Implementation of a Bayesian Inference for Diffusion-Driven Mixed-Effects Models

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1 Introduction

N is the number of stocks we are studying. n is the number of observations of the stock price we have. m is the level of discretization for data augmentation. The inference proposed in [2] is based on the data augmentation throuch a decomposition of the Stochastic variable X :

$$X_t = \eta_t + R_t \tag{1}$$

where η accounts for drift and R is the residual stochastic process and these variables are computed for the augmented points.

2 Model presentation

In order to implement the algorithm proposed by [2], we have to define the stochastic differential equation for stock prices. As suggested in the litterature (for example in [1]), a usual model for stock prices is:

$$\mathrm{d}S_t = \sigma S_t \mathrm{d}W_t + \mu S_t \mathrm{d}t \tag{2}$$

where S_t is the stock price and W_t is a brownian motion.

Under the conventions of stochastic differential models explained in [2] :

$$\mathrm{d}X_t^i = \alpha(X_t^i, \theta, b^i)\mathrm{d}t + \sqrt{\beta(X_t^i, \theta, b^i)}\mathrm{d}W_t^i \tag{3}$$

we thus consider $\alpha(X_t^i, \theta, b^i) = \theta X_t^i$ and $\beta(X_t^i, \theta, b^i) = {b^i}^2 {X_t^i}^2$. To use the framework defined by Whitaker, we have to derive full conditionals for parameter θ and b. $\pi(\theta|b, x) \propto \pi(\theta)\pi(x|\theta, b)$ and $\pi(b|\theta, x) \propto \pi(b)\pi(x|\theta, b)$. Where

$$\pi(x|\theta, b) = \prod_{i=1}^{N} \prod_{j=0}^{n-1} \prod_{k=1}^{m} \pi(x^i_{\tau_{j,k}} | x^i_{\tau_{j,k-1}}, \theta, b^i) \tag{4}$$

and

$$\begin{aligned} \pi(x^i_{\tau_{j,k}} | x^i_{\tau_{j,k-1}}, \theta, b^i) &= N(x^i_{\tau_{j,k}}; x^i_{\tau_{j,k-1}} + \alpha(x^i_{\tau_{j,k-1}}, \theta, b^i)\Delta\tau, \beta(x^i_{\tau_{j,k-1}}, \theta, b^i)\Delta\tau) \\ &= N(x^i_{\tau_{j,k}}; x^i_{\tau_{j,k-1}} + \theta x^i_{\tau_{j,k-1}}\Delta\tau, {b^i}^2 {x^i_{\tau_{j,k-1}}}^2 \Delta\tau) \end{aligned} \tag{5}$$

After some computation we get the form of $\pi(x|\theta, b)$ with respect to θ and b^i:

$$\pi(x|\theta, b) \propto_\theta \exp\left(-\frac{\Delta\tau N n m}{2{b^i}^2}(\theta - \mu^*)^2\right) \tag{6}$$

where $\mu^* = \sum_{i=1}^{N} \sum_{j=0}^{n-1} \sum_{k=1}^{m} \frac{1}{\Delta\tau}\left(\frac{x^i_{\tau_{j,k}}}{x_{\tau^i_{j,k-1}}} - 1\right)$ and we can also define $\sigma^{2*} = \frac{{b^i}^2}{\Delta\tau N n m}$

$$\pi(x|\theta, b) \propto_{b^i} \frac{1}{{b^i}^{2\frac{nm}{2}}} \exp\left(-\frac{1}{{b^i}^2}\beta^*\right) \tag{7}$$

where $\beta^* = \sum_{j=0}^{n-1} \sum_{k=1}^{m} \left(\frac{x^i_{\tau_{j,k}}}{x_{\tau^i_{j,k-1}}} - (1 + \theta \Delta \tau) \right)^2$.

We see that a good prior for θ would be a Normal distribution. And a good prior for b^i would be an Inv-Gamma distribution. These priors are conjugate and leads us to a close form of the posteriors, so we could use Gibbs Sampler in order to update the parameters.

We can now write prior and posterior distributions (the initial parameters are drawn from the observation of the data) :

$$\begin{array}{rcl} \theta & \sim & \text{Normal}(\mu_0 = 0, \sigma_0^2 = 0.01) \\ {b^i}^2 & \overset{i.i.d}{\sim} & \text{Inv-Gamma}(\alpha_0 = 4, \beta_0 = 1) \\ \theta | b^i, x & \sim & \text{Normal} \left(\frac{\sigma_0^2}{\sigma^{*2} + \sigma_0^2} \mu^* + \frac{\sigma^{*2}}{\sigma^{*2} + \sigma_0^2} \mu_0, \left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma^{*2}} \right)^{-1} \right) \\ {b^i}^2 | \theta, x & \overset{ind}{\sim} & \text{Inv-Gamma} \left(\alpha_0 + \frac{nm}{2}, \beta_0 + \beta^* \right) \end{array} \tag{8}$$

3 Algorithm description

Now that our model is clear let us provide an algorithm to implement it :

Algorithm 1: Sampler

Data: $\forall (i,j) \in [\![1,N]\!] \times [\![1,n]\!]$, x_j^i the n simultaneous observations of the N stock prices
Result: Samples from θ and b
n = number of observations;
N = number of stocks;
m = level of discretization;
n_{iter} = number of iterations;
$\Delta\tau = \frac{1}{m}$;
$\Delta t = 1$;

$$x = \begin{pmatrix} m\left\{\begin{matrix} x_1^1 & x_1^2 & \dots & x_1^N \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix}\right. \\ m\left\{\begin{matrix} x_1^2 & x_2^2 & \dots & x_2^N \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix}\right. \\ [\dots] \\ x_n^1 \quad x_n^2 \quad \dots \quad x_n^N \end{pmatrix}$$ Note that x shape is (n-1)*m + 1 , N;

η = copy of x;
R = $0 \times x$;
b = $(b^1, \dots, b^N) * n_{iter}$ sampled from Equation 8 (n_{iter} rows and N colums) ;
$\theta = [\theta] * n_{iter}$ sampled from Equation 8 (n_{iter} rows);
for $l \leftarrow 1$ **to** n_{iter} **do**
  for $j \leftarrow 0$ **to** $n-2$ **do**
    for $k \leftarrow 1$ **to** $m-1$ **do**
      for $i \leftarrow 0$ **to** $N-1$ **do**
        $\eta[m*j+k][i] = (1 + b[l][i]\Delta\tau)\eta[m*j+k-1][i]$;
        $\mu_R = R[m*j+k-1][i] + \Delta\tau \dfrac{-R[m*j+k-1][i]}{j+1-j+\frac{k-1}{m}}$;
        $\sigma_R^2 = \dfrac{j+1-j+\frac{k}{m}}{j+1-j+\frac{k-1}{m}} \Delta\tau\theta[l]^2 x[m*j+k-1][i]$;
        $R[m*j+k][i]$ = sample from Normal $\left(\mu_R, \sigma_R^2\right)$;
      end
      $x[m*j+k] = \eta[m*j+k] + R[m*j+k]$
    end
  end
  Gibbs Sampler step to derive $\theta[l+1]$ and $b[l+1]$;
end

4 Data observation

$$X_t^i = \alpha_i X_t^i \,\mathrm{d}t + \sigma_i X_t \,\mathrm{d}W_t$$

$$\begin{cases} \alpha_i & \sim & \mathrm{N}(0, 0.1) \\ \sigma_i & \sim & \mathrm{N}(0, 0.1) \end{cases}$$

We considered this model in order to observe our data through already existing tools (`Pymc3` on Pyhton). We give an insight of the results in Figure 1

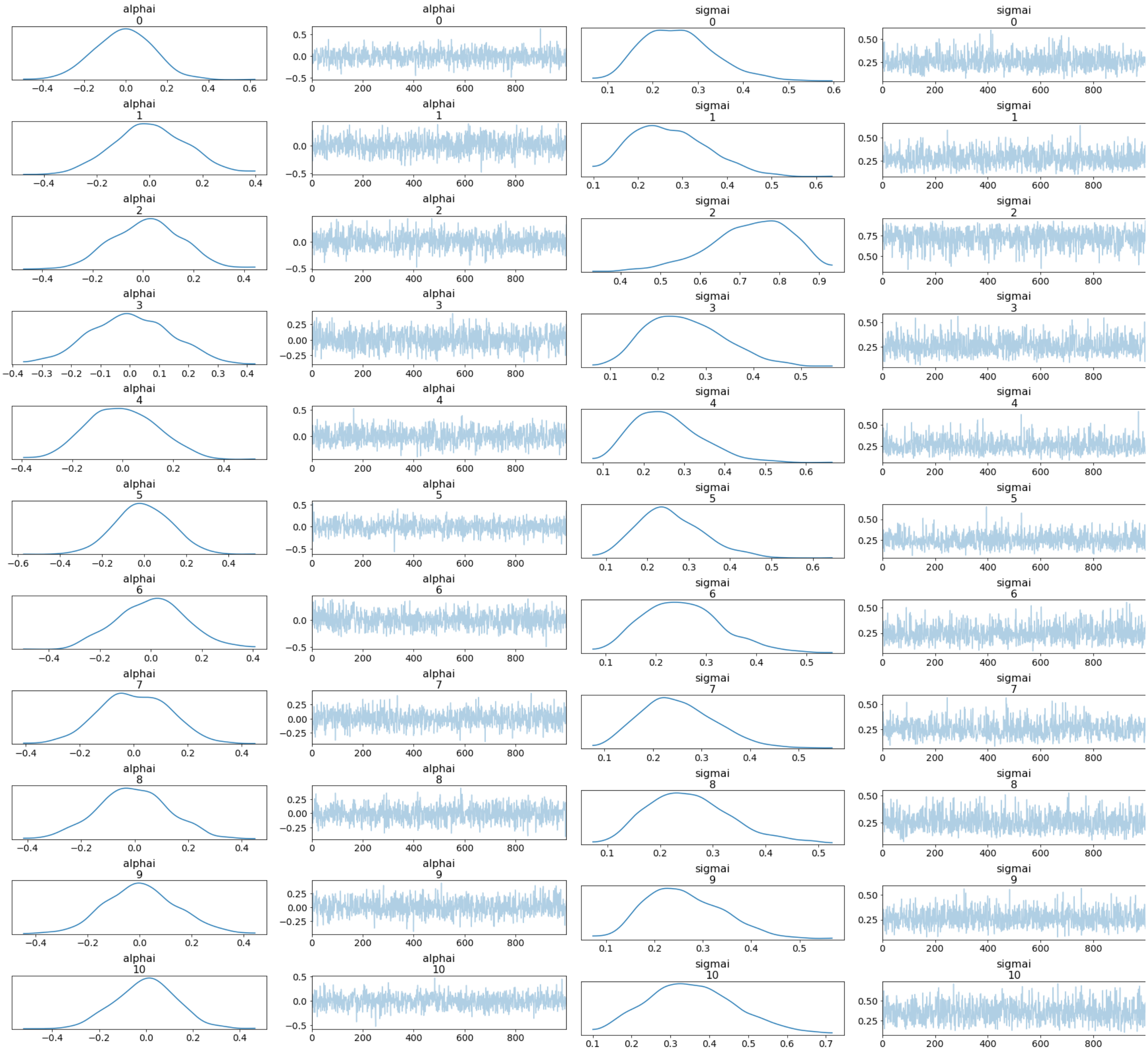

Figure 1: Posterior distribution of some α_i and σ_i

References

[1] Hans Föllmer and Martin Schweizer. A microeconomic approach to diffusion models for stock prices. Mathematical finance, 3(1):1–23, 1993.

[2] Gavin A Whitaker, Andrew Golightly, Richard J Boys, Chris Sherlock, et al. Bayesian inference for diffusion-driven mixed-effects models. Bayesian Analysis, 12(2):435–463, 2017.