Kernel Functions

Sunday, October 28, 2018 4:12 PM

2015:

1. which of the following functions is a kernel and which is not ? Give an argument why.

Suppose that $x_i = \begin{pmatrix} x_i^{(1)} \\ x_i^{(2)} \end{pmatrix}$ are two-dimensional vectors.

$$k(x_1, x_2) = x_1^\top \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix} x_2$$

$$k(x_1, x_2) = x_1^\top \begin{pmatrix} 25 & 0 \\ 0 & -9 \end{pmatrix} x_2$$

$$k(x_1, x_2) = x_1^\top \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} x_2$$

Hint: Recall: if $k(x_1, x_2)$ is a kernel function, then there must exist a mapping ϕ and a Hilbert space such that

$$k(x_1, x_2) = \langle \phi(x_1), \phi(x_2) \rangle$$

where

$$\langle v_1, v_2 \rangle$$

is the inner product in the Hilbert space for two vectors v_1 and v_2, and

$$k(x_1, x_1) = \|\phi(x_1)\|^2$$

is the norm of $\phi(x_1)$ in this Hilbert space. Check whether you can find an explicit mapping ϕ for the euclidean inner product, or whether you can find a contradictions to properties of a kernel.

Ans:

(a) Yes, it is a kernel function.

Let $\phi(x) = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} x$

Then $k(x_1, x_2) = \langle \phi(x_1), \phi(x_2) \rangle$

$= x_1^T \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}^T \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} x_2$

$= x_1^T \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix} x_2 .$

(b) No, it is not a kernel

Let $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Then, the Gram matrix for these two points is K where

$$K_{11} = (1\ 0)\begin{pmatrix} 25 & 0 \\ 0 & -9 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 25$$

$$K_{12} = (1\ 0)\begin{pmatrix} 25 & 0 \\ 0 & -9 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$K_{21} = (0\ 1)\begin{pmatrix} 25 & 0 \\ 0 & -9 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$K_{22} = (0\ 1)\begin{pmatrix} 25 & 0 \\ 0 & -9 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -9$$

$K = \begin{pmatrix} 25 & 0 \\ 0 & -9 \end{pmatrix}$ is not positive definite.
because the eigenvalues are 25 and -9,
and -9 is negative.

(c) No, it is not a kernel
because it is not symmetric.

$$k\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = (1\ 0)\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$k\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = (0\ 1)\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$