Fast algorithms and numerical methods for the solution of Boundary Element Methods

Session 3: Boundary Integral Equations

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Principle of derivation of Boundary Integral Representation

Boundary-value problem over Ω : $\begin{cases} \mathcal{L}u + f = 0 & \text{in } \Omega, \\ u = g_1 & \text{on } \partial\Omega_D, \\ T^n(u) = g_2 & \text{on } \partial\Omega_N. \end{cases}$

where u: unknown; g_1, g_2 and source f given. T^n: first-order partial differential operator, linear with respect to n. $\mathcal{L}$ linear second-order partial differential operator

- $\mathcal{L}$ and T^n assumed to satisfy the reciprocity identity

$$\int_\Omega (\mathcal{L}u.v - \mathcal{L}v.u)dV = \int_{\partial\Omega} (T^n(u).v - T^n(v).u)dS$$

- G: fundamental solution (point source f applied at x $\notin \partial\Omega$)

$$\mathcal{L}G(\boldsymbol{x}, \boldsymbol{y}) + \delta(\boldsymbol{y} - \boldsymbol{x}) = 0 \quad \text{in } \Omega$$

- Property of the Dirac distribution

$$\int_\Omega \delta(\boldsymbol{y} - \boldsymbol{x})u(\boldsymbol{y})dV_y = \kappa u(\boldsymbol{x}) \quad (\kappa = 1 \; if \; \boldsymbol{x} \in \Omega, \quad \kappa = 0 \; if \; \boldsymbol{x} \notin \Omega)$$

- Integral Representation formula: $x \notin \partial\Omega$

$$\kappa u(\boldsymbol{x}) = \int_\Omega f(\boldsymbol{y})G(\boldsymbol{x}, \boldsymbol{y})dV_y + \int_{\partial\Omega} (u(\boldsymbol{y})T^n G(\boldsymbol{x}, \boldsymbol{y}) - T^n u(\boldsymbol{y})G(\boldsymbol{x}, \boldsymbol{y}))dS_y$$

Integral Representation for Helmholtz equation

Scattering problen in Ω_ℓ ($\ell = e$ or $\ell = i$): $u \in H^1(\Omega_1)$

$$\Delta u + \omega^2 u = 0 \text{ dans } \Omega_\ell$$

Rigorously we could only assume $u \in H^1(\Delta; \Omega_1)$ and use duality parings $< .,. >_{H^{-1/2},H^{1/2}}$ instead of integrals on Γ

Notations: $\mathbf{n}$ normal from Ω_i to Ω_e. f^i/f^e interior/exterior traces

$$[f]_\Gamma = f^i - f^e$$

We know that

$$\Delta G(\mathbf{y},\mathbf{x}) + \omega^2 G(\mathbf{y},\mathbf{x}) = -\delta_x(\mathbf{y}) \quad G(\mathbf{y},\mathbf{x}) = G(r) = \frac{1}{4\pi r} e^{i\omega r} \text{ with } r = |\mathbf{x} - \mathbf{y}|$$

Representation Theorem

(a) If u is solution in Ω_i then

$$\int_\Gamma \left(G(\mathbf{x},\mathbf{y}) \left(\frac{\partial u}{\partial n}\right)^i (\mathbf{y}) - \frac{\partial G}{\partial n_y}(\mathbf{x},\mathbf{y}) u^i(\mathbf{y}) \right) d\gamma_y = \begin{cases} u(\mathbf{x}) \text{ if } \mathbf{x} \in \Omega_i \\ \\ 0 \text{ if } \mathbf{x} \in \Omega_e \end{cases}$$

(b) If u is solution in Ω_e then (with the normal still from Ω_i to Ω_e)

$$\int_\Gamma \left(-G(\mathbf{x},\mathbf{y}) \left(\frac{\partial u}{\partial n}\right)^e (\mathbf{y}) + \frac{\partial G}{\partial n_y}(\mathbf{x},\mathbf{y}) u^e(\mathbf{y}) \right) d\gamma_y = \begin{cases} 0 \text{ if } \mathbf{x} \in \Omega_i \\ \\ u(\mathbf{x}) \text{ if } \mathbf{x} \in \Omega_e \end{cases}$$

(c) If u is solution in $\Omega_i \cup \Omega_e$ then

$$\forall \mathbf{x} \in \Omega_i \cup \Omega_e,\ u(\mathbf{x}) = \int_\Gamma \left(G(\mathbf{x},\mathbf{y}) \left[\frac{\partial u}{\partial n}\right]_\Gamma (\mathbf{y}) - \frac{\partial G}{\partial n_y}(\mathbf{x},\mathbf{y}) [u]_\Gamma(\mathbf{y}) \right) d\gamma_y$$

What is next?

For exterior problems, we have

$$u(\mathbf{x}) = \int_\Gamma \left(-G(\mathbf{x},\mathbf{y}) \left(\frac{\partial u}{\partial n}\right)^e (\mathbf{y}) + \frac{\partial G}{\partial n_y}(\mathbf{x},\mathbf{y}) u^e(\mathbf{y}) \right) d\gamma_y, \quad \forall \mathbf{x} \in \Omega_e, \mathbf{x} \notin \Gamma$$

☺ u in the domain is known only from values on boundary

☺ Reduction of computational costs and memory requirements

It is a Boundary Integral Representation (not an equation)

You need to obtain the traces of u on Γ (with an equation)

The Boundary Element Method is a two steps methods

1. Resolution of a boundary integral equation to obtain traces (TP2)
2. Application of the boundary integral representation to obtain solution in the domain (TP1)

Single and double layer potentials

Single layer potential: q with enough regularity, e.g., $q \in C^0(\Gamma)$

$$\mathcal{S}q(\mathbf{x}) = \int_\Gamma G(\mathbf{x}-\mathbf{y})q(\mathbf{y})d\gamma_y, \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e$$

Double layer potential: φ with enough regularity, e.g., $\varphi \in C^0(\Gamma)$

$$\mathcal{D}\varphi(\mathbf{x}) = \int_\Gamma \frac{\partial G}{\partial n_y}(\mathbf{x}-\mathbf{y})\varphi(\mathbf{y})d\gamma_y, \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e$$

Boundary Integral Representations: if u sol. of Helmholtz eq. in $\Omega_i \cup \Omega_e$

$$(a) \quad \mathcal{S}\gamma_1^i u(\mathbf{x}) - \mathcal{D}\gamma_0^i u(\mathbf{x}) = \begin{cases} u(\mathbf{x}) \text{ if } \mathbf{x} \in \Omega_i \\ \\ 0 \text{ if } \mathbf{x} \in \Omega_e \end{cases}$$

$$(b) \quad -\mathcal{S}\gamma_1^e u(\mathbf{x}) + \mathcal{D}\gamma_0^e u(\mathbf{x}) = \begin{cases} 0 \text{ if } \mathbf{x} \in \Omega_i \\ \\ u(\mathbf{x}) \text{ if } \mathbf{x} \in \Omega_e \end{cases}$$

$$(c) \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e,\ u(\mathbf{x}) = \mathcal{S}\left[\frac{\partial u}{\partial n}\right]_\Gamma(\mathbf{x}) - \mathcal{D}[u]_\Gamma(\mathbf{x})$$

with $\mathbf{n}$ normal from Ω_i to Ω_e

Some other boundary integral representations

- If u is solution of the Helmholtz equation in $\Omega_i \cup \Omega_e$ then it can be represented with single and double layer potentials.
- On the other hand, if we pick a function q or φ, then, we can verify that the single $\mathcal{S}q$ and double layer $\mathcal{D}\varphi$ potentials are solutions of the Helmholtz equation in $\Omega_i \cup \Omega_e$.

It is how we can derive various boundary integral representations.

Use of boundary conditions

We need to derive boundary integral equations and use the boundary conditions to solve the correct PDE.

We need the traces of the single and double layer potentials.

Traces of the single layer potential

(i) The single layer potential is continuous across Γ and has the following traces:

$$\boxed{\gamma_0^i(\mathcal{S}q)(\mathbf{x}) = \gamma_0^e(\mathcal{S}q)(\mathbf{x}) = \int_\Gamma G(\mathbf{x}-\mathbf{y})q(\mathbf{y})d\gamma_y := Sq(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma}$$

(ii) The normal derivative of the single layer potential is discontinuous across Γ:

$$\boxed{\begin{aligned} \gamma_1^i(\mathcal{S}q)(\mathbf{x}) &= \frac{1}{2}q(\mathbf{x}) + D'q(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma \\ \gamma_1^e(\mathcal{S}q)(\mathbf{x}) &= -\frac{1}{2}q(\mathbf{x}) + D'q(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma \end{aligned}}$$

with

$$D'q(\mathbf{x}) = \int_\Gamma \frac{\partial G}{\partial n_x}(\mathbf{x},\mathbf{y})q(\mathbf{y})d\gamma_y, \quad \forall \mathbf{x} \in \Gamma$$

Traces of the double layer potential

(i) The double layer potential is discontinuous across Γ :

$$\boxed{\begin{aligned} \gamma_0^i(\mathcal{D}\varphi)(\mathbf{x}) &= -\frac{1}{2}\varphi(\mathbf{x}) + D\varphi(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma \\ \gamma_0^e(\mathcal{D}\varphi)(\mathbf{x}) &= \frac{1}{2}\varphi(\mathbf{x}) + D\varphi(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma \end{aligned}}$$

with

$$D\varphi(\mathbf{x}) = \int_\Gamma \frac{\partial G}{\partial n_y}(\mathbf{x}, \mathbf{y})\varphi(\mathbf{y})d\gamma_y, \quad \forall \mathbf{x} \in \Gamma$$

(ii) The normal derivative of the double layer potential is continuous across Γ :

$$\boxed{\gamma_1^i(\mathcal{D}\varphi)(\mathbf{x}) = \gamma_1^e(\mathcal{D}\varphi)(\mathbf{x}) := N\varphi(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma}$$

Dirichlet Interior problem with natural traces

Boundary Integral representation with single and double layer potentials for which the densities are the natural traces of the solution

$$\boxed{u(\mathbf{x}) = \mathcal{S}\gamma_1^i u(\mathbf{x}) - \mathcal{D}\gamma_0^i u(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_i, \quad \mathbf{x} \notin \Gamma}$$

By using the trace theorems

$$u^i(\mathbf{x}) = S\gamma_1^i u(\mathbf{x}) - (-\frac{I}{2} + D)\gamma_0^i u(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma$$

We use the boundary condition $u^i = u_d$ and set $q = \gamma_1^i u$ such that the Boundary Integral Equation to solve is

$$\boxed{Sq = (\frac{I}{2} + D)u_d \text{ on } \Gamma}$$

Dirichlet Interior problem with a continuation

u is extended in Ω_e such that it still satisfies Helmholtz equation in Ω_e

$$u(\mathbf{x}) = \mathcal{S}\left[\frac{\partial u}{\partial n}\right]_\Gamma(\mathbf{x}) - \mathcal{D}[u]_\Gamma(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e, \quad \mathbf{x} \notin \Gamma$$

Continuation with 0 such that $[u]_\Gamma = \gamma_0^i u$ and $\left[\frac{\partial u}{\partial n}\right]_\Gamma = \gamma_1^i u$ (ends with equation for natural traces)

Dirichlet Interior problem with a continuation

u is extended in Ω_e such that it still satisfies Helmholtz equation in Ω_e

$$u(\mathbf{x}) = \mathcal{S}\left[\frac{\partial u}{\partial n}\right]_\Gamma(\mathbf{x}) - \mathcal{D}[u]_\Gamma(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e, \quad \mathbf{x} \notin \Gamma$$

Continuation with 0 such that $[u]_\Gamma = \gamma_0^i u$ and $\left[\frac{\partial u}{\partial n}\right]_\Gamma = \gamma_1^i u$ (ends with equation for natural traces)

Continuation by continuity: $[u]_\Gamma = 0$, we note $q = \left[\frac{\partial u}{\partial n}\right]_\Gamma(\mathbf{x})$

Representation: $u(\mathbf{x}) = \mathcal{S}q \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e, \quad \mathbf{x} \notin \Gamma$

Equation: $u_d = Sq$

Dirichlet Interior problem with a continuation

u is extended in Ω_e such that it still satisfies Helmholtz equation in Ω_e

$$u(\mathbf{x}) = \mathcal{S}\left[\frac{\partial u}{\partial n}\right]_\Gamma(\mathbf{x}) - \mathcal{D}[u]_\Gamma(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e, \quad \mathbf{x} \notin \Gamma$$

Continuation with 0 such that $[u]_\Gamma = \gamma_0^i u$ and $\left[\frac{\partial u}{\partial n}\right]_\Gamma = \gamma_1^i u$ (ends with equation for natural traces)

Continuation by continuity: $[u]_\Gamma = 0$, we note $q = \left[\frac{\partial u}{\partial n}\right]_\Gamma(\mathbf{x})$

Representation: $u(\mathbf{x}) = \mathcal{S}q \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e, \quad \mathbf{x} \notin \Gamma$

Equation: $u_d = Sq$

Continuation by continuity of normal deriv. $\left[\frac{\partial u}{\partial n}\right]_\Gamma = 0, \varphi = [u]_\Gamma$

Repres.: $u(\mathbf{x}) = -\mathcal{D}\varphi(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e, \quad \mathbf{x} \notin \Gamma$

Equation: $u_d = (\frac{I}{2} - D)\varphi$

Neumann Interior problem with natural traces

Boundary Integral representation with single and double layer potentials for which the densities are the natural traces of the solution

$$u(\mathbf{x}) = \mathcal{S}\gamma_1^i u(\mathbf{x}) - \mathcal{D}\gamma_0^i u(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_i, \quad \mathbf{x} \notin \Gamma$$

By using the trace theorems

$$\gamma_1^i u(\mathbf{x}) = (\frac{I}{2} + D')\gamma_1^i u(\mathbf{x}) - N\gamma_0^i u(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma$$

We use the boundary condition $\gamma_1^i u = u_n$ and set $\varphi = \gamma_0^i u$ such that the Boundary Integral Equation to solve is

$$N\varphi = (-\frac{I}{2} + D')u_n \text{ on } \Gamma$$

Neumann Interior problem with a continuation

u is extended in Ω_e such that it still satisfies Helmholtz equation in Ω_e

$$u(\mathbf{x}) = \mathcal{S}\left[\frac{\partial u}{\partial n}\right]_\Gamma(\mathbf{x}) - \mathcal{D}[u]_\Gamma(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e, \quad \mathbf{x} \notin \Gamma$$

Continuation with 0 such that $[u]_\Gamma = \gamma_0^i u$ and $\left[\frac{\partial u}{\partial n}\right]_\Gamma = \gamma_1^i u$ (ends with equation for natural traces)

Neumann Interior problem with a continuation

u is extended in Ω_e such that it still satisfies Helmholtz equation in Ω_e

$$u(\mathbf{x}) = \mathcal{S}\left[\frac{\partial u}{\partial n}\right]_\Gamma(\mathbf{x}) - \mathcal{D}[u]_\Gamma(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e, \quad \mathbf{x} \notin \Gamma$$

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Representation: $u(\mathbf{x}) = \mathcal{S}q \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e, \quad \mathbf{x} \notin \Gamma$

Equation: $u_n = (\frac{I}{2} + D')q$

Neumann Interior problem with a continuation

u is extended in Ω_e such that it still satisfies Helmholtz equation in Ω_e

$$u(\mathbf{x}) = \mathcal{S}\left[\frac{\partial u}{\partial n}\right]_\Gamma(\mathbf{x}) - \mathcal{D}[u]_\Gamma(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e, \quad \mathbf{x} \notin \Gamma$$

Continuation with 0 such that $[u]_\Gamma = \gamma_0^i u$ and $\left[\frac{\partial u}{\partial n}\right]_\Gamma = \gamma_1^i u$ (ends with equation for natural traces)

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Equation: $u_n = (\frac{I}{2} + D')q$

Continuation by continuity of normal deriv. $\left[\frac{\partial u}{\partial n}\right]_\Gamma = 0, \varphi = [u]_\Gamma$

Repres.: $u(\mathbf{x}) = -\mathcal{D}\varphi(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e, \quad \mathbf{x} \notin \Gamma$

Equation: $u_n = -N\varphi$

TP1: Représentation intégrale pour domaine extérieur

D'après le théorème de représentation intégrale, on a:

$$u(\mathbf{x}) = \mathcal{S}\left[\frac{\partial u}{\partial n}\right]_\Gamma(\mathbf{x}) - \mathcal{D}[u]_\Gamma(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_i \cup \Omega_e$$

On note $u^+(\mathbf{x}) = u(\mathbf{x})$ si $\mathbf{x} \in \Omega_e$ et on choisit un prolongement par continuité $[u]_\Gamma = 0$ et $p = \left[\frac{\partial u}{\partial n}\right]_\Gamma = -\partial_\mathbf{n} u^+ - \partial_\mathbf{n} u^{inc}$

On obtient bien que:

Le champ diffracté dans le cas de conditions à la frontière de type Dirichlet est donné par la représentation intégrale

$$u^+(\mathbf{x}) = \int_\Gamma G(\mathbf{x},\mathbf{y}) p(\mathbf{y}) d\Gamma(\mathbf{y})$$

$p = -\partial_\mathbf{n} u^+ - \partial_\mathbf{n} u^{inc}$, $\mathbf{n}$ normale ext, $G(\mathbf{x},\mathbf{y}) = \frac{i}{4} H_0^{(1)}(k||\mathbf{x}-\mathbf{y}||)$.

TP2: Résolution de l'équation intégrale de frontière

Lors du TP1, p a été obtenu analytiquement. Dans le TP2, nous allons le déterminer numériquement en résolvant l'équation intégrale.

On part de la représentation intégrale

$$u(\mathbf{x}) = \mathcal{S}p(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega_e, \quad \mathbf{x} \notin \Gamma$$

On prend la trace de Dirichlet et on utilise la condition de Dirichlet $[u]_\Gamma = 0$.

La densité p est alors donnée par

$$\text{Trouver } p \in H^{-1/2}(\Gamma) \text{ tel que} \int_\Gamma G(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) d\Gamma(\mathbf{y}) = -u^{inc}(\mathbf{x})$$

Le but de ce TP2 est de (i) vérifier que la solution numérique correspond bien à la solution analytique et (ii) supprimer l'utilisation de la solution analytique pour calculer le champ diffracté dans Ω^+.

Etape 1: Formulation Variationnelle

Ecrire la formulation variationnelle correspondant à l'eq. intégrale:

$$\text{Trouver } p \in H^{-1/2}(\Gamma) \text{ tel que} \int_\Gamma G(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) d\Gamma(\mathbf{y}) = -u^{inc}(\mathbf{x})$$

Dans un premier temps, nous considérons une interpolation $\mathbb{P}^0$.

Etape 1: Formulation Variationnelle

Ecrire la formulation variationnelle correspondant à l'eq. intégrale:

Trouver $p \in H^{-1/2}(\Gamma)$ tel que $\int_\Gamma G(\mathbf{x},\mathbf{y})p(\mathbf{y})d\Gamma(\mathbf{y}) = -u^{inc}(\mathbf{x})$

Dans un premier temps, nous considérons une interpolation $\mathbb{P}^0$.

On suppose que les intégrales de bord ont un sens, sinon il faudrait plutôt considérer des crochets de dualité.

Le problème s'écrit $\forall p' \in H^{-1/2}(\Gamma)$

Trouver $p \in H^{-1/2}(\Gamma)$ tel que $\int_\Gamma Sp(\mathbf{x})\bar{p'}(\mathbf{x})d\Gamma(\mathbf{x}) = -\int_\Gamma u^{inc}(\mathbf{x})\bar{p'}(\mathbf{x})d\Gamma(\mathbf{x})$

Finalement, on cherche $p \in H^{-1/2}(\Gamma)$ tel que $\forall p' \in H^{-1/2}(\Gamma)$

$$\int_\Gamma\int_\Gamma G(\mathbf{x},\mathbf{y})p(\mathbf{y})\bar{p'}(\mathbf{x})d\Gamma(\mathbf{x})d\Gamma(\mathbf{y}) = -\int_\Gamma u^{inc}(\mathbf{x})\bar{p'}(\mathbf{x})d\Gamma(\mathbf{x})$$

Etape 2: Forme discrétisée

Ecrire la forme discrétisée de la formulation variationnelle.

Si on écrit le problème sous la forme matricielle

$$\mathbb{A}\mathbf{p} = \mathbf{b}$$

quelles sont les expressions de $\mathbb{A}_{ij}$ et $\mathbf{b}_i$?

Etape 4: Assemblage de la matrice $\mathbb{A}$

Numériquement, on va voir qu'il y a deux différences par rapport au cas de la représentation intégrale:

1. Il y a une double intégration dans la formulation variationnelle. Il faut donc appliquer deux fois les formules de quadrature de Gauss (sur les segments Γ_e et $\Gamma_{e'}$).
2. Il va falloir gérer la singularité de la fonction de Green

Ecrire le pseudo-code pour l'assemblage de la matrice $\mathbb{A}$.

Etape 5: Gestion de la singularité de la fonction de Green

Il faut dissocier deux cas de figure:

- le cas régulier où $\Gamma_e \neq \Gamma_{e'}$
- le cas singulier où $\Gamma_e = \Gamma_{e'}$: on ne peut pas utiliser une quadrature numérique

Méthode semi-analytique: développement limité autour de 0

$$G(\mathbf{x},\mathbf{y}) = \underbrace{\frac{1}{2\pi}\ln\frac{1}{||\mathbf{x}-\mathbf{y}||}}_{partie\ sing.} + \underbrace{\frac{i}{4} - \frac{1}{2\pi}[\ln(\frac{k}{2}) + \gamma]}_{partie\ reg.} + \mathcal{O}\Big(||\mathbf{x}-\mathbf{y}||^2 \ln\frac{1}{||\mathbf{x}-\mathbf{y}||}\Big)$$

γ: constante d'Euler ≈ 0.5772156649

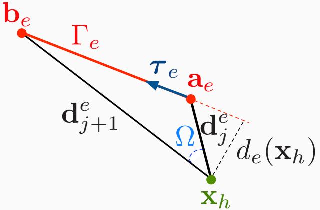
La partie régulière s'intègre encore avec une quadrature de Gauss.

... Etape 5: Gestion de la singularité de la fonction de Green

La partie singulière s'intègre de manière analytique:

$$\int_{\Gamma_e} \frac{-1}{2\pi} \ln ||\mathbf{x_h} - \mathbf{y_h}|| d\Gamma(\mathbf{y_h}) = -\frac{1}{2\pi}\Big(\mathbf{d}^e_{j+1}.\boldsymbol{\tau}_e \ln ||\mathbf{d}^e_{j+1}|| - \mathbf{d}^e_j.\boldsymbol{\tau}_e \ln ||\mathbf{d}^e_j|| - |\Gamma_e| + d_e(\mathbf{x}_h)\Omega\Big)$$

- $\boldsymbol{\tau}_e$ est le vecteur tangent unitaire,
- $d_e(\mathbf{x})_h$ est la distance de $(\mathbf{x})_h$ à Γ_e,
- Ω est l'angle solide sous lequel $(\mathbf{x})_h$ voit Γ_e, $0 \leq \Omega \leq \pi$.



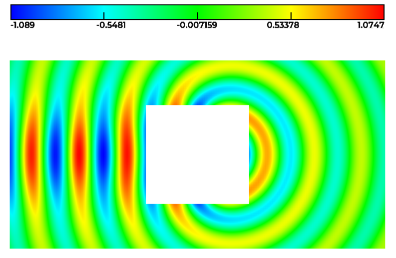
Comme $\Gamma_e = \Gamma_{e'}$, $\Omega = 0$ on a finalement simplement

$$\int_{\Gamma_e} \frac{-1}{2\pi} \ln ||\mathbf{x_h} - \mathbf{y_h}|| d\Gamma(\mathbf{y_h}) = -\frac{1}{2\pi}\Big(\mathbf{d}^e_{j+1}.\boldsymbol{\tau}_e \ln ||\mathbf{d}^e_{j+1}|| - \mathbf{d}^e_j.\boldsymbol{\tau}_e \ln ||\mathbf{d}^e_j|| - |\Gamma_e|\Big)$$

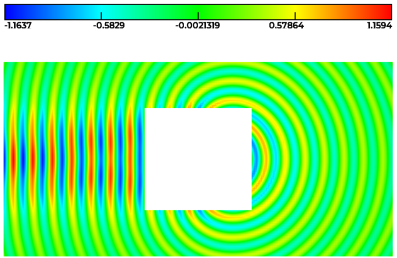
Quelle est finalement l'expression des termes $\mathbb{A}_{ii}$ (attention à la double intégration)? Modifier en conséquence la construction de la diagonale de la matrice $\mathbb{A}$.

Etape 6: Validation

Vérifier la trace de p sur le cas du disque.



$Re(p),\ k = 2\pi$



$Re(p),\ k = 5\pi$

Pour valider le code, il vous faut comparer l'erreur à la solution analytique pour plusieurs fréquences, plusieurs maillages, en plusieurs points, ...
N'oubliez pas de vérifier la convergence de votre code.
Quelle est la complexité de votre temps de calcul par rapport au nombre de points (pour une fréquence fixée)?

Etape 7: Couplage

Coupler ce code de résolution d'une équation intégrale avec le code d'évaluation de la représentation intégrale développé en TP1.

Quelle est la différence au niveau des temps de calcul entre la représentation intégrale et l'équation intégrale?

Bravo: Vous avez votre premier code BEM! Vous pouvez maintenant tester le comportement de cette méthode et illustrer les notions expliquées en cours