Excess supply function

$$e_z^\sigma(p) = \sum_i \frac{\exp\left(\frac{u_{iz}(p_z)}{\sigma}\right)}{\sum_{z'\in Z_0} \exp\left(\frac{u_{iz'}(p_{z'})}{\sigma}\right)} - \sum_j \frac{\exp\left(\frac{-c_{jz}(p_z)}{\sigma}\right)}{\sum_{z'\in Z_0} \exp\left(\frac{-c_{jz'}(p_{z'})}{\sigma}\right)}$$

where

$$Z_0 = Z \cup \{0\}$$

Properties of e^σ :

Gross substitutes: $e_z^\sigma(p)$ is increasing in p_z and decreasing in p_x $(x \neq z)$. Indeed

$$\begin{aligned} e_z^\sigma(p) &= \sum_i \frac{1}{\sum_{z'\in Z_0} \exp\left(\frac{u_{iz'}(p_{z'})}{\sigma} - \frac{u_{iz}(p_z)}{\sigma}\right)} \\ &\quad - \sum_j \frac{1}{\sum_{z'\in Z_0} \exp\left(\frac{c_{jz}(p_z)}{\sigma} - \frac{c_{jz'}(p_{z'})}{\sigma}\right)} \end{aligned}$$

We want to solve

$$e_z^\sigma(p) = 0 \ \forall z \in Z.$$

Coordinate update algorithms. (Jacobi and Gauss-Seidel).

Idea (Jacobi) = given a guess p_z^t of the solution, update p_z^{t+1} such that

$$e_z^\sigma\left(p_z^{t+1}; p_{-z}^t\right) = 0$$

where p_{-z}^t means all the other entries of p but the zth entry.

Great if you have access to parallel computing. For instance if $|Z| = 3$

$$\begin{aligned} e_1^\sigma\left(p_{z_1}^{t+1}, p_{z_2}^t, p_{z_3}^t\right) &= 0 \\ e_2^\sigma\left(p_{z_1}^t, p_{z_2}^{t+1}, p_{z_3}^t\right) &= 0 \\ e_3^\sigma\left(p_{z_1}^t, p_{z_2}^t, p_{z_3}^{t+1}\right) &= 0 \end{aligned}$$

Serial version = Gauss-Seidel. Assume $|Z| = 3$

$$\begin{aligned} e_1^\sigma\left(p_{z_1}^{t+1}, p_{z_2}^t, p_{z_3}^t\right) &= 0 \\ e_2^\sigma\left(p_{z_1}^{t+1}, p_{z_2}^{t+1}, p_{z_3}^t\right) &= 0 \\ e_3^\sigma\left(p_{z_1}^{t+1}, p_{z_2}^{t+1}, p_{z_3}^{t+1}\right) &= 0 \end{aligned}$$

Reference: James Ortega and Werner Rheinboldt (1970). Iterative Solution of Nonlinear Equations in Several Variables. SIAM.

0.1 Coordinate update function

Define the coordinate update function $cu_z(p_{-z})$ as the solution p'_z to

$$e_z(p'_z, p_{-z}) = 0$$

We can describe Jacobi as
$p_z^{t+1} = cu_z\left(p_{-z}^t\right)$ for each z.

1 Convergence of Jacobi

Jacobi can be written as

$$p_z^{t+1} = cu_z\left(p^t\right)$$

Note: we are looking for p^* such that $e(p^*) = 0$. That is, we are looking for p^* such that $p^* = cu(p^*)$.

Property 1. cu_z is increasing, in the sense that

$$p \leq p' \implies cu(p) \leq cu(p')$$

Indeed,

$$e_z(cu_z(p_{-z}); p_{-z}) = 0$$

take derivative wrt p_x where $x \neq z$ and get

$$\frac{\partial e_z}{\partial p_z} \frac{\partial cu_z}{\partial p_x}(p_{-z}) + \frac{\partial e_z}{\partial p_x} = 0$$

hence

$$\frac{\partial cu_z}{\partial p_x}(p_{-z}) = -\frac{\frac{\partial e_z}{\partial p_x}}{\frac{\partial e_z}{\partial p_z}}$$

Now we know that $\frac{\partial e_z}{\partial p_z} > 0$ and $\frac{\partial e_z}{\partial p_x} \leq 0$ (gross substitutes).

Therefore, the coordinate update is a monotone increasing function.

If p^* exists, and if p^0 is such that $p^0 \leq p^*$ for all z, then

$$p^1 = cu\left(p^0\right) \leq cu(p^*) = p^*$$

thus by induction all the Jacobi sequence p^t will be less that p^*.

Assume that $p^0 \leq p^1$. Then $cu\left(p^0\right) = p^1 \leq cu\left(p^1\right) = p^2$ and by induction p^t will be increasing.

Assuming e_z is continuous in p_z, this implies $p^t \to p^*$.

Theorem (Berry, Gandhi, Haile, weak version). Assuming e_z is increasing in p_z, decreasing in p_x for $x \neq z$, and that $\sum_{z \in \mathcal{Z}} e_z(p_z)$ is increasing in each of the p_x, $x \in \mathcal{Z}$ (law of aggregate supply). The the map $p \to e(p)$ is inverse isotone, meaning that for any two price vectors p and p', then

$$e_z(p) \leq e_z(p') \;\; \forall z \in Z$$

implies $p_z \leq p'_z$ for all $z \in Z$.

Why is this useful?
for our purposes, assume that p is a subsolution, i.e.

$$e_z(p) \leq 0 = e_z(p^*) \;\; \forall z$$

then 1) $p \leq p^*$.
2) $e_z(p) \leq 0 = e_z(cu_z(p); p_{-z})$
therefore $p_z \leq cu_z(p)$.