Chapter 1

Bicategorical Models of Linear Logic

In this chapter, we will refine our definition of a model of linear logic by making explicit some information of the logic that is hidden in the usual model. For this, we will need to shift settings going from the simple categorical setting to the more complex bicategorical one. For this, we start be going back to the formal definition of a categorical model of a logic :

Definition 1. *Let $\mathcal{L}$ be a logic. A categorical model of a logic is a category $\mathcal{C}$, along with a mapping $[-]$ such that :*

- *for every formula A of $\mathcal{L}$, there is an object $[A]$ in $\mathcal{C}$*
- *for every proof π of $A \to B$ of $\mathcal{L}$, there is a morphism $[\pi] : [A] \to [B]$ in $\mathcal{C}$*
- *for every pair of proofs π_1, π_2 such that $\pi_1 =_{\mathcal{L}} \pi_2$, we have$[\pi_1] = [\pi_2]$ in $\mathcal{C}$*

This last line is where our work will focus. In Linear Logic, the equality relation $=_{\mathcal{L}}$ is given the relation raising from the cut-elimination process, or more exactly, the transitive reflexive closure of a weaker version introduced by Melliès inTODO: cite. The cut-elimination process introduced by Girard is a formal process taking any proof of linear logic and transforming it into a proof using no cut rules, giving us a normalization of proofs. We will recall the rules of cut-elimination as we encounter them in our construction. What the line then means is simply : whenever two proofs are related by cut elimination in any way, they're modelized by the same morphism. Note that this is slightly stronger than just saying "every proof is modelized by the interpretation of its cutless proof" as the weaker version of the cut elimination process is not confluent, allowing for a proof to produce two different cutless proofs, which must then have the same model. We will see examples of this in what follows.

What we wish to achieve with our model is to refine the existing one by allowing different proofs related by cut-elimination to be interpreted in different but related ways. This means that the proof transformations themselves must be interpreted in the model explicitely. This matches well with the notion of bicategory. Indeed, as proofs are related by proof transformations in our logic, morphisms are related by 2-morphisms in a bicategory. This pushes us towards the following description of a bicategorical model of logic :

Let $\mathcal{L}$ be a logic with proof transformations. A bicategorical model of a logic is a bicategory $\mathcal{C}$, along with a mapping $[-]$ such that :

- for every formula A of $\mathcal{L}$, there is an object $[A]$ in $\mathcal{C}$
- for every proof π of $A \to B$ of $\mathcal{L}$, there is a morphism $[\pi] : [A] \to [B]$ in $\mathcal{C}$
- for every transformation of proofs $t : \pi_1 \Rightarrow \pi_2$ in the logic , there is a 2-morphism $[t] : [\pi_1] \Rightarrow [\pi_2]$ in $\mathcal{C}$

This is not enough though as this definition is missing the equivalent of the last line of the categorical model, which introduced the equalities between proofs required to obtain the coherence structures of categories. We will need something similar to obtain the coherence structures of bicategories. What we will see very soon is that every basic cut-elimination rule can be matched to one of the usual structural 2-morphisms of certain bicategorical setting. As the basic definition of coherence in bicategories can be summed up to :

Every formal 2−diagram consisting of structural 2−morphisms commutes,

one requirement on proof transformations that will allow for this property to hold for their interpretation is the requirement that any two proof transformation between the same proofs are interpreted in the same way. TODO: this is overkill, but the alternative is offering a full stan
giving us the following definition :

Definition 2. *Let $\mathcal{L}$ be a logic with proof transformations. A bicategorical model of a logic is a bicategory $\mathcal{C}$, along with a mapping $[-]$ such that :*

- *for every formula A of $\mathcal{L}$, there is an object $[A]$ in $\mathcal{C}$*
- *for every proof π of $A \to B$ of $\mathcal{L}$, there is a morphism $[\pi] : [A] \to [B]$ in $\mathcal{C}$*
- *for every transformation of proofs $t : \pi_1 \Rightarrow \pi_2$ in the logic , there is a 2-morphism $[t] : [\pi_1] \Rightarrow [\pi_2]$ in $\mathcal{C}$*
- *for every pair of proof transformations $t_1, t_2 : \pi_1 \Rightarrow \pi_2$, we have$[t_1] = [t_2]$ in $\mathcal{C}$*

In this chapter, we will go over the case of Intuitionistic linear logic in full, starting by the core logic , which will include a proper definition of bicategories.

1.1 Core Logic and Bicategories

In this section, we start with the core rules of Linear Logic, with no connector, and take the opportunity to recall some basic bicategorical definitions.

The very core rules of Linear Logic, and indeed of most logics can be reduced to two simple rules :

$$\dfrac{}{A \vdash A}\,\text{ax} \qquad \dfrac{\dfrac{\pi_1}{\Gamma \vdash A} \quad \dfrac{\pi_2}{A, \Delta \vdash B}}{\Gamma, \Delta \vdash B}\,\text{cut}$$

Note that this doesn't give rise to a very interesting logical system as the only proofs that can be built only with these are proofs of $A \vdash A$. They still offer some good insights into the categorical (or rather bicategorical) structure of a model of logic.
Indeed, the axiom rule is modelized by identities and the cut rule by composition of

morphisms as we have seen in Chapter 2. The next part is more interesting. We have mentioned earlier that the rules of the cut elimination process will be modelized by $2-$morphisms in a bicategorical models where they were used to produce the constraints in a categorial model. This lines up with the general presentation of higher-order categories where the constraints of the categorical definition are turned into new data in the higher order definition. The new constraints of the definition are then obtained from a coherence theorem. Let us take a look at this basic case with the rules of cut elimination process involving the axiom and cut rules :

l-ax *cut*

$$\frac{\dfrac{}{A \vdash A}\,\mathrm{ax} \qquad \dfrac{\pi_1}{A, \Delta \vdash B}}{A, \Delta \vdash B}\,\mathrm{cut} \qquad \Rightarrow \qquad \frac{\pi_1}{A, \Delta \vdash B}$$

r-ax *cut*

$$\frac{\dfrac{\pi_1}{\Gamma \vdash B} \qquad \dfrac{}{B \vdash B}\,\mathrm{ax}}{\Gamma \vdash B}\,\mathrm{cut} \qquad \Rightarrow \qquad \frac{\pi_1}{\Gamma \vdash B}$$

commut *cut*

$$\frac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash A} \qquad \dfrac{\pi_2}{A, \Delta \vdash B}}{\Gamma, \Delta \vdash B}\,\mathrm{cut} \qquad \dfrac{\pi_3}{B, \Theta \vdash C}}{\Gamma, \Delta, \Theta \vdash C}\,\mathrm{cut} \qquad \Rightarrow \qquad \frac{\dfrac{\pi_1}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\pi_2}{A, \Delta \vdash B} \qquad \dfrac{\pi_3}{B, \Theta \vdash C}}{A, \Delta, \Theta \vdash C}\,\mathrm{cut}}{\Gamma, \Delta, \Theta \vdash C}\,\mathrm{cut}$$

The first two transformations are called the left and right axiom cut transformations, while the last one is called the cut commutation.
Note that in those three cases, the detail of the branches π_1, π_2, π_3 doesn't matter to the transformation, only the form of their conclusion. We will call those branches the argument branches of the transformations.
Those rules then translate into the $2-$ morphisms :

$$id_A \circ [\pi_1] \Rightarrow [\pi_1]$$

$$[\pi_1] \circ id_B \Rightarrow [\pi_1]$$

$$([\pi_1] \circ [\pi_2]) \circ [\pi_3] \Rightarrow [\pi_1] \circ ([\pi_2] \circ [\pi_3])$$

The coherence conditions of the definition of bicategory then match the requirements of our definition of bicategorical model by definition, but we can still look at it a bit more in depth. We can categorize the required equalities of proof transformations in three classes :

- The natural equalities. Those are of the following form :

  let T be a proof tree with a branch π_1, t_1 a transformation turning T into T' with π_1 as an argument branch and t_2 a transformation turning π_1 into π_1'. We want to be able to say that the two following sequences are equalized in the model :

$$\dfrac{\dfrac{\pi_1}{T}}{\Gamma \vdash A} \Rightarrow^{t_1} \dfrac{\dfrac{\pi_1}{T'}}{\Gamma \vdash A} \Rightarrow^{t_2} \dfrac{\dfrac{\pi_1'}{T'}}{\Gamma \vdash A}$$

$$\dfrac{\dfrac{\pi_1}{T}}{\Gamma \vdash A} \Rightarrow^{t_2} \dfrac{\dfrac{\pi_1'}{T}}{\Gamma \vdash A} \Rightarrow^{t_1} \dfrac{\dfrac{\pi_1'}{T'}}{\Gamma \vdash A}$$

Those particular equalities turn into naturality conditions when translated into the bicategorical model.

- The parallel equalities. Those are of the following form :

  let T be a proof tree with two branches π_1 and π_2, t_1 a transformation turning π_1 into π_1' and t_2 a transformation turning π_2 into π_2'. We want to be able to say that the two sequences $t_1 \cdot t_2$ and $t_2 \cdot t_1$ are equalized in the model. These equations amount to bifunctoriality conditions. For example, with T being the cut rule, the equations become $([t_1] \circ id_{[\pi_2]}) \cdot (id_{[\pi_1]} \circ [t_2]) = (id_{[\pi_1]} \circ [t_2]) \cdot ([t_1] \circ id_{[\pi_2]})$ where $\cdot$ is the vertical composition of $2-$morphisms in the hom-categories.

- The coherent equalities. Those are the ones that actually produce new coherence conditions. They're obtained by looking at proof trees where we can apply two different proof transformations which conflict with each other by sharing part of the proof that is transformed. Let us look at such possible conflicting pairs involving the three cut elimination rules we have introduced so far :

  - In the case of the following proof tree:

    $$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{ax} \quad \dfrac{\pi_1}{A \vdash B}}{A \vdash B}\,\text{cut} \quad \dfrac{\pi_2}{B \vdash C}}{A \vdash C}\,\text{cut}$$

    one can apply both the left axiom cut transformation and the cut commutation. If we apply the cut commutation, we can then apply the left axiom cut again to obtain :

    $$\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\pi_2}{B \vdash C}}{A \vdash C}\,\text{cut}$$

    which is the result we also obtain by applying the left axiom cut in the first place.

  - In the case of the following proof tree:

    $$\dfrac{\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{}{B \vdash B}\,\text{ax}}{A \vdash B}\,\text{cut} \quad \dfrac{\pi_2}{B \vdash C}}{A \vdash C}\,\text{cut}$$

one can apply both the right axiom cut transformation and the cut commutation. If we apply the cut commutation, we can then apply the left axiom cut to obtain :

$$\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\pi_2}{B \vdash C}}{A \vdash C}\text{cut}$$

which is the result we also obtain by applying the right axiom cut in the first place.

- This case is the dual of the first one, with the right axiom cut transformation.

$$\dfrac{\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\pi_2}{B \vdash C}}{A \vdash C}\text{cut} \quad \dfrac{}{C \vdash C}\text{ax}}{A \vdash C}\text{cut}$$

- In the case of the following proof tree :

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\pi_2}{B \vdash C}}{A \vdash C}\text{cut} \quad \dfrac{\pi_3}{C \vdash D}}{A \vdash D}\text{cut} \quad \dfrac{\pi_4}{D \vdash E}}{A \vdash E}\text{cut}$$

We can apply the cut commutation in two different ways, either on the top two cut rules or on the bottom two cut rules. If we start with the top two, we obtain :

$$\dfrac{\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\dfrac{\pi_2}{B \vdash C} \quad \dfrac{\pi_3}{C \vdash D}}{B \vdash D}\text{cut}}{A \vdash D}\text{cut} \quad \dfrac{\pi_4}{D \vdash E}}{A \vdash E}\text{cut}$$

we can then apply it again on the bottom two to obtain :

$$\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\dfrac{\dfrac{\pi_2}{B \vdash C} \quad \dfrac{\pi_3}{C \vdash D}}{B \vdash D}\text{cut} \quad \dfrac{\pi_4}{D \vdash E}}{B \vdash E}\text{cut}}{A \vdash E}\text{cut}$$

And we can apply it once more to the top two to finally obtain :

$$\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\dfrac{\pi_2}{B \vdash C} \quad \dfrac{\dfrac{\pi_3}{C \vdash D} \quad \dfrac{\pi_4}{D \vdash E}}{C \vdash E}\text{cut}}{B \vdash E}\text{cut}}{A \vdash E}\text{cut}$$

On the other end, if we start by applying the cut commutation to the bottom two, we can then apply it once more to the only pair available to obtain the same result.

One can see that there are no other potential conflict pairs TODO: errr left and right axiom ???, due to the structure of the various transformations. Those will give us the four coherence conditions appearing in the proper definition of a bicategory that we now introduce :

Definition 3. *A bicategory $\mathcal{C}$ consists of :*

- *A collections of objects A, B, C.*
- *For each pair of objects A, B, a category $\mathcal{C}(A, B)$ whose objects are called morphisms or $1-$cells and whose morphisms are called 2-morphisms or $2-$cells.*
- *For each object A, a distinguished $1-$cell $id_A \in \mathcal{C}(A, A)$ called the identity morphism.*
- *For each triple of objects A, B, C a functor*

$$\circ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$$

  called horizontal composition.
- *For each pair of objects A, B, two natural isomorphisms called the left and right unitors:*

$$l : id_A \circ - \Rightarrow - : \mathcal{C}(A, B) \to \mathcal{C}(A, B) \text{ and } r : - \circ id_B \Rightarrow - : \mathcal{C}(A, B) \to \mathcal{C}(A, B)$$

- *For each quadruple of objects A, B, C, D a natural isomorphism called the associator*

$$a : (- \circ -) \circ - \Rightarrow - \circ (- \circ -) : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \times \mathcal{C}(C, D) \to \mathcal{C}(A, D)$$

such that the following diagrams commute for any object A, B, C, D, E of $\mathcal{C}$ and f, g, h, i objects of $\mathcal{C}(A, B)$, $\mathcal{C}(B, C)$, $\mathcal{C}(C, D)$, $\mathcal{C}(D, E)$ respectively :

$$\begin{tikzcd}
((f \circ g) \circ h) \circ i \arrow[rr, "a(f,g,h) \circ Id_i"'] \arrow[d, "a(f \circ g,h,i)"'] & & (f \circ (g \circ h)) \circ i \arrow[d, "a(f,g \circ h,i)"'] \\
(f \circ g) \circ (h \circ i) \arrow[dr, "a(f,g,h \circ i)"'] & & f \circ ((g \circ h) \circ i) \arrow[dl, "Id_f \circ a(g,h,i)"] \\
& f \circ (g \circ (h \circ i)) &
\end{tikzcd}$$

$$\begin{tikzcd}
(f \circ Id_B) \circ g \arrow[rr, "a(f,id_B,g)"'] \arrow[dr, "r(f) \circ Id(g)"'] & & f \circ (Id_B \circ g) \arrow[dl, "Id_f \circ l(g)"] \\
& f \circ g &
\end{tikzcd}$$

$$\begin{tikzcd}
(id_A \circ f) \circ g \arrow[rr, "a(id_A,f,g)"'] \arrow[dr, "l(f)\circ Id(g)"'] & & id_A \circ (f \circ g) \arrow[dl, "l(f\circ g)"] \\
& f \circ g &
\end{tikzcd}$$

$$\begin{tikzcd}
(f \circ g) \circ id_C \arrow[rr, "a(f,g,id_C)"'] \arrow[dr, "r(f\circ g)"'] & & f \circ (g \circ id_C) \arrow[dl, "Id_f\circ r(g)"] \\
& f \circ g &
\end{tikzcd}$$

We then introduce further bicategorical notions that will prove useful, such as the notion of pseudofunctor, a morphism between bicategories preserving the coherence conditions :

Definition 4. *Let $\mathcal{C}, \mathcal{D}$ be two bicategories. A pseudofunctor $F : \mathcal{C} \to \mathcal{D}$ is given by :*

- *For each object A of $\mathcal{C}$, an object $F(A)$ of $\mathcal{D}$.*
- *For each hom-category $\mathcal{C}(A, B)$ in $\mathcal{C}$, a functor*
$$F(A, B) : \mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$$
- *For each object A of $\mathcal{C}$, an invertible 2-cell*
$$F_{id_A} : id_{F(A)} \Rightarrow F(A, B)(id_A)$$
- *For each triple of objects A, B, C of $\mathcal{C}$, a natural isomorphism ϕ whose elements are:*
$$\phi_{f,g} : F(f) \circ F(g) \Rightarrow F(f \circ g)$$
for f, g objects of $\mathcal{C}(A, B), \mathcal{C}(B, C)$ respectively

such that, for any object A, B, C, D of $\mathcal{C}$, any objects f, g, h of $\mathcal{C}(A, B)$, $\mathcal{C}(B, C)$, $\mathcal{C}(C, D)$ respectively, the following diagrams commute :

$$\begin{tikzcd}
F(f) \circ (F(g) \circ F(h)) \arrow[d, "Id_{F(f)}\circ\phi_{g,h}"'] & & (F(f) \circ F(g)) \circ F(h) \arrow[ll, "a(F(f),F(g),F(h))"'] \arrow[d, "\phi_{f,g}\circ Id_{F(h)}"'] \\
F(f) \circ F(g \circ h) \arrow[d, "\phi_{f,g\circ h}"'] & & F(f \circ g) \circ F(h) \arrow[d, "\phi_{f\circ g,h}"'] \\
F(f \circ (g \circ h)) & & F((f \circ g) \circ h) \arrow[ll, "F(a(f,g,h))"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(f) & F(f \circ id_B) \arrow[l, "F(r(f))"'] \\
F(f) \circ id_{F(B)} \arrow[u, "r(F(f))"'] \arrow[r, "Id_{F(f)}\circ F_{id_B}"'] & F(f) \circ F(id_B) \arrow[u, "\phi_{f,id_B}"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(f) & F(id_A \circ f) \arrow[l, "F(l(f))"'] \\
id_{F(A)} \circ F(f) \arrow[u, "l(F(f))"'] \arrow[r, "F_{id_A}\circ Id_{F(f)}"'] & F(id_A) \circ F(f) \arrow[u, "\phi_{id_A,f}"']
\end{tikzcd}$$

Note that every space of proof transformations between proofs of a fixed statement must be modelized by a category, which is very natural, as this space is natively a category with composition being the application of proof transformations sequentially and the identity being the proof transformation that does nothing.

Next, like in categorical theory in chapter 2, we introduce morphisms between pseudofunctors in the form of pseudo-natural transformations. Starting from this definition and for the rest of the chapter, we'll represent the coherence conditions between 2− morphisms in a way that is more visual. For example, a 2−morphism $t : f \Rightarrow g$ where $f, g : A \to B$ are morphisms, will be represented :

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "t"] \end{tikzcd}$$

Combinations of 2− morphisms are then represented geometrically in a type of figure called pasting diagrams, and the coherence conditions are described as an equality of pasting diagrams. One thing of note is that those pasting diagrams hide the use of the associator, as composition is simply represented as a sequence of arrows, without any notion of priority.

Definition 5. *Let $\mathcal{C}, \mathcal{D}$ be two bicategories, and $F, G : \mathcal{C} \to \mathcal{D}$ two pseudofunctors. A pseudo-natural transformation $\gamma : F \Rightarrow G$ is given by :*

- *for every object A of $\mathcal{C}$, a 1−cell $\gamma_A : F(A) \to G(A)$*
- *for every pair of objects A, B of $\mathcal{C}$ and every 1−cell f of $\mathcal{C}(A, B)$, an invertible 2−cell γ_f:*

$$\begin{tikzcd} F(A) \arrow[r, "\gamma_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow_{\gamma_f}"] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\gamma_B"'] & G(B) \end{tikzcd}$$

such that the following properties are verified :

- *Naturality : For every 2−cell $\tau : f \Rightarrow g : A \to B$, the 2−cells associated to the following pasting diagrams are equal :*

$$\begin{tikzcd} F(A) \arrow[r, "\gamma_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow_{\gamma_f}"] & G(A) \arrow[d, bend right, "G(f)"'{name=L}] \arrow[d, bend left, "G(g)"{name=R}] \\ F(B) \arrow[r, "\gamma_B"'] & G(B) \arrow[Rightarrow, from=L, to=R, "G(\tau)"] \end{tikzcd} = \begin{tikzcd} F(A) \arrow[r, "\gamma_A"] \arrow[d, bend right, "F(f)"'{name=L}] \arrow[d, bend left, "F(g)"{name=R}] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow_{\gamma_g}"] & G(A) \arrow[d, "G(g)"] \\ F(B) \arrow[r, "\gamma_B"'] \arrow[Rightarrow, from=L, to=R, "F(\tau)"] & G(B) \end{tikzcd}$$

- *Unitality :For every object A of $\mathcal{C}$, the $2-$cells associated to the following pasting diagrams are equal :*

$$\begin{tikzcd}
F(A) \arrow[r, "\gamma_A"] \arrow[d, "Id_{F(A)}"'] \arrow[dr, phantom, "\equiv"] & G(A) \arrow[d, bend right=50, "Id_{G(A)}"'] \arrow[d, bend left=50, "G(Id_A)"] \arrow[d, phantom, "\overset{G_A}{\Rightarrow}"] \\
F(A) \arrow[r, "\gamma_A"'] & G(A)
\end{tikzcd}
=
\begin{tikzcd}
F(A) \arrow[rr, "\gamma_A"] \arrow[d, bend right=50, "Id_{F(A)}"'] \arrow[d, bend left=50, "F(Id_A)"] \arrow[d, phantom, "\overset{F_A}{\Rightarrow}"] \arrow[drr, phantom, "\Nearrow\gamma_{Id_A}"] & & G(A) \arrow[d, "G(Id_A)"] \\
F(A) \arrow[rr, "\gamma_A"'] & & G(A)
\end{tikzcd}$$

- *Compositionality : for every triple of objects A, B, C of $\mathcal{C}$ and every pair of $1-$cells f, g of $\mathcal{C}(A, B)$, $\mathcal{C}(B, C)$ respectively, the $2-$ cells associated to the following pasting diagrams are equal :*

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[rr, bend right, "F(g\circ f)"'] \arrow[d, "\gamma_A"] & F(B) \arrow[r, "F(g)"] \arrow[d, phantom, "\Downarrow F_{g,f}"] & F(C) \arrow[d, "\gamma_C"] \\
G(A) \arrow[rr, "G(g\circ f)"] & & G(C)
\end{tikzcd}
=
\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[dd, "\gamma_A"] \arrow[dr, phantom, "\gamma_f\Swarrow"] & F(B) \arrow[r, "F(g)"] \arrow[d, "\gamma_B"] \arrow[dr, phantom, "\gamma_g\Swarrow"] & F(C) \arrow[dd, "\gamma_C"] \\
 & G(B) \arrow[dr, "G(g)"] \arrow[d, phantom, "\Downarrow G_{g,f}"] & \\
G(A) \arrow[ur, "G(f)"] \arrow[rr, "G(g\circ f)"'] & & G(C)
\end{tikzcd}$$

Finally for this part, we go one step further by introducing a notion of morphism between pseudo-natural transformations in the form of modifications :

Definition 6. *Let $\gamma, \delta : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ be two pseudo-natural transformations., a modification $m : \gamma \Rrightarrow \delta$ is given by a $2-$cell $m_A : \gamma_A \Rightarrow \delta_A$ for every object A of $\mathcal{C}$ such that for every $f : A \to B$ in $\mathcal{C}$, we have :*

$$\begin{tikzcd}
F(A) \arrow[r, bend left, "\delta_A"] \arrow[r, bend right, "\gamma_A"'] \arrow[r, phantom, "m_A\Uparrow"] \arrow[d, "F(f)"] \arrow[dr, phantom, "\Nearrow\gamma_f"] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\gamma_B"] & G(B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \arrow[r, "\delta_A"] \arrow[d, "F(f)"] \arrow[dr, phantom, "\Nearrow\delta_f"] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, bend left, "\delta_B"] \arrow[r, bend right, "\gamma_B"'] \arrow[r, phantom, "m_B\Uparrow"] & G(B)
\end{tikzcd}$$

TODO: move to relevant section

Definition 7. *Let A, B be two objects in a bicategory $\mathcal{C}$. An equivalence from A to B is given by :*

- *a pair of $1-$cells $f : A \to B$ and $g : B \to A$.*

- *a pair of invertible $2-$cells $e : id_A \Rightarrow g \circ f$ and $e' : id_B \Rightarrow f \circ g$.*

We say that f is an equivalence if such g, e, e' exist.

$$\frac{}{A \vdash A}\text{ax} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\text{l}-\otimes \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\text{r}-\otimes \qquad \frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\text{cut}$$

$$\frac{}{\vdash I}\text{const} \qquad \frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\text{l}- \multimap \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\text{r}- \multimap \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\text{exch}$$

Figure 1.1: Rules for MILL

1.2 MILL and monoidal bicategories

We now move up to the first proper fragment of Linear Logic that we will study, multiplicative intuitionistic linear logic (MILL), whose rules we have seen in Chapter 2 and are recalled here in Figure 1.1. What arises very quickly by adapting our study from chapter 2 is the apparent need of $\otimes$ being a pseudo-bifunctor.

Indeed, the required functoriality of $\otimes$ in the hom-categories rises naturally from the categorical structure of proof transformations over proofs of a fixed statement.

Before looking at the proof transformations involving $\otimes$ and the required $2-$morphisms to complete the definition of bifunctor, let us get back at the first proof transformation we introduced in this chapter :

$$\frac{\frac{}{A \vdash A}\text{ax} \quad \frac{\pi_1}{A, \Delta \vdash B}}{A, \Delta \vdash B}\text{cut} \Rightarrow \frac{\pi_1}{A, \Delta \vdash B}$$

We claimed earlier that this proof transformation was modelized by the $2-$ morphism

$$id_A \circ [\pi_1] \Rightarrow [\pi_1]$$

which is a bit of a shortcut as this is only strictly true when Δ is empty. This serves to highlight one of the main difference between the cut rule of linear logic and the categorical notion of composition. For a composition to be allowed in a categorical or bicategorical setting, the domain of one morphism must be equal to the codomain of the other, while in logic, for a cut rule to be applicable, there only needs to be a common formula in the conclusion of one statement and the hypothesis of another. This basically means that cut is akin to some kind of partial composition. Hence, the proper $2-$morphism associated to this proof transformation would intuitively be :

$$(id_A \otimes id_\Delta \circ [\pi_1] \Rightarrow [\pi_1]$$

Note that in this case, this shortcut is of no consequence as all the context data can be handled through use of $\otimes$ and the associated rules. However, it was important to highlight as the proof transformation rules for $\otimes$ make heavy use of the partial composition of cut in ways that don't allow as easy a shortcut. Let us look at those rules :

$$\eta\otimes \qquad \dfrac{}{A\otimes B\vdash A\otimes B}\text{ax} \quad\Rightarrow\quad \dfrac{\dfrac{\dfrac{}{A\vdash A}\text{ax}\quad \dfrac{}{B\vdash B}\text{ax}}{A,B\vdash A\otimes B}r-\otimes}{A\otimes B\vdash A\otimes B}l-\otimes$$

$$\text{r-}\otimes\ \text{l-}\otimes\ cut \qquad \dfrac{\dfrac{\dfrac{\pi_1}{A\vdash B}\quad\dfrac{\pi_2}{C\vdash D}}{A,C\vdash B\otimes D}r-\otimes \quad \dfrac{\dfrac{\pi_3}{B,D\vdash E}}{B\otimes D\vdash E}l-\otimes}{A,C\vdash E}\text{cut} \quad\Rightarrow\quad \dfrac{\dfrac{\pi_1}{A\vdash B}\quad \dfrac{\dfrac{\pi_2}{C\vdash D}\quad\dfrac{\pi_3}{B,D\vdash E}}{B,C\vdash E}\text{cut}}{A,C\vdash E}\text{cut}$$

$$\text{r-}\otimes\ rcut_1 \qquad \dfrac{\dfrac{\pi_1}{A\vdash B}\quad \dfrac{\dfrac{\pi_2}{B\vdash D}\quad\dfrac{\pi_3}{C\vdash E}}{B,C\vdash D\otimes E}r-\otimes}{A,C\vdash D\otimes E}\text{cut} \quad\Rightarrow\quad \dfrac{\dfrac{\dfrac{\pi_1}{A\vdash B}\quad\dfrac{\pi_2}{B\vdash D}}{A\vdash D}\text{cut} \quad \dfrac{\pi_3}{C\vdash E}}{A,C\vdash D\otimes E}r-\otimes$$

$$\text{r-}\otimes\ rcut_2 \qquad \dfrac{\dfrac{\pi_1}{A\vdash B}\quad\dfrac{\dfrac{\pi_2}{C\vdash D}\quad\dfrac{\pi_3}{B\vdash E}}{C,B\vdash D\otimes E}r-\otimes}{C,A\vdash D\otimes E}\text{cut} \quad\Rightarrow\quad \dfrac{\dfrac{\pi_2}{C\vdash D}\quad\dfrac{\dfrac{\pi_1}{A\vdash B}\quad\dfrac{\pi_3}{B\vdash E}}{A\vdash E}\text{cut}}{C,A\vdash D\otimes E}r-\otimes$$

$$\text{commut}\ cut_2 \qquad \dfrac{\dfrac{\pi_1}{A\vdash B}\quad\dfrac{\dfrac{\pi_2}{C\vdash D}\quad\dfrac{\pi_3}{B,D\vdash E}}{A,D\vdash E}\text{cut}}{A,C\vdash E}\text{cut} \quad\Rightarrow\quad \dfrac{\dfrac{\pi_2}{C\vdash D}\quad\dfrac{\dfrac{\pi_1}{A\vdash B}\quad\dfrac{\pi_3}{B,D\vdash E}}{B,C\vdash E}\text{cut}}{A,C\vdash E}\text{cut}$$

$$\text{l-}\otimes\ lcut \qquad \dfrac{\dfrac{\dfrac{\pi_1}{A,B\vdash C}}{A\otimes B\vdash C}l-\otimes\quad\dfrac{\pi_2}{C\vdash D}}{A\otimes B\vdash D}\text{cut} \quad\Rightarrow\quad \dfrac{\dfrac{\dfrac{\pi_1}{A,B\vdash C}\quad\dfrac{\pi_2}{C\vdash D}}{A,B\vdash D}\text{cut}}{A\otimes B\vdash D}l-\otimes$$

$$\text{l-}\otimes\ rcut \qquad \dfrac{\dfrac{\pi_1}{A\vdash B}\quad\dfrac{\dfrac{\pi_2}{B,C,D\vdash E}}{B,C\otimes D\vdash E}\text{l-}\otimes}{A,C\otimes D\vdash E}\text{cut} \quad\Rightarrow\quad \dfrac{\dfrac{\dfrac{\pi_1}{A\vdash B}\quad\dfrac{\pi_2}{B,C,D\vdash E}}{A,C,D\vdash E}\text{cut}}{A,C\otimes D\vdash E}\text{l-}\otimes$$

which turn into the following $2-$ morphisms:

$$id_{A\otimes B}\Rightarrow id_A\otimes id_B$$

$$([\pi_1]\otimes[\pi_2])\circ[\pi_3]\Rightarrow([\pi_1]\otimes id_C)\circ((id_B\otimes[\pi_2])\circ[\pi_3])$$

$$([\pi_1]\otimes id_C)\circ([\pi_2]\otimes[\pi_3])\Rightarrow([\pi_1]\circ[\pi_2])\otimes[\pi_3]$$

$$(id_C\otimes[\pi_1])\circ([\pi_2]\otimes[\pi_3])\Rightarrow[\pi_2]\otimes([\pi_1]\circ[\pi_3])$$

$$([\pi_1]\otimes id_C)\circ((id_B\otimes[\pi_2])\circ\pi_3)\Rightarrow(id_A\otimes[\pi_2])\circ(([\pi_1]\otimes id_D)\circ[\pi_3])$$

$$id_{[\pi_1]\circ[\pi_2]}$$

$$id_{([\pi_1]\otimes(id_C\otimes id_D))\circ[\pi_2]}$$

That first $2-$morphism is exactly one of those required by the definition of pseudofunctor. As for the next three, they can all be seen as specific cases of another, more general $2-$morphism, though with some additional $2-$morphisms composed to handle the added identities. The more general morphism is as such :

$$(f \otimes g) \circ (h \otimes i) \Rightarrow (f \circ h) \otimes (g \circ i)$$

with $f : A \to B, g : C \to D, h : B \to E, i : D \to F$.

This $2-$morphism is exactly the second morphism required by the definition of a pseudo-functor ($F(f) \circ F(g) \Rightarrow F(f \circ g)$). Requiring the existence of this $2-$morphism would thus make $\otimes$ a pseudo-functor in our model, and be enough to modelize the related proof transformations.
Moreover, this requirement is not a tremendous restriction on the logic, as the proof transformation associated to that $2-$morphism, given by :

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\pi_2}{C \vdash D}}{A, C \vdash B \otimes D}\, r-\otimes}{A \otimes C \vdash B \otimes D}\, l-\otimes \quad \dfrac{\dfrac{\dfrac{\pi_3}{B \vdash E} \quad \dfrac{\pi_4}{D \vdash F}}{B, D \vdash E \otimes F}\, r-\otimes}{B \otimes D \vdash E \otimes F}\, l-\otimes}{A \otimes C \vdash E \otimes F}\, \text{cut} \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\pi_3}{B \vdash E}}{A \vdash E}\, \text{cut} \quad \dfrac{\dfrac{\pi_2}{C \vdash D} \quad \dfrac{\pi_4}{D \vdash F}}{C \vdash F}\, \text{cut}}{A, C \vdash E \otimes F}\, r-\otimes}{A \otimes C \vdash E \otimes F}\, l-\otimes$$

can be generated from the proof transformations we have outlined. Indeed, starting from the initial proof, we can use in sequence :

$$\text{l-}\otimes\ cut \cdot \text{r-}\otimes\ \text{l-}\otimes\ cut \cdot \text{r-}\otimes\ cut_2 \cdot \text{r-}\otimes\ cut_1$$

to obtain the second proof.

Next, the commut cut_2 rule can be interpreted as a bifunctorial property, which means we only have two proof transformation rules left to study, l-$\otimes$ *lcut* and l-$\otimes$ *rcut*. Those two outline a glitch in the usual definition of categorical model of LL. Indeed, the choice of interpreting contexts $A_1, ...A_n$ as a tensor product of objects $[A_1] \otimes ... \otimes [A_n]$ creates an impossibility to dinstinguish an interpretation of $A, B \vdash C$ and an interpretation of $A \otimes B \vdash C$. This results in particular in difficulties in interpreting the $l-\otimes$ proof rule and all proof transformations that it induces. The role of the $l-\otimes$ proof rule can be seen as giving an order of priority to the applications of the tensor product, whether effective or implied by context. Indeed, a proof of $A, B, C \vdash D$ can be turned either into a proof of $A \otimes (B \otimes C) \vdash D$ or $(A \otimes B) \otimes C \vdash D$ through simple applications of the $l-\otimes$ rule. Those two proofs would not a priori be interpreted in the same way as we have no prior associativity hypothesis on $\otimes$. we will thus require such associativity in our model, which, in bicategorical terms, take the form of a pseudo-natural transformation

$$a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

equipped with an invertible modification

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "a_{A\otimes B,C,D}"] \arrow[d, "a_{A,B,C}\otimes id_D"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a_{A,B,C\otimes D}"] \arrow[d, phantom, "\Downarrow\pi_{A,B,C,D}"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "a_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "id_A\otimes a_{B,C,D}"']
\end{tikzcd}$$

Similarly to the categorical case, the constant I must be interpreted as a unit of the bifunctor, which, in a bicategorical setting, induces a few $2-$morphisms related to the equalities in the categorical setting.

From this, using the same idea of conflict between two potential proof transformation that we used earlier this chapter, we can infer the equalities of $2-$morphisms required by the pseudo-functor definition, and a few additional ones between our new $2-$morphisms, which are the ones included in the definition of a monoidal bicategory, which follows :

Definition 8. *A monoidal bicategory $\mathcal{C}$ is a bicategory equipped with :*

- *a unit object I.*
- *a pseudo-functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$*
- *three pseudo-natural transformations a, l, r whose components are equivalences and given by :*

$$a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$
$$l_A : I \otimes A \to A$$
$$r_A : A \otimes I \to A$$

- *four invertible modifications π, μ, L, R whose components are given by :*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "a_{A\otimes B,C,D}"] \arrow[d, "a_{A,B,C}\otimes id_D"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a_{A,B,C\otimes D}"] \arrow[d, phantom, "\Downarrow\pi_{A,B,C,D}"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "a_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "id_A\otimes a_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes C \arrow[rr, "a_{A,I,C}"] \arrow[dr, "r_A\otimes id_C"'] & \arrow[d, phantom, "\xRightarrow{\mu_{A,C}}"] & A \otimes (I \otimes C) \arrow[dl, "id_A\otimes l_C"] \\
& A \otimes C &
\end{tikzcd}$$

$$\begin{tikzcd}
(I \otimes B) \otimes C \arrow[rr, "a_{I,B,C}"] \arrow[dr, "l_B\otimes id_C"'] & \arrow[d, phantom, "\xRightarrow{L_{B,C}}"] & I \otimes (B \otimes C) \arrow[dl, "l_{B\otimes C}"] \\
& B \otimes C &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B) \otimes I \arrow[rr, "a_{A,B,I}"] \arrow[dr, "r_{A\otimes B}"'] & \arrow[d, phantom, "\xRightarrow{R_{A,B}}"] & A \otimes (B \otimes I) \arrow[dl, "id_A\otimes r_B"] \\
& A \otimes B &
\end{tikzcd}$$

such that the following conditions are verified :

- *Associativity : For all A, B, C, D, E objects of $\mathcal{C}$, the two following pasting diagrams must be equal:*

$$
\begin{tikzcd}[column sep=small]
 & (((A\otimes B)\otimes C)\otimes D)\otimes E \arrow[rr, "a_{(A\otimes B)\otimes C,D,E}"] \arrow[dl, "(a_{A,B,C}\otimes id_D)\otimes id_E"'] \arrow[dr, "a_{A\otimes B,C,D}\otimes id_E"] & & ((A\otimes B)\otimes C)\otimes(D\otimes E) \arrow[rr, "a_{A\otimes B,C,D\otimes E}"] & & (A\otimes B)\otimes(C\otimes(D\otimes E)) \arrow[dr, "a_{A,B,C\otimes(D\otimes E)}"] & \\
B\otimes C))\otimes D)\otimes E \arrow[dd, "a_{A,B\otimes C,D}\otimes id_E"] & \arrow[d, phantom, "\Leftarrow \pi_{A,B,C,D}\otimes Id_{id_E}"] & & \arrow[d, phantom, "\Downarrow \pi_{A\otimes B,C,D,E}"] & & \arrow[d, phantom, "\Rightarrow \otimes_{id_A,id_B}\otimes Id_{a_{C,D,E}}"] & \\
 & & ((A\otimes B)\otimes(C\otimes D))\otimes E \arrow[rr, "a_{A\otimes B,C\otimes D,E}"] \arrow[dl, "a_{A,B,C\otimes D}\otimes id_E"] & & (A\otimes B)\otimes((C\otimes D)\otimes E) \arrow[ur, "id_{A\otimes B}\otimes a_{C,D,E}"] \arrow[ur, bend right, "(id_A\otimes id_B)\otimes a_{C,D,E}"'] \arrow[dr, "a_{A,B,(C\otimes D)\otimes E}"'] & & A\otimes(B\otimes(C\otimes(D\otimes E))) \\
B\otimes C)\otimes D))\otimes E \arrow[dr, "(id_A\otimes a_{B,C,D})\otimes id_E"] & & & \arrow[r, phantom, "\swarrow \pi_{A,B,C\otimes D,E}"] & & \arrow[u, phantom, "\Downarrow a_{id_A,id_B,a_{C,D,E}}"] & \\
 & (A\otimes(B\otimes(C\otimes D)))\otimes E \arrow[rr, "a_{A,B\otimes(C\otimes D),E}"] & & A\otimes((B\otimes(C\otimes D))\otimes E) \arrow[rr, "id_A\otimes a_{B,C\otimes D,E}"] & & A\otimes(B\otimes((C\otimes D)\otimes E)) \arrow[uur, "id_A\otimes(id_B\otimes a_{C,D,E})"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & (((A\otimes B)\otimes C)\otimes D)\otimes E \arrow[rr, "a_{(A\otimes B)\otimes C,D,E}"] \arrow[dl, "(a_{A,B,C}\otimes id_D)\otimes id_E"'] \arrow[d, phantom, "\Leftarrow a^{-1}_{a_{A,B,C},id_D,id_E}"] & & ((A\otimes B)\otimes C)\otimes(D\otimes E) \arrow[rr, "a_{A\otimes B,C,D\otimes E}"] \arrow[dl, bend right, "a_{A,B,C}\otimes(id_D\otimes id_E)"'] \arrow[dl, bend left, "a_{A,B,C}\otimes id_{D\otimes E}"] \arrow[dl, phantom, "\Leftarrow Id_{a_{A,B,C}}\otimes\otimes_{id_D,id_E}"] & & (A\otimes B)\otimes(C\otimes(D\otimes E)) \arrow[dr, "a_{A,B,C\otimes(D\otimes E)}"] & \\
B\otimes C))\otimes D)\otimes E \arrow[rr, "a_{A\otimes(B\otimes C),D,E}"'] \arrow[dd, "a_{A,B\otimes C,D}\otimes id_E"] & & (A\otimes(B\otimes C))\otimes(D\otimes E) \arrow[dr, "a_{A,B\otimes C,D\otimes E}"] & & & \arrow[l, phantom, "\swarrow \pi_{A,B,C,D\otimes E}"] & \\
 & \arrow[r, phantom, "\swarrow \pi_{A,B\otimes C,D,E}"] & & & A\otimes((B\otimes C)\otimes(D\otimes E)) \arrow[rr, "id_A\otimes a_{B,C,D\otimes E}"] & & A\otimes(B\otimes(C\otimes(D\otimes E))) \\
B\otimes C)\otimes D))\otimes E \arrow[rr, "a_{A,(B\otimes C)\otimes D,E}"] \arrow[dr, "(id_A\otimes a_{B,C,D})\otimes id_E"'] & \arrow[d, phantom, "\Leftarrow a^{-1}_{id_A,a_{B,C,D},id_E}"] & A\otimes(((B\otimes C)\otimes D)\otimes E) \arrow[urr, "id_A\otimes a_{B\otimes C,D,E}"] \arrow[dr, "id_A\otimes(a_{B,C,D}\otimes id_E)"] & & \arrow[u, phantom, "\Downarrow Id_{id_A}\otimes\pi_{B,C,D,E}"] & & \\
 & (A\otimes(B\otimes(C\otimes D)))\otimes E \arrow[rr, "a_{A,B\otimes(C\otimes D),E}"] & & A\otimes((B\otimes(C\otimes D))\otimes E) \arrow[rr, "id_A\otimes a_{B,C\otimes D,E}"] & & A\otimes(B\otimes((C\otimes D)\otimes E)) \arrow[uur, "id_A\otimes(id_B\otimes a_{C,D,E})"'] &
\end{tikzcd}
$$

- *For all A, B, C objects of $\mathcal{C}$, the pasting diagram*

$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[rr, "a_{A,B,C}"] \arrow[dr, phantom, "\Rightarrow a_{r_A,id_B,id_C}"] & & A\otimes(B\otimes C) & \\
((A\otimes I)\otimes B)\otimes C \arrow[ur, "(r_A\otimes id_B)\otimes id_C"] \arrow[rr, "a_{A\otimes I,B,C}"'] \arrow[dr, "a_{A,I,B}\otimes id_C"'] & & (A\otimes I)\otimes(B\otimes C) \arrow[ur, bend left, "r_A\otimes id_B\otimes id_C"] \arrow[ur, bend right, "r_A\otimes id_{B\otimes C}"'] \arrow[ur, phantom, "\Rightarrow Id_{r_A}\otimes\otimes^{-1}_{id_B,id_C}"] \arrow[rr, "a_{A,I,B\otimes C}"'] \arrow[d, phantom, "\Downarrow \pi_{A,I,B,C}"] & & A\otimes(I\otimes(B\otimes C)) \arrow[ul, "id_A\otimes l_{B\otimes C}"'] \arrow[ul, phantom, "\Rightarrow \mu_{A,B\otimes C}"] \\
 & (A\otimes(I\otimes B))\otimes C \arrow[rr, "a_{A,I\otimes B,C}"'] & \, & A\otimes((I\otimes B)\otimes C) \arrow[ur, "id_A\otimes a_{I,B,C}"'] &
\end{tikzcd}$$

must be equal to the pasting diagram

$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[r, "a_{A,B,C}"] \arrow[dr, phantom, "\Rightarrow a_{id_A,l_B,id_C}"] & A\otimes(B\otimes C) & \\
((A\otimes I)\otimes B)\otimes C \arrow[ur, "(r_A\otimes id_B)\otimes id_C"] \arrow[dr, "a_{A,I,B}\otimes id_C"'] \arrow[r, phantom, "\Rightarrow \mu_{A,B}\otimes Id_{id_C}"] & \, & \, & A\otimes(I\otimes(B\otimes C)) \arrow[ul, "id_A\otimes l_{B\otimes C}"'] \arrow[l, phantom, "Id_{id_A}\otimes L_{B,C} \Rightarrow"'] \\
 & (A\otimes(I\otimes B))\otimes C \arrow[r, "a_{A,I\otimes B,C}"'] \arrow[uu, "(id_A\otimes l_B)\otimes id_C"] & A\otimes((I\otimes B)\otimes C) \arrow[ur, "id_A\otimes a_{I,B,C}"'] \arrow[uu, "id_A\otimes(l_B\otimes id_C)"'] &
\end{tikzcd}$$

- *For all A, B, C objects of $\mathcal{C}$, the pasting diagram*

$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[rr, "a_{A,B,C}"] & & A\otimes(B\otimes C) & \\
((A\otimes B)\otimes I)\otimes C \arrow[ur, "r_{A\otimes B}\otimes id_C"] \arrow[ur, phantom, "\Rightarrow \mu_{A\otimes B,C}"'] \arrow[rr, "a_{A\otimes B,I,C}"'] \arrow[dr, "a_{A,B,I}\otimes id_C"'] & & (A\otimes B)\otimes(I\otimes C) \arrow[ul, bend right, "(id_A\otimes id_B)\otimes l_C"'] \arrow[ul, bend left, "id_{A\otimes B}\otimes l_C"] \arrow[ul, phantom, "\otimes_{id_A,id_B}\otimes Id_{l_c} \Rightarrow"] \arrow[rr, "a_{A,B,I\otimes C}"'] \arrow[ur, phantom, "\Rightarrow a_{id_A,id_B,l_C}"] \arrow[d, phantom, "\Downarrow \pi_{A,B,I,C}"] & & A\otimes(B\otimes(I\otimes C)) \arrow[ul, "id_A\otimes(id_B\otimes l_C)"'] \\
 & (A\otimes(B\otimes I))\otimes C \arrow[rr, "a_{A,B\otimes I,C}"'] & \, & A\otimes((B\otimes I)\otimes C) \arrow[ur, "id_A\otimes a_{B,I,C}"'] &
\end{tikzcd}$$

must be equal to the pasting diagram

$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[r, "a_{A,B,C}"] \arrow[dr, phantom, "\Rightarrow a_{id_A,r_B,id_C}"] & A\otimes(B\otimes C) & \\
((A\otimes B)\otimes I)\otimes C \arrow[ur, "r_{A\otimes B}\otimes id_C"] \arrow[dr, "a_{A,B,I}\otimes id_C"'] \arrow[r, phantom, "\Rightarrow R_{A,B}\otimes Id_{id_C}"] & \, & \, & A\otimes(B\otimes(I\otimes C)) \arrow[ul, "id_A\otimes(id_B\otimes l_C)"'] \arrow[l, phantom, "Id_{id_A}\otimes \mu_{B,C} \Rightarrow"'] \\
 & (A\otimes(B\otimes I))\otimes C \arrow[r, "a_{A,B\otimes I,C}"'] \arrow[uu, "(id_A\otimes r_B)\otimes id_C"] & A\otimes((B\otimes I)\otimes C) \arrow[ur, "id_A\otimes a_{B,I,C}"'] \arrow[uu, "id_A\otimes(r_B\otimes id_C)"'] &
\end{tikzcd}$$

Thus, in order to interpret $MILL$, a bicategory must be at least monoidal, to interpret properly the linear conjunction $\otimes$. In what follows, we will be interested in looking at pseudofunctors that preserve the monoidal structure, whose definition follows, alongside the higher order steps like natural transformations and modifications that preserve the natural structure :

Definition 9. *Let $\mathcal{C}, \mathcal{D}$ be two monoidal bicategories. A monoidal pseudofunctor $F : \mathcal{C} \to \mathcal{D}$ is a pseudofunctor equipped with:*

- *a $1-$cell $F^{\otimes}_I : I \to F(I)$*

- *a pseudo-natural transformation* $F^{\otimes}$ *whose components are of the form :*

$$F^{\otimes}_{A,B} : F(A) \otimes F(B) \to F(A \otimes B)$$

- *three invertible modifications* F^a, F^l, F^r *whose components are of the form :*

$$\begin{tikzcd}
(F(A) \otimes F(B)) \otimes F(C) \arrow[rr, "a_{F(A),F(B),F(C)}"] \arrow[d, "F^{\otimes}_{A,B} \otimes id_{F(A)}"] & & F(A) \otimes (F(B) \otimes F(C)) \arrow[d, "id_{F(A)} \otimes F^{\otimes}_{B,C}"] \\
F(A \otimes B) \otimes F(C) \arrow[d, "F^{\otimes}_{A \otimes B,C}"] & \arrow[phantom, "\Nearrow F^a_{A,B,C}"] & F(A) \otimes F(B \otimes C) \arrow[d, "F^{\otimes}_{A,B \otimes C}"] \\
F((A \otimes B) \otimes C) \arrow[rr, "F(a_{A,B,C})"] & & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes F(A) \arrow[r, "F^{\otimes}_I \otimes id_{F(A)}"] \arrow[d, "l_{F(A)}"'] \arrow[dr, phantom, "\Rightarrow F^l_A"] & F(I) \otimes F(A) \arrow[d, "F^{\otimes}_{I,A}"] \\
F(A) & F(I \otimes A) \arrow[l, "F(l_A)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \otimes I \arrow[r, "id_{F(A)} \otimes F^{\otimes}_i"] \arrow[d, "r_{F(A)}"'] \arrow[dr, phantom, "\Rightarrow F^r_A"] & F(A) \otimes F(I) \arrow[d, "F^{\otimes}_{A,I}"] \\
F(A) & F(A \otimes I) \arrow[l, "F(r_A)"]
\end{tikzcd}$$

such that the following properties are verified :

- TODO: *THE DEMONIC DIAGRAM For all* A, B, C, D *objects of* $\mathcal{C}$*, the following pasting diagram*

  must be equal to the pasting diagram

- *For all* A, B *objects of* $\mathcal{C}$*, the following pasting diagram*

$$\begin{tikzcd}
 & (F(A) \otimes I) \otimes F(B) \arrow[r, "a_{F(A),I,F(B)}"] \arrow[d, "(id_{F(A)} \otimes F^{\otimes}_I) \otimes id_{F(B)}"'] \arrow[dr, phantom, "\Swarrow a_{id_{F(A)},F^{\otimes}_I,id_{F(B)}}"] & F(A) \otimes (I \otimes F(B)) \arrow[d, "id_{F(A)} \otimes (F^{\otimes}_I \otimes id_{F(B)})"] \arrow[ddd, bend left, "id_{F(A)} \otimes l_{F(B)}"] \\
F(A \otimes I) \otimes F(B) \arrow[d, "F^{\otimes}_{A \otimes I,B}"'] & (F(A) \otimes F(I)) \otimes F(B) \arrow[l, "F^{\otimes}_{A,I} \otimes id_{F(B)}"'] \arrow[r, "a_{F(A),F(I),F(B)}"'] \arrow[d, phantom, "\Swarrow F^{a\ -1}_{A,I,B}"] & F(A) \otimes (F(I) \otimes F(B)) \arrow[d, "id_{F(A)} \otimes F^{\otimes}_{I,B}"'] \arrow[d, phantom, shift left=8ex, "\Leftarrow Id_{id_A} \otimes F^l_B"] \\
F((A \otimes I) \otimes B) \arrow[r, "F(a_{A,I,B})"] \arrow[dr, "F(r_A \otimes id_B)"'] & F(A \otimes (I \otimes B)) \arrow[d, "F(id_A \otimes l_B)"] \arrow[l, phantom, shift right=2ex, "\Leftarrow F(\mu_{A,I,B})^{-1}"] & F(A) \otimes F(I \otimes B) \arrow[l, "F^{\otimes}_{A,I \otimes B}"'] \arrow[d, "id_{F(A)} \otimes F(l_B)"] \arrow[dl, phantom, "\Leftarrow F^{\otimes}_{id_A,l_B}"] \\
 & F(A \otimes B) & F(A) \otimes F(B) \arrow[l, "F^{\otimes}_{A,B}"]
\end{tikzcd}$$

must be equal to the pasting diagram

$$\begin{tikzcd}
(F(A) \otimes F(I)) \otimes F(B) \arrow[d, "F^{\otimes}_{A,I} \otimes id_{F(B)}"'] & F(A \otimes I) \otimes F(B) \arrow[l, "(id_{F(A)} \otimes F^{\otimes}_I) \otimes id_{F(B)}"'] \arrow[rr, "a_{F(A),I,F(B)}"] \arrow[dr, "r_{F(A)} \otimes id_{F(B)}"] \arrow[d, phantom, "\Leftarrow F^r_A \otimes Id_{id_{F(B)}}"] & & F(A) \otimes (I \otimes F(B)) \arrow[dl, "id_{F(A)} \otimes l_{F(B)}"] \arrow[dll, phantom, shift right=4ex, "\Leftarrow \mu^{-1}_{F(A),F(B)}"] \\
F(A \otimes I) \otimes F(B) \arrow[rr, "F(r_A) \otimes id_{F(B)}"'] \arrow[d, "F^{\otimes}_{A \otimes I,B}"'] & \arrow[d, phantom, "\Swarrow F^{\otimes}_{r_A,id_B}"] & F(A) \otimes F(B) \arrow[d, "F^{\otimes}_{A,B}"] & \\
F((A \otimes I) \otimes B) \arrow[rr, "F(r_A \otimes id_B)"'] & & F(A \otimes B) &
\end{tikzcd}$$

Definition 10. *Let $\mathcal{C}, \mathcal{D}$ be two monoidal bicategories and $F, G : \mathcal{C} \to \mathcal{D}$ two monoidal pseudofunctors. A monoidal pseudonatural transformation $\gamma : F \Rightarrow G$ is a pseudonatural transformation equipped with:*

- *an invertible2−cell*
$$\gamma_I^{\otimes} : F_I^{\otimes} \circ \gamma_I \Rightarrow G_I^{\otimes}$$
- *an invertible modification whose components are of the form :*

$$\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, "F^{\otimes}_{A,B}"] \arrow[d, "\gamma_A \otimes \gamma_B"'] & F(A \otimes B) \arrow[d, "\gamma_{A \otimes B}"] \arrow[dl, phantom, "\Swarrow \gamma^{\otimes}_{A,B}"] \\
G(A) \otimes G(B) \arrow[r, "G^{\otimes}_{A,B}"'] & G(A \otimes B)
\end{tikzcd}$$

such that the following properties are verified :

- *For all A object of $\mathcal{C}$, the following pasting diagram*

$$\begin{tikzcd}
 & F(A) \otimes I \arrow[r, "id_{F(A)} \otimes F^{\otimes}_I"] \arrow[dl] \arrow[d, "r_{F(A)}"] & F(A) \otimes F(I) \arrow[d, "F^{\otimes}_{A,I}"] \arrow[dr, "\gamma_A \otimes \gamma_I"] & \\
G(A) \otimes I \arrow[r, phantom, "\overset{r_{\gamma_A}}{\Longrightarrow}"] \arrow[dr, "r_{G(A)}"'] & F(A) \arrow[d, "\gamma_A"] \arrow[r, phantom, "\Longrightarrow"] & F(A \otimes I) \arrow[l, "F(r_A)"] \arrow[d, "\gamma_{A \otimes I}"] \arrow[r, phantom, "\overset{\gamma^{\otimes}_{A,I}}{\Longrightarrow}"] & G(A) \otimes G(I) \arrow[dl, "G^{\otimes}_{A,I}"] \\
 & G(A) \arrow[r, phantom, "\overset{\gamma_{r_A}}{\Nearrow}"] & G(A \otimes I) \arrow[l, "G(r_A)"] &
\end{tikzcd}$$

is equal to the pasting diagram :

$$\begin{tikzcd}
F(A) \otimes I \arrow[r, "id_{F(A)} \otimes F^{\otimes}_I"] \arrow[d, "\gamma_A \otimes id_I"] & F(A) \otimes F(I) \arrow[d, "\gamma_A \otimes id_{F(I)}"] \arrow[dr, "\gamma_A \otimes \gamma_I"] & \\
G(A) \otimes I \arrow[r, "id_{G(A)} \otimes F^{\otimes}_I"] \arrow[rr, bend right, "id_{G(A)} \otimes G^{\otimes}_I"'] \arrow[d, "r_{G(A)}"] & G(A) \otimes F(I) \arrow[r, "id_{G(A)} \otimes \gamma_I"] \arrow[d, phantom, "\Downarrow Id_{id_{G(A)} \otimes \gamma^{\otimes}_I}"] & G(A) \otimes G(I) \arrow[d, "G^{\otimes}_{A,I}"] \\
G(A) & \arrow[r, phantom, "\overset{G^r_A}{\Longrightarrow}"] & G(A \otimes I) \arrow[ll, "G(r_A)"]
\end{tikzcd}$$

- *For all A, B, C objects of $\mathcal{C}$, the following pasting diagram*

$$\begin{tikzcd}
(F(A) \otimes F(B)) \otimes F(C) \arrow[r, "F^{\otimes}_{A,B} \otimes id_{F(C)}"] \arrow[d, "a_{F(A),F(B),F(C)}"] & F(A \otimes B) \otimes F(C) \arrow[r, "F^{\otimes}_{A \otimes B,C}"] \arrow[d, phantom, "\Swarrow F^a_{A,B,C}"] & F((A \otimes B) \otimes C) \arrow[d, "F(a_{A,B,C})"] \\
F(A) \otimes (F(B) \otimes F(C)) \arrow[r, "id_{F(A)} \otimes F^{\otimes}_{B,C}"] \arrow[d, "\gamma_A \otimes id_{F(B) \otimes F(C)}"] \arrow[dd, bend right=60, "\gamma_A \otimes (\gamma_B \otimes \gamma_C)"'] & F(A) \otimes F(B \otimes C) \arrow[r, "F^{\otimes}_{A,B \otimes C}"] \arrow[d, "\gamma_A \otimes id_{F(B \otimes C)}"] \arrow[dd, bend left=60, "\gamma_A \otimes (\gamma_{B \otimes C})"] & F(A \otimes (B \otimes C)) \arrow[dd, "\gamma_{A \otimes (B \otimes C)}"] \arrow[ddl, phantom, "\Swarrow \gamma^{\otimes}_{A,B \otimes C}"] \\
G(A) \otimes (F(B) \otimes F(C)) \arrow[r, "id_{G(A)} \otimes F^{\otimes}_{B,C}"] \arrow[d, "id_{G(A)} \otimes (\gamma_B \otimes \gamma_C)"] & G(A) \otimes F(B \otimes C) \arrow[d, "id_{G(A)} \otimes \gamma_{B \otimes C}"] \arrow[dl, phantom, "\Swarrow \gamma^{\otimes}_{A \otimes B} \otimes Id_{id_{G(C)}}"] & \\
G(A) \otimes (G(B) \otimes G(C)) \arrow[r, "G^{\otimes}_{A,B \otimes C}"'] & G(A) \otimes G(B \otimes C) \arrow[r, "G^{\otimes}_{A \otimes B,C}"'] & G(A \otimes (B \otimes C))
\end{tikzcd}$$

is equal to the pasting diagram :

$$\begin{tikzcd}
 & (F(A)\otimes F(B))\otimes F(C) \arrow[r, "F^{\otimes}_{A,B}\otimes id_{F(C)}"] \arrow[d, "(\gamma_A\otimes\gamma_B)\otimes id_{F(C)}"'] \arrow[dd, bend right=60, "(\gamma_A\otimes\gamma_B)\otimes\gamma_C"'] \arrow[dddl, bend right, "a_{F(A),F(B),F(C)}"'] \arrow[dr, phantom, "\Swarrow Id_{id_{F(A)}}\otimes\gamma^{\otimes}_{B,C}"] & F(A\otimes B)\otimes F(C) \arrow[r, "F^{\otimes}_{A\otimes B,C}"] \arrow[d, "\gamma_{A\otimes B}\otimes id_{F(C)}"] \arrow[dd, bend left=60, "\gamma_{A\otimes B}\otimes\gamma_C"] \arrow[ddr, phantom, "\Swarrow \gamma^{\otimes}_{A\otimes B,C}"] & F((A\otimes B)\otimes C) \arrow[dd, "\gamma_{(A\otimes B)\otimes C}"] \arrow[dr, "F(a_{A,B,C})"] & \\
 & (G(A)\otimes G(B))\otimes F(C) \arrow[r, "G^{\otimes}_{A,B}\otimes id_{F(C)}"'] \arrow[d, "id_{G(A)\otimes G(B)}\otimes\gamma_C"'] & G(A\otimes B)\otimes F(C) \arrow[d, "id_{G(A\otimes B)}\otimes\gamma_C"] & & F(A\otimes(B\otimes C)) \arrow[ddl, "\gamma_{A\otimes(B\otimes C)}"] \arrow[dl, phantom, "\Swarrow \gamma_{\alpha_{A,B,C}}"] \\
 & (G(A)\otimes G(B))\otimes G(C) \arrow[r, "G^{\otimes}_{A,B}\otimes id_{G(C)}"] \arrow[d, "a_{G(A),G(B),G(C)}"] \arrow[dl, phantom, "\Swarrow a_{\gamma_A,\gamma_B,\gamma_C}"] & G(A\otimes B)\otimes G(C) \arrow[r, "G^{\otimes}_{A\otimes B,C}"] \arrow[d, phantom, "\Swarrow G^{a}_{A,B,C}"] & G((A\otimes B)\otimes C) \arrow[d, "G(a_{A,B,C})"] & \\
F(A)\otimes(F(B)\otimes F(C)) \arrow[r, "\gamma_A\otimes(\gamma_B\otimes\gamma_C)"] & G(A)\otimes(G(B)\otimes G(C)) \arrow[r, "id_{G(A)}\otimes G^{\otimes}_{B,C}"] & G(A)\otimes G(B\otimes C) \arrow[r, "G^{\otimes}_{A,B\otimes C}"] & G(A\otimes(B\otimes C)) &
\end{tikzcd}$$

Definition 11. *A monoidal modification $m : \gamma \Rrightarrow \delta : F \Rightarrow G$ between two monoidal pseudonatural transformations γ and δ is a modification verifying the following property :*

$$\begin{tikzcd}
 & F(I) \arrow[dd, "\delta_I"'] \arrow[dd, bend left=80, "\gamma_I"] \arrow[dd, phantom, shift left=5ex, "\overset{m_I}{\Leftarrow}"] & & \\
I \arrow[ur, "F_I^{\otimes}"] \arrow[dr, "G_I^{\otimes}"'] \arrow[r, phantom, "\Swarrow \delta_I^{\otimes}"] & & = & \gamma_I^{\otimes} \\
 & G(I) & &
\end{tikzcd}$$

and, for every object A, B of $\mathcal{C}$, the following diagram

$$\begin{tikzcd}
F(A)\otimes F(B) \arrow[r, "F^{\otimes}_{A,B}"] \arrow[d, "\gamma_A\otimes\gamma_B"] \arrow[d, bend right=80, "\delta_A\otimes\delta_B"'] \arrow[d, phantom, shift right=5ex, "\underset{m_A\otimes m_B}{\Leftarrow}"] \arrow[dr, phantom, "\Swarrow \gamma^{\otimes}_{A,B}"] & F(A\otimes B) \arrow[d, "\gamma_{A\otimes B}"] \\
G(A)\otimes G(B) \arrow[r, "G^{\otimes}_{A,B}"] & G(A\otimes B)
\end{tikzcd}$$

is equal to the diagram

$$\begin{tikzcd}
F(A)\otimes F(B) \arrow[r, "F^{\otimes}_{A,B}"] \arrow[d, "\delta_A\otimes\delta_B"] \arrow[dr, phantom, "\Swarrow \delta^{\otimes}_{A,B}"] & F(A\otimes B) \arrow[d, "\delta_{A\otimes B}"'] \arrow[d, bend left=80, "\gamma_{A\otimes B}"] \arrow[d, phantom, shift left=5ex, "\underset{m_{A\otimes B}}{\Leftarrow}"] \\
G(A)\otimes G(B) \arrow[r, "G^{\otimes}_{A,B}"] & G(A\otimes B)
\end{tikzcd}$$

TODO: Symmetric stuff, a couple of hundred new diagrams

Let us now get back to the remaining rules of $MILL$ that we have not interpreted yet, ie the ones involving the linear implication $\multimap$. First things first, through a reasoning similar to what we did for $\otimes$ and supported by the following proof transformation rules, we can state that $\multimap$ must be interpreted as a pseudo-bifunctor, though contravariant in its first argument.

$$\eta \multimap \qquad \dfrac{}{A \multimap B \vdash A \multimap B}\,\text{ax} \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{ax} \qquad \dfrac{}{B \vdash B}\,\text{ax}}{A, A \multimap B \vdash B}\,l\!-\!\multimap}{A \multimap B \vdash A \multimap B}\,r\!-\!\multimap$$

$$\text{r-}\!\multimap \text{ l-}\!\multimap \ \mathit{cut} \qquad \dfrac{\dfrac{\dfrac{\pi_1}{A, B \vdash C}}{B \vdash A \multimap C}\,r\!-\!\multimap \qquad \dfrac{\dfrac{\pi_2}{D \vdash A} \quad \dfrac{\pi_3}{C \vdash E}}{D, A \multimap C \vdash E}\,l\!-\!\multimap}{B, D \vdash E}\,\text{cut} \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\pi_2}{D \vdash A} \quad \dfrac{\pi_1}{A, B \vdash C}}{B, D \vdash C}\,\text{cut} \qquad \dfrac{\pi_3}{C \vdash E}}{B, D \vdash E}\,\text{c}$$

$$\text{l-}\!\multimap \ \mathit{lcut} \qquad \dfrac{\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\pi_2}{C \vdash D}}{A, B \multimap C \vdash D}\,l\!-\!\multimap \qquad \dfrac{\pi_3}{D \vdash E}}{A, B \multimap C \vdash E}\,\text{cut} \quad \Rightarrow \quad \dfrac{\dfrac{\pi_1}{A \vdash B} \qquad \dfrac{\dfrac{\pi_2}{C \vdash D} \quad \dfrac{\pi_3}{D \vdash E}}{C \vdash E}\,\text{cut}}{A, B \multimap C \vdash E}\,l\!-\!\multimap$$

$$\text{l-}\!\multimap \ \mathit{rcut}_1 \qquad \dfrac{\dfrac{\pi_1}{A \vdash B} \qquad \dfrac{\dfrac{\pi_2}{B \vdash C} \quad \dfrac{\pi_3}{D \vdash E}}{B, C \multimap D \vdash E}\,l\!-\!\multimap}{A, C \multimap D \vdash E}\,\text{cut} \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\pi_2}{B \vdash C}}{A \vdash C}\,\text{cut} \qquad \dfrac{\pi_3}{D \vdash E}}{A, C \multimap D \vdash E}\,l\!-$$

$$\text{l-}\!\multimap \ \mathit{rcut}_2 \qquad \dfrac{\dfrac{\pi_1}{A \vdash B} \qquad \dfrac{\dfrac{\pi_2}{C \vdash D} \quad \dfrac{\pi_3}{B, E \vdash F}}{C, B, D \multimap E \vdash F}\,l\!-\!\multimap}{C, A, D \multimap E \vdash F}\,\text{cut} \quad \Rightarrow \quad \dfrac{\dfrac{\pi_2}{C \vdash D} \qquad \dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\pi_3}{B, E \vdash F}}{A, E \vdash F}\,\text{cu}}{C, A, D \multimap E \vdash F}\,l\!-\!\multimap$$

$$\text{r-}\!\multimap \ \mathit{rcut} \qquad \dfrac{\dfrac{\pi_1}{A \vdash B} \qquad \dfrac{\dfrac{\pi_2}{B, C \vdash D}}{B \vdash C \multimap D}\,r\!-\!\multimap}{A \vdash C \multimap D}\,\text{cut} \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\pi_1}{A \vdash B} \quad \dfrac{\pi_2}{B, C \vdash D}}{A, C \vdash D}\,\text{cut}}{A \vdash C \multimap D}\,r\!-\!\multimap$$

Definition 12. *A biadjunction between two pseudo-functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ is given by a pair of pseudo-natural transformations $\eta : Id_{\mathcal{C}} \to GF$ and $\epsilon : FG \to Id_{\mathcal{D}}$ along with two invertible modifications with components :*

$$\begin{tikzcd}
G(D) \arrow[r, "\eta_{G(D)}"] \arrow[dr, "id_{G(D)}"'] & G(F(G(D))) \arrow[d, "G(\epsilon_D)"] \arrow[dl, phantom, "\Rightarrow s_D"] \\
& G(D)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F(C) \arrow[r, "F(\eta_C)"] \arrow[dr, "id_{F(C)}"'] & F(G(F(C))) \arrow[d, "\epsilon_{F(C)}"] \arrow[dl, phantom, "\Leftarrow t_C"] \\
& F(C)
\end{tikzcd}$$

such that the following diagram equalities hold for all objects C of $\mathcal{C}$ and D of $\mathcal{D}$:

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, "\eta_C"] & G(F(C)) \arrow[d, "G(F(\eta_C))"] \arrow[dr, bend left, "id_{G(F(C))}"] \arrow[dr, phantom, "\Rightarrow G(t_C)"] & \\
G(F(C)) \arrow[r, "\eta_{G(F(C))}"] \arrow[drr, "id_{G(F(C))}"'] \arrow[dr, phantom, "\Rightarrow s_{F(C)}"] & G(F(G(F(C)))) \arrow[dr, "G(\epsilon_{F(C)})"] & \\
 & & G(F(C))
\end{tikzcd}
\qquad = \qquad Id_{id_C \circ \eta_C}$$

$$\begin{tikzcd}
F(G(D)) \arrow[dr, "F(\eta_{G(D)})"'] \arrow[drr, "id_{F(G(D))}"] \arrow[ddr, bend right, "id_{F(G(D))}"'] \arrow[dr, phantom, "\Rightarrow t_{G(D)}"] & & \\
 & F(G(F(G(D)))) \arrow[r, "\epsilon_{F(G(D))}"] \arrow[d, "F(G(\epsilon_D))"] \arrow[d, phantom, "F(s_D)\Rightarrow"'] & F(G(D)) \arrow[d, "\epsilon_D"] \\
 & F(G(D)) \arrow[r, "\epsilon_D"] & D
\end{tikzcd}
\qquad = \qquad Id_{\epsilon_D \circ id_D}$$

Definition 13. *A monoidal bicategory $\mathcal{C}$ is monoidal closed if the pseudo-functor $_{\otimes}B$: $\mathcal{C} \to \mathcal{C}$ has a right biadjoint for all objects B of $\mathcal{C}$.*

1.2.1 the case of MALL

Definition 14. *A bicategory $\mathcal{C}$ is cartesian if the diagonal pseudofunctor $\Delta_n : \mathcal{C} \to \mathcal{C}^n$ has a right biadjoint.*

Definition 15. *A cartesian bicategory $\mathcal{C}$ is cartsian closed if the pseudo-functor $_{\times}B$: $\mathcal{C} \to \mathcal{C}$ has a right biadjoint for all objects B of $\mathcal{C}$.*

1.3 ILL and linear exponential comonads

Definition 16. *A pseudo-comonoid A in a monoidal bicategory $\mathcal{C}$ is given by an object A of the bicategory, equipped with :*

- *a $1-$cell $J : A \to I$*
- *a $1-$cell $P : C \to C \otimes C$*
- *three invertible $2-$ cells*

$$\begin{tikzcd}
A \otimes A \arrow[d, "P \otimes id_A"] & A \arrow[l, "P"'] \arrow[r, "P"] \arrow[d, phantom, "\Rightarrow \alpha"] & A \otimes A \arrow[d, "id_A \otimes P"] \\
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] & & A \otimes (A \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "P"] \arrow[dr, "l_A^{-1}"'] \arrow[dr, phantom, "\Rightarrow \lambda"] & A \otimes A \arrow[d, "J \otimes id_A"] \\
 & I \otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "P"] \arrow[dr, "r_A^{-1}"'] \arrow[dr, phantom, shift left=3, "\overset{\rho}{\Rightarrow}"] & A \otimes A \arrow[d, "id_A \otimes J"] \\
 & A \otimes I
\end{tikzcd}$$

such that the following properties are verified : The diagram

$$\begin{tikzcd}
A \otimes I \otimes A & & A \otimes A \otimes A \arrow[ll, "id_A \otimes J \otimes id_A"'] \\
A \otimes A \arrow[u, "id_A \otimes l_A^{-1}"] \arrow[urr, "id_A \otimes P"'] \arrow[u, phantom, shift left=6, "\overset{id_A \otimes \lambda}{\Rightarrow}"] \arrow[rr, phantom, "\underset{\alpha}{\Rightarrow}"] & & A \otimes A \arrow[u, "P \otimes id_A"'] \\
A \arrow[u, "P"] \arrow[urr, "P"'] & &
\end{tikzcd}$$

is equal to the diagram

$$\begin{tikzcd}
A \otimes I \otimes A & & A \otimes A \otimes A \arrow[ll, "id_A \otimes J \otimes id_A"'] \\
A \otimes A \arrow[u, "r_A^{-1}"] \arrow[urr, "P \otimes id_A"'] \arrow[u, phantom, shift left=6, "\overset{\rho \otimes id_A}{\Rightarrow}"] & & \\
A \arrow[u, "P"] & &
\end{tikzcd}$$

and the following diagram :

$$\begin{tikzcd}
A \otimes A \otimes A \otimes A & A \otimes A \otimes A \arrow[l, "id_A \otimes id_A \otimes P"'] & A \otimes A \arrow[l, "id_A \otimes P"'] \arrow[dl, phantom, "\overset{\alpha}{\Leftarrow}"] \\
A \otimes A \otimes A \arrow[u, "P \otimes id_A \otimes id_A"] & A \otimes A \arrow[l, "id_A \otimes P"] \arrow[u, "P \otimes id_A"] \arrow[d, phantom, "\alpha\Downarrow"] & A \arrow[l, "P"] \arrow[u, "P"'] \arrow[dl, "P"] \\
 & A \otimes A \arrow[ul, "P \otimes id_A"] &
\end{tikzcd}$$

must be equal to the diagram :

$$\begin{tikzcd}
A \otimes A \otimes A \otimes A & A \otimes A \otimes A \arrow[l, "id_A \otimes id_A \otimes P"'] \arrow[d, phantom, "\Downarrow Id_{id_A} \otimes \alpha"] & A \otimes A \arrow[l, "id_A \otimes P"'] \arrow[dl, "id_A \otimes P"] \\
A \otimes A \otimes A \arrow[u, "P \otimes id_A \otimes id_A"] & A \otimes A \otimes A \arrow[ul, "id_A \otimes P \otimes id_A"'] \arrow[l, phantom, "\overset{\alpha \otimes Id_{id_A}}{\Leftarrow}"] \arrow[r, phantom, "\overset{\alpha}{\Leftarrow}"] & A \arrow[u, "P"'] \arrow[dl, "P"] \\
 & A \otimes A \arrow[ul, "P \otimes id_A"] \arrow[u, "P \otimes id_A"] &
\end{tikzcd}$$

Definition 17. *A pseudo-comonad on a bicategory $\mathcal{C}$ is given by a pseudo-functor $F : \mathcal{C} \to \mathcal{C}$, two pseudo-natural transformations $v : F \Rightarrow Id_{\mathcal{C}}$ and $n : F \Rightarrow F \circ F$ called the counit and comultiplications, and three invertible modifications α, λ, ρ whose components are given by the following diagrams :*

$$\begin{tikzcd}
F(A) \arrow[r, "n_A"] \arrow[d, "n_A"] & F(F(A)) \arrow[d, "n_{F(A)}"] \arrow[dl, phantom, "\overset{\alpha_A}{\Leftarrow}"] \\
F(F(A)) \arrow[r, "F(n_A)"] & F(F(F(A)))
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(A) \arrow[dl, "id_{F(A)}"'] \arrow[d, "n"] \arrow[dr, "id_{F(A)}"] & \\
F(A) & F(F(A)) \arrow[l, "F(v_A)"] \arrow[r, "v_{F(A)}"'] \arrow[ul, phantom, "\overset{\lambda_A}{\Leftarrow}"] \arrow[ur, phantom, "\overset{\rho_A}{\Leftarrow}"] & F(A)
\end{tikzcd}$$

such that the following properties are verified : TODO:

Definition 18. *The Kleisli bicategory $\mathcal{C}_F$ associated to a pseudo-comonad F on a bicategory $\mathcal{C}$ is defined as having the same 0-cells as $\mathcal{C}$, and whose hom-category $\mathcal{C}_F(A, B)$ is given by $\mathcal{C}(F(A), B)$.*
The composition in $\mathcal{C}_F$ of $f : F(A) \to B$ and $g : F(B) \to C$ is defined as

$$g \circ_F f := g \circ f(F) \circ n_A$$

. This definition can easily be extended to provide the required composition functors. The identities in $\mathcal{C}_F$ are given by the components of the counit of the comonad. The 2-isomorphisms and additional properties of the bicategory come directly from the pseudo-comonad structure.

Definition 19. *A linear exponential pseudo-comonad*

And thus we have the following definition for a model of ILL :

Definition 20. *A bicategorical model of ILL is a symmetric monoidal bicategory with a linear exponential pseudo-comonad*

We can then recover one of the most interesting properties from categorical models of linear logic in the bicategorical settings with the following theorem :

Theorem 1. *Let $\mathcal{C}, !$ a bicategorical model of ILL. Then the Kleisli bicategory $\mathcal{C}_!$ is cartesian closed.*

Proof. The proof will proceed in two steps, first we will show that $\mathcal{C}_!$ is cartesian, and then that it is cartesian closed.

- To prove that $\mathcal{C}_!$ is cartesian, we have to prove that the diagonal pseudofunctors $\Delta^n : \mathcal{C}_! \to \mathcal{C}_!^n$ have a right pseudo adjoint, given by $\Pi^n : A_1, ...A_n \to A_1\&...\&A_n$.

- Next, to prove that $\mathcal{C}_!$ is cartesian closed, we also have to prove the existence of a family of pseudo-adjunctions in $\mathcal{C}_!$, this time between the pseudo-functors $\Pi_B : A \to B\&A$ and $\Rightarrow_B : A \to !B \multimap A$ indexed by B object of $\mathcal{C}$.

  Let us start with a full description of both functors on morphisms and 2−morphisms: First, for $\Pi_B : \mathcal{C}_! \to \mathcal{C}_!$, we need the effect on morphisms to produce, from a morphism $f : A_1 \to_{\mathcal{C}_!} A_2$, a morphism $\Pi_B(f) : B\&A_1 \to_{\mathcal{C}_!} B\&A_2$, meaning, when looking at the original category $\mathcal{C}$, we need to turn a morphism $f : !A_1 \to A_2$ into a morphism $\Pi_B(f) : !(B\&A_1) \to B\&A_2$. This is obtained through the following construction :

$$\begin{tikzcd} !(B\&A_1) \arrow[r, "s_{B,A_1}"] & !B\otimes !A_1 \arrow[r, "id_{!B}\otimes n_{A_1}"] & !B\otimes !!A_1 \arrow[d, "id_{!B}\otimes !f"] \\ B\&A_2 & !(B\&A_2) \arrow[l, "v_{B\&A_2}"'] & !B\otimes !A_2 \arrow[l, "s^{-1}_{B,A_2}"'] \end{tikzcd}$$

  A consequence of this construction is that the effect on a 2−morphism $\tau : f \Rightarrow g$ is very obvious, applying the initial 2−morphism at the only point where the morphism f appears, with the remaining space between morphisms being filled by identities.

Next, for $\Rightarrow_B: \mathcal{C}_! \to \mathcal{C}_!$, we need the effect on morphisms to produce, from a morphism $f : A_1 \to_{\mathcal{C}_!} A_2$, a morphism $\Rightarrow_B (f) :!B \multimap A_1 \to_{\mathcal{C}_!} !B \multimap A_2$, meaning, when looking at the original category $\mathcal{C}$, we need to turn a morphism $f :!A_1 \to A_2$ into a morphism $\Rightarrow_B (f) :!(!B \multimap A_1) \to !B \multimap A_2$. This is obtained through the following construction :

$$\begin{tikzcd}
!(!B \multimap A_1) \arrow[r, "\eta^{\otimes,\mathcal{C}}_{!B,!(!B\multimap A_1)}"] & !B \multimap (!B \otimes !(!B \multimap A_1)) \arrow[r, "!B\multimap s_{B,!B\multimap A_1}"] & !B \multimap !(B \& (!B \multimap A_1)) \arrow[d, "!B\multimap n_{B\&(!B\multimap A_1)}"] \\
 & & !B \multimap !!(B \& (!B \multimap A_1)) \arrow[d, "!B\multimap !s^{-1}_{B,(!B\multimap A_1)}"] \\
!B \multimap !A_1 \arrow[d, "!B\multimap f"] & !B \multimap !(!B \otimes !B \multimap A_1) \arrow[l, "!B\multimap !(\epsilon^{\otimes,\mathcal{C}}_{!B,A_1}"'] & !B \multimap !(!B \otimes !(!B \multimap A_1)) \arrow[l, "!B\multimap !(id_{!B}\otimes v_{!B\multimap A_1})"'] \\
!B \multimap A_2 & &
\end{tikzcd}$$

In a similar way, the effect on 2−morphisms is easy to build.

Let us now proceed step by step to build this pseudo-adjunction. First, we need candidates for the pseudo-natural transformations $\eta : id_{\mathcal{C}_!} \to \Rightarrow_B (\Pi_B)$ and $\epsilon : \Pi_B(\Rightarrow_B) \to id_{\mathcal{C}_!}$.
So we need $\eta_A : A \to_{\mathcal{C}_!} !B \multimap (B \& A)$ and $\epsilon_A : B\&(!B \multimap A) \to_{\mathcal{C}_!} A$. They are obtained through the following constructions using the monoidal closed structure of the underlying bicategory $\mathcal{C}$:

for η_A :

$$!A \xrightarrow{\eta^{\mathcal{C}}_{!B,!A}} !B \multimap (!B \otimes !A) \xrightarrow{!B\multimap s^{-1}_{B,A}} !B \multimap !(B\&A) \xrightarrow{!B\multimap v_{B\&A}} !B \multimap (B\&A)$$

and for ϵ_A :

$$!(B\&(!B \multimap A)) \xrightarrow{s_{B,!B\multimap A}} !B \otimes !(!B \multimap A) \xrightarrow{id_{!B}\otimes v_{!B\multimap A}} !B \otimes !B \multimap A \xrightarrow{\epsilon^{\mathcal{C}}_{!B,A}} A$$

This handles the morphism components of the two transformations, we now need to describe their 2−morphism components which need to be of the form:

$$\begin{tikzcd}
A_1 \arrow[r, "\eta_{A_1}"] \arrow[d, "f"] \arrow[dr, phantom, "\eta_f \Nearrow"] & !B \multimap (B\&A_1) \arrow[d, "\Rightarrow_B(\Pi_B(f))"] \\
A_2 \arrow[r, "\eta_{A_2}"] & !B \multimap (B\&A_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
B\&!B \multimap A_1 \arrow[r, "\epsilon_{A_1}"] \arrow[d, "\Pi_B(\Rightarrow_B(f))"] \arrow[dr, phantom, "\epsilon_f \Nearrow"] & A_1 \arrow[d, "f"] \\
B\&!B \multimap A_2 \arrow[r, "\epsilon_{A_2}"] & A_2
\end{tikzcd}$$

Those are diagrams in $\mathcal{C}_!$, and thus, when expanding them to describe η_f and ϵ_f properly, we will need to remember to use the specific composition of $\mathcal{C}_!$. Let us now look at the full diagram for η

□