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Exercise 1 (LP duality)

1. Problem (P) : $\begin{cases} \min\limits_{x \in \mathbb{R}^d} c^T x \\ \text{s.t.} \left| \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right. \end{cases}$

To compute the dual of problem P, let us derive the following function : $g(\lambda, \mu) = \min\limits_{x \in \mathbb{R}^d} L(x, \lambda, \mu)$

where $L(x, \lambda, \mu) = c^T x + \lambda^T (Ax - b) - \mu^T x$

Let $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}_+^d$.

$L(\cdot, \lambda, \mu)$ is differentiable,

$$\frac{\partial L}{\partial x}(x, \lambda, \mu) = c + A^T \lambda - \mu \quad \text{not a function of } x.$$

Indeed, $L(\cdot, \lambda, \mu)$ is an affine function. It's minimum is $-\infty$ unless $\frac{\partial L}{\partial x}$ is null.

Thus, $g(\lambda, \mu) = \left| \begin{array}{ll} -\infty & \text{if } c + A^T \lambda - \mu \neq 0 \\ -\lambda^T b & \text{if } c + A^T \lambda - \mu = 0 \end{array} \right.$

dual problem : $\max\limits_{\substack{\lambda \in \mathbb{R}^m \\ \mu \in \mathbb{R}_+^d}} g(\lambda, \mu) = \max\limits_{\lambda \in \mathbb{R}^m} b^T(-\lambda)$

s.t. $A^T \lambda + c \geq 0$

Replacing $-\lambda$ by $y = -\lambda$, one obtains :

$$\boxed{\text{dual of problem (P)} : \begin{cases} \max\limits_{y \in \mathbb{R}^m} b^T y \\ \text{s.t. } A^T y \leqslant c \end{cases}}$$

which is exactly problem (D).

2. Problem (D) : $\begin{cases} \max\limits_{y \in \mathbb{R}^m} b^T y \\ \text{s.t. } A^T y \leqslant c \end{cases} \iff \begin{cases} \min\limits_{y \in \mathbb{R}^m} -b^T y \\ \text{s.t. } A^T y - c \leqslant 0 \end{cases}$

The Lagrangian is
$$\begin{aligned} L(y, \mu) &= -b^T y + \mu^T (A^T y - c) \\ &= (A\mu - b)^T y - \mu^T c \end{aligned}$$

Let $g(\mu) = \min\limits_{y \in \mathbb{R}^m} L(y, \mu)$

$$g(\mu) = \begin{vmatrix} -\infty & \text{if } A\mu \neq b \\ -\mu^T c & \text{if } A\mu = b \end{vmatrix}$$

Therefore, the dual of problem (D) is :

$$\max_{\mu \in \mathbb{R}^n_+} g(\lambda) \iff \max_{\substack{\mu \in \mathbb{R}^n \\ A\mu = b \\ \mu \geqslant 0}} -\mu^T c$$

$$\boxed{\text{dual of problem (D)} : \begin{cases} \min\limits_{x \in \mathbb{R}^n} c^T x \\ \text{s.t. } \begin{vmatrix} Ax = b \\ x \geqslant 0 \end{vmatrix} \end{cases}}$$

which is exactly problem (P).

3. Let the following problem be : $\begin{cases} \min\limits_{x,y} & c^T x - b^T y \\ \text{s.t.} & \begin{vmatrix} Ax = b \\ x \geqslant 0 \\ A^T y \leq c \end{vmatrix} \end{cases}$

Let us derive the dual of this problem.

the Lagrangian is :

$$L(x, y, \lambda, \mu_1, \mu_2) = c^T x - b^T y + \lambda^T (Ax - b) - \mu_1^T x + \mu_2^T (A^T y - c)$$
$$= (A^T y + c - \mu_1)^T x + (A\mu_2 - b)^T y - \lambda^T b - \mu_2^T c$$

Thus, $g(\lambda, \mu_1, \mu_2) = \min\limits_{x,y} L(x, y, \lambda, \mu_1, \mu_2)$

$$g(\lambda, \mu_1, \mu_2) = \begin{vmatrix} -\infty & \text{if } A^T y + c - \mu_1 \neq 0 \text{ or } A\mu_2 - b \neq 0 \\ -\lambda^T b - \mu_2^T c & \text{if } \begin{vmatrix} A^T y + c - \mu_1 = 0 \\ A\mu_2 - b = 0 \end{vmatrix} \end{vmatrix}$$

The dual of the problem is therefore :

$$\max_{\substack{\lambda, \mu_1, \mu_2 \\ \mu_1 \geqslant 0 \\ \mu_2 \geqslant 0}} g(\lambda, \mu_1, \mu_2) \iff \begin{array}{ll} \min\limits_{\lambda, \mu_1, \mu_2} & c^T \mu_2 + b^T \lambda \\ \text{s.t.} & \mu_1 = A^T y + c \\ & \mu_1 \geqslant 0 \\ & A\mu_2 = b \\ & \mu_2 \geqslant 0 \end{array}$$

replacing λ by $(-y)$ and μ_2 by x, we obtain the following <u>dual</u> :

$$\boxed{\begin{cases} \min\limits_{x,y} & c^T x - b^T y \\ \text{s.t.} & \begin{vmatrix} Ax = b \\ x \geqslant 0 \\ A^T y \leq c \end{vmatrix} \end{cases}}$$

which is exactly the initial (self-dual) problem.

<u>The problem is self-dual.</u>

4. • Since the conditions $\begin{Bmatrix} Ax = b \\ x \geq 0 \end{Bmatrix}$ and $\{A^T y \leq c\}$ are independant, $\begin{cases} \min\limits_{x,y} & c^T x - b^T y \\ \text{s.t.} & \left| \begin{array}{l} Ax = b \\ x \geq 0 \\ A^T y \leq c \end{array} \right. \end{cases} \Leftrightarrow \begin{cases} \min\limits_{x} & c^T x \\ \text{s.t} & \left| \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right. \end{cases} + \begin{cases} \min\limits_{y} & -b^T y \\ \text{s.t} & A^T y \leq c \end{cases}$

but $\min\limits_{y} -b^T y = \max\limits_{y} b^T y$

Therefore, $\boxed{(\text{self dual}) = (P) + (D)}$

(x^*, y^*) optimal solution of (self-dual) $\Longleftrightarrow$ $\begin{cases} x^* \text{ optimal solution of } (P) \\ y^* \text{ optimal solution of } (D) \end{cases}$

• let p^* be the minimum of problem (P) and d^* be the maximum of problem (D).

Then, $\begin{cases} p^* = c^T x^* \\ d^* = b^T y^* \end{cases}$

$[x^*, y^*]$ is assumed to be the optimal solution of the (self-dual) problem.

Its optimal value is thus $c^T x^* - b^T y^* = p^* - d^*$

Using questions 1 & 2 or the strong duality property of linear programs, one obtains $\underline{p^* = d^*}$.

Therefore, $\boxed{\text{the optimal value of (self-dual) is } 0}$.

Exercise 2 (Regularized Least - Square)

1. $f(x) = \|x\|_1 = \sum_{i=1}^{d} |x_i|$

conjugate of norm L_1 : $f^*(y) = \sup_{x \in \mathbb{R}^d} \{ y^T x - \|x\|_1 \}$

Let $\|\cdot\|_* : y \mapsto \sup_{\|x\|_1 \leq 1} y^T x$ be the dual of the L_1-norm.

$*$ if $\|y\|_* \leq 1$, then $y^T x \leq \|x\|_1$ by definition of the dual norm

Thus, $y^T x - \|x\|_1 \leq 0$ with equality for $x = 0$.

so $f^*(y) = 0$

$*$ if $\|y\|_* > 1$, there exists $x_0 \in \mathbb{R}^n$, $\left| \begin{array}{l} \|x_0\| \leq 1 \\ y^T x_0 > 1 \geq \|x_0\|_1 \end{array} \right.$

$$\forall t > 0, \; f^*(y) \geq y^T(t x_0) - \|t x_0\|_1$$
$$\geq t\,(y^T x_0 - \|x_0\|_1) \xrightarrow[t \to +\infty]{} +\infty$$

Thus $f^*(y) = +\infty$

In conclusion, $\boxed{f^*(y) = \left| \begin{array}{ll} +\infty & \text{if } \|y\|_\infty > 1 \\ 0 & \text{if } \|y\|_\infty \leq 1 \end{array} \right.}$

Since the dual of the L_1-norm is the infinity norm

2. (RLS): $\min_x \|Ax-b\|_2^2 + \|x\|_1$

$$\Leftrightarrow \min_{x,z} \|z\|_2^2 + \|x\|_1 \quad \text{s.t. } z = Ax - b$$

The Lagrangian is : $L(x,z,\lambda) = \|z\|_2^2 + \|x\|_1 + \lambda^T(z - Ax + b)$

Let g be : $g(\lambda) = \min_{x,z} L(x,z,\lambda)$

$$g(\lambda) = \min_{x,z} \left\{ \|x\|_1 - (A^T\lambda)^T x + \|z\|_2^2 + \lambda^T z + \lambda^T b \right\}$$

- $\forall \lambda$, $\min_x \{ \|x\|_1 - (A^T\lambda)^T x \} = -\max_x \left((A^T\lambda)^T x - \|x\|_1 \right)$

$$= -f^*(A^T\lambda) \text{ with } f : y \mapsto \|y\|_1.$$

Thanks to question 1, one can replace with the value of f^*.

$$\min_x \left(\|x\|_1 - (A^T\lambda)^T x \right) = \begin{cases} -\infty & \text{if } \|A^T\lambda\|_\infty > 1 \\ 0 & \text{if } \|A^T\lambda\|_\infty \leq 1 \end{cases}$$

- $\forall \lambda$, $\|z\|_2^2 + \lambda^T z = z^T z + \lambda^T z$

Its minimum is reached for $z = -\frac{\lambda}{2}$

Thus, $\min (\|z\|_2^2 + \lambda^T z) = -\frac{\lambda^T\lambda}{4}$

Therefore, $g(\lambda) = \begin{cases} -\infty & \text{if } \|A^T\lambda\|_\infty > 1 \\ \lambda^T b - \frac{\lambda^T\lambda}{4} & \text{if } \|A^T\lambda\|_\infty \leq 1 \end{cases}$

The dual of (RLS) is : $\boxed{\begin{cases} \max_\lambda b^T\lambda - \frac{\lambda^T\lambda}{4} \\ \text{s.t. } \|A^T\lambda\|_\infty \leq 1 \end{cases}}$

Exercise 3 (Data Separation)

1. (Sep 2) : $\begin{cases} \min\limits_{w,z} \ \frac{1}{n\tau} 1^T z + \frac{1}{2}\|w\|_2^2 \\ \text{s.t.} \left| \begin{array}{l} \forall i \in [\![1,n]\!],\ z_i \geq 1 - y_i(w^T x_i) \\ z \geq 0 \end{array} \right. \end{cases}$

Let us prove that for all w, and with constraints $\begin{cases} z \geq 0 \\ z_i \geq 1 - y_i(w^T x_i) \end{cases}$

$$\min_z \frac{1}{n\tau} 1^T z + \frac{1}{2}\|w\|_2^2 = \frac{1}{n}\sum_{i=1}^{n} \max(0 ; 1 - y_i(w^T x_i)) + \frac{\tau}{2}\|w\|_2^2$$

Thus, solving problem (Sep 2) would solve problem (Sep 1).

Let us solve the problem (P_w) : $\begin{cases} \min\limits_{z} \ \frac{1}{n\tau} 1^T z + \frac{1}{2}\|w\|_2^2 \\ \text{s.t. } z \geq 0 \\ \qquad z_i \geq 1 - y_i(w^T x_i) \ \forall i \in [\![1,n]\!] \end{cases}$

Its Lagrangian G is :

$$G_w(z, \lambda, \mu) = \frac{1}{n\tau} 1^T z + \frac{1}{2}\|w\|_2^2 - \lambda^T z + \sum_{i=1}^{n} \mu_i (1 - y_i(w^T x_i) - z_i)$$

$$= \left(\frac{1}{n\tau} 1 - \lambda - \mu\right)^T z + \frac{1}{2}\|w\|_2^2 + \sum_{i=1}^{n} \mu_i (1 - y_i(w^T x_i))$$

G_w is an affine function of z.

So, $g_w(\lambda, \mu) = \min\limits_z G_w(z, \lambda, \mu)$

$$g_w(\lambda, \mu) = \begin{cases} -\infty & \text{if } \lambda + \mu \neq \frac{1}{n\tau} 1 \\ \frac{1}{2}\|w\|_2^2 + \sum_{i=1}^{n} \mu_i (1 - y_i(w^T x_i)) & \text{if } \lambda + \mu = \frac{1}{n\tau} 1 \end{cases}$$

Problem (P_w) is thus equivalent to

$$\begin{cases} \max\limits_{\lambda, \mu} \ \frac{1}{2}\|w\|_2^2 + \sum_{i=1}^{n} \mu_i (1 - y_i(w^T x_i)) \\ \text{s.t.} \left| \begin{array}{l} \lambda \geq 0 \\ \mu \geq 0 \\ \lambda + \mu = \frac{1}{n\tau} 1 \end{array} \right. \end{cases}$$

$$\Leftrightarrow \begin{cases} \max\limits_{\mu} \frac{1}{2}\|\omega\|_2^2 + \sum_{i=1}^{m} \mu_i (1 - y_i(\omega^T x_i)) \\ \text{s.t. } 0 \leq \mu_i \leq \frac{1}{\tau m} \quad \forall i \in [\![1, n]\!]. \end{cases}$$

The sum is maximized when $\mu_i = \begin{cases} 0 & \text{if } (1 - y_i(\omega^T x_i)) \leq 0 \\ \frac{1}{\tau m} & \text{if } (1 - y_i(\omega^T x_i)) > 0 \end{cases}$

Thus the solution of P_ω is $\frac{1}{2}\|\omega\|_2^2 + \frac{1}{m\tau} \sum_{i=1}^{m} \mathcal{L}(\omega, x_i, y_i)$

$$(\text{Sep}2): \min_{\omega} \begin{pmatrix} \min\limits_{z} \frac{1}{m\tau} 1^T z + \frac{1}{2}\|\omega\|_2^2 \\ \text{st. } z_i \geq 1 - y_i(\omega^T x_i) \\ z \geq 0 \end{pmatrix} = \min_{\omega} \frac{1}{2}\|\omega\|_2^2 + \frac{1}{m\tau} \sum_{i=1}^{m} \mathcal{L}(\omega, x_i, y_i)$$

$$= \frac{1}{\tau}\left[\min_{\omega} \frac{1}{m} \sum_{i=1}^{m} \mathcal{L}(\omega, x_i, y_i) + \frac{\tau}{2}\|\omega\|_2^2\right]$$

$$= \frac{1}{\tau} (\text{Sep } 1).$$

Therefore, $\boxed{\text{problem (Sep 2) solves problem (Sep 1)}}$

2. Let G be the Lagrangian of (Sep 2).

$$G(w, z, \lambda, \pi) = \frac{1}{n\tau} 1^T z + \frac{1}{2} w^T w + \sum \lambda_i (1 - y_i (w^T x_i) - z_i) - \pi^T z$$

$$G(w, z, \lambda, \pi) = \left(\frac{1}{n\tau} 1 - \lambda - \pi\right)^T z + \frac{1}{2} w^T w - \sum_{i=1}^{n} \lambda_i y_i (w^T x_i) + \sum_{i=1}^{n} \lambda_i$$

Let (λ, π) be the function $(\lambda, \pi) = \min_{z, w} G(w, z, \lambda, \pi)$

G is affine in z and quadratic in w.

if $\left(\frac{1}{n\tau} 1 - \lambda - \pi \neq 0\right)$, $g(\lambda, \pi)$ is $-\infty$.

if $\left(\frac{1}{n\tau} 1 - \lambda - \pi = 0\right)$, $g(\lambda, \pi) = \min_{w} \left[\frac{1}{2} w^T w - w^T \left(\sum_{i=1}^{n} \lambda_i y_i x_i\right) + \sum_{i=1}^{n} \lambda_i\right]$

$$= -\frac{1}{2} \left\| \sum_{i=1}^{n} \lambda_i y_i x_i \right\|_2^2 + \sum_{i=1}^{n} \lambda_i$$

So $g(\lambda, \pi) = \begin{cases} -\infty & \text{if } \exists i_0, \lambda_{i_0} + \pi_{i_0} \neq \frac{1}{n\tau} \\ \sum_{i=1}^{n} \lambda_i - \frac{1}{2} \left\| \sum_{i=1}^{n} \lambda_i y_i x_i \right\|_2^2 & \text{if } \frac{1}{n\tau} = \lambda_i + \pi_i \ \forall i \in [\![1, n]\!] \end{cases}$

Thus, the dual of (Sep 2) is:

$$\max_{\lambda, \pi} \sum_{i=1}^{n} \lambda_i - \frac{1}{2} \left\| \sum \lambda_i y_i x_i \right\|_2^2$$

$$\text{s.t. } \lambda \geq 0, \quad \pi \geq 0, \quad \lambda + \pi = \frac{1}{n\tau} 1$$

$$\Longleftrightarrow \boxed{\begin{aligned} &\max_{\lambda} \sum_{i=1}^{n} \lambda_i - \frac{1}{2} \left\| \sum_{i=1}^{n} \lambda_i y_i x_i \right\|_2^2 \\ &\text{s.t. } 0 \leq \lambda_i \leq \frac{1}{n\tau} \quad \forall i \in [\![1, n]\!] \end{aligned}}$$

Exercise 4 : (Robust linear programming)

Show that $\begin{cases} \min\limits_{x} \; c^T x \\ \text{s.t.} \;\; \sup\limits_{a \in P} a^T x \leq b \end{cases} \Leftrightarrow \begin{cases} \min \; c^T x \\ \text{s.t.} \left| \begin{array}{l} d^T z \leq b \\ C \, z = x \\ z \geq 0 \end{array} \right. \end{cases}$ (no transpose)

with $P = \{ a \mid C^T a \preceq d \}$

Let us first find the dual of $\begin{cases} \max\limits_{a} \; a^T x \\ \text{s.t.} \;\; a \in P \end{cases} \Leftrightarrow \begin{cases} \min\limits_{a} \; -a^T x \\ \text{s.t.} \;\; C^T a \preceq d \end{cases}$

Its Lagrangian is $\mathcal{L}(a, \lambda) = -a^T x + \lambda^T (C^T a - d)$

$$\Leftrightarrow \mathcal{L}(a, \lambda) = (-x + C\lambda)^T a - \lambda^T d$$

thus, $g(\lambda) = \min\limits_{a} \mathcal{L}(a, \lambda) = \left| \begin{array}{ll} -\infty & \text{if } C\lambda \neq x \\ -\lambda^T d & \text{if } C\lambda = x \end{array} \right.$

The dual is $\begin{cases} \max\limits_{\lambda} \; -\lambda^T d \\ \text{s.t.} \left| \begin{array}{l} \lambda \geq 0 \\ C\lambda = x \end{array} \right. \end{cases} \Leftrightarrow \begin{cases} \min\limits_{\lambda} \; \lambda^T d \\ \text{s.t.} \left| \begin{array}{l} \lambda \geq 0 \\ C\lambda = x \end{array} \right. \end{cases}$

Therefore the initial problem is equivalent to :

$$\begin{cases} \min\limits_{x} \; c^T x \\ \text{s.t.} \left| \begin{array}{l} \min \; z^T d \leq b \\ \quad \text{s.t.} \left| \begin{array}{l} z \geq 0 \\ Cz = x \end{array} \right. \end{array} \right. \end{cases} \Leftrightarrow \boxed{\begin{cases} \min\limits_{x} \; c^T x \\ \text{s.t.} \left| \begin{array}{l} d^T z \leq b \\ z \geq 0 \\ Cz = x \end{array} \right. \end{cases}}$$

Excercise 5 : (Boolean LP)

1. Lagrangian relaxation.

$$\begin{cases} \min\limits_x \ c^T x \\ \text{s.t.} \ \left| \begin{array}{l} Ax \leq b \\ x_i(1-x_i) = 0 \quad \forall i \in [\![1,n]\!] \end{array} \right. \end{cases}$$

Lagrangian : $\mathcal{L}(x, \lambda, \mu) = c^T x + \lambda^T (Ax - b) + \sum_{i=1}^{n} \mu_i x_i (1-x_i)$

$$\mathcal{L}(x, \lambda, \mu) = (A^T\lambda + c + \mu)^T x - x^T D_\mu x - \lambda^T b$$

with $D_\mu = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix}$

$\mathcal{L}$ is a quadratic function of x. It is minimum for $\bar{x} = \frac{1}{2} D_\mu^{-1}(A^T\lambda + c + \mu)$ if $\mu_i \neq 0 \ \forall i \in [\![1,n]\!]$.

The minimum is then $\frac{1}{4}(A^T\lambda + c + \mu)^T D_\mu^{-1}(A^T\lambda + c + \mu) - \lambda^T b$

⋮