RADIOMETRY AND HDR

Light transport can be modeled with geometric (or ray) optics by treating light as particles rather than waves, except in cases such as polarization. Basic properties of geometric optics include linearity (no nonlinear interactions when multiple light sources are switched on/off) and energy conservation (exemplified by the Phong reflection model).

Basic Quantities:

Radiant Flux or Power (Φ) is the energy flowing through a surface per unit time. *Irradiance* (E) is the area density of incoming flux, in W/m^2. For a sphere of radius r, $E = \frac{\Phi}{4\pi r^2}$. *Intensity* ($I$) is flux per solid angle, in W/sr: $I = \frac{d\Phi}{d\omega}$. *Radiance* ($L$) is radiant flux density per unit area per unit solid angle, in W/(m^2 sr): $L = \frac{d^2\Phi}{(dA\cos\theta)\,d\omega}$, and remains constant along a straight path in free space.

Lambert's Law: Irradiance E on a surface is proportional to the cosine of the angle θ between the light direction $\mathbf{l}$ and the surface normal $\mathbf{n}$. If $E_1 = \frac{\Phi}{A}$ and $E_2 = \frac{\Phi\cos\theta}{A}$, then E_2 accounts for the angle of incidence.

Incident and Exitant Radiance: $L_i(x, \omega_i)$ is the incident radiance at point x and direction ω_i, while $L_r(x, \omega_r)$ is the exitant (reflected) radiance. Generally, $L_i \neq L_r$ because real materials are not perfectly reflective.

BRDF (Bidirectional Reflectance Distribution Function): $f_r(x, \omega_r, \omega_i) = \frac{dL_r(x,\omega_r)}{dE_i(x,\omega_i)} = \frac{dL_r(x,\omega_r)}{L_i(x,\omega_i)\cos\theta\,d\omega_i}$, with units of 1/sr. It relates reflected radiance to incident radiance.

Cameras: Each pixel measures radiance, and camera ISO sets the gain. Aperture changes affect depth of field and can introduce vignetting. Exposure time is often the ideal parameter for adjusting HDR captures. Clipping occurs when sensor capacity is exceeded. ND 1.0 factor 10 reduction (ND 0.3 50% reduction, 0.6 75% reduction ...), ND 2.0 100, etc. Aperture decrease by one stop, halves; stops: f/1 → f/1.4 → f/2 → f/2.8 → f/4 → f/5.6 → f/8 → f/11 → f/16 → f/22. ISO x 4 = Irradiance x 4.

HDR Imaging: Combining multiple exposures (e.g. bracketing by several stops) can capture a higher dynamic range. The camera response function f may be nonlinear, but modern cameras are often close to linear. The standard model is $Z_{ij} = f(E_i\,\Delta t_j)$, where Z_{ij} are pixel values, E_i is irradiance, and Δt_j is exposure time. When f is linear, $E_i = Z_{ij}/\Delta t_j$. To compute HDR values from LDR, remove over and under exposed pixel values, then map to [0,1] range by dividing by 255, compute exposure factors (2 stops apart - /2, /4, /16 ...), compute weights based off function.

HDR Formats: Popular formats include PFM (Portable FloatMap), storing 12 bytes per pixel, and OpenEXR (from ILM), which supports compression and is widely used. Radiance .HDR format packs the three colour components into 3x8b mantissa plus shared 8 bit exponent for a total of 32 bits per pixel. Offers wide dynamic range while using only one-third as many bits as 96 bit float.

Tone Mapping: To display HDR content on LDR devices, global operators compress the overall range (e.g. simple scaling or histogram-based approaches), but can reduce local contrast. Local operators (e.g. bilateral filters or gradient-domain methods) preserve detail and avoid halos.

Inverse Tone Mapping: Methods (often CNN-based) can reconstruct approximate HDR from LDR images by "hallucinating" detail in saturated regions. A mask can preserve original LDR values in dark regions and replace bright regions with the network's HDR prediction.

IMAGE BASED RELIGHTING

4D Reflectance Field: A 4D reflectance field describes how incoming light directions map to outgoing radiance for a surface, ignoring positional variation. One practical step is to downsample and convert from the mirror-probe representation to a latitude–longitude format so that dot products can be computed more easily.

8D Generalized Reflectance Field (BSSRDF): The Bidirectional Scattering–Surface Reflectance Distribution Function (BSSRDF) accounts for spatial and angular variation in reflectance, forming an 8D function. In many cases, the positional dependence is small enough that only angle matters, effectively reducing the dimensionality by two.

Normalization of Intensities: When converting to latitude–longitude maps, the regions near the poles become stretched and appear brighter if uncorrected.

Solid Angle Compensation: The solid-angle element changes with latitude in spherical coordinates, so a Jacobian factor corrects for stretching at the poles compared to the equator. The conversion from spherical angles (θ, φ) to Cartesian (X, Y, Z) can be expressed as: $X = \cos\varphi\,\sin\theta, \quad Y = \cos\theta, \quad Z = \sin\varphi\,\sin\theta$.

Diffuse vs. Obstructed Reflection: In an example with a helmet, the yellow region corresponds to diffuse body reflection, whereas the black region indicates obstruction or reflections from the ground.

Free-Form Light Stage: By identifying the brightest pixel in each captured sphere image, one can determine a light direction from the corresponding surface normal. Using four reference spheres provides relative angles. A Voronoi diagram is constructed over the directions, and the energy in each cell is allocated to the corresponding light. Combining these with a weighted sum of photographs approximates a reflectance function with a finite set of light sources (e.g. 30 lights).

Neural Network Relighting: Highly specular or complex reflectance functions can be difficult to sample via standard methods. Neural networks can interpolate from a smaller set of images, overcoming Nyquist sampling limits for surfaces that exhibit strong anisotropy or high-frequency reflections.

Measuring sun saturation: Find the diffuse sphere highlight. Brightest point; use BRDF to find incident radiance ($L_r = \rho L_i/\pi$). Compare to lossless perfect mirror; mult. L_i by reflectance. Then subtract measured value from lossless L_r. **Light Transport Matrix.** $p = Tl$. T is the light transport matrix; l is a vector of new lighting coefficients (intensities of each basis light). Each row corresponds to a single pixel in the output image. It tells us how that pixel's intensity depends on the different light contributions (columns). Each column corresponds to one light and contains the response of all pixels to that particular light.

RECIPROCITY AND LIGHT FIELDS

Dual Light Stages: A "dual" setup uses a dense array of lights and a camera. Sharp specular reflections can introduce aliasing if the lights are not sampled densely (a fully continuous dome prevents such artifacts). Helmholtz reciprocity for BRDFs can generalize to more complex transport. By capturing 120 k images—one per pixel (i.e. pixel tracing)—each pixel's reflectance function can be recovered. We reparameterize a light probe to that function and take a dot product of the incoming radiance with the reflectance function to get the final pixel color under novel lighting.

Dual Photography: Consider a scene lit by an $M \times N$ pixel projector and recorded by a camera (or even a single-pixel photodiode). With enough measurements, one can invert the light transport matrix to predict images as if the camera and projector were swapped. Adaptive measurement strategies can reduce the number of required captures by grouping or parallelizing certain measurements. Such methods also allow "seeing around corners" or revealing hidden geometry by using indirect light paths.

Light Fields: Radiance in 3D space can be represented as a function of 3D position plus 2D direction, giving a 5D plenoptic function on surfaces or a 4D representation in free space. Each 2D slice of this 4D light field corresponds to one image. In practice, sampling a set of rays at different positions and directions enables *synthetic refocusing*: by appropriately re-binning rays, one can reconstruct images focused at different depths. Within the convex hull of the captured viewpoints, one can also do limited viewpoint synthesis. However, optical flow or correspondence estimates may introduce artifacts if they fail in specular or textureless regions.

RADIOMETRY AND FRESNEL REFLECTANCE

Light Transport Using Geometric Optics. Light is treated as rays, ignoring wave phenomena except in special cases (e.g. polarization). Two key properties are *linearity* (no nonlinear interactions when multiple lights are combined) and *energy conservation* (no net gain or loss of radiant energy).

Assumptions. (1) Polarization is ignored. (2) No fluorescence (no wavelength shift to longer wavelengths). (3) Steady-state (no slow re-emissions like phosphorescence).

Lambert's Law. Irradiance $E = \frac{d\Phi}{dA}$ is proportional to $\cos\theta$, where θ is the angle between the incident light direction and the surface normal. Hence $E_1 = \frac{\Phi}{A}$ and $E_2 = \frac{\Phi\cos\theta}{A}$.

Radiometric Integrals. Integrals typically run over the upper hemisphere Ω, as $E(x, \mathbf{n}) = \int_\Omega L_i(x, \omega)\cos\theta\,d\omega$. For back-facing directions, $\cos\theta$ is zero.

Spherical Coordinates. A direction (x, y, z) maps to (θ, φ) with $x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, $z = \cos\theta$. A hemispherical integral is $E = \int_0^{2\pi}\int_0^{\pi/2} L_i\cos\theta\,\sin\theta\,d\theta\,d\varphi$.

Area Integral (Area Lights). If the radiance comes from an area A, $E = \int_\Omega L\cos\theta_i\,d\omega = \int_A L\cos\theta_i\cos\theta_o\,\frac{dA}{r^2}$, where r is the distance, θ_o the subtended angle.

BRDF. The bidirectional reflectance distribution function f_r is $f_r = \frac{dL_r}{dE_i} = \frac{dL_r}{L_i\cos\theta_i\,d\omega_i}$ (units 1/sr). Under *Helmholtz reciprocity*, $f_r(\omega_r, \omega_i) = f_r(\omega_i, \omega_r)$. *Energy conservation* requires $\int_\Omega f_r\cos\theta_i\,d\omega_i \leq 1$.

Rendering Equation. The reflected radiance $L_r(x, \omega_r) = \int_\Omega f_r(x, \omega_r, \omega_i)\,L_i(x, \omega_i)\cos\theta_i\,d\omega_i$.

Snell's Law & Specular. At an interface with refractive indices η_i, η_t, perfect reflection implies $\theta_i = \theta_r$, while refraction satisfies $\eta_i\sin\theta_i = \eta_t\sin\theta_t$. Dispersion arises if η depends on wavelength.

Polarization. Light may be unpolarized or polarized; Fresnel reflectance depends on refractive index and polarization.

Fresnel Reflectance (Dielectrics). For parallel ($R_\parallel$) and perpendicular ($R_\perp$) polarizations at an interface, $R_\parallel = \left|\frac{\eta_t\cos\theta_i - \eta_i\cos\theta_t}{\eta_t\cos\theta_i + \eta_i\cos\theta_t}\right|^2$, $R_\perp = \left|\frac{\eta_i\cos\theta_i - \eta_t\cos\theta_t}{\eta_i\cos\theta_i + \eta_t\cos\theta_t}\right|^2$, and $F_r = \frac{1}{2}(R_\parallel + R_\perp)$. *Brewster's angle* $\theta_B = \tan^{-1}(\frac{\eta_t}{\eta_i})$ gives $R_\parallel = 0$. Total internal reflection occurs at/above θ_c with $\sin(\theta_c) = \frac{\eta_t}{\eta_i}$ (Critical Angle).

Schlick's Approximation. A polynomial approximation for dielectrics is $F_r(\cos\theta) = R_0 + (1 - R_0)(1 - \cos\theta)^5$, where $R_0 = (\frac{\eta_i - \eta_t}{\eta_i + \eta_t})^2$.

Fresnel for Conductors. A complex refractive index $\eta + i\,k$ yields reflection coefficients $R_\parallel = \frac{(\eta^2+k^2)\cos^2\theta_i - 2\eta\cos\theta_i + 1}{(\eta^2+k^2)\cos^2\theta_i + 2\eta\cos\theta_i + 1}$ and $R_\perp = \frac{(\eta^2+k^2) - 2\eta\cos\theta_i + \cos^2\theta_i}{(\eta^2+k^2) + 2\eta\cos\theta_i + \cos^2\theta_i}$. For normal incidence, if $k = 0$ and R_0 is known, then $\eta = \frac{1+\sqrt{R_0}}{1-\sqrt{R_0}}$.

REFLECTION MODELS AND MEASUREMENT

Types of BRDF Models. (1) *Phenomenological*: simple equations (Lambert, Phong) capturing qualitative surface behaviors. (2) *Physically Based*: derived from surface microgeometry (microfacet models). (3) *Data-Driven*: learned or fitted directly from measured reflectance. Parameter fitting uses either empirical formulas or actual measured data.

Lambertian Reflection $f_r(\omega_r, \omega_i) = \frac{\rho_d}{\pi}$, where $\rho_d \in [0, 1]$ is the diffuse reflectance and π arises from hemisphere integration.

Phong Model $f_r(\omega_o, \omega_i) = \frac{\rho_d}{\pi} + \frac{\rho_s\,(\omega_r\cdot\omega_o)^s}{(\mathbf{n}\cdot\omega_i)} = \frac{\rho_d}{\pi} + \frac{\rho_s\,(\cos\theta)^s}{(\mathbf{n}\cdot\omega_i)}$, where ρ_s is specular strength, s controls lobe width, and ω_r is the reflection direction.

Blinn-Phong Model $f_r(\omega_o, \omega_i) = \frac{\rho_d}{\pi} + \frac{\rho_s\,(\mathbf{n}\cdot\omega_h)^s}{(\mathbf{n}\cdot\omega_i)}$, with ω_h the halfway vector. This is often preferred for its convenient halfway representation.

LaFortune Model
$f_r(\omega_r, \omega_i) = \frac{\rho_d}{\pi} + \sum_j\left[C_{x,j}(\omega_{i,x}\omega_{r,x}) + C_{y,j}(\omega_{i,y}\omega_{r,y}) + C_{z,j}(\omega_{i,z}\omega_{r,z})\right]^{s_j}$, allowing off-specular peaks, retro-reflection, or anisotropy by adjusting $\{C_{x,j}, C_{y,j}, C_{z,j}, s_j\}$.

Ward Anisotropic Model
$f_r(\omega_r, \omega_i) = \frac{\rho_d}{\pi} + \rho_s\,\frac{1}{\sqrt{\cos\theta_i\,\cos\theta_r}}\,\frac{\exp[-\tan^2\delta\,(\frac{\cos^2\phi}{\alpha_x^2} + \frac{\sin^2\phi}{\alpha_y^2})]}{4\pi\,\alpha_x\,\alpha_y}$, where δ is the angle between the normal and the projected halfway vector, α_x, α_y are anisotropic roughness, and ϕ is the azimuth.

Ashikhmin–Shirley Phong Model
$f_r(\omega_r, \omega_i) = \frac{\sqrt{(n_u+1)(n_v+1)}}{8\pi}\,\frac{(\mathbf{n}\cdot\omega_h)^s}{(\omega_i\cdot\omega_h)\,\max(\mathbf{n}\cdot\omega_i, \mathbf{n}\cdot\omega_r)}\,F_r(\omega_i\cdot\omega_h)$, where F_r is often Schlick's Fresnel and n_u, n_v set anisotropy.

Physically Based Models (Microfacet Theory). A rough surface is treated as numerous tiny specular facets. The BRDF has $f_r(\omega_r, \omega_i) = \frac{D(\omega_h)\,G(\omega_r,\omega_i)\,F_r(\omega_h)}{4(\mathbf{n}\cdot\omega_i)(\mathbf{n}\cdot\omega_r)}$. Here, D is the facet distribution, G the geometric masking/shadowing term, and F_r the Fresnel factor.

Torrance–Sparrow Model uses a Beckmann distribution $D(\omega_h) = \frac{\exp[-(\tan\delta/m)^2]}{\pi\,m^2\,\cos^4\delta}$, where δ is the angle between $\mathbf{n}$ and ω_h and m the RMS slope. The geometric term $G(\omega_r, \omega_i)$ often uses $\min\{1, \frac{2(\mathbf{n}\cdot\omega_h)(\mathbf{n}\cdot\omega_r)}{\omega_r\cdot\omega_h}, \frac{2(\mathbf{n}\cdot\omega_h)(\mathbf{n}\cdot\omega_i)}{\omega_r\cdot\omega_h}\}$.

Blinn Microfacet Distribution Instead of a Gaussian, use a cosine-lobe $D(\omega_h) = \frac{(s+2)(\mathbf{n}\cdot\omega_h)^s}{2\pi}$, with s akin to a Phong exponent.

GGX Distribution A modern distribution with sharper peak and heavier tails: $D(\omega_h) = \frac{\alpha^2}{\pi\,\cos^4\delta\,(\alpha^2+\tan^2\delta)^2}$, where δ is the angle $\mathbf{n} \wedge \omega_h$.

Poulin–Fournier (Anisotropy) Models cylindrical grooves (e.g. satin, velvet) with parameters $\{d, h, \sigma_x, \sigma_y\}$. Special cases capture high reflectivity at grazing angles.

BRDF Parameterizations (Rusinkiewicz). Instead of $(\theta_i, \phi_i, \theta_o, \phi_o)$, use halfway/difference angles $(\theta_h, \phi_h, \theta_d, \phi_d)$, simplifying microfacet-based integration.

Data-Driven Microfacet Models may store or fit D, F, G in low-dimensional lookup tables. A final form can look like $\rho_M(\theta_h, \theta_d, \phi_d) = \rho_d + \rho_s\left[\frac{D(\theta_h)\,F(\theta_d)\,G(\theta_i)\,G(\theta_o)}{\cos\theta_i\,\cos\theta_o}\right]$.

Neural BRDF uses a small neural net with ReLUs and exponential outputs to represent measured data, preserving real-world reflectance nuances with only hundreds of weights.

MONTE CARLO INTEGRATION

Direct Illumination Rendering Equation. $L_r(x, \omega_r) = \int_\Omega f_r(x, \omega_r, \omega_i)\,L_i(x, \omega_i)\cos\theta_i\,V(\omega_i)\,d\omega_i$. Brute-force solutions are expensive, especially under environment-map (EM) lighting.

Monte Carlo Integration. An integral $I(f) = \int_S f(x)\,p(x)\,dx$ can be estimated by $I_N(f) = \frac{1}{N}\sum_{i=1}^N f(x_i)$ where $x_i \sim p(x)$. As $N \to \infty$, $I_N(f) \to I(f)$. The variance decreases as $1/N$, so the standard deviation (image noise) decreases as $1/\sqrt{N}$.

Importance Sampling. For $I(f) = \int_S f(x)\,dx$, we write $I(f) = \int_S \frac{f(x)}{p(x)}\,p(x)\,dx$ and use $\frac{1}{N}\sum\frac{f(x_i)}{p(x_i)}$ with $x_i \sim p(x)$. Good choices of p that mimic f reduce variance. In direct illumination, p can match the BRDF or incoming radiance distribution.

CDF Inversion. If $p(x) \geq 0$ and $C(x) = \int_0^x p(t)\,dt$, then for a uniform $u \in [0, 1]$, $x = C^{-1}(u)$ has PDF $p(x)$. *Example:* To sample a disk of radius R uniformly, $(r, \theta) = (R\sqrt{u_1}, 2\pi\,u_2)$.

BRDF Sampling. Many BRDF lobes allow analytic sampling. For a Phong lobe with exponent n, $p(\theta, \varphi) = \frac{n+1}{2\pi}\cos^n\theta$. Invert to get $\theta = \arccos[(1 - u_1)^{\frac{1}{n+1}}]$, $\varphi = 2\pi\,u_2$. Often, we sample a halfway vector ω_h and reflect the view vector about ω_h to get the sampled direction.

Isotropic Gaussian (Ward). $p(\omega_h) \propto e^{-(\tan^2\theta_h/\alpha^2)}$. One obtains $\theta_h = \arctan(\alpha\sqrt{-\ln(u_1)})$, $\varphi_h = 2\pi\,u_2$.

GGX. $p(\omega_h) \propto \frac{\alpha^2}{(\alpha^2+\tan^2\theta_h)^2}$. Thus $\theta_h = \arctan(\frac{\alpha\sqrt{u_1}}{\sqrt{1-u_1}})$, $\varphi_h = 2\pi\,u_2$.

Anisotropic Gaussian (Ward). $p(\omega_h) \propto \exp[-\tan^2\theta_h(\frac{\cos^2\varphi_h}{\alpha_x^2} + \frac{\sin^2\varphi_h}{\alpha_y^2})]$.

Sample $\varphi_h = \arctan(\frac{\alpha_y}{\alpha_x}\tan(2\pi\,u_2))$, then $\theta_h = \arctan\left(\sqrt{\frac{-\ln(u_1)}{\cos^2\varphi_h/\alpha_x^2 + \sin^2\varphi_h/\alpha_y^2}}\right)$. To draw a sample from the anisotropic Ward half-vector distribution, compute the raw azimuth $\phi = 2\pi\mu_2$ then use the formula.

Anisotropic Phong (Ashikhmin–Shirley).
$p(\omega_h) = \frac{\sqrt{(n_u+1)(n_v+1)}}{2\pi}\cos^s\theta_h$, $s = \cos^2\varphi_h\,n_u^2 + \sin^2\varphi_h\,n_v^2$. Then $\theta_h = \arccos[(1 - u_1)^{\frac{1}{s+1}}]$, $\varphi_h = \arctan(\sqrt{(n_u+1)(n_v+1)}\,\tan(2\pi\,u_2))$.

Microfacet Sampling. After sampling ω_h, reflection is $\omega_i = 2(\omega_h\cdot\omega_r)\,\omega_h - \omega_r$. PDF conversion is $p(\omega_r) = \frac{p(\omega_h)}{4(\omega_h\cdot\omega_i)}$.

Environment-Map Sampling. One can build a 2D CDF over the environment map to sample directions proportionally to EM intensity. This reduces variance by focusing more samples in bright regions.

DIRECT ILLUMINATION

Rendering Equation. $L_r(x,\omega_r) = \int_\Omega f_r(x,\omega_r,\omega_i)\, L_i(x,\omega_i)\, \cos\theta_i\, V(\omega_i)\, d\omega_i$. Here, L_r is the reflected radiance at point x in direction ω_r, f_r is the BRDF, L_i is the incoming radiance from direction ω_i, $\cos\theta_i$ is the foreshortening factor, $V(\omega_i)$ is visibility, and Ω is the hemisphere of directions. Direct evaluation is expensive, so we use Monte Carlo (MC) integration.
Monte Carlo Integration. For an integral $I(f) = \int_S f(x)\, dx$, an unbiased MC estimator is $I(f) \approx \frac{1}{N}\sum_{i=1}^N \frac{f(x_i)}{p(x_i)}$ where $x_i \sim p(x)$, and p is a chosen proposal PDF. Good proposal distributions (ones that track f) reduce variance. As $N \to \infty$, $I(f)$ converges, but the variance decreases only as $1/N$, so quadrupling samples halves the noise standard deviation.
BRDF Sampling. In rendering, we often importance-sample directions according to BRDFs or environment-map (EM) intensities. A Phong lobe with exponent n can be sampled using $p(\theta,\varphi) = \frac{n+1}{2\pi}\cos^n\theta$, yielding $\theta = \arccos[(1-u_1)^{1/(n+1)}]$, $\varphi = 2\pi u_2$. For microfacet models, one typically samples the halfway vector ω_h and reflects ω_r about ω_h via $\omega_i = 2(\omega_h\cdot\omega_r)\,\omega_h - \omega_r$. The PDF conversion is $p(\omega_r) = \frac{p(\omega_h)}{4(\omega_h\cdot\omega_i)}$. Ward's rough-surface model uses a Gaussian form $p(\omega_h) \propto e^{-(\tan^2\theta_h/\alpha^2)}$, while GGX employs $p(\omega_h) \propto \frac{\alpha^2}{(\alpha^2+\tan^2\theta_h)^2}$. Anisotropic variants adjust the distribution in ϕ_h.
Environment-Map Sampling. One can define $q_L(\omega_i) = \frac{L_i(\omega_i)}{\int_\Omega L_i(\omega)\, d\omega}$ to focus samples where L_i is bright, improving efficiency. The direct-illumination MC estimator becomes $\frac{1}{N}\sum_{j=1}^N \frac{f_r(\omega_r,\omega_{i,j})\, \cos\theta_{i,j}\, L_i(\omega_{i,j})\, V(\omega_{i,j})}{q_L(\omega_{i,j})}$.
Multiple Importance Sampling (MIS). MIS reduces variance further by combining samples from different proposals q_m, q_n with a weighted approach. An MIS estimator of $I = \int f(x)\, dx$ is

$$I_{\text{MIS}} = \frac{1}{M+N}\Big(\sum_{i=1}^M \frac{f(x_i)\, w_m(x_i)}{q_m(x_i)} + \sum_{j=1}^N \frac{f(x_j)\, w_n(x_j)}{q_n(x_j)}\Big),$$ where

$w_m(x) = \frac{M\, q_m(x)}{M\, q_m(x)+N\, q_n(x)}$ and $w_n(x) = \frac{N\, q_n(x)}{M\, q_m(x)+N\, q_n(x)}$. This is widely used (e.g. in PBRT).
Rejection Sampling. If sampling from the target distribution $p(x)$ is difficult, one can sample from a simpler $q(x)$ and accept each $x \sim q(x)$ with probability $p(x)/(f_{\max}\, q(x))$, provided $p(x) \le f_{\max}\, q(x)$. This helps generate samples in proportion to $p(x)$ without needing a closed-form inversion.
Sampling Importance Resampling. Instead of discarding rejected samples, one can reweight and resample them according to their importance, thus allocating more new samples to regions of higher weight.
Median Cut / Variance Minimization. To approximate an environment map with a finite set of "lights," one recursively subdivides the map to form 2^n regions, each containing a fraction of total energy (or minimizing variance in each partition). A point light is placed at each region's centroid with color/intensity equal to the sum of pixel values in that region, yielding an efficient approximate lighting setup.

SPHERICAL HARMONICS

Basis Functions and Spherical Harmonics (SH). We can represent 3D functions (such as environment maps) via a frequency decomposition akin to Fourier analysis. In spherical coordinates (θ,ϕ) with $(x,y,z) = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$, the Spherical Harmonics are defined by $Y_l^m(\theta,\phi) = K_l^m\, e^{i\,m\phi}\, P_l^{|m|}(\cos\theta)$, where $K_l^m = \sqrt{\frac{2l+1}{4\pi}\ \frac{(l-|m|)!}{(l+|m|)!}}$ and $P_l^{|m|}$ are the associated Legendre polynomials.
SH Reconstruction of Environment Maps. An environment map $L(\theta,\phi)$ can be approximated by $L(\theta,\phi) \approx \sum_{l=0}^L \sum_{m=-l}^l L_{lm}\, Y_l^m(\theta,\phi)$. The coefficients L_{lm} are computed via $L_{lm} = \int_\Omega L(\theta,\phi)\, Y_l^m(\theta,\phi)\, \sin\theta\, d\theta\, d\phi$, or discretely by $L_{lm} \approx \sum_i L(\theta_i,\phi_i)\, Y_l^m(\theta_i,\phi_i)\, w_i$. Reconstruction is then $L'(\theta,\phi) = \sum_{l,m} L_{lm}\, Y_l^m(\theta,\phi)$.
Irradiance via SH. The diffuse irradiance at a surface with normal $\mathbf{n}$ is $E(\mathbf{n}) = \int_{\Omega(\mathbf{n})} L(\omega)\, (\mathbf{n}\cdot\omega)\, d\omega$. By projecting the cosine factor onto the SH basis, we get $E_{lm} = \sqrt{\frac{4\pi}{2l+1}}\, A_l\, L_{lm}$, where A_l accounts for the cosine kernel. This means $E(\theta,\phi) \approx \sum_{l,m}(\hat{A}_l\, L_{lm})\, Y_l^m(\theta,\phi)$, often using just the first 9 coefficients for a good diffuse approximation.
Reflected Radiance. For a BRDF $\rho(\omega_i,\omega_o)$, the outgoing radiance is $B(\mathbf{N};\omega_o) = \int_\Omega L(\mathbf{N};\omega_i)\, \rho(\omega_i,\omega_o)\, (\omega_i\cdot\mathbf{z})\, d\omega_i$. When the BRDF or environment is also expressed in SH, one can form products of their respective coefficients to obtain a fast evaluation for real-time shading. For isotropic BRDFs, reparameterizations reduce dimensionality further, and we can store the reflection in a small set of SH coefficients.
Precomputed Radiance Transfer (PRT). In global illumination, we expand both the environment lighting $L(\omega) = \sum_{l,m} L_{lm} Y_l^m(\omega)$ and the transport function $T(\mathbf{x},\omega) = \sum_{l,m} T_{lm}(\mathbf{x})\, Y_l^m(\omega)$ into SH. The final radiance at point $\mathbf{x}$ is $B(\mathbf{x}) = \sum_{l,m} L_{lm}\, T_{lm}(\mathbf{x})$. At runtime, this becomes a dot product of the lighting and transport coefficients, enabling efficient real-time evaluation of indirect lighting.

MEASUREMENT AND DATA-DRIVEN BRDFS

Measurements. Dense measurements use a gonioreflectometer, which rotates a light source and sensor around a sample to capture the BRDF. Missing data is then interpolated. For isotropic materials, one can use a spherical sample: a single photograph captures many directions of incident and reflected light by rotating the light along one axis. Alternatively, image-based measurements and mirror-based or basis-illumination approaches also collect detailed reflectance data. **Rusinkiewicz Reparameterization.** Traditional BRDFs are expressed via (θ_i,ϕ_i) for the incident direction and (θ_o,ϕ_o) for the exitant direction. We define the halfway vector $\omega_h = (\theta_h,\phi_h)$, the normalized bisector of ω_i and ω_o, and the difference vector $\omega_d = (\theta_d,\phi_d)$, which captures the angular difference relative to ω_h. Because many BRDFs exhibit high symmetry around the halfway vector, this representation often simplifies fitting and data storage. BRDF depends on θ_h, the angle between the surface normal and the halfway vector, θ_d, the difference angle between incoming and outgoing directions around the halfway vector, ϕ_d the azimuth difference. Reduces 4D BRDF to a 3D table for isotropic reflectance (there is no dependence on the azimuth of the halfway vector itself; material is same in all orientations). **Data-Driven BRDFs.** One approach is to record a large set of measured BRDFs, flatten each into a high-dimensional vector, and apply PCA to find a lower-dimensional linear subspace. While effective for compact storage, this can produce non-physical BRDFs if the manifold is highly nonlinear. More advanced techniques (e.g. local linear embedding, kernel-based mixture models, or neural networks) better preserve physical constraints. Another method is to express a novel BRDF as a linear combination of previously measured BRDFs, solving an overdetermined system to fit the unknown reflectance, ensuring realistic material behavior.

SPATIALLY VARYING BRDFS

Spatially Varying BRDF (SVBRDF). An SVBRDF is a 6D function representing reflectance variation across different surface positions and viewing angles; it describes planar surfaces where each surface point can have a distinct BRDF.
SVBRDF Capture. *Linear Light Source:* A moving linear light source scans a planar sample, requiring fewer images than a point-light setup and no HDR. *Reflectance Trace Analysis:* Tracks how light sweeps across the surface over time, revealing diffuse peaks (aligned with the surface normal) and specular peaks (aligned with the reflection direction). Fitting a diffuse component and subtracting it simplifies specular-parameter estimation. *Pocket Reflectometry:* Portable devices measure BRDF over time; time-shift compensation and dynamic time warping align the traces. Geometry can also be inferred for bumpy surfaces. **Statistical Modelling.** 0th, 1st, and 2nd moments characterize reflectance (total energy, mean direction, and roughness). Isotropic materials have uniform reflectance in all directions; anisotropic materials require more advanced descriptors. Spherical harmonics (SH) provide a steerable basis over spherical domains. Low-order SH captures broad diffuse behavior; higher orders capture sharp specular features. **Hardware-based Reflectometry.** *SH-based setups:* LED spheres can create near-continuous spherical illumination; long-exposure captures separate diffuse and specular components using higher SH terms. *Stereo:* Multiple camera angles reconstruct both geometry and reflectance, enabling more accurate rendering. **Mobile Surface Reflectometry.** Handheld devices can acquire SVBRDFs in real time. *Harris Corners* guide feature tracking to compensate for motion, and homography transformations align images to a canonical frame. Recovered SVBRDF maps typically include normals (geometry), diffuse albedo, specular albedo, and roughness. **Deep Learning.** A U-Net architecture encodes input images and decodes diffuse/specular albedo, roughness, and normals; skip connections preserve spatial detail. A parallel branch may isolate saturated specular highlights. *Rendering-based losses* improve perceptual quality over simple L1/L2 losses by directly comparing rendered outputs. Large synthetic datasets (with procedural or augmented data) assist in training networks for robust SVBRDF estimation.

GLOBAL ILLUMINATION

Light Transport Equation (Kajiya). A common starting point is the integral form: $L_r(x,\omega_r) = L_e(x,\omega_r) + \int_\Omega f_r(x,\omega_r,\omega_i)\, L_i(x,\omega_i)\, \cos\theta_i\, d\omega_i$, where L_r is the outgoing radiance at point x in direction ω_r, L_e is emitted radiance (for light sources), f_r is the BRDF, L_i is the incident radiance from direction ω_i, and $\cos\theta_i$ accounts for foreshortening.
Surface-Based Reformulation. We can rewrite light transport in terms of points on surfaces rather than directions. Let $L(x'\to x)$ denote radiance traveling from x' to x, and define a geometric coupling term $G(x'\leftrightarrow x) = V(x'\leftrightarrow x)\, \frac{|\cos\theta|\, |\cos\theta'|}{\|x-x'\|^2}$, where $V(\cdot)$ is visibility (1 if unoccluded, 0 otherwise), and the cosines plus $\|x-x'\|^2$ handle angular/distance falloff. The area-integral version becomes $L(x'\to x) = L_e(x'\to x) + \int_A f_r(x''\to x'\to x)\, L(x''\to x')\, G(x''\leftrightarrow x')\, dA(x'')$.
Monte Carlo Path Sampling. Because these integrals can be very costly, we use a stochastic approach. One views $L(x_1\to x_2)$ as a sum over potentially infinite paths: $L(x_1\to x_2) = \sum_{i=0}^\infty P(x_i)$. In practice, we truncate or use Russian Roulette to avoid infinite recursion.
Russian Roulette (RR). Each bounce is stopped with probability q or continued with probability $1-q$. Continuing scales the contribution by $\frac{1}{1-q}$ to remain unbiased. If the integrand is F and c is some constant, we define a new integrand $\begin{cases} c, & \text{with prob } q \\ \frac{F-q\,c}{1-q}, & \text{with prob } 1-q \end{cases}$ so $\mathrm{E}(F') = \mathrm{E}(F)$ but with increased variance. General formula for prob. to stop after N bounces $p(n) = (1-q)^{n-1}q$
Solid-Angle vs. Area PDFs. When picking path vertices, one may sample directions (ω) or points on a surface (area). The two PDFs relate via $p_A = p_\omega\, \frac{|\cos\theta_i|}{\|x_i-x_{i-1}\|^2}$.
Path Contribution. The radiance from an i-vertex path can be written schematically as $\frac{L_e(x_i\to x_{i-1})}{p_A(x_i)} \prod_{j=1}^{i-1} \frac{f_r(x_{j+1}\to x_j\to x_{j-1})\, |\cos\theta_j|}{p_\omega(x_{j+1}-x_j)}$.
Bidirectional Path Tracing (BDPT). Standard path tracing sends rays from the camera only. BDPT also traces from lights, then connects partial paths. This is more robust for caustics or specular interactions that camera-based tracing may miss. Multiple importance sampling (MIS) combines different connections to reduce variance.
Metropolis Light Transport (MLT). BDPT may still be inefficient in scenes with extremely rare paths (e.g. small apertures). MLT uses MCMC to mutate existing high-contribution paths. If x is current and x' a proposed path, $a(x\to x') = \min(1, \frac{f(x')\, T(x'\to x)}{f(x)\, T(x\to x')})$. If accepted, we move to x', favoring paths that yield large contributions.
Radiance/Irradiance Caching. For diffuse indirect lighting, irradiance $E(x,n) = \int_\Omega L_i(x,\omega_i)\, \cos\theta_i\, d\omega_i$ changes slowly, so we compute and cache it at sparse sample points $\{p_j, n_j, E_j, r_j\}$. Nearby query points interpolate as $E(x) = \frac{\sum_j w_j(x)\, E_j}{\sum_j w_j(x)}$.
Photon Mapping. Caustics are difficult for caching alone. Photon mapping uses two passes: (1) *Photon emission*, storing photons in a KD-tree; (2) *Gathering* at each shading point within radius r. The approximate radiance is $L(x,\omega) \approx \sum_{p=1}^n f_r(x,\omega_p,\omega)\, \frac{\Delta\Phi_p}{\pi r^2}$. *Progressive Photon Mapping (PPM)* refines by gradually shrinking r.
Virtual Point Lights (VPLs). *Instant Radiosity* places VPLs at bounce vertices to approximate indirect illumination. Each VPL acts as a small light; summing many VPLs enables GPU-friendly rendering.
Lightcuts. When there are many lights or VPLs, evaluating all is expensive. *Lightcuts* builds a hierarchy of lights, clustering them adaptively. We start at the root (all lights). If approximating that cluster is sufficiently accurate, we stop; if not, we split. This yields near-logarithmic complexity in the number of lights.

SCATTERING

Volumetric Scattering. Participating media are modeled as particles in a volume. The medium's absorption $\sigma_a(x,\omega)$, out-scattering $\sigma_s(x,\omega)$, and emission $Q(x,\omega)$ define how radiance changes along a differential distance dt. The *extinction coefficient* is $\sigma_t = \sigma_a + \sigma_s$. Integrating yields a transmittance (Beer's Law, for homogenous medium: $T_r = \exp(-\sigma_t d)$):$T_r(x\to x') = \exp\Big(-\int_0^d \sigma_t(x+t\,d,\omega)\, dt\Big)$..
Single-scattering albedo $\alpha = \frac{\sigma_s}{\sigma_a+\sigma_s} = \frac{\sigma_s}{\sigma_t}$
In-scattering is governed by a phase function $p(\omega\to\omega')$, which is the volumetric analog of a BRDF. An isotropic phase function has $p(\omega\to\omega') = \frac{1}{4\pi}$, while Henyey–Greenstein (Anisotropic) is
$p_{HG}(\cos\theta) = \frac{1-g^2}{4\pi(1+g^2-2g\, \cos\theta)^{3/2}}$, $\quad -1 \le g \le 1$. Added radiance is
$S(x,\omega) = Q(x,\omega) + \sigma_S(x,\omega)\int_\Omega p(x,-\omega'\to\omega)L_i(x,\omega')\, d\omega'$. Overall radiance transfer is $dL_o(x,\omega) = -\sigma_t(x,\omega)L_i(x,-\omega)dt + S(x,\omega)dt$.
The *radiative transport equation* can be written as
$(\vec{\omega}\cdot\nabla)L(x,\vec{\omega}) = -\sigma_t\, L(x,\vec{\omega}) + \sigma_s\int_{4\pi} p(\vec{\omega},\vec{\omega}')\, L(x,\vec{\omega}')\, d\vec{\omega}' + s(x,\vec{\omega})$,
where the last term represents emission or sources. Highly scattering media require multiple bounces of light, making computation expensive.
Subsurface Scattering (BSSRDF). Translucent materials (e.g. skin, marble) allow light to penetrate the surface before reemitting. A BSSRDF is an 8D function generalizing the 4D BRDF. A common approximation for diffuse subsurface scattering is the *Dipole Diffusion* model:$R_d(r) = \frac{A}{4\pi D}\Big[\frac{z_r(1+\sigma_{\text{tr}} r)}{r^3}e^{-\sigma_{\text{tr}} r} - \frac{z_v(1+\sigma_{\text{tr}} r_v)}{r_v^3}e^{-\sigma_{\text{tr}} r_v}\Big]$,
where r is radial distance, σ_{tr} an effective transport coefficient, z_r, z_v the real/virtual source depths, A a scaling factor, and D the diffusion coefficient. When MS is so frequent that directions get randomised we can use a diffusion approximation. In highly scattering, low absorption material photons quickly lose memory of incoming direction after a few scattering events. By default, general BSSRDF $S(x_i,\omega_i;x_o,\omega_o)$ is 8D (2 2D spatial locations on the surface plus two 2D directions). Diffusion - outgoing radiance at some point on the surface depends mostly on distance from where photons entered rather than precise angles. End up with radial profile $R_d(r)$ where $r = \|x_i - x_o\|$
Scattering in Skin. A multi-layer diffusion approach models epidermis and dermis, each with separate scattering and absorption properties. *Hanrahan–Krueger* includes single scattering described by a Henyey–Greenstein phase function ($g > 0$ for forward scattering). For real-time rendering, one may approximate diffusion with Gaussian blurs in texture space (e.g. screen-space subsurface scattering). In measurement, *polarization-based methods* or LED probes help fit scattering parameters (e.g. melanin or hemoglobin concentrations for spectral models).
Kubelka-Munk Theory. Consider two partially-reflective/partially-transmissive boundaries, with T_1, R_1 for the first boundary, T_2, R_2 for the second boundary. Light can take infinitely many bounce paths within the two-layer system, so the total *transmittance* T_{12} and *reflectance* R_{12} can be written as geometric series:$T_{12} = T_1\, T_2\, \big[1 + R_2R_1 + (R_2R_1)^2 + \dots\big] = \frac{T_1\, T_2}{1-R_2\, R_1}$, $R_{12} = R_1 + T_1\, R_2\, T_1\, \big[1 + R_2R_1 + (R_2R_1)^2 + \dots\big] = R_1 + \frac{T_1\, R_2\, T_1}{1-R_2\, R_1}$.
Hair Scattering. Hair fibers are roughly cylindrical, causing strongly anisotropic reflection. *Kajiya–Kay* or *Marschner* hair models treat single-scattering highlights plus multiple forward/backward scattering for lighter hair. Photon mapping or specialized 6D data structures can handle multiple scattering in dense hair volumes. Fur vs. hair differs in fiber diameter and internal structure, affecting reflection lobes.
Misc. Albedo ρ_d is the fraction of incident light a surface reflects. A $\rho_d = 0.5$ means that the surface reflects 50% of incoming light. For a diffuse (Lambertian) surface, the BRDF is ρ_d/π; $L_r = \rho L_i$, $L_r = (\rho/\pi)E_i$. The weight of the sample $p(\omega_r)$ in MC rendering in DI is $\frac{p(\omega_h)}{4\omega_h\cdot\omega_i}$ - ω_h halfway vector, ω_i incident vector.
Direction ω on a unit sphere surface area element is $d\omega = \sin\theta\, d\theta\, d\phi$. To convert ω_h to ω_r use chain rule. In the standard Lambertian (diffuse) lighting setup, only the directions above the surface normal contribute to the irradiance. In spherical coordinates, this "upper" hemisphere corresponds to polar angles θ running from 0 (the surface normal) to $\pi/2$ (the horizon). Hence, you see the integration domain for θ restricted to $[0,\pi/2]$. **Mirror Ball** surface normal: $n = \frac{(v+l)}{\|v+l\|}$; v=(0,0,1).
$(u,v) = \frac{1}{2}(1+\frac{N_x}{N_z+1}, 1+\frac{N_y}{N_z+1})$ to mark on MB (0,0) botleft. Reflected directions $R = 2(N\cdot V) - V$. Paramaterise vector to LatLong: θ = Polar Angle = $\arccos(z)$ in $[0,\pi]$, ϕ = Azimuth = $\arctan 2(y,x)$ in $[0,2\pi]$, map to u,v via $u = \phi/2\pi$ and $v = \theta/\pi$. **1O ODE.** Convert to normal form $dy/dx + P(x)y = Q(x)$. Integrating Factor is $\mu(x) = e^{\int P(x)}$. Mult. $\mu(x)$ with 1O ODE LHS. Convert to $d/dx(P(x)y) = Q(x)$. Integrate, subst. IC. **NeRF 3DGS.** NeRF is a continuous volumetric representation learned by a MLP. For each 3D point (x,y,z) and viewing direction, NeRF predicts density and emitted colour. 3DGS is a point based representation that stores a cloud of 3D splat primitives each with its position, shape (covariance), colour and opacity. Rendered by splatting each Gaussian directly into the image plane and blending their contributions NeRF: High fidelity, continuous geometry and colour. But slow if many samples per pixel needed. 3DGS: fast rendering, faster training. But high memory over head and has no guaranteed topological continuity (unlike a mesh) if coverage is incomplete.