Numerical Linear Algebra: Homework Practical Lanczos

Clémence Duplat

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1 Introduction

In Numerical Linear Algebra, the Lanczos method is an efficient method for finding an eigenvalue approximiation. The method initially formulated was not useful due to numerical instabilities but a lot of solutions were showed for making this method numerically stable. The first goal of this report is to see why and how to improve the stability of the Lanczos method. Additionally, we will implement this practical Algorithm. The second goal of this report is to understand the benefits of tridiagonalization. This will lead us to a tridiagonal eigenvalue algorithm. Finally, we will apply this two methods to a realistic application, the Atomic force microscopy. Effectively, finding the eigenvalues of a this system can tell us the frequencies at which it vibrates, which is crucial to analyze it correctly.

2 Semi-orthogonality and strategic reorthogonalization

In this section we will recall how the initial Lanczos algorithm works. The Lanczos algorithm can be given as follows:

Algorithm 1: Lanczos in exact arithmetic

input : Linear, symmetric real operator A on $\mathbb{R}$
output: Approximately orthogonally similar tridiagonal $T \sim A$, given by diagonals α, β

init *choose starting vector* $\mathbf{r}_0$,$\beta_0 := \|\mathbf{r}_0\|$, $\mathbf{q}_0 = 0$
for $j = 1, 2, \ldots$ **do**
  $\mathbf{q}_j := \mathbf{r}_{j-1}/\beta_{j-1}$
  $\mathbf{v}_j = A\mathbf{q}_j - \beta_{j-1}\mathbf{q}_{j-1}$
  $\alpha_j = \mathbf{v}_j^* \mathbf{q}_j$
  $\mathbf{r}_j = \mathbf{v}_j - \alpha_j \mathbf{q}_j$
  $\beta_j = \|\mathbf{r}_j\|$
end

The matrix T then has the form

$$T = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \beta_{j-2} & \alpha_{j-1} & \beta_{j-1} \\ 0 & \cdots & 0 & \beta_{j-1} & \alpha_j \end{pmatrix}$$

One step in the main loop of algorithm **??** is called a Lanczos step, and is usually written as

$$\beta_j \mathbf{q}_{j+1} = A\mathbf{q}_j - \alpha_j \mathbf{q}_j - \beta_{j-1}\mathbf{q}_{j-1}. \tag{1}$$

As you know, with $Q_j := [\mathbf{q}_1, \ldots, \mathbf{q}_j]$, we have the relation

$$AQ_j = Q_j T_j + \beta_j \mathbf{q}_{j+1}\mathbf{e}^*. \tag{2}$$

It terminates when $\beta_j = 0$ for some j smaller or equal to n and if it does not stop early, then the columns od Q_k form the basis in the Krylov space $K_k(A, r)$. In exact arithmetic the above is a great procedure for tridiagonalization. Numerically this is not the case unfortunately and this is due to the influence of the roundoff errors on the algorithm in finite precision. This will cause loss of orthogonality.

3 Loss of orthogonality

As me just said, although these routines are commonly used in practice, they might not be theoretically optimal. Therefore orthogonalization becomes a key aspect of robust Lanczos variants that cope with loss of orthogonality. Indeed when orthogonality loss is encounter practical routines either orthogonalise against a portion of the Lanczos or either against all (named full orthogonalisation). Before applying partial or full orthogonalisation, we first need to know where and how orthogonality is lost.
We can do that by analysing the elements w_{ik} of the matrix $W_j := Q_j^* Q_j$, which measures the orthogonality of the Lanczos vectors. In order to have orthogonal vectors, we know the matrix W_j should look like that:

- 1 for the diagonal elements (dot product of a vector orthogonal to itself)
- 0 anywhere else (dot product of 2 vectors orthogonal)

If there is any orthogonality loss, we will observe deviations from this ideal values. Indeed the vectors are no longer mutually orthogonal.

We will analyse different startegies.

First of all we will run the simple *Lanczos* algorithm on the system Test.mtx. If we compute the matrix W_j and plot it on a heatmap, we can see that the matrix begins to differ from the identity matrix at a small iteration step. Therefore this first algorithm loses orthogonality very early in the process.

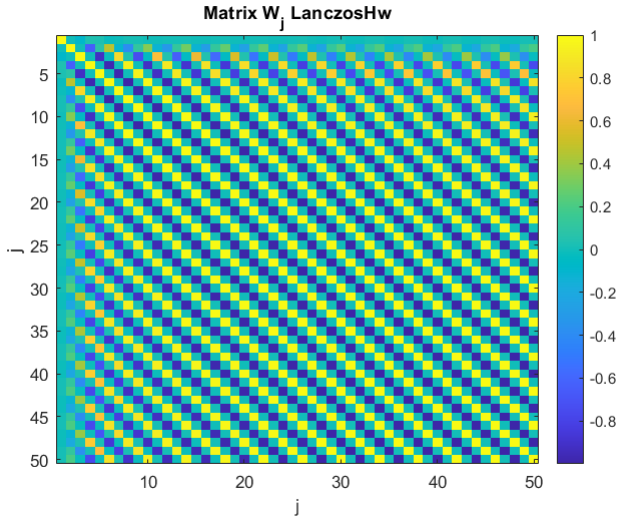


Figure 1: Heatmap initial Lanczos

Additionaly, we can plot the loss orthogonality by comparing each value to the identity matrix. We recall that the orthogonality loss is due to the influence of roundoff errors but it can also occur when eigenvalues are close to each other. The corresponding eigenvectors will then tend to be nearly linearly independent. The convergence of the ritz values will then start to faulter. Effectively, the loss of orthogonality can cause copies of Ritz pairs, the computed eigenvalues will not be correct anymore.

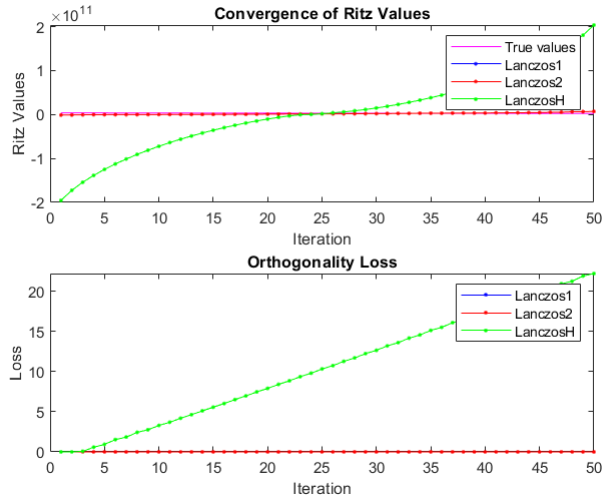


Figure 2: Orthogonality loss and ritz convergence plot

Secondly, this can lead us to very simple modified Lanczos scheme implemented

in *Lanczos1.m*. This scheme will keep track of

$$w_{j,\infty} := \|W_j - I_j\|_{max}$$

(see how it differs from the identity matrix and detect when reorthogonalization is necessary) and as soon some threshold is reached for Lanczos vector $\mathbf{q}'_j$, we reorthogonalize $\mathbf{q}'_j$ before computing β_j and follow up by reorthogonalizing $\mathbf{q}'_{j+1}$ as well. Reorthogonalizing against $\mathbf{q}'_{j+1}$ is necessary because loss of orthogonality can occur at any time and so it ensure that the newly Lanczos vectors are orthogonal to the current Krylov subspace. After implementing this new algorithm in *Lanczos1.m*, we can compute the matrix W_j and plot it on a heatmap. We can see that this time the matrix W_j corresponds to the identity matrix and that there is no loss of orthogonality in the process.

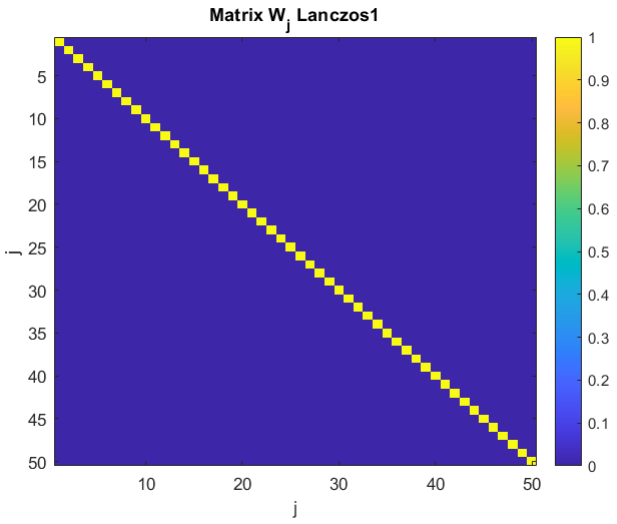


Figure 3: Heatmap Lanczos1

Additionaly, we can see that there is no more orthogonality loss in figure 2. When it comes to the threshold to use, we can introduce the concept of *semi-orthogonality*: we say Q_j is semiorthogonal if $w_{j,\infty} = \mathcal{O}(\sqrt{\epsilon})$. Why this is sufficient is captured in the following theorem.

Theorem 1 *With notation from before, let P_j denote the orthogonal basis for span(Q_j). If $w_{j,\infty} < \sqrt{\frac{\epsilon}{j}}$ then*

$$\|P_j^* A P_j - T_j\|_{max} = \mathcal{O}(\epsilon \|A\|).$$

This compution has although a very high cost due to the calculation of $w_{j,\infty}$ that requires the calculation of $Wj = Q_k(:, 1:j)' * Q_k(:, 1:j)$;

Thirdly, we can improve this algorithm by gaining more insight into W. To study W, we begin by re-writing equations 1 and 2 to

$$\beta_j \mathbf{q}_{j+1} = A\mathbf{q}_j - \alpha_j \mathbf{q}_j - \beta_{j-1}\mathbf{q}_{j-1} - \mathbf{f}_j \tag{3}$$

and

$$AQ_j = Q_j T_j + \beta_j \mathbf{q}_{j+1} \mathbf{e}_j^* + F_j \tag{4}$$

where $\mathbf{f}_j$ is an error vector and F_j a corresponding error matrix. We assume $\|F_j\| \leq \epsilon \|A\|$, with ϵ machine precision. For simplicity we make the following two assumptions for all $k \in \{1, \ldots, j\}$:

- $\|\mathbf{q}_k\|_2 = 1$
- $\beta_k \mathbf{q}_{k+1}^* \mathbf{q}_k = C\epsilon\|A\|$, with C a modest constant.

The first assumption make in sort that the Lanczos vector is normalized in order to have a unit norm. The second assumption indicates that the orthogonality between two consecutive Lanczos vectors is maintained up to a small error proportional to $||A||$(norm of A) and ϵ (error machine). These errors are likely to maintain numerical stability. Strictly speaking they are not necessary, but they greatly simplify the derivations. These two assumptions can help us find some information for our new algorithm. Indeed, with the normalization assumption we can find the following:

- Diagonal elements w_{kk} must be equal to 1 because the Lanczos vector is orthogonal to itself due to the normalization
- Off-diagonal elements $w_{kk-1} = \underbrace{\mathbf{q}_k^* \mathbf{q}_{k-1}}_{:=\psi_k}$ for $k = 2, \ldots, j$ represent the inner production wich should be zero but might not be due to numerical errors.

We also know that W is symmetric so we can find : $w_{jk+1} = w_{k+1j}$ If we rewrite the equation given in 3 for j and k we find the 2 following equations:

$$\beta_j \mathbf{q}_{j+1} = A\mathbf{q}_j - \alpha_j \mathbf{q}_j - \beta_{j-1} \mathbf{q}_{j-1} - \mathbf{f}_j \tag{5}$$

$$\beta_k \mathbf{q}_{k+1} = A\mathbf{q}_k - \alpha_k \mathbf{q}_k - \beta_{k-1} \mathbf{q}_{k-1} - \mathbf{f}_k \tag{6}$$

By simplifying the results of $q_k^* - q_j^*$ we finally find the following recurrence equation:

Theorem 2 *Let $W_j := Q_j^* Q_j$, with Q_j the matrix of Lanczos vectors found by algorithm* **??**. *Then, with $w_{j0} := 0$:*

$$\begin{aligned}
w_{kk} &= 1 && \text{for } k = 1, \ldots, j \\
w_{kk-1} &= \underbrace{\mathbf{q}_k^* \mathbf{q}_{k-1}}_{:=\psi_k} && \text{for } k = 2, \ldots, j \\
w_{jk+1} &= w_{k+1j} && \text{for } k = 1, \ldots, j-1 \\
\beta_j w_{j+1k} &= \beta_k w_{jk+1} + (\alpha_k - \alpha_j) w_{jk} + \beta_{k-1} w_{jk-1} - \beta_{j-1} w_{j-1k} + \underbrace{\mathbf{q}_j^* \mathbf{f}_k - \mathbf{q}_k^* \mathbf{f}_j}_{:=\theta_{j,k}}
\end{aligned}$$

where the last equation holds for $k = 1, \ldots, j-1$.

Thanks to this recurrence, we can identify when reorthogonalization is needed. Indeed, it quantifies how the local error is propagated and how the level of orthogonality rises. If the off-diagonals elements becomes too large, reorthogonalization can be done. This is implemented in a new algorithm called *Lanczos2.m.*

By plotting the heatmap of W_j, we see that, because W_j looks like the identity matrix, there is no loss of orthogonality.

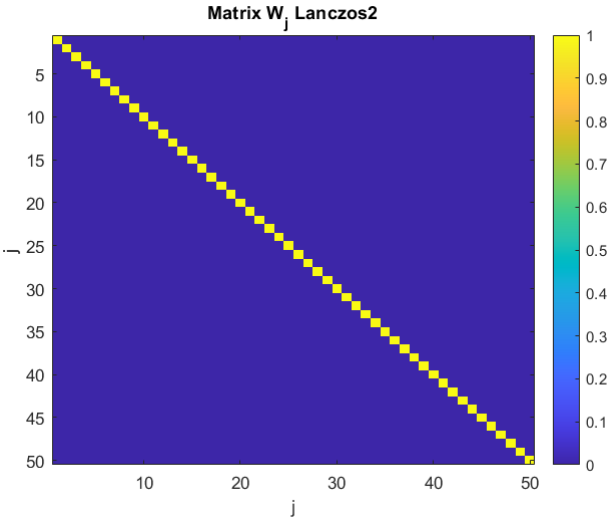


Figure 4: Heatmap Lanczos2

We can also interpret it on Figure 2.

4 The anavoidability of orthogonality loss

Now suppose that $W_j := I_j + U_j + U_j^*$ with U_j strictly upper triangular. Suppose the columns of U are given by $U = [\mathbf{u}_1, \ldots, \mathbf{u}_j]$. We can show that we can write the above as

$$\beta_j Q_j^* \mathbf{q}_{j+1} = T_j \mathbf{u}_j - \alpha_j \mathbf{u}_j - \beta_{j-1}\mathbf{u}_{j-1} + \underbrace{(F_j^* \mathbf{q_j} - Q_j^* \mathbf{f}_j)}_{:=\mathbf{g}_j} + \mathbf{e}_j(\beta_j \mathbf{q}_{j+1}^* \mathbf{q}_j - \beta_{j-1}\mathbf{q}_j^* \mathbf{q}_{j-1})$$

Proof:
If we multiply the equation 3 by Q_j^* we obtain:

$$Q_j^* \beta_j \mathbf{q}_{j+1} = Q_j^* A \mathbf{q}_j - \alpha_j Q_j^* \mathbf{q}_j - \beta_{j-1} Q_j^* \mathbf{q}_{j-1} - Q_j^* \mathbf{f}_j$$

Because we know we also have equation 4, we can say that:

$$Q_j^* A = T_j Q_j^* + \beta_j \mathbf{e}_j \mathbf{q}_{j+1}^* + F_j^*$$

By replacing it in the first equation of the proof we find:

$$\beta_j Q_j^* \mathbf{q}_{j+1} = T_j Q_j^* q_j + \beta_j e_j q_{j+1}^* q_j + F_j^* q_j - \alpha_j Q_j^* q_j - \beta_{j-1} Q_j^* q_{j-1} - Q_j^* f_j$$

Because we suppose that $W_j := I_j + U_j + U_j^*$ and we know that $W_j := Q_j^* Q_j$, we can easily find that $Q_j^* q_j = u_j$ and $Q_j^* q_{j-1} = u_{j-1}$. By replacing this knowledge in our last equation we juste prove it.

We can group these expressions, indexed over j, together into an expression:

$$\beta_j Q_j^* \mathbf{q}_{j+1} - \beta_j e_j q_{j+1}^* q_j = T_j u_j + F_j^* q_j - \alpha_j u_j - \beta_{j-1} u_{j-1} - Q_j^* f_j$$

We see that the left side of the equation represents the jth column of the matrix product $\beta_j Q_j^* \mathbf{q}_{j+1} e_{j+1}^*$ so if want to reaarnage it like that :

$$\beta_j Q_j^* \mathbf{q}_{j+1} \mathbf{e}_j^* = T_j U_j - U_j T_j + G_j$$

G_j will correspond to the 'extra' terms in order to make it possible to rearrange the right term as : $T_j U_j - U_j T_j$. It will also correspond to the numerical inaccuracies, the deviations of the ideal Lanczos behavior.
We are now ready to prove Paige's lemma:

Lemma 1 (Paige, 1971) *Let T_j denote the Lanczos matrix for some given A at iteration j. Suppose T_j has an eigendecomposition $T_j S_j = S_j \Theta_j$, with associated Ritz vectors $Y = Q_j S_j = Q_j[\mathbf{s}_1, \ldots, \mathbf{s}_j]$. Say $s_{ji} = \mathbf{e}_j^* \mathbf{s}_i$. Let (θ_i, y_i) be a Ritz pair at this iteration for T_j. Define*

$$\beta_{ji} := \|Ay_i - \theta_i y_i\|_2 = \beta_j |s_{ji}|$$

Then

$$|y_i^* q_{j+1}| = \frac{|\gamma_{ii}|}{\beta_{ji}}$$

with $\gamma_{ii} = \mathbf{s}_i^* G_j \mathbf{s} \approx \epsilon \|A\|$.

Proof:
***Firstly**, we will prove the equation*

$$\beta_{ji} := \|Ay_i - \theta_i y_i\|_2 = \beta_j |s_{ji}|$$

.
We know that $Y_i = Q_j S_i$ and we have the equation 2.
Therefore, we can write

$$AY_i = AQ_j S_i = Q_j T_j S_i + \beta_j q_{j+1} e_j^*$$

.
Using the eigenvalue decomposition $T_j S_i = \theta_i S_i$, we find:

$$Ay_i = Q_j \theta_i S_i + \beta_j q_{j+1} S_{ji}$$

Finally, since $\|q_{j+1}|\_2 = 1$, we can rewrite

$$\beta_{ji} := \|Ay_i - \theta_i y_i\|_2 = \|Q_j \theta_i s_i + \beta_j q_{j+1} s_{ji} - \theta_i y_i\|_2 = \|\beta_j q_{j+1} s_{ji}\|_2 = \beta_j |s_{ji}|$$

***Secondly**, we will prove the equation*

$$|y_i^* q_{j+1}| = \frac{|\gamma_{ii}|}{\beta_{ji}}$$

with $\gamma_{ii} = \mathbf{s}_i^* G_j \mathbf{s} \approx \epsilon \|A\|$.
First of all we recall that $Y_i = Q_j S_j$ *so*

$$Y_i^* q_{j+1} = S_j^* Q_j^* q_{j+1}$$

.
Using the equation 2 and multiplying booth sides by $S_j^* Q_j^*$*, we get:*

$$S_j^* Q_j^* A Q_j = S_j^* Q_j^* Q_j T_j + \beta_j S_j^* Q_j^* q_{j+1} e_j^*$$

We can simplify this by knowing that $Q_j^* Q_j = I$ *and* $Q_j^* A Q_j = T_j$*:*

$$S_j^* T_j = S_j^* T_j + \beta_j y_i^* q_{j+1} s_{ji}$$

We know that the error will be

$$\beta_j y_i^* q_{j+1} s_{ji} = \gamma_{ii}$$

Knowing that

$$\beta_{ji} := \beta_j |s_{ji}|$$

we can obtain our final result:

$$|y_i^* q_{j+1}| = \frac{|\gamma_{ii}|}{\beta_{ji}}$$

Orthogonality is lost only, and always, in the direction of the converging Ritz vectors. Indeed, as Ritz pair converges, s_{ij} become significant and the residual β_{ji} becomes smaller, indicating convergence.

5 The eigenvalues of a tridiagonal matrix

To see how tridiagonalization is desired, suppose a tridiagonal symmetric (real) matrix T given by:

$$T = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ 0 & \cdots & 0 & \beta_{n-1} & \alpha_n \end{pmatrix}$$

Let $T_j := T(1:j, 1:j)$ as before. Now let p_j denote the characteristic polynomial of T_j (with $p_0 = 1$). We know that the characteristic polynomial of a matrix T is defined by

$$p(x) = det(T - xI)$$

Now let's consider the tridiagonal matrix T_j, his characteristic polynomial will be

$$p(x) = det(T_j - xI)$$

If we split the matrix T into smaller sub-matrices, we will find a recurrence relation. Indeed, let's take the first case where T is one dimension and equal to $[\alpha_1]$ and so

$$p_1(x) = \alpha_1 - x$$

For T_2 a 2 by 2 matrix we have

$$p_2(x) = (\alpha_1 - x) * (\alpha_2 - x) - (\beta_1)2$$

Now we will make a generality and take the matrix T_j of dimension j by j. We can compute it's determinant by using Laplace expansion along the last row. By definition, the Laplace expansion is the expression of the determinant of my matrix T_j as a weighted sum of minors(determinant of $(j-1)*(j-1)$ submatrices of T_j). The equaltity is given by :

$$det(T_j) = \sum_{i=1}^{j} (-1)^{i+j} b_{j,i} m_{j,i}$$

where $b_{j,i}$ is the entry of the jth row and ith column and $m_{j,i}$ is the determinant of the submatrix obtained by remowing the jth row and the ith column of T_j. Knowing this we can write the polynomial characteristic as:

$$p_j(x) = (\alpha_j - x) * det(T_{j-1} - xI) - \beta_{j-1}^2 * det(T_{j-2} - xI)$$

and so

$$p_j(x) = (\alpha_j - x)p_{j-1}(x) - \beta_{j-1}^2 p_{j-2}(x)$$

for $j \geq 2$ and for all $x \in \mathbb{R}$.

Let's consider $p_n(x)$, we know it can be computed in a sequential way, starting from $p_o(x)$ and $p_1(x)$ and upgrading until $p_n(x)$. We also know that computing a determinant only involves basic operations as multiplications, additions and subtractions, so the complexity of each is $O(1)$. To arrive to n we will have to calculate (n-1) determinant, so it can be evaluated in $O(n)$ floating point operations.

An important property of tridiagonal symmetric matrices is the Sturm Sequence Property:

Theorem 3 (Sturm sequence property) *Suppose T is such that $\{\beta_i\}_{i=1}^{n-1}$ contains no zeros. Then for any $j \in \{1, \ldots, n\}$, the eigenvalues of T_j and T_{j-1} interlace as follows:*

$$\lambda_j(T_j) < \lambda_{j-1}(T_{j-1}) < \lambda_{j-1}(T_j) < \cdots < \lambda_2(T_j) < \lambda_1(T_{j-1}) < \lambda_1(T_j)$$

.

If one of the β is equal to zero, this will imply that T is block diagonal. The matrix will then be 'splitted' into independent smaller matrices. The eigenvalues of each block can be found independently without affecting the eigenvalues of the other blocks.

From the Sturm sequence property, that tell us that the eigenvalues of the submatrices T_j and T_{j-1} interlaces, another important result can be derived.
Suppose $s(x)$ denotes the number of sign changes in the sequence $p_0(x), \ldots, p_n(x)$. The convention is adopted that if $p_j(x) = 0$, then it has opposite sign to $p_{j-1}(x)$. The result is then that for any x, $s(x)$ equals the number of eigenvalues of T that are less than or equal to x.
Effectively, if we take a polynomial $p_j(x)$ associated with the determinant $xI - T_j$ and that for a given x this polynomial changes of sign, we can interpret this as x that has passed through an eigenvalue of T_j. Because T_j and T_{j-1} interlaces (from Sturm Sequence Pproperty), x will pass through the increased eigenvalues in an alternating way. This way if we keep a track of the sign changes (zero is consedered as a sign change), it will correspond to the eigenvalues that are less or equal to x.

If we now take the Gershgorin circle theorem given by:

Theorem 4 (Gershgorin) *For $i \in \{1, \ldots, n\}$ let R_i be the sum of the absolute values of the non-diagonal entries in the i-th row:*

$$R_i = \sum_{j \neq i} |a_{ij}| .$$

Let $D(a_{ii}, R_i) \subseteq \mathbb{C}$ be a closed disc centered at a_{ii} with radius R_i. Such a disc is called a Gershgorin disc.
Then, every eigenvalue of A lies within at least one of the Gershgorin discs $D(a_{ii}, R_i)$.

Knowing this theorem, we construct an algorithm named *findKthEigenvalue.m* that finds the kth eigenvalue of a given trdiagonal matrix. Indeed this theorem will help us to find an estimation of where the eigenvalues of our matrix will be. Each diagonal element of our matrix will be the center of the disc with a radius equal to the sum of the absolute values of the non-diagonal elements of that row. Therefore, we will initialize an interval [a,b] with the lowest lower bound and the highest upper bound of the Gershgorin discs of the matrix. When this is done, we will use the *bissection method* to find the midpoint. Using the the properties of sturm sequence and the fact that '$s(x)$ equals the number of eigenvalues of T that are less than or equal to x' we can compute an algorithm that count the eigenvalues below m. Thanks to that we can apply (untill a condition is met) bissection the following way:

- if m is smaller then k, we know that the kth eigenvalue is greater than m so the lower-bound will be m
- if m is greater or equal to k, we know that the kth eigenvalue is smaller than m si the upper-bound will be m

A more sophisticated algorithm *findFirstKEigenvalues.m*, find the first k eigenvalues efficiently. It is working the same way but it recycles information by working in a recursive manner. However, this recursive procedure is not always stable. Effectively, in recursive procedures when a small error is introduces, it is propaged to the rest of the algorithm. Additionaly, the bissection method can converge very slowly. When we have large matrices, it must be usefull to use a matrix decomposition before.

6 Atomic force microscopy

6.1 What is AFM?

Atomic force microscopy (AFM) is a microscopy technique that works on the same length scales a scanning tunneling microscopy (STM), but does not need its subject to be electrically conductive.The driving transmitted of information is not tunneling electons, but various atomic forces. This makes AFM better suited to study proteins, blood cells, etc. than STM.
The basic priciple is that a cantilever with a small tip at its free end is brought close to, or in contact with a material or material surface. A concentrated light beam is bounced off of the cantilever and captured on a light sensitive receiver. Then the movement of the cantilever due to atomic forces can be recorded and information on the sample can be deduced. In non-contact atomic force microscopy the cantilever tip does not come in contact with the sample surface, but rather sits at some distance, where chosen non-contact forces are strongest. For instance the van der Waals forces are strongest from 1 nm to 10 nm above the surface. The cantilever is oscillated at one of its resonant frequencies (frequency modulation) or just above one (amplitude modulation). The long-range forces above the surface decrease the resonance frequency of the cantilever. By moving the tip closer and further this decrease can be counteracted. As such, a constant oscillation (in amplitude or frequency) is created, which can be used to measure the distance to the sample. Since this mode of AFM operation does not actually contact the sample, it is much more suited for the observation of delicate systems.

Additionally, the cantilever oscillation frequencies and modes can provide insight into the types of forces acting on the AFM system and even on the indiviual atoms that the sample is made up of. It should be clear to you from these considerations that a rigorous modal analysis of the AFM cantilever is absolutely vital.

6.2 Modal analysis using the finite element method

In a modal analysis the typical assumption is that of undamped harmonic motion without driving forces. Whenever the given volume is discretized to a mesh and a choice of basis functions is made (taking boundary conditions into account) this leads to a system

$$(K - \omega^2 M)\delta\mathbf{u} = 0$$

with

$$\delta\mathbf{u} = \begin{pmatrix} \delta\mathbf{u}_x \\ \delta\mathbf{u}_y \\ \delta\mathbf{u}_z \end{pmatrix}$$

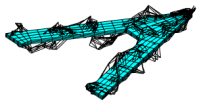
a vector of displacements. Such a problem is called a generalized eigenvalue problem. The matrix K is called the stiffness matrix, and the matrix M is called the mass matrix. Note that each node of the given grid produces three degrees of freedom (dof). Elements in the stiffness matrix correspond to the stiffness at dof-dof interactions (e.g. the xy-stiffness at a given node), and the elements of the mass.

The generalized eigenvalue problem can be converted into a **standard eigenvalue** problem by applying the following assumptions:

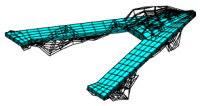
$$(M^{-1}K)\delta\mathbf{u} = \omega^2\delta\mathbf{u}$$

We have to assume that M is non-singular(invertible). Generally, we know that M is block diagonal matrix and thus invertible because the diagonal elements of the matrix are non-zero.
The boundary conditions seems very clear: Fixed at one end and free at the other end to vibrate.
To visualize the fundamental modes of vibrations, we will compute the smallest eigenvalues. Therefore, we will first apply a Lanczos Tridiagonalization (implemented in *Lanczos.m*) on $M^{-1}K$. This we lead us to a tridiagonal matrix. We can then compute the th kth first eigenvalues (implemented in *findFirstKEigenvalues.m*. We can see that there are equal to the true eigenvalues of our matrix (using *eigs)*. Finally, we can the visualize the associated eigenmodes as follow:

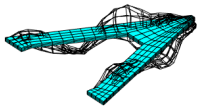
Dominant eigenmodes of the cantillever



Fundamental mode



Mode 1



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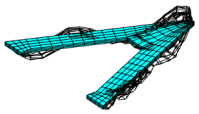
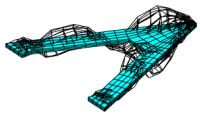


Mode 3



Mode 4

We see that it seems quiet logic because it oscillates the way a cantillever would do.

7 References

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