From the power law to extreme value mixture distributions

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Introduction

Discrete power law = Zipf distribution

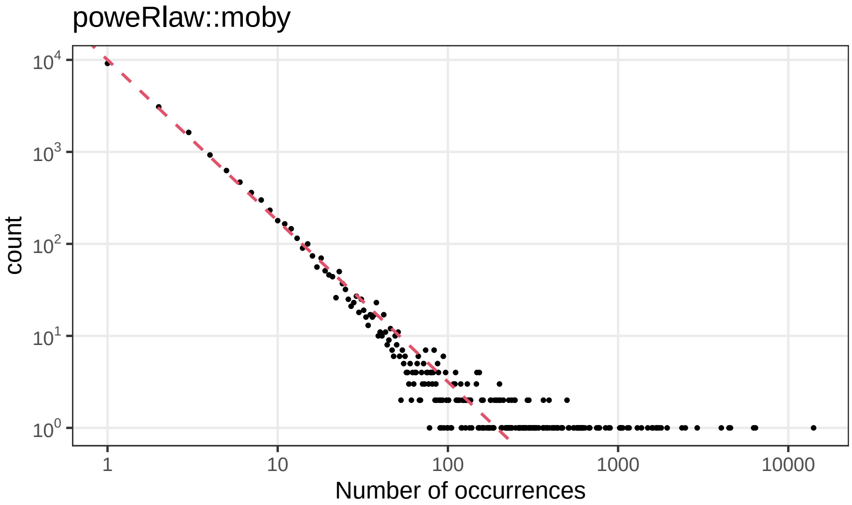
$$p(x) = \frac{x^{-\alpha}}{\zeta(\alpha, u+1)}, \qquad x = u+1, u+2, \ldots$$

- u is a non-negative integer
- $\alpha > 1$ is the **exponent**
- $\zeta(\alpha, z) = \sum_{i=0}^{\infty} (z+i)^{-\alpha}$ is the Hurwitz zeta function

$$\log p(x) = -\alpha \log x + \text{constant}$$

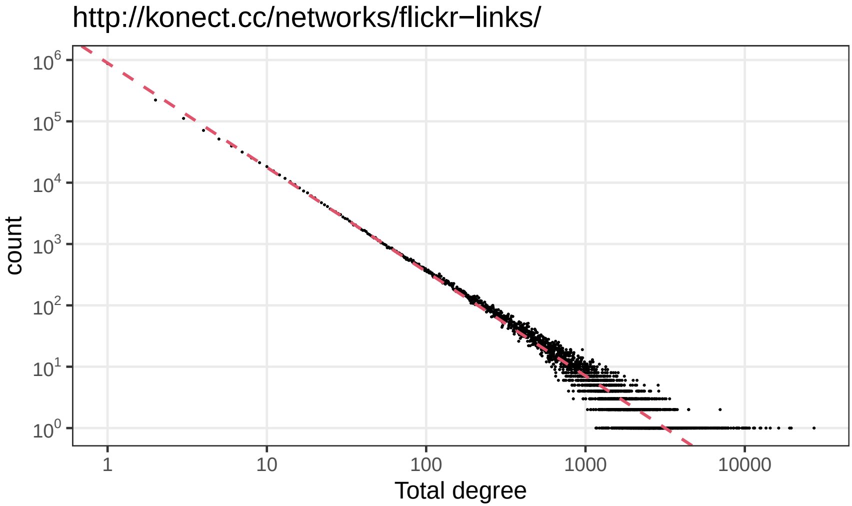
The log-log plot

- Frequency of occurrence of (sampled) unique words in the novel Moby Dick



Another example

- The social network of Flickr users



Particular interest in networks

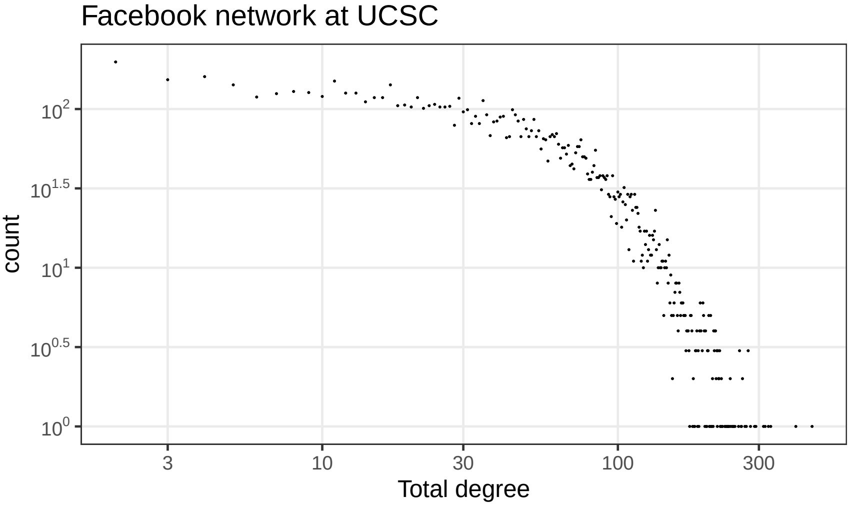
- Total degrees (undirected) or in-degrees (directed)
- Preferential attachment model
  - Barabási and Albert (1999, Science)
  - Generating networks using simple rules
  - The-rich-get-richer
- Resulting degree distribution follows the power law
  - Barabási, Albert and Jeong (1999, Physica A)
  - Bollobás et al. (2001, Random Structures & Algorithms)

Related works

- Wang and Resnick (2023, Extremes)
  - Reciprocity associated with extremal dependence between in-degrees & out-degrees
  - Original model underestimates reciprocity in real-life networks, hence unrealistic
- Here we focus on the degree distribution
  - Does the data really follow the power law?

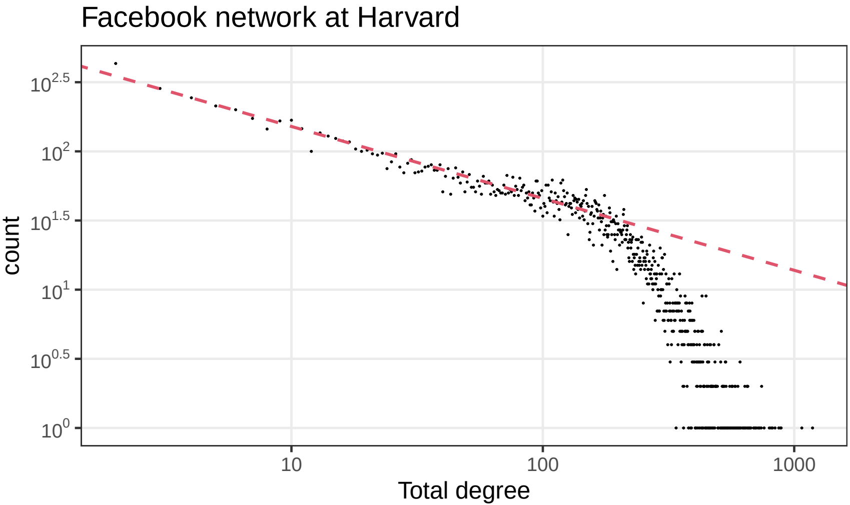
“Close to” power law?

- Analysed by Valero et al. (2022, Physica A)



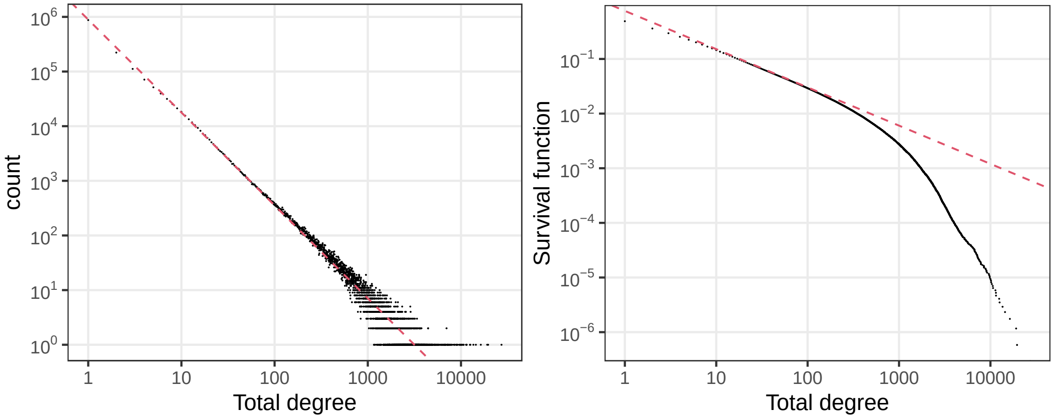
What about this

- Partial power law?



Empirical frequencies mask the tail fit

- The social network of Flickr users (again)



Goals

- A distribution that fits the tail more adequately
- While retaining the (partial) power law
- And covering the possibility of curvature
- Simultaneously determine between the two

Outline

- The Zipf-polylog distribution & its DoA
- Our mixture model
- Selection between power law or not
- Applications to real data

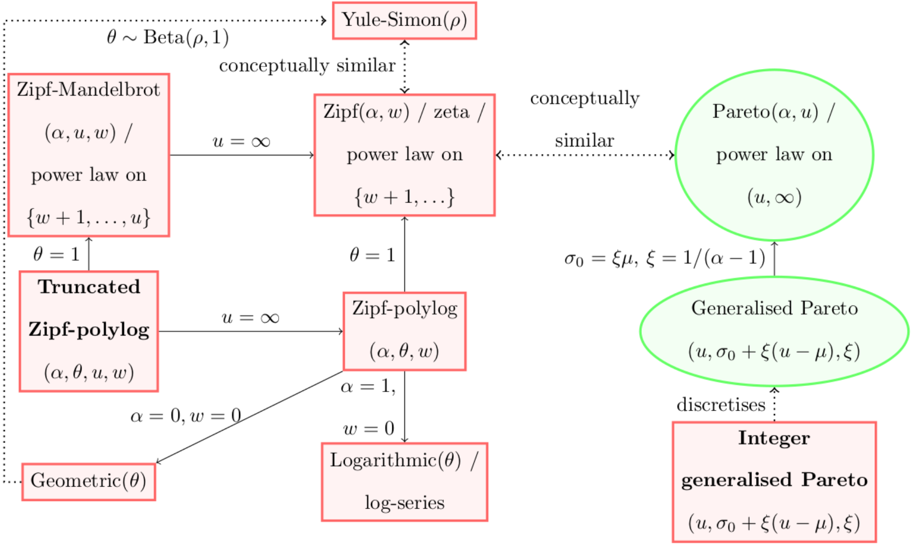
The Zipf-polylog distribution & its DoA

The Zipf / zeta distribution / discrete power law

$$p(x;\alpha) = \frac{x^{-\alpha}}{\sum\limits_{k=w+1}^{\infty} k^{-\alpha}}, \qquad x = w+1, w+2, \ldots$$

- Aka the zeta distribution
- The **continuous** counterpart - the Pareto distribution
- No direct relationship between the two

Relationships



The Zipf-polylog (ZP) distribution (Valero et al., 2022, Physica A)

$$p_{\mathrm{ZP}}(x;\alpha,\theta) = \frac{x^{-\alpha}\theta^{x}}{\sum_{k=w+1}^{\infty} k^{-\alpha}\theta^{k}}, \qquad x = w+1, w+2, \ldots,$$

- $(\alpha,\theta) \in ((-\infty,\infty)\times(0,1)) \cup ((1,\infty)\times\{1\})$
- Looks like a discrete version of Gamma, but not quite
- A disjoint union of Zipf ($\theta = 1$) and polylog ($\theta \in (0,1)$) distributions
- Accommodating curved data when $\theta \in (0,1)$

ZP inadequate for tails

- Going from $\theta = 1$ to $\theta \in (0, 1]$ is still insufficient for the right tail
- Consider the maximum domain of attraction (DoA) of ZP distribution

Domain of attraction

- A distribution F is in the DoA of an extreme value distribution H if there exists $a_n > 0,\ b_n \in R$ such that

  $$\lim_{n\to\infty} |F^n(a_n x + b_n) - H(x)| = 0,$$

  where H must be a negative Weibull, Gumbel, or Fréchet distribution.
- Applies to continuous & discrete distributions
- Poisson and geometric distributions do not belong to a DoA according to the definition

Recovery to DoA for discrete distributions

- Shimura (2012, Extremes)
- If discrete F is the discretisation of continuous F_0, and F_0 is in a DoA, then F is **recoverable** to the same DoA
- Geometric and Poisson are recoverable to the Gumbel DoA

Key results for recovery (Shimura, 2012)

$$\Omega(F, x) := \left(\log \frac{\bar{F}(x+1)}{\bar{F}(x+2)}\right)^{-1} - \left(\log \frac{\bar{F}(x)}{\bar{F}(x+1)}\right)^{-1},$$

- If $\lim_{x \to \infty} \Omega(F, x) = 0$, then F is recoverable to the Gumbel DoA
- If $\lim_{x \to \infty} \Omega(F, x) = \xi > 0$, then F is **in** the Fréchet DoA with tail index ξ

DoA of geometric(θ) distribution

- $\theta \in (0, 1)$

$$
\begin{aligned}
\bar{F}(x) &= \theta^x \\
\frac{\bar{F}(x+1)}{\bar{F}(x+2)} &= \frac{\bar{F}(x)}{\bar{F}(x+1)} = \frac{1}{\theta} \\
\lim_{x \to \infty} \Omega(F, x) &= \lim_{x \to \infty} \left[\left(\log \frac{1}{\theta} \right)^{-1} - \left(\log \frac{1}{\theta} \right)^{-1} \right] = 0
\end{aligned}
$$

- Recoverable to the Gumbel DoA

DoA of ZP(α, θ) distribution

- When $\theta \in (0, 1)$ i.e. the polylog distribution

$$\lim_{x \to \infty} \Omega(F, x) = 0$$

- Same limit i.e. also recoverable to the Gumbel DoA
- Proof similar to geometric case

DoA of ZP(α, θ) distribution

- When $\theta = 1$ i.e. the Zipf distribution

$$\lim_{x \to \infty} \Omega(F, x) = 1/(\alpha - 1)$$

- **In** Fréchet DoA with tail index $1/(\alpha - 1)$
- Proof in the appendices of the paper

Some remarks

- Voitalov et al. (2019, Physical Review Research) gave the result for the continuous version i.e. the Pareto distribution
- Regular variation arguments rather than GP distribution used
- Can't use result for our proof as Zipf $\neq$ discretisation of Pareto

Practically

- Approximate right tail of ZP(α, θ) by (discrete version of) GP distribution with shape parameter ξ
$$\xi = \mathbb{I}_{\{\theta=1\}}/(\alpha - 1)$$
- Can't quite capture heavy tails of a different heaviness other than $1/(\alpha - 1)$

c.f.

- For bivariate Gaussian(ρ),
$$\chi := \Pr(X > u | Y > u) = \mathbb{I}_{\{|\rho|=1\}}$$
- Can't quite capture the spectrum of asymptotic independence

Discrete version of GP distribution

- Integer generalised Pareto (IGP) distribution
- Prieto et al. (2014, Accident Analysis and Prevention)
- Rohrbeck et al. (2018, Annals of Applied Statistics)

Mixture model

General framework

$$f(z) = \pi_1 f_1(z) + \pi_2 f_2(z) + \cdots + \pi_m f_m(z)$$

- Subject to $\sum_{i=1}^{m} \pi_i = 1$, and $0 < \pi_i < 1$
- Usually same support for all components

In extremes

$$f(z) = \begin{cases} (1-\phi_u) \times \dfrac{h(z)}{H(u)}, & z \le u, \\ \phi_u \times g_u(z), & z > u, \end{cases}$$

- Disjoint support for the components
- Comprehensive review by Scarrott and MacDonald (2012, REVSTAT)
- R package evmix by Hu and Scarrot (2018, JSS)

From continuous . . .

$$f(z) = \begin{cases} (1-\phi_u) \times \dfrac{h(z)}{H(u)}, & z \leq u, \\ \phi_u \times g_u(z), & z > u, \end{cases}$$

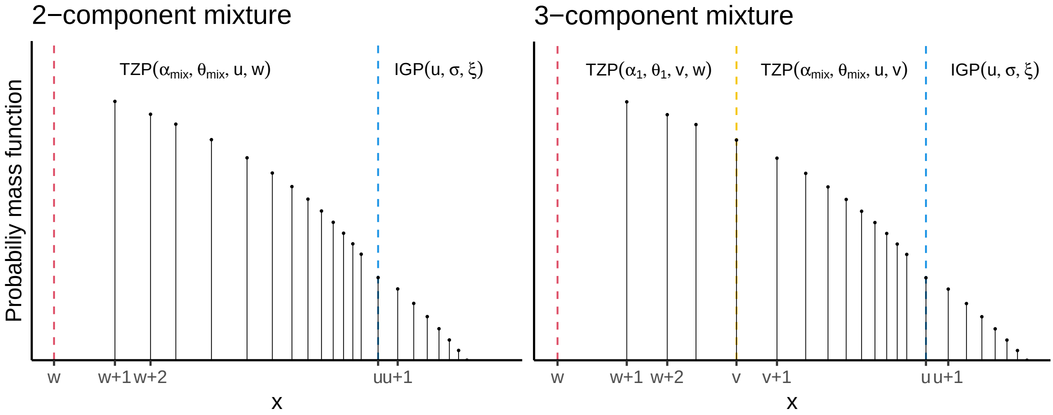
- $h(z)$: bulk / body distribution
- $g_u(z)$: GP density
- ϕ_u: exceedances rate

. . . to discrete

$$p(x) = \begin{cases} (1-\phi_u) \times p_{\mathrm{TZP}}(x; \alpha_{\mathrm{mix}}, \theta_{\mathrm{mix}}, u, w), & x = w+1, w+2, \ldots, u, \\ \phi_u \times \left[G_u(x; \sigma, \xi) - G_u(x-1; \sigma, \xi)\right], & x = u+1, u+2, \ldots \end{cases}$$

- $p_{\mathrm{TZP}}(x)$: density of **truncated** ZP distribution
- $G_u(x)$: CDF of GP distribution
- u: a parameter, allowing threshold uncertainty
- w: fixed, as low as possible

Schematic



2−component mixture
TZP(αmix, θmix, u, w)
IGP(u, σ, ξ)
Probabiliy mass function
w
w+1
w+2
u
u+1
x
3−component mixture
TZP(α1, θ1, v, w)
TZP(αmix, θmix, u, v)
IGP(u, σ, ξ)
w
w+1
w+2
v
v+1
u
u+1
x

Inference & selection between power law or not

Bayesian inference

- To accommodate the threshold uncertainty
- Markov chain Monte Carlo (MCMC)
- Samples of α_{mix}, θ_{mix}, u, σ and ξ
- Interest in if $\theta_{\text{mix}} = 1$ or $\theta_{\text{mix}} \in (0, 1)$

How to test / select?

- θ_{mix} is continuous, never exactly 1 in the samples
- At the boundary makes it even more tricky
- Can't look at the proportion of $\theta_{\text{mix}} = 1$ in the samples
- Proximity is insufficient as different tail behaviours implied

Bayesian model *selection*

1. Define M which equals 0 if $\theta_{\text{mix}} \in (0, 1)$, and 1 if $\theta_{\text{mix}} = 1$
2. Assign $\Pr(M = 0)$ and $\Pr(M = 1)$
3. Select between $M = 0$ and $M = 1$ in the MCMC
   - Gibbs variable selection (Carlin and Chib, 1995, JRSSB) or
   - Reversible jump MCMC (Green, 1995, Biometrika)
4. Calculate $\hat{\Pr}(M = 0|\text{data})$ and $\hat{\Pr}(M = 1|\text{data})$ from MCMC samples
5. Calculate the Bayes factor

Bayes factor

$$B_{10} = \frac{\hat{\Pr}(M = 1|\text{data})}{\hat{\Pr}(M = 0|\text{data})} \bigg/ \frac{\Pr(M = 1)}{\Pr(M = 0)}$$

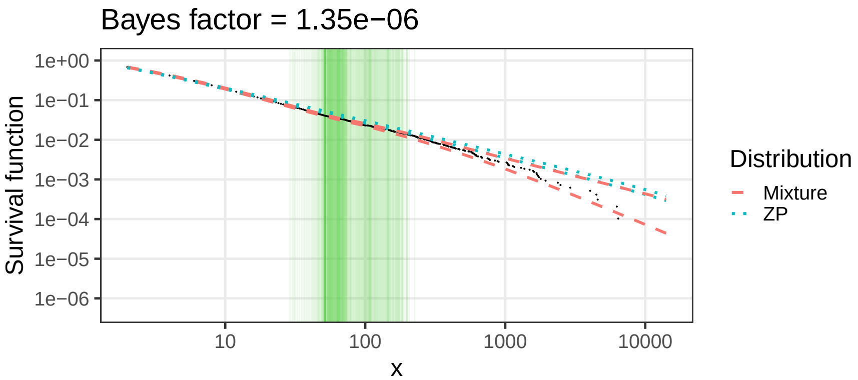
- $B_{10} > 1$: evidence of "the **body** of the data follows the power law"
- $B_{10} < 1$: evidence of "the **body** of the data does not follow the power law"

For ZP(α, θ) distribution as well

- Can apply model selection to determine $\theta = 1$ or not
- Not ZP vs mixture though - can determine visually
- $B_{10} > 1$: evidence of "the **whole** of the data follows the power law"
- $B_{10} < 1$: evidence of "the **whole** of the data does not follow the power law"

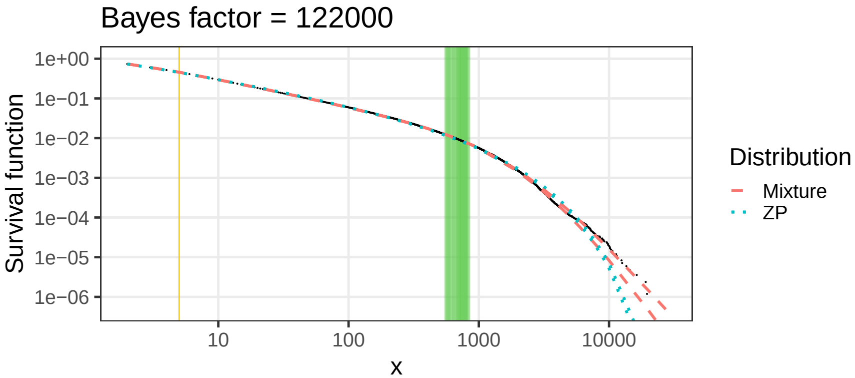
Applications

Moby Dick (`poweRlaw::moby`)



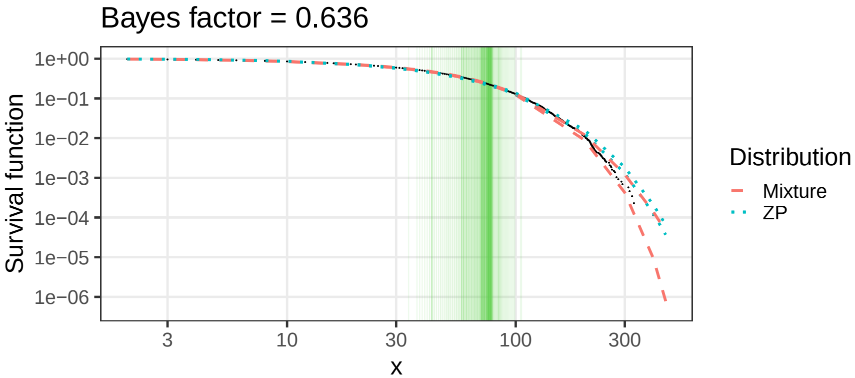
- u: Moderate uncertainty but identified
- Not power law for body (left tail)

Flickr users (Voitalov et al., 2019)



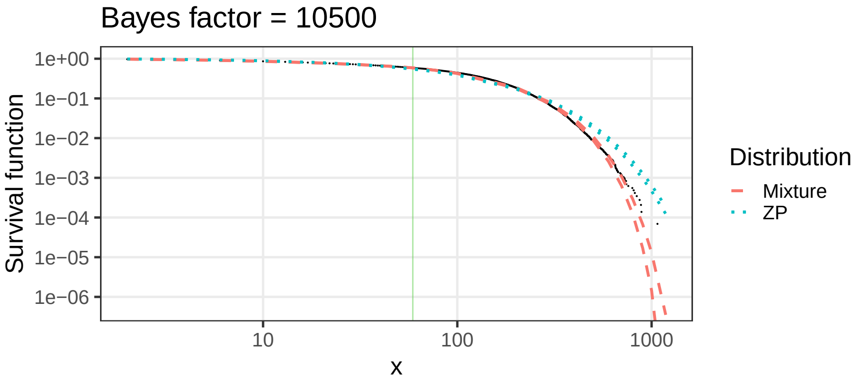
- 3-component mixture required
- Partial power law otherwise overlooked by ZP fit

Facebook network at UCSC (Valero et al., 2022)



- Mixture (IGP) better than ZP in the right tail
- Could be power law or not for body

Facebook network at Harvard (Valero et al., 2022)



- Similar adequacy but strong evidence for partial power law
- $\alpha_{\text{mix}} < 1$, would not be possible for Zipf fit over the whole of data

Main takeaway

- For ZP, $\theta \in (0,1)$ (polylog) almost always preferred to $\theta = 1$ (Zipf)
- "Concavity" due to lighter right tail than implied *had the power law in the body been extended*
- Mixture resolves by replacing ZP by IGP for the tail

Summary

- ZP distribution useful starting point for data that seems to follow the power law
- Generalises Zipf distribution, but inadequate for right tail
- Mixture model uses integer GP distribution instead
- Bayesian model selection decides if body follows the power law or not
- Applications show good fit and varying degrees of threshold uncertainty

Next steps

- More formal model comparison (ZP, 2-component mixture, 3-component mixture) via e.g. marginal likelihood
- Modified preferential attachment model that leads to such degree distributions

Thank you for listening!