

1 Some bicategorical definitions

In this section, we recall a few definitions required by our bicategorical setting.

Definition 1. A bicategory \mathcal{C} consists of :

- A collections of objects A, B, C .
- For each pair of objects A, B , a category $\mathcal{C}(A, B)$ whose objects are called morphisms or 1-cells and whose morphisms are called 2-morphisms or 2-cells.
- For each object A , a distinguished 1-cell $id_A \in \mathcal{C}(A, A)$ called the identity morphism.
- For each triple of objects A, B, C a functor

$$\circ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \rightarrow \mathcal{C}(A, C)$$

called horizontal composition.

- For each pair of objects A, B , two natural isomorphisms called the left and right unitors:

$$l : id_A \circ - \Rightarrow - : \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, B) \text{ and } r : - \circ id_B \Rightarrow - : \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, B)$$

- For each quadruple of objects A, B, C, D a natural isomorphism called the associator

$$a : (- \circ -) \circ - \Rightarrow - \circ (- \circ -) : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \times \mathcal{C}(C, D) \rightarrow \mathcal{C}(A, D)$$

such that the following diagrams commute for any object A, B, C, D, E of \mathcal{C} and f, g, h, i objects of $\mathcal{C}(A, B), \mathcal{C}(B, C), \mathcal{C}(C, D), \mathcal{C}(D, E)$ respectively :

$$\begin{array}{ccc}
 ((f \circ g) \circ h) \circ i & \xrightarrow{a(f, g, h) \circ Id_i} & (f \circ (g \circ h)) \circ i \\
 \downarrow a(f \circ g, h, i) & & \downarrow a(f, g \circ h, i) \\
 (f \circ g) \circ (h \circ i) & & f \circ ((g \circ h) \circ i) \\
 \searrow a(f, g, h \circ i) & & \swarrow Id_f \circ a(g, h, i) \\
 & f \circ (g \circ (h \circ i)) & \\
 \\
 (f \circ Id_B) \circ g & \xrightarrow{a(f, Id_B, g)} & f \circ (Id_B \circ g) \\
 \searrow r(f) \circ Id(g) & & \swarrow Id_f \circ l(g) \\
 & f \circ g &
 \end{array}$$

Definition 2. Let \mathcal{C}, \mathcal{D} be two bicategories. A pseudofunctor $F : \mathcal{C} \rightarrow \mathcal{D}$ is given by :

- For each object A of \mathcal{C} , an object $F(A)$ of \mathcal{D} .

- For each hom-category $\mathcal{C}(A, B)$ in \mathcal{C} , a functor

$$F(A, B) : \mathcal{C}(A, B) \rightarrow \mathcal{D}(F(A), F(B))$$

- For each object A of \mathcal{C} , an invertible 2-cell

$$F_{id_A} : id_{F(A)} \Rightarrow F(A, B)(id_A)$$

- For each triple of objects A, B, C of \mathcal{C} , a natural isomorphism ϕ whose elements are:

$$\phi_{f,g} : F(f) \circ F(g) \Rightarrow F(f \circ g)$$

for f, g objects of $\mathcal{C}(A, B), \mathcal{C}(B, C)$ respectively

such that, for any object A, B, C, D of \mathcal{C} , any objects f, g, h of $\mathcal{C}(A, B), \mathcal{C}(B, C), \mathcal{C}(C, D)$ respectively, the following diagrams commute :

$$\begin{array}{ccc}
F(f) \circ (F(g) \circ F(h)) & \xleftarrow{a(F(f), F(g), F(h))} & (F(f) \circ F(g)) \circ F(h) \\
\downarrow Id_{F(f)} \circ \phi_{g,h} & & \downarrow \phi_{f,g} \circ Id_{F(h)} \\
F(f) \circ F(g \circ h) & & F(f \circ g) \circ F(h) \\
\downarrow \phi_{f,g \circ h} & & \downarrow \phi_{f \circ g, h} \\
F(f \circ (g \circ h)) & \xleftarrow{F(a(f,g,h))} & F((f \circ g) \circ h)
\end{array}$$

$$\begin{array}{ccc}
F(f) & \xleftarrow{F(r(f))} & F(f \circ id_B) \\
\uparrow r(F(f)) & & \uparrow \phi_{f, id_B} \\
F(f) \circ id_{F(B)} & \xrightarrow{Id_{F(f)} \circ F_{id_B}} & F(f) \circ F(id_B)
\end{array}$$

$$\begin{array}{ccc}
F(f) & \xleftarrow{F(l(f))} & F(id_A \circ f) \\
\uparrow l(F(f)) & & \uparrow \phi_{id_A, f} \\
id_{F(A)} \circ F(f) & \xrightarrow{F_{id_A} \circ Id_{F(f)}} & F(id_A) \circ F(f)
\end{array}$$

Definition 3. Let \mathcal{C}, \mathcal{D} be two bicategories, and $F, G : \mathcal{C} \rightarrow \mathcal{D}$ two pseudofunctors. A pseudo-natural transformation $\gamma : F \Rightarrow G$ is given by :

- for every object A of \mathcal{C} , a 1-cell $\gamma_A : F(A) \rightarrow G(A)$
- for every pair of objects A, B of \mathcal{C} and every 1-cell f of $\mathcal{C}(A, B)$, an invertible 2-cell γ_f :

$$\begin{array}{ccc}
F(A) & \xrightarrow{\gamma_A} & G(A) \\
F(f) \downarrow & \nearrow \gamma_f & \downarrow G(f) \\
F(B) & \xrightarrow{\gamma_B} & G(B)
\end{array}$$

such that the following properties are verified :

- *Naturality* : For every 2-cell $\tau : f \Rightarrow g : A \rightarrow B$, the 2-cells associated to the following pasting diagrams are equal :

$$\begin{array}{ccc}
 F(A) & \xrightarrow{\gamma_A} & G(A) \\
 \downarrow F(f) & \nearrow \gamma_f G(f) \left(\begin{array}{c} \xrightarrow{G(\tau)} \\ \xrightarrow{\quad} \end{array} \right) & \downarrow G(g) \\
 F(B) & \xrightarrow{\gamma_B} & G(B)
 \end{array} = \begin{array}{ccc}
 F(A) & \xrightarrow{\gamma_A} & G(A) \\
 \downarrow F(f) \left(\begin{array}{c} \xrightarrow{F(\tau)} \\ \xrightarrow{\quad} \end{array} \right) & \nearrow \gamma_g & \downarrow G(g) \\
 F(B) & \xrightarrow{\gamma_B} & G(B)
 \end{array}$$

- *Unitality* : For every object A of \mathcal{C} , the 2-cells associated to the following pasting diagrams are equal :

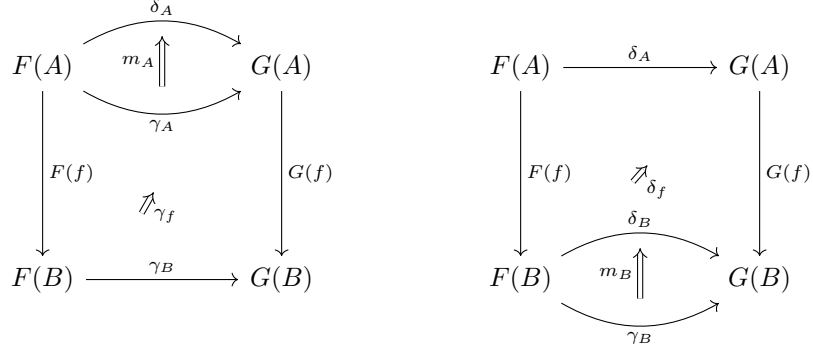
$$\begin{array}{ccc}
 F(A) & \xrightarrow{\gamma_A} & G(A) \\
 \downarrow Id_{F(A)} & \equiv Id_{G(A)} \left(\begin{array}{c} \xrightarrow{G_A} \\ \xrightarrow{\quad} \end{array} \right) & \downarrow G(Id_A) \\
 F(A) & \xrightarrow{\gamma_A} & G(A)
 \end{array} = \begin{array}{ccc}
 F(A) & \xrightarrow{\gamma_A} & G(A) \\
 \downarrow Id_{F(A)} \left(\begin{array}{c} \xrightarrow{F_A} \\ \xrightarrow{\quad} \end{array} \right) & \nearrow \gamma_{Id_A} & \downarrow G(Id_A) \\
 F(A) & \xrightarrow{\gamma_A} & G(A)
 \end{array}$$

- *Compositionality* : for every triple of objects A, B, C of \mathcal{C} and every pair of 1-cells f, g of $\mathcal{C}(A, B)$, $\mathcal{C}(B, C)$ respectively, the 2-cells associated to the following pasting diagrams are equal :

$$\begin{array}{ccc}
 F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{F(g)} & F(C) \\
 \downarrow \gamma_A & \searrow & \downarrow F_{g,f} & \nearrow & \downarrow \gamma_C \\
 & & F(g \circ f) & & \\
 G(A) & \xrightarrow{G(g \circ f)} & G(C) & &
 \end{array} = \begin{array}{ccccc}
 F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{F(g)} & F(C) \\
 \downarrow \gamma_A & \searrow \gamma_f & \downarrow \gamma_B & \searrow \gamma_g & \downarrow \gamma_C \\
 & & G(B) & & \\
 G(A) & \nearrow G(f) & \downarrow G_{g,f} & \searrow G(g) & G(C) \\
 & & G(g \circ f) & &
 \end{array}$$

Definition 4. Let $\gamma, \delta : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{D}$ be two pseudo-natural transformations., a modification $m : \gamma \Rightarrow \delta$ is given by a 2-cell $m_A : \gamma_A \Rightarrow \delta_A$ for every

object A of \mathcal{C} such that for every $f : A \rightarrow B$ in \mathcal{C} , we have :



Definition 5. Let A, B be two objects in a bicategory \mathcal{C} . An equivalence from A to B is given by :

- a pair of 1-cells $f : A \rightarrow B$ and $g : B \rightarrow A$.
- a pair of invertible 2-cells $e : id_A \Rightarrow g \circ f$ and $e' : id_B \Rightarrow f \circ g$.

We say that f is an equivalence if such g, e, e' exist.

Definition 6. A monoidal bicategory \mathcal{C} is a bicategory equipped with :

- a unit object I .
- a pseudo-functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$
- three pseudo-natural transformations a, l, r whose components are equivalences and given by :

$$a_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$$

$$l_A : I \otimes A \rightarrow A$$

$$r_A : A \otimes I \rightarrow A$$

- four invertible modifications π, μ, L, R whose components are given by :

$$\begin{array}{ccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{a_{A \otimes B, C, D}} & (A \otimes B) \otimes (C \otimes D) \xrightarrow{a_{A, B, C \otimes D}} A \otimes (B \otimes (C \otimes D)) \\ \downarrow a_{A, B, C} \otimes id_D & & \downarrow \pi_{A, B, C, D} \quad id_A \otimes a_{B, C, D} \uparrow \\ (A \otimes (B \otimes C)) \otimes D & \xrightarrow{a_{A, B \otimes C, D}} & A \otimes ((B \otimes C) \otimes D) \end{array}$$

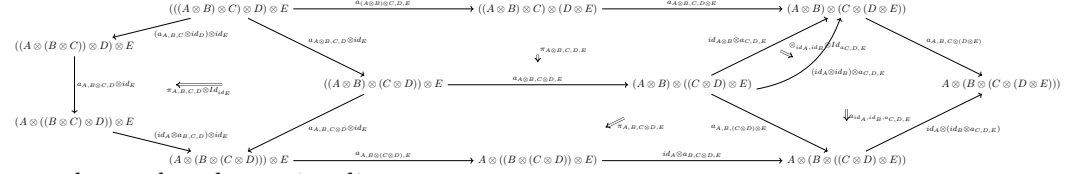
$$\begin{array}{ccc} (A \otimes I) \otimes C & \xrightarrow{a_{A, I, C}} & A \otimes (I \otimes C) \\ \searrow r_A \otimes id_C & \xrightarrow{\mu_{A, C}} & \swarrow id_A \otimes l_C \\ & A \otimes C & \end{array}$$

$$\begin{array}{ccc} (I \otimes B) \otimes C & \xrightarrow{a_{I, B, C}} & I \otimes (B \otimes C) \\ \searrow l_B \otimes id_C & \xrightarrow{L_{B, C}} & \swarrow l_B \otimes C \\ & B \otimes C & \end{array}$$

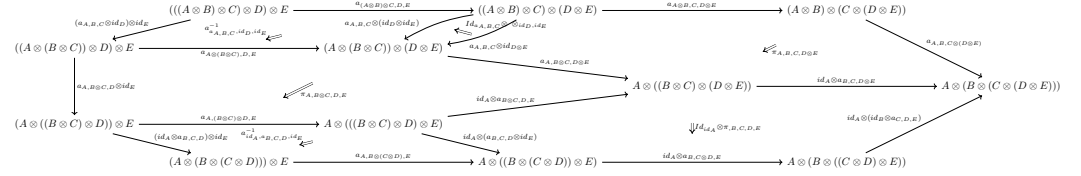
$$\begin{array}{ccc} (A \otimes B) \otimes I & \xrightarrow{a_{A, B, I}} & A \otimes (B \otimes I) \\ \searrow r_A \otimes B & \xrightarrow{R_{A, B}} & \swarrow id_A \otimes r_B \\ & A \otimes B & \end{array}$$

such that the following conditions are verified :

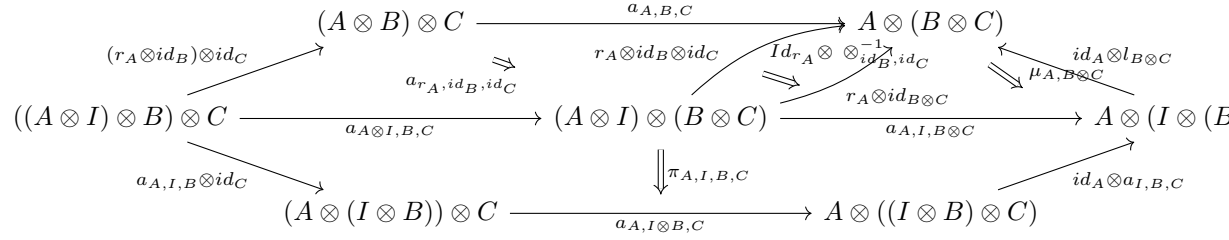
– *Associativity* : For all A, B, C, D, E objects of \mathcal{C} , the pasting diagram



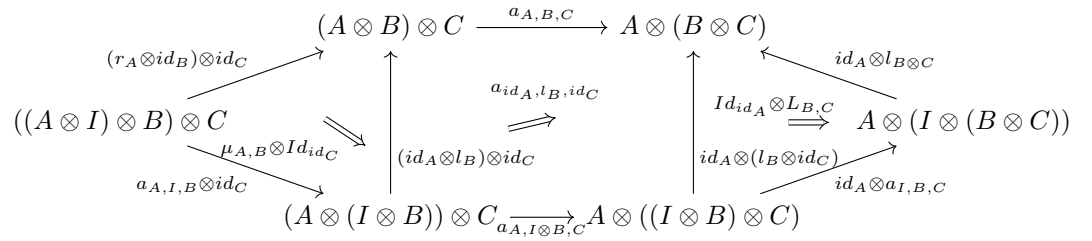
must be equal to the pasting diagram



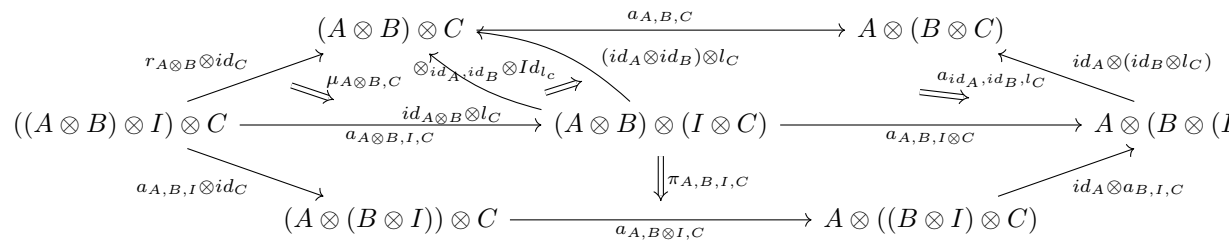
– For all A, B, C objects of \mathcal{C} , the pasting diagram



must be equal to the pasting diagram



– For all A, B, C objects of \mathcal{C} , the pasting diagram



must be equal to the pasting diagram

$$\begin{array}{ccccc}
& & (A \otimes B) \otimes C & \xrightarrow{a_{A,B,C}} & A \otimes (B \otimes C) \\
& \nearrow r_{A \otimes B} \otimes id_C & \uparrow & \nearrow a_{id_A, r_B, id_C} & \nwarrow id_A \otimes (id_B \otimes l_C) \\
((A \otimes B) \otimes I) \otimes C & \xRightarrow{R_{A,B} \otimes Id_{id_C}} & (id_A \otimes r_B) \otimes id_C & \xRightarrow{} & A \otimes (B \otimes (I \otimes C)) \\
& \searrow a_{A,B,I} \otimes id_C & \downarrow & \searrow id_A \otimes (r_B \otimes id_C) & \nwarrow id_A \otimes a_{B,I,C} \\
& & (A \otimes (B \otimes I)) \otimes C & \xrightarrow{a_{A,B \otimes I,C}} & A \otimes ((B \otimes I) \otimes C)
\end{array}$$

Definition 7. Let \mathcal{C}, \mathcal{D} be two monoidal bicategories. A monoidal pseudofunctor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a pseudofunctor equipped with:

- a 1-cell $F_I^\otimes : I \rightarrow F(I)$
- a pseudo-natural transformation F^\otimes whose components are of the form :

$$F_{A,B}^\otimes : F(A) \otimes F(B) \rightarrow F(A \otimes B)$$

- three invertible modifications F^a, F^l, F^r whose components are of the form :

$$\begin{array}{ccc}
(F(A) \otimes F(B)) \otimes F(C) & \xrightarrow{a_{F(A), F(B), F(C)}} & F(A) \otimes (F(B) \otimes F(C)) \\
\downarrow F_{A,B}^\otimes \otimes id_{F(A)} & & \downarrow id_{F(A)} \otimes F_{B,C}^\otimes \\
F(A \otimes B) \otimes F(C) & \xrightarrow{F^a_{A,B,C}} & F(A) \otimes F(B \otimes C) \\
\downarrow F_{A \otimes B, C}^\otimes & & \downarrow F_{A, B \otimes C}^\otimes \\
F((A \otimes B) \otimes C) & \xrightarrow{F(a_{A,B,C})} & F(A \otimes (B \otimes C))
\end{array}$$

$$\begin{array}{ccc}
I \otimes F(A) & \xrightarrow{F_I^\otimes \otimes id_{F(A)}} & F(I) \otimes F(A) \\
\downarrow l_{F(A)} & \xRightarrow{F^l_A} & \downarrow F_{I,A}^\otimes \\
F(A) & \xleftarrow{F(l_A)} & F(I \otimes A)
\end{array}
\qquad
\begin{array}{ccc}
F(A) \otimes I & \xrightarrow{id_{F(A)} \otimes F_I^\otimes} & F(A) \otimes F(I) \\
\downarrow r_{F(A)} & \xRightarrow{F^r_A} & \downarrow F_{A,I}^\otimes \\
F(A) & \xleftarrow{F(r_A)} & F(A \otimes I)
\end{array}$$

such that the following properties are verified :

- **TODO: THE DEMONIC DIAGRAM** For all A, B, C, D objects of \mathcal{C} , the following pasting diagram

must be equal to the pasting diagram

- For all A, B objects of \mathcal{C} , the following pasting diagram

$$\begin{array}{c}
(F(A) \otimes I) \otimes F(B) \xrightarrow{a_{F(A), I, F(B)}} F(A) \otimes (I \otimes F(B)) \\
\downarrow (id_{F(A)} \otimes F_I^\otimes) \otimes id_{F(B)} \quad \swarrow a_{id_{F(A)}, F_I^\otimes, id_{F(B)}} \quad \downarrow id_{F(A)} \otimes (F_I^\otimes \otimes id_{F(B)}) \\
F(A \otimes I) \otimes F(B) \xleftarrow{F_{A, I}^\otimes \otimes id_{F(B)}} (F(A) \otimes F(I)) \otimes F(B) \xrightarrow{a_{F(A), F(I), F(B)}} F(A) \otimes (F(I) \otimes F(B)) \\
\downarrow F_{A \otimes I, B}^\otimes \quad \swarrow F_{A, I, B}^a \quad \downarrow id_{F(A)} \otimes F_{I, B}^\otimes \quad \swarrow Id_{id_A} \otimes F_B^l \\
F((A \otimes I) \otimes B) \xrightarrow{F(a_{A, I, B})} F(A \otimes (I \otimes B)) \xleftarrow{F_{A, I \otimes B}^\otimes} F(A) \otimes F(I \otimes B) \\
\downarrow F(r_A \otimes id_B) \quad \swarrow F(\mu_{A, I, B}^{-1}) \quad \downarrow F(id_A \otimes l_B) \quad \swarrow F_{id_A, l_B}^\otimes \quad \downarrow id_{F(A)} \otimes F(l_B) \\
F(A \otimes B) \xleftarrow{F_{A, B}^\otimes} F(A) \otimes F(B)
\end{array}$$

must be equal to the pasting diagram

$$\begin{array}{c}
(F(A) \otimes F(I)) \otimes F(B) \xleftarrow{(id_{F(A)} \otimes F_I^\otimes) \otimes id_{F(B)}} F(A \otimes I) \otimes F(B) \xrightarrow{a_{F(A), I, F(B)}} F(A) \otimes (I \otimes F(B)) \\
\downarrow F_{A, I}^\otimes \otimes id_{F(B)} \quad \swarrow F_A^r \otimes Id_{id_{F(B)}} \quad \swarrow r_{F(A)} \otimes id_{F(B)} \quad \swarrow \mu_{F(A), F(B)}^{-1} \quad \swarrow id_{F(A)} \otimes l_{F(B)} \\
F(A \otimes I) \otimes F(B) \xrightarrow{F(r_A) \otimes id_{F(B)}} F(A) \otimes F(B) \\
\downarrow F_{A \otimes I, B}^\otimes \quad \swarrow F_{r_A, id_B}^\otimes \quad \downarrow F_{A, B}^\otimes \\
F((A \otimes I) \otimes B) \xrightarrow{F(r_A \otimes id_B)} F(A \otimes B)
\end{array}$$

Definition 8. Let \mathcal{C}, \mathcal{D} be two monoidal bicategories and $F, G : \mathcal{C} \rightarrow \mathcal{D}$ two monoidal pseudofunctors. A monoidal pseudonatural transformation $\gamma : F \Rightarrow G$ is a pseudonatural transformation equipped with:

- an invertible 2-cell

$$\gamma_I^\otimes : F_I^\otimes \circ \gamma_I \Rightarrow G_I^\otimes$$

- an invertible modification whose components are of the form :

$$\begin{array}{ccc}
F(A) \otimes F(B) & \xrightarrow{F_{A, B}^\otimes} & F(A \otimes B) \\
\gamma_A \otimes \gamma_B \downarrow & \swarrow \gamma_{A, B}^\otimes & \downarrow \gamma_{A \otimes B} \\
G(A) \otimes G(B) & \xrightarrow{G_{A, B}^\otimes} & G(A \otimes B)
\end{array}$$

such that the following properties are verified :

- For all A object of \mathcal{C} , the following pasting diagram

$$\begin{array}{ccccc}
& & F(A) \otimes I & \xrightarrow{id_{F(A)} \otimes F_I^\otimes} & F(A) \otimes F(I) \\
& \swarrow & \downarrow r_{F(A)} & \xrightarrow{\quad} & \downarrow F_{A, I}^\otimes \\
G(A) \otimes I & \xleftarrow{\gamma_A \otimes id_I} & F(A) & \xleftarrow{F(r_A)} & F(A \otimes I) & \xrightarrow{\gamma_{A, I}^\otimes} & G(A) \otimes G(I) \\
& \searrow r_{G(A)} & \downarrow \gamma_A & \xrightarrow{\gamma_{r_A}} & \downarrow \gamma_{A \otimes I} & \swarrow G_{A, I}^\otimes \\
& & G(A) & \xleftarrow{G(r_A)} & G(A \otimes I) & &
\end{array}$$

is equal to the pasting diagram :

$$\begin{array}{ccccc}
F(A) \otimes I & \xrightarrow{id_{F(A)} \otimes F_I^\otimes} & F(A) \otimes F(I) & & \\
\downarrow \gamma_A \otimes id_I & & \downarrow \gamma_A \otimes id_{F(I)} & \searrow \gamma_A \otimes \gamma_I & \\
G(A) \otimes I & \xrightarrow{id_{G(A)} \otimes F_I^\otimes} & G(A) \otimes F(I) & \xrightarrow{id_{G(A)} \otimes \gamma_I} & G(A) \otimes G(I) \\
\downarrow r_{G(A)} & \searrow & \downarrow Id_{id_{G(A)}} \otimes \gamma_I^\otimes & \searrow & \downarrow G_{A,I}^\otimes \\
& & id_{G(A)} \otimes G_I^\otimes & \xRightarrow{G_A^r} & \\
G(A) & \xleftarrow{G(r_A)} & G(A \otimes I) & &
\end{array}$$

– For all A, B, C objects of \mathcal{C} , the following pasting diagram

$$\begin{array}{ccccc}
(F(A) \otimes F(B)) \otimes F(C) & \xrightarrow{F_{A,B}^\otimes \otimes id_{F(C)}} & F(A \otimes B) \otimes F(C) & \xrightarrow{F_{A \otimes B, C}^\otimes} & F((A \otimes B) \otimes C) \\
\downarrow a_{F(A), F(B), F(C)} & & \swarrow F_{A,B,C}^a & & \downarrow F(a_{A,B,C}) \\
F(A) \otimes (F(B) \otimes F(C)) & \xrightarrow{id_{F(A)} \otimes F_{B,C}^\otimes} & F(A) \otimes F(B \otimes C) & \xrightarrow{F_{A, B \otimes C}^\otimes} & F(A \otimes (B \otimes C)) \\
\downarrow \gamma_A \otimes id_{F(B) \otimes F(C)} & & \downarrow \gamma_A \otimes id_{F(B \otimes C)} & & \downarrow \gamma_A \otimes (a_{A,B,C}) \\
G(A) \otimes (F(B) \otimes F(C)) & \xrightarrow{id_{G(A)} \otimes F_{B,C}^\otimes} & G(A) \otimes F(B \otimes C) & \xrightarrow{\gamma_A \otimes \gamma_{B \otimes C}} & G(A \otimes (B \otimes C)) \\
\downarrow id_{G(A)} \otimes (\gamma_B \otimes \gamma_C) & \swarrow \gamma_{A \otimes B}^\otimes \otimes Id_{id_{G(C)}} & \downarrow id_{G(A)} \otimes \gamma_{B \otimes C} & & \downarrow \gamma_{A \otimes (B \otimes C)} \\
G(A) \otimes (G(B) \otimes G(C)) & \xrightarrow{G_{A, B \otimes C}^\otimes} & G(A) \otimes G(B \otimes C) & \xrightarrow{G_{A \otimes B, C}^\otimes} & G(A \otimes (B \otimes C))
\end{array}$$

is equal to the pasting diagram :

$$\begin{array}{ccccc}
(F(A) \otimes F(B)) \otimes F(C) & \xrightarrow{F_{A,B}^\otimes \otimes id_{F(C)}} & F(A \otimes B) \otimes F(C) & \xrightarrow{F_{A \otimes B, C}^\otimes} & F((A \otimes B) \otimes C) \\
\downarrow a_{F(A), F(B), F(C)} & & \downarrow \gamma_A \otimes id_{F(B \otimes C)} & & \downarrow F(a_{A,B,C}) \\
(G(A) \otimes G(B)) \otimes F(C) & \xrightarrow{G_{A,B}^\otimes \otimes id_{F(C)}} & G(A \otimes B) \otimes F(C) & \xrightarrow{\gamma_{A \otimes B} \otimes \gamma_C} & G((A \otimes B) \otimes C) \\
\downarrow id_{G(A) \otimes G(B)} \otimes \gamma_C & \swarrow \gamma_{A \otimes B}^\otimes \otimes Id_{id_{G(C)}} & \downarrow id_{G(A \otimes B)} \otimes \gamma_C & & \downarrow \gamma_{A \otimes (B \otimes C)} \\
(G(A) \otimes G(B)) \otimes G(C) & \xrightarrow{G_{A,B}^\otimes \otimes id_{G(C)}} & G(A \otimes B) \otimes G(C) & \xrightarrow{G_{A \otimes B, C}^\otimes} & G((A \otimes B) \otimes C) \\
\downarrow a_{G(A), G(B), G(C)} & \swarrow \gamma_{A \otimes B}^\otimes \otimes Id_{id_{G(C)}} & \downarrow id_{G(A \otimes B)} \otimes \gamma_C & & \downarrow \gamma_{A \otimes (B \otimes C)} \\
F(A) \otimes (F(B) \otimes F(C)) & \xrightarrow{\gamma_A \otimes \gamma_{B \otimes C}} & G(A) \otimes (G(B) \otimes G(C)) & \xrightarrow{id_{G(A)} \otimes G_{B,C}^\otimes} & G(A \otimes (B \otimes C))
\end{array}$$

Definition 9. A monoidal modification $m : \gamma \Rightarrow \delta : F \Rightarrow G$ between two monoidal pseudonatural transformations γ and δ is a modification verifying the following property :

$$\begin{array}{c}
\begin{array}{ccc}
I & \xrightarrow{F_I^\otimes} & F(I) \\
& \searrow \delta_I^\otimes & \downarrow \delta_I \\
& & G(I)
\end{array}
\end{array}$$

and, for every object A, B of \mathcal{C} , the following diagram

$$\begin{array}{ccccc}
 F(A) \otimes F(B) & \xrightarrow{F_{A,B}^\otimes} & F(A \otimes B) & & \\
 \delta_A \otimes \delta_B \swarrow & \downarrow \gamma_A \otimes \gamma_B & \swarrow \gamma_{A,B}^\otimes & \downarrow \gamma_{A \otimes B} & \\
 G(A) \otimes G(B) & \xrightarrow{G_{A,B}^\otimes} & G(A \otimes B) & &
 \end{array}$$

is equal to the diagram

$$\begin{array}{ccccc}
 F(A) \otimes F(B) & \xrightarrow{F_{A,B}^\otimes} & F(A \otimes B) & & \\
 \delta_A \otimes \delta_B \downarrow & \swarrow \delta_{A,B}^\otimes & \delta_{A \otimes B} \downarrow & \swarrow \overline{m}_{A \otimes B} & \gamma_{A \otimes B} \\
 G(A) \otimes G(B) & \xrightarrow{G_{A,B}^\otimes} & G(A \otimes B) & &
 \end{array}$$

TODO: Symmetric stuff, a couple of hundred new diagrams

Definition 10. A biadjunction between two pseudo-functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ is given by a pair of pseudo-natural transformations $\eta : Id_{\mathcal{C}} \rightarrow GF$ and $\epsilon : FG \rightarrow Id_{\mathcal{D}}$ along with two invertible modifications with components :

$$\begin{array}{ccc}
 G(D) \xrightarrow{\eta_{G(D)}} G(F(G(D))) & & F(C) \xrightarrow{F(\eta_C)} F(G(F(C))) \\
 \searrow id_{G(D)} \quad \xrightarrow{s_D} \downarrow G(\epsilon_D) & & \searrow id_{F(C)} \quad \xleftarrow{t_C} \downarrow \epsilon_{F(C)} \\
 & G(D) & F(C)
 \end{array}$$

such that the following diagram equalities hold for all objects C of \mathcal{C} and D of \mathcal{D} :

$$\begin{array}{ccc}
 C \xrightarrow{\eta_C} G(F(C)) & & \\
 \downarrow \eta_C & \downarrow G(F(\eta_C)) & \xrightarrow{G(t_C)} id_{G(F(C))} \\
 G(F(C)) \xrightarrow{\eta_{G(F(C))}} G(F(G(F(C)))) & & \\
 \searrow id_{G(F(C))} \quad \xrightarrow{s_{F(C)}} \downarrow G(\epsilon_{F(C)}) & & \\
 & G(F(C)) &
 \end{array}
 = Id_{id_C \circ \eta_C}$$

$$\begin{array}{ccc}
F(G(D)) & \xrightarrow{id_{F(G(D))}} & F(G(D)) \\
\downarrow F(\eta_{G(D)}) & \searrow t_{G(D)} & \downarrow \epsilon_{F(G(D))} \\
F(G(F(G(D)))) & \xrightarrow{\epsilon_{F(G(D))}} & F(G(D)) \\
\downarrow F(s_D) & & \downarrow \epsilon_D \\
F(G(D)) & \xrightarrow{\epsilon_D} & D
\end{array}
\quad \begin{array}{c} \\ \\ \\ \\ \end{array}
\quad \begin{array}{c} \\ \\ \\ \\ \end{array}
\quad Id_{\epsilon_D \circ id_D}$$

$\begin{array}{c} \xrightarrow{id_{F(G(D))}} \\ \xrightarrow{F(\eta_{G(D)})} \\ \xrightarrow{F(s_D)} \\ \xrightarrow{id_{F(G(D))}} \end{array}$

Definition 11. A monoidal bicategory \mathcal{C} is monoidal closed if the pseudo-functor $\otimes B : \mathcal{C} \rightarrow \mathcal{C}$ has a right biadjoint for all objects B of \mathcal{C} .

Definition 12. A bicategory \mathcal{C} is cartesian if the diagonal pseudofunctor $\Delta_n : \mathcal{C} \rightarrow \mathcal{C}^n$ has a right biadjoint.

Definition 13. A cartesian bicategory \mathcal{C} is cartesian closed if the pseudo-functor $\times B : \mathcal{C} \rightarrow \mathcal{C}$ has a right biadjoint for all objects B of \mathcal{C} .

Definition 14. A pseudo-comonoid A in a monoidal bicategory \mathcal{C} is given by an object A of the bicategory, equipped with :

- a 1-cell $J : A \rightarrow I$
- a 1-cell $P : C \rightarrow C \otimes C$
- three invertible 2-cells

$$\begin{array}{ccc}
A \otimes A & \xleftarrow{P} & A \xrightarrow{P} A \otimes A \\
\downarrow P \otimes id_A & \searrow \alpha & \downarrow id_A \otimes P \\
(A \otimes A) \otimes A & \xrightarrow{a_{A,A,A}} & A \otimes (A \otimes A)
\end{array}$$

$$\begin{array}{ccc}
A & \xrightarrow{P} & A \otimes A \\
\searrow l_A^{-1} & \searrow \lambda & \downarrow J \otimes id_A \\
& & I \otimes A
\end{array}$$

$$\begin{array}{ccc}
A & \xrightarrow{P} & A \otimes A \\
\searrow r_A^{-1} & \searrow \rho & \downarrow id_A \otimes J \\
& & A \otimes I
\end{array}$$

such that the following properties are verified : The diagram

$$\begin{array}{ccc}
A \otimes I \otimes A & \xleftarrow{id_A \otimes J \otimes id_A} & A \otimes A \otimes A \\
\uparrow id_A \otimes l_A^{-1} & \searrow id_A \otimes \lambda & \uparrow P \otimes id_A \\
A \otimes A & \xrightarrow{id_A \otimes P} & A \otimes A \\
\uparrow P & \searrow \alpha & \uparrow P \\
A & \xrightarrow{P} & A \otimes A
\end{array}$$

is equal to the diagram

$$\begin{array}{ccc}
 A \otimes I \otimes A & \xleftarrow{id_A \otimes J \otimes id_A} & A \otimes A \otimes A \\
 \uparrow r_A^{-1} & \Rightarrow \rho \otimes id_A & \nearrow P \otimes id_A \\
 A \otimes A & & \\
 \uparrow P & & \\
 A & &
 \end{array}$$

and the following diagram :

must be equal to the diagram :

Definition 15. *pseudo-comonad*

Definition 16. *A linear exponential pseudo-comonad*