Tipping points in macroeconomic Agent-Based models

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The aim of this work is to explore the possible types of *phenomena* that simple macroeconomic Agent-Based models (ABM) can reproduce. Our motivation is to understand the large macro-economic fluctuations observed in the "Mark I" ABM devised by D. Delli Gatti and collaborators. Our central finding is the generic existence of a *phase transition* between a "good economy" where unemployment is low, and a "bad economy" where unemployment is high. We show that this transition is robust against many modifications of the model, and is induced by an asymmetry between the rate of hiring and the rate of firing of the firms. This asymmetry is, in Mark I, due to the fact that as the interest rate increases, firms become more and more reluctant to take further loans and have to reduce their workforce. The unemployment level remains small until a tipping point, beyond which the economy suddenly collapses. If the parameters are such that the system is close to this transition, any small fluctuation is amplified as the system jumps between the two equilibria. We have explored several natural extensions of the model. One is to introduce a bankruptcy threshold, limiting the leverage of firms. This leads to a rich phase diagram with, in particular, a region where *acute endogenous crises* occur, during which the unemployment rate shoots up before the economy can recover. We allow the hiring/firing propensity to depend on the financial fragility of firms, and introduce simple wage update policies. This leads to inflation (in the "good" phase) or deflation (in the "bad" phase), but leaves the overall phase diagram of the model essentially unchanged. We have finally explored the effect of simple monetary policies that attempt to contain rising unemployment and defang crises. We end the paper with general comments on the usefulness of ABMs to model macroeconomic phenomena, in particular in view of the time needed to reach a steady state that raises the issue of *ergodicity* in these models.

It is human nature to think wisely and to act absurdly – Anatole France

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Contents

I. INTRODUCTION

A. From micro-rules to macro-behaviour

Inferring the behaviour of large assemblies from the behaviour of its elementary constituents is arguably one of the most important problems in a host of different disciplines: physics, material sciences, biology, computer sciences, sociology and, of course, economics and finance. It is also a notoriously hard problem. Statistical physics has developed in the last 150 years essentially to understand this micro-macro link. Clearly, when interactions are absent or small enough, the system as a whole merely reflects the properties of individual entities. This is the canvas of traditional macro-economic approaches. Economic agents are assumed to be identical, non-interacting, rational agents, an idealization known as the "Representative Agent" (RA). In this framework, micro and macro trivially coincide. However, we know (in particular from physics) that discreteness, heterogeneities and/or interactions can lead to totally unexpected phenomena. Think for example of super-conductivity or super-fluidity[1]: before their experimental discovery, it was simply beyond human imagination that individual electrons or atoms could "conspire" to create a collective state that can flow without friction. Micro and macro behaviour not only do not coincide in general, but genuinely *surprising* behaviour can emerge through aggregation. From the point of view of economic theory, this is interesting, because financial and economic history is strewn with bubbles, crashes, crises and upheavals of all sorts. These are very hard to fathom within a Representative Agent framework [2], within which crises would require large aggregate shocks, when in fact small local shocks can trigger large systemic effects when heterogeneities, interactions and network effects are taken into account [3–5]. The need to include these effects has spurred a large activity in "Agent-Based models" (ABMs) [6–8]. These models need numerical simulations, but are extremely versatile because any possible behavioural rules, interactions, heterogeneities can be taken into account.

In fact, these models are so versatile that they suffer from the "wilderness of high dimensional spaces" (paraphrasing Sims [9]). The number of parameters and explicit or implicit choices of behavioural rules is so large (~ 10 or more, even in the simplest models, see below) that the results of the model may appear unreliable and arbitrary, and the calibration of the parameters is an hopeless (or highly unstable) task. Mainstream RA models, on the other hand, are simple enough to lead to closed form analytical results, with simple narratives and well-trodden calibration avenues [10]. In spite of their unrealistic character, these traditional models are still favoured by most economists, both in academia and in institutional and professional circles. ABMs are seen at best as a promising research direction and at worst as an unwarranted "black box" (see [11] for an enlightening discussion on the debate between traditional DSGE models and ABMs, and [12, 13] for further insights).

B. A methodological manifesto

At this stage, it seems to us that some clarifications are indeed needed, concerning both the objectives and methodology of Agent-Based models. ABMs do indeed suffer from the wilderness of high dimensional spaces, and some guidance is necessary to put these models on a firm footing. In this respect, statistical physics offers a key concept: the *phase diagram* in parameter space [14]. A classic example, shown in Fig. 1, is the phase diagram of usual substances as a function of two parameters, here temperature and pressure. The generic picture is that the number of distinct phases is usually small (e.g. three in the example of Fig. 1: solid, liquid, gas). Well within each phase, the properties are qualitatively similar and small changes of parameters have a small effect. Macroscopic (aggregate) properties do not fluctuate any more for very large systems and are robust against changes of microscopic details. This is the "nice" scenario, where the dynamics of the system can be described in terms of a small number of macroscopic (aggregate) variables, with some effective parameters that encode the microscopic details. But other scenarios are of course possible; for example, if one sits close to the boundary between two phases, fluctuations can remain large even for large systems and small changes of parameters can radically change the macroscopic behaviour of the system. There may be mechanisms naturally driving the system close to criticality (like in Self Organized Criticality [15]), or, alternatively, situations in which whole phases are critical, like for "spin-glasses" [16].

In any case, the very first step in exploring the properties of an Agent-Based model should be, we believe, to identify the different possible phases in parameter space and the location of the phase boundaries. In order to do this, numerical simulations turn out to be very helpful [17, 18]: aggregate behaviour usually quickly sets in, even for small sizes. Some parameters usually turn out to be crucial, while others are found to play little role. This is useful

[1] See e.g. Ref. [1] for an history of the discovery of super-fluidity and a list of references.

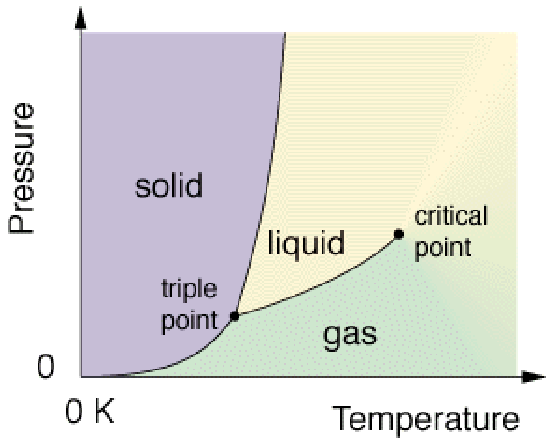

FIG. 1: A typical "phase diagram", here the solid-liquid-gas phases in the temperature-pressure plane. Far from phase transitions, within a given phase, the behaviour of the system is qualitatively similar for all values of the parameters. Close to phase boundaries, on the other hand, small changes of parameters can lead to dramatic macroscopic changes.

to establish a qualitative *phenomenology* of the model – what kind of behaviour can the model reproduce, which basic mechanisms are important, which effects are potentially missing? This first, qualitative step allows one to unveil the "skeleton" of the ABM. Simplified models that retain most of the phenomenology can then be constructed and perhaps solved analytically, enhancing the understanding of the important mechanisms, and providing some narrative to make sense of the observed effects. In our opinion, calibration of an ABM using real data can only start to make sense after this initial qualitative investigation is in full command – which is in itself not trivial when the number of parameters is large. The phase diagram of the model allows one to restrict the region of parameters that lead to "reasonable" outcomes (see for example the discussion in [19, 20]).

C. Outline and results of the paper

The aim of this paper is to put these ideas into practice in the context of a well-studied macroeconomic Agent-Based model (called "Mark I" below), devised by Delli Gatti and collaborators [21, 22]. This model is at the core of the European project "CRISIS", which justifies our attempt to shed some theoretical light on this framework.[2] In the first part of the paper, we briefly recall the main ingredients of the model and show that as one increases the baseline interest rate, there is a phase transition between a "good" state of the economy, where unemployment is low and a "bad" state of the economy where production and demand collapse. In the second part of the paper, we study the phase diagram of a highly simplified version of Mark I, dubbed 'Mark 0' that aims at capturing the main drivers of this phase transition. We show that the most important parameter is the asymmetry between the firms' propensity to hire (when business is good) or to fire (when business is bad). In Mark I, this asymmetry is induced by the reaction of firms to the level of interest rates, but other plausible mechanisms would lead to the same effect. The simplest version of the model is amenable to an analytic treatment and exhibits a "tipping point" (i.e. a discontinuous transition) between high employment and high unemployment. Enriching the model by endowing firms with a strategy that depends on their financial wealth only slightly modifies the picture. The level of unemployment is now continuous, and low frequency fluctuations of the employment rate/production appear due to the proximity of a second order transition. Finally, we allow wages to adapt (whereas they are kept fixed in Mark I) and allow inflation or deflation to set in. The model reveals an extremely rich phenomenology, with endogenous crises, for which we provide a full phase diagram. Open questions and future directions, in particular concerning macroeconomic models in general, are discussed in our final section.

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[2] see www.crisis-economics.eu

II. A PHASE TRANSITION IN “MARK I”

A. Description of the model in a nutshell

The Mark I family of agent-based models was proposed by Delli Gatti and collaborators as a family of simple stylized macroeconomic models [21, 22]; it is the cornerstone of the more comprehensive Agent-Based model currently put together by the members of the CRISIS project. Note that several other macroeconomic Agent-Based models have been put forth in the recent years, see [23–28]. Mark I is interesting in part because large fluctuations in unemployment and output seem to persist in the stationary state (a feature in fact shared by many ABMs cited above).

The Mark I economy [21] is made up of a set of firms, households, firms owners and a bank. Firms produce a certain quantity of a single (and not storable) good, proportional to the number of their employees, that is sold at a time-dependent and firm-dependent price. Firms pay identical time-independent wages. When the cash owned by a firm is not enough to pay the wages, it asks banks for a loan. The bank provides loans to the firms at an interest rate that depends on the financial fragility of the firm. Households provide workforce in exchange of a salary and want to spend a fixed fraction of their savings. The owners of the firms do not work but receive dividends if the firms make profits. Firms are adaptive, in the sense that they continuously update their production (i.e. they hire/fire workers) and their prices, in an attempt to match their production with the demand of goods issued by the households. They also choose how much extra loan they want to take on, as a function of the offered interest rate. This last feature, combined with the price and production update rules, will turn out to be crucial in the dynamics of the model.

The above description defines the basic structure of the Mark I family, but it is of course totally insufficient to code the model, since many additional choices have to be made, leading to several different possible implementations of the model. Here we will use as a baseline model one of the simplest implementation of Mark I, whose description can be found in [29]; the total number of parameters in this version is 10 (but some parameters are actually implicitly fixed from the beginning). We have recoded this basic version and also a slightly different version that we call “Mark I+”, which differs on minor details (some that we will specify below) but also on one major aspect: our version of the model *strictly conserves the amount of money in circulation*, i.e. the money in the bank + total firm assets + total households savings, in order to avoid – at this stage – any effect due to uncontrolled money creation. A detailed pseudo-code of Mark I+ is provided in Appendix A.

B. State variables

In short (see Appendix A for a complete description), the dynamic evolution of the model is defined by the following state variables. The state of each firm $i = 1 \cdots N_{\rm F}$ is specified by its price $p_i(t)$, the salary it offers $W_i(t)$, its production $Y_i(t)$, its target production $Y_i^T(t)$, its demand $D_i(t)$, its liquidity $\mathcal{L}_i(t)$, its total debt $\mathcal{D}_i^T(t)$. Moreover, each firm is owned by a household and has a list of employees that is dynamically updated. The state of each household $a = 1 \cdots N_{\rm H}$ is specified by its savings $S_a(t)$ and by the firm for which it works (if any).

C. Update rules for prices and production

Among all the micro-rules that any Agent-Based model has to postulate, some seem to be more crucial than others. An important item in Mark I is the behavioural rule for firms adaptation to their economic environment. Instead of the standard, infinite horizon, utility optimizing firm framework (that is both unrealistic and intractable), Mark I postulates a heuristic rule for production $Y_i(t)$ and price $p_i(t)$ update, which reads as follows:

$$\begin{aligned}
Y_i(t) = D_i(t)\ \&\ p_i(t) > \bar{p}(t) &\Rightarrow Y_i^T(t+1) = Y_i(t)[1+\gamma_y \xi_i(t)] \\
Y_i(t) = D_i(t)\ \&\ p_i(t) < \bar{p}(t) &\Rightarrow p_i(t+1) = p_i(t)[1+\gamma_p \xi_i(t)] \\
Y_i(t) > D_i(t)\ \&\ p_i(t) < \bar{p}(t) &\Rightarrow Y_i^T(t+1) = Y_i(t)[1-\gamma_y \xi_i(t)] \\
Y_i(t) > D_i(t)\ \&\ p_i(t) > \bar{p}(t) &\Rightarrow p_i(t+1) = p_i(t)[1-\gamma_p \xi_i(t)],
\end{aligned} \tag{1}$$

where $D_i(t)$ is the total demand for the goods produced by firm i at time t, and

$$\bar{p}(t) = \frac{\sum_i p_i(t) D_i(t)}{\sum_i D_i(t)} \tag{2}$$

is the average price of sold goods at time t, $\xi_i(t)$ a $U[0,1]$ random variable, independent across firms and across times, and γ_y, γ_p two parameters in $[0,1]$. The quantity $Y_i^T(t)$ is the *target* production at time t, not necessarily the realized one, as described below. These heuristic rules can be interpreted as a plausible *tâtonnement* process of the firms, that attempt to guess their correct production level and price based on the information on the last time step. In spirit, each unit time step might correspond to a quarter, so the order of magnitude of the γ parameters should be a few percent. Note that in the version of Mark I that we consider, wages are fixed to a constant value $W_i(t) \equiv 1$, for all times and all firms.

D. Debt and loans

The model further assumes linear productivity, hence the target production corresponds to a target workforce $Y_i^T(t)/\alpha$, where α is a constant coefficient that can always be set to unity (gains in productivity are not considered at this stage). The financial need of the firm is $\max[0, Y_i^T(t)W_i(t) - \mathcal{L}_i(t)]$, where $\mathcal{L}_i(t)$ is the cash available. The total current debt of the firm is $\mathcal{D}_i^T(t)$. The financial fragility of the firm $\ell_i(t)$ is defined in Mark I as the ratio of debt over cash. The offered rate by the banks for the loan is given by:

$$\rho_i(t) = \rho_0 G\left(\ell_i(t)\right) \times (1 + \xi_i'(t)), \tag{3}$$

where ρ_0 is the baseline (central bank) interest rate, G is an increasing function (taken to be $G(\ell) = 1 + \tanh(\ell)$ in the reference Mark I and $G(\ell) = 1 + \ln(1+\ell)$ in Mark I+), and ξ' another noise term. Depending on the rate offered, firms decide to take the full loan they need or only a fraction $F(\rho)$ of it, where $F \leq 1$ is a decreasing function of ρ, called "credit contraction". For example, in the reference Mark I, $F(\rho \leq 5\%) = 1$ and $F(\rho > 5\%) = 0.8$. We have played with the choice of the two functions F, G and the phase transition reported below is in fact robust whenever these functions are reasonable. In Mark I+, we chose a continuous function, such as to avoid built in discontinuities:

$$F(x) = \begin{cases} 1 \quad \text{if } x < 5\% \\ 1 - \frac{x-5\%}{5\%} \quad \text{if } 5\% < x < 10\% \\ 0 \quad \text{if } x > 10\%. \end{cases} \tag{4}$$

The important feature here is that when $F < 1$, the firm does not have enough money to hire the target workforce $Y_i^T(t)$ and is therefore obliged to hire less, or even to start firing in order to match its financial constraints. Note that financial constraints induce an *asymmetry* in the hiring/firing process: when firms are indebted, hiring will be slowed down by the cost of further loans. As we will see later, this asymmetry is responsible for an abrupt change in the steady state of the economy.

E. Spending budget and bankruptcy

Firms pay salaries to workers and households determine their budget as a fraction c (constant in time and across households) of their total savings (including the latest salary). Each household then selects M firms at random and sorts them according to the price; it then buys all it can buy from each firm sequentially, from the lower price to the highest price[3]. The budget left-over is added to the savings. Each firm sells a quantity $D_i(t) \leq Y_i(t)$, compute its profits (that includes interests paid on debt), and updates its cash and debt accordingly. Moreover, each firm pays back to the bank a fraction τ of its total debt $\mathcal{D}_i^T(t)$. It also pays dividends to the firm owners if profits are positive. Firms with negative liquidity $\mathcal{L}_i(t) < 0$ go bankrupt. In Mark I+, the cost of the bankruptcy (i.e. $-\mathcal{L}_i(t)$) is spread over healthy firms and on households. Once a firm is bankrupt it is re-initialized in the next time step with the owner's money, to a firm with a price and production equal to their corresponding average values at that moment in time, and zero debt (see Appendix A for more precise statements).

F. Numerical results: a phase transition

When exploring the phase space of Mark I, it soon becomes clear that the baseline interest rate ρ_0 plays a major role. In order not to mix different effects, we remove altogether the noise term ξ in Eq. (3) that affects the actual

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[3] In this sense, the good market is not efficient since the household demand is not necessarily satisfied when M is small. The job market, instead, is efficient because all the workers can contact all the firms until all the open positions are filled.

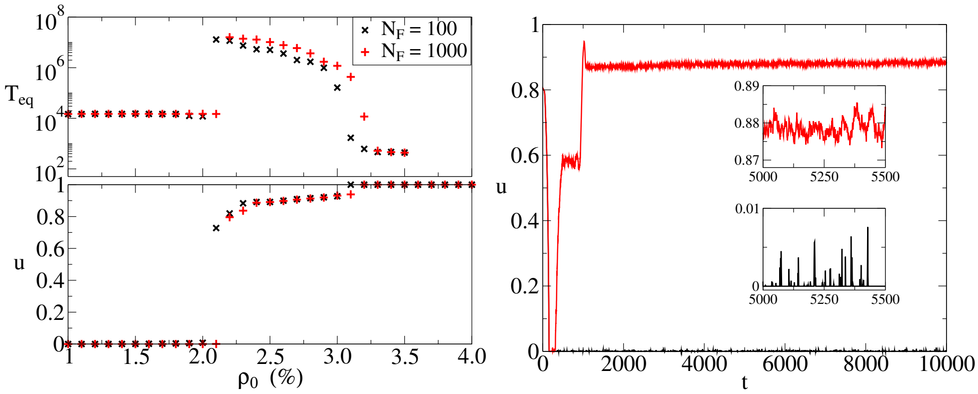

FIG. 2: *Left:* Average unemployment rate u as a function of the interest rate ρ_0 for two system sizes (with $N_{\rm H} = 10N_{\rm F}$ in both cases). The average is over 100 000 time steps discarding the first 50 000 time steps. The phase transition at $\rho_c = 2.1\%$ is of first order with u jumping discontinuously from small values to 1 (intermediate values obtained for $\rho_0 \in [2.1\%, 3.1\%]$ are only due to a much longer equilibration time near the critical point). Note that in the bottom graph we show averages for $T_{eq} = 50\,000$ regardless of ρ_0 while an estimate of the time needed to reach the steady state as a function of ρ_0 is plotted in logarithmic scale in the top graph. *Right:* Two trajectories of the unemployment rate with $N_{\rm F} = 1000$ at $\rho_0 = 1.9\%$ and $\rho_0 = 2.5\%$.

rate offered to the firms. We find that as long as ρ_0 is smaller than a certain threshold ρ_c, firms are on average below the credit contraction threshold and always manage to have enough loans to pay wages. In this case the economy is stable and after few (~ 100) time steps reaches a stationary state where the unemployment rate is low. If on the other hand the baseline interest rate ρ_0 exceeds a critical value ρ_c, firms cannot afford to take as much loans as they would need to hire (or keep) the desired amount of workers. Surprisingly, this induces a sudden, catastrophic breakdown of the economy. Production collapses to very small values and unemployment sky-rockets. This transition between two states of the economy takes place in both the reference Mark I and in the modified Mark I+; as we shall show in the next section, this transition is actually generic and occurs in simplified models as well. Note that ρ_c is *different* from the value at which $F(x)$ starts decreasing.

The data we show in Fig. 2 corresponds to Mark I+ with parameters $\gamma_p = \gamma_y = 0.1$ and $M = 3$ (see Appendix A for the general parameter setting of the model). While the qualitative behaviour of the model is robust, the details of the transition may change with other parameter settings. For example, smaller values of γ_p, γ_y lead to lower critical thresholds ρ_c (as well as smaller values of M) and to longer equilibration times (T_{eq} scales approximately as $1/\gamma_{y,p}$ for $\rho_0 < \rho_c$). Increasing the size of the economy only affects the magnitude of the fluctuations within one phase leaving the essential features of the transition unchanged. Interestingly, although it is not clear from Fig. 2, the model exhibits oscillatory patterns of the employment rate. The presence of these oscillations can be seen in the frequency domain of the employment rate time series (not shown here), which is essentially characterized by a white noise power spectrum with a well defined peak at intermediate frequencies. All these effects will become clearer within the reduced model described in the next section.

To sum up, our most salient finding is that Mark I (or Mark I+) has essentially two stationary states, with a first order (discontinuous) transition line separating the two. If the parameters are such that the system lies close to this critical line, then it is plausible that any small modulation of these parameters – such as the noise term that appears in Eq. (3) – will be *amplified* by the proximity of the transition, and lead to interesting boom/bust oscillations, of the kind originally observed in Mark I [29], and perhaps of economic relevance. The question, of course, is how generic this scenario is. We will now show, by studying much simplified versions of Mark I, that this transition is indeed generic, and is induced by any asymmetry between hiring and firing. We will then progressively enrich our watered-down model (call "Mark 0" below) and see how the qualitative picture that we propose is affected by additional features.

III. HYBRID ABM'S: THE MINIMAL "MARK 0" MODEL

Moving away from the RA framework, Agent-Based Modeling bites the bullet and attempts to represent in details *all* the individual components of the economy (as, for example, in [23, 24]). This might however be counter-productive, at least in the research stage we are still in: keeping too many details is not only computer-time consuming, it may also hobble the understanding of the effects that ABMs attempt to capture. It may well be that some sectors of the economy can be adequately represented in terms of aggregate variables, while discreteness, heterogeneities and interactions are crucial in other sectors. In our attempt to simplify Mark I, we posit that the whole household sector can be represented by aggregate variables: total savings $S(t)$, total income $W(t)$ and total consumption budget $B(t)$. We also remove the banking sector and treat the loans in the simplest possible way – see below. While the interest rate is zero in the simplest version, the incentive to hire/fire provided by the interest rate, that was at play in Mark I, will be encoded in a phenomenological way in the update rule for production. The firms, on the other hand, are kept as individual entities (but the above simplifications will allow us to simulate very large economies, with $N_{\rm F} = 100\,000$ firms or more).

A. Set-up of the model

The minimal version of the Mark 0 model is defined as follows. The salient features are:

- There are $N_{\rm F}$ firms in total and $\mu N_{\rm F}$ households, $\mu > 1$.[4] Each firm $i = 1 \cdots N_{\rm F}$ pays a salary $W_i(t)$ and has a production $Y_i(t)$. Productivity is fixed to $\alpha = 1$, therefore $Y_i(t)$ is also equal to the number of employees of firm i. Therefore the employment $\varepsilon(t)$ and unemployment $u(t)$ are:

$$\begin{aligned} \varepsilon(t) &= \frac{1}{\mu N_{\rm F}} \sum_i Y_i(t) \ , \\ u(t) &= 1 - \varepsilon(t) \ . \end{aligned} \tag{5}$$

- Households are described by their total savings $S(t)$ (which at this stage are always non-negative) and by their total income $\sum_i W_i(t) Y_i(t)$. At each time step, they set a total consumption budget[5]

$$B(t) = c\,[S(t) + \sum_i W_i(t) Y_i(t)] \tag{6}$$

  which is distributed among firms using an intensity of choice model [30]. The demand of goods for firm i is therefore:

$$D_i(t) = \frac{B(t)}{p_i(t)} \frac{e^{-\beta p_i(t)/\overline{p}(t)}}{\sum_i e^{-\beta p_i(t)/\overline{p}(t)}} \ , \tag{7}$$

  where β is the price sensitivity parameter ($\beta = 0$ corresponds to complete price insensitivity and $\beta \to \infty$ means that households select only the firm with the lowest price). The normalization is such that $B(t) = \sum_i p_i(t) D_i(t)$, as it should be.

- Firms are described by their price $p_i(t)$, their salary $W_i(t)$, and their production $Y_i(t)$.
  - For simplicity, we fix the salary $W_i(t) \equiv 1$ – an extension that includes wage dynamics is discussed below.
  - For the price, we keep the Mark I price update rule (1), with the average production-weighted price:

$$\overline{p}(t) = \frac{\sum_i p_i(t) Y_i(t)}{\sum_i Y_i(t)} \ . \tag{8}$$

  - For production, we assume that firms are more careful with the way they deal with their workforce than posited in Mark I. Independently of their price level, firms try to adjust their production to the observed demand. When firms want to hire, they open positions on the job market; we assume that the total number of unemployed workers, which is $\mu N_{\rm F} u(t)$, is distributed among firms according to an intensity of choice of model which depends on both the wage offered by the firm[6] and on the same parameter β as it is for Eq. (7); therefore the maximum number of available workers to each firm is:

$$\mu \tilde{u}_i(t) = \frac{e^{\beta W_i(t)/\overline{w}(t)}}{\sum_i e^{\beta W_i(t)/\overline{w}(t)}} \mu N_{\rm F} u(t) \ . \tag{9}$$

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[4] Actually, households are treated as a unique aggregate variable, therefore μ is not a relevant parameter: one can see that its value is irrelevant and one can always set $\mu = 1$ for simplicity. Yet it is useful to think that the aggregate variables represents in an effective way a certain number of individual households, hence we keep the parameter μ explicit in the following.

[5] Of course, one could choose different c's for the fraction of savings and the fraction of wages devoted to spending, or any other non-linear spending schedule.

[6] Since at this stage wages are equal among firms the distribution is uniform. Below we allow firms to update their wage. A higher wage will then translate in the availability of a larger share of unemployed workers in the hiring process.

where

$$\overline{w}(t) = \frac{\sum_i W_i(t) Y_i(t)}{\sum_i Y_i(t)} \ . \tag{10}$$

In summary, we have

$$\begin{aligned}
&\text{If } Y_i(t) < D_i(t) \quad \Rightarrow \quad \begin{cases} Y_i(t+1) = Y_i(t) + \min\{\eta_+(D_i(t) - Y_i(t)), \mu \tilde{u}_i(t)\} \\ \text{If } p_i(t) < \overline{p}(t) \quad \Rightarrow \quad p_i(t+1) = p_i(t)(1 + \gamma_p \xi_i(t)) \end{cases} \\
&\text{If } Y_i(t) > D_i(t) \quad \Rightarrow \quad \begin{cases} Y_i(t+1) = \max\{Y_i(t) - \eta_-[Y_i(t) - D_i(t)], 0\} \\ \text{If } p_i(t) > \overline{p}(t) \quad \Rightarrow \quad p_i(t+1) = p_i(t)(1 - \gamma_p \xi_i(t)) \end{cases}
\end{aligned} \tag{11}$$

where $\eta_\pm \in [0,1]$ are the hiring/firing propensity of the firms: for example say $\eta_- = 0.5$ means that half of the excess production force of a given firm is fired at each time step. Note that this rule ensures that there is no overshoot in production; furthermore the max in the second rule is not necessary mathematically when $\eta_- \leq 1$, but we kept it for clarity.

- Accounting is done as follows. Each firm $i = 1 \cdots N_{\mathrm{F}}$ pays a total wage bill $W_i(t) Y_i(t)$ and receives an amount $p_i(t) \min\{Y_i(t), D_i(t)\}$ from the goods sold. Moreover, if the total profit $\mathcal{P}_i(t) = p_i(t) \min\{Y_i(t), D_i(t)\} - W_i(t) Y_i(t)$ is positive, the firm pays a dividend $\delta \times \mathcal{P}_i(t)$ to the households.[7] In summary, the accounting equations for total savings $S(t)$ and firms' liquidities $\mathcal{L}_i(t)$ are the following (here $\theta(x)$ is the Heaviside step function):

$$\begin{aligned}
\mathcal{P}_i(t) &= p_i(t) \min\{Y_i(t), D_i(t)\} - W_i(t) Y_i(t) \ , \\
\mathcal{L}_i(t+1) &= \mathcal{L}_i(t) + \mathcal{P}_i(t) - \delta \mathcal{P}_i(t) \theta(\mathcal{P}_i(t)) \ , \\
S(t+1) &= S(t) - \sum_i \mathcal{P}_i(t) + \delta \sum_i \mathcal{P}_i(t) \theta(\mathcal{P}_i(t)) \ ,
\end{aligned} \tag{12}$$

  and total money $S(t) + \sum_i \mathcal{L}_i(t)$ is clearly conserved.

- Note that we allow for negative firms' liquidities, which we interpret as the firm being indebted. We measure the indebtment level of a firm through the ratio

$$\Phi_i = -\mathcal{L}_i / (W_i Y_i) \tag{13}$$

  which we interpret as a measure of financial fragility. If $\Phi_i(t) < \Theta$, i.e. when the level of debt is not too high compared to the size of the company (measured as the total payroll), the firm is allowed to continue its activity. If on the other hand $\Phi_i(t) \geq \Theta$, the firm defaults. The default resolution we choose is the following. We first define the set $\mathcal{H}$ of financially healthy firms[8] j, that are potential buyers for the defaulted firm i. The condition for this is that $\mathcal{L}_j(t) > -\mathcal{L}_i(t)$, meaning that the firm j can take on the debt of i without going under water.

  – With probability $1 - f$, $f \in [0,1]$ being a new parameter, a firm j is chosen at random in $\mathcal{H}$; j transfers to i the needed money to pay the debts, hence $\mathcal{L}_j \to \mathcal{L}_j + \mathcal{L}_i$ (remember that $\mathcal{L}_i$ is negative) and $\mathcal{L}_i \to 0$ after the transaction. Furthermore, we set $p_i = p_j$ and $W_i = W_j$, and the firm i keeps its employees and its current level of production.

  – With probability f, or whenever $\mathcal{H} = \emptyset$, the firm i is not bailed out, goes bankrupt and its production is set to zero. In this case its debt $\mathcal{L}_i(t) < 0$ is transferred to the households' savings, $S(t) \to S(t) + \mathcal{L}_i(t)$ in order to keep total money fixed[9].

  Hence, when f is large, most of the bankruptcies load weighs on the households, reducing their savings, whereas when f is small, bankruptcies tend to fragilise the firm sector.

  Note that although the total amount of "real" money M is strictly conserved in Mark 0 (as discussed above), the amount of money in circulation may exceed M since we allow firms with negative equities (i.e. indebted firms).

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[7] We have also considered the case where firms distribute a fraction δ of the profits *plus* the reserves. See below.

[8] $j \in \mathcal{H}$ whenever $\mathcal{L}_j(t) > \Theta Y_j(t) W_j(t)$.

[9] One can interpret this by imagining the presence of a bank that collects the deposits of households and lend money to firms, at zero interest rate. If a firm goes bankrupt, the bank looses its loan, which means that its deposits (the households' savings) are reduced.

- A defaulted firm has a finite probability φ per unit time to get revived; when it does so its price is fixed to $p_i(t) = \overline{p}(t)$, its workforce is the available workforce, $Y_i(t) = \mu u(t)$, and its liquidity is the amount needed to pay the wage bill, $\mathcal{L}_i(t) = W_i(t)Y_i(t)$. The liquidity is provided by the households, therefore $S(t) \to S(t) - W_i(t)Y_i(t)$ when the firm is revived, again to ensure total money conservation. Note that during this bankrupt/revival phase, the households' savings $S(t)$ might become negative: if this happens, then we set $S(t) = 0$ and the debt of households is spread over the firms with positive liquidity[10], proportionally to their current value of $\mathcal{L}_i$, again in order to ensure total money conservation and $S(t) \geq 0$.

- Note that because of the $\min\{D_i(t) - Y_i(t), \mu u(t)\}$ term in Eq. (11), the total production of the model is bounded by $\sum_i Y_i(t) = \mu N_{\mathrm{F}}\varepsilon(t) \leq \mu N_{\mathrm{F}}$, as it should be because $\varepsilon(t) = 1$ corresponds to full employment and in that case $\sum_i Y_i(t) = \mu N_{\mathrm{F}}$. However, in the following we will sometimes (when stated) remove this bound for a better mathematical tractability. This amounts to replace $\min\{D_i(t) - Y_i(t), \mu u(t)\} \to D_i(t) - Y_i(t)$ in Eq. (11) (it corresponds to choosing $\mu = \infty$). When the bound is removed the high employment phase translates into an exponential explosion of the employment rate which is of course unrealistic.

The above description contains all the details of the definition of the model, however for full clarification a pseudo-code of this minimal Mark 0 model is provided in Appendix B (together with the extensions discussed in Sec. V). The total number of relevant parameter of Mark 0 is equal to 9: $c, \beta, \gamma_p, \eta_\pm, \delta, \Theta, \varphi, f$ plus the number of firms N_{F}. However, most of these parameters end up playing very little role in determining the *qualitative*, long-time aggregate behaviour of the model. The two most important quantities in this respect turn out to be:

1. the ratio $R = \eta_+/\eta_-$ between η_+ and η_-, which is meant to capture any asymmetry in the hiring/firing process. As noted above, a rising interest rate leads to such a hiring/firing asymmetry in Mark I and Mark I+; but other sources of asymmetry can be envisaged: for example, overreaction of the firms to bad news and under-reaction to good news, leading to an over-prudent hiring schedule. Capital inertia can also cause a delay in hiring, whereas firing can be immediate.

2. the default threshold Θ, which controls the ratio between total debt and total circulating currency. In our minimal setting with no banks, it plays the role of a money multiplier. Monetary policy within Mark 0 boils down to setting of the maximum acceptable debt to payroll ratio Θ.

The other parameters change the phase diagram of the model quantitatively but not qualitatively. In order of importance, the most notable ones are f (the redistribution of debt over households or firms upon bankruptcies) and β (the sensitivity to price), see below.

B. Numerical results & Phase diagram

When running numerical simulations of Mark 0, we find (Fig. 3) that after a transient that can be surprisingly long[11], the unemployment rate settles around a well defined average value, with some fluctuations (except in some cases where endogenous crises appear, see below). We find the same qualitative phase diagram for all parameters $\beta, \gamma_p, \varphi, \delta, f$. For a given set of parameters, there is a critical value R_c of $R = \eta_+/\eta_-$ separating a full unemployment phase for $R < R_c$ from a phase where some of the labour force is employed for $R > R_c$. Here $R_c \leq 1$ is a value that depends on all other parameters, that we compute analytically in the next section in a special case.

The transition is particularly abrupt in the limit $\Theta \to \infty$ (no indebtment limit), in which the unemployment rate jumps from 0 to 1 at R_c, see Fig. 3. The figure shows that indeed only the ratio $R = \eta_+/\eta_-$ is relevant, the actual values of $\eta_\pm$ only change the time scale over which the production fluctuates. Moreover, the phase diagram is almost independent of N_{F}, which confirms that we are effectively in a limit where the number of firms can be considered to be very large[12].

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[10] Again, this can be interpreted by imagining that the firms with positive liquidity deposit their cash in the bank. When the bank needs to provide a loan to a revived firm, or loses money due to a bankrupt, it prefers to take this money from households' deposits, but if these are not available, then it takes the money from firms' deposits.

[11] Think of one time step as a quarter, which seems reasonable for the frequency of price and workforce updates. The equilibration time is then 20 years or so, or even much longer as for the convergence to the 'bad' state in Mark I, see Fig. 2. Albeit studying a very different ABM, similarly long time scales can be observed in the plots shown in [28]. See also the discussion in the conclusion on this point

[12] Actually, in the specific case $\Theta = \infty$, even $N_{\mathrm{F}} = 1$ provides a similar phase diagram, although the critical R_c slightly depends on N_{F} when this number is small

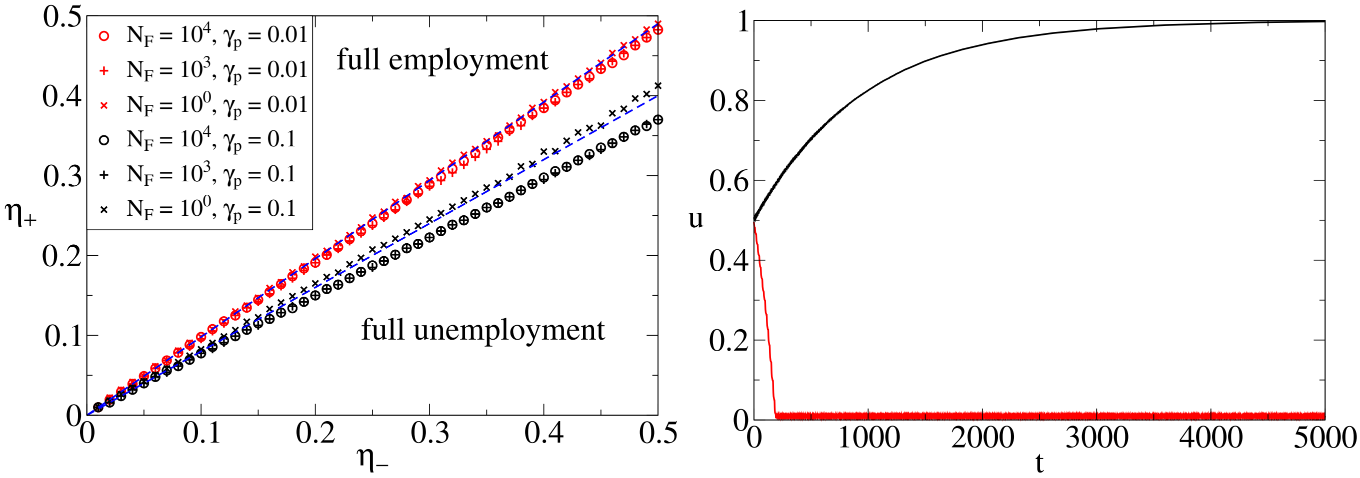

FIG. 3: (*Left*) Phase diagram of the basic Mark 0 model with $\beta = 0$ and $\Theta = \infty$. There are two distinct phases separated by a critical line which depends mostly on the parameter γ_p and is less and less sensitive to the system size as long as $\gamma_p \to 0$. The dashed lines correspond to the analytical result in Eq. (24) which agrees well with numerical simulations for small values of η_+, η_- where the perturbative method of Section III-C and Appendix C 1 is justified. (*Right*) Two typical trajectories of $u(t)$ in the two phases ($R = 5/3$ in the full employment line and $R = 3/5$ in the full unemployment line). The other parameters are: $N_{\rm F} = 10\,000$, $\gamma_p = 0.1$, $\Theta = \infty$, $\varphi = 0.1$, $\beta = 0$, $c = 0.5$. $\delta = 0.02$ and $\varphi = 1$.

For finite values of Θ, the phase diagram is more complex (see Fig. 4) and, besides the full unemployment phase (region 1, which always prevails when $R < R_c$, we find three other different regions for $R > R_c$, that actually survive many extensions of Mark 0 that we have considered (see section V):

- At very large Θ (region 4, "FE"), the full employment phase persists, although a small value of the unemployment appears in a narrow region around $R \approx R_c$. The width of this region of small unemployment vanishes as Θ increases.
- At very low Θ (region 2, "RU"), one finds persistent "Residual Unemployment" in a large region of $R > R_c$. The unemployment rate decreases continuously with R and Θ but reaches values as large as 0.5 close to $R = R_c$ (see Fig. 4, Bottom Left).
- A very interesting "endogenous crises" phase appears for intermediate values of Θ (region 3, "EC"), where the unemployment rate is most of the time very close to zero, but endogenous crises occur, which manifest themselves as sharp spikes of the unemployment that can reach quite large values. These spikes appear almost periodically, and their frequency and amplitude depend on some of the other parameters of the model, in particular f and β, see Fig. 4, Bottom Right.

The phase diagram in the plane $R - \Theta$ is presented in Fig. 4, together with typical time series of the unemployment rate, for each of the four phases.

We also show in Fig. 5 a trajectory of $u(t)$ in the good phase of the economy (FE, region 4) and zoom in on the small fluctuations of $u(t)$ around its average value. These fluctuations reveal a clear periodic pattern in the low unemployment phase; recall that we had already observed these oscillations within Mark I+. Oscillating patterns (perhaps related to the so-called business cycle?) often appear in simplified first order differential models of the macro-economy; one of the best known examples is provided by the Goodwin model [31, 32], which is akin to a predator-prey model where these oscillations are well known. But these oscillations do also show up in other ABMs, see [26, 28]. Note that these fluctuations/oscillations around equilibrium *do not regress* when the number of firms get larger. We have simulated the model with $N_{\rm F} = 10\,000$ firms and $N_{\rm F} = 1\,000\,000$ firms with nearly identical amplitude and frequencies for these fluctuations. We will offer some insight on the origin of these oscillations in the next section.

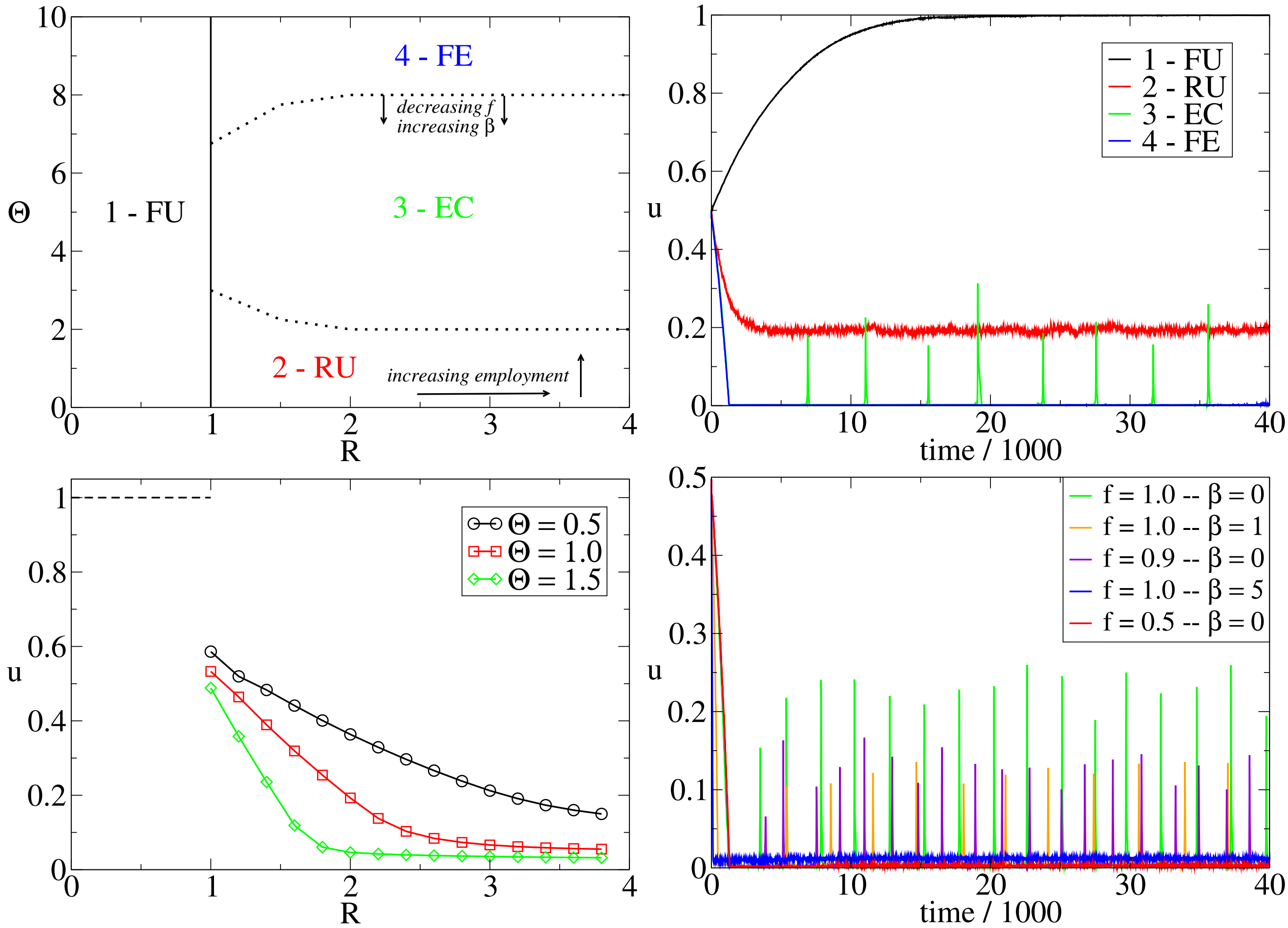

FIG. 4: (*Top Left*) Phase diagram of the basic Mark 0 model in the $R - \Theta$ plane, here with $N_{\rm F} = 5000$, $c = 0.5$, $\gamma_p = 0.05$, $\delta = 0.02$, $\varphi = 0.1$, $f = 1$, $\beta = 0$. There are four distinct phases separated by critical lines. We report schematically the evolution of the critical lines with the parameters f and β; the other parameters are irrelevant. (*Top Right*) Typical time series of $u(t)$ for each of the phases. (*Bottom Left*) Stationary value of the unemployment rate as a function of R for different values of Θ in phase 2. (*Bottom Right*) Typical trajectories of $u(t)$ in the Endogenous Crisis phase (region 3) for different values of f and β, the other parameters are kept fixed. Note that increasing β or decreasing f lead to small crisis amplitudes.

C. Qualitative interpretation. Position of the phase boundaries

An important quantity that characterizes the behavior of the model in the "good" phase of the economy (i.e. for $R > R_c$) is the leverage (debt-to-equity) ratio k:

$$k(t) = -\frac{\sum_i \min\{\mathcal{L}_i(t), 0\}}{S(t) + \sum_i \max\{\mathcal{L}_i(t), 0\}} \tag{14}$$

where, due to money conservation, $k \leq 1$. The good state of the economy is characterized by a large value of k reflecting the natural tendency of the economy towards indebtment, the level of which being controlled in Mark 0 by the parameter Θ (the ratio k increases with Θ).[13] Interestingly, in regions 2-RU and 4-FE $k(t)$ reaches a stationary state, whereas in region 3-EC its dynamics is characterized by an intermittent behavior corresponding to the appearance of endogenous crises during which indebtment is released through bankruptcies.

This phenomenology can be qualitatively explained by the dynamics of the distribution of firms fragilities Φ_i. For $R > R_c$, firms overemploy and make on average negative profits, which means that the Φ_i's are on average drifting towards the bankruptcy threshold Θ.

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[13] In a further version of the model, a central bank will be in charge of controlling k through a proper monetary policy.

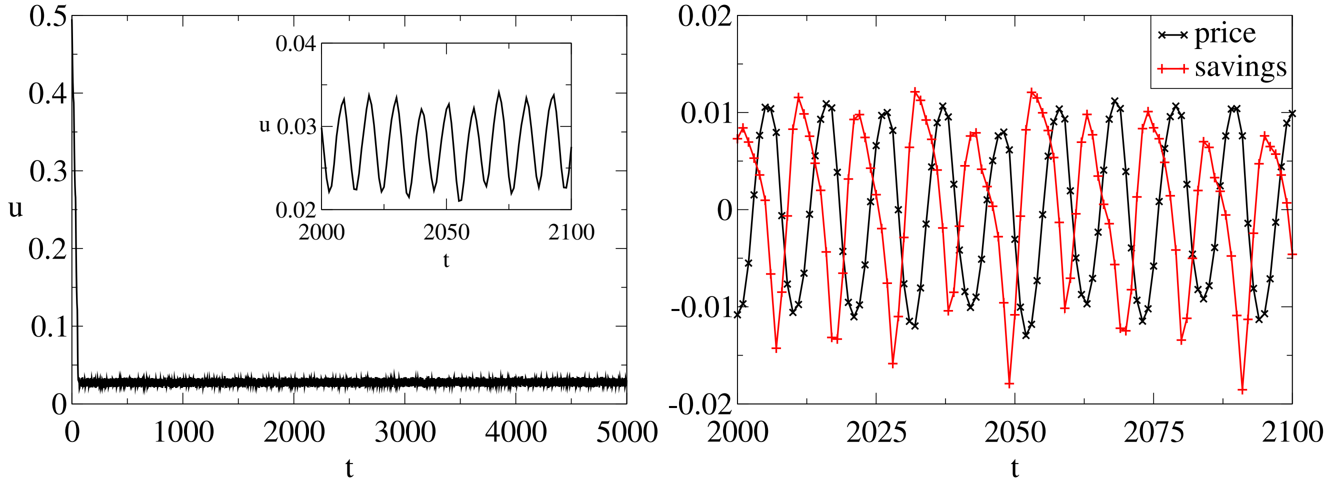

FIG. 5: *Left*: Typical time evolution of the unemployment u, starting from an initial condition $u = 0.5$, for the basic Mark 0 model in the Full Employment phase (region 4). The trajectory leading to a "good" state of the economy is obtained for $\eta_+ = 0.5$ and $\eta_- = 0.3$. Note the clear endogenous "business cycles" that appear in that case. These runs are performed with $N_{\rm F} = 10\,000$ firms, $c = 0.5$, $\beta = 2$, $\gamma_p = 0.1$, $\varphi = 0.1$, $\Theta = 5$, $\delta = 0.02$.
Right: Oscillations for the average price and the average savings per household (each shifted by their average values for clarity) for the same run as in the *Left* figure. When prices are low, savings increase, while when prices are high, savings decrease. See text for a more detailed description.

- When Θ is small enough (i.e. in region 2-RU) the drift is continuously compensated by the reinitialization of bankrupted firms and the fragility distribution reaches a stationary state.
- For intermediate values of Θ, however, (i.e. in region 3-EC) the number of bankruptcies per unit of time becomes intermittent. Firms fragilities now collectively drift towards the bankruptcy threshold; as soon as firms with higher fragilities reach Θ, bankruptcies start to occur. Since for f large enough, bankruptcies are mostly financed by households, demand starts falling which has the effect of increasing further the negative drift. This feedback mechanism gives rise to an avalanche of bankruptcies after which most of the firms are reinitialized with positive liquidities. This mechanism has the effect of *synchronizing* the fragilities of the firms, therefore leading to cyclical waves of bankruptcies, corresponding to the unemployment spikes showed in Fig. 4. The distribution of firms fragilities does not reach a stationary state in this case.
- When $\Theta \gg 1$ (i.e. region 4-FE), households are wealthy enough to absorb the bankruptcy cost without pushing the demand of goods below the maximum level of production reached by the economy. [14] Hence, the economy settles down to a full employment phase with a constant (small) rate of bankruptcies.

The above interpretation is supported by a simple one-dimensional random walk model for the firms assets, with a drift that depends on the number of firms that fail. This highly simplified model accurately reproduces the above phenomenology, and is amenable to a full analytical solution, which will be published separately.

The dependence of the phase boundaries on the different parameters is in general quite intuitive. The dependence of aggregate variables on β, for example, is interesting: everything else being kept equal, we find that increasing β (i.e. increasing the price selectivity of buyers), increases the level of unemployment (see inset of Fig. 6) and the amplitude of the fluctuations around the average value (a similar effect was noted in [24]). Increasing γ_p increases the dispersion of prices around the average value and is thus similar to increasing β.

Increasing β has however also counter-intuitive effects: it increases both the average price compared to wages and the profits of firms, hence stabilizing the FE phase and shifting its boundary with EC to lower values of Θ (see Fig. 6 for the amplitude of region 3-EC as a function of β). This effect can be understood by considering the demand-production gap $\delta Y_i = D_i - Y_i$ as a function of the price difference $\delta p_i = p_i - \bar{p}$. For a fixed value of δY_i the rules for price and production updates are independent of β; however, the response of the demand to a price change is

[14] The amount of money circulating in the economy increases with Θ and for $R > R_c$ it is largely channelled to households savings since firms have on average negative liquidities.

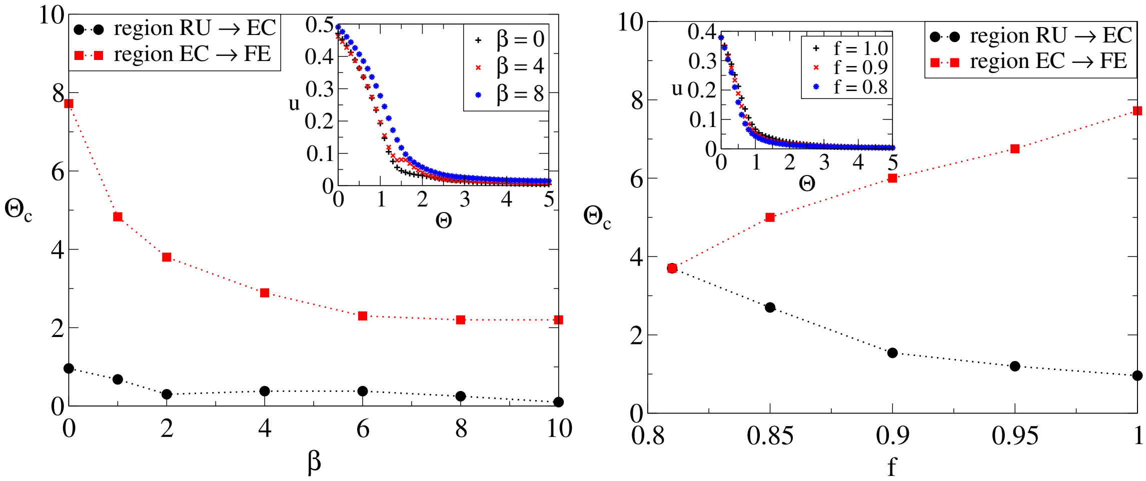

FIG. 6: Phase boundary location Θ_c as a function of the price sensitivity β (left) and of the debt share parameter f (right). Circles correspond to the location of the boundary between regions 2-RU and 3-EC while squares correspond to the boundary between regions 3-EC and 4-FE. In the inset of both figures we plot the average unemployment rate as a function of Θ for different values of β and f. For $f \leq 0.81$ region 4-EC disappears and the average value of the unemployment continuously goes to 0 without the appearance of endogenous crises. Increasing β shrinks the amplitude of region 3-EC which however remains finite for $\beta \gg 1$. As a criterion for being in region 3-EC we require that the amplitude of the crises $\max(u) - \min(u)$ stays above 5%. Parameters are: $N = 5000$, $R = 3$ ($\eta_m = 0.1$), $\delta = 0.02$, $\varphi = 0.1$, $\beta = 0$, $\gamma_p = 0.05$.

stronger for higher values of β. As a consequence, the absolute value of the gap δY_i for a given value of δp_i is on average increasing with β.[15]. Therefore, the total amount of unsold goods $\sum_i \max[D_i - Y_i, 0]$ and households savings S (households involuntarily save money) also increase with β. Such larger unsatisfied demand is responsible for the average price increase while households increased wealth shifts the boundary of region FE to lower values of Θ.

Decreasing f (i.e. the financial load taken up by households when bankruptcies occur) also stabilize, as expected, the full employment phase. Such a stabilization can also be achieved by distributing a fraction δ of the profit plus the total liquidity of the firms (instead of just a fraction δ of the profits), which has the obvious effect of supporting the demand. In fact, we find that as soon as $f \leq 0.81$ or $\delta \geq 0.25$, the Endogenous Crisis phase disappears and is replaced by a continuous cross-over between the Residual Unemployment phase and the Full Employment phase (see Fig. 6).

D. Intermediate conclusion

The main message of the present section is that in spite of many simplifications, and across a broad range of parameters, the phase transition observed in Mark I as a function of the baseline interest rate is present in Mark 0 as well. We find that these macroeconomic ABMs display quite generically two very different phases – high demand/low unemployment vs. low demand/high unemployment, with a boundary between the two that is essentially controlled by the asymmetry between the hiring and firing propensity of the firms. Moreover, in the Mark 0 model there is an additional splitting of the low unemployment phase in several regions characterized by a different dynamical behaviour of the unemployment rate, depending on the level of debt firms can accumulate before being forced into bankruptcy. We find in particular an intermediate debt region where endogenous crises appear, characterized by acute unemployment spikes. This Endogenous Crisis phase disappears when households are spared from the financial load of bankruptcies and/or when capital does not accumulate within firms, but is transferred to households through dividends.

We now explain how the phase transition at R_c can be understood, in a still simplified setting, using analytical calculations. We will then turn to several extensions of Mark 0, in particular allowing for some wage dynamics that lead to long term inflation or deflation, absent in the above version of the model.

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[15] For small values of δp_i one finds approximately $\delta Y_i \approx -C(\beta)\delta p_i$, where $C(\beta)$ is increasing with β.

IV. ANALYTICAL DESCRIPTION

We attempt here to describe analytically some aspects of the dynamics of Mark 0 in its simplest version, namely without bankruptcies ($\Theta = \infty$ for which φ and f become irrelevant), with $\beta = 0$, fixed wages $W = 1$, and no dividends ($\delta = 0$). For simplicity, we also fix $\mu = 1$ and $c = 1/2$. The only relevant parameters are therefore γ_p and $\eta_\pm$. The equations of motion of this very minimal model are:

$$\begin{aligned}
&\text{If } Y_i(t) < D_i(t) \quad \Rightarrow \quad \begin{cases} Y_i(t+1) = Y_i(t) + \min\{\eta_+[D_i(t) - Y_i(t)], 1 - \overline{Y}(t)\} \\ \text{If } p_i(t) < \overline{p}(t) \quad \Rightarrow \quad p_i(t+1) = p_i(t)(1 + \gamma_p \xi_i(t)) \end{cases} \\
&\text{If } Y_i(t) > D_i(t) \quad \Rightarrow \quad \begin{cases} Y_i(t+1) = \max\{Y_i(t) - \eta_-[Y_i(t) - D_i(t)], 0\} \\ \text{If } p_i(t) > \overline{p}(t) \quad \Rightarrow \quad p_i(t+1) = p_i(t)(1 - \gamma_p \xi_i(t)) \end{cases} \\
&D_i(t) = \frac{1}{2p_i(t)}[\max\{s(t), 0\} + \overline{Y}(t)] \\
&\mathcal{L}_i(t+1) = \mathcal{L}_i(t) - Y_i(t) + p_i(t)\min\{Y_i(t), D_i(t)\} \\
&s(t) = M_0 - \overline{\mathcal{L}}(t) \ .
\end{aligned} \tag{15}$$

Here M_0 is the total money in circulation, whose precise value is irrelevant for this discussion, and $s = S/N_\mathrm{F}$ are the savings per agent. Overlines denote an average over firms, which is flat for $\overline{Y} = N_\mathrm{F}^{-1} \sum_i Y_i(t)$ and $\overline{\mathcal{L}} = N_\mathrm{F}^{-1} \sum_i \mathcal{L}_i(t)$ while it is weighted by production for $\overline{p}$, see Eq. (8). Note that the basic variables here are $\{p_i, Y_i, \mathcal{L}_i\}$, all the other quantities are deduced from these ones.

In the high employment phase, the model admits a stationary state with $\overline{Y}_{st} \sim 1$. To show this, let us focus on the case where γ_p is very small, but not exactly zero, otherwise of course the price dynamics is frozen. The stationary state is attained after a transient of duration $\sim 1/\gamma_p$, and in the stationary state fluctuations between firms are very small in such a way that $p_i \sim \overline{p}$ and $Y_i \sim \overline{Y}$. A stationary state of Eq. (15) has $Y_i(t+1) = Y_i(t)$ and therefore $Y_i = D_i$, hence $Y_i = D_i = \overline{Y}$. Furthermore, from $\mathcal{L}_i(t+1) = \mathcal{L}_i(t)$ we deduce that $p_i = \overline{p} = 1$. Finally, we have $D_i = \overline{Y} = (s + \overline{Y})/2$, which gives $s = \overline{Y}$ and $\mathcal{L} = M_0 - \overline{Y}$. One obtains therefore a continuum of stationary states, because the production $\overline{Y}$ or equivalently the employment are not determined by requiring stationarity. However, we will show in the following that these equilibria become unstable as soon as fluctuations are taken into account ($\gamma_p > 0$). We will see that fluctuations can induce either an exponentially fast decrease of $\overline{Y}$ towards zero (corresponding to the full unemployment phase), or an exponentially fast grow of $\overline{Y}$, which is therefore only cutoff by the requirement $\overline{Y} \leq \mu = 1$, corresponding to full employment. It is therefore natural to choose the stationary state with $\overline{Y} = 1$ as a reference, and study the effect of fluctuations around this state. We therefore consider the stationary state with $Y_i = D_i = p_i = 1$, and $\mathcal{L}_i = M_0 - 1$ in such a way that $s = \overline{Y} = 1$, and we consider the limit in which $\gamma_p \sim 0$, in such a way that we can expand around the high employment stationary state and obtain analytical results for the phase transition point. Numerically, the fact that γ_p is small does not change the qualitative behavior of the model.

In order to consider small fluctuations around the high employment stationary state, we define the following variables:

$$\begin{aligned}
Y_i(t) &= 1 - \gamma_p \zeta_i(t) \ , \\
p_i(t) &= e^{\gamma_p \lambda_i(t)} \ , \\
\mathcal{L}_i(t) &= M_0 - 1 - \gamma_p \alpha_i(t) \ .
\end{aligned} \tag{16}$$

Note that in the following, for ζ, α, λ, overlines always denote flat averages over firms.

A. Stability of the high employment phase

To study the stability of the high employment phase, we make two further simplifications. First, we neglect the fluctuations of the savings and fix $\alpha_i \equiv 0$. This is justified if the employment rate varies slowly over the time scale $\tau_c = -1/\ln(1-c)$ that characterizes the dynamics of the savings. But, as we shall show below, the dynamics of employment becomes very slow in the vicinity of the phase transition, hence this assumption seems justified. Second, we consider that $\eta_+ = R\eta_-$ with R of order one and η_- small. Then, it is not difficult to see that ζ_i is a random variable of order η_+ and is therefore also very small. We define $\zeta_i = \eta_+ z_i$. We believe that these approximations should be quite accurate, as is confirmed by the comparison of the theoretical transition line with numerical data.

With these simplifications (i.e. setting $\alpha_i = 0$ and assuming that $\eta_\pm$ and $\zeta_i = \eta_+ z_i$ are small), we have, together with Eq. (16):

$$\begin{aligned} D_i(t) &= 1 - \gamma_p \lambda_i(t) + \frac{1}{2}\gamma_p^2 \lambda_i^2 + O(\gamma_p \eta_+) + O(\gamma_p^3) \ , \\ \overline{p}(t) &= 1 + \gamma_p \overline{\lambda}(t) + \frac{1}{2}\gamma_p^2 \overline{\lambda^2}(t) + O(\gamma_p \eta_+) + O(\gamma_p^3) \ , \end{aligned} \tag{17}$$

In the following we keep the terms of order γ_p^2 while we neglect the terms of order $\gamma_p \eta_+$ and γ_p^3, or in other words we assume that $\eta_+ = o(\gamma_p)$ and we neglect all terms that are $o(\gamma_p^2)$. Then the first two lines of Eq. (15) become

$$\begin{aligned} &\text{If } \lambda_i(t) - \frac{1}{2}\gamma_p \lambda_i(t)^2 < 0 \quad \Rightarrow \quad \begin{cases} z_i(t+1) = z_i(t) - \min\{-\lambda_i(t) + \frac{1}{2}\gamma_p \lambda_i(t)^2, \overline{z}(t)\} \\ \lambda_i(t+1) = \lambda_i(t) + \xi_i(t) - \frac{1}{2}\gamma_p \xi_i(t)^2 \end{cases} \\ &\text{If } \lambda_i(t) - \frac{1}{2}\gamma_p \lambda_i(t)^2 > 0 \quad \Rightarrow \quad \begin{cases} z_i(t+1) = z_i(t) + \frac{1}{R}[\lambda_i(t) - \frac{1}{2}\gamma_p \lambda_i(t)^2] \\ \lambda_i(t+1) = \lambda_i(t) - \xi_i(t) - \frac{1}{2}\gamma_p \xi_i(t)^2 \end{cases} \end{aligned} \tag{18}$$

Therefore, in this limit, the evolution of λ decouples from that of ζ (or z). Note also that if we assume that $|\lambda_i|$ is bounded by a quantity of order 1 (i.e. not of order $1/\gamma_p$), then the condition $\lambda_i(t) - \frac{1}{2}\gamma_p \lambda_i(t)^2 > 0$ is equivalent to $\lambda_i(t) > 0$. Finally we have

$$\begin{aligned} &\text{If } \lambda_i(t) < 0 \quad \Rightarrow \quad \begin{cases} z_i(t+1) = z_i(t) - \min\{-\lambda_i(t) + \frac{1}{2}\gamma_p \lambda_i(t)^2, \overline{z}(t)\} \\ \lambda_i(t+1) = \lambda_i(t) + \xi_i(t) - \frac{1}{2}\gamma_p \xi_i(t)^2 \end{cases} \\ &\text{If } \lambda_i(t) > 0 \quad \Rightarrow \quad \begin{cases} z_i(t+1) = z_i(t) + \frac{1}{R}[\lambda_i(t) - \frac{1}{2}\gamma_p \lambda_i(t)^2] \\ \lambda_i(t+1) = \lambda_i(t) - \xi_i(t) - \frac{1}{2}\gamma_p \xi_i(t)^2 \end{cases} \end{aligned} \tag{19}$$

From this equation we can write evolution equations for the probability distribution $P_t(\lambda)$ and for $\overline{z}(t)$, which are

$$P_{t+1}(\lambda) = \int_0^1 d\xi \int_0^\infty d\lambda' P_t(\lambda') \delta\left(\lambda - \lambda' + \xi + \frac{\gamma_p}{2}\xi^2\right) + \int_0^1 d\xi \int_{-\infty}^0 d\lambda' P_t(\lambda') \delta\left(\lambda - \lambda' - \xi + \frac{\gamma_p}{2}\xi^2\right) \ , \tag{20}$$

and

$$\overline{z}(t+1) = \overline{z}(t) + \frac{1}{R}\int_0^\infty d\lambda\, P_t(\lambda) \left(\lambda - \frac{1}{2}\gamma_p \lambda^2\right) - \int_{-\infty}^0 d\lambda\, P_t(\lambda) \, \min\left\{-\lambda + \frac{1}{2}\gamma_p \lambda^2, \overline{z}(t)\right\} \ . \tag{21}$$

Since the dynamics of λ is decoupled from the one of $\overline{z}$, we can assume that $P_t(\lambda)$ reaches a stationary state. In Appendix C 1 we show that, at first order in γ_p, we have for the stationary distribution

$$P_{\rm st}(\lambda) = \left(1 - |\lambda| - \frac{\gamma_p}{2}\mathrm{sgn}(\lambda)\lambda^2\right) \theta\left(1 - |\lambda| - \frac{\gamma_p}{2}\mathrm{sgn}(\lambda)\lambda^2\right) \ , \tag{22}$$

where the Heaviside theta-function ensures that λ is limited to the region where $P_{\rm st}(\lambda) \geq 0$. Hence we find that $|\lambda| \leq 1 + O(\gamma_p)$, consistently with the assumptions we made above.

Let us now discuss the dynamics of $\overline{z}(t)$. If at some time t we have $\overline{z}(t) = 0$, then it is clear from Eq. (21) that $\overline{z}(t)$ will grow with time, and it will continue to do so unless the last term in the equation becomes sufficiently large. Recalling that $|\lambda| \leq 1 + O(\gamma_p)$, it is clear that even if $\overline{z}(t)$ becomes very large, the last term in Eq. (21) can be at most given by $\int_{-\infty}^0 d\lambda\, P_{\rm st}(\lambda)\, \left(-\lambda + \frac{1}{2}\gamma_p \lambda^2\right)$. Therefore, if

$$\frac{1}{R}\int_0^\infty d\lambda\, P_{\rm st}(\lambda) \left(\lambda - \frac{1}{2}\gamma_p \lambda^2\right) > \int_{-\infty}^0 d\lambda\, P_{\rm st}(\lambda) \left(-\lambda + \frac{1}{2}\gamma_p \lambda^2\right) \ , \tag{23}$$

then $\overline{z}(t)$ will continue to grow with time, and $\overline{Y}(t) = 1 - \gamma_p \eta_+ \overline{z}(t)$ will become very small and the economy will collapse. Conversely, if the condition in Eq. (23) is not satisfied, then Eq. (21) admits a stationary solution with $0 < \overline{z}_{\rm st} < \infty$. Using Eq. (22), the critical boundary line finally reads:[16]

$$R_c = \frac{\eta_+}{\eta_-} = \frac{\int_0^\infty d\lambda\, P_{\rm st}(\lambda)\, \left(\lambda - \frac{1}{2}\gamma_p \lambda^2\right)}{\int_{-\infty}^0 d\lambda\, P_{\rm st}(\lambda)\, \left(-\lambda + \frac{1}{2}\gamma_p \lambda^2\right)} \approx 1 - 2\gamma_p + ... \tag{24}$$

[16] It would be interesting to compute the first corrections in η. We leave this for a later study.

in good agreement with the numerical result, see Fig. 3.

In other words, the ζ_i perform a *biased* random walk, in presence of a noise whose average is given by the last two terms in Eq. (21) (of course, when $\overline{\zeta}$ is too small the minimum in the last term is important, because it is there to prevent $\overline{\zeta}$ from becoming negative). The system will evolve towards full employment, with $\overline{\zeta} = O(\eta_+)$ and $\overline{Y} = 1 - O(\gamma_p \eta_+)$, whenever the average noise is negative, and to full collapse, with $\overline{\zeta} \to \infty$ and $\overline{Y} = 0$, otherwise. The critical line is given by the equality condition, such that the average noise vanishes. Therefore, right at the critical point, the unemployment rate makes an unbiased random walk in time, meaning that its temporal fluctuations are large and slow. This justifies the adiabatic approximation made above, that lead us to neglect the dynamics of the savings.

B. Oscillations in the high employment phase

As discussed above, in the FE phase macroeconomic variables display an oscillatory dynamics, see Fig. 5. Intuitively, the mechanism behind these oscillations is the following. When prices are low, demand is higher than production and firms increase the prices. But at the same time, households cannot consume what they demand, so they involuntarily save: savings increase when prices are low. These savings keep the demand high for a few rounds even while prices are increasing, therefore prices keep increasing above their equilibrium value. When prices are too high, households need to use their savings to consume, and therefore savings start to fall. Increase of prices and decrease of savings determine a contraction of the demand. At some point demand falls below production and prices start to decrease again, with savings decreasing at the same time. When prices are low enough, demand becomes again higher then production and the cycle is restarted. An example is shown in Fig. 5.

Based on this argument, it is clear that to study these oscillations, we need to take into account the dynamics of the savings so we cannot assume $\alpha_i = 0$ as in the previous section. However, here it is enough to consider the first order terms in γ_p. In terms of the basic variables in Eq. (16), the other variables that appear in Eq. (15) are easily written as follows:

$$\begin{aligned} \overline{Y}(t) &= 1 - \gamma_p \overline{\zeta}(t) \ , \\ s(t) &= 1 + \gamma_p \overline{\alpha}(t) \ , \\ \overline{p}(t) &= 1 + \gamma_p \overline{\lambda}(t) \ , \end{aligned} \tag{25}$$

where for ζ, α, λ, overlines denote flat averages over firms, and

$$D_i(t) = 1 + \gamma_p \left(\frac{1}{2}\overline{\alpha}(t) - \frac{1}{2}\overline{\zeta}(t) - \lambda_i(t) \right) \ . \tag{26}$$

Inserting this in Eq. (15), we arrive to the following equations that hold at the lowest order in γ_p:

$$\begin{aligned} &\text{If } -\zeta_i(t) < \frac{1}{2}\overline{\alpha}(t) - \frac{1}{2}\overline{\zeta}(t) - \lambda_i(t) \quad \Rightarrow \quad \begin{cases} \zeta_i(t+1) = \zeta_i(t) - \min\{\eta_+[\frac{1}{2}\overline{\alpha}(t) - \frac{1}{2}\overline{\zeta}(t) - \lambda_i(t) + \zeta_i(t)], \overline{\zeta}(t)\} \\ \text{If } \lambda_i(t) < \overline{\lambda}(t) \quad \Rightarrow \quad \lambda_i(t+1) = \lambda_i(t) + \xi_i(t) \\ \alpha_i(t+1) = \alpha_i(t) - \lambda_i(t) \end{cases} \\ &\text{If } -\zeta_i(t) > \frac{1}{2}\overline{\alpha}(t) - \frac{1}{2}\overline{\zeta}(t) - \lambda_i(t) \quad \Rightarrow \quad \begin{cases} \zeta_i(t+1) = \zeta_i(t) - \eta_-[\frac{1}{2}\overline{\alpha}(t) - \frac{1}{2}\overline{\zeta}(t) - \lambda_i(t) + \zeta_i(t)] \\ \text{If } \lambda_i(t) > \overline{\lambda}(t) \quad \Rightarrow \quad \lambda_i(t+1) = \lambda_i(t) - \xi_i(t) \\ \alpha_i(t+1) = \alpha_i(t) - \zeta_i(t) - \frac{1}{2}\overline{\alpha}(t) + \frac{1}{2}\overline{\zeta}(t) \end{cases} \end{aligned} \tag{27}$$

From an analytical point of view, the above model is still to complex to make progress. We make therefore a further simplification, by assuming that

$$\zeta_i = C\lambda_i \ , \tag{28}$$

where C is a certain numerical constant. In terms of the original Mark 0 variables, this approximation is equivalent to, roughly speaking,

$$p_i - \overline{p} \propto (Y_i - D_i) - (\overline{Y} - \overline{D}). \tag{29}$$

i.e. fluctuations of the prices are proportional to supply-demand gaps. Although numerical simulations only show a weak correlation, the approximation (28) allows us to obtain a more tractable model that retains the basic phenomenology of the oscillatory cycles and reads:

$$\begin{aligned}
&\text{If } 2(1-C)\lambda_i(t) < \overline{\alpha}(t) - C\overline{\lambda}(t) \quad \Rightarrow \quad \begin{cases} \text{If } \lambda_i(t) < \overline{\lambda}(t) \quad \Rightarrow \quad \lambda_i(t+1) = \lambda_i(t) + \xi_i(t) \\ \alpha_i(t+1) = \alpha_i(t) - \lambda_i(t) \end{cases} \\
&\text{If } 2(1-C)\lambda_i(t) > \overline{\alpha}(t) - C\overline{\lambda}(t) \quad \Rightarrow \quad \begin{cases} \text{If } \lambda_i(t) > \overline{\lambda}(t) \quad \Rightarrow \quad \lambda_i(t+1) = \lambda_i(t) - \xi_i(t) \\ \alpha_i(t+1) = \alpha_i(t) - C\lambda_i(t) - \frac{1}{2}\overline{\alpha}(t) + \frac{1}{2}C\overline{\lambda}(t) \end{cases}
\end{aligned} \tag{30}$$

This very minimal model, when simulated numerically, indeed gives persistent oscillations, independent on N, when $C > C^* \approx 0.45$, and can also be partially investigated analytically, see Appendix C 1.

C. The "representative firm" approximation

To conclude this section, we observe that there is a further simplification that allows to retain some of the phenomenology of Mark 0. It consists in describing the firm sector by a unique "representative firm", $N_{\rm F} = 1$, with production $\overline{Y}(t)$, price $p(t)$ and demand $\overline{Y}(t)/p(t)$. The dynamics of the production and price are given by the same rule as above, but now the dynamics of the price completely decouples:

$$\begin{aligned}
p(t) < 1 \;\Rightarrow\; &\begin{cases} \overline{Y}(t+1) = \overline{Y}(t)(1 + \eta_+(\frac{1}{p(t)} - 1) \\ p(t+1) = p(t)(1 + \gamma\xi(t)) \end{cases} \\
p(t) > 1 \;\Rightarrow\; &\begin{cases} \overline{Y}(t+1) = \overline{Y}(t)(1 + \eta_-(\frac{1}{p(t)} - 1) \\ p(t+1) = p(t)(1 - \gamma\xi(t)). \end{cases}
\end{aligned} \tag{31}$$

Of course, this simple model misses several important effects: most notably those associated to Θ, hence the transition between the Full Employment, Endogenous Crises, and Residual Unemployment phases in Fig. 4. In particular, endogenous crises are never present in this case, because of the absence of a bankrupt/revival mechanism, and also the oscillatory pattern in the Full Employment phase disappears, because in this model savings are not considered. Still, this model is able to capture the transition between the Full Unemployment and Full Employment regions as a function of R (see Fig. 4), as confirmed by the analytical solution, which is identical to the one of the model with $N_{\rm F} > 1$ when η is small. And since the model is so simple, one can hope that some of the extensions discussed in the next section can be understood analytically within this "representative firm" framework (we will give a few explicit examples in the following). This would be an important step to put the rich phenomenology that we observe on a firm basis.

V. EXTENSIONS OF MARK 0

A. Extended Mark 0: Adaptation & Wage Update

As emphasized above, the Mark I+ and Mark 0 models investigated up to now both reveal a generic phase transition between a "good" and a "bad" state of the economy. However, many features are clearly missing to make these models plausible – setting up a full-blown, realistic macroeconomic Agent-Based Model is of course a long and thorny endeavour which is precisely what we want to avoid at this stage, focusing instead on simple mechanisms. Still, it is interesting to enrich these simplified models not only to test for robustness of our phase transition but also to investigate new effects that may be of economic significance. We consider here two meaningful (or so we think) ingredients: Adaptation and Wage dynamics, that lead to some important modifications of the above simple picture, for example the appearance of inflation, and the evolution of the *first order transition* reported above into a *second order* phase transition.

With respect to the basic model, we modify the following aspects:

- **Adaptation:** The strategy of firms in Mark 0 is to hire/fire proportionally to the difference between demand for their goods and production. The coefficients $\eta_\pm$ give the amplitude of this adjustment, and were chosen above to be time independent and homogeneous across firms. However, one should expect that depending on the financial fragility of the firm, the production adjustment might be more or less aggressive. Firms that are close to bankruptcy are probably faster to fire and slower to hire, and viceversa for healthy firms. In order to model this effect, we posit that the coefficients $\eta^i_\pm$ for firm i depend on its financial fragility $\Phi_i = -\mathcal{L}_i/(W_iY_i)$, as:

$$\eta^i_\pm = \eta^0_\pm \times \begin{cases} 2 & \text{if } (1 \mp \Gamma\Phi_i) > 2 \ , \\ 0 & \text{if } (1 \mp \Gamma\Phi_i) < 0 \ , \\ (1 \mp \Gamma\Phi_i) & \text{otherwise} \ , \end{cases} \tag{32}$$

where $\eta^0_\pm$ are fixed coefficients, identical for all firms, and Γ measures the strength of the feedback[17]. Note that in this case we define $R = \eta^0_+/\eta^0_-$.

- **Wage update:** A very important item missing from the basic Mark 0 and Mark I models considered above is the dynamics of wages, which are fixed across time and across firms in both these models. Clearly, the ability to modulate the wages is complementary to deciding whether to hire or to fire, and should play a central role in the trajectory of the economy as well as in determining inflation rates.

  Introducing wages in Mark 0 again involves a number of arbitrary assumptions and choices. Here, we follow (in spirit) the choices made in Mark I for price and production update, and propose that at each time step firm i updates its wage as:

$$\begin{aligned} W_i^T(t+1) &= W_i(t)[1 + \gamma_w(1-\Gamma\Phi_i)\varepsilon\xi_i'(t)] \quad \text{if} \quad \begin{cases} Y_i(t) & < D_i(t) \\ \mathcal{P}_i(t) & > 0 \end{cases} \\ W_i(t+1) &= W_i(t)[1 - \gamma_w(1+\Gamma\Phi_i)u\xi_i'(t)] \quad \text{if} \quad \begin{cases} Y_i(t) & > D_i(t) \\ \mathcal{P}_i(t) & < 0 \end{cases} \end{aligned} \tag{33}$$

  where $u = 1-\varepsilon$ is the unemployment rate and γ_w a certain parameter; $\mathcal{P}_i(t) = \min(D_i(t), Y_i(t))p_i(t) - W_i(t)Y_i(t)$ is the profit of the firm at time t and $\xi_i'(t)$ an independent $U[0,1]$ random variable. If $W_i^T(t+1)$ is such that the profit of firm i at time t with this amount of wages would have been negative, $W_i(t+1)$ is chosen to be exactly at the equilibrium point where $\mathcal{P}_i = 0$, hence $W_i(t+1) = \min(D_i(t), Y_i(t))p_i(t)/Y_i(t)$; otherwise $W_i(t+1) = W_i^T(t+1)$. As before, the fragility Φ is defined as $\Phi_i = -\mathcal{L}_i/(Y_iW_i)$, i.e. it is the ratio of the negative of the liquidity of the firm to its total payroll. The corresponding condition for default is $\Phi_i > \Theta$.

  The above rules are intuitive: if a firm makes a profit and it has a large demand for its good, it will increase the pay of its workers. The pay rise is expected to be large if the firm is financially healthy and/or if unemployment is low (i.e. if ε is large) because pressure on salaries is high. Conversely, if the firm makes a loss and has a low demand for its good, it will reduce the wages. This reduction is drastic is the company is close to bankruptcy, and/or if unemployment is high, because pressure on salaries is then low. In all other cases, wages are not updated.

  When a firm is revived from bankruptcy (with probability φ per unit time), its wage level is set to the production weighted average wage of all firms in activity.

  The parameters $\gamma_{p,w}$ allow us to simulate different price/wage update timescales. In the following we set $\gamma_p = 0.05$ and $\gamma_w = z\gamma_p$ with $z \in [0,1]$. The case $z = 0$ clearly corresponds to removing completely the wage update rule, such that the basic version of Mark 0 is recovered.

The extended version of Mark 0 that we consider below is therefore characterized by two additional parameters: Γ, describing firm adaptation and $\gamma_w = z\gamma_p$, describing the frequency of wage updates.

B. Results: adaptation of firms and second order phase transition

We start by analyzing the effect of adaptation of firms, hence we consider the extended Mark 0 model with $\Gamma \neq 0$ and $\gamma_w = 0$. Intuitively, the coupling between financial fragility and hiring/firing propensity should have a stabilizing effect on the economy[18]. This is indeed what we find numerically: the line separating the full employment phase from the endogenous crises phase is shifted to lower values of Θ when $\Gamma \neq 0$, see Fig. 7.

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[17] Another equivalent choice would have been $\eta^i_\pm = \eta^0_\pm\,[1 \mp \tanh(\Gamma\Phi_i)]$.

[18] As mentioned above, the mere presence of a finite default threshold Θ is enough to induce similar effects, through the stabilizing redistribution of capital between households and firms that occurs when firms are revived.

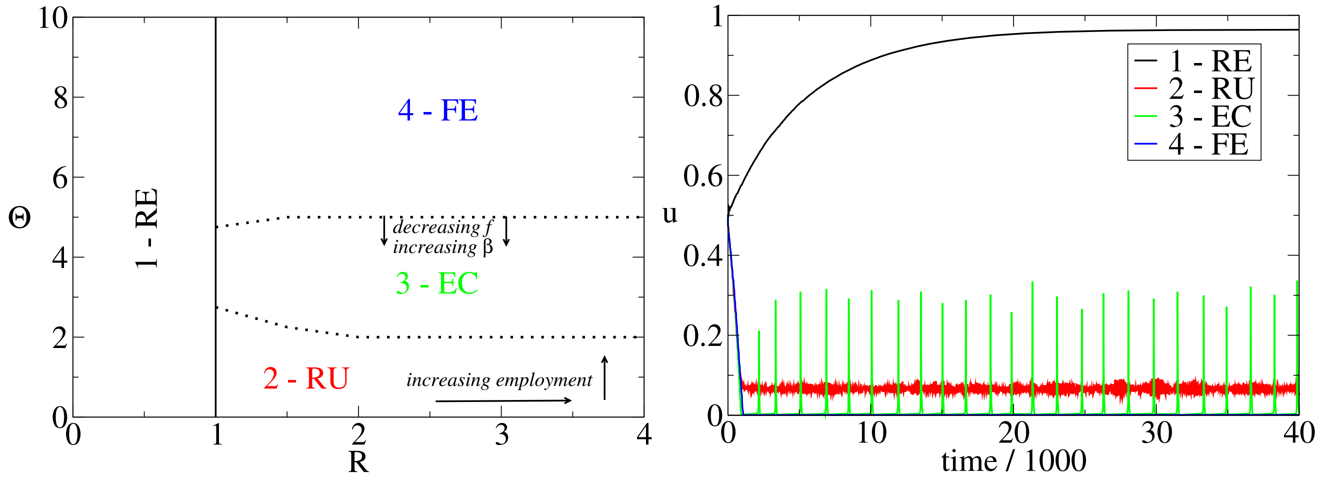

FIG. 7: (*Left*) Phase diagram of the extended Mark 0 model in the $R-\Theta$ plane, with $\Gamma = 0.01$ and $\gamma_w = 0$, i.e. with adaptation but no wage update. The parameters of the basic model are the same as in Fig. 4: $N_{\rm F} = 5000$, $c = 0.5$, $\gamma_p = 0.05$, $\delta = 0.02$, $\varphi = 0.1$, $f = 1$, $\beta = 0$. As for the basic model, there are four distinct phases separated by critical lines. The full employment phase is now stabilized by adaptation, and at the same time the phase with endogenous crises is reduced. Note that the phase boundary between FE and EC goes down as f or β are increased (similar to Fig. 4); the other parameters are to a large degree irrelevant. (*Right*) A typical time series of $u(t)$ for each of the phases.

Moreover, the full unemployment phase at $R < R_c$ is deeply affected by the presence of Γ: for $\Gamma \neq 0$ the unemployment rate in this phase is no longer one, but becomes smaller than one and continuously changing with R. In order to derive an estimate of these continuous values we use an intuitive argument (at $\Theta = \infty$) which is justified a posteriori by the good match with numerical results. Given the critical ratio $R = \eta_+^0/\eta_-^0 = R_c$ separating the high/low unemployment phases when there is no adaptation (i.e. $\Gamma = 0$) one can expect that equilibrium values of the unemployment rate different from 0 and 1 can only be stable if η_+^i/η_-^i remains around the critical value R_c at $\Gamma = 0$, given in Eq. (24). Near criticality therefore we enforce that:

$$\frac{\eta_+^i}{\eta_-^i} = \frac{\eta_+^0(1-\Gamma\Phi_i)}{\eta_-^0(1+\Gamma\Phi_i)} = R_c \quad \Rightarrow \quad -\Gamma\Phi \approx \frac{R_c\eta_-^0 - \eta_+^0}{R_c\eta_-^0 + \eta_+^0}. \tag{34}$$

An explicit form of Φ in terms of the employment rate $\varepsilon = \overline{Y}$ can be obtained with the additional assumption that the system is always close to equilibrium (i.e. $p \approx 1$ and $D \approx Y$, at least when $\eta_+^i/\eta_-^i \sim R_c$), which allows one to express households savings in terms of the firms' production. Indeed (recall the discussion at the beginning of Sec. IV) at equilibrium $W = B = N_{\rm F}Y = c(W + S)$, from which it follows that $N_{\rm F}Y = W = Sc/(1-c)$. For $c = 0.5$ as in our simulations one thus has $S = NY$. Since the total amount of money is conserved (in our simulations $N_{\rm F}\overline{\mathcal{L}} + S = N_{\rm F}\overline{\mathcal{L}} + N_{\rm F}Y = N_{\rm F}$, see Appendix B) one finally obtains that $\overline{\mathcal{L}} = 1 - Y$ and $\Phi = (Y-1)/Y = (\varepsilon - 1)/\varepsilon$ (recall that in our simulations $\mu = 1$), hence

$$\frac{\Gamma}{\varepsilon} = \frac{R_c\eta_-^0 - \eta_+^0}{R_c\eta_-^0 + \eta_+^0} + \Gamma = \frac{R_c - R}{R_c + R} + \Gamma\ . \tag{35}$$

Note that according to this formula the employment goes to $\varepsilon = 1$ at the critical point $R = R_c$. Above this value, the economy is in the "good" state and employment sticks to $\varepsilon = 1$ (this is because in the argument the effect of Θ has been neglected). Moreover, when $R = R_c$, ε is proportional to Γ and therefore in the limit $\Gamma \to 0$ one has $\varepsilon = 0$ for all $R < R_c$. This is the "bad" phase of full unemployment at $\Gamma = 0$, which becomes in this case a phase where employment grows steadily but remains of order Γ except very close to the critical point.

Eq. (35) is plotted in Fig. 8 together with numerical results. Note that in this case the representative firm approximation ($N_{\rm F} = 1$) is in good agreement with numerical results also for $N_{\rm F} = 10,000$. In the inset of Fig. 8 one can see that the variance of the fluctuations of employment rate is diverging as long as the critical value of R is approached. This is confirmed by a spectral analysis of the unemployment time series (see Fig. 9). In order to obtain the power spectrum we apply the GSL Fast Fourier Transform algorithm to the time series $\varepsilon(t) - \langle\varepsilon\rangle$. As one can see in Fig. 9 the power spectrum is well approximated by an Ornstein-Uhlenbeck form:

$$I(\omega) = I_0 \frac{\omega_0^2}{\omega_0^2 + \omega^2} \tag{36}$$

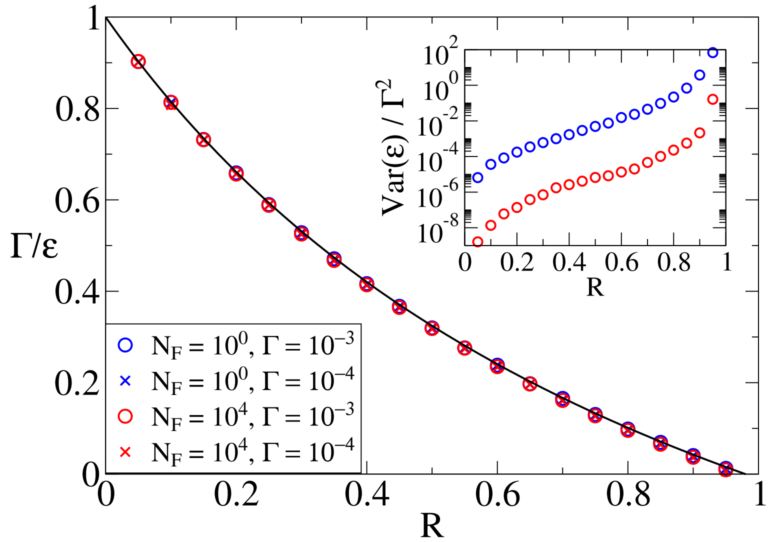

FIG. 8: Inverse of the average employment rate (over time) as a function of the ratio $R = \eta_+^0/\eta_-^0$ with $\eta_-^0 = 0.1$ and $\gamma = 0.01$ when $\Gamma > 0$. When the employment rate is rescaled with the parameter Γ (here $\Gamma = 10^{-3},\ 10^{-4}$) the different lines collapse and Eq. (35) agrees with numerical simulations. In the inset we also plot the rescaled variance, still as a function of η_+^0. Approaching the critical point the variance of the unemployment fluctuations diverges, together with their relaxation time going to infinity. The other parameters are: $\delta = 0.02$, $\Theta = 5$, $\gamma_w = 0$, $c = 0.5$, $\beta = 0$ and $\varphi = 0.1$

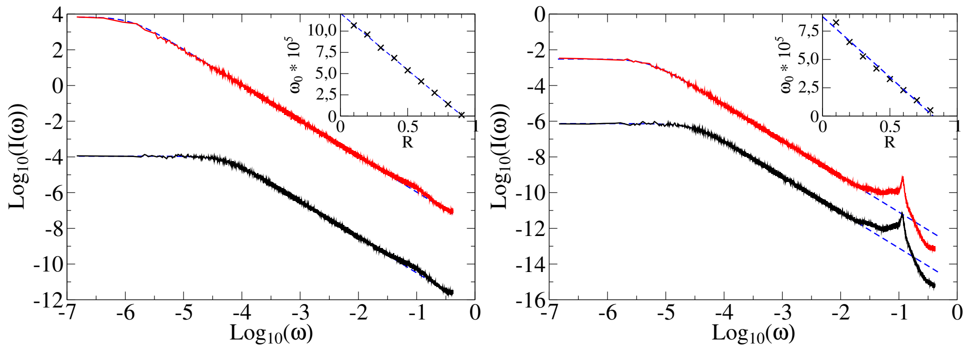

FIG. 9: Logarithm of the normalized power spectrum for Mark 0 with adaptive firms ($\Gamma = 10^{-3}$), $\gamma_p = 0.05$ and $N_{\rm F} = 1$ (left) and $N_{\rm F} = 1000$ (right). The other parameters are set as in Fig. 8. The main plot show two examples of the spectrum for $\eta_-^0 = 0.1$ and $\eta_+^0 = 0.05$ (black line) and $\eta_+^0 = 0.09$ (red line) in the left plot, $\eta_+^0 = 0.05$ (black line) and $\eta_+^0 = 0.08$ (red line) in the right plot. The time series is made of 2^{28} time steps after $T_{eq} = 500\,000$ and the logarithm of the spectrum is averaged over a moving window of 100 points for a better visualization. With both system sizes the fit with Eq. (36) (blue dashed lines) is good with the only difference that when $N_{\rm F} > 1$ a clear oscillatory patterns appear at high frequencies, that becomes sharper and sharper as $N_{\rm F}$ increases. In the inset of each figure we plot the value of ω_0 in Eq. (36) obtained from the fit as a function of the ratio $R = \eta_+^0/\eta_-^0$. In both cases ω_0 goes linearly to 0 as the critical value is approached.

with ω_0 going linearly to 0 when η_+^0 approaches its critical value, meaning that the relaxation time ω_0^{-1} diverges as one approaches the critical point. Note that this is not the case for the Mark I model (or Mark 0 with $\Gamma = 0$) which instead has a white noise power spectrum even in proximity of the transition line. The first order (discontinuous) transition for $\Gamma = 0, \Theta = \infty$ is thus replaced by a second order (continuous) transition when the firms adapt their behaviour as a function of their financial fragility.

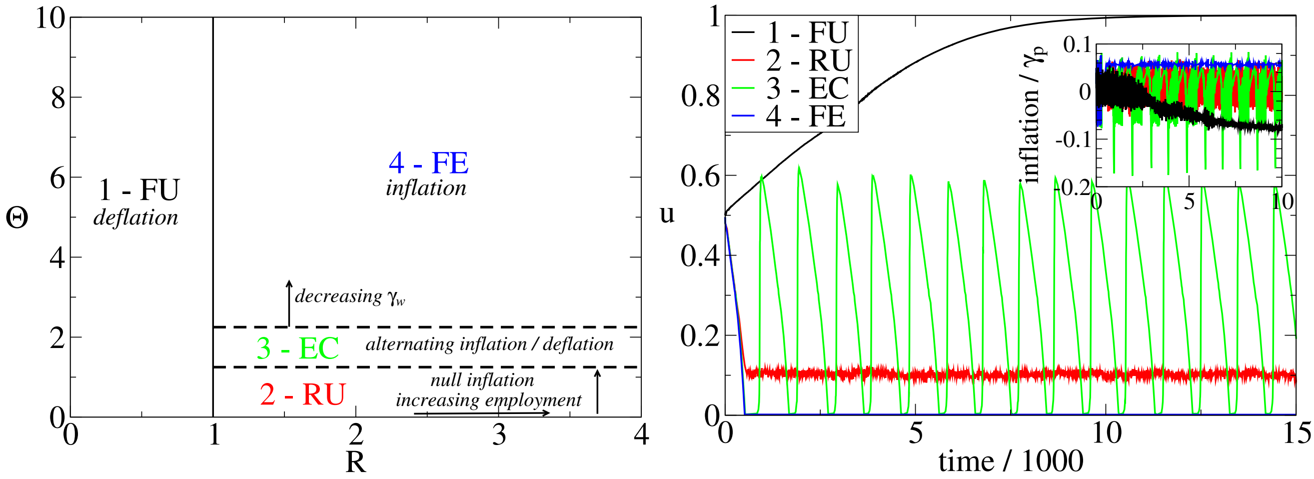

FIG. 10: (*Left*) Phase diagram in the $R - \Theta$ plane of the extended Mark 0 model with wage update, with $\gamma_w = \gamma_p$ and $\Gamma = 0$. The parameters of the basic model are the same as in Fig. 4: $N_{\rm F} = 5000$, $c = 0.5$, $\gamma_p = 0.05$, $\delta = 0.02$, $\varphi = 0.1$, $f = 1$, $\beta = 0$. As for the basic model, there are four distinct phases separated by critical lines. Wage update brings in inflation in the Full Employment phase, and deflation in the full unemployment phase. Endogenous crises are characterized by alternating cycles of inflation and deflation. We report schematically the evolution of the critical lines with the parameters f and β; the other parameters are irrelevant. Note that the wage update strongly stabilizes the full employment phase. (*Right*) A typical trajectory of $u(t)$ for each of the phases. In the inset, the price dynamics is shown, displaying inflation and deflation.

C. Results: variable wages and the appearance of inflation

In our *money conserving* toy economies[19], a stationary inflation rate different from zero is possible as long as the ratio $\overline{p}(t)/\overline{W}(t)$ fluctuates around a steady state. In absence of wage update, we have a fixed $W \equiv 1$ and inflation is therefore impossible. The main effect introduced by letting wages fluctuate is therefore the possibility of inflation.

Using the wage update rules defined below, we found that the average inflation rate depends on parameters such as the households propensity to consume c and the price/wage adjustment parameters $\gamma_{p,w}$. Most interestingly, we observe a strong dependence of the inflation rate upon the bankruptcy threshold Θ, with large Θ's triggering high inflation and low Θ's corresponding to zero inflation. For intermediate Θ's, periods of inflation and deflation may alternate and the model displays interesting instabilities. A promising research direction, that would address some crucial policy problems usually discussed within DSGE models, is to study the coupling of inflation with the households propensity to consume (higher inflation resulting in higher c and vice-versa), with the interest rate on loans (affecting the effective bankruptcy threshold), etc. We however defer the analysis of these situations for later investigations, and here only analyze the influence of wage adjustments on the phase transitions discussed in the previous sections.

In order to only focus on the impact of wage update on the phase diagram, we set the adaptation parameter $\Gamma = 0$, and $\gamma_w = \gamma_p$. The corresponding phase diagram is reported in Fig. 10. In this case, the phenomenology that we find is again very similar to the simple Mark 0 without wage update, except for inflation. The most interesting effect is the appearance of inflation in the "good" phases of the economy, and deflation in the "bad" phases, as shown in Fig. 10. When $\Theta \gg 1$, we find again a first order critical boundary at $R = R_c$ that separates a high unemployment phase (with deflation) from a low unemployment phase (with inflation). For $R > R_c$ we see again two additional phases: "EC - region 3", with endogenous crises and, correspondingly, alternating periods of inflation and deflation but stable prices on the long run, and "RU-region 2", for small Θ, where there is no inflation but a substantial residual unemployment rate (see Fig. 10).

The appearance of endogenous crises is consistent with what discussed in section III B and is related to situations in which the debt-to-savings ratio $k(t)$ grows faster than prices. From this point of view, increasing γ_w allows firms to better adapt wages (and thus prices) and to absorb the indebtment through inflation. (Note that region 3-EC indeed shrinks when increasing γ_w). We also observe that the oscillatory pattern found for the basic model persists as long as $\gamma_w \ll \gamma_p$, i.e. when wage updates are much less frequent that price updates. The power spectrum of the model for $R > R_c$ and $\Theta \gg 1$ is still characterized by the appearance of a peak (roughly corresponding to a period

[19] Recall that the physical money is conserved in the model, but virtual money creation is still possible through firms' indebtment.

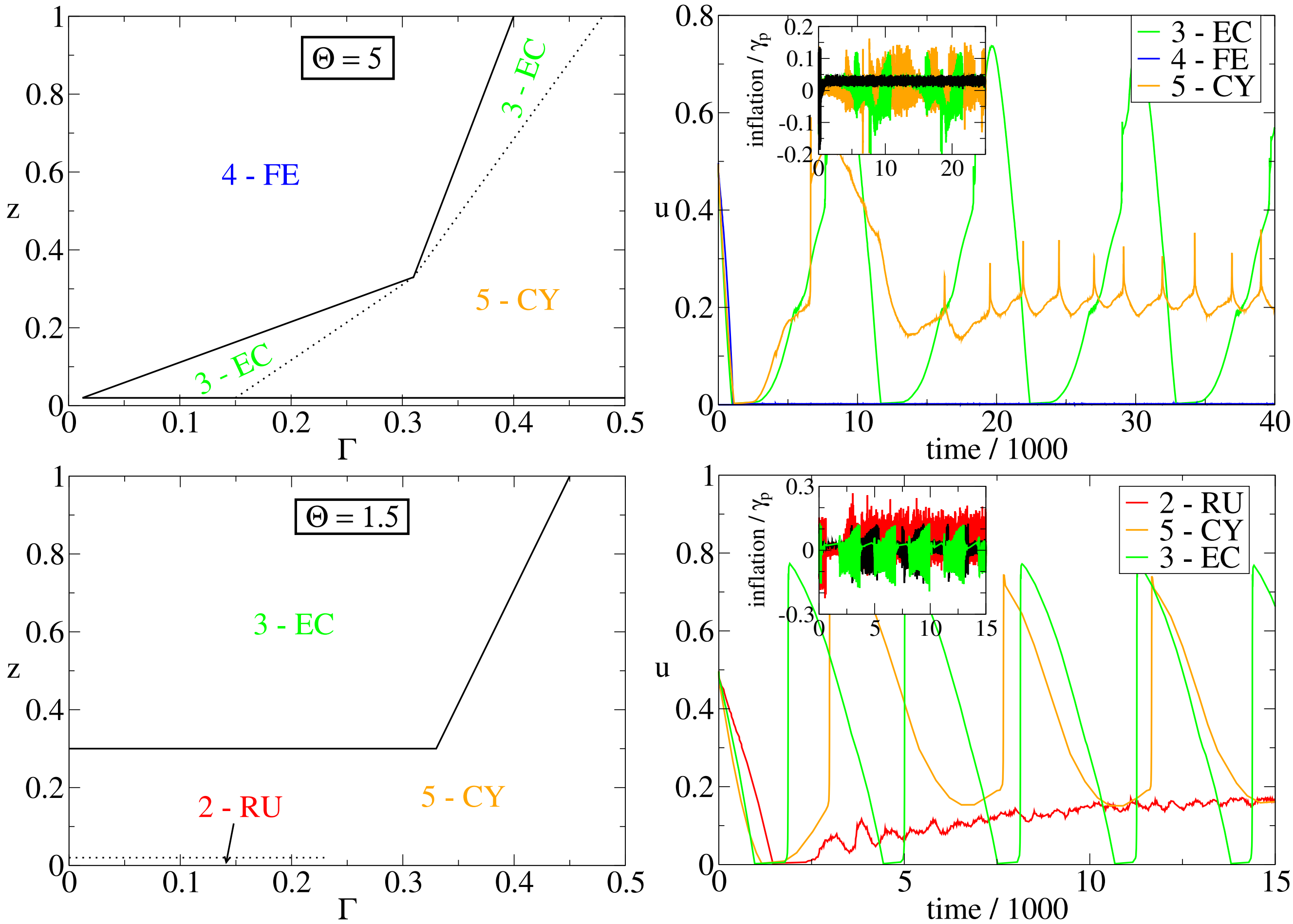

FIG. 11: (*Top Left*) Phase diagram of the Mark 0 model in the $z-\Gamma$ plane at $\Theta = 5$ with $z = \gamma_w/\gamma_p$. Here $N_{\rm F} = 5000$, $c = 0.5$, $\gamma_p = 0.05$, $\delta = 0.02$, $\varphi = 0.1$, $f = 1$, $\beta = 0$. There are three distinct phases separated by critical lines. (*Top Right*) A typical trajectory of $u(t)$ for each of the phases and corresponding inflation in the inset. (*Bottom Left*) Phase diagram of the Mark 0 model in the $z-\Gamma$ plane at $\Theta = 1.5$ (other parameters as above). (*Bottom Right*) Typical trajectories of $u(t)$ in each of the four phases at $\Theta = 1.5$ and corresponding inflation in the inset.

of 7 time steps). Interestingly, we find that upon increasing the ratio $z = \gamma_w/\gamma_p$, the peak in the frequency spectrum disappears but not in a monotonous fashion; intermediate values of z give rise to even more pronounced oscillations than for $z = 0$, before these oscillations disappear for $z > z_c \approx 0.25$.

Finally, we give another "cut" of the phase diagram, in the $z-\Gamma$ plane, for a fixed value of $R = 2 > R_c$ and for two values of Θ. We observe a new type of cyclical behaviour (region CY), similar to endogenous crises, but with a positive base-line level of unemployment and less acute spikes.

In conclusion, the comparison between Figs. 10 and 4 demonstrates the robustness of our phase diagram against changes; introducing wages is a rather drastic modification since it allows inflation to set in, but still does not affect the phase transition at R_c, nor the overall topology of the phase diagram, which confirms its relevance. Interestingly, inflation is present in the good phases of the economy and deflation in the bad phases. Our analytical understanding of these effects is however still very poor; we feel it would be important to bolster the above numerical results by solving simpler "toy models" as we did for the basic Mark 0 (see section IV and Appendix C 1).

D. Extensions and policy experiments

We have also explored other potentially interesting extensions of Mark 0, for example adding trust or confidence, that may appear and disappear on time scales much shorter than the evolution time scale of any "true" economic factor, and can lead to market instabilities and crises (see e.g. [5, 33, 34]). There are again many ways to model the potentially destabilizing feedback of confidence. One of the most important channel is the loss of confidence

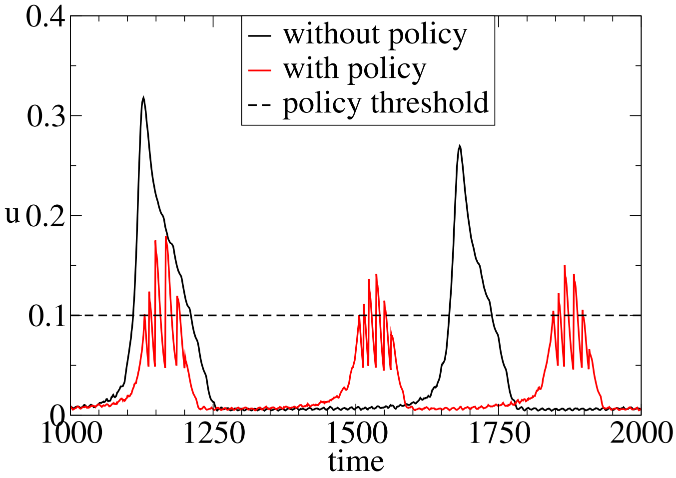

FIG. 12: Here we show an example of a "toy" policy experiment in the Mark 0 model without wage update (i.e $\gamma_w = 0$ and $\Gamma = 0$). We first run a simulation (the "without policy" line) with a constant value $\Theta = 2$ lying in the 3-EC region of Fig. 3. We then run the same simulation with a prototype central bank which increases Θ to 10 as long as u exceeds the threshold of 10%. Note that unemployment is partially contained, but the crisis frequency concomitantly increases. The other parameters values are: $N = 10\,000$, $R = 2$, $\gamma_p = 0.05$, $\varphi = 0.1$, $\delta = 0.02$, $f = 1$.

induced by raising unemployment, that increases the saving propensity of households and reduces the demand. The simplest way to encode this in Mark 0 is to let the "c" parameter, that determines the fraction of wages and savings that is devoted to consumption, be an increasing function of the employment rate $\varepsilon = 1 - u$. We indeed find that the confidence feedback loop can again induce purely endogenous swings of economic activity. Similarly, a strong dependence of c on the recent inflation can induce instabilities.

In order to run meaningful policy experiments, a central bank that decides on the interest rate level and the amount of money in circulation will be needed. This is the project on which we are currently working on. However, even within the simplistic framework of Mark 0, one can envisage a prototype policy experiment which consists in allowing the "central bank" to temporarily increase the bankruptcy threshold Θ in times of high unemployment. Fig. 12 shows an example of this: the economy is, in its normal functioning mode, in the EC (region 3) of the phase diagram, with $R = 2$ and (say) $\Theta = 2$. This leads in general to a rather low unemployment rate u but, as repeatedly emphasized above, this is interrupted by acute endogenous crises. The central bank then decides that whenever u exceeds 10%, its monetary policy becomes accommodating, and amounts to raising Θ from its normal value 2 to – say – $\Theta = 10$. As shown in Fig. 12, this allows the bank to partially contain the unemployment bursts. However, quite interestingly, it also increases the crisis frequency, as if it did not allow the economy to fully release the accumulated stress. We expect that this phenomenology will survive in a more realistic framework.

VI. SUMMARY, CONCLUSION

The aim of our work (which is part of the CRISIS project and still ongoing) was to explore the possible types of *phenomena* that simple macroeconomic Agent-Based Models can reproduce, and to map out the corresponding *phase diagram* of these models, as Figs. 10 and 4 exemplify. The precise motivation for our study was to understand in detail the nature of the macro-economic fluctuations observed in the "Mark I" model devised by D. Delli Gatti and collaborators, which is at the core of the CRISIS project. One of our central findings is the generic existence, in Mark I (and variations around that model) of a first order, discontinuous phase transition between a "good economy" where unemployment is low, and a "bad economy" where unemployment is high. By studying a simpler hybrid model (Mark 0), where the household sector is described by aggregate variables and not at the level of agents[20], we have shown that this transition is induced by an *asymmetry between the rate of hiring and the rate of firing* of the firms. This asymmetry can have many causes at the micro-level – in Mark I, for example, it reflects the reluctance of firms to take loans when the interest rate is too high. As the interest rate increases, the unemployment level remains small

[20] For a recent study exploring the idea of hybrid models, see [35].

until a tipping point beyond which the economy suddenly collapses. If the parameters are such that the system is close to this transition, any small fluctuations (for example in the level of interest rates) is amplified as the system jumps between the two equilibria. It is actually plausible that the central bank policy (absent in our current model) does bring the system close to this transition, since too low an interest rate leads to overheating and inflation, and too high an interest rate leads to large unemployment. The aim of the central bank is therefore to control the system in the vicinity of an instability and could therefore be a natural realization of the 'self-organized criticality' scenario recently proposed in [36] (see also [15]).

Mark 0 is simple enough to be partly amenable to analytic treatments, that allow us to compute approximately the location of the transition line as a function of the hiring/firing propensity of firms, and characterize the oscillations and the crises that are observed. Mark 0 can furthermore be extended in several natural directions. One is to allow this hiring/firing propensity to depend on the financial fragility of firms – hiring more when firms are financially healthy and firing more when they are close to bankruptcy. We find that in this case, the above transition survives but becomes *second order*. As the transition is approached, unemployment fluctuations become larger and larger, and the corresponding correlation time becomes infinite, leading to very low frequency fluctuations. There again, we are able to give some analytical arguments to locate the transition line. Other stabilizing mechanisms, such as the bankruptcy of indebted firms and their replacement by healthy firms (financed by the savings of households), lead to a similar phenomenology.

The role of the bankruptcy threshold Θ, which is the only "proto-monetary" effect in Mark 0, turns out to be crucial in the model, and leads to the phase diagram shown in Fig. 4. We generically find not one but three "good" phases of the economy: one is where full employment prevails (FE), the second one is where some residual unemployment is present (RU), and the third, intermediate one is prone to *acute endogenous crises* (EC), during which the unemployment rate shoots up before the economy recovers. The existence of purely endogenous crises driven by feedback loops in such simple settings is quite interesting from a general standpoint (see also [24, 26]), and reinforces the idea that many economic and financial crises may *not* require large exogenous shocks (see [5, 22, 37–42] for related discussions). We have shown that the endogenous crises can be defanged if the household sector does not carry the full burden of firms bankruptcies and/or when the profits of firms is efficiently re-injected in the economy (see e.g. Fig. 6).

Two further extensions have been considered: one is to allow firms to vary the wages of their employees according to some plausible rules of thumb (wages in Mark I and Mark 0 are fixed); this leads to inflation or deflation but leaves the above picture essentially unchanged (see Fig. 10). The second is to introduce some confidence feedback effects, whereby higher unemployment increases the saving propensity of households and decreases the demand for goods, possibly leading to an unstable loop. Several other extensions are of obvious interest, and we plan to study them in the near future in the same stylized way as above. The most obvious ones is to understand how a central bank that prints money and sets exogenously the interest rate can control the unemployment rate and the inflation rate in the vicinity of an unstable point, as we just mentioned. Other interesting topics are: modeling research and innovation, allowing firms to produce different types of goods, and introducing a financial sector and a housing market [43].

Beyond the generic phase diagram discussed in the whole paper, we found another notable, robust feature: the low unemployment phase of all the ABMs we considered are characterized by endogenous oscillations that do not vanish as the system size becomes large, with a period corresponding, in real time, to $\sim 5 - 10$ years [26]. It is tempting to interpret these oscillations as real and corresponding to the "business cycle", as they arise from a very plausible loop between prices, demand and savings. These oscillations actually also appear in highly simplified models, where both the household and the firm sectors are represented by aggregate variables [31, 44].

Building on this last remark, a very important question, it seems to us, is how much can be understood of the phenomenology of ABMs using "mean-field" approaches, i.e. dynamical equations for aggregate variables of the type considered, for example, in [31, 32]? A preliminary analysis reveals that the dynamical equations corresponding to Mark 0 or Mark I already lead to an amazingly complex phase-diagram [44]. Are these mean-field descriptions quantitatively accurate? When do we really need agents and when is an aggregate description sufficient? The answer to this question is quite important, since it would allow one to devise faithful "hybrid" ABMs, where whole sectors of the economy would be effectively described in terms of these aggregate variables, only keeping agents where they are most needed.

Another nagging question concerns the calibration of macroeconomic ABMs. It seems to us that before attempting any kind of quantitative calibration, exploring and making a catalogue of the different possible qualitative "phases" of the model is mandatory. Is the model qualitatively plausible or is the dynamics clearly unrealistic? In what "phase" is the true economy likely to be? On this point, one of the surprise of the present study is the appearance of very long time scales. For example, even in the case of perfectly stable economy with wage update rule (33) and all γ parameters equal to 10% (a rather large value), the equilibrium state of the economy (starting from an arbitrary initial condition) is only reached after ≈ 200 time steps. If one thinks that the elementary time scale in these models is of the order of three months, this means that the physical equilibration time of the economy is 20-50 years, or even much longer, see e.g. Figs. 4 & 10. But there is no reason to believe that on these long periods all the micro-rules and their associated parameters are stable in time. Therefore, studying the *stationary state* of macroeconomic ABMs might be completely irrelevant to understand the real economy. The economy could be in a perpetual transient state (aka "non ergodic"), unless one is able to endogenise the time evolution of all the relevant parameters governing its evolution (see the conclusion of [11] for a related discussion).

If this is the case, is there any use in studying ABMs at all? We strongly believe that ABMs would still be genuinely helpful. ABMs allow us to experiment and scan the universe of possible outcomes – not missing important scenarios is already very good macroeconomics. Human imagination turns out to be very limited, and that is the reason we like models and equations, that help us guessing what *can* happen, especially in the presence of collective effects that are often very counterintuitive. In this respect, ABMs provide extremely valuable tools for generating scenarios, that can be used to test the effect of policy decisions (see e.g. the pleas by Buchanan [17], and Farmer & Foley [18]). In order to become more quantitative, we think that ABMs will have to be calibrated at the micro-level, using detailed behavioural experiments and field studies to fine tune all the micro-rules needed to build the economy from bottom up (see [43, 45] for work in this direction.) Calibrating on historical data the emergent macro-dynamics of ABMs will most probably fail, because of the dimensionality curse and of the Lucas critique (i.e. the feedback between the trajectory of the economy and policy decisions that dynamically change the parameters of the model).

Acknowledgements

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Appendix A: Pseudo-code for Mark I+

We describe here the pseudo-code of our version of Mark I, which we call Mark I+. To keep the length reasonable, a few irrelevant details will be omitted, but the information given here is enough to reproduce the results presented in the paper. In particular, we describe here only the part of the code that is needed to generate the dynamical evolution of the model, and we omit the part that is needed to generate the data output. The source code is available on the site of the CRISIS project (www.crisis-economics.eu).

1. Notations

We describe the algorithm in an object-oriented fashion, where the different agents are described as representatives of a few classes. We use an object oriented syntax. This syntax should be very intuitive and easy to follow. However, it is useful to clarify a few notational conventions:

- The declaration of a variable a (for example of integer type `int`) will be written in C syntax as `int` a.
- If a is an object of some class, then $a.f(x)$ means that we are calling the method (function) $f(x)$ of object a with argument x.
- For simplicity, in the **for** loops we will use the C syntax where, for example, **for**$(t \leftarrow 1; t \leq T; t \leftarrow t+1)$ means that t is set to one before the loop starts, t is increased by 1 at the end of each iteration, and the loop continues if the condition $t \leq T$ is true.
- Instead of arrays we will use *vectors* of objects, and we will follow the notation of C++[21]. For example, `vector<int>` will denote an ordered set (array) of integers. Moreover:
  - A vector of size N will be declared as `vector<int>` $A(N)$, and by default the declaration `vector<int>` A means that the set A is initiated as empty.
  - $|A|$ will denote the size of the set A
  - The notation $A[i]$ will denote the i-th element of the set (with $i = 0, \cdots, |A|-1$).
  - The notation $A \leftarrow a$ will denote the operation of adding the element a to the set A, therefore increasing its size by one (this correspond to the "push_back" operation in C++).
  - The notation $a \leftarrow_R A$ will denote the extraction of a random element from A, which is set equal to a. Note that the element is not removed from A so the size of A remains constant. The notation $A' \leftarrow_{R,M} A$ denotes extraction at random of M different elements from the set A, that constitute the set A'.
  - The notation $A \not\leftarrow a$ will denote the removal of element a from set A, therefore reducing the size of A by one.
  - To denote an ordered iteration over an ordered set A, we will use a loop **for**$(a \in A)$.
  - We define a function average$(a.f(),\ a \in A)$ that returns the average of the $|A|$ numbers $a[i].f(),\ i = 0, \cdots, |A|-1$
- For logical operators, we use the C++ convention[22] which should be quite transparent.

2. Classes

The classes are:

- *Firm*
- *Household*
- *Bank*

[21] See for example `http://www.cplusplus.com/reference/vector/vector/`
[22] `http://en.wikipedia.org/wiki/Operators_in_C_and_C++#Logical_operators`

The role of the bank is very limited at this level and this class is mainly included for the purpose of future extension. One household has a special role as it is the "owner" of the firms, which will pay dividends to this one household. The main loop is described in Algorithm 1 below. The implementation of the firm class is in Algorithm 3, the household class in Algorithm 6 and the bank class in Algorithm 7.

Algorithm 1 Main loop of Mark I+

Require: N_{F} Number of firms; N_{H} Number of households; ρ_0 baseline interest rate; T total evolution time;

```
                                                                  ▷ Initialisation
vector<household> H(N_H)
household O                                                       ▷ The owner of all firms
vector<firm> F(N_F)
bank B

p̄ ← 1                                                             ▷ Average price
for f ∈ F do
    f.set_owner(O)
end for
B.set_ρ(ρ_0)
                                                                  ▷ Main loop
for (t ← 1; t ≤ T; t ← t + 1) do

    vector<firm> E,D
    for f ∈ F do                                  ▷ Firm decide new strategy on prices and production
        f.set_new_strategy(p̄)
        f.get_loans(B)
        f.compute_interests()
        f.define_labor_demand()
        if f.n_vacancies() > 0 then D ← f                        ▷ Firms in D demand workforce
        else if f.n_vacancies() < 0 then E ← f                   ▷ Firms in E have an excess of workforce
        end if
    end for

    for f ∈ E do                                                  ▷ Job market and production
        while f.n_vacancies() < 0 do f.fire_random_worker()      ▷ Firms with excess workforce fire random workers
        end while
    end for
    vector<household> U
    for h ∈ H do
        if !h.working() then U ← h                                ▷ U is the set of unemployed households
        end if
    end for
    while |U| > 0 && |D| > 0 do                  ▷ Random match of unemployed households and demanding firms
        h ←_R U
        f ←_R D
        f.hire(h)
        U ↛ h
        if f.n_vacancies()==0 then D ↛ f
        end if
    end while
```

Algorithm 2 Main loop of Mark I+ (continued)

```
    for f ∈ F do                                                    ▷ Firms produce and pay workers
        f.produce()
        f.pay_workers()
        if f.age()<100 then f.markup_rule()        ▷ Young firms apply a markup rule to avoid bankrupt
        end if
    end for
                                                                                    ▷ Goods market
    H ←random_permutation(H)
    for h ∈ H do h.consume(F)                                            ▷ Consume in random order
    end for
                                                                        ▷ Accounting and bankrupts
    bad_debts ← 0
    vector<firm> L
    for f ∈ F do
        f.accounting(B)
        if f.liquidity() < 0 then                       ▷ Firms with negative liquidity go bankrupt
            bad_debts ← bad_debts + f.liquidity()                   ▷ Note: bad_debts is negative!
            L̄ ← f                                                   ▷ L̄ is the sent of bankrupt firms
        else L ← f                                                   ▷ L is the set of healthy firms
        end if
    end for
    if |L| == 0 then break                              ▷ If all firms are bankrupt, exit the program
    end if
```

$\overline{p}_b \leftarrow$ average($f[i]$.price(), $f \in L$)

$\overline{Y}^T \leftarrow$ average($f[i]$.target_production(), $f \in L$)

$\overline{Y} \leftarrow$ average ($f[i]$.production(), $f \in L$)

for $f \in \overline{L}$ **do** f.reinit($\overline{p}_b, \overline{Y}^T, \overline{Y}$) ▷ Bankrupt firms are reinitialized with the average parameters of healthy firms

end for

total_liquidity $\leftarrow \sum_{i=0}^{N_{\mathrm{F}}} f[i]$.equity() $+ \sum_{i=0}^{N_{\mathrm{H}}} h[i]$.wealth()

for $f \in F$ **do** f.get_money(bad_debts * $f[i]$.equity() / total_liquidity)

end for

for $h \in H$ **do** h.get_money(bad_debts * $h[i]$.wealth() / total_liquidity)

end for ▷ Bad debt is spread over firms and households proportionally to their wealth

$\overline{p} \leftarrow \frac{\sum_{i=0}^{|F|-1} f[i].\mathrm{price}()\, f[i].\mathrm{sales}()}{\sum_{i=0}^{|F|-1} f[i].\mathrm{sales}()}$ ▷ Update average price

end for

Algorithm 3 The class `firm`

Parameters : $W = 1$, $\alpha = 1$, $\gamma_p = 0.1$, $\gamma_y = 0.1$, $\mu = 0$, $\delta = 0.2$, $\tau = 0.05$
Dynamic variables : `vector<household>` E; `household` O; $p, Y, Y^T, D, \mathcal{L}, v, \mathcal{D}^T, t$
Dynamic variables (auxiliary): $L_d, \rho, \mathcal{I}$

▷ **Initialization methods**

```
function INIT
    E ←empty                        ▷ The set E is the list of employees and is initialized as empty
    p ← 1                           ▷ Price
    Y ← 1                           ▷ Production
    Y^T ← 1                         ▷ Target production
    D ← 1                           ▷ Demand
    ℒ ← 50                          ▷ Liquidity
    v ← 0                           ▷ Number of vacancies
    𝒟^T ← 0                         ▷ Total debt
    t ← 0                           ▷ Internal clock
end function

function SET_OWNER(household Õ)
    O ← Õ
end function

function REINIT(p̃, Ỹ^T, Ỹ)
    p ← p̃
    Y ← Ỹ
    Y^T ← Ỹ^T
    D ← 0
    ℒ ← min{O.wealth(), Y/α}        ▷ The owner injects money to restart the bankrupt firm
    O.get_money(-ℒ)
    v ← 0
    𝒟^T ← 0
    t ← 0
    v ← 0
    for h ∈ E do
        fire(h)
    end for
end function
```

▷ **Output methods**

```
function PRICE
    return p
end function

function PRODUCTION
    return Y
end function

function STOCK
    return Y − D
end function

function SALES
    return D
end function

function TARGET_PRODUCTION
    return Y^T
end function

function EQUITY
    return ℒ − 𝒟^T
end function

function LIQUIDITY
    return ℒ
end function

function N_VACANCIES
    return v
end function
```

Algorithm 4 The class `firm` (continued)

function AGE
  return t
end function

▷ **Accounting methods**

function GET_MONEY($\widetilde{m}$)
  $\mathcal{L} \leftarrow \mathcal{L} + \widetilde{m}$
end function

function GET_LOANS(`bank` $\widetilde{B}$)
  $\mathcal{L}_n \leftarrow WL_d - \mathcal{L}$ ▷ Financial need
  if $\mathcal{L}_n > 0$ **then**
    $\ell \leftarrow (\mathcal{D}^T + \mathcal{L}_n)/(\mathcal{L} + 0.001)$ ▷ This is the leverage
    $\rho_{\text{offer}} \leftarrow \widetilde{B}$.compute_offer_rate($\ell$) ▷ New offered interest rate
    $\mathcal{D}^c \leftarrow \mathcal{L}_n\, F(\rho_{\text{offer}})$ ▷ The function $F(\rho)$ can be whatever decreasing function of ρ, see Eq. (4)
    if $\mathcal{D}^c > 0$ **then**
      $\rho \leftarrow \rho_{\text{offer}}$ ▷ If new credit is contracted, the interest rate is updated
      $\mathcal{D}^T \leftarrow \mathcal{D}^T + \mathcal{D}^c$ ▷ Total debt is increased by current debt $\mathcal{D}^c$
      $\mathcal{L} \leftarrow \mathcal{L} + \mathcal{D}^c$
      $\widetilde{B}$.get_money(-$\mathcal{D}^c$)
    end if
  end if
end function

function COMPUTE_INTERESTS
  $\mathcal{I} \leftarrow \rho\mathcal{D}^T$ ▷ Interests to be paid in this round
end function

function PAY_WORKERS
  for $h \in E$ **do**
    h.get_money(W)
  end for
  $\mathcal{L} \leftarrow \mathcal{L} - W|E|$
end function

function MARKUP_RULE
  if $Y > 0$ **then**
    $p_{\text{markup}} \leftarrow (1+\mu)(W|E| + \mathcal{I})/Y$
    $p \leftarrow \max\{p, p_{\text{markup}}\}$
  end if
end function

function ACCOUNTING(`bank` $\widetilde{B}$)
  $\mathcal{L} \leftarrow \mathcal{L} - \mathcal{I} - \tau\mathcal{D}^T$ ▷ Firm pays interests and repays a fraction τ of its debt
  $\widetilde{B}$.get_money($\mathcal{I} + \tau\mathcal{D}^T$)
  $\mathcal{D}^T \leftarrow (1-\tau)\mathcal{D}^T$
  $\mathcal{P} \leftarrow pD - W|E| - \mathcal{I}$ ▷ Profit
  if $\mathcal{P} > 0$ **then**
    O.get_money($\delta\mathcal{P}$) ▷ Firm pays dividends to the owner
    $\mathcal{L} \leftarrow \mathcal{L} - \delta\mathcal{P}$
  end if
end function

function SELL($\widetilde{q}$)
  $D \leftarrow D + \widetilde{q}$
  $\mathcal{L} \leftarrow \mathcal{L} + p\widetilde{q}$
end function

Algorithm 5 The class `firm` (continued)

▷ **Production and job market methods**

function SET_NEW_STRATEGY($\widetilde{p}$)
    $t \leftarrow t + 1$
    if $Y = D$ && $p \geq \widetilde{p}$ **then** $Y^T \leftarrow Y(1 + \gamma_y$ `random`$)$ ▷ This is Eq. (1) in the main text
    else if $Y = D$ && $p < \widetilde{p}$ **then** $p \leftarrow p(1 + \gamma_p$ `random`$)$
    else if $Y > D$ && $p \geq \widetilde{p}$ **then** $p \leftarrow p(1 - \gamma_p$ `random`$)$
    else if $Y > D$ && $p < \widetilde{p}$ **then** $Y^T \leftarrow Y(1 - \gamma_y$ `random`$)$
    end if
    $Y^T \leftarrow \max\{Y^T, \alpha\}$
    $L_d \leftarrow$ceil(Y^T/α)
end function

function DEFINE_LABOR_DEMAND
    $L_d \leftarrow \min\{L_d, \text{floor}(\mathcal{L}/W)\}$
    $L_d \leftarrow \max\{L_d, 0\}$
    $v \leftarrow L_d - |E|$
end function

function PRODUCE
    $Y = \min\{Y^T, \alpha|E|\}$
    $D = 0$ ▷ The demand is reset to zero at each production cycle
end function

function HIRE(`household` h)
    $E \leftarrow h$
    h.get_job(W)
    $v \leftarrow v - 1$
end function

function FIRE(`household` h)
    h.lose_job()
    $E \not\to h$
    $v \leftarrow v + 1$
end function

function FIRE_RANDOM_WORKER
    if $|E| > 0$ **then**
        $h \leftarrow_R E$
        fire(h)
    end if
end function

Algorithm 6 The class `household`

```
Parameters : M = 3, c = 0.8
Dynamic variables : S, W
                                                              ▷ Initialization methods
  function INIT
      S ← 0                                                                ▷ Savings
      W ← 0                                                                 ▷ Salary
  end function
                                                                ▷ Accounting methods
  function GET_MONEY(m̃)
      S ← S + m̃
  end function
                                                                    ▷ Output methods
  function WEALTH
      return S
  end function

  function WORKING
      if W > 0 then return True
      else return False
      end if
  end function
                                                      ▷ Job and goods market methods
  function GET_JOB(W̃)
      W ← W̃
  end function

  function LOSE_JOB
      W ← 0
  end function

  function CONSUME( vector<firm> F̃ )
      budget ← cS
      if budget > 0 then
          F_c ←_{R,M} F̃              ▷ Extract M random firms from F̃ and put them in the set F_c
          F_c ← sort(f ∈ F_c, f.price())       ▷ Order the set F_c according to firms' prices
          spent ← 0
          for (i ← 0; i < |F_c| && spent < budget; i ← i + 1) do
              s ← f[i].stock()                      ▷ s is the stock available from this firm
              if s > 0 then
                  q ← (budget - spent)/f[i].price()  ▷ Maximum possible consumption from this firm
                  if s > q then
                      f[i].sell(q)
                      spent ← budget        ▷ The household has finished the budget, the loop ends
                  else
                      f[i].sell(s)
                      spent ← spent + s f[i].price()
                  end if
              end if
          end for
      end if
      S ← S−spent
  end function
```

Algorithm 7 The class `bank`

Dynamic variables : E, ρ_b

```
function INIT
    E ← 0                                        ▷ Bank liquidity
    ρ_b ← 0                                      ▷ Baseline interest rate
end function

function SET_ρ(ρ̃)
    ρ_b ← ρ̃
end function

function COMPUTE_OFFER_RATE(ℓ̃)
    return ρ_b G(ℓ̃)                             ▷ We chose G(ℓ) = 1 + log(1 + ℓ)
end function

function GET_MONEY(m̃)
    E ← E + m̃
end function
```

Appendix B: Pseudo-code of Mark 0

We present here the pseudo-code for the Mark 0 model discussed in Sec. III A and in Sec. V. The source code is available on the site of the CRISIS project (www.crisis-economics.eu).

Algorithm 8 The basic Mark 0

Require: N_F Number of firms; $\mu, c, \beta, \gamma_p, \eta_+^0, \eta_-^0, \delta, \Theta, \varphi, f$; T total evolution time;

▷ **Initialization**

for $(i \leftarrow 0; i < N_\mathrm{F}; i \leftarrow i+1)$ **do**
  $W[i] \leftarrow 1$ ▷ Salaries are always fixed to one
  $p[i] \leftarrow 1 + 0.2(\texttt{random} - 0.5)$
  $Y[i] \leftarrow \mu[1 + 0.2(\texttt{random} - 0.5)]/2$ ▷ Initial employment is 0.5
  $\mathcal{L}[i] \leftarrow W[i]Y[i]\, 2\,\texttt{random}$
  $a[i] \leftarrow 1$
end for

$S \leftarrow N_\mathrm{F} - \sum_i \mathcal{L}[i]$

▷ **Main loop**

for $(t \leftarrow 1; t \leq T; t \leftarrow t+1)$ **do**
  $u \leftarrow 1 - \frac{1}{\mu N_\mathrm{F}} \sum_i Y[i]$
  $\varepsilon \leftarrow 1 - u$
  $\overline{p} \leftarrow \frac{\sum_i p[i]Y[i]}{\sum_i Y[i]}$
  $\overline{w} \leftarrow \frac{\sum_i W[i]Y[i]}{\sum_i Y[i]}$
  $\tilde{u}[i] \leftarrow \frac{\exp(\beta W[i]/\overline{w})}{\sum_i a[i] \exp(\beta W[i]/\overline{w})} N_\mathrm{F} u$

  ▷ **Firms update prices, productions and wages**

  for $(i \leftarrow 0; i < N_\mathrm{F}; i \leftarrow i+1)$ **do**

    if $a[i] == 1$ **then**
      $x \leftarrow -\frac{\mathcal{L}[i]}{W[i]Y[i]}$
      $x \leftarrow \min\{x, 1\}$
      $x \leftarrow \max\{x, -1\}$

      $\eta_+ \leftarrow \eta_+^0(1 - \Gamma x)$
      $\eta_- \leftarrow \eta_-^0(1 + \Gamma x)$

      if $Y[i] < D[i]$ **then** ▷ Wage update
        if $\mathcal{P}[i] > 0$ **then**
          $W[i] \leftarrow W[i][1 + \gamma_w(1 - \Gamma x)\varepsilon\, \texttt{random}]$
          $W[i] \leftarrow \min\{W[i], P[i] \min[D[i], Y[i]]/Y[i]\}$
        end if ▷ This is Eq. (11) in the main text
        $Y[i] \leftarrow Y[i] + \min\{\eta_+(D[i] - Y[i]), \mu\tilde{u}[i]\}$
        if $p[i] < \overline{p}$ **then** $p[i] \leftarrow p[i](1 + \gamma_p\, \texttt{random})$
        end if

      else if $Y[i] > D[i]$ **then**
        if $\mathcal{P}[i] < 0$ **then**
          $W[i] \leftarrow W[i][1 - \gamma_w(1 + \Gamma x)u\, \texttt{random}]$
        end if
        $Y[i] \leftarrow \max\{0, Y[i] - \eta_-(D[i] - Y[i])\}$
        if $p[i] < \overline{p}$ **then** $p[i] \leftarrow p[i](1 - \gamma_p\, \texttt{random})$
        end if
      end if
    end if
  end for

  $u \leftarrow 1 - \frac{1}{\mu N_\mathrm{F}} \sum_i Y[i]$ ▷ Update u and $\overline{p}$
  $\overline{p} \leftarrow \frac{\sum_i p[i]Y[i]}{\sum_i Y[i]}$

Algorithm 9 The basic Mark0 (continued)

```
                                                        ▷ Households decide the demand
B ← c(max{S, 0} + Σ_i W[i]Y[i])

for (i ← 0; i < N_F; i ← i + 1) do
    D[i] ← B a[i] exp(−β p[i]/p̄) / (p[i] Σ_i a[i] exp(−β p[i]/p̄))   ▷ Inactive firms have no demand
end for
                                                        ▷ Accounting
for (i ← 0; i < N_F; i ← i + 1) do
    if a[i] == 1 then
        𝒫[i] ← p[i] min{Y[i], D[i]} − W[i]Y[i]
        S ← S − 𝒫[i]
        ℒ[i] ← ℒ[i] + 𝒫[i]
        if 𝒫[i] > 0 && ℒ[i] > 0 then                   ▷ Pay dividends
            S ← S + δ 𝒫[i]
            ℒ[i] ← ℒ[i] − δ 𝒫[i]
        end if
        if ℒ[i] > Θ W[i]Y[i] then                       ▷ Set of healthy firms
            ℋ ← i
        end if
    end if
end for
                                                        ▷ Defaults
deficit = 0
for (i ← 0; i < N_F; i ← i + 1) do
    if a[i] == 1 && ℒ[i] < −Θ Y[i]W[i] then
        j ←_R ℋ
        if random < 1 − f && ℒ[j] > −ℒ[i] then          ▷ Bailed out
            ℒ[j] ← ℒ[j] + ℒ[i]
            ℒ[i] ← 0
            p[i] ← p[j]
            W[i] ← W[j]
        else                                            ▷ Bankrupted
            deficit ← deficit −ℒ[i]
            a[i] ← 0
            Y[i] ← 0
            ℒ[i] ← 0
        end if
    end if
end for
                                                        ▷ Revivals
A⁺ ← 0
for (i ← 0; i < N_F; i ← i + 1) do
    if a[i] == 0 && random < φ then                     ▷ Reactivate firm
        a[i] ← 1
        p[i] ← p̄
        Y[i] ← μu random
        ℒ[i] ← W[i]Y[i]
        deficit ← deficit +ℒ[i]
    end if
    if a[i] == 1 && ℒ[i] > 0 then                       ▷ Firms total savings
        A⁺ ← A⁺ + ℒ[i]
    end if
end for
                                                        ▷ Debt
if deficit > S then                                     ▷ Households cannot be indebted
    deficit ← deficit −S
    S ← 0
    for (i ← 0; i < N_F; i ← i + 1) do
        if a[i] == 1 && ℒ[i] > 0 then
            ℒ[i] ← ℒ[i]− ℒ[i]/A⁺ deficit
        end if
    end for
else
    S ← S− deficit
end if

end for
```

Appendix C: Perturbative solution of the schematic model

1. Stationary distribution at first order in γ_p

From Eq. (20), the stationary distribution $P_{\rm st}(\lambda)$ satisfies

$$P_{\rm st}(\lambda) = \int_0^1 d\xi \int_0^\infty d\lambda' P_{\rm st}(\lambda')\delta\left(\lambda - \lambda' + \xi + \frac{\gamma_p}{2}\xi^2\right) + \int_0^1 d\xi \int_{-\infty}^0 d\lambda' P_{\rm st}(\lambda')\delta\left(\lambda - \lambda' - \xi + \frac{\gamma_p}{2}\xi^2\right) \ . \tag{C1}$$

We recall that the dynamics of $\lambda_i(t)$ is

$$\begin{cases} \lambda_i(t) < 0 & \lambda_i(t+1) = \lambda_i(t) + \xi_i(t) - \frac{1}{2}\gamma_p \xi_i(t)^2 \\ \lambda_i(t) > 0 & \lambda_i(t+1) = \lambda_i(t) - \xi_i(t) - \frac{1}{2}\gamma_p \xi_i(t)^2 \end{cases} \tag{C2}$$

with $\xi_i(t)$ a random variable in $[0,1]$. From these equations it is clear that positive and negative λ_i are pushed towards the origin, and after some transient time one necessarily has

$$-1 - \frac{1}{2}\gamma_p \leq \lambda_i(t) \leq 1 - \frac{1}{2}\gamma_p \ . \tag{C3}$$

We first set $\gamma_p = 0$ and therefore restrict $\lambda \in [-1,1]$ in Eq. (C1). We obtain

$$P_{\rm st}(\lambda) = \int_{\max(0,-\lambda)}^{1-\max(\lambda,0)} {\rm d}\xi P_{\rm st}(\lambda+\xi) + \int_{\max(0,\lambda)}^{1-\max(-\lambda,0)} {\rm d}\xi P_{\rm st}(\lambda-\xi) = \mathcal{L}_0 P_{\rm st} \ , \tag{C4}$$

where we call $\mathcal{L}_0$ the linear operator that appears on the right hand side. When one computes the action of $\mathcal{L}_0$ on the functions $g_n(\lambda) = {\rm sign}(\lambda)\lambda^n$ and $h_n(\lambda) = \lambda^n$, one finds, for $\lambda > 0$:

$$(n+1)\mathcal{L}_0 g_n(\lambda) = \begin{cases} 1 - (-1)^n - \sum_{k=0}^{n-1} D_n^k (-1)^k \lambda^{n-k} & \lambda > 0 \\ 1 - (-1)^n + \sum_{k=0}^{n-1} D_n^k \lambda^{n-k} & \lambda < 0 \end{cases} \tag{C5}$$

with C_n^k the binomial coefficients and $D_n^k = \sum_{j=k}^n C_j^k$, and

$$(n+1)\mathcal{L}_0 h_n(\lambda) = \begin{cases} 1 - \lambda^{n+1} - (\lambda-1)^{n+1} = 1 + (-1)^n - 2\lambda^{n+1} - \sum_{k=1}^n C_{n+1}^k (-1)^k \lambda^{n+1-k} & \lambda > 0 \ , \\ (1+\lambda)^{n+1} + \lambda^{n+1} + (-1)^n = 1 + (-1)^n + 2\lambda^{n+1} + \sum_{k=1}^n C_{n+1}^k \lambda^{n+1-k} & \lambda < 0 \ . \end{cases} \tag{C6}$$

Let us focus on small values of n, useful in the following:

$$\begin{aligned} n = 0 \ &\rightarrow \ \mathcal{L}_0 g_0 = 0 \ , & \mathcal{L}_0 h_0 &= 2(h_0 - g_1) \ , \\ n = 1 \ &\rightarrow \ \mathcal{L}_0 g_1 = 1 - g_1 \ , & \mathcal{L}_0 h_1 &= -g_2 + h_1 \ , \\ n = 2 \ &\rightarrow \ \mathcal{L}_0 g_2 = h_1 - g_2 \ , & \mathcal{L}_0 h_2 &= \frac{2}{3}(h_0 - g_3) + h_2 - g_1 \ . \end{aligned} \tag{C7}$$

In particular, one has:

$$\mathcal{L}_0(h_0 - g_1) = 2(h_0 - g_1) - 1 + g_1 = (h_0 - g_1). \tag{C8}$$

This shows that $P_0(\lambda) = h_0(\lambda) - g_1(\lambda) = 1 - |\lambda|$ (called the "tent") is an eigenvector of $\mathcal{L}_0$ with eigenvalue 1, i.e. this is the stationary state for $\gamma_p = 0$.

This basic solution allows us to obtain perturbatevely the stationary solution for small γ_p. Recalling that the support of $P_{\rm st}(\lambda)$ is given by Eq. (C3), Eq. (C1) can be written as

$$P_{\rm st}(\lambda) = \int_{a_-}^{a_+} {\rm d}\xi P_{\rm st}(\lambda + \xi + \frac{1}{2}\gamma_p\xi^2) + \int_{b_-}^{b_+} {\rm d}\xi P_{\rm st}(\lambda - \xi + \frac{1}{2}\gamma_p\xi^2) = \mathcal{L}_{\gamma_p} P_{\rm st} \ , \tag{C9}$$

with:

$$\begin{aligned} a_- &= \max\left\{0, -\lambda(1 + \frac{\gamma_p}{2}\lambda)\right\} \ , \quad a_+ = \min\left\{1, 1 - \lambda - \frac{\gamma_p}{2}(1 + (1-\lambda)^2)\right\} \ , \\ b_- &= \max\left\{0, \lambda(1 + \frac{\gamma_p}{2}\lambda)\right\} \ , \quad b_+ = \min\left\{1, 1 + \lambda + \frac{\gamma_p}{2}(1 + (1+\lambda)^2)\right\} \ , \end{aligned} \tag{C10}$$

these integration bounds coming from the combination of the support in Eq. (C3) and the integration bounds on λ' in Eq. (C1). Writing $P_{\rm st} = P_0 + \gamma_p P_1$ where P_0 is the tent solution, and $\mathcal{L}_{\gamma_p} = \mathcal{L}_0 + \gamma_p \mathcal{L}_1$, we obtain at first order in γ_p an equation of the form:

$$(1 - \mathcal{L}_0) P_1 = \mathcal{L}_1 P_0 = S \ . \tag{C11}$$

The source term S can be most easily computed by computing $\mathcal{L}_{\gamma_p} P_0$ and expanding the result at first order in γ_p. We find $S = \lambda/2 - \mathrm{sign}(\lambda)\lambda^2 = \frac{1}{2} h_1 - g_2$. This is nice because one can look for a solution involving only $n = 1$ and $n = 2$ that closes the equation:

$$P_1 = \alpha h_1 + \beta g_2 \tag{C12}$$

Using: $\mathcal{L}_0 g_2 = h_1 - g_2 = \mathcal{L}_0 h_1$, one finds:

$$(1 - \mathcal{L}_0) P_1 = \alpha h_1 + \beta g_2 - (\alpha + \beta)(h_1 - g_2) = -\beta h_1 + (\alpha + 2\beta) g_2 = S = \frac{1}{2} h_1 - g_2 \ , \tag{C13}$$

so the result for the coefficients is

$$\beta = -\frac{1}{2} \ , \qquad \alpha = 0 \ . \tag{C14}$$

The final solution for the stationary state to first order in γ is:

$$P = 1 - |\lambda| - \frac{1}{2} \gamma_p \, \mathrm{sign}(\lambda) \lambda^2. \tag{C15}$$

Note that P is still normalized, as it should and goes to zero (to first order in γ_p) at the boundary of the support interval given in Eq. (C3).

2. Perturbative analysis of the oscillations

In order to understand oscillations, we start from Eqs. (30) and in the following we assume that $0 < C < 1$. We introduce $\mathcal{E}(t) = \frac{\overline{\alpha}(t) - C\overline{\lambda}(t)}{2(1-C)}$, $\Lambda(t) = \min\{\overline{\lambda}(t), \mathcal{E}(t)\}$ and $\Omega(t) = \max\{\overline{\lambda}(t), \mathcal{E}(t)\}$, and we write Eq. (30) equivalently as

$$\begin{aligned} &\alpha_i(t+1) = \alpha_i(t) - \min\{\lambda_i(t), C\lambda_i(t) + \frac{1}{2}(\overline{\alpha}(t) - C\overline{\lambda}(t))\} \\ &\begin{cases} \text{If } \lambda_i(t) < \Lambda(t) & \Rightarrow \quad \lambda_i(t+1) = \lambda_i(t) + \xi_i(t) \\ \text{If } \Omega(t) < \lambda_i(t) < \Lambda(t) & \Rightarrow \quad \lambda_i(t+1) = \lambda_i(t) \\ \text{If } \lambda_i(t) > \Omega(t) & \Rightarrow \quad \lambda_i(t+1) = \lambda_i(t) - \xi_i(t) \end{cases} \end{aligned} \tag{C16}$$

The master equation for the distribution of λ reads:

$$\begin{aligned} P_{t+1}(\lambda') &= \int_{-\infty}^{\Lambda(t)} d\lambda \int_0^1 d\xi \, P_t(\lambda) \, \delta(\lambda' - \lambda - \xi) + \int_{\Lambda(t)}^{\Omega(t)} d\lambda \, P_t(\lambda) \delta(\lambda - \lambda') + \int_{\Omega(t)}^{\infty} d\lambda \int_0^1 d\xi \, P_t(\lambda) \, \delta(\lambda' - \lambda + \xi) \\ &= \int_{-\infty}^{\Lambda(t)} d\lambda \int_0^1 d\xi \, P_t(\lambda) \, \delta(\lambda' - \lambda - \xi) + \theta(\Lambda(t) \le \lambda' \le \Omega(t)) P_t(\lambda') + \int_{\Omega(t)}^{\infty} d\lambda \int_0^1 d\xi \, P_t(\lambda) \, \delta(\lambda' - \lambda + \xi) \ . \end{aligned} \tag{C17}$$

while the evolution equation for $\overline{\alpha}$ is

$$\overline{\alpha}(t+1) = \overline{\alpha}(t) - \int_{-\infty}^{\infty} d\lambda \, P_t(\lambda) \min\{\lambda, C\lambda + \frac{1}{2}(\overline{\alpha}(t) - C\overline{\lambda}(t))\} \ . \tag{C18}$$

a. Stationary state

Numerically, we observe that $\Lambda(t) \sim \Omega(t)$ and their variations are much smaller than the width of the distributions of λ and α. In the limit where

$$\Lambda(t) = \Omega(t) = \overline{\lambda}(t) = \mathcal{E}(t) = \frac{\overline{\alpha}(t) - C\overline{\lambda}(t)}{2(1-C)} \tag{C19}$$

we obtain

$$P_{t+1}(\lambda') = \int_{-\infty}^{\Lambda(t)} d\lambda \int_0^1 d\xi \, P_t(\lambda)\, \delta(\lambda' - \lambda - \xi) + \int_{\Lambda(t)}^{\infty} d\lambda \int_0^1 d\xi \, P_t(\lambda)\, \delta(\lambda' - \lambda + \xi) \;, \tag{C20}$$

whose stationary solution is $P(\lambda) = P_0[\lambda - \overline{\lambda}^*]$ where $P_0(x) = (1-|x|)\theta(1-|x|)$ is the tent function, with $\overline{\lambda}^*$ undetermined at this stage. Plugging this result in Eq. (C18), we have

$$\overline{\alpha}(t+1) = \overline{\alpha}(t) - \overline{\lambda}^* + \frac{1-C}{6} \;. \tag{C21}$$

The fixed point is therefore $\overline{\lambda}^* = \frac{1-C}{6}$ and from the condition (C19) we get $\overline{\alpha}^* = (2-C)\overline{\lambda}^* = (2-C)(1-C)/6$.

b. Oscillation around the stationary state – a simple approximation

In order to study the small oscillations we can make a very simple approximation, namely that at each time t we have $P_t(\lambda) = P_0[\lambda - \bar{\lambda}(t)]$. If we inject this approximation in Eq. (C18) we get, provided $\bar{\lambda}(t) - \bar{\lambda}^*$ is not too large, that

$$\begin{aligned} \bar{\alpha}(t+1) &= \bar{\alpha}(t) - \int_{-\infty}^{\infty} d\lambda \min\{\lambda, C\lambda + (1-C)\mathcal{E}(t)\} P_0[\lambda - \bar{\lambda}(t)] \\ &= \bar{\alpha}(t) - \bar{\lambda}(t) - \frac{1-C}{6}\left[-1 + 3A - 3A^2 + A^3\,\mathrm{sign}(A)\right]_{A = \mathcal{E}(t) - \bar{\lambda}(t)} \end{aligned} \tag{C22}$$

Next, injecting this approximation in Eq. (C17) we get

$$\begin{aligned} \bar{\lambda}(t+1) &= \bar{\lambda}(t) + \frac{1}{2}\int_{-\infty}^{\Lambda(t)} d\lambda\, P_0[\lambda - \bar{\lambda}(t)] - \frac{1}{2}\int_{\Omega(t)}^{\infty} d\lambda\, P_0[\lambda - \bar{\lambda}(t)] \\ &= \frac{1}{2}\left[\Lambda(t) + \Omega(t) - \frac{1}{2}[\Lambda(t) - \bar{\lambda}(t)]^2 \mathrm{sgn}[\Lambda(t) - \bar{\lambda}(t)] - \frac{1}{2}[\Omega(t) - \bar{\lambda}(t)]^2 \mathrm{sgn}[\Omega(t) - \bar{\lambda}(t)]\right] \end{aligned} \tag{C23}$$

We therefore get a closed system of two equations for $\bar{\lambda}(t)$ and $\bar{\alpha}(t)$. For small C, this system of equations converges quickly to the fixed point. However, for C sufficiently close to 1 ($C \sim 0.94$) it has a limit cycle of period 2, followed by an exponential divergence.

c. Oscillation around the stationary state – perturbative computation

Another strategy to characterize the oscillations is to look again for a perturbed solution around the stationary state, with: $\overline{\lambda}(t) = \frac{1-C}{6} + \overline{\delta\lambda}_1(t)$ and $\overline{\alpha}(t) = (2-C)\overline{\lambda} + 2(1-C)\overline{\delta\lambda}_2(t) + C\overline{\delta\lambda}_1(t)$. The distribution of λ_i at time t is now $P_t(\lambda) = P_0(\lambda - \frac{1-C}{6}) + \delta P_t(\lambda - \frac{1-C}{6})$. The shifted variable $\lambda - \frac{1-C}{6}$ will be denoted x. The aim is to write an evolution for δP_{t+1} after one time step. The $\mathcal{L}_0$ operator is the same as before, as well as the set of functions g_n and h_n. We also introduce $\Delta(x)$ as a δ-function slightly spread out, but of unit area (its precise width is irrelevant if it is small enough).

To lowest order, one has:

$$\mathcal{L}_0 \Delta \approx \frac{1}{2} h_0. \tag{C24}$$

We will also need, as found above:

$$\mathcal{L}_0 h_0 = 2(h_0 - g_1), \qquad \mathcal{L}_0 g_0 = 0, \qquad \mathcal{L}_0 g_1 = h_0 - g_1. \tag{C25}$$

We now introduce $d = |\overline{\delta\lambda}_2 - \overline{\delta\lambda}_1|$ and $s = \overline{\delta\lambda}_2 + \overline{\delta\lambda}_1$. Assuming $d, s \ll 1$, one finds, to first order:

$$\delta P_{t+1} = \mathcal{L}_0 \delta P_t - \frac{1}{2} d_t h_0 + \frac{1}{2} s_t g_0 + d_t \Delta \tag{C26}$$

Note that the integral $[\delta P] = \int_{-1}^{1} dx \delta P(x)$ is zero and conserved in time, as it should be (one has $[h_0] = 2$, $[g_0] = 0$ and $[\Delta] = 1$).

The idea now is to expand δP in terms of the h, g functions. It turns out that only three of them are needed, plus the Δ contribution. Indeed, assume:

$$\delta P_t = A_t (h_0 - 2\Delta) + B_t g_0 + C_t (g_1 - \frac{1}{2} h_0) \tag{C27}$$

Then, using the dynamical equation and the algebra above, one finds:

$$\mathcal{L}_0 \delta P_t = A_t (2(h_0 - g_1) - h_0) = -2 A_t (g_1 - \frac{1}{2} h_0) \tag{C28}$$

Hence, the evolution is closed on itself, with the following evolution rules:

$$A_{t+1} = -\frac{1}{2} d_t, \qquad B_{t+1} = \frac{1}{2} s_t, \qquad C_{t+1} = -2 A_t. \tag{C29}$$

Note that $A_t - B_t = -\frac{1}{2}(d_{t-1} + s_{t-1}) = -\max(\overline{\delta\lambda}_{1t-1}, \overline{\delta\lambda}_{2t-1}) = -M_{t-1}$.

Now, by definition $\overline{\lambda}_t = \frac{1-C}{6} + \int_{-1}^{1} dx x \delta P_t$, but since h_0 and g_1 are even, the only contribution comes from the g_0 component. Therefore:

$$\overline{\delta\lambda}_{1t} = B_t \int_{-1}^{1} dx x g_0(x) = B_t. \tag{C30}$$

This leads to a first evolution equation:

$$\overline{\delta\lambda}_{1t+1} = \frac{1}{2} s_t = \frac{1}{2}(\overline{\delta\lambda}_{1t} + \overline{\delta\lambda}_{2t}). \tag{C31}$$

The second equation comes from the evolution of the α_i's. From:

$$\overline{\alpha}(t+1) = \overline{\alpha}(t) - C \overline{\delta\lambda}_1(t) - \frac{1-C}{2} \overline{\delta\lambda}_2(t) - (1-C) \int_{-1}^{0} dx\, x \delta P_t(x), \tag{C32}$$

we finally find:

$$\overline{\delta\lambda}_{2t+1} = \frac{3}{4} \overline{\delta\lambda}_{2t} - \frac{1}{4} M_{t-1} + \frac{7}{24} d_{t-2} - \frac{C}{4(1-C)} (\overline{\delta\lambda}_{1t} + \overline{\delta\lambda}_{2t}). \tag{C33}$$

The coupled set of iterations for $\overline{\delta\lambda}_{1t}$ and $\overline{\delta\lambda}_{2t}$, Eqs. (C31) and (C33), lead to damped oscillations for $C < C^{**}$ and sustained oscillations for $C > C^{**}$, with $C^{**} \approx 0.91$. The numerical value of C^{**} however does not coincide with that of $C^* \approx 0.45$, obtained from the direct simulation of Eqs. (30). This might be due to neglecting higher order corrections, which appear to be numerically large: the oscillations generated by Eqs. (30) are of amplitude ~ 0.1 at the onset, suggesting a sub-critical bifurcation. For small C, on the other hand, the oscillation amplitude is small and the above equations appear to be quantitatively correct, validating the above calculations.

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