1 Some bicategorical definitions

In this section, we recall a few definitions required by our bicategorical setting.

Definition 1. *A bicategory $\mathcal{C}$ consists of :*

- *A collections of objects A, B, C.*
- *For each pair of objects A, B, a category $\mathcal{C}(A, B)$ whose objects are called morphisms or $1-$cells and whose morphisms are called 2-morphisms or $2-$cells.*
- *For each object A, a distinguished $1-$cell $id_A \in \mathcal{C}(A, A)$ called the identity morphism.*
- *For each triple of objects A, B, C a functor*

$$\circ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$$

  called horizontal composition.
- *For each pair of objects A, B, two natural isomorphisms called the left and right unitors:*

$$l : id_A \circ - \Rightarrow - : \mathcal{C}(A, B) \to \mathcal{C}(A, B) \text{ and } r : - \circ id_B \Rightarrow - : \mathcal{C}(A, B) \to \mathcal{C}(A, B)$$

- *For each quadruple of objects A, B, C, D a natural isomorphism called the associator*

$$a : (- \circ -) \circ - \Rightarrow - \circ (- \circ -) : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \times \mathcal{C}(C, D) \to \mathcal{C}(A, D)$$

such that the following diagrams commute for any object A, B, C, D, E of $\mathcal{C}$ and f, g, h, i objects of $\mathcal{C}(A, B)$, $\mathcal{C}(B, C)$, $\mathcal{C}(C, D)$, $\mathcal{C}(D, E)$ respectively :

$$\begin{tikzcd}
((f \circ g) \circ h) \circ i \arrow[rr, "a(f,g,h) \circ Id_i"'] \arrow[d, "a(f \circ g,h,i)"'] & & (f \circ (g \circ h)) \circ i \arrow[d, "a(f,g \circ h,i)"'] \\
(f \circ g) \circ (h \circ i) \arrow[dr, "a(f,g,h \circ i)"'] & & f \circ ((g \circ h) \circ i) \arrow[dl, "Id_f \circ a(g,h,i)"] \\
& f \circ (g \circ (h \circ i)) &
\end{tikzcd}$$

$$\begin{tikzcd}
(f \circ Id_B) \circ g \arrow[rr, "a(f,id_B,g)"'] \arrow[dr, "r(f) \circ Id(g)"'] & & f \circ (Id_B \circ g) \arrow[dl, "Id_f \circ l(g)"] \\
& f \circ g &
\end{tikzcd}$$

Definition 2. *Let $\mathcal{C}, \mathcal{D}$ be two bicategories. A pseudofunctor $F : \mathcal{C} \to \mathcal{D}$ is given by :*

- *For each object A of $\mathcal{C}$, an object $F(A)$ of $\mathcal{D}$.*

– *For each hom-category* $\mathcal{C}(A,B)$ *in* $\mathcal{C}$*, a functor*

$$F(A,B) : \mathcal{C}(A,B) \to \mathcal{D}(F(A), F(B))$$

– *For each object* A *of* $\mathcal{C}$*, an invertible* 2*-cell*

$$F_{id_A} : id_{F(A)} \Rightarrow F(A,B)(id_A)$$

– *For each triple of objects* A, B, C *of* $\mathcal{C}$*, a natural isomorphism* ϕ *whose elements are:*

$$\phi_{f,g} : F(f) \circ F(g) \Rightarrow F(f \circ g)$$

for f, g *objects of* $\mathcal{C}(A,B), \mathcal{C}(B,C)$ *respectively*

such that, for any object $A,\ B,\ C,\ D$ *of* $\mathcal{C}$*, any objects* $f,\ g,\ h$ *of* $\mathcal{C}(A,B),\ \mathcal{C}(B,C),\ \mathcal{C}(C,D)$ *respectively, the following diagrams commute :*

$$\begin{tikzcd}
F(f) \circ (F(g) \circ F(h)) \arrow[d, "Id_{F(f)} \circ \phi_{g,h}"'] & (F(f) \circ F(g)) \circ F(h) \arrow[l, "a(F(f),F(g),F(h))"'] \arrow[d, "\phi_{f,g} \circ Id_{F(h)}"'] \\
F(f) \circ F(g \circ h) \arrow[d, "\phi_{f,g \circ h}"'] & F(f \circ g) \circ F(h) \arrow[d, "\phi_{f \circ g,h}"'] \\
F(f \circ (g \circ h)) & F((f \circ g) \circ h) \arrow[l, "F(a(f,g,h))"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(f) & F(f \circ id_B) \arrow[l, "F(r(f))"'] \\
F(f) \circ id_{F(B)} \arrow[u, "r(F(f))"'] \arrow[r, "Id_{F(f)} \circ F_{id_B}"'] & F(f) \circ F(id_B) \arrow[u, "\phi_{f,id_B}"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(f) & F(id_A \circ f) \arrow[l, "F(l(f))"'] \\
id_{F(A)} \circ F(f) \arrow[u, "l(F(f))"'] \arrow[r, "F_{id_A} \circ Id_{F(f)}"'] & F(id_A) \circ F(f) \arrow[u, "\phi_{id_A,f}"']
\end{tikzcd}$$

Definition 3. *Let* $\mathcal{C}, \mathcal{D}$ *be two bicategories, and* $F, G : \mathcal{C} \to \mathcal{D}$ *two pseudofunctors. A pseudo-natural transformation* $\gamma : F \Rightarrow G$ *is given by :*

– *for every object* A *of* $\mathcal{C}$*, a* $1-$*cell* $\gamma_A : F(A) \to G(A)$
– *for every pair of objects* A, B *of* $\mathcal{C}$ *and every* $1-$*cell* f *of* $\mathcal{C}(A,B)$*, an invertible* $2-$*cell* γ_f*:*

$$\begin{tikzcd}
F(A) \arrow[r, "\gamma_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\Nearrow \gamma_f"] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\gamma_B"'] & G(B)
\end{tikzcd}$$

such that the following properties are verified :

– *Naturality : For every* $2-$*cell* $\tau : f \Rightarrow g : A \to B$*, the* $2-$*cells associated to the following pasting diagrams are equal :*

$$
\begin{tikzcd}
F(A) \arrow[rr, "\gamma_A"] \arrow[d, "F(f)"'] \arrow[drr, phantom, "\Nearrow\gamma_f"] & & G(A) \arrow[d, bend right=40, "G(f)"'] \arrow[d, bend left=40, "G(g)"] \arrow[d, phantom, "\overset{G(\tau)}{\Longrightarrow}"] \\
F(B) \arrow[rr, "\gamma_B"'] & & G(B)
\end{tikzcd}
=
\begin{tikzcd}
F(A) \arrow[rr, "\gamma_A"] \arrow[d, bend right=40, "F(f)"'] \arrow[d, bend left=40, "F(g)"] \arrow[d, phantom, "\overset{F(\tau)}{\Longrightarrow}"] \arrow[drr, phantom, "\Nearrow\gamma_g"] & & G(A) \arrow[d, "G(g)"] \\
F(B) \arrow[rr, "\gamma_B"'] & & G(B)
\end{tikzcd}
$$

– *Unitality :For every object* A *of* $\mathcal{C}$*, the* $2-$*cells associated to the following pasting diagrams are equal :*

$$
\begin{tikzcd}
F(A) \arrow[rr, "\gamma_A"] \arrow[d, "Id_{F(A)}"'] \arrow[drr, phantom, "\equiv"] & & G(A) \arrow[d, bend right=40, "Id_{G(A)}"'] \arrow[d, bend left=40, "G(Id_A)"] \arrow[d, phantom, "\overset{G_A}{\Longrightarrow}"] \\
F(A) \arrow[rr, "\gamma_A"'] & & G(A)
\end{tikzcd}
=
\begin{tikzcd}
F(A) \arrow[rr, "\gamma_A"] \arrow[d, bend right=40, "Id_{F(A)}"'] \arrow[d, bend left=40, "F(Id_A)"] \arrow[d, phantom, "\overset{F_A}{\Longrightarrow}"] \arrow[drr, phantom, "\Nearrow\gamma_{Id_A}"] & & G(A) \arrow[d, "G(Id_A)"] \\
F(A) \arrow[rr, "\gamma_A"'] & & G(A)
\end{tikzcd}
$$

– *Compositionality : for every triple of objects* A, B, C *of* $\mathcal{C}$ *and every pair of* $1-$*cells* f, g *of* $\mathcal{C}(A,B)$*,* $\mathcal{C}(B,C)$ *respectively, the* $2-$ *cells associated to the following pasting diagrams are equal :*

$$
\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[rr, bend right, "F(g\circ f)"'] \arrow[d, "\gamma_A"] & F(B) \arrow[r, "F(g)"] \arrow[d, phantom, "\Downarrow F_{g,f}"] & F(C) \arrow[d, "\gamma_C"] \\
G(A) \arrow[rr, "G(g\circ f)"] & & G(C)
\end{tikzcd}
=
\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[dd, "\gamma_A"] \arrow[d, phantom, "\gamma_f \Swarrow"] & F(B) \arrow[r, "F(g)"] \arrow[d, "\gamma_B"] \arrow[d, phantom, "\gamma_g\Swarrow", shift left=8] & F(C) \arrow[dd, "\gamma_C"] \\
& G(B) \arrow[dr, "G(g)"] \arrow[d, phantom, "\Downarrow G_{g,f}"] & \\
G(A) \arrow[ur, "G(f)"] \arrow[rr, "G(g\circ f)"'] & & G(C)
\end{tikzcd}
$$

Definition 4. *Let* $\gamma, \delta : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ *be two pseudo-natural transformations., a modification* $m : \gamma \Rrightarrow \delta$ *is given by a* $2-$*cell* $m_A : \gamma_A \Rightarrow \delta_A$ *for every* *object A of $\mathcal{C}$ such that for every $f : A \to B$ in $\mathcal{C}$, we have :*

$$\begin{tikzcd}
F(A) \arrow[r, bend left, "\delta_A"{name=U}] \arrow[r, bend right, "\gamma_A"'{name=D}] \arrow[d, "F(f)"] \arrow[dr, phantom, "\nearrow\gamma_f"] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\gamma_B"] & G(B)
\arrow[Rightarrow, from=D, to=U, "m_A"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \arrow[r, "\delta_A"] \arrow[d, "F(f)"] \arrow[dr, phantom, "\nearrow\delta_f"] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, bend left, "\delta_B"{name=U}] \arrow[r, bend right, "\gamma_B"'{name=D}] & G(B)
\arrow[Rightarrow, from=D, to=U, "m_B"]
\end{tikzcd}$$

Definition 5. *Let A, B be two objects in a bicategory $\mathcal{C}$. An equivalence from A to B is given by :*

- *a pair of $1-$cells $f : A \to B$ and $g : B \to A$.*
- *a pair of invertible $2-$cells $e : id_A \Rightarrow g \circ f$ and $e' : id_B \Rightarrow f \circ g$.*

We say that f is an equivalence if such g, e, e' exist.

Definition 6. *A monoidal bicategory $\mathcal{C}$ is a bicategory equipped with :*

- *a unit object I.*
- *a pseudo-functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$*
- *three pseudo-natural transformations a, l, r whose components are equivalences and given by :*

$$a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$$l_A : I \otimes A \to A$$

$$r_A : A \otimes I \to A$$

- *four invertible modifications π, μ, L, R whose components are given by :*

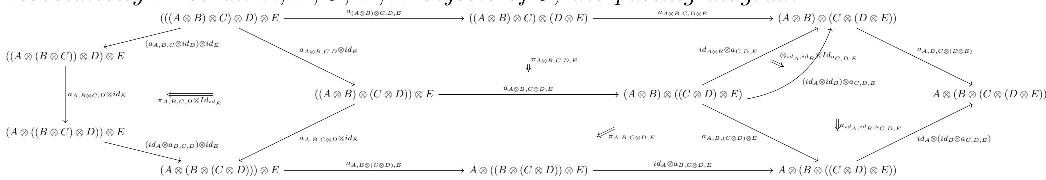
$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "a_{A\otimes B,C,D}"] \arrow[d, "a_{A,B,C}\otimes id_D"] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a_{A,B,C\otimes D}"] \arrow[d, phantom, "\Downarrow\pi_{A,B,C,D}"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "a_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "id_A\otimes a_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes C \arrow[rr, "a_{A,I,C}"] \arrow[dr, "r_A\otimes id_C"'] & \arrow[d, phantom, "\overset{\mu_{A,C}}{\Longrightarrow}"] & A \otimes (I \otimes C) \arrow[dl, "id_A\otimes l_C"] \\
& A \otimes C &
\end{tikzcd}$$

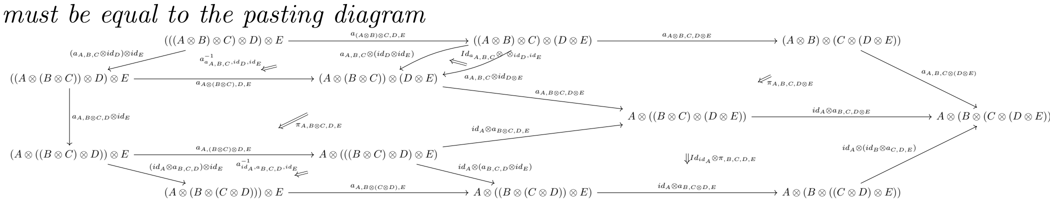
$$\begin{tikzcd}
(I \otimes B) \otimes C \arrow[rr, "a_{I,B,C}"] \arrow[dr, "l_B\otimes id_C"'] & \arrow[d, phantom, "\overset{L_{B,C}}{\Longrightarrow}"] & I \otimes (B \otimes C) \arrow[dl, "l_{B\otimes C}"] \\
& B \otimes C &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B) \otimes I \arrow[rr, "a_{A,B,I}"] \arrow[dr, "r_{A\otimes B}"'] & \arrow[d, phantom, "\overset{R_{A,B}}{\Longrightarrow}"] & A \otimes (B \otimes I) \arrow[dl, "id_A\otimes r_B"] \\
& A \otimes B &
\end{tikzcd}$$

such that the following conditions are verified :

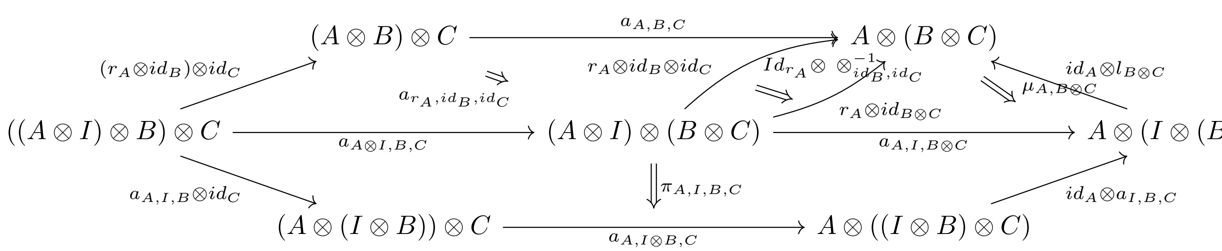
- *Associativity : For all A, B, C, D, E objects of $\mathcal{C}$, the pasting diagram*



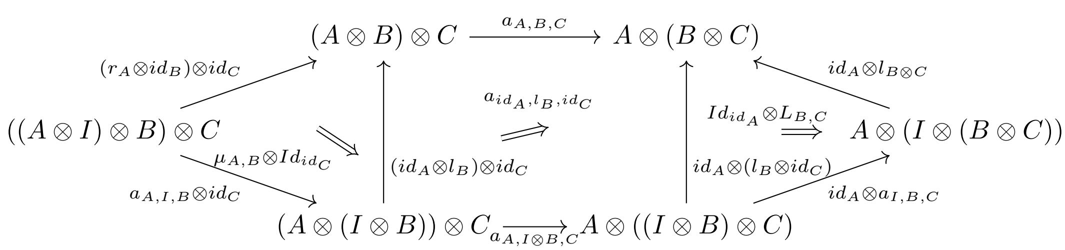
must be equal to the pasting diagram



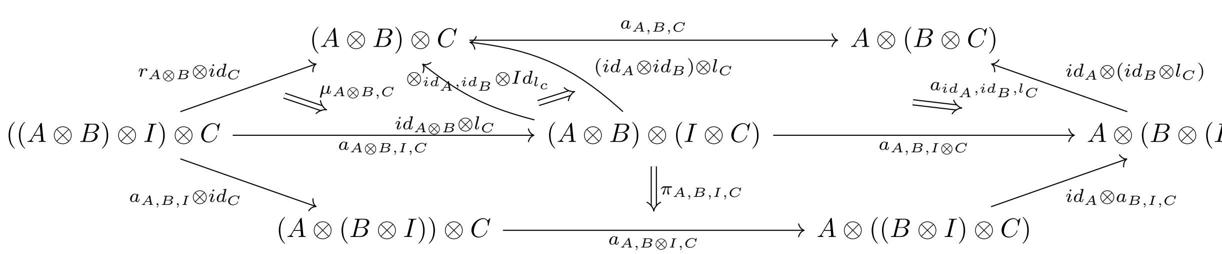
- *For all A, B, C objects of $\mathcal{C}$, the pasting diagram*



must be equal to the pasting diagram



- *For all A, B, C objects of $\mathcal{C}$, the pasting diagram*



must be equal to the pasting diagram

$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[r, "a_{A,B,C}"] & A\otimes(B\otimes C) & \\
((A\otimes B)\otimes I)\otimes C \arrow[ur, "r_{A\otimes B}\otimes id_C"] \arrow[dr, "a_{A,B,I}\otimes id_C"'] \arrow[r, phantom, "\Searrow R_{A,B}\otimes Id_{id_C}"] & \arrow[r, phantom, "\Nearrow a_{id_A,r_B,id_C}"] & \arrow[r, phantom, "\Rightarrow Id_{id_A}\otimes\mu_{B,C}"] & A\otimes(B\otimes(I\otimes C)) \arrow[ul, "id_A\otimes(id_B\otimes l_C)"'] \\
 & (A\otimes(B\otimes I))\otimes C \arrow[uu, "(id_A\otimes r_B)\otimes id_C"'] \arrow[r, "a_{A,B\otimes I,C}"'] & A\otimes((B\otimes I)\otimes C) \arrow[uu, "id_A\otimes(r_B\otimes id_C)"] \arrow[ur, "id_A\otimes a_{B,I,C}"'] &
\end{tikzcd}$$

Definition 7. *Let $\mathcal{C},\mathcal{D}$ be two monoidal bicategories. A monoidal pseudofunctor $F:\mathcal{C}\to\mathcal{D}$ is a pseudofunctor equipped with:*

- *a $1-$cell $F_I^{\otimes}:I\to F(I)$*
- *a pseudo-natural transformation $F^{\otimes}$ whose components are of the form :*

$$F^{\otimes}_{A,B}:F(A)\otimes F(B)\to F(A\otimes B)$$

- *three invertible modifications F^a,F^l,F^r whose components are of the form :*

$$\begin{tikzcd}
(F(A)\otimes F(B))\otimes F(C) \arrow[rr, "a_{F(A),F(B),F(C)}"] \arrow[d, "F^{\otimes}_{A,B}\otimes id_{F(A)}"] & & F(A)\otimes(F(B)\otimes F(C)) \arrow[d, "id_{F(A)}\otimes F^{\otimes}_{B,C}"] \\
F(A\otimes B)\otimes F(C) \arrow[d, "F^{\otimes}_{A\otimes B,C}"] & \Nearrow F^a_{A,B,C} & F(A)\otimes F(B\otimes C) \arrow[d, "F^{\otimes}_{A,B\otimes C}"] \\
F((A\otimes B)\otimes C) \arrow[rr, "F(a_{A,B,C})"] & & F(A\otimes(B\otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
I\otimes F(A) \arrow[r, "F_I^{\otimes}\otimes id_{F(A)}"] \arrow[d, "l_{F(A)}"] \arrow[dr, phantom, "\overset{F^l_A}{\Rightarrow}"] & F(I)\otimes F(A) \arrow[d, "F^{\otimes}_{I,A}"] \\
F(A) & F(I\otimes A) \arrow[l, "F(l_A)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A)\otimes I \arrow[r, "id_{F(A)}\otimes F_i^{\otimes}"] \arrow[d, "r_{F(A)}"] \arrow[dr, phantom, "\Rightarrow F^r_A"] & F(A)\otimes F(I) \arrow[d, "F^{\otimes}_{A,I}"] \\
F(A) & F(A\otimes I) \arrow[l, "F(r_A)"]
\end{tikzcd}$$

such that the following properties are verified :

- *TODO*: <u>*THE DEMONIC DIAGRAM*</u> *For all A,B,C,D objects of $\mathcal{C}$, the following pasting diagram*

must be equal to the pasting diagram

- *For all A, B objects of $\mathcal{C}$, the following pasting diagram*

$$
\begin{tikzcd}
 & (F(A)\otimes I)\otimes F(B) \arrow[r, "a_{F(A),I,F(B)}"] \arrow[d, "(id_{F(A)}\otimes F_I^{\otimes})\otimes id_{F(B)}"'] \arrow[dr, phantom, "\swarrow a_{id_{F(A)},F_I^{\otimes},id_{F(B)}}"] & F(A)\otimes(I\otimes F(B)) \arrow[d, "id_{F(A)}\otimes(F_I^{\otimes}\otimes id_{F(B)})"] \arrow[ddd, bend left, "id_{F(A)}\otimes l_{F(B)}"] \\
F(A\otimes I)\otimes F(B) \arrow[d, "F^{\otimes}_{A\otimes I,B}"'] \arrow[dr, phantom, "\swarrow F^{a\ -1}_{A,I,B}"] & (F(A)\otimes F(I))\otimes F(B) \arrow[l, "F^{\otimes}_{A,I}\otimes id_{F(B)}"'] \arrow[r, "a_{F(A),F(I),F(B)}"'] & F(A)\otimes(F(I)\otimes F(B)) \arrow[d, "id_{F(A)}\otimes F^{\otimes}_{I,B}"] \arrow[d, phantom, shift left=12, "\Leftarrow Id_{id_A}\otimes F^l_B"] \\
F((A\otimes I)\otimes B) \arrow[r, "F(a_{A,I,B})"] \arrow[dr, "F(r_A\otimes id_B)"'] \arrow[dr, phantom, shift left=3, "\Leftarrow F(\mu_{A,I,B})^{-1}"] & F(A\otimes(I\otimes B)) \arrow[d, "F(id_A\otimes l_B)"] \arrow[dr, phantom, "\Leftarrow F^{\otimes}_{id_A,l_B}"] & F(A)\otimes F(I\otimes B) \arrow[l, "F^{\otimes}_{A,I\otimes B}"'] \arrow[d, "id_{F(A)}\otimes F(l_B)"] \\
 & F(A\otimes B) & F(A)\otimes F(B) \arrow[l, "F^{\otimes}_{A,B}"]
\end{tikzcd}
$$

must be equal to the pasting diagram

$$
\begin{tikzcd}
(F(A)\otimes F(I))\otimes F(B) \arrow[d, "F^{\otimes}_{A,I}\otimes id_{F(B)}"'] \arrow[drr, phantom, "\Leftarrow F^r_A\otimes Id_{id_{F(B)}}"] & F(A\otimes I)\otimes F(B) \arrow[l, "(id_{F(A)}\otimes F_I^{\otimes})\otimes id_{F(B)}"'] \arrow[r, "a_{F(A),I,F(B)}"] \arrow[dr, "r_{F(A)}\otimes id_{F(B)}"'] \arrow[dr, phantom, shift left=6, "\Leftarrow \mu^{-1}_{F(A),F(B)}"] & & F(A)\otimes(I\otimes F(B)) \arrow[dl, "id_{F(A)}\otimes l_{F(B)}"] \\
F(A\otimes I)\otimes F(B) \arrow[rr, "F(r_A)\otimes id_{F(B)}"'] \arrow[d, "F^{\otimes}_{A\otimes I,B}"'] \arrow[drr, phantom, "\swarrow F^{\otimes}_{r_A,id_B}"] & & F(A)\otimes F(B) \arrow[d, "F^{\otimes}_{A,B}"] & \\
F((A\otimes I)\otimes B) \arrow[rr, "F(r_A\otimes id_B)"'] & & F(A\otimes B) &
\end{tikzcd}
$$

Definition 8. *Let $\mathcal{C}, \mathcal{D}$ be two monoidal bicategories and $F, G : \mathcal{C} \to \mathcal{D}$ two monoidal pseudofunctors. A monoidal pseudonatural transformation $\gamma : F \Rightarrow G$ is a pseudonatural transformation equipped with:*

- *an invertible2$-$cell*

$$\gamma_I^{\otimes} : F_I^{\otimes} \circ \gamma_I \Rightarrow G_I^{\otimes}$$

- *an invertible modification whose components are of the form :*

$$
\begin{tikzcd}
F(A)\otimes F(B) \arrow[r, "F^{\otimes}_{A,B}"] \arrow[d, "\gamma_A\otimes\gamma_B"'] \arrow[dr, phantom, "\swarrow \gamma^{\otimes}_{A,B}"] & F(A\otimes B) \arrow[d, "\gamma_{A\otimes B}"] \\
G(A)\otimes G(B) \arrow[r, "G^{\otimes}_{A,B}"'] & G(A\otimes B)
\end{tikzcd}
$$

such that the following properties are verified :

- *For all A object of $\mathcal{C}$, the following pasting diagram*

$$
\begin{tikzcd}
 & F(A)\otimes I \arrow[r, "id_{F(A)}\otimes F_I^{\otimes}"] \arrow[dl, "\gamma_A\otimes id_I"] \arrow[d, "r_{F(A)}"] \arrow[dr, phantom, "\Longrightarrow"] & F(A)\otimes F(I) \arrow[d, "F^{\otimes}_{A,I}"] \arrow[dr, "\gamma_A\otimes\gamma_I"] & \\
G(A)\otimes I \arrow[r, phantom, "\overset{r_{\gamma_A}}{\Longrightarrow}"] \arrow[dr, "r_{G(A)}"'] & F(A) \arrow[d, "\gamma_A"] \arrow[dr, phantom, "\nearrow \gamma_{r_A}"] & F(A\otimes I) \arrow[l, "F(r_A)"] \arrow[d, "\gamma_{A\otimes I}"] \arrow[r, phantom, "\overset{\gamma^{\otimes}_{A,I}}{\Longrightarrow}"] & G(A)\otimes G(I) \arrow[dl, "G^{\otimes}_{A,I}"] \\
 & G(A) & G(A\otimes I) \arrow[l, "G(r_A)"] &
\end{tikzcd}
$$

is equal to the pasting diagram :

$$
\begin{tikzcd}
F(A)\otimes I \arrow[r, "id_{F(A)}\otimes F_I^{\otimes}"] \arrow[d, "\gamma_A\otimes id_I"] & F(A)\otimes F(I) \arrow[d, "\gamma_A\otimes id_{F(I)}"] \arrow[dr, "\gamma_A\otimes\gamma_I"] & \\
G(A)\otimes I \arrow[r, "id_{G(A)}\otimes F_I^{\otimes}"] \arrow[d, "r_{G(A)}"] \arrow[rr, bend right, "id_{G(A)}\otimes G_I^{\otimes}"'] & G(A)\otimes F(I) \arrow[r, "id_{G(A)}\otimes\gamma_I"] \arrow[d, phantom, "\Downarrow Id_{id_{G(A)}\otimes\gamma_I^{\otimes}}"] & G(A)\otimes G(I) \arrow[d, "G^{\otimes}_{A,I}"] \\
G(A) & \arrow[r, phantom, "\Longrightarrow G^r_A"] & G(A\otimes I) \arrow[ll, "G(r_A)"]
\end{tikzcd}
$$

– *For all A, B, C objects of $\mathcal{C}$, the following pasting diagram*

$$
\begin{tikzcd}
(F(A)\otimes F(B))\otimes F(C) \arrow[r, "F^{\otimes}_{A,B}\otimes id_{F(C)}"] \arrow[d, "a_{F(A),F(B),F(C)}"] & F(A\otimes B)\otimes F(C) \arrow[r, "F^{\otimes}_{A\otimes B,C}"] \arrow[d, phantom, "\Leftarrow F^a_{A,B,C}"] & F((A\otimes B)\otimes C) \arrow[d, "F(a_{A,B,C})"] \\
F(A)\otimes(F(B)\otimes F(C)) \arrow[r, "id_{F(A)}\otimes F^{\otimes}_{B,C}"] \arrow[d, "\gamma_A\otimes id_{F(B)\otimes F(C)}"] \arrow[dd, bend right=60, "\gamma_A\otimes(\gamma_B\otimes\gamma_C)"'] & F(A)\otimes F(B\otimes C) \arrow[r, "F^{\otimes}_{A,B\otimes C}"] \arrow[d, "\gamma_A\otimes id_{F(B\otimes C)}"] \arrow[dd, bend left=60, "\gamma_A\otimes(\gamma_{B\otimes C})"] \arrow[dr, phantom, "\Swarrow \gamma^{\otimes}_{A,B\otimes C}"] & F(A\otimes(B\otimes C)) \arrow[dd, "\gamma_{A\otimes(B\otimes C)}"] \\
G(A)\otimes(F(B)\otimes F(C)) \arrow[r, "id_{G(A)}\otimes F^{\otimes}_{B,C}"] \arrow[d, "id_{G(A)}\otimes(\gamma_B\otimes\gamma_C)"] \arrow[dr, phantom, "\Leftarrow \gamma^{\otimes}_{A\otimes B}\otimes Id_{id_{G(C)}}"] & G(A)\otimes F(B\otimes C) \arrow[d, "id_{G(A)}\otimes\gamma_{B\otimes C}"] & \\
G(A)\otimes(G(B)\otimes G(C)) \arrow[r, "G^{\otimes}_{A,B\otimes C}"'] & G(A)\otimes G(B\otimes C) \arrow[r, "G^{\otimes}_{A\otimes B,C}"'] & G(A\otimes(B\otimes C))
\end{tikzcd}
$$

is equal to the pasting diagram :

$$
\begin{tikzcd}
& (F(A)\otimes F(B))\otimes F(C) \arrow[r, "F^{\otimes}_{A,B}\otimes id_{F(C)}"] \arrow[d, "(\gamma_A\otimes\gamma_B)\otimes id_{F(C)}"] \arrow[dd, bend right=60, "(\gamma_A\otimes\gamma_B)\otimes\gamma_C"'] \arrow[dddl, bend right, "a_{F(A),F(B),F(C)}"'] \arrow[dr, phantom, "\Leftarrow Id_{id_{F(A)}}\otimes\gamma^{\otimes}_{B,C}"] & F(A\otimes B)\otimes F(C) \arrow[r, "F^{\otimes}_{A\otimes B,C}"] \arrow[d, "\gamma_{A\otimes B}\otimes id_{F(C)}"] \arrow[dd, bend left=60, "\gamma_{A\otimes B}\otimes\gamma_C"] \arrow[dr, phantom, "\Swarrow \gamma^{\otimes}_{A\otimes B,C}"] & F((A\otimes B)\otimes C) \arrow[dd, "\gamma_{(A\otimes B)\otimes C}"] \arrow[dr, "F(a_{A,B,C})"] & \\
& (G(A)\otimes G(B))\otimes F(C) \arrow[r, "G^{\otimes}_{A,B}\otimes id_{F(C)}"'] \arrow[d, "id_{G(A)\otimes G(B)}\otimes\gamma_C"] & G(A\otimes B)\otimes F(C) \arrow[d, "id_{G(A\otimes B)}\otimes\gamma_C"] & \arrow[r, phantom, "\Swarrow \gamma_{\alpha_{A,B,C}}"] & F(A\otimes \arrow[ddl, "\gamma_{A\otimes(B\otimes C)}"] \\
\arrow[r, phantom, "\Swarrow a_{\gamma_A,\gamma_B,\gamma_C}"] & (G(A)\otimes G(B))\otimes G(C) \arrow[r, "G^{\otimes}_{A,B}\otimes id_{G(C)}"] \arrow[d, "a_{G(A),G(B),G(C)}"] & G(A\otimes B)\otimes G(C) \arrow[r, "G^{\otimes}_{A\otimes B,C}"] \arrow[d, phantom, "\Leftarrow G^a_{A,B,C}"] & G((A\otimes B)\otimes C) \arrow[d, "G(a_{A,B,C})"] & \\
F(A)\otimes(F(B)\otimes F(C)) \arrow[r, "\gamma_A\otimes(\gamma_B\otimes\gamma_C)"] & G(A)\otimes(G(B)\otimes G(C)) \arrow[r, "id_{G(A)}\otimes G^{\otimes}_{B,C}"] & G(A)\otimes G(B\otimes C) \arrow[r, "G^{\otimes}_{A,B\otimes C}"] & G(A\otimes(B\otimes C)) &
\end{tikzcd}
$$

Definition 9. *A monoidal modification $m : \gamma \Rrightarrow \delta : F \Rightarrow G$ between two monoidal pseudonatural transformations γ and δ is a modification verifying the following property :*

$$
\begin{tikzcd}
& F(I) \arrow[dd, "\delta_I"'] \arrow[dd, bend left=80, "\gamma_I"] & \\
I \arrow[ur, "F_I^{\otimes}"] \arrow[dr, "G_I^{\otimes}"'] \arrow[r, phantom, "\Swarrow \delta_I^{\otimes}"] & \arrow[r, phantom, "\Leftarrow m_I"] & \\
& G(I) &
\end{tikzcd}
\quad = \quad \gamma_I^{\otimes}
$$

and, for every object A, B of $\mathcal{C}$, the following diagram

$$\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, "F^{\otimes}_{A,B}"] \arrow[d, "\gamma_A \otimes \gamma_B"] \arrow[d, bend right=60, "\delta_A \otimes \delta_B"'] \arrow[d, phantom, shift right=6ex, "\overset{\Leftarrow}{m_A \otimes m_B}"] \arrow[dr, phantom, "\swarrow \gamma^{\otimes}_{A,B}"] & F(A \otimes B) \arrow[d, "\gamma_{A \otimes B}"] \\
G(A) \otimes G(B) \arrow[r, "G^{\otimes}_{A,B}"'] & G(A \otimes B)
\end{tikzcd}$$

is equal to the diagram

$$\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, "F^{\otimes}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"] \arrow[dr, phantom, "\swarrow \delta^{\otimes}_{A,B}"] & F(A \otimes B) \arrow[d, "\delta_{A \otimes B}"'] \arrow[d, bend left=60, "\gamma_{A \otimes B}"] \arrow[d, phantom, shift left=5ex, "\overset{\Leftarrow}{m_{A \otimes B}}"] \\
G(A) \otimes G(B) \arrow[r, "G^{\otimes}_{A,B}"'] & G(A \otimes B)
\end{tikzcd}$$

TODO: Symmetric stuff, a couple of hundred new diagrams

Definition 10. *A biadjunction between two pseudo-functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ is given by a pair of pseudo-natural transformations $\eta : Id_{\mathcal{C}} \to GF$ and $\epsilon : FG \to Id_{\mathcal{D}}$ along with two invertible modifications with components :*

$$\begin{tikzcd}
G(D) \arrow[r, "\eta_{G(D)}"] \arrow[dr, "id_{G(D)}"'] & G(F(G(D))) \arrow[d, "G(\epsilon_D)"] \arrow[dl, phantom, "\overset{s_D}{\Rightarrow}"] \\
& G(D)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F(C) \arrow[r, "F(\eta_C)"] \arrow[dr, "id_{F(C)}"'] & F(G(F(C))) \arrow[d, "\epsilon_{F(C)}"] \arrow[dl, phantom, "\underset{t_C}{\Leftarrow}"] \\
& F(C)
\end{tikzcd}$$

such that the following diagram equalities hold for all objects C of $\mathcal{C}$ and D of $\mathcal{D}$:

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, "\eta_C"] & G(F(C)) \arrow[d, "G(F(\eta_C))"'] \arrow[ddr, bend left, "id_{G(F(C))}"] \arrow[dr, phantom, "\overset{G(t_C)}{\Rightarrow}"] & \\
G(F(C)) \arrow[r, "\eta_{G(F(C))}"] \arrow[drr, "id_{G(F(C))}"'] & G(F(G(F(C)))) \arrow[dr, "G(\epsilon_{F(C)})"] \arrow[d, phantom, "\overset{s_{F(C)}}{\Rightarrow}"] & \\
& & G(F(C))
\end{tikzcd}
\quad = \quad Id_{id_C \circ \eta_C}$$

$$\begin{tikzcd}
F(G(D)) \arrow[dr, "F(\eta_{G(D)})"'] \arrow[drr, "id_{F(G(D))}"] \arrow[ddr, bend right, "id_{F(G(D))}"'] \arrow[r, phantom, "\Rightarrow t_{G(D)}"] & {} & \\
{} \arrow[dr, phantom, "\Rightarrow F(s_D)"] & F(G(F(G(D)))) \arrow[r, "\epsilon_{F(G(D))}"] \arrow[d, "F(G(\epsilon_D))"] & F(G(D)) \arrow[d, "\epsilon_D"] \\
& F(G(D)) \arrow[r, "\epsilon_D"] & D
\end{tikzcd} \quad = \quad Id_{\epsilon_D \circ id_D}$$

Definition 11. *A monoidal bicategory $\mathcal{C}$ is monoidal closed if the pseudo-functor ${}_{\otimes}B : \mathcal{C} \to \mathcal{C}$ has a right biadjoint for all objects B of $\mathcal{C}$.*

Definition 12. *A bicategory $\mathcal{C}$ is cartesian if the diagonal pseudofunctor $\Delta_n : \mathcal{C} \to \mathcal{C}^n$ has a right biadjoint.*

Definition 13. *A cartesian bicategory $\mathcal{C}$ is cartsian closed if the pseudo-functor ${}_{\times}B : \mathcal{C} \to \mathcal{C}$ has a right biadjoint for all objects B of $\mathcal{C}$.*

Definition 14. *A pseudo-comonoid A in a monoidal bicategory $\mathcal{C}$ is given by an object A of the bicategory, equipped with :*

- *a $1-$cell $J : A \to I$*
- *a $1-$cell $P : C \to C \otimes C$*
- *three invertible $2-$ cells*

$$\begin{tikzcd}
A \otimes A \arrow[d, "P \otimes id_A"] & A \arrow[l, "P"'] \arrow[r, "P"] \arrow[d, phantom, "\Rightarrow \alpha"] & A \otimes A \arrow[d, "id_A \otimes P"] \\
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] & & A \otimes (A \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "P"] \arrow[dr, "l_A^{-1}"'] & A \otimes A \arrow[d, "J \otimes id_A"] \arrow[dl, phantom, "\Rightarrow \lambda"] \\
& I \otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "P"] \arrow[dr, "r_A^{-1}"'] & A \otimes A \arrow[d, "id_A \otimes J"] \arrow[dl, phantom, "\Rightarrow \rho"] \\
& A \otimes I
\end{tikzcd}$$

such that the following properties are verified : The diagram

$$\begin{tikzcd}
A \otimes I \otimes A & A \otimes A \otimes A \arrow[l, "id_A \otimes J \otimes id_A"'] \\
A \otimes A \arrow[u, "id_A \otimes l_A^{-1}"] \arrow[ur, "id_A \otimes P"'] \arrow[r, phantom, "\Rightarrow \alpha"] \arrow[u, phantom, shift left=6, "\Rightarrow id_A \otimes \lambda"] & A \otimes A \arrow[u, "P \otimes id_A"'] \\
A \arrow[u, "P"] \arrow[ur, "P"']
\end{tikzcd}$$

is equal to the diagram

$$\begin{tikzcd}
A\otimes I\otimes A & & A\otimes A\otimes A \arrow[ll, "id_A\otimes J\otimes id_A"'] \\
A\otimes A \arrow[u, "r_A^{-1}"] \arrow[urr, "P\otimes id_A"'] \arrow[r, phantom, "\overset{\rho\otimes id_A}{\Longrightarrow}"] & & \\
A \arrow[u, "P"] & &
\end{tikzcd}$$

and the following diagram :

$$\begin{tikzcd}
A\otimes A\otimes A\otimes A & A\otimes A\otimes A \arrow[l, "id_A\otimes id_A\otimes P"'] & A\otimes A \arrow[l, "id_A\otimes P"'] \\
A\otimes A\otimes A \arrow[u, "P\otimes id_A\otimes id_A"] & A\otimes A \arrow[u, "P\otimes id_A"] \arrow[l, "id_A\otimes P"] \arrow[d, phantom, "\Downarrow\alpha"] \arrow[ur, phantom, "\overset{\alpha}{\Swarrow}"] & A \arrow[u, "P"'] \arrow[l, "P"] \arrow[dl, "P"] \\
 & A\otimes A \arrow[ul, "P\otimes id_A"] &
\end{tikzcd}$$

must be equal to the diagram :

$$\begin{tikzcd}
A\otimes A\otimes A\otimes A & A\otimes A\otimes A \arrow[l, "id_A\otimes id_A\otimes P"'] \arrow[d, phantom, "\Downarrow Id_{id_A}\otimes\alpha"] & A\otimes A \arrow[l, "id_A\otimes P"'] \arrow[dl, "id_A\otimes P"] \\
A\otimes A\otimes A \arrow[u, "P\otimes id_A\otimes id_A"] & A\otimes A\otimes A \arrow[ul, "id_A\otimes P\otimes id_A"'] \arrow[l, phantom, "\overset{\alpha\otimes Id_{id_A}}{\Longleftarrow}"] & A \arrow[u, "P"'] \arrow[l, phantom, "\overset{\alpha}{\Swarrow}"] \arrow[dl, "P"] \\
 & A\otimes A \arrow[u, "P\otimes id_A"] \arrow[ul, "P\otimes id_A"] &
\end{tikzcd}$$

Definition 15. *A pseudo-comonad on a bicategory $\mathcal{C}$ is given by a pseudo-functor $F : \mathcal{C} \to \mathcal{C}$, two pseudo-natural transformations $v : F \Rightarrow Id_{\mathcal{C}}$ and $n : F \Rightarrow F \circ F$ called the counit and comultiplications, and three invertible modifications α, λ, ρ whose components are given by the following diagrams :*

$$\begin{tikzcd}
F(A) \arrow[r, "n_A"] \arrow[d, "n_A"] & F(F(A)) \arrow[d, "n_{F(A)}"] \arrow[dl, phantom, "\overset{\alpha_A}{\Swarrow}"] \\
F(F(A)) \arrow[r, "F(n_A)"'] & F(F(F(A)))
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(A) \arrow[dl, "id_{F(A)}"'] \arrow[d, "n"] \arrow[dr, "id_{F(A)}"] & \\
F(A) \arrow[ur, phantom, "\overset{\lambda_A}{\Leftarrow}", near end] & F(F(A)) \arrow[l, "F(v_A)"] \arrow[r, "v_{F(A)}"'] \arrow[ur, phantom, "\overset{\rho_A}{\Leftarrow}", near start] & F(A)
\end{tikzcd}$$

such than the following properties are verified : TODO:

Definition 16. *The Kleisli bicategory $\mathcal{C}_F$ associated to a pseudo-comonad F on a bicategory $\mathcal{C}$ is defined as having the same 0-cells as $\mathcal{C}$, and whose hom-category $\mathcal{C}_F(A, B)$ is given by $\mathcal{C}(F(A), B)$.*
The composition in $\mathcal{C}_F$ of $f : F(A) \to B$ and $g : F(B) \to C$ is defined as

$$g \circ_F f := g \circ f(F) \circ n_A$$

. This definition can easily be extended to provide the required composition functors. The identities in $\mathcal{C}_F$ are given by the components of the counit of the

comonad. The 2*-isomorphisms and additional properties of the bicategory come directly from the pseudo-comonad structure.*

Definition 17. *A linear exponential pseudo-comonad*