Endogenous growth, externalities and social planner

Reminder from tutorials 3, 4, and 5

Clément Montes and Nina Stizi
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Some disclaimers about this reminder. *This recap is provided as an additional support to help you connect the equations covered in the course to graphical representation and economic interpretation. The course and the tutorials remain the only reference. This reminder is not necessary, and by all means not sufficient, to study Macroeconomics 1. Should you have any question or remark, please reach out to* `clement.montes@ensae.fr` *or* `nina.stizi@ensae.fr`

Objectives of the reminder

1. Overview how technological change can be endogenized (Römer's 1986 and 1990).
2. Recall the first welfare theorem.
3. Overview of the costs of having endogenized technological change (Römer's 1986 and 1990).
4. Compare the decentralized allocation with the social optimum (Römer's 1986 and 1990).
5. Define, interpret and analyze the departures from social optimum (Römer's 1986 and 1990; DICE; CKR with a Hotelling resource constraint) and how to solve for it.

1 Endogenizing technological change in Römer's models

The determinant of growth in the exogenous growth models of Solow-Swan and Cass-Koopmans-Ramsey (CKR) is g, the exogenous growth rate of technological progress. Endogenizing technological change brings a possibility to optimize and control the growth rate of the economy. Even if technological progress is not the full story, understanding the dynamics of technological progress brings us closer to understanding the drivers of growth. The two models from Römer show different ways of endogenizing growth:

- **Römer's 1986**: technological progress arises from the growth rate of the **per-capita capital stock** $A_t \equiv \frac{K_t}{L_t}$.
- **Römer's 1990**: technological progress is driven by the growth rate of the **size of the continuum of inventions** N_t, which enters the production process of final goods as differentiated intermediate inputs.

The focus on technological change explains why, in both models, the household side is identical to that of the CKR framework: preferences and intertemporal optimization remain unchanged, while all innovations occur on the production side of the economy.

1.1 Römer 1986: Learning by doing

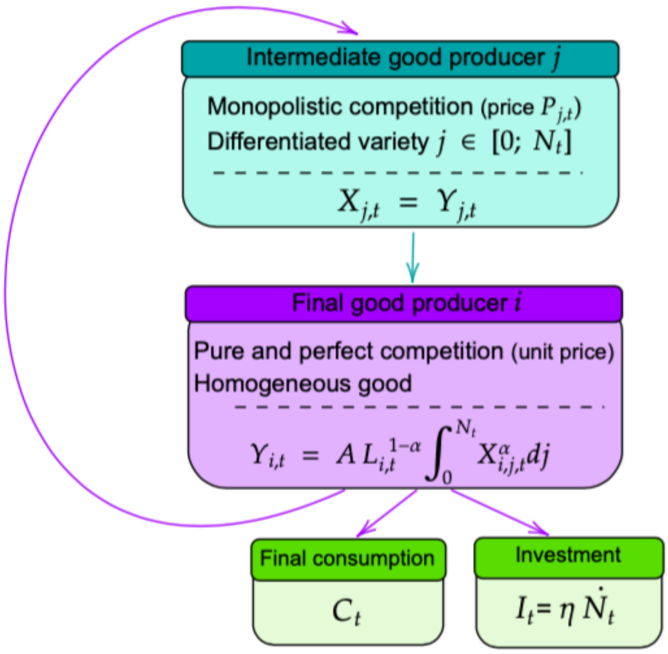
The key idea in **Römer's 1986 model (Tutorial 4)** is that the aggregate capital stock K_t acts as a proxy for accumulated experience. Intuitively, producing goods is not just about owning machines; it is also about knowing how to use them efficiently. Formally, Römer models technological progress as an endogenous stock $A_t \equiv \frac{K_t}{L_t}$.

This formulation captures that **the higher the per-capita capital stock, the more efficient a worker**. Put otherwise, if a worker has access to a larger stock of capital for herself, then she will be more productive. In other words, the repeated use of productive capital—such as machinery, software, or equipment—improves workers' skills and efficiency, allowing them to produce more output within a given period. This mechanism is commonly referred to as **learning by doing**.

1.2 Römer 1990: Inventions and Expanding Varieties

The key idea in Römer's 1990 model (Chapter 5) is that final good Y_t is producing using a continuum of differentiated intermediate goods indexed by $j \in [0, N_t]$. Each intermediate good is produced monopolistically by its inventor, generating profits that provide an incentive to innovate.[1]. In this framework, **technological progress is measured by the size of the set of available inventions**, that is N_t. The larger N_t, the more distinct intermediate inputs are available for production, increasing the economy's productive capacity. Consequently, **the rate of expansion of varieties, that is the growth rate of the size of the varieties set, is the growth rate of technological progress.** Henceforth, economic growth is driven by the expansion of varieties.

Figure 1: Scheme Showing Demand and Supply for Firms in Römer 1990 Model



Notes: The parameter η arises from the assumption of constant investment costs, which ensures a finite number of intermediate-good producers and allows inventors to earn monopoly profits. This assumption makes sense because it allows the intermediate good market to have a finite number of firms, thus allowing inventors to earn profits. Intermediate-good producers operate a linear technology, while final-good producers use a Cobb–Douglas production function. The use of an integral reflects the continuum of intermediate inputs, a standard modeling device in growth theory.

Figure 1 illustrates the production side of the economy. The final-good sector operates under perfect competition and therefore takes prices as given, choosing inputs to meet demand. Demand for the homogeneous final good comes from consumption, investment, and intermediate-good production.[2]. From its production function, which respects the standard neoclassic assumption (see the five usual properties in the Solow-Swan recap), it follows that if N_t increases, the set of available inputs expands, thereby raising final output.

A complementary intuition for why an increase in N_t is socially desirable comes from the expression for aggregate investment[3]. In this model, investment is devoted entirely to the creation of new intermediate varieties, that is, to expanding the stock of available inventions in the economy.

2 First welfare theorem

The First welfare theorem addresses the conditions under which government intervention in markets is unnecessary. It formalizes Adam Smith's idea of the *invisible hand*, according to which decentralized markets can lead to socially efficient outcomes.

[1] This structure can be interpreted as a model of patent protection: once an invention is created, its inventor enjoys monopoly power over its use. In the lecture notes, this monopoly lasts indefinitely. Tutorial 5 relaxes this assumption by introducing a probability π that inventors lose monopoly rights, thereby allowing for finite patent duration.

[2] Although this may appear unintuitive if the final good is interpreted as, for example, cars, it becomes more plausible if the final good is energy: energy is required as an input even in research and innovation activities.

[3] This expression derives from equilibrium in the asset and loan markets. Household assets correspond to the market value of firms (i.e., the benefits of firms V_t) $B_t = \int_0^{N_t} V_t \, dt = \int_0^{N_t} \eta \, dt = \eta N_t$. Bonds and loans markets clearing implies $B_t = K_t$. Since there is no depreciation of intermediate goods, nor final goods in that model $I_t = \dot{K}_t$, which gives the formula for aggregate investment.

First Welfare Theorem

Assume:

1. There is no externality[a]
2. All markets are in pure and perfect competition (i.e. markets are atomistic; products are homogeneous; information is free; entry and exit in the market is free; inputs are freely traded).
3. Markets are complete (i.e. demand and supply are not zero, and there exists a market for every good, service, and contingency).
4. The number of agents in the model is finite.

then the decentralized competitive equilibrium is Pareto efficient. Equivalently, the optimal policy is *laissez-faire*.

[a]Externality: when the actions of one agent directly affect the utility or production of another agent without being mediated through prices. An externality can be positive positive if one "helps the other unknowingly", or negative if it has a negative impact.

This theorem is especially valuable because, in its absence, one would have to explicitly derive both the decentralized equilibrium and the allocation selected by a Benevolent, Omniscient, and Omnipotent Planner (BOOP), and then show that they coincide. The First Welfare Theorem circumvents this exercise by allowing efficiency to be established directly, as long as its underlying assumptions are satisfied.

3 Costs of endogenizing growth in Römer's models

Endogenizing technological progress comes at a cost: the loss of social optimality of the decentralized equilibrium.

3.1 Römer's 1986 model

Studying A_t as being the stock of capital per capita introduces a **positive externality** (due to knowledge spillovers). Therefore, it breaks the first assumption of the first welfare theorem and the decentralized equilibrium is sub-optimal. Technological progress A_t considers the aggregated capital rather than the firm's specific capital stock, and is therefore identical across firms. This captures the idea that **accumulation of knowledge by one firm benefits all other firms**. Because knowledge is considered non-rival and non-excludable, it spills over across firms. This is **diffusion of knowledge** (or **knowledge spillovers**).

Figure 2: Scheme Showing How Individual Decisions Impact Other Firms in Römer 1986 Model

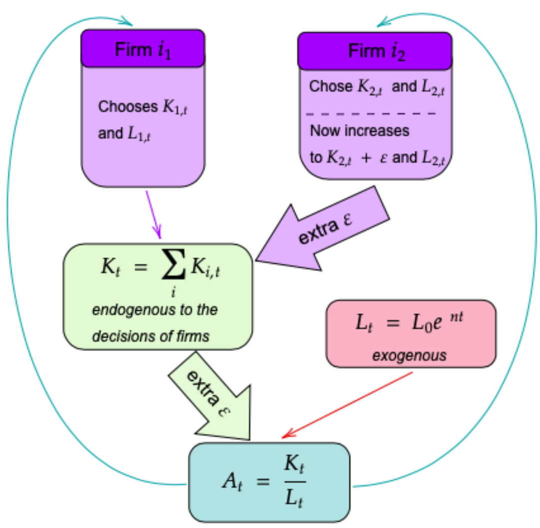


Figure 2 illustrates the diffusion of knowledge in the model. In this example, the economy consists of 2 firms operating in pure and perfect competition. Firms only choose their input to meet demand. They take the productivity A_t as given. If one firm unilaterally increases its capital stock (firm i_2 in the figure), the aggregate capital stock, which is the sum of the capital stocks of every firms will increase as well (by ε as well as the purple arrow shows). Ultimately, since productivity is assumed

to be the stock of capital per-capita, the **productivity, common to both firms**, also increases. As a result, a unilateral increase in capital by firm i_2 positively affects firm i_1, even if i_1 does not alter its own input choices.
Since A_t has been interpreted as *learning by doing* beforehand, we naturally call this externality a knowledge spillover. It is a **positive** externality because an increase (decrease) in $K_{i_2,t}$ raises (lowers) the productivity of firm i_1 the relationship between the two is therefore positive.

As a consequence, the first assumption of the first welfare theorem is violated in that framework and the decentralized equilibrium does not coincide with the social optimum. The deviation from Pareto optimality is driven by all firms in the model $i \in \mathcal{I}$,rather than by any single firm in isolation. This follows the fact that we demonstrated that all firms use the same quantity of inputs $\forall i \in \mathcal{I}$, $\frac{K_{i,t}}{N_{i,t}} = \frac{K_t}{N_t}$.

Römer's 1986 model intuition summary (step-by-step)

Technological progress is the growth rate of an endogenous stock, defined by $A_t \equiv \frac{K_t}{L_t}$ (yielding learning-by-doing), and equal across firms (yielding knowledge spillovers).

1. **Capital accumulation creates experience**: operating machines, organizing production, and solving problems generate tacit knowledge → *Learning-by-doing*
2. **Knowledge diffuses across firms**: through worker mobility, imitation, observation, and informal communication → *Knowledge spillovers (to other firms).*
3. **Firms cannot appropriate all the benefits**: knowledge accumulated by one firm leaks, so no firm captures the full productivity gain from its own learning → *(Positive) externality*
4. **Aggregate productivity increases**: economy-wide efficiency rises as a by-product of decentralized investment decisions. Put differently, learning by doing and knowledge spillovers imply that aggregate experience in production raises productivity economy-wide, even though individual firms take productivity as given. Growth comes from the accumulation of per-capita capital.
5. **The (positive) externality from knowledge accumulation and spillovers creates an inefficiency**: Because of the positive externality of knowledge spillovers, the private return from capital is lower than the social return from capital. This wedge is the source an inefficiency in the decentralized equilibrium. → *Society benefits from knowledge diffusion more than firms which produce the accumulation of knowledge.*

3.2 Römer's 1990 model

An important contribution of this model is its explicit treatment of production as a multi-stage process, distinguishing between intermediate-goods producers (inventors) and final-goods producers. In practice intermediate goods account for roughly two thirds of trade flows[4]. Production processes are thus more complex than the simple choice of an input bundle.

To simplify the model, final-good producers are assumed to produce a homogeneous good under pure and perfect competition. As a result, they optimally choose their input quantities to meet demand. These optimal input choices, in turn, determine the demand faced by inventors. The timing of the inventor in the model is as follows:

1. The inventor computes its expected profit and compares it η, the cost of inventing. If the expected benefit exceeds the cost, the firm chooses to enter the market...
2. ... conditional on entry, the inventor chooses its price and input in each period to maximize profits.

Because invention (and thus market entry) is costly, no firm would enter under perfect competition, where profits are driven to zero. To provide incentives for innovation, inventors are therefore granted monopoly power, which generates a **monopolistic distortion** and leads to the sub-optimality of the decentralized equilibrium. Indeed, the second assumption of the first welfare theorem is violated and thus the best policy is not the *laissez-faire*. Note that this inefficiency does not stem from an externality.

[4]This statistic refers to an open-economy setting and therefore lies outside the scope of the present closed-economy model, but it remains a useful motivation.

Römer's 1990 model intuition summary (step-by-step)

Technological progress is the growth rate of an endogenous stock: the size of the continuum of inventors-producers N_t that each offer a single variety j. The more varieties (i.e., the more inventions), the larger the technological capacity of an economy.

1. **Intermediate good producers intentionally create new ideas**: inventors devote resources (measured in final goods) to discovering new designs (varieties) that describe how to produce new intermediate goods. $\rightarrow$ *Intentional R&D.*
2. **Ideas are non-rival and build on existing knowledge**: once an idea is created, it is used by all final producers at the same time, and existing ideas make it easier to create new ones[a]. $\rightarrow$ *Knowledge accumulation.*
3. **Inventors obtain monopoly power**: each new invention grants its inventor exclusive rights to produce the corresponding intermediate good, allowing the inventor to earn profits. $\rightarrow$ *Monopoly rents as innovation incentives.*
4. **A larger set of inventors raises productivity in final production**: each invention enters the final-good production function as a specialized intermediate input. A larger set of specialized intermediate inputs enables final-good firms to produce more efficiently. $\rightarrow$ *Growth through expanding varieties.*
5. **Innovators do not internalize all knowledge spillovers**: while inventors earn monopoly profits, their ideas also facilitate future innovation by others, an effect they do not take into account. $\rightarrow$ *(Positive) knowledge externality.*
6. **The positive externality creates an inefficiency**: because inventors do not fully internalize the contribution of their ideas to future knowledge, the private return to R&D is lower than the social return, leading to sub-optimality. $\rightarrow$ *Underinvestment in innovation in the decentralized equilibrium.*

[a] $\dot{N}_t$ depends on N_t, so the more inventions are done, the more new inventions are done.

4 Reaching the social optimum

4.1 Comparing allocations

How do we compare allocations? To compare allocations, consider two economies that start from the same initial conditions. If they evolve according to the same dynamic equations, their equilibria will coincide. Since both the BOOP problem and the decentralized equilibrium begin from identical initial conditions, the two equilibria coincide whenever the behavioral equations governing agents are the same. In practice, in this lecture, one needs to compare the equations describing the behavior of the household (Euler equation), describing the production of firms (usually law of motion of capital) and terminal conditions.
If a discrepancy arises, assess whether production in the decentralized equilibrium is higher or lower than in the social optimum. This comparison informs the choice of the appropriate policy instrument. The final step is to re-solve the model with the policy in place in order to determine its optimal value.

This semester we have seen that:

- a **negative externality** imposed by firms typically leads to **overproduction**: firms do not internalize the harm their production imposes on others, so private returns exceed social returns.
- a **positive externality** typically leads to **underproduction**: firms do not internalize the benefits their actions confer on others, so private returns are lower than social returns.

4.2 Römer 1986: Private versus social return to capital

Comparing allocations

Decentralized equilibrium

1. Euler condition: $\frac{\dot{c}_t}{c_t} = \frac{f'(1)-(\delta+\rho)}{\theta}$
2. Law of motion of capital: $\dot{k}_t = [f(1)-(\delta+n)]k_t - c_t$
3. Initial condition: $k_0 = \frac{K_0}{L_0}$
4. Transversality $\lim_{t\to+\infty} k_t e^{-[f'(1)-(n+\delta)]t} = 0$

Social planner allocation

1. Euler condition: $\frac{\dot{c}_t}{c_t} = \frac{f(1)-(\delta+\rho)}{\theta}$
2. Ressource constraint: $\dot{k}_t = [f(1)-(\delta+n)]k_t - c_t$
3. Initial condition: $k_0 = \frac{K_0}{L_0}$
4. Transversality $\lim_{t\to+\infty} k_t e^{-[f(1)-(n+\delta)]t} = 0$

When comparing the two sets of equations, the only difference arises from the term $f'(1)$ in the decentralized equilibrium which is replaced by $f(1)$ in the BOOP's problem.

On the left hand side, $f'(1)$ represents the price of capital paid by firms in their private optimization. It corresponds to the marginal productivity of capital at the firm level - that is, how much an individual firm values an additional unit of capital. On the right hand side $f(1)$ originates from the resource constraint[5]. It captures how much aggregate production is enhanced when the economy accumulates additional capital.

We have shown that $f(1) > f'(1)$. The impact of capital on aggregate production—and thus on social welfare—is therefore valued more highly by society than by individual firms. Put differently, the social return to capital exceeds the private return to capital. This discrepancy arises because firms do not internalize the positive effects of their capital accumulation on the production of other firms. Addressing the inefficiency in the Römer 1986 model therefore requires closing the gap between these two valuations.

4.3 Römer 1990: Private versus social return to inventions

Comparing allocations

Decentralized equilibrium

1. Euler condition: $\frac{\dot{c}_t}{c_t} = \frac{r-\rho}{\theta}$
2. Instant. budget constraint: $\dot{N}_t = \frac{(1+\alpha)r}{\alpha} N_t - \frac{L}{\eta} c_t$
3. Initial condition: N_0 given
4. Transversality $\lim_{t\to+\infty} N_t e^{-rt} = 0$

Social planner allocation

1. Euler condition: $\frac{\dot{c}_t}{c_t} = \frac{r^p-\rho}{\theta}$
2. Technology & ressource: $\dot{N}_t = \frac{(1+\alpha)r^p}{\alpha} N_t - \frac{L}{\eta} c_t$
3. Initial condition: N_0 given
4. Transversality $\lim_{t\to+\infty} N_t e^{-r^p t} = 0$

   with $r^p = \alpha^{\frac{-1}{1-\alpha}} r > r$

The two allocations differ because interest rates are different. The growth rate of the economy (given by the right hand side of the Euler equation) is higher in the BOOP allocation.

As highlighted earlier, the difference is that in the decentralized equilibrium inventions are too few because of monopoly power. Thus production of final goods lacks inputs and the aggregate production is sub-optimal. However, an important distinction is that we can no longer interpret Y_t as the aggregate production, since some of it is used as input by inventors. We will instead get interested in the GDP which is the production that can be used either for investments or for consumption (purple arrows toward green boxed in figure 1). The GDP is now defined as $GDP_t = Y_t - \int_0^{N_t} Y_{j,t} dj = Y_t - X_t = \frac{(1+\alpha)\eta r}{\alpha} N_t$.

To get a sense of whether it is possible to increase the GDP, we are interested in $\frac{\partial \text{GDP}}{\delta X_t}$. It measures the marginal productivity of intermediate inputs net of the marginal cost of intermediate inputs.

- In the decentralized equilibrium: $\frac{\partial \text{GDP}}{\delta X_t} = \frac{1-\alpha}{\alpha} N_t > 0$
  - *Proof*: Final firms are perfectly competitive: they remunerate their inputs at their marginal productivity. To produce, final firm i uses a quantity $X_{i,j,t}$ of intermediate input of variety j. So, marginal product of $X_{i,j,t}$ equals the unit price of $X_{i,j,t}$ which is $p_{j,t}$ (i.e., $\frac{\partial Y_{i,t}}{\partial X_{i,j,t}} = p_{j,t}$). In the decentralized equilibrium, each variety j of intermediate inputs is produced monopolistically at unit price $p_{j,t} = \frac{1}{\alpha}$. All the variety within the continuum of

[5] Recall that there are no prices in the BOOP's program

varieties of size N_t are used by firm i. Hence, aggregating across final firms, we get that the marginal product of intermediate inputs in the final output is $\frac{\partial Y_t}{\partial X_t} = \frac{1}{\alpha} N_t$.

Because the GDP is defined as the final production net of intermediate inputs, the marginal product of intermediate inputs in the GDP is equal to the marginal product of intermediate inputs in the final output (i.e., $\frac{1}{\alpha} N_t$) net of the marginal cost of intermediate inputs (i.e., N_t)[6]. Therefore: $\frac{\partial \text{ GDP}}{\partial X_t} = \frac{1}{\alpha} N_t - N_t = \frac{1-\alpha}{\alpha} N_t$

- In the BOOP allocation: $\frac{\partial \text{GDP}}{\delta X_t} = 0$
  - in the BOOP allocation, everything is as if: $p_{jt} = 1$. Prices do not exist in her program, however intuitively, once she solves for the monopolistic distortion, she corrects for the sub-optimal quantity of inputs and thus the price should adjust to that change. Intuitively it is as if: $\frac{\partial Y_t}{\partial X_t} = N_t$. Since the second term of the GDP is only impacted by the production technology of the intermediate input (which is unchanged), the result follows.

All in all, in the decentralized equilibrium, because of monopolistic competition, the marginal product of inventors is strictly larger than their marginal cost. On the reverse, the two are equals in the BOOP allocation.

5 Departure from social optimum in Römer's model, CKR with non-renewable inputs and DICE

Model	Inefficiency	Interpretation	Associated problem	Solution
Römer 1986 (Tutorial 4)	Knowledge Spillovers	*Diffusion of knowledge*: technological progress is measured by the aggregated capital stock and not the individual firm's capital, to reflect the fact that past accumulated knowledge within a firm not only benefits the individual firm but also all the other firms.	Firms do not internalize that their private return to capital is lower than the social return: they under-invest in capital	Subvention to investment for firms funded by lump-sum tax on the households or Subvention of savings for households funded by complete labor taxe.
Römer 1990 (Tutorial 5)	Monopolistic competition of inventor producers	Inventors are remunerated for their invention by a monopoly situation. Additionally in Tutorial 5, inventors have a probability to fall in pure and perfect competition. This lower their expected benefit, which lowers even more their production.	Inventors underproduce because of their monopoly power and their possible fall of profits. As a consequence, final good producers do not have enough inputs to produce enough for social optimum.	Subvention for inventors to produce more or subvention for final firms to increase demand to inventors. If there are two distortions, the government needs two policy tools.
DICE model (Chapter 3)	Pollution $\Omega_t \equiv \frac{1-b_1\mu_t^{b_2}}{1-\theta_1 T_t^{\theta_2}}$ $\frac{\partial \Omega_t}{\partial T_t} < 0$ economic cost of climate change	When producing, firms emit pollution that will impact other firms negatively through the increase of temperature, that will lower Ω_t for everyone.	Firms do not internalize that they should produce less. They take Ω_t as given in their program, and thus do not consider the impact of their production on technology.	Pigouvian tax implementing the polluter-pays principle

[6] Producing one additional unit of variety j requires solely one unit of final good. So marginal cost of producing one additional unit of $X_{j,t}$ is then the price of one unit of final good, which is 1. Therefore, the marginal cost of the "sum" of intermediate inputs over the continuum of variety of size N_t is $N_t \times 1 = N_t$.

CKR with finite resources (Tutorial 3)	Finite stock of input	Generating growth decreases gradually the stock of non renewable energy. The production is sustainable in the long run but suffering decreasing productivity of factors.	Consumption for next generation will decrease as the production decreases.	Rawlsian planner imposing that consumption is identical for every generations

Recall from Tutorial 5 that the number of policy instruments required must match the number of inefficiencies present in the economy. If there is a single inefficiency (for example, inventors operating under monopoly power), one policy instrument (such as a tax or subsidy) is sufficient to implement the social planner's allocation. However, once multiple inefficiencies arise (for instance, inventors not only hold monopoly power but also anticipate that this power may be temporary and therefore further restrict production), a single policy instrument is no longer sufficient. In such cases, at least two distinct policy instruments are required to replicate the social planner's outcome.