# Lecture 2: Regression

# Machine Learning

Task

Performance

Experience

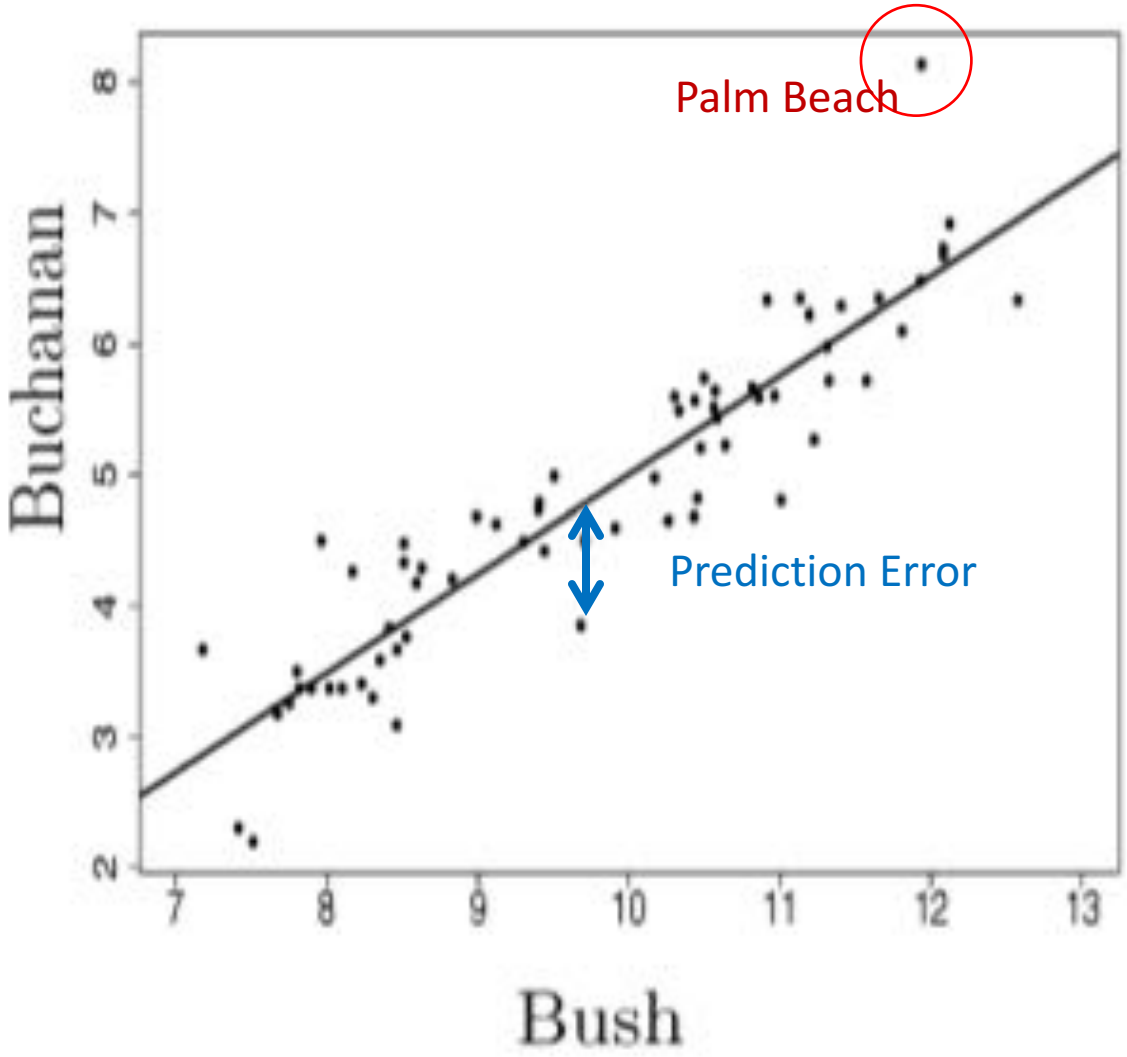
Algorithms that improve their performance
at some task with experience
– Tom Mitchell (1998)

# Regression

Machine Learning
> Supervised Learning
> Regression

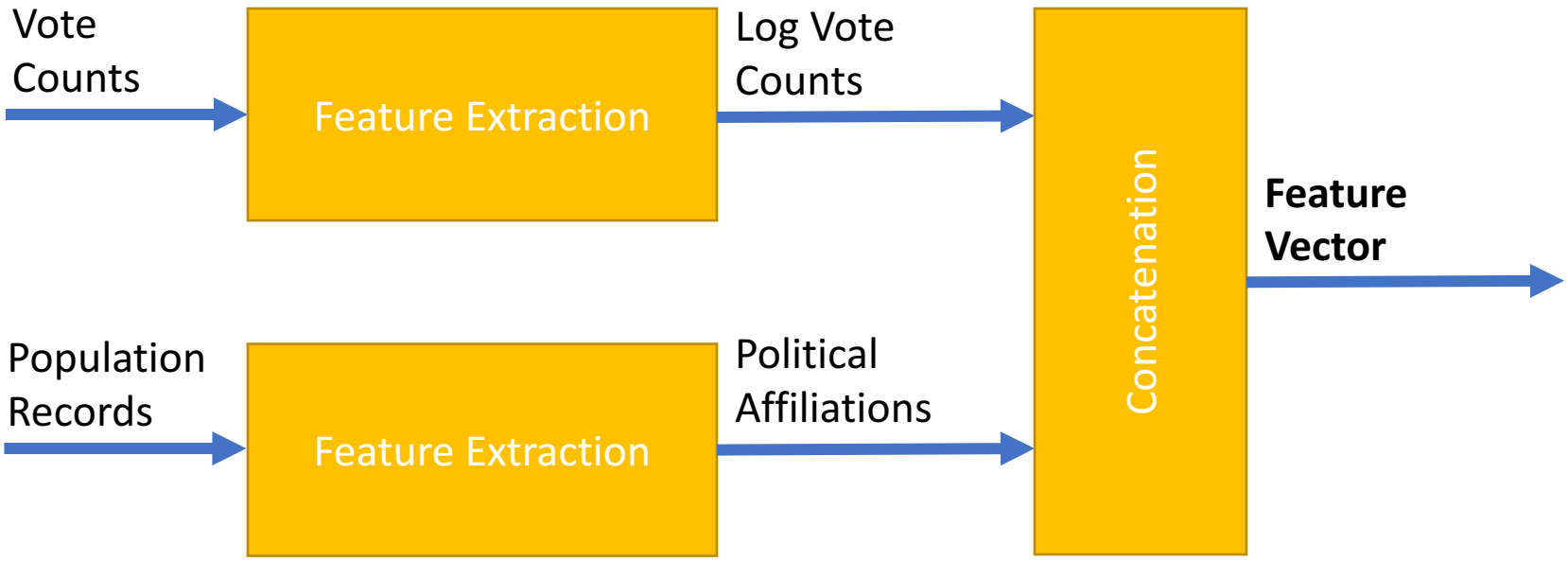
- **Task.** Find function $f: \mathbb{R}^d \to \mathbb{R}$ such that $y \approx f(x; \theta)$
- **Experience.** Training data $\left(x^{(1)}, y^{(1)}\right), \ldots, \left(x^{(n)}, y^{(n)}\right)$
- **Performance.** Prediction error $y - f(x; \theta)$ on test data

# Working Example



# Methodology

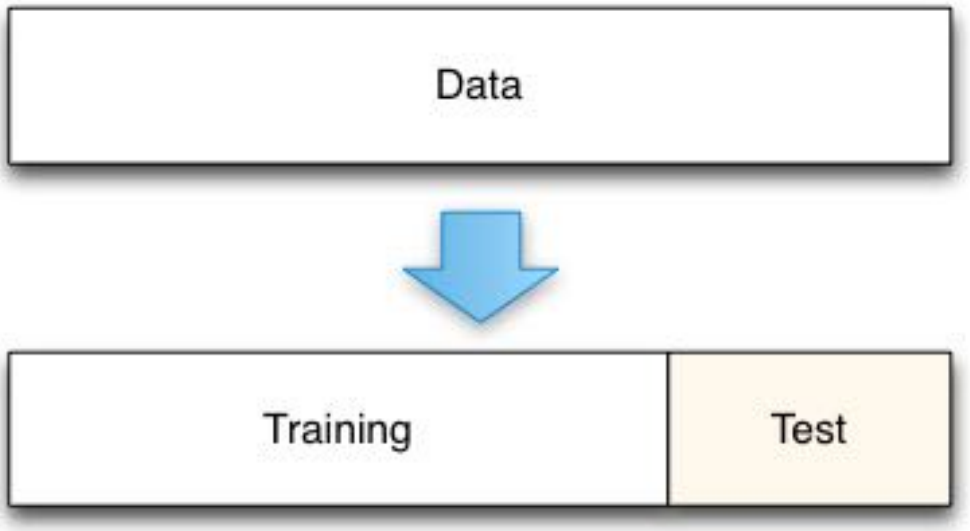
# Features


Vote Counts
Feature Extraction
Log Vote Counts
Concatenation
Feature Vector
Population Records
Feature Extraction
Political Affiliations

# Training data vs test data

Partition data into:

- Training data set $\mathcal{S}_n$
- Test data set $\mathcal{S}_*$



**Training data**

$$\mathcal{S}_n = \left\{ \left( x^{(i)}, y^{(i)} \right) \middle| \, i = 1, \ldots, n \right\}$$

- Features/Inputs $x^{(i)} = \left( x_1^{(i)}, \ldots, x_d^{(i)} \right)^\top \in \mathbb{R}^d$
- Response/Output $y^{(i)} \in \mathbb{R}$

Each $f$ is a *predictor* or *hypothesis*

**Model** (or Hypothesis Class) $\mathcal{H}$

Set of *linear* functions $f: \mathbb{R}^d \to \mathbb{R}$

$$f(x; \theta, \theta_0) = \theta_d x_d + \cdots + \theta_1 x_1 + \theta_0 = \theta^\top x + \theta_0$$

**Model Parameters**

$$\theta \in \mathbb{R}^d, \theta_0 \in \mathbb{R}$$

**Test Loss/Objective**

$$\mathcal{R}(\hat{f}; \mathcal{S}_*) = \frac{1}{n} \sum_{(x,y) \in \mathcal{S}_*} \frac{1}{2} \left( y - \hat{f}(x) \right)^2$$

Sometimes, we write $\mathcal{R}(\hat{\theta}; \mathcal{S}_*)$ instead of $\mathcal{R}(\hat{f}; \mathcal{S}_*)$

Given a predictor $\hat{f}$, we use the test loss $\mathcal{R}(\hat{f}; \mathcal{S}_*)$ to measure how well $\hat{f}$ generalizes to new data.

Sometimes, we write $\mathcal{L}(\theta; \mathcal{S}_n)$ instead of $\mathcal{L}(f; \mathcal{S}_n)$

**Training Loss/Objective**

$$\mathcal{L}(f; \mathcal{S}_n) = \frac{1}{n} \sum_{(x,y) \in \mathcal{S}_n} \frac{1}{2} \big(y - f(x)\big)^2$$

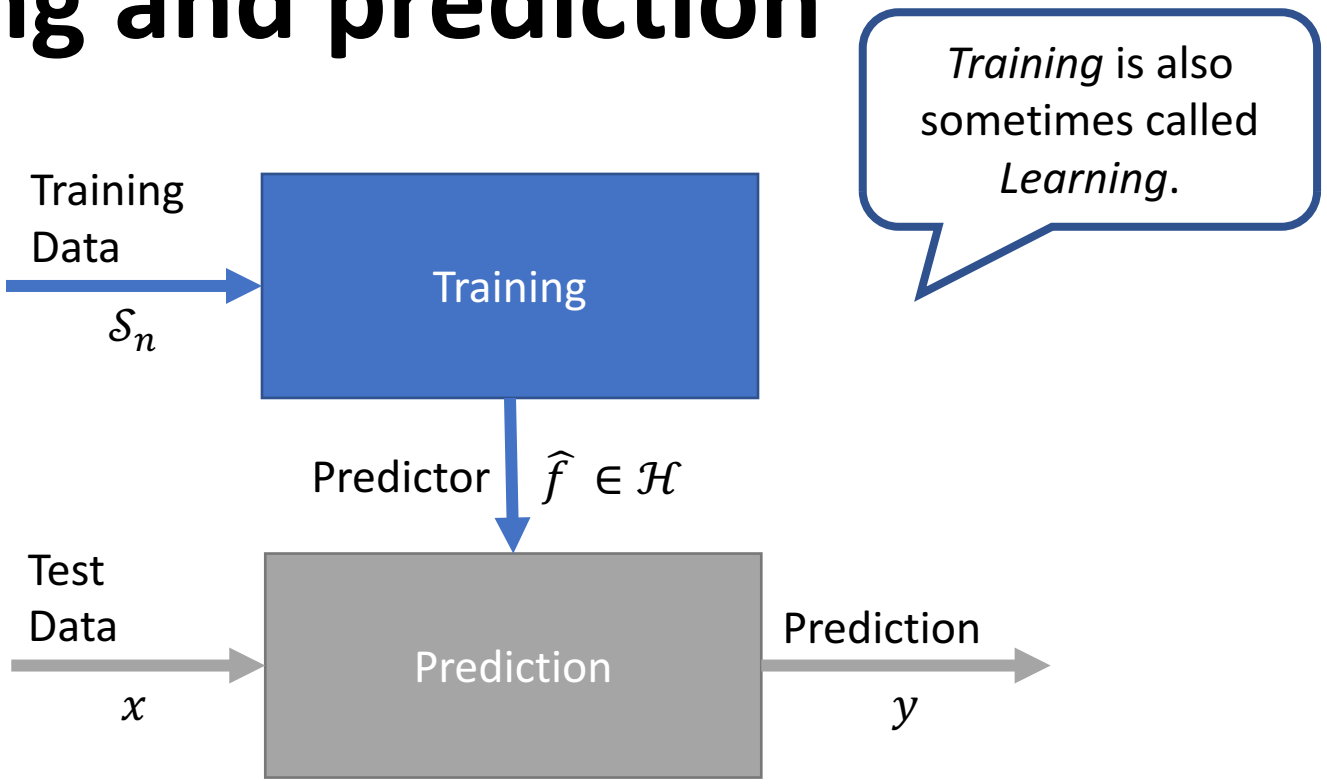
Find predictor $\hat{f} \in \mathcal{H}$ that minimizes $\mathcal{L}(f; \mathcal{S}_n)$.

**Training Algorithm**

Set gradient to zero, and solve equations.

The test loss and training loss can be different.

# Training and prediction



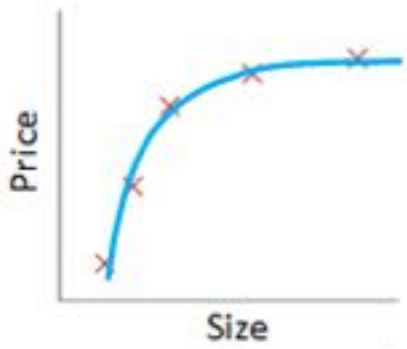

**Assumption.** Test data and training data are identically distributed.

# Generalization

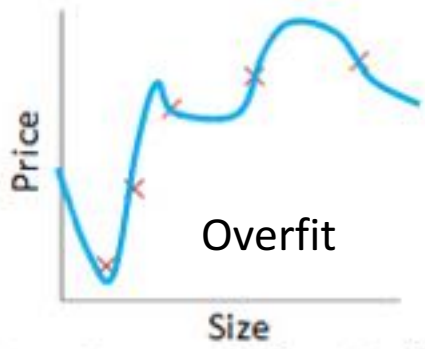
The goal of machine learning is to find a predictor $\hat{f} \in \mathcal{H}$ that generalizes well, i.e. that predicts well on test data $\mathcal{S}_*$.

# UnderFitting and Overfitting

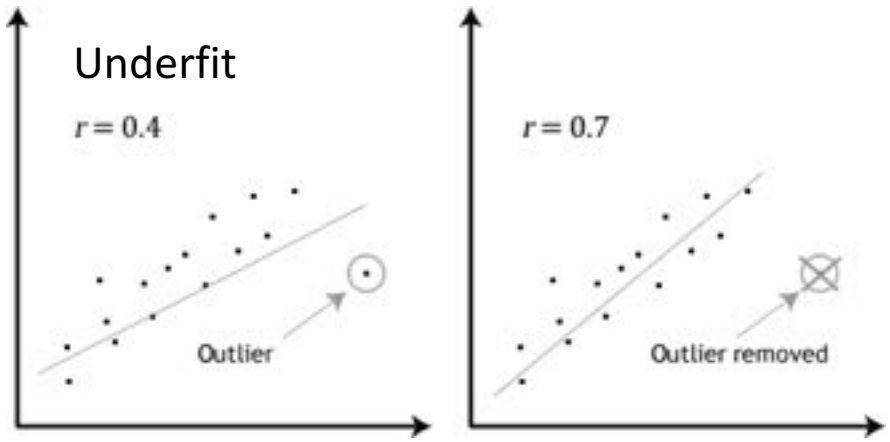

$\theta_0 + \theta_1 x$




$\theta_0 + \theta_1 x + \theta_2 x^2$




$\theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4$



# Model Selection

**Overfitting.** If model $\mathcal{H}$ is too big, then $\hat{f} \in \mathcal{H}$ performs

- well on training data, but poorly on test data.

**Underfitting.** If model $\mathcal{H}$ is too small, then $\hat{f} \in \mathcal{H}$ performs

- poorly on training data, and poorly on test data.

Finding a model with the right size is called model selection.

# Optimization

# Loss and Risk

**Loss Function** $\text{Loss}(z) = \frac{1}{2}z^2$ Squared error.
Penalize big errors more heavily.

**Empirical Risk / Training Loss**

$$\mathcal{L}_1(\theta; x, y) = \text{Loss}\big(y - f(x;\theta)\big)$$ Point loss

$$\mathcal{L}_n(\theta; \mathcal{S}_n) = \frac{1}{n}\sum_{(x,y)\in\mathcal{S}_n} \mathcal{L}_1(\theta; x, y)$$ Average loss

$$= \frac{1}{n}\sum_{(x,y)\in\mathcal{S}_n} \frac{1}{2}\big(y - f(x;\theta)\big)^2$$

The training loss is the average of the point losses.

Risk = “Expected Loss”
Empirical = “of the Data”

# Gradient

$$\nabla \mathcal{L}_n(\theta; \mathcal{S}_n) = \begin{pmatrix} \frac{\partial \mathcal{L}_n}{\partial \theta_1}(\theta; \mathcal{S}_n) \\ \frac{\partial \mathcal{L}_n}{\partial \theta_2}(\theta; \mathcal{S}_n) \\ \vdots \\ \frac{\partial \mathcal{L}_n}{\partial \theta_d}(\theta; \mathcal{S}_n) \end{pmatrix} = \frac{1}{n} \sum_{(x,y) \in \mathcal{S}_n} \begin{pmatrix} \frac{\partial \mathcal{L}_1}{\partial \theta_1}(\theta; x, y) \\ \frac{\partial \mathcal{L}_1}{\partial \theta_2}(\theta; x, y) \\ \vdots \\ \frac{\partial \mathcal{L}_1}{\partial \theta_d}(\theta; x, y) \end{pmatrix}$$

$$\nabla \mathcal{L}_n(\theta; \mathcal{S}_n) = \frac{1}{n} \sum_{(x,y) \in \mathcal{S}_n} \nabla \mathcal{L}_1(\theta; x, y)$$

The training gradient is the average of the point gradients.

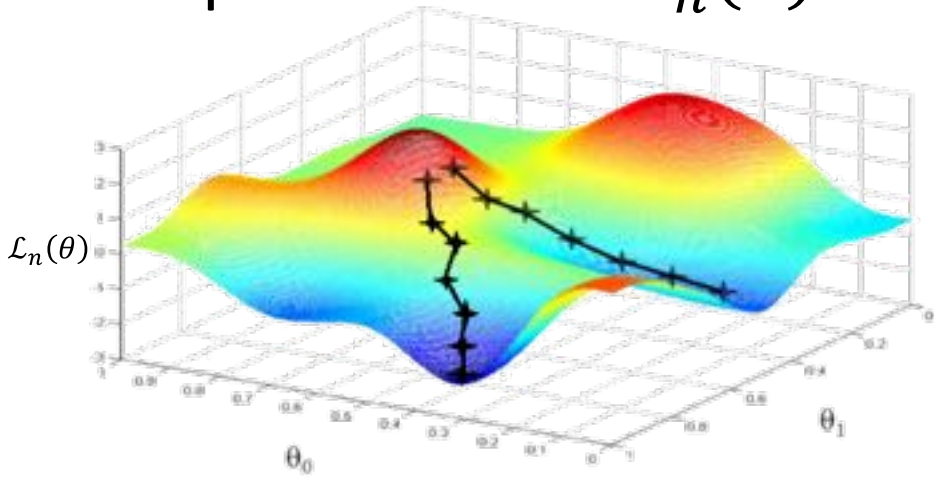
# EXACT SOLUTION

If there are no constraints on the parameters,

1. Set the gradient to zero, and solve for the parameters.
2. Run through all the solutions to find the parameter that has the smallest training loss.

# Gradient Descent

1. Initialize $\theta$ randomly.
2. Update $\theta \longleftarrow \theta - \eta_k \nabla \mathcal{L}_n(\theta)$ ,  $\eta_k$ learning rate, $k$ iteration number.
3. Repeat (2) until convergence.
   (e.g. when improvement in $\mathcal{L}_n(\theta)$ is small enough)



# Gradient Descent

- Gradient descent is based on the observation
  - if the multi-variable function $\mathcal{L}(\theta)$ is defined and differentiable in a neighborhood of $\theta$
  - then decreases fastest if one goes from $\theta$ in the direction of the negative gradient of $\mathcal{L}$ at $\theta$, i.e., $-\nabla\mathcal{L}(\theta)$

- It follows that, if
  - $\theta_{k+1} = \theta_k - \eta\,\nabla\mathcal{L}(\theta_k)$ for $\eta$ small enough
  - then $\mathcal{L}(\theta_k) \geq \mathcal{L}(\theta_{k+1})$

- The term $\eta\,\nabla\mathcal{L}(\theta)$ is subtracted from $\theta$ because we want to move against the gradient, toward the minimum

# Gradient Descent

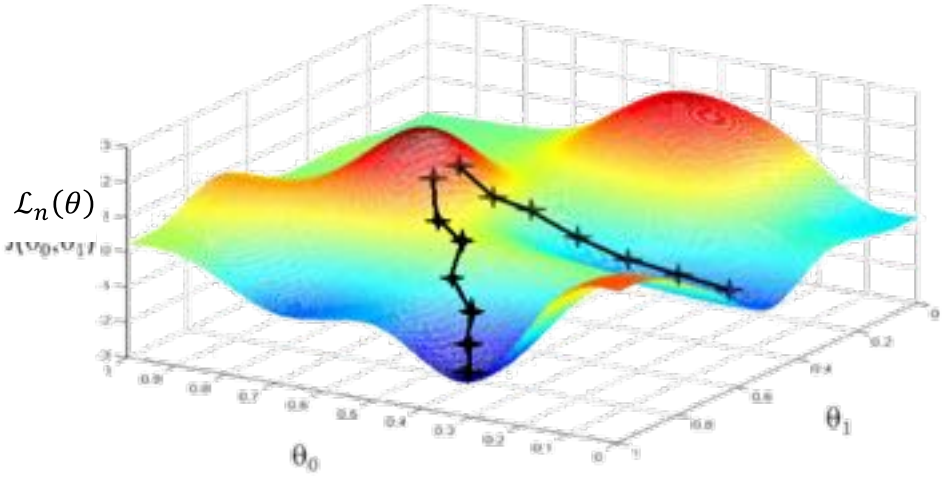
- With this in mind, we start with a guess $\theta_0$.
- Consider the sequence $\theta_0, \theta_1, \theta_2, \ldots$, such that
  - $\theta_{k+1} = \theta_k - \eta_k \nabla \mathcal{L}(\theta_k), k \geq 0,$
- We have
  - $\mathcal{L}(\theta_0) \geq \mathcal{L}(\theta_1) \geq \mathcal{L}(\theta_2) \geq \cdots$

- So hopefully the sequence $(\theta_k)$ converges to the desired local minimum.

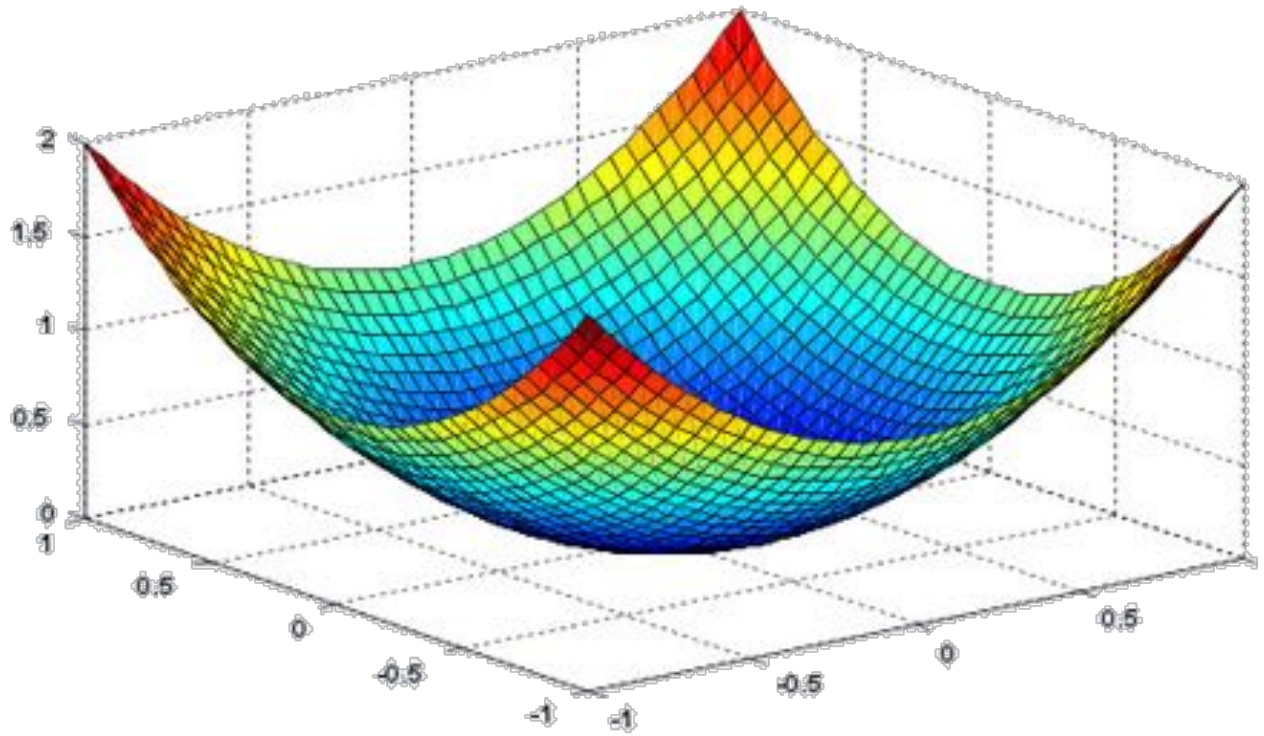
- Note that the value of the step size (or learning rate) $\eta$ is allowed to change at every iteration

# Local Minima

- Gradient descent leads us to a local minimum, which is not necessarily the global minimum. Different starting points may lead to different local minima.
- Typically, we perform gradient descent from several starting points, and run through all the local minima to find the parameter that has the smallest training loss.



# Convex Optimization



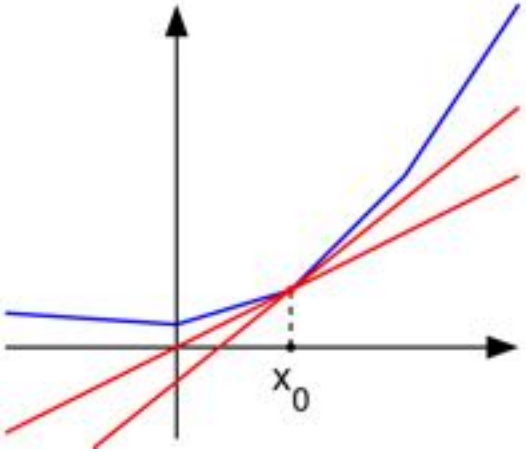

Local Minimum = Global Minimum.
Fast Algorithms.

# Sub-Gradients

- A sub-gradient $v \in \partial f(x)$ is a vector such that for all $y$,

$$f(y) - f(x) \geq v^{\mathsf{T}}(y - x).$$

- At non-differentiable points of the training objective function, the gradient does not exist, but we can use any sub-gradient instead for descent.



# Stochastic Gradient Descent

training gradient = average of point gradients

$$\nabla \mathcal{L}_n(\theta; \mathcal{S}_n) = \frac{1}{n} \sum_{(x,y) \in \mathcal{S}_n} \nabla \mathcal{L}_1(\theta; x, y)$$

This average can take a long time to compute for large data sets.

**Trick**

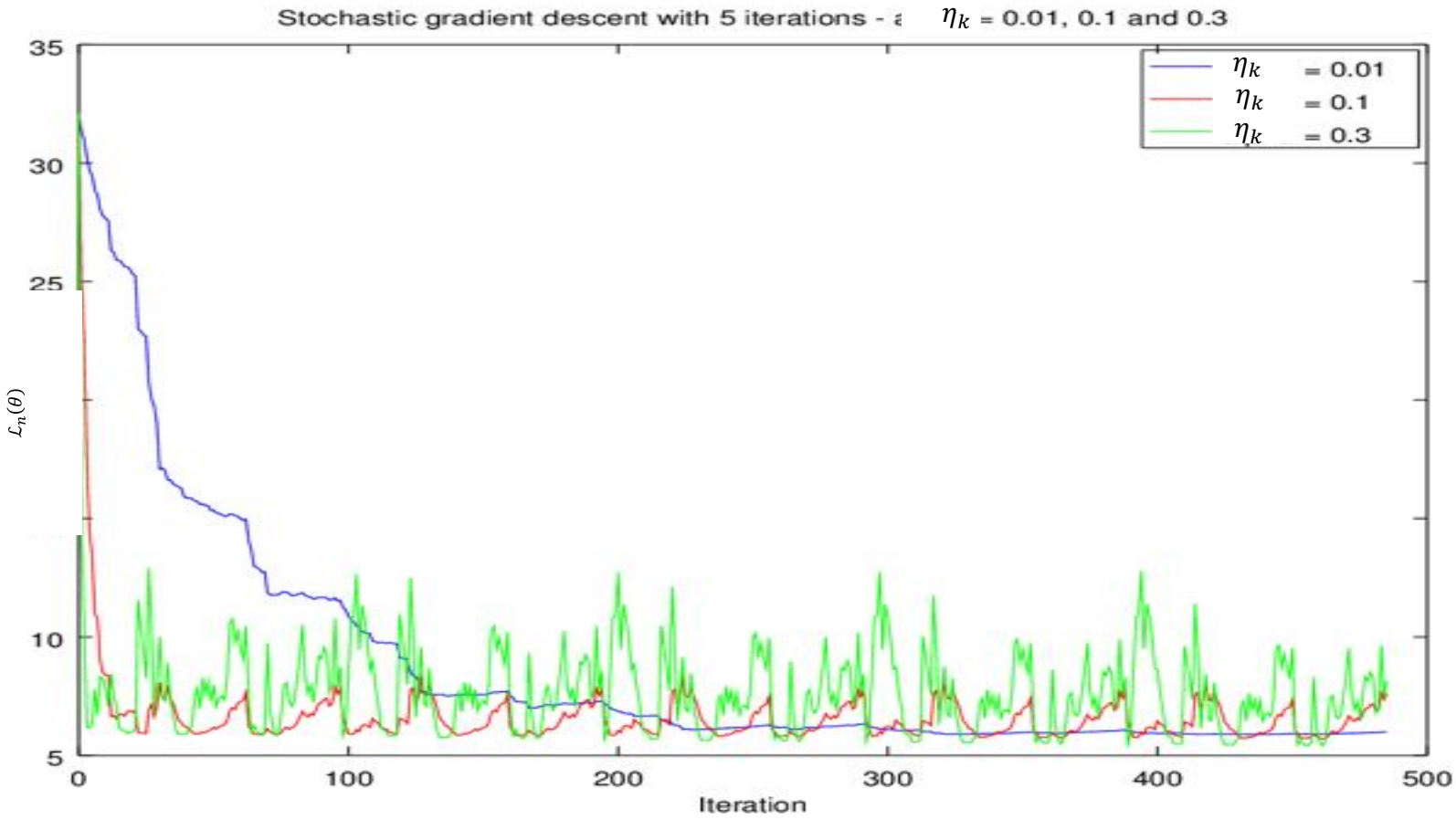
Estimate the gradient by averaging over a smaller *minibatch* (subset of the training data).

$$\nabla \mathcal{L}_n(\theta; \mathcal{S}_n) \approx \nabla \mathcal{L}_m(\theta; \mathcal{B}_m)$$
$$= \frac{1}{m} \sum_{(x,y) \in \mathcal{B}_m} \nabla \mathcal{L}_1(\theta; x, y)$$

# Stochastic Gradient Descent

1. Initialize $\theta$ randomly.
2. Select minibatch $\mathcal{B}_m$ of data from $\mathcal{S}_n$ at random.
   a. $\theta \longleftarrow \theta - \eta_k \nabla \mathcal{L}_m(\theta; \mathcal{B}_m)$.
3. Repeat Step (2) until convergence.

# Stochastic Gradient Descent



# Stochastic Gradient Descent

**Learning Rate**

Small learning rates help convergence, but big learning rates speed up descent. We want the best of both worlds, so we choose a learning rate that starts big and ends small, e.g. $\eta_k = 1/(k+1)$.

**Momentum**

Reduce fluctuations in gradient by taking a weighted sum of the previous update $\Delta^{(t-1)}$ with the current gradient.

$$\theta^{(t+1)} = \theta^{(t)} - \eta_k \Delta^{(t)}, \quad \Delta^{(t)} = (1-\epsilon)\Delta^{(t-1)} + \epsilon \nabla \mathcal{L}_m(\theta; \mathcal{B}_m)$$

**Software**

All these tricks are implemented in the ADAM optimizer.

# Multivariate Linear Regression

# Least Squares

**Data**

$$\left(x^{(1)}, y^{(1)}\right), \left(x^{(2)}, y^{(2)}\right), \ldots, \left(x^{(n)}, y^{(n)}\right), \; x \in \mathbb{R}^d, y \in \mathbb{R}$$

**Model**

$$f(x; \theta, \theta_0) = \theta_1 x_1 + \cdots + \theta_d x_d + \theta_0 = \theta^\top x + \theta_0$$

$$\theta \in \mathbb{R}^d, \theta_0 \in \mathbb{R}$$

**Training Loss and Objective**

$$\mathcal{L}_1(\theta, \theta_0; x, y) = \frac{1}{2}(y - (\theta^\top x + \theta_0))^2$$

$$\mathcal{L}_n(\theta, \theta_0; \mathcal{S}_n) = \frac{1}{n}\sum_{(x,y)\in\mathcal{S}_n} \mathcal{L}_1(\theta, \theta_0; x, y)$$

# Constant Feature Trick

Define $x_0 = 1$ and set $\tilde{x} = (x_d, \dots, x_1, x_0) \in \mathbb{R}^{d+1}$

**Data**

$$\left(\tilde{x}^{(1)}, y^{(1)}\right), \left(\tilde{x}^{(2)}, y^{(2)}\right), \dots, \left(\tilde{x}^{(n)}, y^{(n)}\right), \; \tilde{x} \in \mathbb{R}^{d+1}, y \in \mathbb{R}$$

**Model**

$$f(x; \theta, \theta_0) = \theta_1 x_1 + \cdots + \theta_d x_d + \theta_0 x_0 = \tilde{\theta}^\top \tilde{x}$$

$$\tilde{\theta} = (\theta, \theta_0) \in \mathbb{R}^{d+1}$$

**Training Loss and Objective**

$$\mathcal{L}_1\left(\tilde{\theta}; \tilde{x}, y\right) = \frac{1}{2}\left(y - \tilde{\theta}^\top \tilde{x}\right)^2$$

$$\mathcal{L}_n\left(\tilde{\theta}; \mathcal{S}_n\right) = \frac{1}{n} \sum_{(x,y) \in \mathcal{S}_n} \mathcal{L}_1\left(\tilde{\theta}; \tilde{x}, y\right)$$

# Point Gradient

Compute the point gradient

$$\nabla \mathcal{L}_1(\theta; x, y) = \begin{pmatrix} \frac{\partial \mathcal{L}_1}{\partial \theta_1}(\theta; x, y) \\ \frac{\partial \mathcal{L}_1}{\partial \theta_2}(\theta; x, y) \\ \vdots \\ \frac{\partial \mathcal{L}_1}{\partial \theta_d}(\theta; x, y) \end{pmatrix}$$

$$\mathcal{L}_1(\theta; x, y) = \frac{1}{2}(y - \theta^\top x)^2$$

$$\mathcal{L}_n(\theta; \mathcal{S}_n) = \frac{1}{n}\sum_{(x,y)\in\mathcal{S}_n} \mathcal{L}_1(\theta; x, y)$$

# Point Gradient

$$\frac{\partial \mathcal{L}_1}{\partial \theta_i}(\theta; x, y) = -x_i\,(y - \theta^\top x)$$

$$\nabla \mathcal{L}_1(\theta; x, y) = \begin{pmatrix} -x_1(y - \theta^\top x) \\ \vdots \\ -x_d(y - \theta^\top x) \end{pmatrix}$$

$$= -\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}(y - \theta^T x) = -x(y - \theta^\top x)$$

$$\mathcal{L}_1(\theta; x, y) = \frac{1}{2}(y - \theta^\top x)^2$$

$$\mathcal{L}_n(\theta; \mathcal{S}_n) = \frac{1}{n}\sum_{(x,y)\in\mathcal{S}_n} \mathcal{L}_1(\theta; x, y)$$

# Training Gradient

Let $X = \left[x^{(1)}, \ldots, x^{(n)}\right]^\top$, $Y = \left[y^{(1)}, \ldots, y^{(n)}\right]^\top$

Write the training gradient in terms of $X, Y$.

$$\nabla \mathcal{L}_n(\theta; x, y) = \frac{1}{n} \sum_{(x,y) \in \mathcal{S}_n} \nabla \mathcal{L}_1(\theta; x, y)$$

**Hints.**

$$\frac{1}{\mathrm{n}} \sum_{t=1}^{n} x^{(t)} y^{(t)} = \frac{1}{n} \left[x^{(1)}, \ldots, x^{(n)}\right] \left[y^{(1)}, \ldots, y^{(n)}\right]^\top = \frac{1}{n} X^\top Y$$

$$\theta^\top x = x^\top \theta$$

$$\mathcal{L}_1(\theta; x, y) = \frac{1}{2}(y - \theta^\top x)^2$$

$$\mathcal{L}_n(\theta; \mathcal{S}_n) = \frac{1}{n} \sum_{(x,y) \in \mathcal{S}_n} \mathcal{L}_1(\theta; x, y)$$

# Training Gradient

$$\nabla \mathcal{L}_n(\theta) = \frac{1}{n}\sum_{(x,y)\in\mathcal{S}_n} -x(y - \theta^\top x)$$
$$= \frac{1}{n}\sum_{(x,y)\in\mathcal{S}_n} -xy + x(\theta^\top x)$$
$$= \frac{1}{n}\sum_{(x,y)\in\mathcal{S}_n} -xy + x(x^\top \theta) = -B + A\theta$$

where

$$B = \frac{1}{\mathrm{n}}\sum_{t=1}^{n} x^{(t)} y^{(t)} = \frac{1}{n}\left[x^{(1)}, \ldots, x^{(n)}\right]\left[y^{(1)}, \ldots, y^{(n)}\right]^\top = \frac{1}{n} X^\top Y$$

$$\mathrm{A} = \frac{1}{\mathrm{n}}\sum_{t=1}^{n} x^{(t)} x^{(t)\top} = \frac{1}{n}\left[x^{(1)}, \ldots, x^{(n)}\right]\left[x^{(1)}, \ldots, x^{(n)}\right]^\top = \frac{1}{n} X^\top X$$

# Gradient Descent

$$\theta \longleftarrow \theta - \eta_k \left[ \frac{1}{n} (X^\top X)\theta - \frac{1}{n} X^\top Y \right]$$

# Exact Solution

Optimization problem is convex, so the minimum is attained when the gradient is zero.

$$\nabla \mathcal{L}_n\left(\hat{\theta}\right) = 0 \quad \Leftrightarrow \quad \frac{1}{n}(X^\top X)\,\hat{\theta} = \frac{1}{n}X^\top Y$$
$$\Leftrightarrow \quad \hat{\theta} = (X^\top X)^{-1}X^\top Y$$

**Issues.**

1. Need $X^\top X$ to be invertible
   - Feature vectors $x^{(1)}, \ldots, x^{(n)}$ must span $\mathbb{R}^d$
   - Must have more data than features, $n \geq d$
   - Use regularization if $X^\top X$ is not invertible
2. What if $X^\top X \in \mathbb{R}^{d \times d}$ is a large matrix?
   - Takes long time to invert
   - Use stochastic gradient descent if $X^\top X$ is too large

# Regularization

# Ridge Regression

Weight $y \approx \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_d x_d$

Height Age Temp. on Mars

For simplicity, we ignore $\theta_0$.

How do we ensure that $\theta_k = 0$ when feature $x_k$ is irrelevant?

Pick simplest model that explains data ➔ **generalization**

# Ridge Regression

**Add a penalty.**

Pressure to fit data

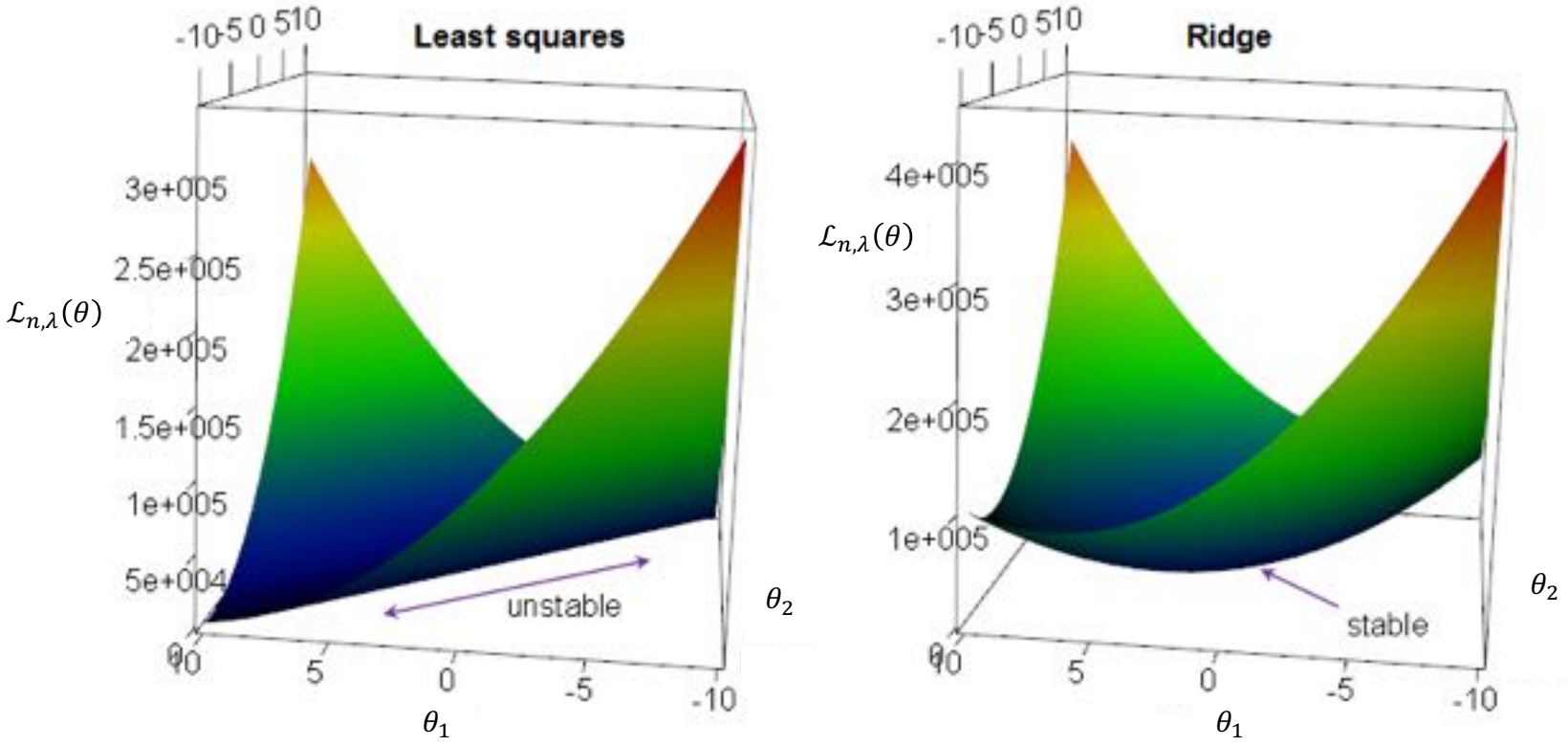
Pressure to simplify model

$$\mathcal{L}_{n,\lambda}(\theta) = \frac{1}{n}\sum_{(x,y)\in\mathcal{S}_n} \frac{1}{2}(y - \theta^\top x)^2 + \frac{\lambda}{2}\|\theta\|^2$$

Regularization parameter $\lambda \geq 0$

Regularizer

# Ridge Regression



# Training Algorithms

**Gradient**

$$\nabla \mathcal{L}_{n,\lambda}(\theta) = \lambda\theta + \frac{1}{n}(X^\top X)\theta - \frac{1}{n}X^\top Y$$

**Exact Solution**

$$\nabla \mathcal{L}_{n,\lambda}(\hat{\theta}) = 0 \quad \Leftrightarrow \quad \lambda\hat{\theta} + \frac{1}{n}(X^\top X)\,\hat{\theta} = \frac{1}{n}X^\top Y$$

$$\Leftrightarrow \quad \hat{\theta} = (n\lambda I + X^\top X)^{-1}X^\top Y$$

This matrix is always invertible when $\lambda > 0$.

# Training Algorithms

**Gradient**

$$\nabla \mathcal{L}_{n,\lambda}(\theta) = \lambda\theta + \frac{1}{n}(X^\top X)\theta - \frac{1}{n}X^\top Y$$

**Gradient Descent**

$$\theta \longleftarrow (1 - \eta_k\lambda)\,\theta - \eta_k\left[\frac{1}{n}(X^\top X)\theta - \frac{1}{n}X^\top Y\right]$$

Without regularization, i.e. $\lambda = 0$, this shrinkage factor equals 1.

# Training Loss vs Test Loss

**Training Loss**

$$\mathcal{L}_{n,\lambda}(\theta; \mathcal{S}_n) = \frac{1}{n} \sum_{(x,y) \in \mathcal{S}_n} \frac{1}{2} (y - \theta^\top x)^2 + \frac{\lambda}{2} \|\theta\|^2$$

**Test Loss/Error**

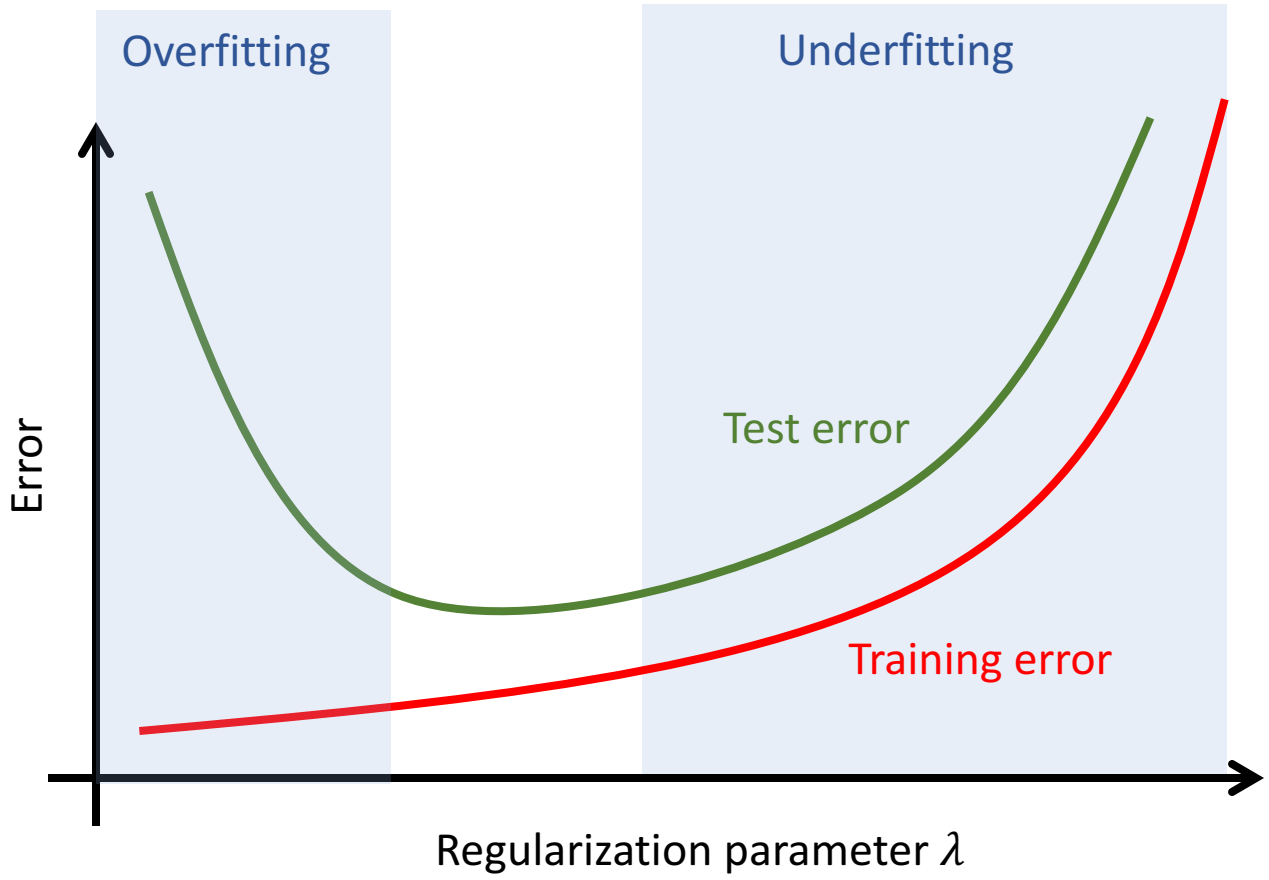
$$\mathcal{R}(\hat{\theta}; \mathcal{S}_*) = \frac{1}{n} \sum_{(x,y) \in \mathcal{S}_*} \frac{1}{2} \left(y - \hat{\theta}^\top x\right)^2$$

**Training Error**

$$\mathcal{R}(\hat{\theta}; \mathcal{S}_n) = \frac{1}{n} \sum_{(x,y) \in \mathcal{S}_n} \frac{1}{2} \left(y - \hat{\theta}^\top x\right)^2$$

The *training error* is the test loss applied to the training set, and it may be different from the training loss.

# Effect of Regularization



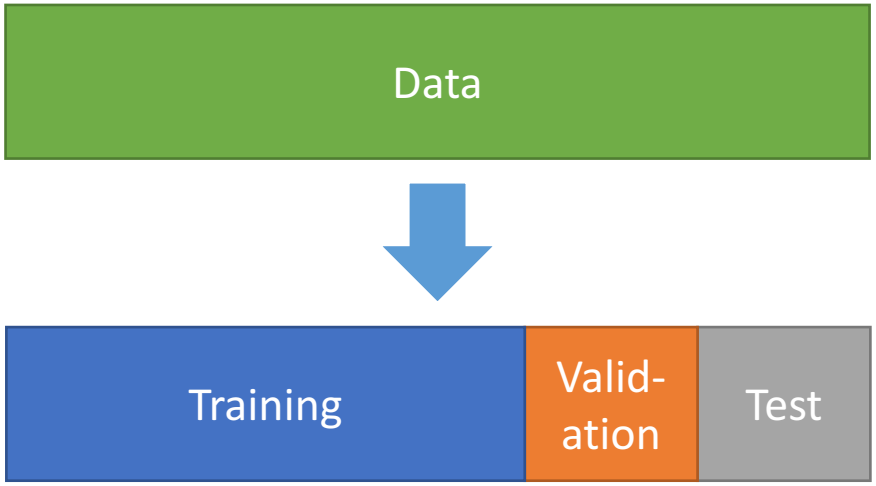
# Picking Hyperparameters

- The regularization parameter $\lambda$ is an example of a *hyperparameter*, which affects the model complexity.
- We don't usually have access to the test data. How do we know if the value of $\lambda$ minimizes the test loss?
- The solution is to create a *validation* data set, as a proxy to the test data, and to compute the *validation loss*.

# Validation Set

Split the data into

- **Test set** $\mathcal{S}_*$
  For evaluating, reporting performance at the end
- **Training set** $\mathcal{S}_n$
  For training optimal parameters in a model
- **Validation set** $\mathcal{S}_{\mathrm{val}}$
  For model selection, e.g. picking $\lambda$ in ridge regression.
  Acts as a proxy for test set.



# Validation Loss

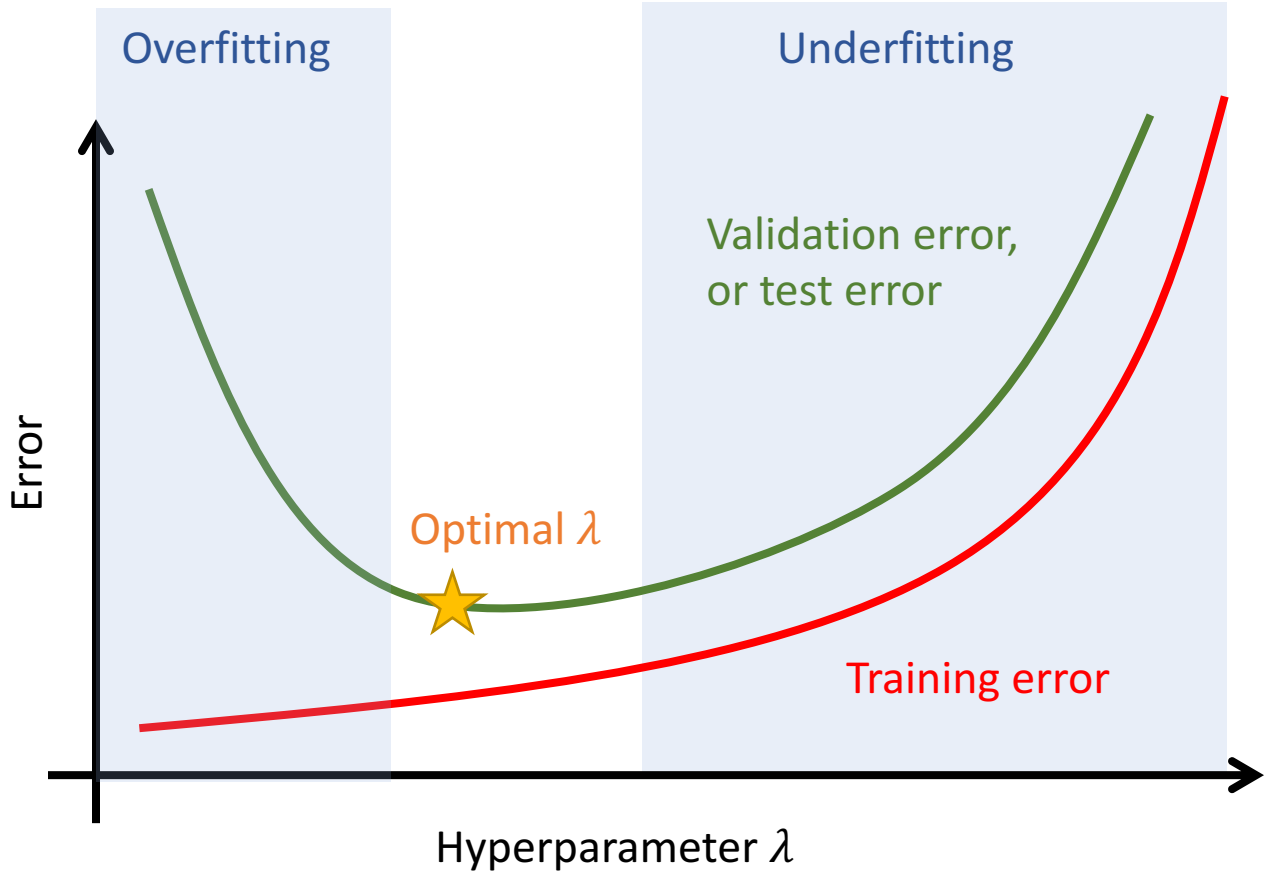
The *validation loss* is the test loss applied to the validation set.

**Example.** Ridge Regression

Test loss/error $\mathcal{R}\left(\hat{\theta}; \mathcal{S}_*\right) = \frac{1}{n}\sum_{(x,y)\in\mathcal{S}_*} \frac{1}{2}\left(y - \hat{\theta}^\top x\right)^2$

Validation loss/error $\mathcal{R}\left(\hat{\theta}; \mathcal{S}_{\mathrm{val}}\right) = \frac{1}{n}\sum_{(x,y)\in\mathcal{S}_{\mathrm{val}}} \frac{1}{2}\left(y - \hat{\theta}^\top x\right)^2$

# Model Selection



# Automatic Differentiation

In optimization, we often need to differentiate the objective function by hand to obtain the gradients for a descent algorithm.

*Wouldn't it be nice if this step was automated?* :)

Packages like Theano, Python, Caffe, TensorFlow automatically differentiate the functions.

# Summary

- Methodology
  - Features, Response
  - Training, Prediction
  - Training, Test Data
  - Model, Hypothesis, Parameters
  - Training, Test Loss
  - Training Algorithm
  - Generalization
  - Underfitting, Overfitting
  - Model Selection

- Optimization
  - Loss Functions
  - Empirical Risk
  - Exact Solution
  - Gradient Descent
  - Convex Optimization
  - Stochastic Gradient Descent

# Summary

- Multivariate Linear Regression
  - Model
  - Training Loss
  - Constant Feature Trick
  - Gradient
  - Gradient Descent
  - Exact Solution
  - Issues with Exact Solution

- Regularization
  - Generalization
  - Ridge Regression
  - Regularizer
  - Regularization Parameter
  - Exact Solution and Invertibility
  - Gradient Descent and Shrinkage
  - Training Loss vs Test Loss
  - Effect on Test Loss

# Intended Learning Outcomes

## Methodology

- Given a machine learning example, identify the components:
  - Features, Response
  - Training, Prediction
  - Training data, Test data
  - Model, Hypothesis, Parameters
  - Training loss, Test loss
  - Training algorithm
- State that the goal of machine learning is generalization.
- Give an example of underfitting, overfitting and model selection.

# Intended Learning Outcomes

## Optimization

- Give examples of loss functions, and define empirical risk in terms of the loss function.
- List two general types of algorithms used in optimization, e.g. exact solution, and gradient descent. Outline the broad steps involved in each of them.
- Explain why framing a problem as convex optimization is highly desirable, in terms of speed and local minima.
- Explain the motivation behind performing stochastic gradient descent, rather than traditional gradient descent.

# Intended Learning Outcomes

## Multivariate Linear Regression

- State the model and the training loss.
- Explain how the ‘constant feature’ trick can be used to reduce the problem to one without the constant parameter $\theta_0$.
- Describe two training algorithms that may be applied.
- Derive the gradient of the training loss.
- Derive the formula for the exact solution.
- Describe two potential weaknesses of the exact solution, and possible solutions for these weaknesses.
- Apply the above algorithms to a given data set.

# Intended Learning Outcomes

## Regularization

- Explain why regularization can help with generalization.
- State the training loss and test loss in ridge regression. Identify the regularizer and regularization parameter in the training loss of a given machine learning problem.
- Explain why regularization solves the invertibility problem in traditional linear regression.
- Describe the difference in gradient descent between traditional and regularized linear regression.
- Describe how the test error and training error varies with the regularization parameter. Describe how the model selection can be performed using a validation set.