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2.12. (a) $X = \{ x \in \mathbb{R}^n \mid \alpha \leq a^T x \leq \beta \}$ for $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$, $a \in \mathbb{R}^n$

Let $x \in X$, $y \in X$ and $t \in [0;1]$.

$$a^T(tx + (1-t)y) = t\,a^T x + (1-t)\,a^T y$$

since $x \in X$, $t\alpha \leq t a^T x \leq t\beta$

as well, $y \in X$, $(1-t)\alpha \leq (1-t)a^T y \leq (1-t)\beta$

Thus, $\alpha \leq a^T(tx + (1-t)y) \leq \beta$

In conclusion, $tx + (1-t)y \in X$. $\boxed{X \text{ is convex}}$

2.12 (b) $X = \{ x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, \ i = 1 \dots n \}$

A rectangle is an intersection of n slabs of the form $X_i = \{ x \in \mathbb{R}^n \mid \alpha_i \leq a_i^T x \leq \beta_i \}$ with $a_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ position i.

$X_i = \{ x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i \}$ convex

Thus we have $X = \bigcap_{i=1}^{n} X_i$ <u>is convex</u> since a finite intersection of convex is convex.

2.12.(c) $X = \{x \in \mathbb{R}^n \mid a_1^T x \leq b_1 ,\ a_2^T x \leq b_2\}$

let $x \in X$, $y \in X$ and $t \in [0;1]$.

$$a_1^T(tx + (1-t)y) \leq t b_1 + (1-t) b_1$$
$$\leq b_1$$

same for $a_2^T(tx + (1-t)y) \leq b_2$

Thus $tx + (1-t)y \in X$

X is connex.

2.12(d) $X = \{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$

For fixed $y = y_0 \in S$, $\|x - x_0\|_2 \leq \|x - y_0\|_2$

$$\iff (x - x_0)^T(x - x_0) \leq (x - y_0)^T(x - y_0)$$
$$\iff 2(y_0 - x_0)^T x \leq y_0^T y_0 - x_0^T x_0$$
$$\iff \alpha^T x \leq b \quad \text{with } \begin{vmatrix} \alpha = 2(y_0 - x_0) \\ b = y_0^T y_0 - x_0^T x_0 \end{vmatrix}$$

Hence, for fixed y_0, $\{x \mid \|x - x_0\|_2 \leq \|x - y_0\|_2\}$ is a halfspace and it is thus convex.

$X = \bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$ is an intersection of convex sets.

Hence $\boxed{X \text{ is convex}}$.

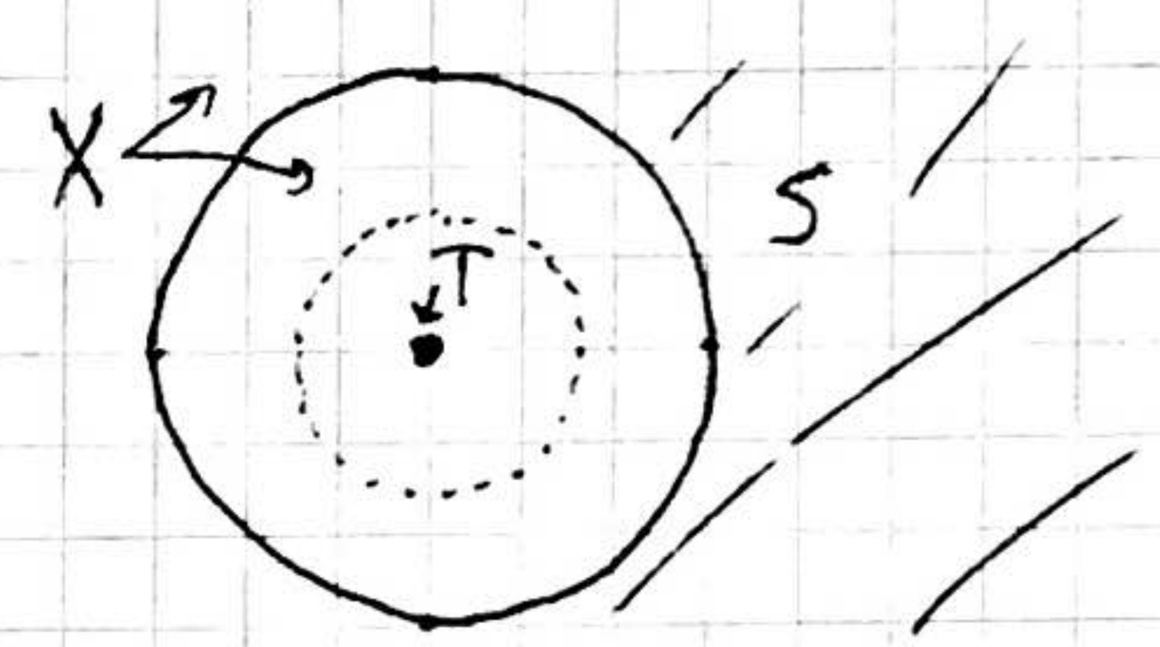
2.12. (e) $X = \{ x \in \mathbb{R}^n \mid dist(x, S) \leq dist(x, T) \}$

where $dist(x, S) = \inf \{ \|x - s\|_2 \mid s \in S \}$

X is not convex in general. A counter example can be found if S is not convex.

For instance let $T = \{0\}$ and $S = \mathbb{R}^n \setminus B(0,1)$

Then $X = B(0; \frac{1}{2})^c = \mathbb{R}^n \setminus B(0; \frac{1}{2})$



X is clearly not convex.

$X = \{ x \in \mathbb{R}^n \mid \|x\| \geq \frac{1}{2} \}$

for $x \in X$ and $y = -x \in X$, $t = \frac{1}{2}$,

$tx + (1-t)y = \frac{1}{2}x - \frac{1}{2}x = 0 \notin X$.

Hence, $\boxed{\text{X is not convex.}}$

2.12. (f) $X = \{ x \in \mathbb{R}^n \mid x + S_2 \subseteq S_1 \}$, S_1 convex.

$$X = \bigcap_{y \in S_2} \{ x \in \mathbb{R}^n \mid x + y \in S_1 \} = \bigcap_{y \in S_2} \Lambda(y)$$

with $\Lambda(y) = \{ x \in \mathbb{R}^n \mid x + y \in S_1 \}$

Let's show that for fixed y, $\Lambda(y)$ is convex.

if $(x_1, x_2) \in \Lambda(y)^2$ and $t \in [0; 1]$,

$tx_1 + (1-t)x_2 + y = t\underbrace{(x_1 + y)}_{\in S_1} + (1-t)\underbrace{(x_2 + y)}_{\in S_1} \in S_1$ since S_1 is convex.

Thus $\Lambda(y)$ is convex and X is an intersection of convex sets.

Hence, $\boxed{\text{X is convex}}$

PS: $\Lambda(y)$ is convex because it is an affine transformation of S_1 convex.

2.12. (g) $X = \{ x \in \mathbb{R}^n \mid \|x-a\|_2 \leq \theta \|x-b\|_2 \}$ with $a \neq b$ and $\theta \in [0,1]$

$x \in X \Leftrightarrow \|x-a\|_2^2 \leq \theta^2 \|x-b\|_2^2$

$\Leftrightarrow (1-\theta^2) x^T x + 2(\theta^2 b - a)^T x + a^T a - \theta^2 b^T b \leq 0$

$\Leftrightarrow f(x) \leq 0$

with f $\left| \begin{array}{l} \text{a quadratic function if } \theta \neq 1. \\ \text{an affine function if } \theta = 1. \end{array} \right.$

* if $\theta = 1$, f is affine and X is a halfspace, thus it is convex.

* if $\theta < 1$, let us show X is a ball.

$$x \in X \Leftrightarrow x^T x + \frac{2(\theta^2 b - a)^T}{1-\theta^2} x + \frac{(a^T a - \theta^2 b^T b)}{1-\theta^2} \leq 0$$

$$\Leftrightarrow \left(x + \frac{\theta^2 b - a}{1-\theta^2}\right)^T \left(x + \frac{\theta^2 b - a}{1-\theta^2}\right) \leq \frac{(\theta^2 b - a)^T(\theta^2 b - a)}{(1-\theta^2)^2} + \frac{\theta^2 b^T b - a^T a}{1-\theta^2}$$

$$\Leftrightarrow \left\| x - \frac{a - \theta^2 b}{1-\theta^2} \right\|_2^2 \leq R^2$$

$$\text{with } R = \sqrt{\frac{\theta^2 \|b\|_2^2 - \|a\|_2^2}{1-\theta^2} + \frac{\|\theta^2 b - a\|_2^2}{(1-\theta^2)^2}}$$

thus $X = B\left(\frac{a - \theta^2 b}{1-\theta^2}, R\right)$

$\boxed{X \text{ is convex.}}$

3.21. (a) $f(x) = \max_{i=1\dots k} \|A^{(i)}x - b^{(i)}\|$, $A^{(i)} \in \mathbb{R}^{m\times n}$, $b^{(i)} \in \mathbb{R}^m$

let $f_i(x) = \|A^{(i)}x - b^{(i)}\|$.

f_i is convex since it is the transformation of a norm of an affine function.

f is thus the pointwise maximum of k convex functions f_i.

Hence, $\boxed{f \text{ is convex}}$.

3.21. (b) $f(x) = \sum_{i=1}^{r} |x|_{[i]}$, $|x|_{[i]}$ is the ith largest component of $|x|$.

$$f(x) = \max_{1\leq i_1 < \dots < i_r \leq n} \{|x_{i_1}| + \dots + |x_{i_r}|\}$$

$$f(x) = \max_{1\leq i_1 < \dots < i_r \leq n} f_{i_1,\dots,i_r}(x) \quad \text{with } f_{i_1,\dots,i_r}(x) = |x_{i_1}| + \dots + |x_{i_r}|$$

$f_{i_1,\dots,i_r}$ is convex, thus f is the pointwise maximum of $\binom{n}{r}$ convex functions $f_{i_1,\dots,i_r}$.

Hence, $\boxed{f \text{ is convex}}$

3.32. (a) f, g convex, nondecreasing, positive.

Let $(x,y) \in \mathbb{R}^2$ and $t \in [0;1]$.

$$(fg)(tx+(1-t)y) = f(tx+(1-t)y)\, g(tx+(1-t)y)$$
$$\leq [t f(x) + (1-t) f(y)][t g(x) + (1-t) g(y)]$$
$$= t^2 f(x)g(x) + (1-t)^2 f(y)g(y) + t(1-t)[f(x)g(y) + f(y)g(x)]$$
$$= [t - t(1-t)] f(x)g(x) + (1-t)(1-t) f(y)g(y) + t(1-t)[f(x)g(y) + f(y)g(x)]$$
$$= t f(x)g(x) + (1-t) f(y)g(y) + t(1-t)\Big[\underbrace{f(x)g(y) + f(y)g(x) - f(x)g(x) - f(y)g(y)}_{f(x)[g(y)-g(x)] + f(y)[g(x)-g(y)]}\Big]$$
$$= t(fg)(x) + (1-t)(fg)(y) + t(1-t)\underbrace{[f(x)-f(y)]}_{\leq 0}\underbrace{[g(y)-g(x)]}_{\geq 0}$$

since f and g non decreasing, $f(x) \leq f(y)$ and $g(x) \leq g(y)$,

hence, $(fg)(tx+(1-t)y) \leq t(fg)(x) + (1-t)(fg)(y)$

$\boxed{fg \text{ is convexe.}}$

3.32. (b) f, g concave, positive, g nondecreasing, f nonincreasing

Let $(x, y) \in \mathbb{R}^2$ and $t \in [0;1]$.

As we did in (a),

$$fg(tx + (1-t)y) \geq t\,(fg)(x) + (1-t)(fg)(y) + t(1-t)\underbrace{[f(x) - f(y)]}_{\geq 0}\underbrace{[g(y) - g(x)]}_{\geq 0}$$

$$\geq t\,(fg)(x) + (1-t)(fg)(y)$$

Hence $\boxed{fg \text{ is concave.}}$

3.32. c) f convex, nondecreasing, positive.

g concave, nonincreasing, positive.

Let $t \in [0;1]$ and $x < y$

The goal is to show that $h(t,x,y) = t\left(\frac{f}{g}\right)(x) + (1-t)\left(\frac{f}{g}\right)(y) - \left(\frac{f}{g}\right)(tx+(1-t)y)$ is positive. Hence $\frac{f}{g}$ will be convexe.

Since f is convexe, $f(tx+(1-t)y) \leq t f(x) + (1-t) f(y)$

Since g is concave, $\dfrac{1}{g(tx+(1-t)y)} \leq \dfrac{1}{t g(x) + (1-t) g(y)}$

Thus, $$h(t,x,y) \geq t\,\frac{f(x)}{g(x)} + (1-t)\,\frac{f(y)}{g(y)} - \frac{t f(x) + (1-t) f(y)}{t g(x) + (1-t) g(y)}$$

$$= \frac{1}{g(x)g(y)}\left[t f(x) + (1-t) f(y) - \frac{t f(x) g(x) g(y) + (1-t) f(y) g(x) g(y)}{t g(x) + (1-t) g(y)} \right]$$

$$= \frac{1/g(x)g(y)}{t g(x) + (1-t) g(y)}\Big[t^2 f(x)g(x)g(y) + t(1-t) f(x) g(y)^2 + t(1-t) f(y) g(x)^2 + (1-t)^2 f(y) g(x) g(y) - t f(x) g(x) g(y) - (1-t) f(y) g(x) g(y) \Big]$$

$$= \alpha\left[-f(x)g(x)g(y) + f(x) g(y)^2 + f(y) g(x)^2 - f(y) g(x) g(y) \right]$$

with $\alpha = \dfrac{t(1-t)}{g(x)g(y)\left(t g(x) + (1-t) g(y)\right)} \geq 0$ since g is positive

$$= \alpha \underbrace{\left(g(x) - g(y)\right)}_{\substack{\geq 0 \text{ because} \\ g \text{ nonincreasing}}} \underbrace{\left(f(y) g(x) - f(x) g(y)\right)}_{\substack{\geq 0 \text{ because } f \text{ nondecreasing, } f \text{ positive,} \\ g \text{ nonincreasing, } g \text{ positive.}}} \geq 0$$

Indeed, $\begin{matrix} 0 \leq f(x) \leq f(y) \\ 0 \leq g(y) \leq g(x) \end{matrix} \Rightarrow f(y) g(x) \geq f(x) g(y)$

Hence, $h(t,x,y) \geq 0$ and $\left(\frac{f}{g}\right)(tx + (1-t)y) \leq t\left(\frac{f}{g}\right)(x) + (1-t)\left(\frac{f}{g}\right)(y)$

$\boxed{\frac{f}{g} \text{ is convexe.}}$

3.36. (a) Max function: $f(x) = \max_{i=1...n} x_i$ on $\mathbb{R}^n$.

$f^* : y \mapsto \sup_{x \in \mathbb{R}^n} g(y,x)$ with $g(y,x) = y^T x - f(x)$

$g(y,x) = \sum_{i=1}^{m} x_i y_i - \max_{i \in [\![1,m]\!]} x_i$

* if there exists k so that $y_k < 0$,
then for fixed x such that $\begin{cases} x_k = -t \\ x_{i \neq k} = 0 \end{cases}$, $t > 0$,

$$f^*(y) \geq \sum x_i y_i - \max x_i$$
$$\geq -t y_k \xrightarrow[t \to +\infty]{} +\infty \quad \text{since } y_k < 0$$

Thus, $f^*(y) = +\infty$.

* if $y \succeq 0$, which means $y_k \geq 0$ for all $k \in [\![1,n]\!]$,
then for x fixed so that $x_k = t \in \mathbb{R}$ for all k,

$$f^*(y) \geq \sum x_i y_i - \max x_i$$
$$\geq t \sum y_i - t$$
$$\geq t \left(\sum y_i - 1 \right)$$

if $\sum y_i < 1$, with $t < 0$, $t(\sum y_i - 1) \xrightarrow[t \to -\infty]{} +\infty$

if $\sum y_i > 1$, with $t > 0$, $t(\sum y_i - 1) \xrightarrow[t \to +\infty]{} +\infty$

thus, if $y \succeq 0$ and $\sum y_i \neq 1$, $f^*(y) = +\infty$.

if $\sum y_i = 1$, then for all $x \in \mathbb{R}^n$, $\sum x_i y_i - \max x_i \leq \max x_i (\sum y_i - 1)$
≤ 0

thus $f^*(y) \leq 0$.

And $f^*(y) \geq \sum 0 \times y_i - \max_{i=1..m} 0$ for $x = 0$

$f^*(y) \geq 0$. Hence, $f^*(y) = 0$.

In conclusion, $\boxed{f^*(y) = \begin{cases} 0 & \text{if } \sum_{i=1}^{m} y_i = 1 \text{ and } y \succeq 0 \\ +\infty & \text{otherwise} \end{cases}}$

3.36. (b) : sum of largest elements: $f(x) = \sum_{i=1}^{r} x_{[i]}$ on $\mathbb{R}^n$

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \left(\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{r} x_{[i]} \right)$$

* if $r = n$, $f^*(y) = \sup \left(\sum_{i=1}^{n} x_i (y_i - 1) \right) = \sup \left((y-1)^T x \right)$

$$f^*(y) = \begin{cases} +\infty & \text{if } y \neq 1 \\ 0 & \text{if } y = 1 \Leftrightarrow \forall i \in [\![1,n]\!],\ y_i = 1 \end{cases}$$

* if $r < n$,
  * if there exists k, $y_k < 0$, for fixed x so that $\begin{cases} x_k = -t < 0 \\ x_{i \neq k} = 0 \end{cases}$,

    $f^*(y) \geq \sum x_i y_i - 0$

    $\geq -t y_k \xrightarrow[t \to +\infty]{} +\infty$ because $y_k < 0$.

    Thus $f^*(y) = +\infty$.

  * if $y_k \geq 0$ for all $k \in [\![1,n]\!]$, for fixed $x = t1 = \begin{pmatrix} t \\ \vdots \\ t \end{pmatrix}$, $t \in \mathbb{R}$,

    $f^*(y) \geq \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{r} x_{[i]}$

    $\geq t \left(\sum_{i=1}^{n} y_i - r \right) \xrightarrow[t \to +\infty]{} +\infty$ if $\sum y_i > r$ $\quad f^*(y) = +\infty$

    $\xrightarrow[t \to -\infty]{} +\infty$ if $\sum y_i < r$

  * if $y \succeq 0$ and $\sum_{i=1}^{n} y_i = r$,

    if there exists k, $y_k > 1$, for x so that $\begin{cases} x_k = t > 0 \\ x_{i \neq k} = 0 \end{cases}$,

    $f^*(y) \geq \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{r} x_{[i]} \geq t y_k - t$

    $\geq t(y_k - 1) \xrightarrow[t \to +\infty]{} +\infty$ thus, $f^*(y) = \infty$.

  * if $y \succeq 0$, $y \preceq 1$ and $\sum_{i=1}^{n} y_i = r$,

    $\forall x \in \mathbb{R}^n$, $\sum_{i=1}^{n} y_i x_i \leq \sum_{i=1}^{r} x_{[i]}$, thus, $f^*(y) \leq 0$.

    for $x = 0$, $f^*(y) \geq \sum 0 \times y_i - \sum 0 \geq 0$, so $f^*(y) = 0$

In conclusion, $\boxed{f^*(y) = \begin{cases} 0 & \text{if } 0 \preceq y \preceq 1 \text{ and } \sum_{i=1}^{n} y_i = r \\ +\infty & \text{otherwise} \end{cases}}$

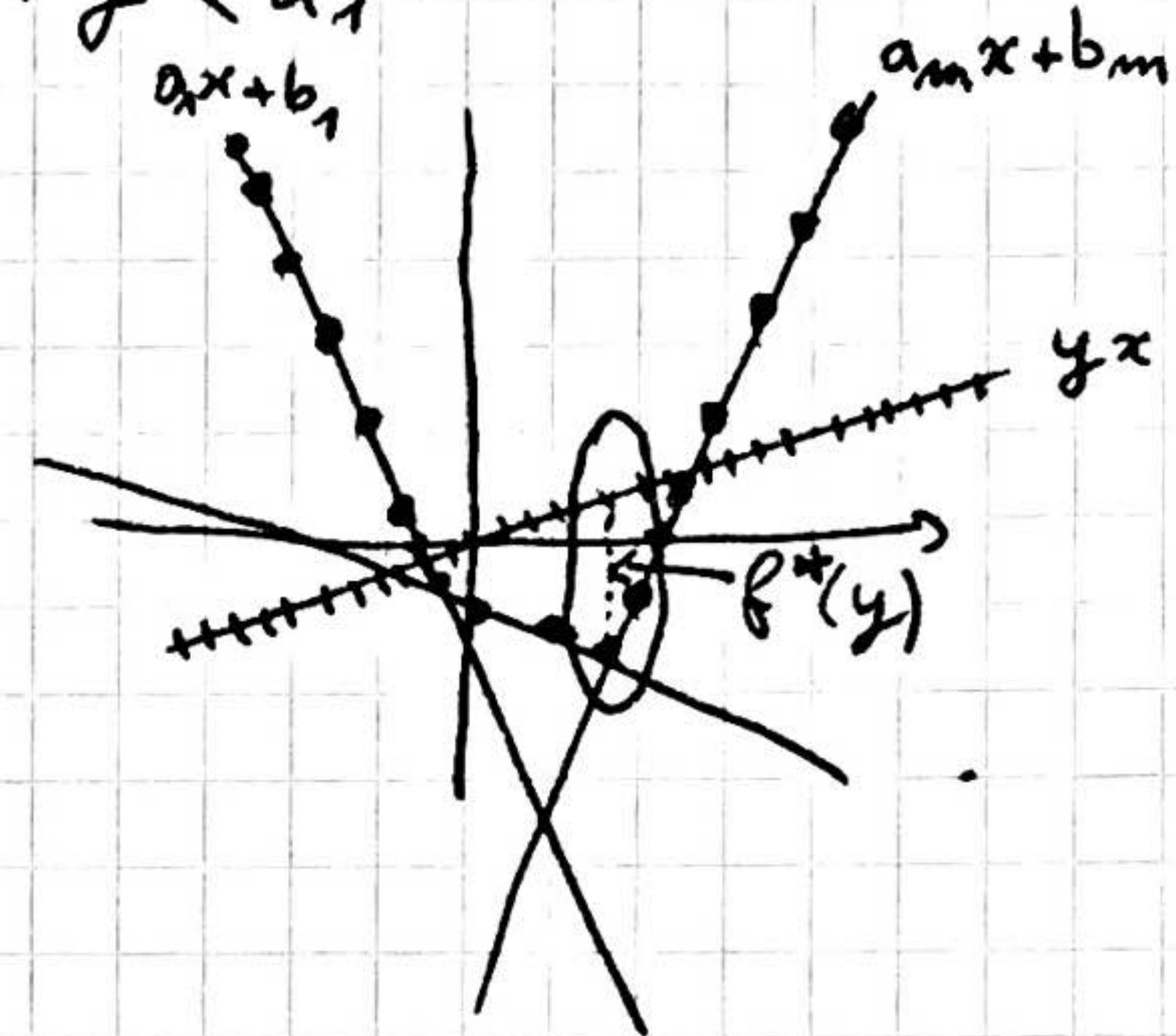
works also for $r = n$.

3.36 (c) $f(x) = \max_{i=1\dots m} (a_i x + b_i)$ on $\mathbb{R}$. $a_1 \leq \dots \leq a_m$.

$$f^*(y) = \sup_{x \in \mathbb{R}} \left\{ yx - \max_{i \in [\![1,m]\!]} (a_i x + b_i) \right\}$$

* if $y > a_m$, $yx - \max_{i \in [\![1,m]\!]}(a_i x + b) \underset{+\infty}{\sim} (y - a_m)x \xrightarrow[x \to +\infty]{} +\infty$
* if $y < a_1$, $yx - \max(a_i x + b) \underset{-\infty}{\sim} (y - a_1)x \xrightarrow[x \to -\infty]{} +\infty$

Thus, $f^*(y) = +\infty$ for $y > a_m$ or $y < a_1$.



* if $y \in [a_1 ; a_m]$,

let i be such that $a_i \leq y \leq a_{i+1}$,

$f^*(y)$ is reached for x_0 such that $a_i x_0 + b_i = a_{i+1} x_0 + b_{i+1}$

$a_{i+1} \neq a_i$, so $x_0 = \dfrac{b_i - b_{i+1}}{a_{i+1} - a_i}$

$$f^*(y) = y x_0 - (a_i x_0 + b_i)$$
$$= (y - a_i) x_0 - b_i$$
$$f^*(y) = \left(\frac{b_i - b_{i+1}}{a_{i+1} - a_i}\right)(y - a_i) - b_i$$

$$\boxed{f^*(y) = \begin{cases} +\infty & \text{if } y < a_1 \text{ or } y > a_m \\ \left(\dfrac{b_i - b_{i+1}}{a_{i+1} - a_i}\right)(y - a_i) - b_i & \text{if } y \in [a_1 ; a_m] \end{cases}}$$

with i such that $a_i \leq y \leq a_{i+1}$

3.36. (d) $f(x) = x^p$ on $\mathbb{R}^{++}$ where $p > 1$.

* For $y \in \mathbb{R}$, let $g : x \longmapsto yx - x^p$

g is differentiable and $g'(x) = y - p x^{p-1}$

g is concave on $\mathbb{R}^{++}$, so a necessary condition for its supremum is $g'(\bar{x}) = 0$.

$$g'(\bar{x}) = 0 \iff \bar{x}^{p-1} = \frac{y}{p}$$

$$\iff \bar{x} = \left(\frac{y}{p}\right)^{\frac{1}{p-1}}$$

for $y \geq 0$,

thus, g is maximum in $\bar{x} = \left(\frac{y}{p}\right)^{\frac{1}{p-1}}$

So, $f^*(y) = \sup_{x \in \mathbb{R}^{++}} \{yx - x^p\} = \sup_{x \in \mathbb{R}^{++}} g(x) = g(\bar{x})$

$$f^*(y) = y\bar{x} - \bar{x}^p = (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}}$$

if $y < 0$, $\sup g(x) = 0$

$$\boxed{\text{In conclusion, for } p > 0 \quad f^*(y) = \begin{cases} 0 & \text{if } y < 0 \\ (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}} & \text{if } y \geq 0 \end{cases}}$$

* if $p < 0$,

then the behavior of g depends on the value of y.

if $y > 0$, $g(x) = yx - \frac{1}{x^{-p}} \xrightarrow[x \to +\infty]{} +\infty$

Thus $f^*(y) = +\infty$.

if $y \leq 0$, $\frac{y}{p} \geq 0$ and the same calculation holds

$$\bar{x} = \left(\frac{y}{p}\right)^{\frac{1}{p-1}} \quad \text{and} \quad f^*(y) = y\left(\frac{y}{p}\right)^{\frac{1}{p-1}}\left(1 - \frac{1}{p}\right) = (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}}$$

Therefore,

$$\boxed{f^*(y) = \begin{cases} +\infty & \text{if } y > 0 \\ (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}} & \text{if } y \leq 0 \end{cases}}$$

3.36.(e) $f(x) = -\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$ on $\mathbb{R}^n_{++}$

$$f^*(y) = \max_{x \in \mathbb{R}^n_{++}} g(x) \quad \text{where } g : x \longmapsto y^T x + \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$$

Let's distinguish 2 cases depending on y. Let y be a real.

* if there exists $i_0 \in [\![1, n]\!]$, such that $y_{i_0} > 0$,
then let's choose $x_t = t\,\mathbb{1} + (\alpha - 1) e_{i_0}$ with $\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and $e_{i_0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$,

$x_t = \begin{pmatrix} t \\ \vdots \\ t \\ \alpha t \\ t \\ \vdots \\ t \end{pmatrix}$ for $t > 0$ and α to be chosen shortly.

$x_t \in \mathbb{R}^n_{++}$, so $f^*(y) \geqslant g(x_t)$

$$= (\alpha - 1) y_{i_0} t + t \sum_{i=1}^{n} y_i + t\,\alpha^{\frac{1}{n}}$$

$$= t\left[(\alpha - 1) y_{i_0} + \sum_{i=1}^{n} y_i + \alpha^{\frac{1}{n}}\right]$$

if one chooses α such that $(\alpha - 1) y_{i_0} + \sum_{i=1}^{n} y_i + \alpha^{\frac{1}{n}} > 0$,
then $g(x_t) \xrightarrow[t \to +\infty]{} +\infty$.

Indeed, if we choose $\alpha = 2 + \dfrac{\left|\sum_{i=1}^{n} y_i\right|}{y_{i_0}}$, $\alpha > 0$.

Then $\alpha > -\dfrac{\sum y_i}{y_{i_0}} + 1 > -\dfrac{\sum y_i + \alpha^{\frac{1}{n}}}{y_{i_0}} + 1$

And $(\alpha - 1) y_{i_0} + \sum_{i=1}^{n} y_i + \alpha^{\frac{1}{n}} > 0$

Thus g is unbounded and $f^*(y) = +\infty$.

* if $y_i \leq 0$ for all $i \in [\![1, n]\!]$.

* if $\left(\prod_{i=1}^{n} (-y_i)\right)^{\frac{1}{n}} < \frac{1}{n}$, then for x, $x_i = \begin{cases} -\dfrac{t}{y_i}, & t \in \mathbb{R}, \\ 0 \text{ if } y_i = 0 \end{cases}$

$$f^*(y) \geqslant -nt + t\left(\prod_{i=1}^{n}\left(-\frac{1}{y_i}\right)\right)^{\frac{1}{n}}$$

$$\geqslant t\left[\underbrace{\left(\prod\left(-\frac{1}{y_i}\right)\right)^{\frac{1}{n}} - n}_{\geqslant 0 \text{ since } \left(\prod(-y_i)\right)^{\frac{1}{n}} < \frac{1}{n}}\right] \xrightarrow[t \to +\infty]{} +\infty$$

* if $y \preceq 0$ and $\left(\prod_{i=1}^{m}(-y_i)\right)^{\frac{1}{m}} \geq \frac{1}{m}$,

one can use the mean inequality $\left(\frac{x+y}{2} \geq \sqrt{xy}\right)$,

$$\frac{1}{m}\sum_{i=1}^{m} x_i \geq \left(\prod_{i=1}^{m}(x_i)\right)^{\frac{1}{m}}$$

so for $-x_i y_i > 0$, $\frac{1}{m}\sum_{i=1}^{m} -x_i y_i \geq \left(\prod_{i=1}^{m}(-x_i y_i)\right)^{\frac{1}{m}}$

$$\geq \left(\prod(x_i)\right)^{\frac{1}{m}} \cdot \left(\prod(-y_i)\right)^{\frac{1}{m}}$$

$$\geq \frac{1}{m}\left(\prod(x_i)\right)^{\frac{1}{m}}$$

$$\geq \frac{1}{m} f(x) \iff -f(x) \geq \sum x_i y_i$$

Thus, $\sum x_i y_i - f(x) \leq 0$ for all $x \in \mathbb{R}^n_{++}$.

Hence, $f^*(y) \leq 0$

For $x_i = -\frac{1}{y_i}$, $f^*(y) \geq \sum_{i=1}^{m} x_i y_i + \prod_{i=1}^{m}(x_i)^{\frac{1}{m}}$

$$\geq -m + \left(\prod\left(-\frac{1}{y_i}\right)\right)^{\frac{1}{m}}$$

$$\geq 0 \quad \text{because } \left(\prod\left(-\frac{1}{y_i}\right)\right)^{\frac{1}{m}} \leq m$$

So $f^*(y) = 0$

In conclusion, $\boxed{f^*(y) = \begin{cases} 0 & \text{if } y \preceq 0 \text{ and } \left(\prod_{i=1}^{m}(-y_i)\right)^{\frac{1}{m}} \geq \frac{1}{m} \\ +\infty & \text{otherwise} \end{cases}}$

3.32. (f) $\quad f(x,t) = -\log(t^2 - x^Tx)$ on $\{(x,t) \mid \|x\|_2 < t\}$

$$f^*(y,\lambda) = \sup_{(x,t)\in \mathrm{dom} f} \left\{ y^Tx + \lambda t + \log(t^2 - x^Tx) \right\}$$

Let $(y,\lambda) \in \mathbb{R}^n \times \mathbb{R}$.

$*$ if $\|y\|_2 \geq -\lambda$,

for fixed $x = \alpha y$ and fixed t, ~~so~~ that $\|x\| = t - \alpha$, $t = \alpha(1 + \|y\|_2)$, $\alpha > 0$

$$f^*(y,\lambda) \geq y^Tx + \lambda t + \log(t^2 - x^Tx)$$

$$= \cancel{\alpha\|y\|_2^2 + \lambda\alpha + \lambda\alpha\|y\|_2 + \log(\alpha^2 + \alpha^2\|y\|_2^2 + 2\alpha^2\|y\|_2 - \alpha^2\|y\|_2^2)}$$

$$\cancel{= \alpha(\|y\|_2^2 + \lambda(1 + \|y\|_2)) + 2\log\alpha + \log(1 + 2\|y\|_2)}$$

$$t = \|x\| + \alpha = \alpha\|y\| + \alpha > \alpha\|y\|$$

thus, $$f^*(y,\lambda) \geq \alpha\|y\|_2^2 + \alpha\lambda\|y\|_2 + \log(\|x\|^2 + \alpha^2 + 2\alpha\|x\| - \|x\|^2)$$

$$\geq \alpha\|y\|_2 \underbrace{\left[\|y\|_2 + \lambda\right]}_{\geq 0} + 2\log(\alpha) + \log(1 + \|y\|_2)$$

$$\xrightarrow[\alpha \to +\infty]{} +\infty$$

Thus, $f^*(y,\lambda) = +\infty$

$*$ if $\|y\|_2 < -\lambda$, $\quad g(x,t) = y^Tx + \lambda t + \log(t^2 - x^Tx)$

let's solve the max for x and t,

$$\nabla g(x,t) = \begin{pmatrix} y - \frac{2}{t^2 - x^Tx} x \\ \lambda + \frac{2t}{t^2 - x^Tx} \end{pmatrix}. \quad \nabla g(x,t) = 0 \Leftrightarrow \begin{cases} y = \frac{2}{t^2 - x^Tx} x \\ -\lambda = \frac{2t}{t^2 - x^Tx} \end{cases}$$

we find $x = \frac{2}{\lambda^2 - y^Ty} y$ and $t = -\frac{2\lambda}{\lambda^2 - y^Ty}$

this gives us $$f^*(y,\lambda) = \frac{2y^Ty}{\lambda^2 - y^Ty} - \frac{2\lambda^2}{\lambda^2 - y^Ty} + \log\left(\frac{4\lambda^2 - 4y^Ty}{(\lambda^2 - y^Ty)^2}\right)$$

$$= -2 + \log 4 - \log(\lambda^2 - y^Ty)$$

In conclusion, $$\boxed{f^*(y,\lambda) = \begin{cases} +\infty & \text{if } \|y\|_2 \geq -\lambda \\ -2 + \log 4 - \log(\lambda^2 - y^Ty) & \text{if } \|y\|_2 < -\lambda \end{cases}}$$