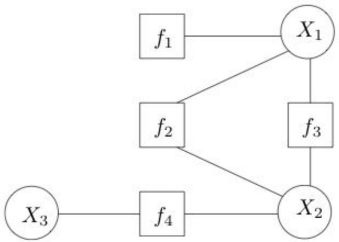
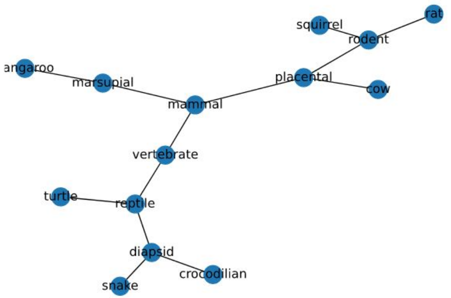
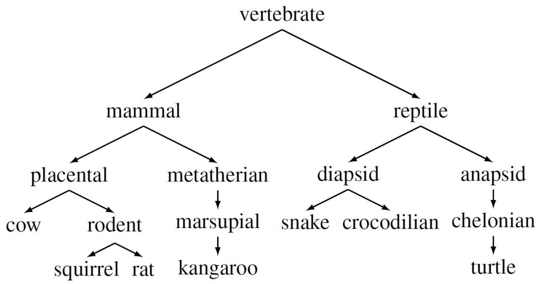




Graphical Models
Belief Propagation Algorithm

Clément Bonnet, Charbel-Raphaël Segerie

March 31st, 2021



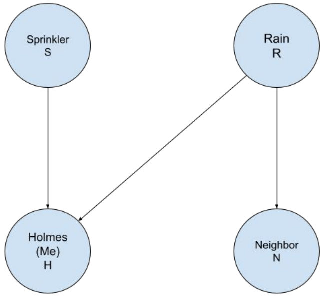
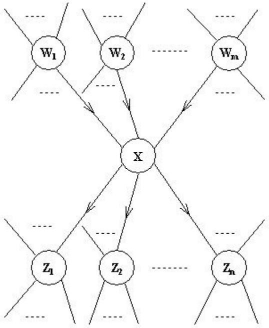
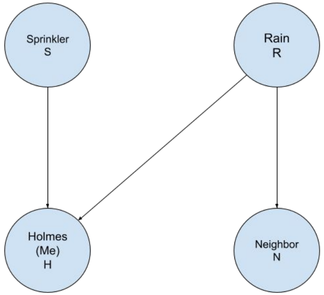
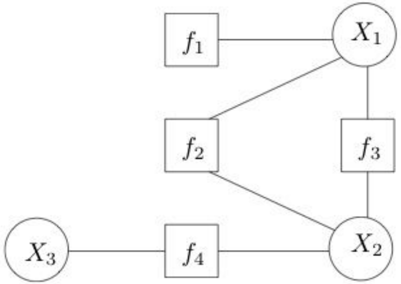
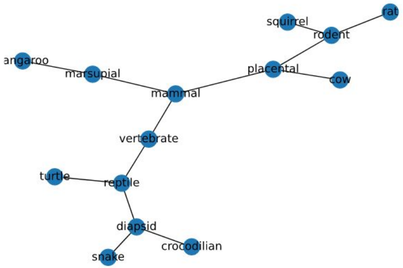


Table of Contents

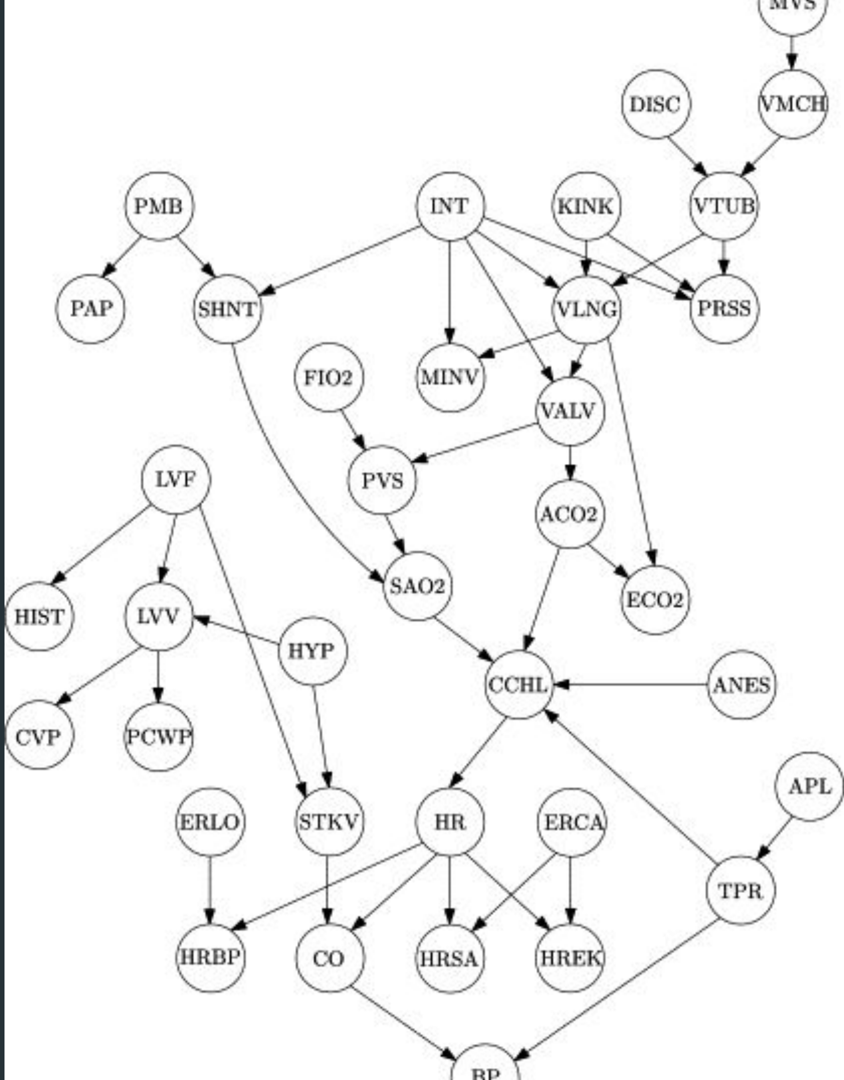




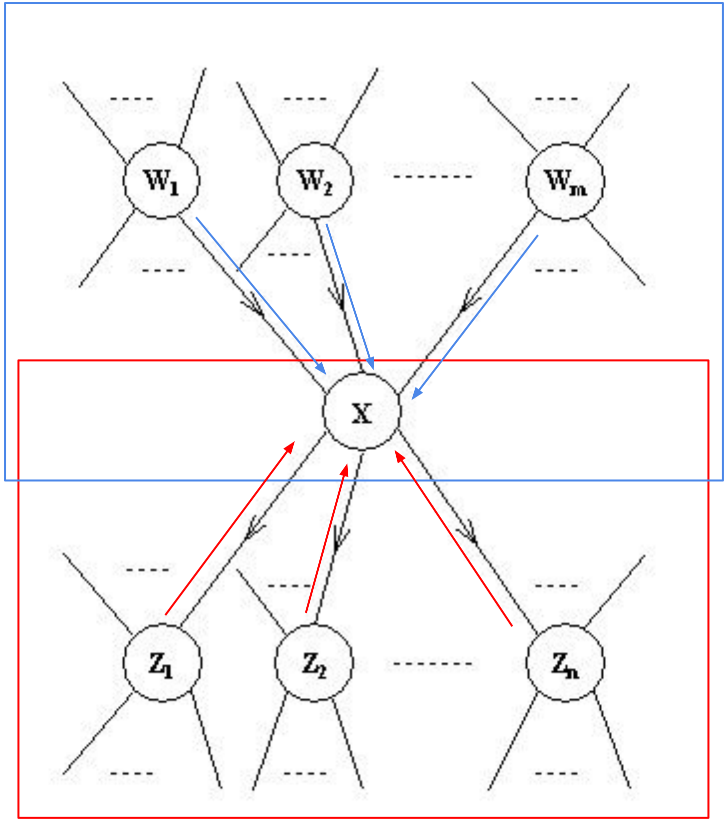




Bayesian Networks



Bayesian Networks: Propagation Rules



Acyclic graph, but messages flow in both direction.

message Child -> Parent

Likelihood $\lambda(x) = \mathbb{P}(Z_1, \ldots, Z_n | X = x) = \prod_{j=1}^{n} \lambda_{Z_j}(x)$

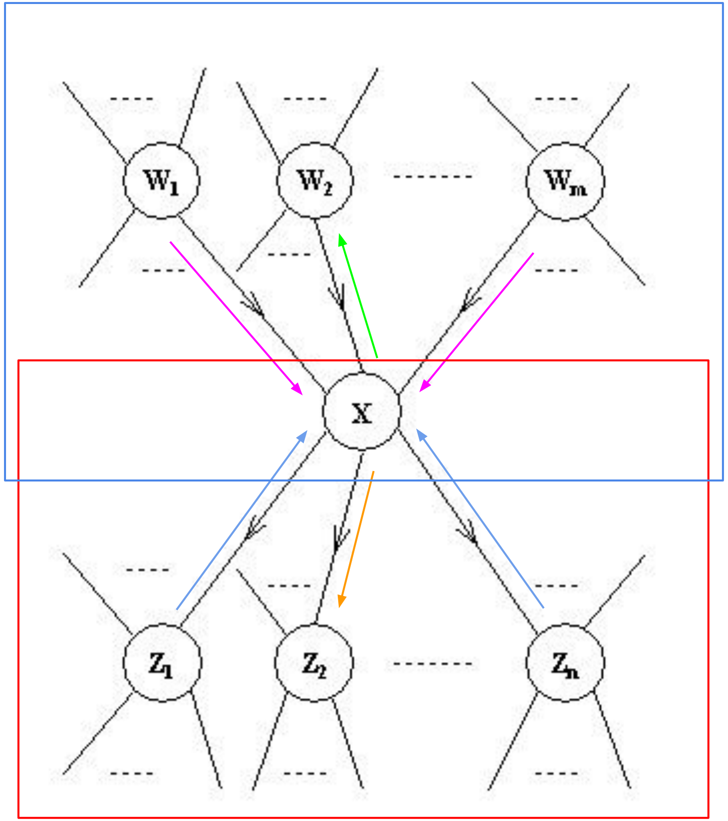
message Parent-> Child

Prior $\pi(x) = \sum_{\mathbf{w} \in \{0,1\}^m} \mathbb{P}(X = x | \mathbf{w}) \prod_{k=1}^{m} \pi_X(w_k)$

Belief

$$\begin{aligned} BEL(x) &\equiv \alpha \lambda(x) \pi(x) \\ &= \alpha \boxed{\left[\prod_{j=1}^{n} \lambda_{Z_j}(x) \right]} \boxed{\left[\sum_{\mathbf{w}} P(x \mid \mathbf{w}) \prod_{k=1}^{n} \pi_X(w_k) \right]} \end{aligned}$$

Bayesian Networks: Updates



Likelihood:

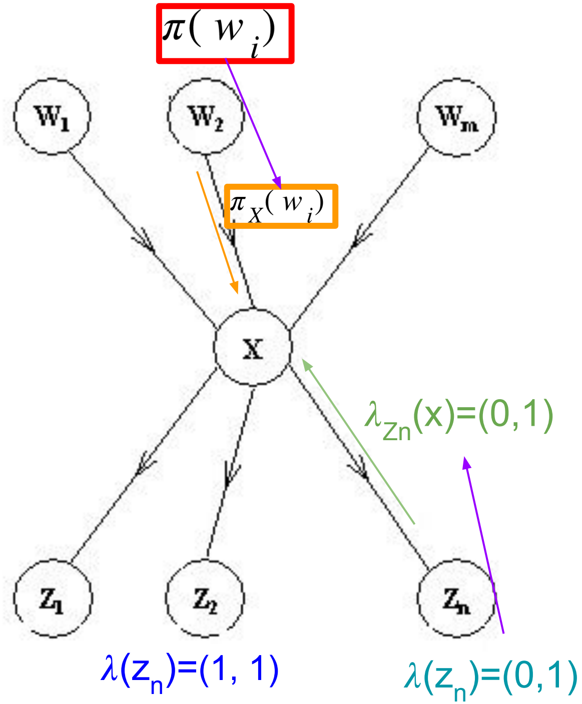
Model of the world specified by the user

$$\overrightarrow{\lambda_X(w_i)} \propto \sum_{x} \boxed{\lambda(x)} \sum_{\substack{\mathbf{w}' \in \{0,1\}^m \\ w'_i = w_i}} \mathbb{P}(X = x | \mathbf{w}') \prod_{k \neq i} \overrightarrow{\pi_X(w'_k)}$$

Prior:

$$\overrightarrow{\pi_{Z_j}(x)} \propto \boxed{\pi(x)} \prod_{k \neq j} \overrightarrow{\lambda_{Z_k}(x)}$$

Bayesian Networks: Boundary Conditions



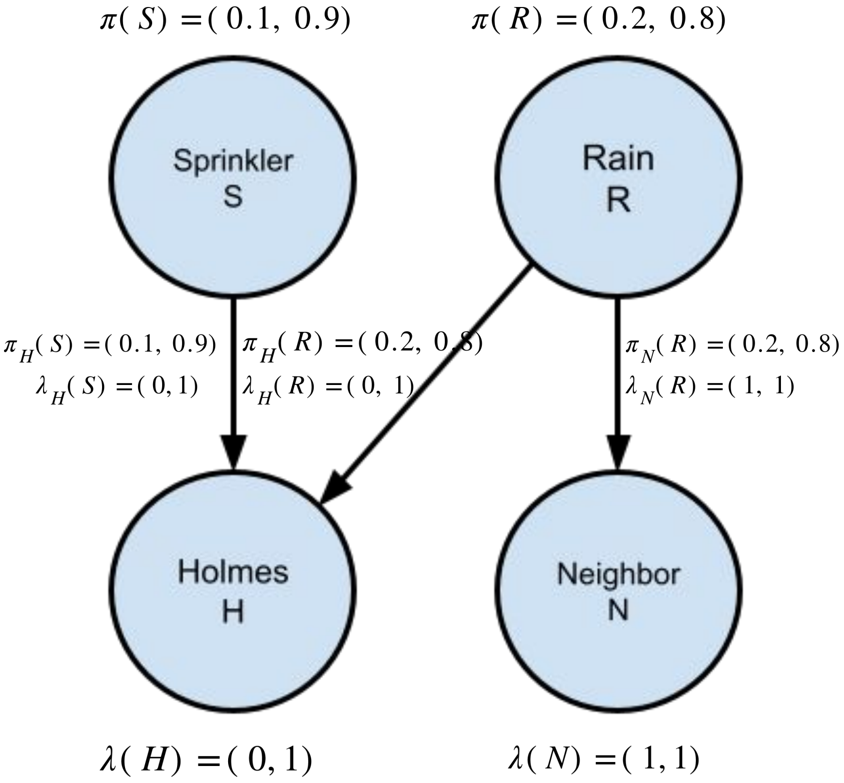
Example of binary states

Root nodes: If W is a node with no parents, we set the value of the prior message $\boldsymbol{\pi}_X(w)$ equal to the prior probabilities.

Anticipatory nodes: If X is a childless node that has not been instantiated, we set the $\lambda(z)$ value as a vector of all 1's

Evidence nodes: If evidence X=i is obtained, we set the $\lambda(z)$ value to (0,...,0,1,0,...0) with 1 at the i_{th} position, and we set the likelihood message equal to the evidence node.

Bayesian Networks: Case Study → The Wet Grass Example: Priors



$\mathbb{P}(N\|R)$	$N=0$	$N=1$
$R=0$	0.8	0.2
$R=1$	0	1

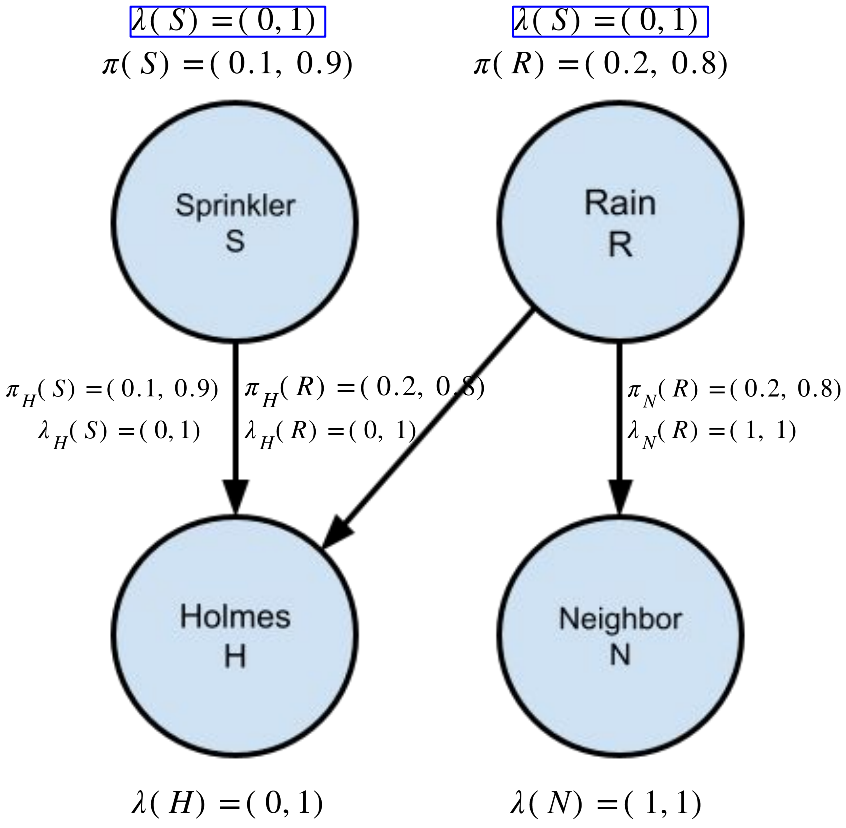
Model of the world

$\mathbb{P}(H\|R,S)$	$H=0$	$H=1$
$R=1, S=1$	0	1
$R=0, S=1$	0.1	0.9
$R=1, S=0$	0	1
$R=0, S=0$	1	0

X	$\mathrm{BEL}(x)$	$\pi(x)$	$\lambda(x)$
R	-	(0.2, 0.8)	-
S	-	(0.1, 0.9)	-
N	-	-	(1, 1)
H	-	-	(0, 1)

Boundary Conditions

Bayesian Networks: Case Study → The Wet Grass Example: Likelihood



message Child -> Parent

Likelihood $$\boxed{\lambda(x) = \mathbb{P}(Z_1, \ldots, Z_n | X = x) = \prod_{j=1}^{n} \lambda_{Z_j}(x)}$$

X	BEL(x)	$\pi(x)$	$\lambda(x)$
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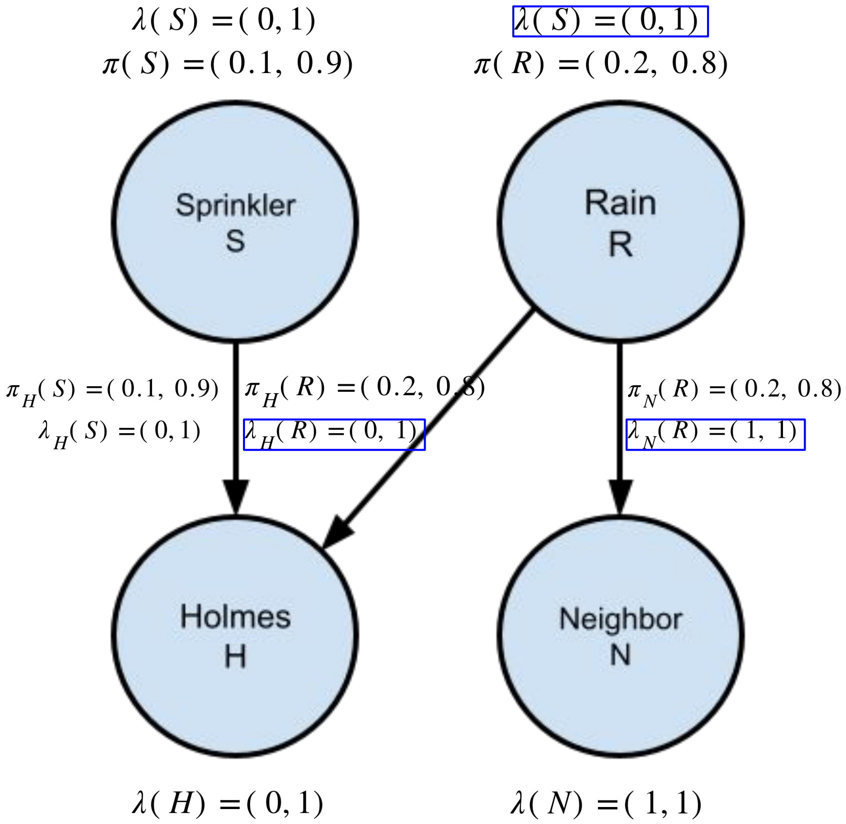
R	(0, 0.8)	(0.2, 0.8)	(0, 1)
S	(0, 0.9)	(0.1, 0.9)	(0, 1)
N	(0.16, 0.84)	(0.16, 0.84)	(1, 1)
H	(0, 0.962)	(0.038, 0.962)	(0, 1)

Propagation of belief

Bayesian Networks: Case Study → The Wet Grass Example: Likelihood



message Child -> Parent

Likelihood $\boxed{\lambda(x) = \mathbb{P}(Z_1, \ldots, Z_n | X = x) = \prod_{j=1}^{n} \lambda_{Z_j}(x)}$

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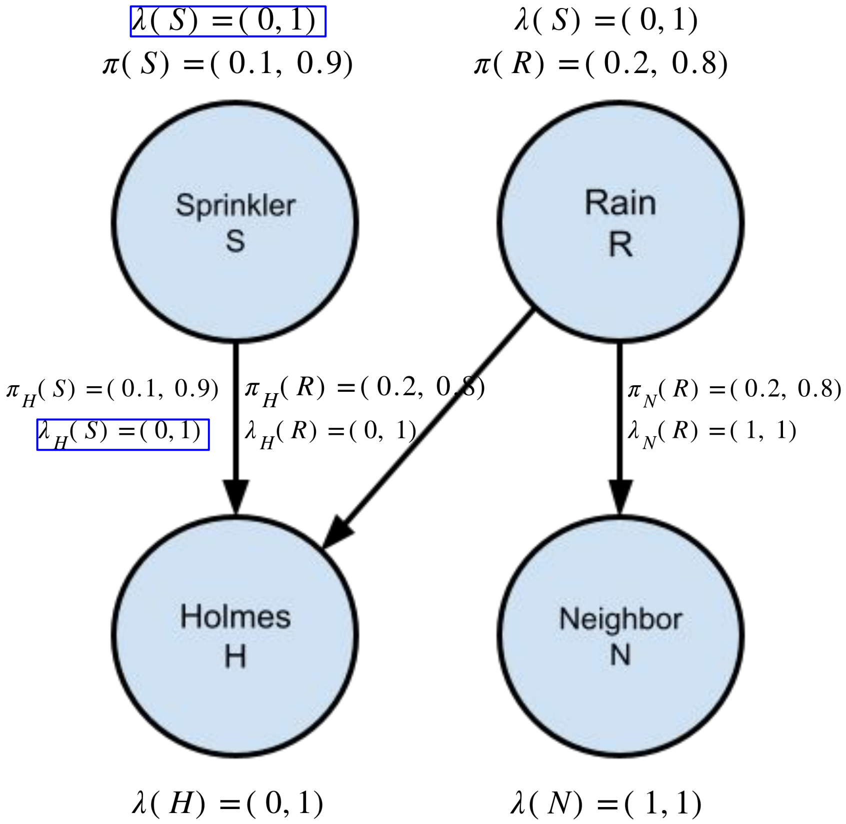
Propagation of belief

Bayesian Networks: Case Study → The Wet Grass Example: Likelihood



message Child -> Parent

Likelihood $\boxed{\lambda(x)=\mathbb{P}(Z_1,\ldots,Z_n|X=x)=\prod_{j=1}^{n}\lambda_{Z_j}(x)}$

X	BEL(x)	$\pi(x)$	$\lambda(x)$
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S	-	(0.1, 0.9)	-
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Propagation of belief

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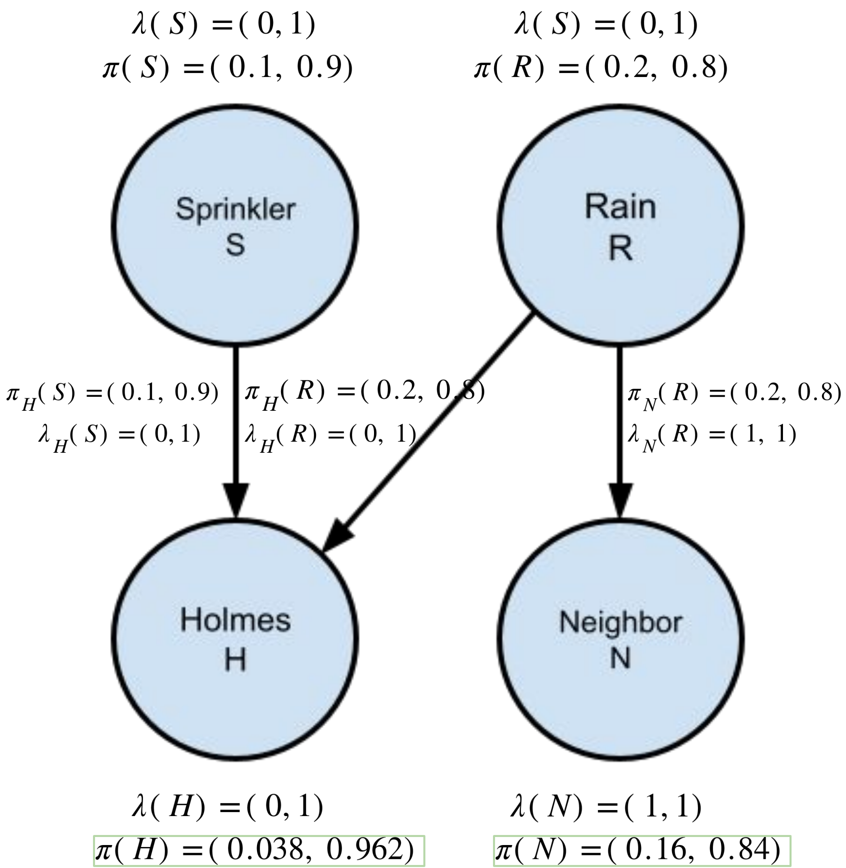
H	(0, 0.962)	(0.038, 0.962)	(0, 1)

Bayesian Networks: Case Study: The Wet Grass Example: Priors messages



$\mathbb{P}(N\|R)$	$N=0$	$N=1$
$R=0$	0.8	0.2
$R=1$	0	1

Model of the world

$\mathbb{P}(H\|R,S)$	$H=0$	$H=1$
$R=1, S=1$	0	1
$R=0, S=1$	0.1	0.9
$R=1, S=0$	0	1
$R=0, S=0$	1	0

Prior $\pi(x) = \displaystyle\sum_{\mathbf{w}\in\{0,1\}^m} \mathbb{P}(X=x|\mathbf{w}) \prod_{k=1}^{m} \pi_X(w_k)$

X	BEL(x)	$\pi(x)$	$\lambda(x)$
R	-	(0.2, 0.8)	-
S	-	(0.1, 0.9)	-
N	-	-	(1, 1)
H	-	-	(0, 1)

X	BEL(x)	$\pi(x)$	$\lambda(x)$
R	(0, 0.8)	(0.2, 0.8)	(0, 1)
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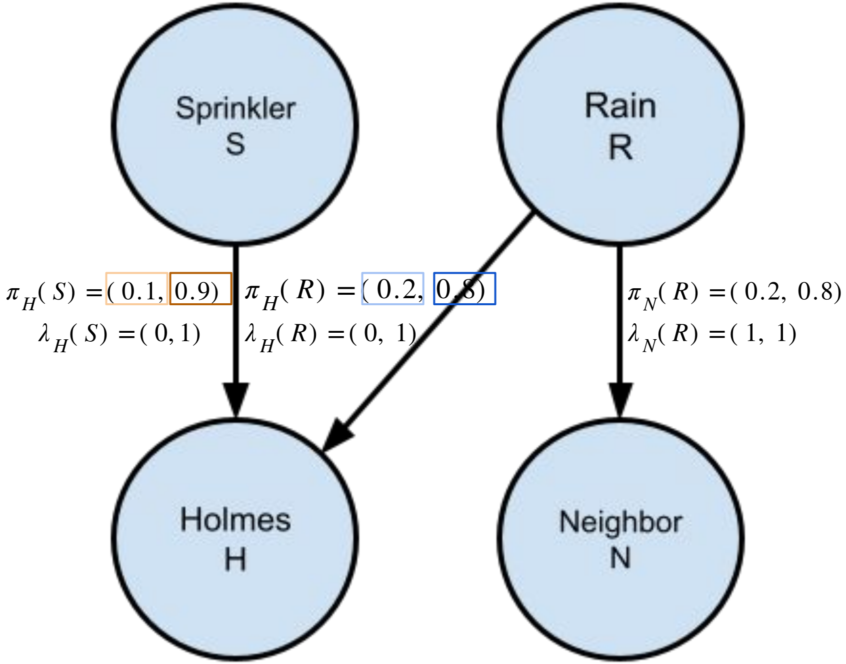
N	(0.16, 0.84)	(0.16, 0.84)	(1, 1)
H	(0, 0.962)	(0.038, 0.962)	(0, 1)

Propagation of belief

Bayesian Networks: Case Study: The Wet Grass Example: Priors messages

$\lambda(S) = (0, 1)$
$\pi(S) = (0.1,\ 0.9)$

$\lambda(S) = (0, 1)$
$\pi(R) = (0.2,\ 0.8)$



$\lambda(H) = (0, 1)$
$\pi(H) = (\boxed{0.038},\ 0.962)$

$\lambda(N) = (1, 1)$
$\pi(N) = (0.16,\ 0.84)$

$\mathbb{P}(N\|R)$	$N=0$	$N=1$
$R=0$	0.8	0.2
$R=1$	0	1

Model of the world

$\mathbb{P}(H\|R,S)$	$H=0$	$H=1$
$R=1, S=1$	0	1
$R=0, S=1$	0.1	0.9
$R=1, S=0$	0	1
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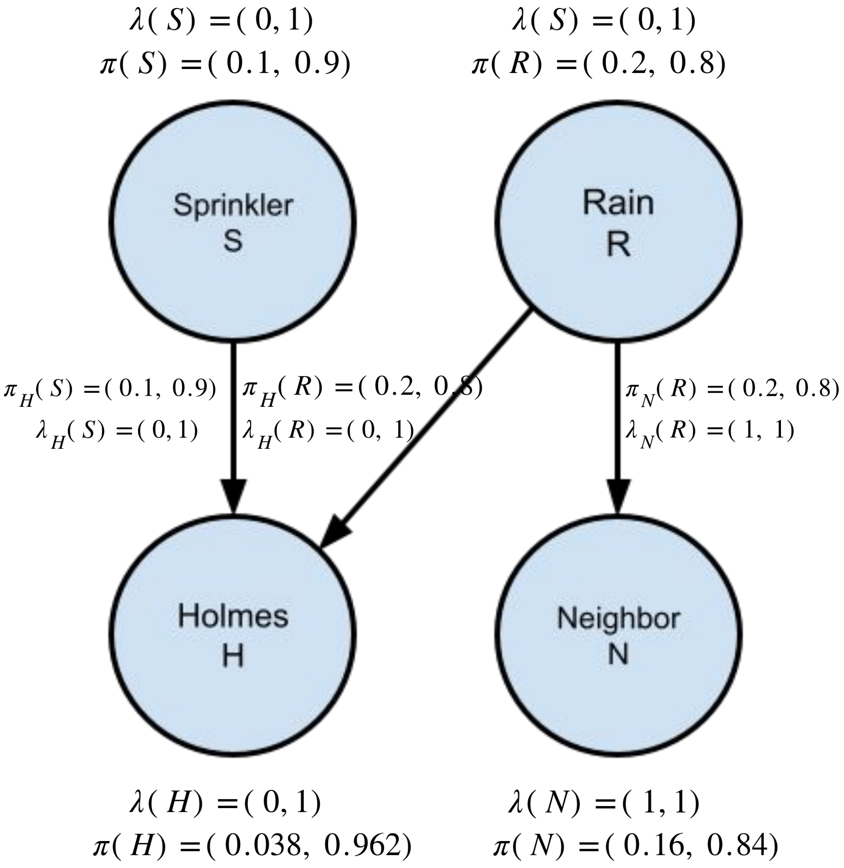
Prior $\pi(x) = \sum_{\mathbf{w} \in \{0,1\}^m} \mathbb{P}(X{=}x|\mathbf{w}) \prod_{k=1}^{m} \pi_X(w_k)$

$$\begin{aligned}
\boxed{\pi(H=0)} &= \mathbb{P}(H=0|R=0,S=0)\pi_H(R=0)\pi_H(S=0) \\
&+ \mathbb{P}(H=0|R=0,S=1)\pi_H(R=0)\pi_H(S=1) \\
&+ \mathbb{P}(H=0|R=1,S=0)\pi_H(R=1)\pi_H(S=0) \\
&+ \mathbb{P}(H=0|R=1,S=1)\pi_H(R=1)\pi_H(S=1) \\
&= 1 \times (0.2 \times 0.1) + 0.1 \times (0.2 \times 0.9) \\
&+ 0 \times (0.8 \times 0.1) + 0 \times (0.8 \times 0.9) \\
&= 0.02 + 0.018 = 0.038
\end{aligned}$$

Bayesian Networks: Case Study: The Wet Grass Example: Beliefs



Model of the world

$\mathbb{P}(N\|R)$	$N = 0$	$N = 1$
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$R = 1$	0	1

$\mathbb{P}(H\|R, S)$	$H = 0$	$H = 1$
$R = 1, S = 1$	0	1
$R = 0, S = 1$	0.1	0.9
$R = 1, S = 0$	0	1
$R = 0, S = 0$	1	0

Belief $\boxed{BEL(x) \equiv \alpha\lambda(x)\pi(x)}$

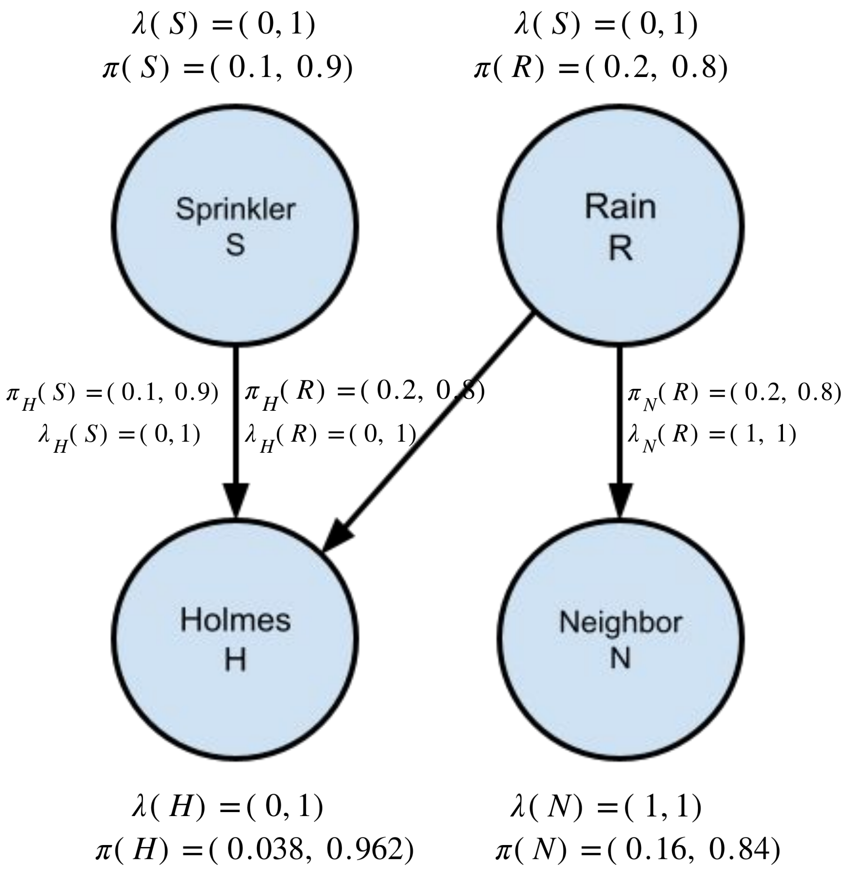
X	BEL(x)	$\pi(x)$	$\lambda(x)$
R	-	(0.2, 0.8)	-
S	-	(0.1, 0.9)	-
N	-	-	(1, 1)
H	-	-	(0, 1)

Propagation of belief

Product coordinate by coordinate

X	BEL(x)	$\pi(x)$	$\lambda(x)$
R	$(\boxed{0}, 0.8)$	$(\boxed{0.2}, 0.8)$	$(\boxed{0}, 1)$
S	(0, 0.9)	(0.1, 0.9)	(0, 1)
N	(0.16, 0.84)	(0.16, 0.84)	(1, 1)
H	(0, 0.962)	(0.038, 0.962)	(0, 1)

Bayesian Networks: Case Study: The Wet Grass Example: Beliefs



$\mathbb{P}(N \mid R)$	$N=0$	$N=1$
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Model of the world

$\mathbb{P}(H \mid R, S)$	$H=0$	$H=1$
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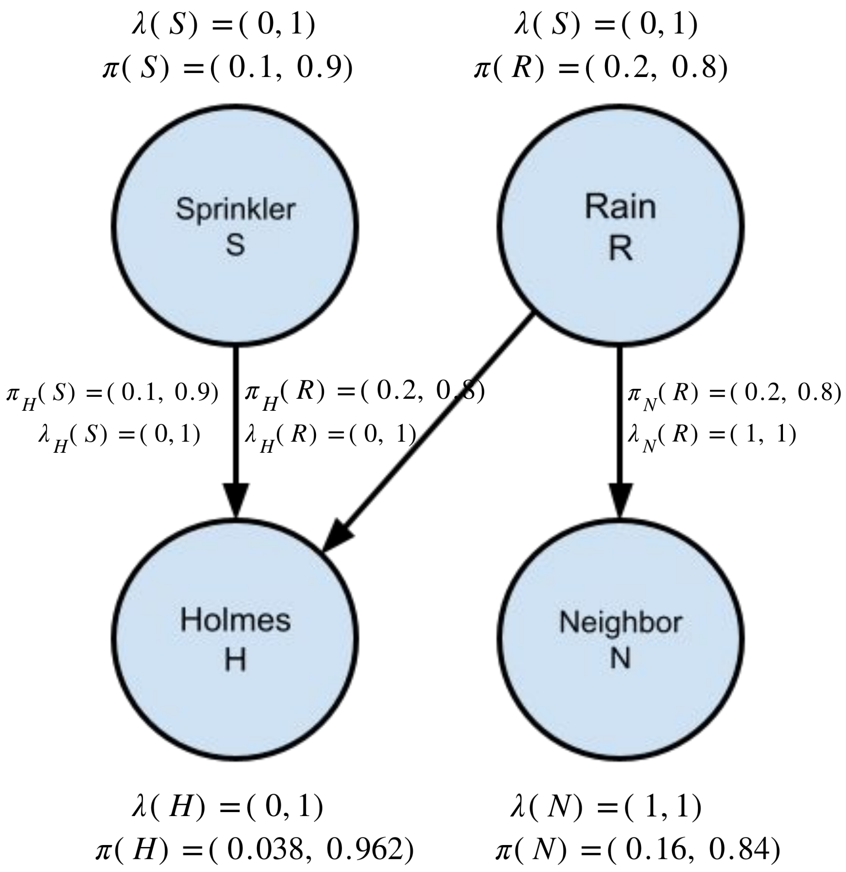
Propagation of belief

Bayesian Networks: Case Study: The Wet Grass Example: Beliefs



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Model of the world

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Belief $\boxed{BEL(x) \equiv \alpha\lambda(x)\pi(x)}$

X	BEL(x)	$\pi(x)$	$\lambda(x)$
R	-	(0.2, 0.8)	-
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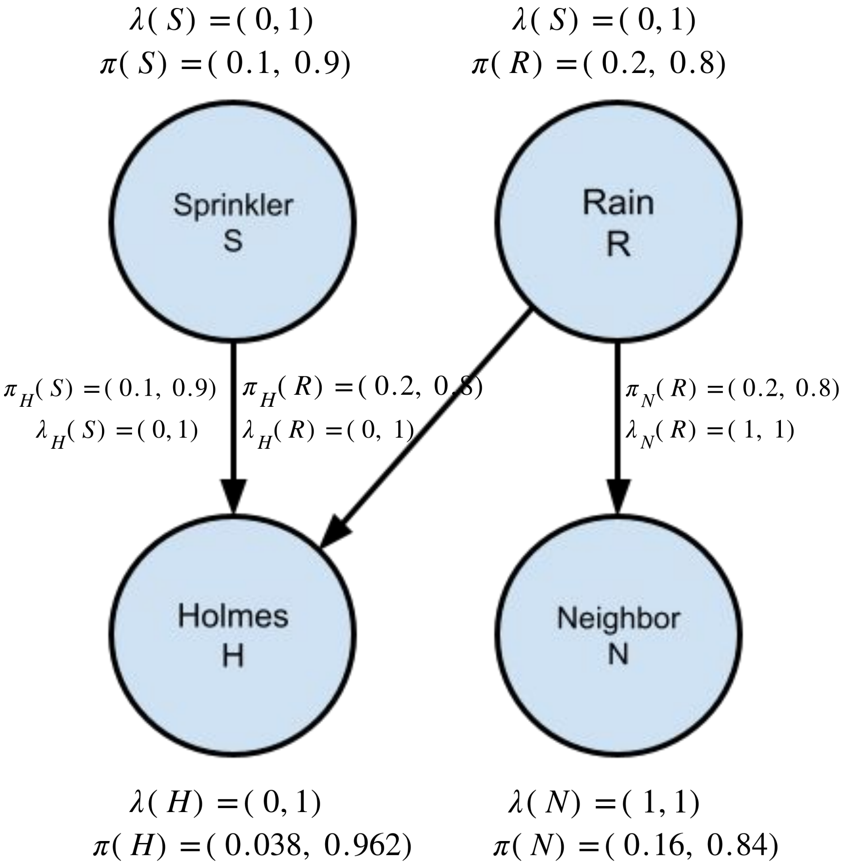
Propagation of belief

Bayesian Networks: Case Study: The Wet Grass Example: Beliefs



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$R=0$	0.8	0.2
$R=1$	0	1

Model of the world

$\mathbb{P}(H \mid R, S)$	$H=0$	$H=1$
$R=1, S=1$	0	1
$R=0, S=1$	0.1	0.9
$R=1, S=0$	0	1
$R=0, S=0$	1	0

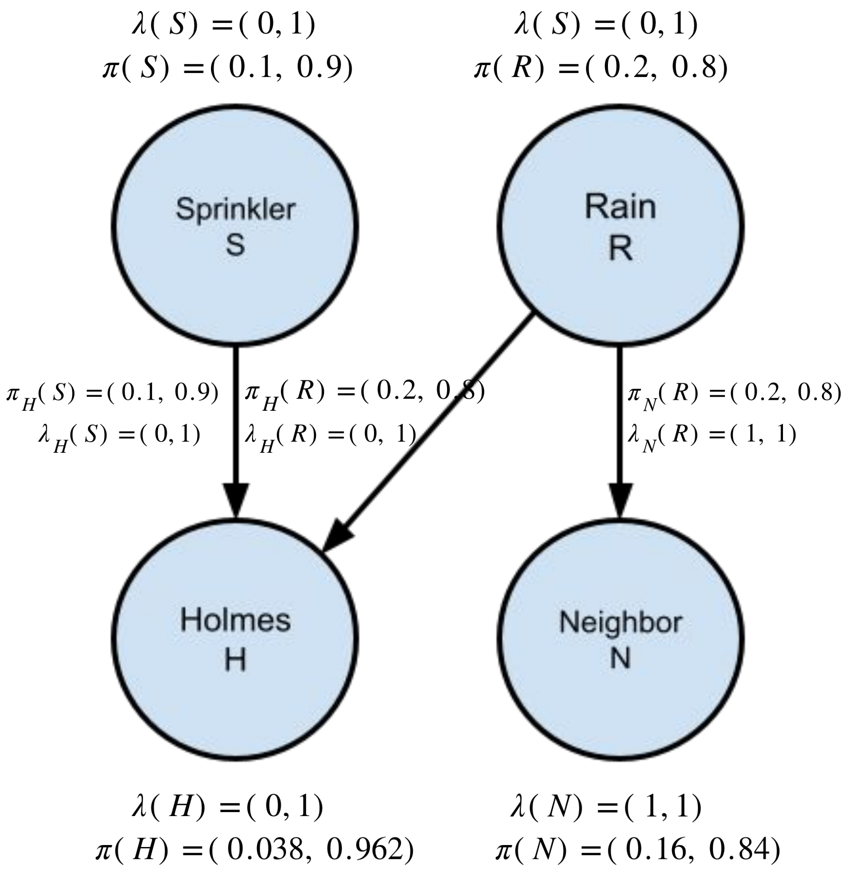
Belief $\boxed{BEL(x) \equiv \alpha\lambda(x)\pi(x)}$

X	BEL(x)	$\pi(x)$	$\lambda(x)$
R	-	(0.2, 0.8)	-
S	-	(0.1, 0.9)	-
N	-	-	(1, 1)
H	-	-	(0, 1)

Propagation of belief

X	BEL(x)	$\pi(x)$	$\lambda(x)$
R	(0, 0.8)	(0.2, 0.8)	(0, 1)
S	(0, $\boxed{0.9}$)	(0.1, $\boxed{0.9}$)	(0, $\boxed{1}$)
N	(0.16, 0.84)	(0.16, 0.84)	(1, 1)
H	(0, 0.962)	(0.038, 0.962)	(0, 1)

Bayesian Networks: Case Study: The Wet Grass Example: Beliefs



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Model of the world

$\mathbb{P}(H\|R,S)$	$H=0$	$H=1$
$R=1, S=1$	0	1
$R=0, S=1$	0.1	0.9
$R=1, S=0$	0	1
$R=0, S=0$	1	0

Belief $\boxed{BEL(x) \equiv \alpha\lambda(x)\pi(x)}$

etc...

X	BEL(x)	$\pi(x)$	$\lambda(x)$
R	-	(0.2, 0.8)	-
S	-	(0.1, 0.9)	-
N	-	-	(1, 1)
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X	BEL(x)	$\pi(x)$	$\lambda(x)$
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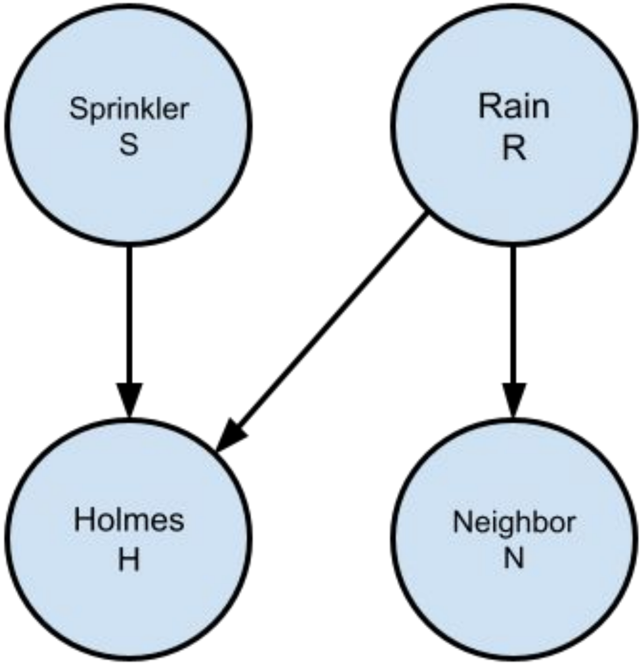
N	(0.16, 0.84)	(0.16, 0.84)	(1, 1)
H	(0, 0.962)	(0.038, 0.962)	(0, 1)

Propagation of belief

Bayesian Networks: Case Study: The Wet Grass Example: Results



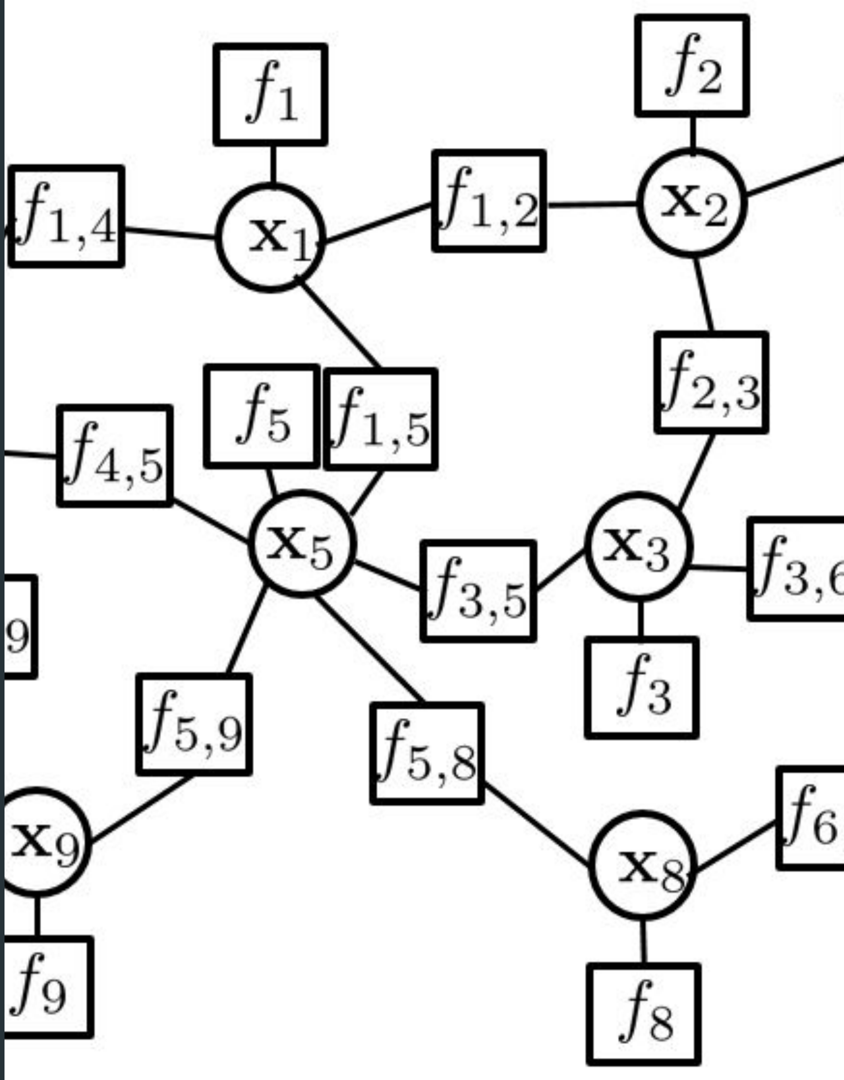
Holmes sees only his grass

- Watson's grass is wet: 0.865
- Holmes' grass is wet: 1
- It has rained: 0.832
- The sprinkler was left on: 0.917

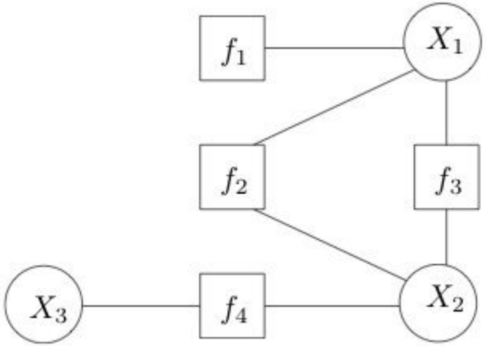
If Holmes also sees his neighbor's wet lawn

- Watson's grass is wet: 1
- Holmes' grass is wet: 1
- It has rained: 0.961
- The sprinkler was left on: 0.904

Factor Graphs



Factor Graphs: Theory



Factor Graph = Bipartite Graph

□ Factor node

○ Variable node

Total likelihood:

$$p(\mathbf{x}) = \prod_{f \in F} \mathbf{f}_f(\mathbf{x}_f)$$

Propagation:

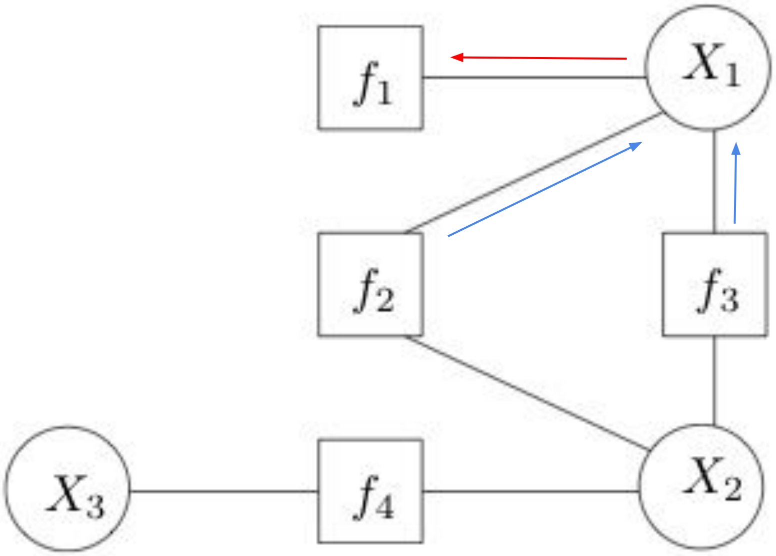
$$\mu_{v \to f}(x_v) = \prod_{f^\star \in N(v) \setminus \{f\}} \mu_{f^\star \to v}(x_v)$$

$$\mu_{f \to v}(x_v) = \sum_{\mathbf{x}'_f : x'_v = x_v} \mathbf{f}_f(\mathbf{x}'_f) \prod_{v^* \in N(f) \setminus \{v\}} \mu_{v^* \to f}(x'_{v^*})$$

Marginals:

$$p_{X_v}(x_v) \propto \prod_{f \in N(v)} \mu_{f \to v}(x_v)$$

Factor Graphs: Example of Propagation: Variable-to-Factor Node



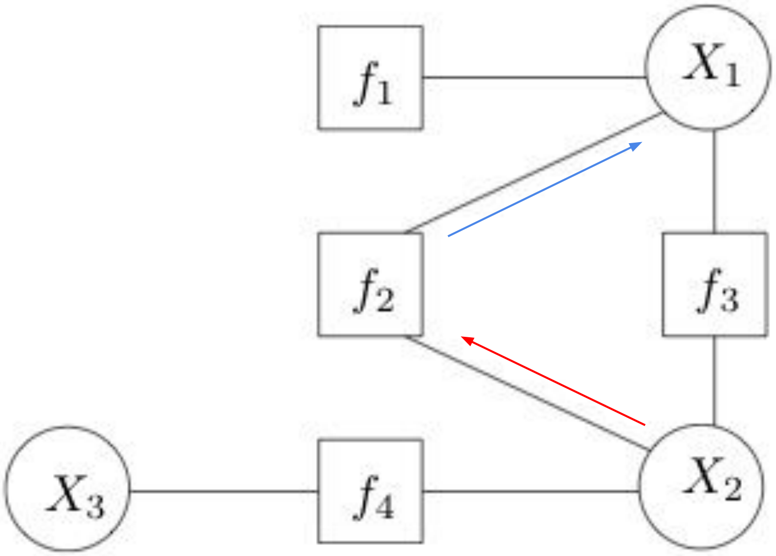
$$\overrightarrow{\mu_{v \to f}}(x_v) = \prod_{f^\star \in N(v) \setminus \{f\}} \overrightarrow{\mu_{f^\star \to v}}(x_v)$$

Messages from variable X to factor nodes f are vectors of size NbStates(X)

Messages from factor f to variable X are also vectors of size NbStates(X)

We just multiply coordinate by coordinate the two incoming messages

Factor Graphs: Example of Propagation: Factor-to-Variable Node



$$\overrightarrow{\mu_{f\to v}}(x_v) = \sum_{\mathbf{x}'_f : x'_v = x_v} \mathrm{f}_f(\mathbf{x}'_f) \prod_{v^* \in N(f) \setminus \{v\}} \overrightarrow{\mu_{v^*\to f}}(x'_{v^*})$$

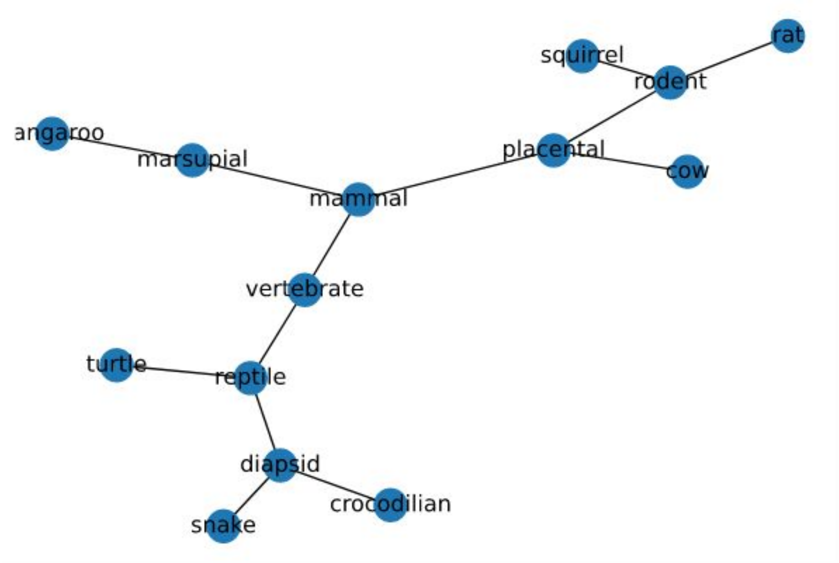
Messages from variable X to factor nodes f are vectors of size NbStates(X)

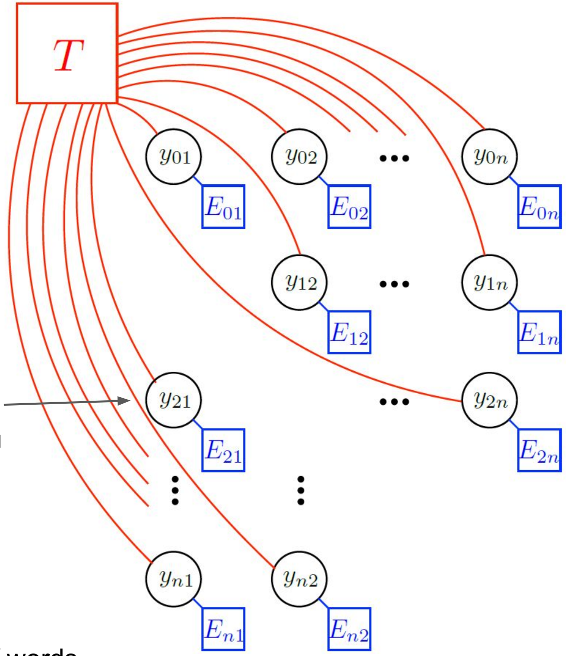
Messages from factor f to variable X are also vectors of size NbStates(X)

We just multiply coordinate by coordinate the incoming messages

Factor Graphs: Taxonomy Induction

From a corpus of abstracts on Wikipedia, and a selection of words (x_1, …, x_n), the objective is to discover the taxonomic structure.





1 if word x_2 is the parent of x_1

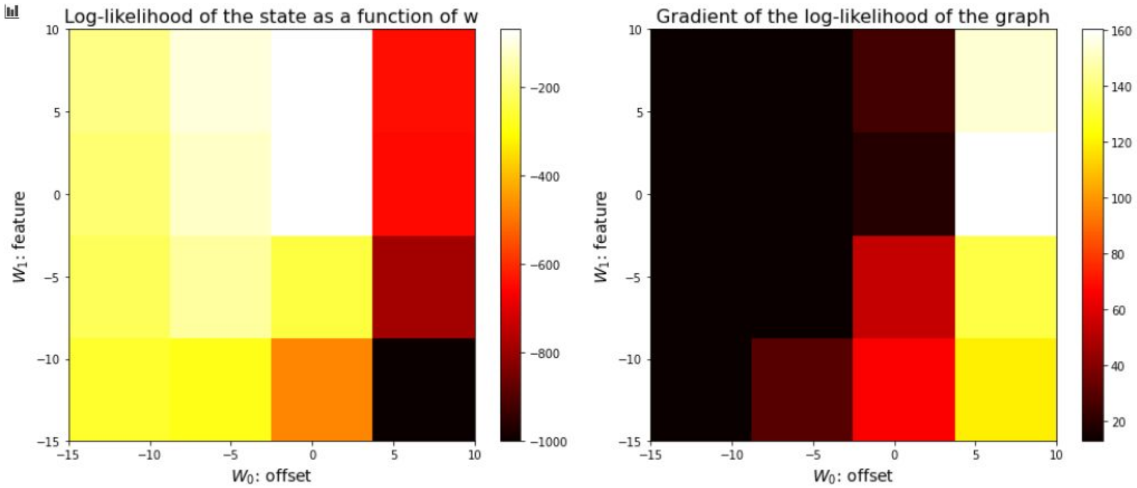
$$P(y|x) \propto \prod_F \phi_F(y)$$

Embedding of the pair of words

$$\boxed{\phi_{E_{i,j}}(y_{i,j})} = \begin{cases} \exp(\mho(x_i, x_j) \cdot w) & \text{if } y_{i,j} = 1 \\ \exp(0) = 1 & \text{if } y_{i,j} = 0 \end{cases}$$

$$\boxed{\phi_T(y_{i,j})} = \begin{cases} 1 & \text{if } y \text{ forms a legal taxonomy tree} \\ 0 & \text{otherwise} \end{cases}$$

Factor Graphs: Taxonomy Induction → Training



- Trained on a taxonomic structure by maximizing its likelihood
- Gradient Descent computed by finite difference
- Trained on a few iterations upon convergence
- Yet, too many edges are drawn

Conclusion

Implemented belief propagation on two types of graphs:

- Bayesian networks → case study with the wet grass example
- Factor graphs → taxonomy induction