Derivation of the Mathematical Identity Used to Solve Project Euler 152

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1 Problem Description

Project Euler problem #152 is written as follows:

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Sums of Square Reciprocals

Problem 152

There are several ways to write the number $\frac{1}{2}$ as a sum of square reciprocals using *distinct* integers.

For instance, the numbers $\{2, 3, 4, 5, 7, 12, 15, 20, 28, 35\}$ can be used:

$$\frac{1}{2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{12^2} + \frac{1}{15^2} + \frac{1}{20^2} + \frac{1}{28^2} + \frac{1}{35^2}$$

In fact, only using integers between 2 and 45 inclusive, there are exactly three ways to do it, the remaining two being: $\{2, 3, 4, 6, 7, 9, 10, 20, 28, 35, 36, 45\}$ and $\{2, 3, 4, 6, 7, 9, 12, 15, 28, 30, 35, 36, 45\}$.

How many ways are there to write $\frac{1}{2}$ as a sum of reciprocals of squares using distinct integers between 2 and 80 inclusive?

The following proof establishes that for each prime number $p > 2$, terms that include p as a factor in their denominator must appear in specific combinations that satisfy the property $(p^2 \mid \gamma)$, where p is the prime and γ is the numerator after all terms with the factor p have had p factored out and have been summed together into a single reduced fraction.

Corollary: Prime numbered terms may not appear by themselves, since the combined fraction will be $1/1$, so $\gamma = 1$, and p^2 trivially does not divide 1.

2 Lemma: Correct Solutions Must Include n = 2

To start, we will prove that all correct solutions must include the term $\frac{1}{2^2}$. This is very simple. If we compute the partial sum of all terms from n = 3 to n = 80, we get 0.3825, which is less than 0.5, so it follows that the term k = 2 must be included in order to reach the target sum.

Indeed, it can be shown that the p-series $\sum_1^\infty \frac{1}{n^2}$ converges to $\frac{\pi^2}{6} = 1.64$, so it follows that summing all the terms from 3 to infinity would only yield 0.3949, which is still less than the target sum.

Thus, n = 2 must always be included in the sum, and from this point forward we will assume its inclusion and solve the sub problem:

$$\frac{1}{4} = \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{80^2}$$

where each term in the sum may or may not actually be included in any given solution.

3 Main Proof

Now, consider an arbitrary solution:

$$\frac{1}{4} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \ldots$$

and let p be a prime number such that $p > 2$. We can separate the sum into terms that include p as a factor and terms that don't as follows:

$$\frac{1}{4} = (\frac{1}{a^2} + \frac{1}{b^2} + \ldots) + \frac{1}{p^2}(\frac{1}{c^2} + \frac{1}{d^2} + \ldots)$$

where we have factored out $\frac{1}{p^2}$ in the terms on the right. Multiplying by 4 gives:

$$1 = 4(\frac{1}{a^2} + \frac{1}{b^2} + \ldots) + \frac{4}{p^2}(\frac{1}{c^2} + \frac{1}{d^2} + \ldots)$$

and remember these sums are finite, so we can combine each of them into a single rational number:

$$1 = 4(\frac{\alpha}{\beta}) + \frac{4}{p^2}(\frac{\gamma}{\delta})$$

By definition, a^2 and b^2 and all the terms in the left sum, had no factors of p, so it follows that the common denominator β has no factors of p either. On the right hand side, the denominator δ CAN include factors of p if $\frac{1}{p^4}$ or similar terms were included in the original sum, but this won't be important.

Multiply both sides by $\beta\delta$:

$$\delta\beta = 4\alpha\delta + \frac{4\gamma\beta}{p^2}$$

Lastly, note that this in an integer equation. Since $\delta\beta$ and $4\alpha\delta$ are integers, it follows that $\frac{4\gamma\beta}{p^2}$ must be as well. For that to be true, the numerator of this fraction must contain a factor of p^2, but we've shown that β is not a multiple of p^2, and 4 is not either since $p > 2$. Therefore, it follows that γ is divisible by p^2 ($p^2 \mid \gamma$) for all primes larger than 2, which is the main result I will use in my code. This must be true for each prime and its set of multiples (excluding 2) in any valid solution, and this observation will reduce the search space drastically.