Implementation of a Bayesian Inference for Diffusion-Driven Mixed-Effects Models

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1 Introduction

N is the number of stocks we are studying. n is the number of observations of the stock price we have. m is the level of discretization for data augmentation. The inference proposed in [2] is based on the data augmentation throuch a decomposition of the Stochastic variable X :

$$X_t = \eta_t + R_t \tag{1}$$

where η accounts for drift and R is the residual stochastic process and these variables are computed for the augmented points.

2 Model presentation

In order to implement the algorithm proposed by [2], we have to define the stochastic differential equation for stock prices. As suggested in the litterature (for example in [1]), a usual model for stock prices is:

$$\mathrm{d}S_t = \sigma S_t \mathrm{d}W_t + \mu S_t \mathrm{d}t \tag{2}$$

where S_t is the stock price and W_t is a brownian motion.

Under the conventions of stochastic differential models explained in [2] :

$$\mathrm{d}X_t^i = \alpha(X_t^i, \theta, b^i)\mathrm{d}t + \sqrt{\beta(X_t^i, \theta, b^i)}\mathrm{d}W_t^i \tag{3}$$

we thus consider $\alpha(X_t^i, \theta, b^i) = \theta X_t^i$ and $\beta(X_t^i, \theta, b^i) = {b^i}^2 {X_t^i}^2$. To use the framework defined by Whitaker, we have to derive full conditionals for parameter θ and b. $\pi(\theta|b, x) \propto \pi(\theta)\pi(x|\theta, b)$ and $\pi(b|\theta, x) \propto \pi(b)\pi(x|\theta, b)$. Where

$$\pi(x|\theta, b) = \prod_{i=1}^{N} \prod_{j=0}^{n-1} \prod_{k=1}^{m} \pi(x^i_{\tau_{j,k}} | x^i_{\tau_{j,k-1}}, \theta, b^i) \tag{4}$$

and

$$\begin{aligned} \pi(x^i_{\tau_{j,k}} | x^i_{\tau_{j,k-1}}, \theta, b^i) &= N(x^i_{\tau_{j,k}}; x^i_{\tau_{j,k-1}} + \alpha(x^i_{\tau_{j,k-1}}, \theta, b^i)\Delta\tau, \beta(x^i_{\tau_{j,k-1}}, \theta, b^i)\Delta\tau) \\ &= N(x^i_{\tau_{j,k}}; x^i_{\tau_{j,k-1}} + \theta x^i_{\tau_{j,k-1}}\Delta\tau, {b^i}^2 {x^i_{\tau_{j,k-1}}}^2 \Delta\tau) \end{aligned} \tag{5}$$

After some computation we get the form of $\pi(x|\theta, b)$ with respect to θ and b^i:

$$\begin{aligned} \pi(x|\theta, b) &\propto_\theta \exp\left(-\frac{\Delta\tau}{2} \sum_{i=1}^{N} \sum_{j=0}^{n-1} \sum_{k=1}^{m} \frac{(\theta - \mu^i_{j,k})^2}{{b^i}^2}\right) \\ &\propto_\theta \exp\left(-\sum_{i=1}^{N} \frac{\Delta\tau n m}{2{b^i}^2}(\theta - \mu^i)^2\right) \end{aligned} \tag{6}$$

where $\mu^i_{j,k} = \frac{1}{\Delta\tau}\left(\frac{x^i_{\tau_{j,k}}}{x^i_{\tau_{j,k-1}}} - 1\right)$ and $\mu_i = \sum_{j=0}^{n-1} \sum_{k=1}^{m} \mu^i_{j,k}$. We can also define $\sigma_i^2 = \frac{{b^i}^2}{\Delta\tau n m} = \frac{1}{\tau_i}$.

$$\pi(x|\theta, b) \propto_{b^i} \frac{1}{{b^i}^{2\frac{nm}{2}}} \exp\left(-\frac{1}{{b^i}^2}\beta^*\right) \tag{7}$$

where $\beta^* = \displaystyle\sum_{j=0}^{n-1}\sum_{k=1}^{m}\left(\frac{x^i_{\tau_{j,k}}}{x_{\tau^i_{j,k-1}}} - (1+\theta\Delta\tau)\right)^2$.
We see that a good prior for θ would be a Normal distribution. And a good prior for b^i would be an Inv-Gamma distribution. These priors are conjugate and leads us to a close form of the posteriors, so we could use Gibbs Sampler in order to update the parameters.

As the posterior distributions of θ is a product of normal density functions with different means and variances, we will show the computation to derive the posterior law. We denote here μ_0 and $\sigma_0^2 = \dfrac{1}{\tau_0}$ the parameters of the prior normal distribution of θ.

$$
\begin{aligned}
\pi(\theta|x,b) &\propto \prod_{i=1}^{N} \exp\left(-\frac{\tau_i}{2}(\theta-\mu_i)^2\right) \times \exp\left(-\frac{\tau_0}{2}(\theta-\mu_0)^2\right) \\
&\propto \exp\left(-\frac{1}{2}\left[\theta^2\sum_{i=0}^{N}\tau_i - 2\theta\sum_{i=0}^{N}\mu_i\tau_i\right] + \mathrm{K}\right) \\
&\propto \exp\left(-\frac{1}{2}\left(\theta^2\tau^* - 2\theta\mu_\tau\right)\right) \\
&\propto \exp\left(-\frac{\tau^*}{2}\left(\theta - \frac{\mu_\tau}{\tau}\right)\right)
\end{aligned}
$$

where $\tau^* = \displaystyle\sum_{i=0}^{N}\tau_i$, $\mu_\tau = \displaystyle\sum_{i=0}^{N}\mu_i\tau_i$ and we can define $\mu^* = \dfrac{\mu_\tau}{\tau^*}$.

We can now write prior and posterior distributions (the initial parameters are drawn from the observation of the data) :

$$
\begin{aligned}
\theta &\sim \text{Normal}(\mu_0 = 0, \sigma_0^2 = 0.01) \\
{b^i}^2 &\overset{i.i.d}{\sim} \text{Inv-Gamma}(\alpha_0 = 4, \beta_0 = 1) \\
\theta|b^i, x &\sim \text{Normal}\left(\mu^*, {\sigma^2}^* = \frac{1}{\tau^*}\right) \\
{b^i}^2|\theta, x &\overset{ind}{\sim} \text{Inv-Gamma}\left(\alpha_0 + \tfrac{nm}{2}, \beta_0 + \beta^*\right)
\end{aligned}
\tag{8}
$$

3 Algorithm description

Now that our model is clear let us provide an algorithm to implement it :

Algorithm 1: Sampler

Data: $\forall(i,j) \in [\![1,N]\!] \times [\![1,n]\!]$, x_j^i the n simultaneous observations of the N stock prices
Result: Samples from θ and b
n = number of observations;
N = number of stocks;
m = level of discretization;
n_{iter} = number of iterations;
$\Delta\tau = \frac{1}{m}$;
$\Delta t = 1$;

$$x = \begin{pmatrix} m\left\{\begin{matrix} x_1^1 & x_1^2 & \dots & x_1^N \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix}\right. \\ m\left\{\begin{matrix} x_1^2 & x_2^2 & \dots & x_2^N \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix}\right. \\ [\dots] \\ x_n^1 \quad x_n^2 \quad \dots \quad x_n^N \end{pmatrix}$$ Note that x shape is (n-1)*m + 1 , N;

η = copy of x;
R = $0 \times x$;
b = $(b^1, \dots, b^N) * n_{iter}$ sampled from Equation 8 (n_{iter} rows and N colums) ;
$\theta = [\theta] * n_{iter}$ sampled from Equation 8 (n_{iter} rows);
for $l \leftarrow 1$ **to** n_{iter} **do**
    for $j \leftarrow 0$ **to** $n-2$ **do**
        for $k \leftarrow 1$ **to** $m-1$ **do**
            for $i \leftarrow 0$ **to** $N-1$ **do**
                $\eta[m*j+k][i] = (1 + b[l][i]\Delta\tau)\eta[m*j+k-1][i]$;
                $\mu_R = R[m*j+k-1][i] + \Delta\tau \frac{-R[m*j+k-1][i]}{j+1-\left(j+\frac{k-1}{m}\right)}$;
                $\sigma_R^2 = \frac{j+1-\left(j+\frac{k}{m}\right)}{j+1-\left(j+\frac{k-1}{m}\right)} \Delta\tau\theta[l]^2 x[m*j+k-1][i]^2$;
                $R[m*j+k][i]$ = sample from Normal $\left(\mu_R, \sigma_R^2\right)$;
            end
            $x[m*j+k] = \eta[m*j+k] + R[m*j+k]$
        end
    end
    Gibbs Sampler step to derive $\theta[l+1]$ and $b[l+1]$;
end

4 Data observation

$$X_t^i = \alpha_i X_t^i \,\mathrm{d}t + \sigma_i X_t \,\mathrm{d}W_t$$

$$\begin{cases} \alpha_i & \sim & \mathrm{N}(0, 0.1) \\ \sigma_i & \sim & \mathrm{N}(0, 0.1) \end{cases}$$

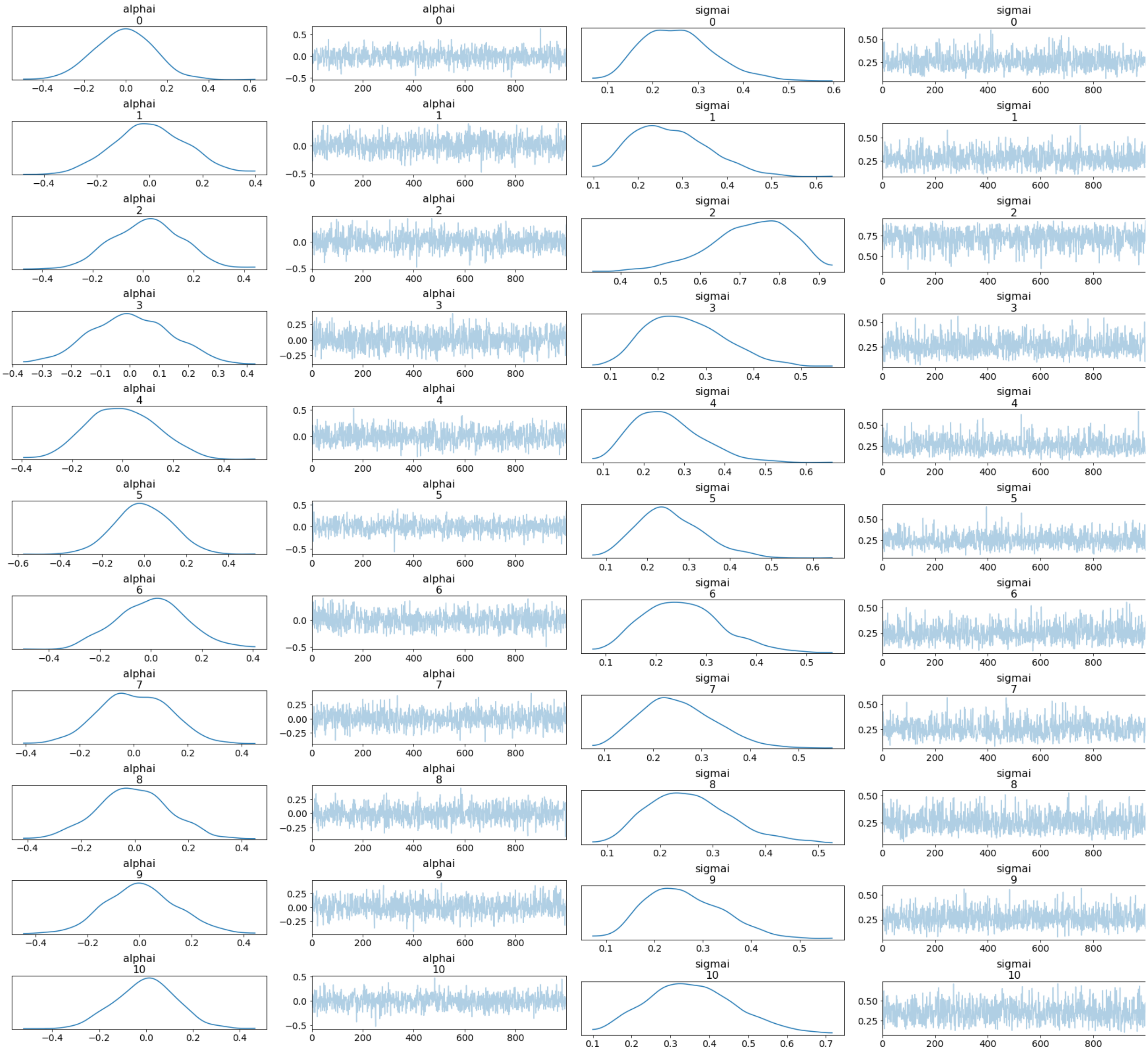

Figure 1: Posterior distribution of some α_i and σ_i

We considered this model in order to observe our data through already existing tools (`Pymc3` on Pyhton). We give an insight of the results in Figure 1

References

[1] Hans Föllmer and Martin Schweizer. A microeconomic approach to diffusion models for stock prices. Mathematical finance, 3(1):1–23, 1993.

[2] Gavin A Whitaker, Andrew Golightly, Richard J Boys, Chris Sherlock, et al. Bayesian inference for diffusion-driven mixed-effects models. Bayesian Analysis, 12(2):435–463, 2017.