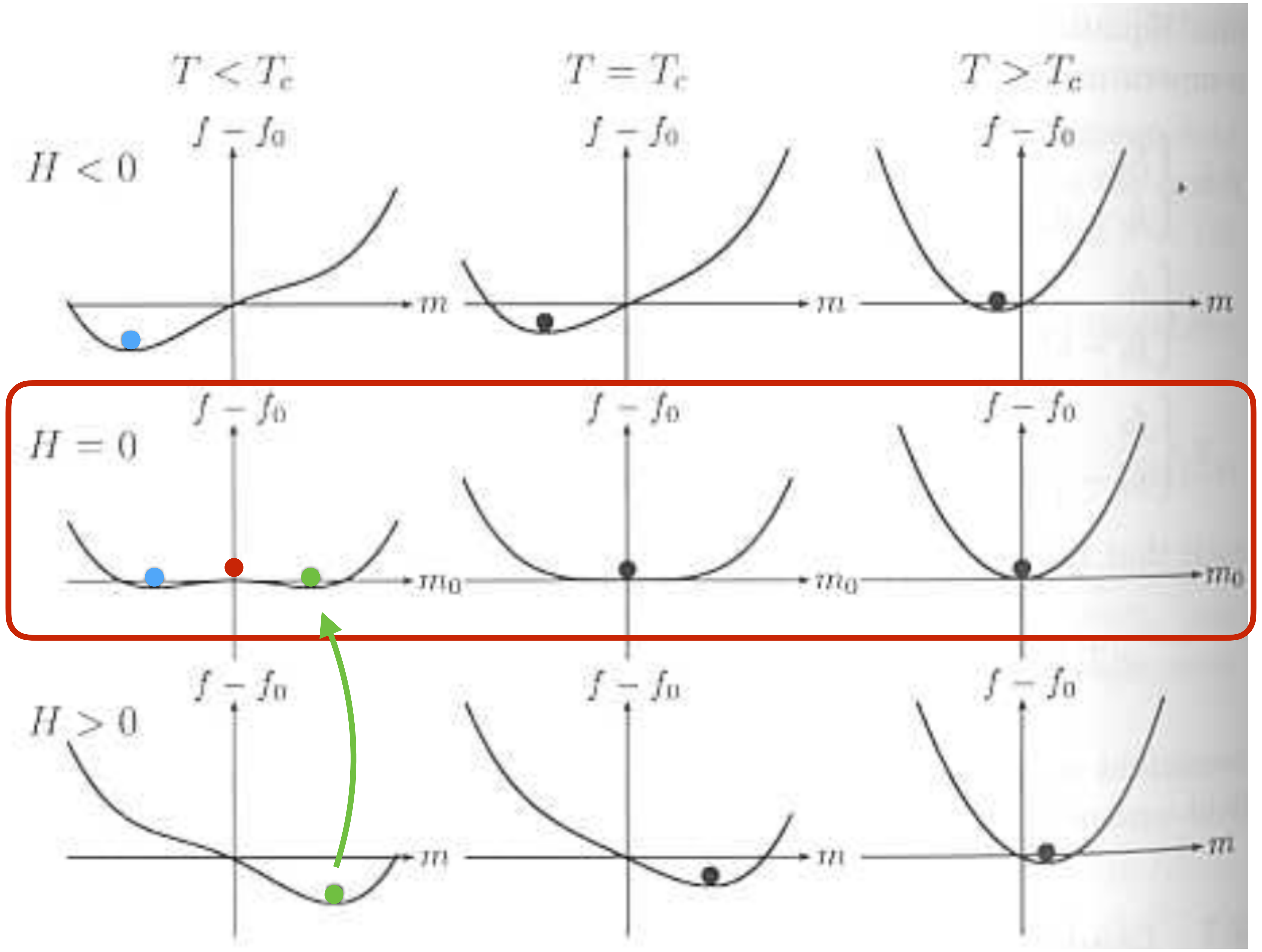
Unique solution

Stable solution

Free energy: analyzing stability

To check which solutions are stable:

$$F = -k_B T \log Z \qquad \text{where} \qquad Z = \exp\left(-\frac{\beta N J m^2 q}{2}\right) \prod_{i=1}^{N} \left[2\cosh[\beta(qJm + H)]\right]$$



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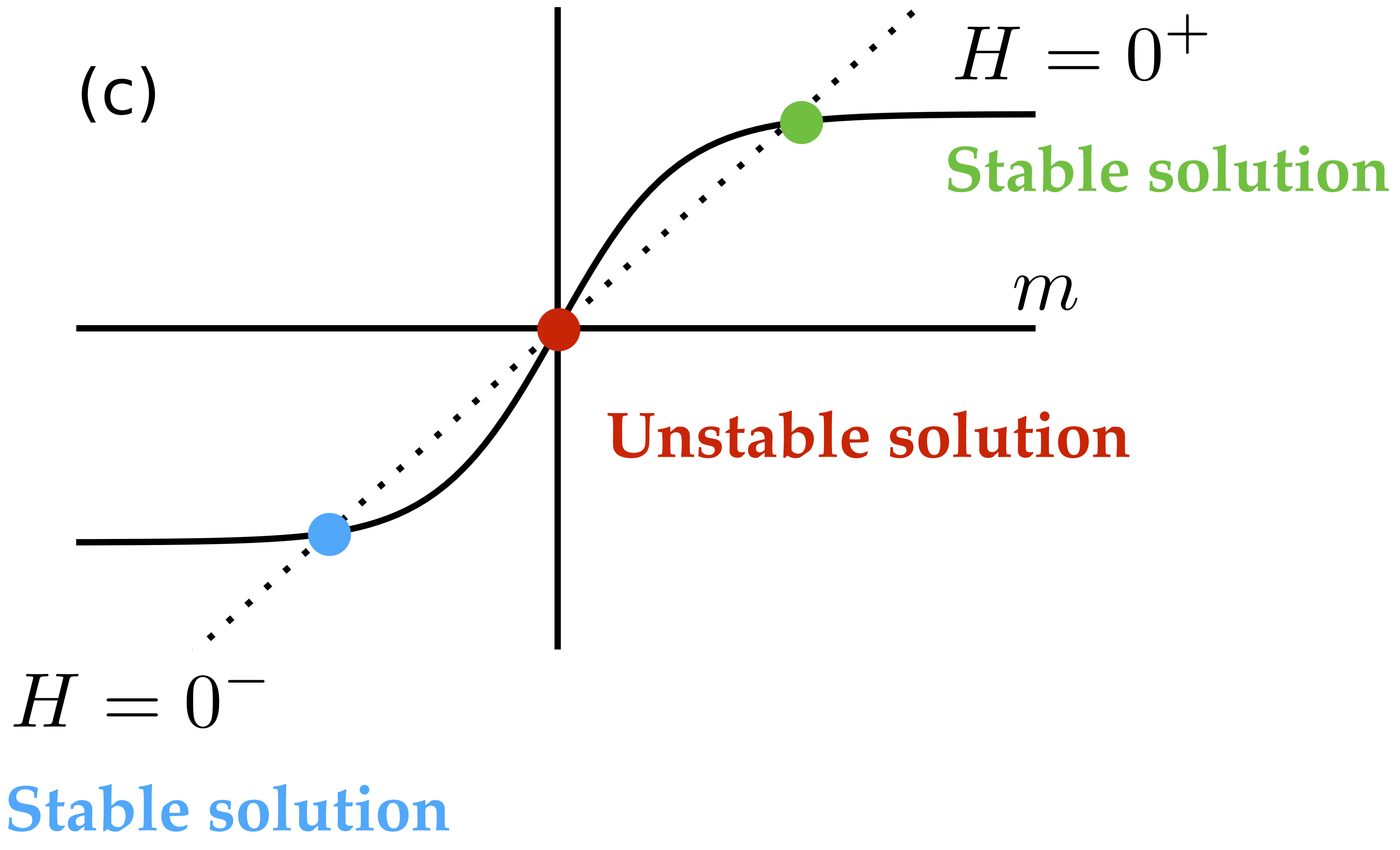


Continuous phase transition and Critical temperature

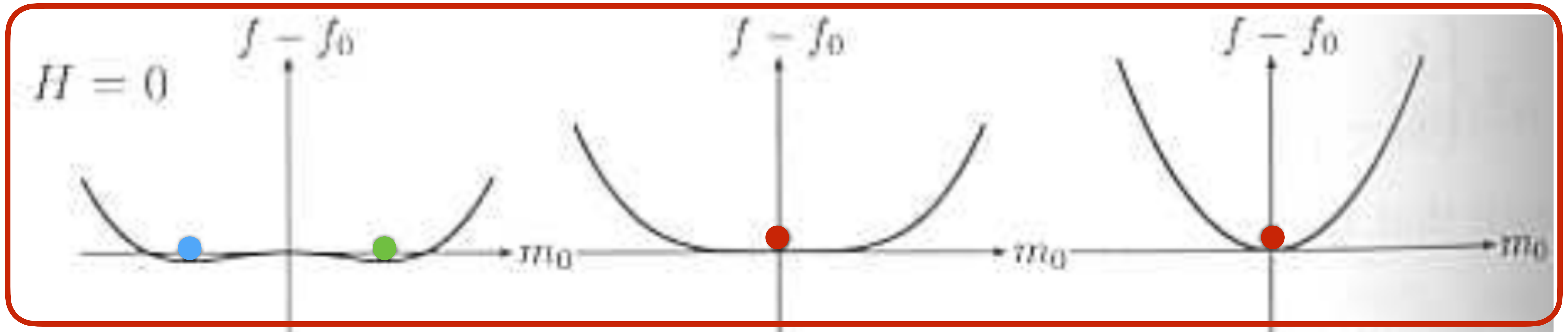
For $H = 0$: Continuous phase transition when varying the temperature T

$\boxed{H = 0}$

Critical temperature

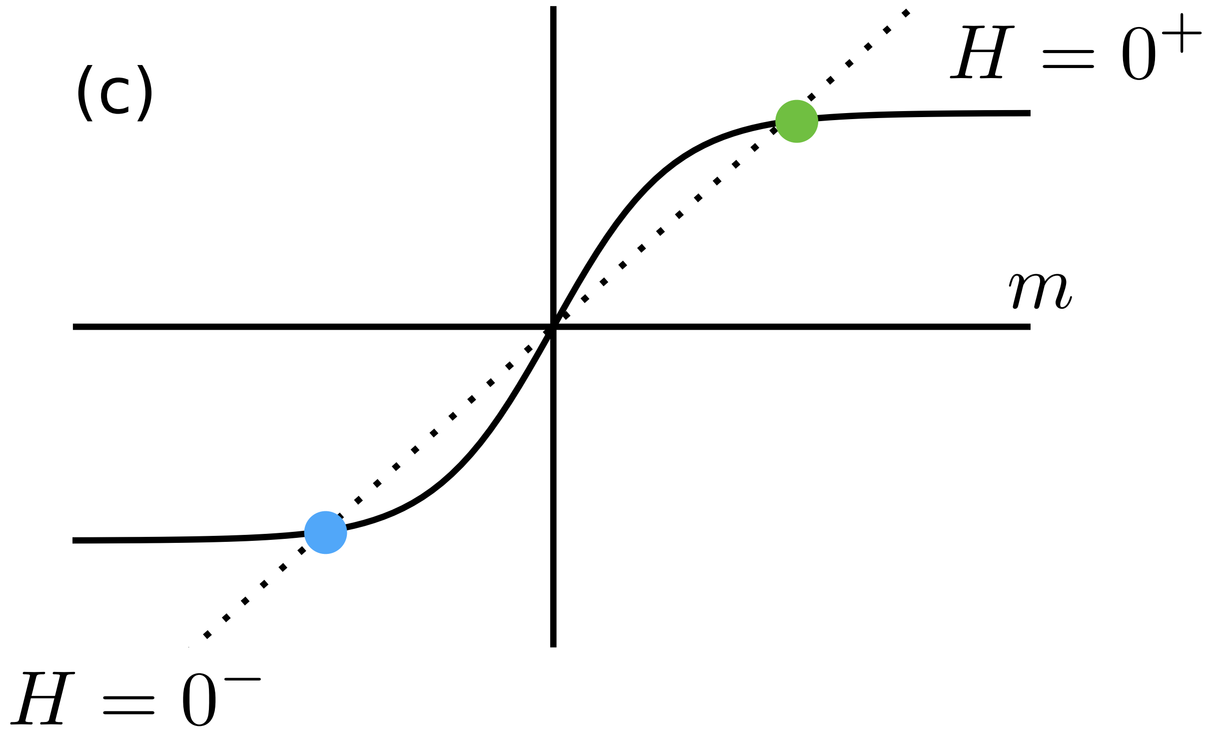
$$T_c = \frac{qJ}{k_B}$$

$T < T_c$ $\quad$ $T > T_c$

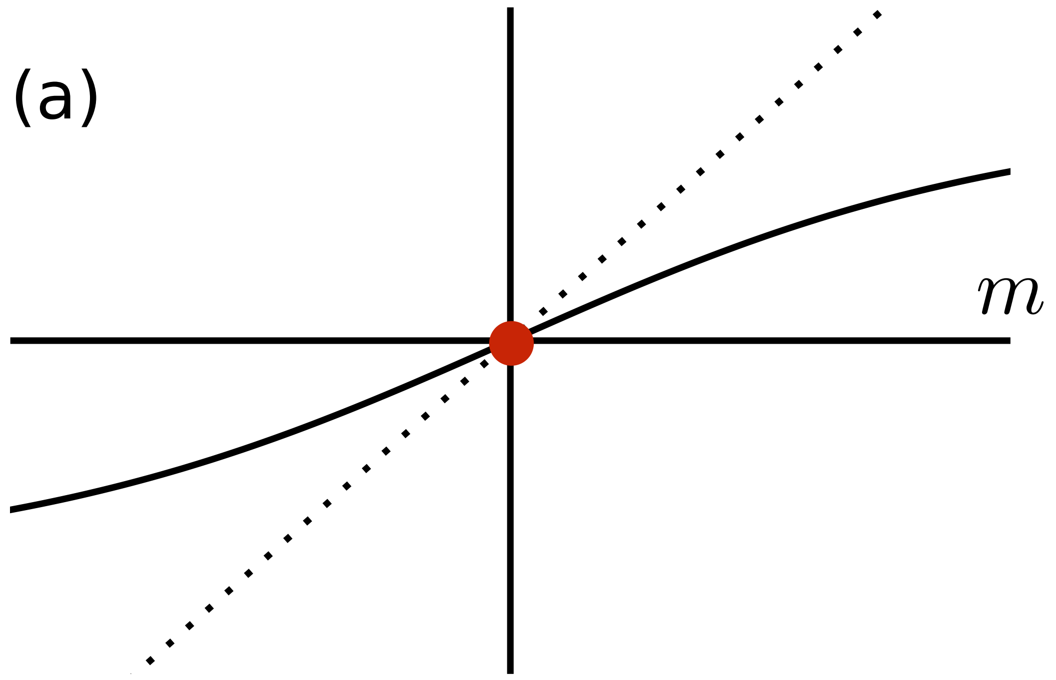


$$m = \tanh[\beta(qJm + H)]$$

(c)



(a)



Critical temperature

Critical temperature

$$T_c = \frac{qJ}{k_B}$$

Dimension: ***d* = 1:** transition at

Critical temperature

Critical temperature

$$T_c = \frac{qJ}{k_B}$$

Dimension: ***d* = 1:** transition at $T_c = \frac{2J}{k_B}$

Critical temperature

Critical temperature

$$T_c = \frac{qJ}{k_B}$$

Dimension: *$d = 1$:* transition at $T_c = \frac{2J}{k_B}$ —> **Not good!** **In 1d Ising model, no transition at finite positive T**

Critical temperature

Critical temperature

$$T_c = \frac{qJ}{k_B}$$

Dimension: ***d* = 1:** transition at $T_c = \dfrac{2J}{k_B}$ —> **Not good!** **In 1d Ising model, no transition at finite positive T**

***d* = 2:**

Critical temperature

Critical temperature

$$T_c = \frac{qJ}{k_B}$$

Dimension: *d = 1:* transition at $T_c = \frac{2J}{k_B}$ —> **Not good!** **In 1d Ising model, no transition at finite positive T**

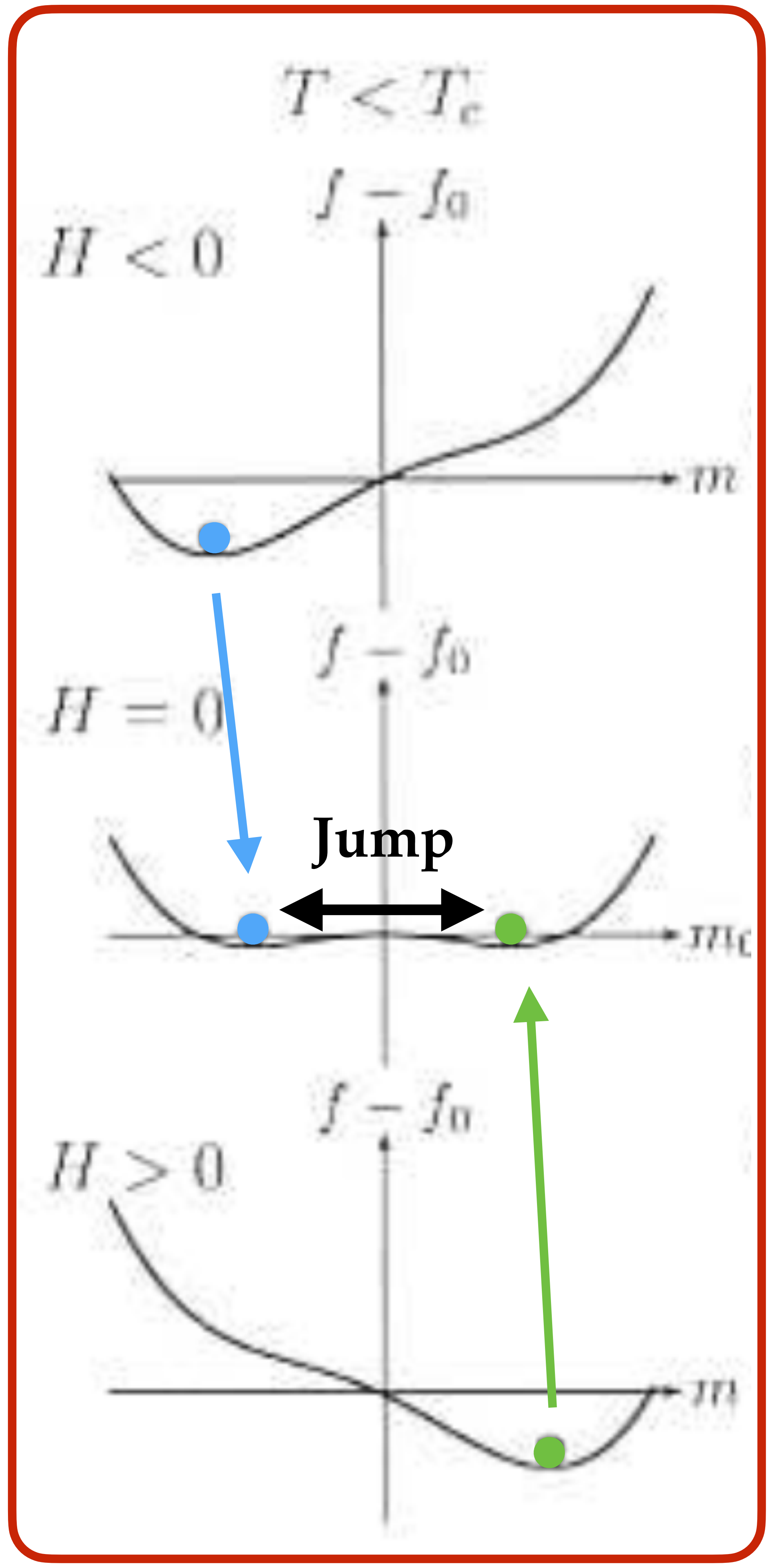
d = 2: $T_c = \frac{4J}{k_B}$ Onsager: $k_B T_c = \frac{2J}{\log(1+\sqrt{2})}$ $T_c \simeq \frac{2.269J}{k_B}$

Larger Tc ==>> neglecting the fluctuations, go against phase transition to paramagnetic

Discontinuous phase transition when varying H at T<Tc

For $T < Tc$: Discontinuous phase transition when varying the external field H

$T < T_c$



See online interactive demo: here

Jump at H = 0 from m_0 to $-m_0$

Mean-field
Critical exponents

Reminder: 2D Ising model

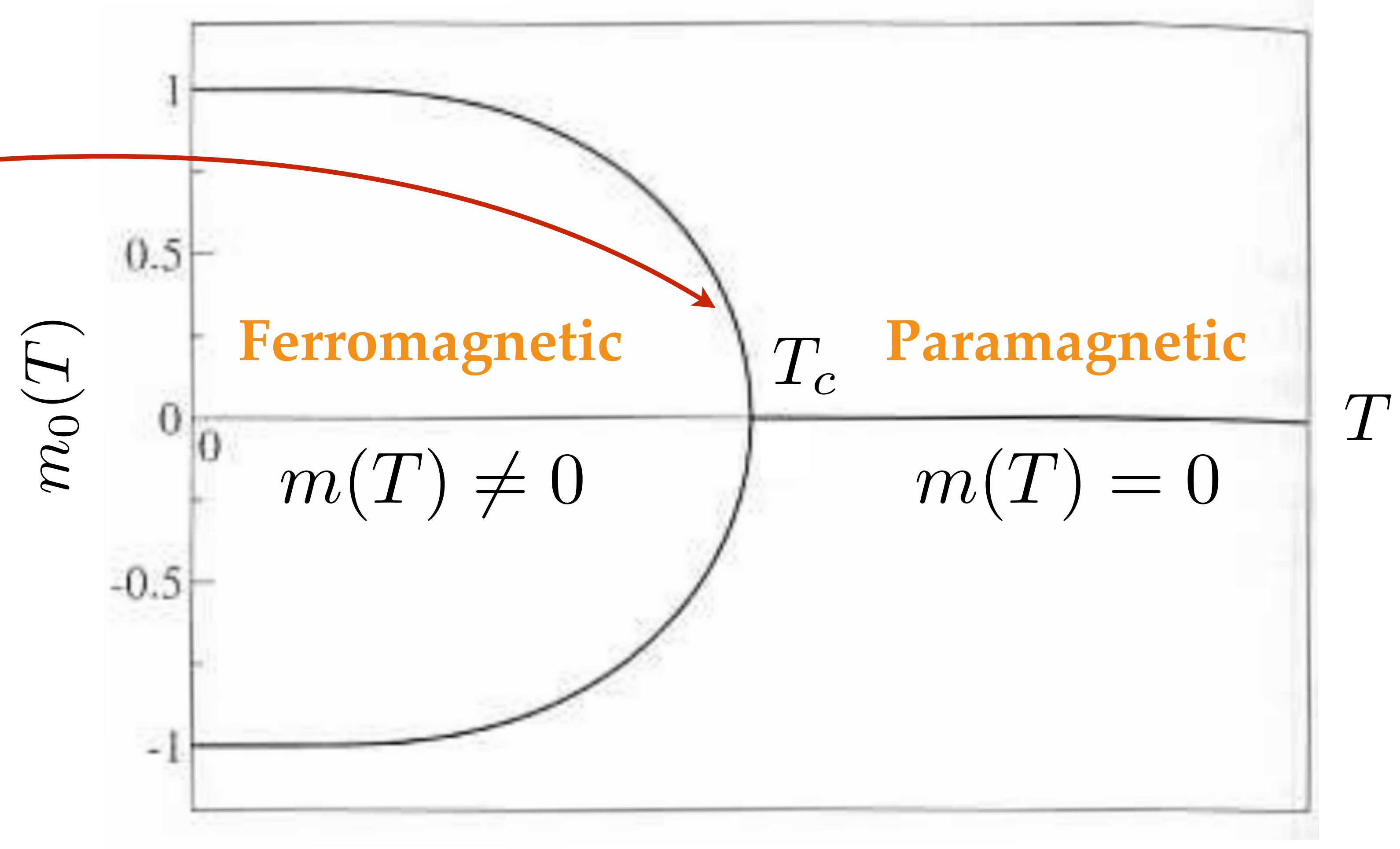
Exact solution for H=0 by Onsager (1944):

proves the existence of a critical phase transition at $$k_B T_c = \frac{2J}{\log(1+\sqrt{2})} \simeq 2.269\, J$$

with a magnetization: $$m(T) = \left(1 - \sinh^{-4}\left(\frac{2\,J}{k_B T}\right)\right)^{1/8}$$

Critical exponent

Universality: $$m(T) \underset{T_c}{\sim} (T - T_c)^{1/8}$$



Difficult to derive exact results with field and in larger dimensions

Different approaches: **Mean-field theory** **Landau theory** **Renormalisation group**

Finding the critical exponents (cf. Exercise in tutorial)

Self-consistency relation: m is solution of: $$m = \tanh\left[\frac{T_c}{T}m + \frac{H}{k_B T}\right] \qquad (1)$$

Free energy: $$f = \frac{k_B T_c}{2}m^2 - k_B T \log\left[2\cosh\left(\frac{T_c}{T}m + \frac{H}{k_B T}\right)\right] \qquad (2)$$

Exposant	Champ moyen
α	
β	
γ	
δ	

- $m_0 \sim (T_c - T)^\beta \qquad (T < T_c)$ **for *H*=0 ?**

- **magnetization** $m \sim H^{1/\delta}$ **at *T*=*Tc* and *H* small ?**

- **Susceptibility per spin:** $\chi \sim |T - T_c|^{-\gamma}$ **?**

- **Heat capacity:** $C \sim |T - T_c|^{-\alpha}$ **?**

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Exposant	Champ moyen
α	
β	0.5
γ	
δ	

- $m_0 \sim (T_c - T)^\beta$ $\quad (T < T_c)$ **for *H*=0** ?
  - Start from **Eq. (1) with H=0**
  - Expand for small m (as m is continuous and m=0 at T_c)
- **magnetization** $m \sim H^{1/\delta}$ **at T=T_c and H small ?**
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α	
β	0.5
γ	1
δ	

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  — Start from **Eq. (1) with H non 0**

  — Apply derivative on both sides of Eq.(1), then take the limit ***H=0***

  — Expand for small m

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α	
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- **Heat capacity:** $C \sim |T - T_c|^{-\alpha}$ **?** $$C = -T\left(\frac{\partial^2 f}{\partial T^2}\right)_{H=0}$$

Finding the critical exponents (cf. Exercise in tutorial)

Exposant	Champ moyen
α	discont.
β	0.5
γ	1
δ	

Self-consistency relation: m is solution of: $$m = \tanh\left[\frac{T_c}{T}m + \frac{H}{k_B T}\right] \qquad (1)$$

Free energy: $$f = \frac{k_B T_c}{2}m^2 - k_B T \log\left[2\cosh\left(\frac{T_c}{T}m + \frac{H}{k_B T}\right)\right] \qquad (2)$$

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  - Start from **Eq. (1) with H=0**
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  - Start from **Eq. (1) with H non 0**
  - Apply derivative on both sides of Eq.(1), then take the limit ***H=0***
  - Expand for small m

- **Heat capacity:** $C \sim |T - T_c|^{-\alpha}$ **?** $$C = -T\left(\frac{\partial^2 f}{\partial T^2}\right)_{H=0}$$
  - Start from **Eq. (2) with H=0**
  - Replace **m** by its expression **m(T) for H=0. Treat T>Tc and T<Tc separately**
  - Derive twice by ***T***
  - Expand for small m

Finding the critical exponents (cf. Exercise in tutorial)

Self-consistency relation: m is solution of: $$m = \tanh\left[\frac{T_c}{T}m + \frac{H}{k_B T}\right] \quad (1)$$

Free energy: $$f = \frac{k_B T_c}{2}m^2 - k_B T \log\left[2\cosh\left(\frac{T_c}{T}m + \frac{H}{k_B T}\right)\right] \quad (2)$$

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α	discont.
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  - Start from **Eq. (2) with H=0**
  - Replace **m** by its expression **m(T) for H=0. Treat T>Tc and T<Tc separately**
  - Derive twice by ***T***
  - Expand for small m

Finding the critical exponents (cf. Exercise in tutorial)

Exposant	Champ moyen
α	discont.
β	0.5
γ	1
δ	3

Self-consistency relation: m is solution of:

$$m = \tanh\left[\frac{T_c}{T}m + \frac{H}{k_BT}\right] \qquad (1)$$

Free energy:

$$f = \frac{k_BT_c}{2}m^2 - k_BT\log\left[2\cosh\left(\frac{T_c}{T}m + \frac{H}{k_BT}\right)\right] \qquad (2)$$

- $m_0 \sim (T_c - T)^\beta \quad (T < T_c)$ **for H=0 ?**
  - Start from **Eq. (1) with H=0**
  - Expand for small m (as m is continuous and m=0 at Tc)
- **Susceptibility per spin:** $\chi \sim |T - T_c|^{-\gamma}$ **?** $\qquad \chi = \lim_{H\to 0}\left(\frac{\partial m}{\partial H}\right)_T$
  - Start from **Eq. (1) with H non 0**
  - Apply derivative on both sides of Eq.(1), then take the limit ***H=0***
  - Expand for small m
- **Heat capacity:** $C \sim |T - T_c|^{-\alpha}$ **?** $\qquad C = -T\left(\frac{\partial^2 f}{\partial T^2}\right)_{H=0}$
  - Start from **Eq. (2) with H=0**
  - Replace **m** by its expression **m(T) for H=0. Treat T>Tc and T<Tc separately**
  - Derive twice by ***T***
  - Expand for small m
- **magnetization** $m \sim H^{1/\delta}$ **at *T=Tc* and *H* small ?**
  - Start from **Eq. (1) with *T=Tc***
  - Expand for small m (as m is small, for T=Tc and H small)

Critical exponents and Universality

$$C \sim |T - T_c|^{-\alpha}$$
$$m_0 \sim (T_c - T)^{\beta} \qquad (T < T_c)$$
$$\chi \sim |T - T_c|^{-\gamma}$$
$$m \sim H^{1/\delta} \qquad (T = T_c)$$

Exponents	*d* = 2	*d* = 3	Mean-field
α	$\ln\|T - T_c\|$	0.01 ± 0.01	0 (discont.)
β	0.125	0.312 ± 0.003	0.5
γ	1.75	1.250 ± 0.002	1
δ	15 (*)	5.0 ± 0.05	3

Critical exponents seem to get closer to MF for larger dimensions d

Critical exponents and Universality

$$C \sim |T - T_c|^{-\alpha}$$
$$m_0 \sim (T_c - T)^{\beta} \qquad (T < T_c)$$
$$\chi \sim |T - T_c|^{-\gamma}$$
$$m \sim H^{1/\delta} \qquad (T = T_c)$$

Exponents	***d*** **= 2**	***d*** **= 3**	***d*** **>= 4**	**Mean-field**
α	$\ln\|T - T_c\|$	0.01 ± 0.01	0	0 (discont.)
β	0.125	0.312 ± 0.003	0.5	0.5
γ	1.75	1.250 ± 0.002	1	1
δ	15 (*)	5.0 ± 0.05	3	3

Upper critical dimension = 4

Critical exponents seem to get closer to MF for larger dimensions ***d***

Critical exponents for ***d*** **>= 4** remains unchanged

Mean-field approximation is exact in infinite dimension *d* !

Exponents of Mean-field Ising are the same as for d >= 4

Critical exponents and Universality

$$C \sim |T - T_c|^{-\alpha}$$
$$m_0 \sim (T_c - T)^{\beta} \qquad (T < T_c)$$
$$\chi \sim |T - T_c|^{-\gamma}$$
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Exponents	$d = 2$	$d = 3$	$d >= 4$	**Mean-field**
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β	0.125	0.312 ± 0.003	0.5	0.5
γ	1.75	1.250 ± 0.002	1	1
δ	15 (*)	5.0 ± 0.05	3	3

Independent of the lattice type

Critical exponents are Universal

Critical Temperatures are Non-Universal:

Lattice	z	$k_B T_c / J$
$d = 1$ line	2	0
$d = 2$ hexagonal	3	$2/\ln(2+\sqrt{3})$[a]
square	4	$2/\ln(1+\sqrt{2})$[b] ≈ 2.269185
triangular	6	$4/\ln 3$[a]
$d = 3$ diamond	4	2.70[c]
simple cubic	6	4.51152[d]
body-centred cubic	8	6.40[e]
face-centred cubic	12	9.79[e]
Mean-field	z	z

Mean-field Ising Summary

Mean-field Ising model: ~ System of non-interacting spins, immersed in the effective field $H + Jqm$

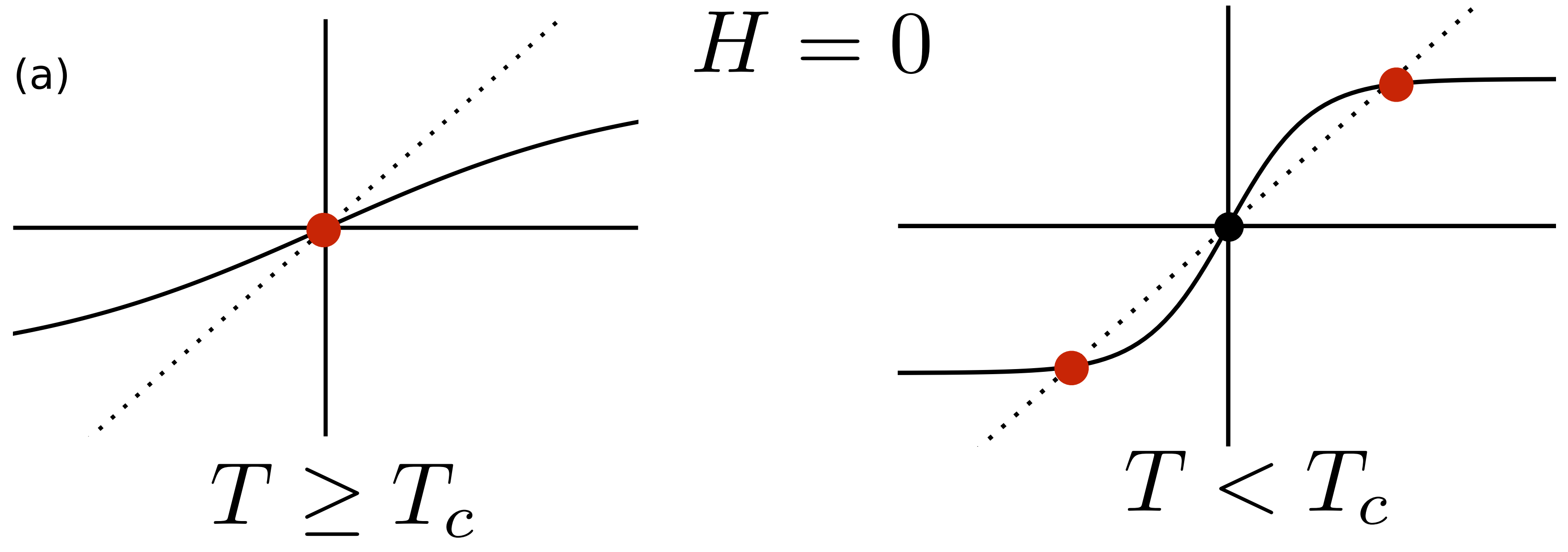
External field (→ H); local effective field created by their direct neighbors (→ Jqm)

Self-consistency relation: m is solution of:

$$m = \tanh[\beta(qJm + H)]$$

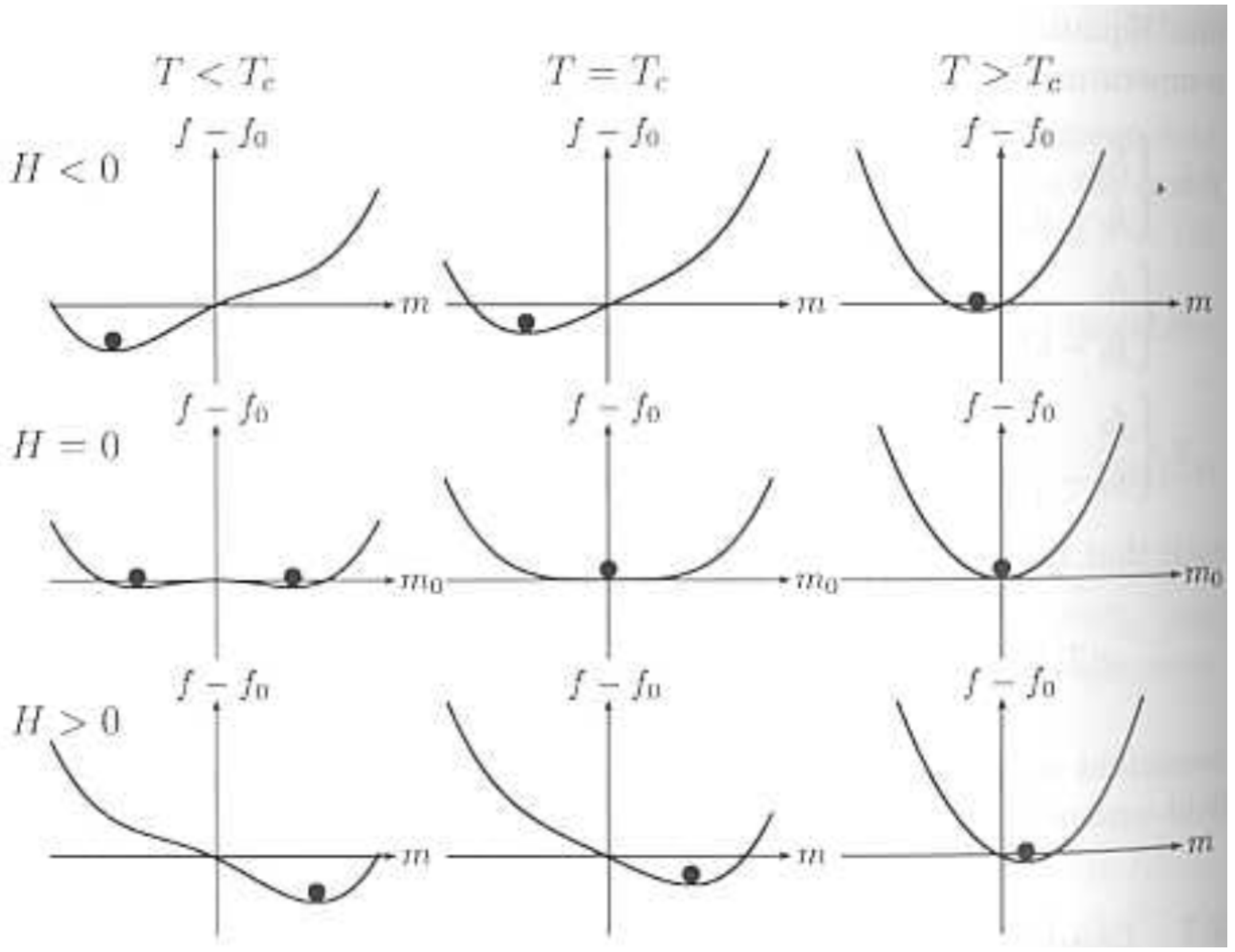
Continuous PT at $T_c = \dfrac{qJ}{k_B}$

● Stable
● Unstable



Stability analysis using

Free energy: $F = -k_B T \log Z$



Universality of the critical exponents:

$C \sim |T - T_c|^{-\alpha}$

$m_0 \sim (T_c - T)^{\beta}$ $(T < T_c)$

$\chi \sim |T - T_c|^{-\gamma}$

$m \sim H^{1/\delta}$ $(T = T_c)$

Exponents	$d = 2$	$d = 3$	$d >= 4$	Mean-field
α	$\ln \lvert T - T_c \rvert$	0.01 ± 0.01	0	0 (discont.)
β	0.125	0.312 ± 0.003	0.5	0.5
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Upper critical dimension = 4