1. Phase-Portrait via Nullcline Analysis

System form: $\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$.

Goal: Sketch global behavior by finding equilibrium and flow directions.

Recipe:

1. **Compute nullclines:** solve $f_1(x_1, x_2) = 0$ for N_1, and $f_2(x_1, x_2) = 0$ for N_2.
   - Express each as explicit curves (e.g. x_2 as function of x_1 or vice versa).
   - Identify branches, asymptotes, turning points.
2. **Equilibria:** find intersections $N_1 \cap N_2$.
   - Solve analytically or numerically.
   - Label each (x_{1e}, x_{2e}).
3. **Region partition:** nullclines divide plane into regions.
   - In each region pick a test point; compute $\dot{x}_1$ and $\dot{x}_2$ signs.
   - Draw arrows indicating flow.
4. **Invariant sets:** identify regions where arrows point inward on all boundaries—trajectories cannot leave.
5. **Local linearization:** at each equilibrium compute Jacobian

$$Df(x_e) = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix}_{x_e}.$$

   - Compute eigenvalues $\lambda_{1,2}$.
   - Classify: node (real same sign), saddle (real opposite), focus (complex), center (pure imaginary).
6. **Global portrait:** combine local linear behavior with region arrows.
   - Sketch trajectories qualitatively.
   - Mark stable/unstable manifolds of saddles.
   - Identify limit cycles or ω-limit sets.

2. Lyapunov Stability & LaSalle's Invariance

System form: $\dot{x} = f(x)$, equilibrium at $x = 0$.

Goal: Prove (global) asymptotic stability (GAS).

Recipe:

1. **Choose candidate $V(x)$:**
   - Energy-like or quadratic form.
   - Ensure $V(0) = 0$, $V(x) > 0$ for $x \neq 0$.
2. **Radial unboundedness:** verify $\lim_{\|x\|\to\infty} V(x) = \infty$. Equiv. $\forall M > 0$, $\exists R > 0$ such that $\|x\| > R \implies V(x) > M$.
3. **Compute derivative:** $\dot{V}(x) = \nabla V(x) \cdot f(x)$.
   - Show $\dot{V}(x) \leq -W(x)$, with $W(x)$ positive semidefinite.
4. **LaSalle's invariance:**
   - Identify set $E = \{x : \dot{V}(x) = 0\}$.
   - Show largest invariant subset of E is $\{0\}$.
   - Conclude $x(t) \to 0$ as $t \to \infty$.
5. **(Optional) Exponential stability:** if $\dot{V} \leq -cV$, then V decays exponentially.

3. Passivity–Based Storage Function

System form: $\dot{z}_1 = z_2$, $\dot{z}_2 = -(2 + \sin z_1)\, z_1 - z_2 + u$, $y = z_2$.

Goal: Show input–output passivity.

Recipe:

1. **Input-affine form:** $x = [z_1, z_2]^T$, write $\dot{x} = f(x) + g(x)\, u$, $y = h(x)$.
2. **Storage function $S(x)$:** impose $L_g S(x) = h(x)$.

$$\frac{\partial S}{\partial z_2} = z_2 \implies S = \tfrac{1}{2} z_2^2 + P(z_1).$$

3. **Determine $P(z_1)$:** require $\dot{S} \leq y\, u$.

$$\dot{S} = z_2 \dot{z}_2 + P'(z_1) z_2 = z_2\big[-(2+\sin z_1) z_1 - z_2 + u\big] + P'(z_1) z_2.$$

   Choose $P'(z_1) = (2 + \sin z_1) z_1$ to cancel terms.

4. **Conclude passivity:** $\dot{S} \leq z_2\, u = y\, u$.

Passivity and Losslessness

Consider the SISO control-affine system

$$\dot{x} = f(x) + g(x)\, u, \qquad y = h(x). \tag{1}$$

A continuously differentiable function $S\colon \mathbb{R}^n \to \mathbb{R}$ is a lossless storage function (and the system is lossless passive) if and only if

$$\frac{\partial S}{\partial x}(x)\, f(x) = 0, \tag{2}$$

$$\frac{\partial S}{\partial x}(x)\, g(x) = h(x). \tag{3}$$

4. Input-to-State Stability (ISS) Analysis

Form: $\dot{x} = f(x, d)$, disturbance d.

Goal: Bound state by input magnitude.

Recipe:

1. **Lyapunov candidate:** $V(x) = \frac{1}{2}\|x\|^2$. $\alpha_1(\|x\|) = \frac{1}{2}\|x\|^2 \le V(x) \le \alpha_2(\|x\|) = \frac{1}{2}\|x\|^2$.
2. **Compute $\dot{V}$:** $\dot{V} = \nabla V \cdot f(x, d)$.
3. **Derive ISS condition:** find functions $\chi, \rho > 0$ such that
$$\|x\| \ge \chi(\|d\|) \implies \dot{V} \le -\rho(\|x\|).$$
Then $\|x(t)\| \le \beta(\|x(0)\|, t) + \gamma(\sup_{s \le t} \|d(s)\|)$.
4. **Small-gain for interconnection:** for two subsystems with gains γ_1, γ_2, require $\gamma_1 \circ \gamma_2(r) < r$ for all $r > 0$ to ensure overall ISS.

5. Input-Output Linearization & Zero Dynamics

Form: $\dot{x} = f(x) + g(x)\,u,\ y = h(x)$.

Goal: Transform input–output behavior to a linear chain.

Recipe:

1. **Lie derivatives:** compute $y^{(k)} = L_f^k h(x)$ until $L_g L_f^{r-1} h(x) \ne 0$. That r is the relative degree.
2. **Feedback law:**
$$u = \frac{-L_f^r h(x) + v}{L_g L_f^{r-1} h(x)}$$
yields $y^{(r)} = v$.
3. **Normal form:** define $\xi = [y, \dot{y}, \ldots, y^{(r-1)}]^T$, let z be remaining coordinates.
$$\dot{\xi} = A\,\xi + B\,v, \quad \dot{z} = q(z, \xi).$$
The z-subsystem is zero dynamics.
4. **Zero dynamics stability:** analyze $\dot{z} = q(z, 0)$. If asymptotically stable (or ISS), then full closed-loop can be stabilized by choosing v.

6. PD Stabilization of Linearized Dynamics

Context: after I-O linearization $y^{(r)} = v$.

Goal: choose v to stabilize output dynamics.

Recipe:

1. **Double integrator ($r = 2$):** set
$$v = -k_p\,y - k_d\,\dot{y}, \quad k_p, k_d > 0.$$
2. **Characteristic equation:** $\ddot{y} + k_d\dot{y} + k_p y = 0$. Choose k_p, k_d to place poles in left half–plane.
3. **Higher relative degree:** use v as high-order PD or pole-placement on ξ-dynamics.
4. **Guarantee:** with stable zero dynamics, overall system is GAS.

7. Routh–Hurwitz Stability Criterion

System form: $\dot{x} = A\,x$, with $A \in \mathbb{R}^{n \times n}$.

Goal: Determine whether all eigenvalues of A lie in the open left half-plane, without explicitly computing them.

Recipe:

1. **Characteristic polynomial:**
$$P(s) = \det\bigl(sI - A\bigr) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0.$$
2. **Build Routh array:**
   - 1st row: $[a_n,\ a_{n-2},\ a_{n-4}, \ldots]$
   - 2nd row: $[a_{n-1},\ a_{n-3},\ a_{n-5}, \ldots]$
   - Subsequent rows by $b_{i,j} = \frac{b_{i-2,1}\,b_{i-1,j+1} - b_{i-1,1}\,b_{i-2,j+1}}{b_{i-1,1}}$.
3. **Stability test:** all entries in the first column $> 0 \iff$ all $\Re(\lambda_i) < 0$.

Example: consider the second-order system
$$A = \begin{pmatrix} -2 & 1 \\ -5 & -3 \end{pmatrix}.$$
Its characteristic polynomial is
$$P(s) = \det(sI - A) = \det\begin{pmatrix} s+2 & -1 \\ 5 & s+3 \end{pmatrix}$$
$$= (s+2)(s+3) + 5 = s^2 + 5s + 11,$$
so $a_2 = 1,\ a_1 = 5,\ a_0 = 11$.

Routh array:
$$\begin{array}{c|cc} s^2 & 1 & 11 \\ s^1 & 5 & 0 \\ s^0 & 11 & --- \end{array} \implies \text{1st column } [1,\ 5,\ 11] > 0$$

All entries are positive, so $\Re(\lambda_{1,2}) < 0$ and the system is asymptotically stable.