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On MacLane's Conditions for Coherence of Natural Associativities, Commutativities, etc.

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Let there be given a category $\mathscr{A}$ and a covariant bifunctor T from $\mathscr{A} \times \mathscr{A}$ to $\mathscr{A}$, and let us for simplicity write AB for $T(A, B)$ and fg for $T(f, g)$, where A, B are objects and f, g are morphisms. By a *natural associativity* for T is meant a natural *isomorphism* $a : A(BC) \to (AB)C$, and if such an a exists we say that T is naturally associative, or that it is associative to within natural isomorphism. Similarly we can contemplate natural isomorphisms $c : AB \to BA$, $e : KA \to A$, $f : AK \to A$, expressing respectively that T is (naturally) commutative, or admits the fixed object K of $\mathscr{A}$ as a left identity, or admits K as a right identity.

Suppose that one or more of a, c, e, f are given; then MacLane [*1*, *2*] has raised the question of their *coherence*: the given isomorphisms are said to be coherent if any natural automorphism manufactured from them and their inverses alone (together with identity morphisms) is the identity automorphism. A formal definition is given in [*2*], but the reader will see well enough what is meant by glancing at the numbered statements (1)–(10) below, which belong to the infinite set of statements whose conjunction is the assertion of coherence. MacLane has shown in [*2*] that in fact coherence is equivalent to the truth of a number of the statements in this finite list; and our present object is to point out that some of his results can be slightly improved, in that they contain redundant conditions. The statements we wish to list are:

(1)
$$\begin{tikzcd} A[B(CD)] \arrow[r, "a"] \arrow[d, "1a"'] & (AB)(CD) \arrow[r, "a"] & {[(AB)C]D} \\ A[(BC)D] \arrow[rr, "a"'] & & {[A(BC)]D} \arrow[u, "a1"'] \end{tikzcd}$$ commutes.

(2)
$$\begin{tikzcd} AB \arrow[r, "c"] \arrow[dr, "1"'] & BA \arrow[d, "c"] \\ & AB \end{tikzcd}$$
commutes.

(3)
$$\begin{tikzcd} A(BC) \arrow[r, "a"] \arrow[d, "1c"'] & (AB)C \arrow[r, "c"] & C(AB) \arrow[d, "a"] \\ A(CB) \arrow[r, "a"'] & (AC)B \arrow[r, "c1"'] & (CA)B \end{tikzcd}$$
commutes.

(4) $e = f : KK \longrightarrow K.$

(5)
$$\begin{tikzcd} K(BC) \arrow[dr, "e"] \arrow[dd, "a"'] & \\ & BC \\ (KB)C \arrow[ur, "e1"'] & \end{tikzcd}$$
commutes.

(6)
$$\begin{tikzcd} A(KC) \arrow[dr, "1e"] \arrow[dd, "a"'] & \\ & AC \\ (AK)C \arrow[ur, "f1"'] & \end{tikzcd}$$
commutes.

(7)
$$\begin{tikzcd} A(BK) \arrow[dr, "1f"] \arrow[dd, "a"'] & \\ & AB \\ (AB)K \arrow[ur, "f"'] & \end{tikzcd}$$
commutes.

(8)
$$\begin{tikzcd} AK \arrow[dr, "f"] \arrow[dd, "c"'] & \\ & A \\ KA \arrow[ur] & \end{tikzcd}$$
commutes.

(9)
$$\begin{tikzcd} A(KC) \arrow[r, "1e"] \arrow[d, "a"'] & AC \\ (AK)C \arrow[r, "c1"'] & (KA)C \arrow[u, "e1"'] \end{tikzcd}$$
commutes.

(10)
$$\begin{tikzcd} KK \arrow[dr, "e"] \arrow[dd, "c"'] & \\ & K \\ KK \arrow[ur, "e"'] & \end{tikzcd}$$
commutes.

Note that (9) and (10) are what (6) and (4), respectively, become if f is not given but is *defined* in terms of e and c by (8). Finally, we list a variant form of (3), which is obviously equivalent to it in the presence of (2):

(3′)
$$\begin{tikzcd} A(BC) \arrow[r, "a"] & (AB)C \arrow[r, "c"] & C(AB) \arrow[d, "a"] \\ A(CB) \arrow[u, "1c"] \arrow[r, "a"'] & (AC)B & (CA)B \arrow[l, "c1"] \end{tikzcd}$$
commutes.

MacLane's results in [2] can now be stated as:

THEOREM 1. *a is coherent* $\Leftrightarrow$ (1).

THEOREM 2. *a and c are coherent* $\Leftrightarrow$ (1), (2), *and* (3).

THEOREM 3. *a, e, and f are coherent* $\Leftrightarrow$ (1), (4), (5), (6), *and* (7).

THEOREM 4. *a, c, and e are coherent* $\Leftrightarrow$ (1), (2), (3), (5), (9), *and* (10).

The case where a, c, e and f are all given is not considered in [2], but clearly Theorem 4 is equivalent to:

THEOREM 5. *a, c, e, and f are coherent* $\Leftrightarrow$ (1), (2), (3), (4), (5), (6), *and* (8).

Now Theorems 1 and 2 are best possible, as we shall see below. But Theorem 3 can be improved to:

THEOREM 3′. *a, e, and f are coherent* $\Leftrightarrow$ (1) *and* (6); because

THEOREM 6. (5) *and* (6) $\Rightarrow$ (4); *whence by symmetry* (6) *and* (7) $\Rightarrow$ (4); and

THEOREM 7. (1) *and* (6) $\Rightarrow$ (5); *whence by symmetry* (1) *and* (6) $\Rightarrow$ (7). (Note that (1) is symmetric in the sense that it remains unchanged if a is replaced by a^{-1} and AB by BA).

It follows that Theorem 5 can also be improved by dropping (4) and (5) from the hypotheses; but in fact still more is true:

THEOREM 5′. *a, c, e, and f are coherent* $\Leftrightarrow$ (1), (2), (3), *and* (6), because

THEOREM 8. (2), (3), (6), *and* (7) $\Rightarrow$ (8).

Theorem 5′ now gives, *a fortiori*,

THEOREM 4′. *a, c, and e are coherent* $\Leftrightarrow$ (1), (2), (3), *and* (9).

It is also true that

THEOREM 4″. *a, c, and e are coherent* $\Leftrightarrow$ (1), (2), (3), *and* (5), because

THEOREM 9. (2), (3), (5) *and* (8) $\Rightarrow$ (6).

Finally, if we replace (3) by (3′), we can do a little better than Theorem 4′:

THEOREM 4‴. *a, c, and e are coherent* $\Leftrightarrow$ (1), (3′), *and* (9), because

THEOREM 10. (3′), (6), (7), *and* (8) $\Rightarrow$ (2).

We now prove Theorems 6–10.

THEOREM 6. The naturality of e gives the commutativity of

$$\begin{tikzcd}
K(KK) \arrow[r, "e"] \arrow[d, "1e"'] & KK \arrow[d, "e"] \\
KK \arrow[r, "e"'] & K,
\end{tikzcd}$$

whence $1e = e : K(KK) \to KK$, since e is an isomorphism. If we put $B = C = K$ in (5) and $A = C = K$ in (6), and use this, we get $e1 = f1 : (KK)K \to KK$. The naturality of f now allows us to conclude $e = f : KK \to K$.

THEOREM 7. Consider the diagram

$$\begin{tikzcd}
A[K(CD)] \arrow[rr, "a"] \arrow[dd, "1a"'] \arrow[dr, "1e"'] & & (AK)(CD) \arrow[rr, "a"] \arrow[dl, "f1"] & \arrow[d, phantom, "\mathrm{III}"] & {[(AK)C]D} \arrow[dl, "(f1)1"] \\
\arrow[r, phantom, "\mathrm{I}"] & A(CD) \arrow[u, phantom, "\mathrm{II}"] \arrow[rr, "a"] \arrow[d, phantom, "\mathrm{V}"] & & (AC)D \arrow[r, phantom, "\mathrm{IV}"] & {} \\
A[(KC)D] \arrow[ur, "1(e1)"'] \arrow[rrrr, "a"'] & {} & & & {[A(KC)]D}. \arrow[ul, "(1e)1"] \arrow[uu, "a1"']
\end{tikzcd}$$

The outside commutes by (1), regions II and IV by (6), and regions III and V by the naturality of a. It follows that region I commutes. Putting $A = K$ we get (5), because, e being both natural and an isomorphism, we can conclude $h = k : \underset{\sim}{P} \to \underset{\sim}{Q}$ from $1h = 1k : K\underset{\sim}{P} \to K\underset{\sim}{Q}$.

THEOREM 8. The argument is similar to that used to prove Theorem 7: we put $B = K$ in (3), use (6), (7), and the naturality of c, and infer the commutativity of a diagram differing from (8) in that the sense of c is reversed; (2) now allows us to infer (8).

THEOREM 9. Put $A = K$ in (3); using (5), (8) with the sense of c reversed —which is a consequence of (8) and (2)—and the naturality of c, we get (6).

THEOREM 10. Put $B = K$ in (3′); using (6), (7), (8), and the naturality of c, we easily get (2).

We end by showing that Theorems 1, 2, 3′, 4′, 4″, 4‴, 5′ are best possible, in that the conditions given therein for coherence are in each case independent. For this purpose we take $\mathscr{A}$ to be the category of graded abelian groups and homogeneous maps of degree 0, and take the functor T to be the usual tensor product of graded abelian groups; here K is the infinite cyclic group, lying in degree 0. We define a, c, e, f by

$$\begin{aligned} a(x \otimes (y \otimes z)) &= (-1)^{\alpha(\xi,\eta,\zeta)}(x \otimes y) \otimes z, \\ c(x \otimes y) &= (-1)^{\gamma(\xi,\eta)} y \otimes x, \\ e(1 \otimes x) &= (-1)^{\epsilon(\xi)} x, \\ f(x \otimes 1) &= (-1)^{\phi(\xi)} x; \end{aligned}$$

where ξ, η, ζ are the degrees of x, y, z respectively, and α, γ, ϵ, ϕ are functions which will change from example to example. For brevity we shall write $\alpha = 1$, $\gamma = \xi$, etc., to mean $\alpha(\xi, \eta, \zeta) = 1$ for all ξ, η, ζ, $\gamma(\xi, \eta) = \xi$ for all ξ, η, etc. We shall also write (1) T, (2) F, to denote respectively the truth of (1) and the falsehood of (2).

Theorem 1 *is best possible.* Take $\alpha = 1$; then (1)F.

Theorem 2 *is best possible.*

(i) $\alpha = 1, \gamma = 1$; (1)F, (2)T, (3)T.
(ii) $\alpha = 0, \gamma = \xi$; (1)T, (2)F, (3)T.
(iii) $\alpha = 0, \gamma = 1$; (1)T, (2)T, (3)F.

Theorem 3′ *is best possible.*

(i) $\alpha = 1, \epsilon = 1, \phi = 0$; (1)$F$, (6)$T$.
(ii) $\alpha = 0, \epsilon = 0, \phi = 1$; (1)$T$, (6)$F$.

Theorem 4′ *is best possible.*

(i) $\alpha = 1$, $\gamma = 1$, $\epsilon = 0$; (1)F, (2)T, (3)T, (9)T.
(ii) $\alpha = 0$, $\gamma = \frac{1}{2}\xi(\eta^2 - \eta)$, $\epsilon = 0$; (1)T, (2)F, (3)T, (9)T.
(iii) $\alpha = 0$, $\gamma = \frac{1}{2}\xi\eta(\xi + \eta)$, $\epsilon = 0$; (1)T, (2)T, (3)F, (9)T.
(iv) $\alpha = 0$, $\gamma = 0$, $\epsilon = \xi$; (1)T, (2)T, (3)T, (9)F.

Theorem 4″ *is best possible.*

(i) $\alpha(1, 2, 7) = \alpha(2, 7, 1) = \alpha(7, 2, 1) = \alpha(1, 7, 2) = 1$, $\alpha(\xi, \eta, \zeta) = 0$ for all other triples (ξ, η, ζ); $\gamma = 0$, $\epsilon = 0$; (1)F (apply to $x \otimes [y \otimes (z \otimes t)]$ where x, y, z, t have degrees 1, 2, 3, 4 respectively), (2)T, (3)T, (5)T.
(ii) $\alpha = 0$, $\gamma = \xi$, $\epsilon = 0$; (1)T, (2)F, (3)T, (5)T.
(iii) $\alpha = \xi$, $\gamma = 0$, $\epsilon = 0$; (1)T, (2)T, (3)F, (5)T.
(iv) $\alpha = 0$, $\gamma = 0$, $\epsilon = \xi$; (1)T, (2)T, (3)T, (5)F.

Theorem 4‴ *is best possible.*

(i) $\alpha = 1$, $\gamma = 1$, $\epsilon = 0$; (1)F, (3′)T, (9)T.
(ii) $\alpha = 0$, $\gamma = \eta$, $\epsilon = 0$; (1)T, (3′)F, (9)T.
(iii) $\alpha = 0$, $\gamma = 0$, $\epsilon = \xi$; (1)T, (3′)T, (9)F.

Theorem 5′ *is best possible.*

(i) $\alpha = 1$, $\gamma = 1$, $\epsilon = 0$, $\phi = 1$; (1)F, (2)T, (3)T, (6)T.
(ii) $\alpha = 0$, $\gamma = \xi$, $\epsilon = 0$, $\phi = 0$; (1)T, (2)F, (3)T, (6)T.
(iii) $\alpha = 0$, $\gamma = 1$, $\epsilon = 0$, $\phi = 0$; (1)T, (2)T, (3)F, (6)T.
(iv) $\alpha = 0$, $\gamma = 0$, $\epsilon = 0$, $\phi = 1$; (1)T, (2)T, (3)T, (6)F.

References

1. MacLane, S., "Categorical Algebra." Boulder Colloquium Lectures, 1963.
2. MacLane, S., Natural associativity and commutativity. *Rice. Univ. Stud.* **49** (1963), 28-46.