math+econ+code on equilibrium
virtual whiteboard, day 4

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1 NTU stability in the Gale-Shapley sense

Consider a labor market with fixed wages.
Worker i matched with firm j has utility α_{ij}
Firm j matched with worker i has utility γ_{ij}.
Unassigned individuals get 0.
Important assumption: no ties
ie $\alpha_{ij} \neq \alpha_{ij'}$ for $j \neq j'$ and $\gamma_{ij} \neq \gamma_{i'j}$ for $i \neq i'$
and $\alpha_{ij} \neq 0$ and $\gamma_{ij} \neq 0$

A matching outcome is a vector $\left(\mu_{ij}, u_i, v_j\right)$ where $\mu_{ij} \in \{0, 1\}$
$\mu_{ij} = 1$ iff i and j are matched
$\mu_{i0} = 1$ iff i unmatched
μ_{0j} etc
u_i=payoff of i
v_j= etc

A matching outcome $\left(\mu_{ij}, u_i, v_j\right)$ is stable in the sense of Gale-Shapley iff
(i) population counting
$\sum_j \mu_{ij} + \mu_{i0} = 1$
$\sum_i \mu_{ij} + \mu_{0j} = 1$
(ii) pairwise stability holds
$\max\left(u_i - \alpha_{ij}, v_j - \gamma_{ij}\right) \geq 0$ i.e. $u_i < \alpha_{ij}$ and $v_j < \gamma_{ij}$ do not occur simulataneously, ie ij is not a blocking pair.
(iii) strong complementarity holds:
today (Gale-Shapley): $\mu_{ij} > 0 \implies u_i = \alpha_{ij}, v_j = \gamma_{ij}$

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(iii) weak complementarity holds:

$\mu_{ij} > 0 \implies \max\left(u_i - \alpha_{ij}, v_j - \gamma_{ij}\right) = 0$

GS stability implies GH stability.

2 Deferred acceptance algorithm, Gale-Shapley view

Principle: Workers make offers to firms that have not rejected them yet.

$A^t(i) \subseteq J$ =set of avaible firms to worker i at time t

$P^t(i) \subseteq J$ =set of proposals made by worker i at time t [either a singleton or the empty set]

$K^t(j) \subseteq I$ =set of proposals kept by firm j at the end of round t [either a singleton or the empty set]

$t = 0$ all firms are avaiable to anyone $A^0(i) = J$

Iterate over t:

$P^t(i) = \arg\max_j \{\alpha_{ij}, 0 : j \in A^t(i)\}$, with the understanding that the arg max is empty if 0 is maxizer.

$K^t(j) = \arg\max\left\{\gamma_{ij}, 0 : i \in \left(P^t\right)^{-1}(j)\right\}$ where $\left(P^t\right)^{-1}(j)$ is the set of i such that $j \in P^t(i)$.

$A^{t+1}(i) = A^t(i) \setminus \left\{P^t(i) \setminus \left(K^t\right)^{-1}(i)\right\}$

Repeat until $A^{t+1}(i) = A^t(i)$.

Claim: This converges to a stable matching in the Gale-Shapley sense.

Proof: deferred.

3 Reformulation of Gale-Shapley stability using Adachi's point of view

Adachi's theorem. Consider (u_i, v_j) vectors of payoffs. The following are equivalent:

(i) (μ, u, v) is stable in the Gale-Shapley sense

(ii) One has

$$\begin{aligned} u_i &= \max_j \left\{\alpha_{ij}, 0 : \gamma_{ij} \geq v_j\right\} \\ v_j &= \max_i \left\{\gamma_{ij}, 0 : \alpha_{ij} \geq u_i\right\} \end{aligned}$$

[where $\max_j \left\{\alpha_{ij}, 0 : \gamma_{ij} \geq v_j\right\}$ is a shortcut notation for $\max\left\{\max_j \left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\}, 0\right\}$]

For simplicity, we present the proof when $|I| = |J|$ and all α and γ are positive.

Assume that (μ, u, v) is stable in the Gale-Shapley sense, and let's show (ii).

By contradiction, assume (ii) does not hold, wlog

$$u_i \neq \max_j \left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\}$$

for some i. i is matched with $J(i)$, and $u_i = \alpha_{iJ(i)}$. We know that $v_{J(i)} = \gamma_{iJ(i)}$. As a result

$$\max_j \left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\} \geq \alpha_{iJ(i)} = u_i$$

thus we have $\max_j \left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\} > u_i$. Denote j^* the maximizer of $\max_j \left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\}$, we have

$\alpha_{ij^*} = \max_j \left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\} > u_i$.

We have $\gamma_{ij^*} \geq v_{j^*}$ and $\alpha_{ij^*} > u_i = \alpha_{iJ(i)}$ this is going to imply to $j^* \neq J(i)$ and $\gamma_{ij^*} > v_{j^*}$, therefore i, j^* is ablocking pair. Contradition, thus (ii) in fact holds.

Conversely, let's assume

$$\begin{aligned} u_i &= \max_j \left\{\alpha_{ij}, 0 : \gamma_{ij} \geq v_j\right\} \\ v_j &= \max_i \left\{\gamma_{ij}, 0 : \alpha_{ij} \geq u_i\right\} \end{aligned}$$

Define $\mu_{ij} = 1$ iff $u_i = \alpha_{ij}$. If $u_i = \alpha_{ij}$, then i is in the set of i's such that $\alpha_{ij} \geq u_i$, thus $v_j \geq \gamma_{ij}$.

But $u_i = \alpha_{ij}$, then j must be in the feasible set for i, and hence $\gamma_{ij} \geq v_j$

As a result, $\gamma_{ij} = v_j$.

Let us now show that (μ, u, v) is Gale-Shapley stable. Assume ij is a blocking pair. Then

$\alpha_{ij} > u_i$ and $\gamma_{ij} > v_j$. But $\gamma_{ij} > v_j$ implies $\gamma_{ij} \geq v_j$, thus $u_i \geq \alpha_{ij}$, which is a contradiction.

4 Reformulation of Gale-Shapley stability using m+e+c of view

Question: how to build on Adachi's formulation in order to reformulate Gale-Shapley stability as an equilibrium problem w Gross substitutes?

Change-of-sign trick

Set $p_i = -u_i$ and $p_j = v_j$

$e_i(p) = p_i + \max_j \left\{\alpha_{ij}, 0 : \gamma_{ij} \geq p_j\right\} = p_i - \min_j \left\{-\alpha_{ij}, 0 : \gamma_{ij} \geq p_j\right\}$

$e_j(p) = p_j - \max_i \left\{\gamma_{ij}, 0 : \alpha_{ij} \geq -p_i\right\}$

Write $c_{ij} = -\alpha_{ij}$

$e_i(p) = p_i - \min_j \left\{c_{ij}, 0 : \gamma_{ij} \geq p_j\right\}$

$e_j(p) = p_j - \max_i \left\{\gamma_{ij}, 0 : p_i \geq c_{ij}\right\}$

Consider n which is a generic index for either i or j
We have
$e_n(p)$ is increasing in p_n and weakly decreasing in p_{-n}.

Gale-Shapley stable matchings reformulate as
$e_n(p) = 0$.

4.1 Adachi's algorithm

Adachi's algorithm is nothing else than Gauss-Seidel applied to the above problem.

Start from p_i^0 small enough ie $p_i^0 = \min_j \{c_{ij}, 0\}$ and $p_j^0 = \min_i \left\{\gamma_{ij}, 0\right\}$
$p_i^{t+1} = \min_j \left\{c_{ij}, 0 : \gamma_{ij} \geq p_j^t\right\}$
$p_j^{t+1} = \max_i \left\{\gamma_{ij}, 0 : p_i^{t+1} \geq c_{ij}\right\}$
Actually, can rewrite as:
$p_i^{t+1} = \min_j \left\{c_{ij}, 0 : \gamma_{ij} \geq p_j^t\right\}$
$p_j^{t+1} = \max_i \left\{\gamma_{ij}, 0 : p_i^{t+1} = c_{ij}\right\}$

Property: The set of p's that are solution is a lattice. The algorithm described above picks the smallest point of the lattice – ie the most favorable to workers, the most disfavorable to firms.

Question: if we start from any staring point p^0 and run Adachi, do we converge to a fixed point?

(The anwer would be yes as per D1 with inverse isotonicity, but it's not clear we have it).

5 Aggregate stable matchings

Consider n_x identical workers and m_y identical firms.

What can be a stable matching?

Assume a very simple example with 1 worker and 2 identical firms.

A match between worker and firm is going to generate a utility of 1 both for the worker and the matched firm.

Unmatched workers or firms get 0.

Gale-Shapley stable matching: one of the firms is matched with the worker.

Then one firm will get a utility of one, the unmatched firm will get utility zero.

The worker will get utility one.

Consider the model with utility burning. An equilibrium with money-burning
(i) population counting
$\sum_y \mu_{xy} + \mu_{x0} = n_x$
$\sum_x \mu_{xy} + \mu_{0y} = m_y$

(ii) pairwise stability holds
$\max\left(u_x - \alpha_{xy}, v_y - \gamma_{xy}\right) \geq 0$
$u_x \geq 0$ and $v_y \geq 0$
(iii) weak complementarity holds:
$\mu_{xy} > 0 \implies \max\left(u_x - \alpha_{xy}, v_y - \gamma_{xy}\right) = 0$
$\mu_{x0} \implies u_x = 0$ and $\mu_{0y} > 0 \implies v_y = 0$

THis is part of the theory seen yesterday with distance function
$D_{xy}\left(u_x, v_y\right) = \max\left(u_x - \alpha_{xy}, v_y - \gamma_{xy}\right)$

As a result, if we incoprorate random utilities $\alpha_{xy} + T\varepsilon_{iy}$ for a worker i of type x matched with a firm of type y and $\gamma_{xy} + T\eta_{xj}$ for a firm j of type y matched with a worker of type x, then the equilibrium in this model will be such that

$$\begin{aligned}
M_{xy}\left(\mu_{x0}, \mu_{0y}\right) &= \exp\left(-\frac{1}{T} D_{xy}\left(-T\ln\mu_{x0}, -T\ln\mu_{0y}\right)\right) \\
&= \exp\left(-\frac{1}{T}\max\left(-T\ln\mu_{x0} - \alpha_{xy}, -T\ln\mu_{0y} - \gamma_{xy}\right)\right) \\
&= \min\left\{\begin{array}{c} \exp\left(\ln\mu_{x0} + \frac{\alpha_{xy}}{T}\right) \\ \exp\left(\ln\mu_{0y} + \frac{\gamma_{xy}}{T}\right) \end{array}\right\} \\
&= \min\left\{\mu_{x0}\exp\left(\frac{\alpha_{xy}}{T}\right), \mu_{0y}\exp\left(\frac{\gamma_{xy}}{T}\right)\right\}
\end{aligned}$$

and the μ_{x0} and μ_{0y}'s satisfy

$$\begin{aligned}
\mu_{x0} + \sum_y \min\left\{\mu_{x0}\exp\left(\frac{\alpha_{xy}}{T}\right), \mu_{0y}\exp\left(\frac{\gamma_{xy}}{T}\right)\right\} &= n_x \\
\mu_{0y} + \sum_x \min\left\{\mu_{x0}\exp\left(\frac{\alpha_{xy}}{T}\right), \mu_{0y}\exp\left(\frac{\gamma_{xy}}{T}\right)\right\} &= m_y
\end{aligned}$$

6 DARUM (deferred acceptance for random utility models

Consider an aggregate version of Gale-Shapley.
$\mu_{xy}^{A,t}$ is the number of positions of type y available to workers of type x at time t.
Initially, there is a maximum number of positions y available to workers x.
$\mu_{xy}^{A,0} = \min\left\{n_x, m_y\right\}$

At step t
Proposal phase:

$$
\begin{aligned}
\mu^{P,t} \in \arg\max_{\mu} \quad & \left\{ \sum_{xy} \mu_{xy} \alpha_{xy} \right\} \\
s.t. \quad & \sum_{y} \mu_{xy} \leq n_x \\
& \mu_{xy} \leq \mu_{xy}^{A,t}
\end{aligned}
$$

Disposal phase:

$$
\begin{aligned}
\mu^{K,t} \in \arg\max_{\mu} \quad & \left\{ \sum_{xy} \mu_{xy} \gamma_{xy} \right\} \\
s.t. \quad & \sum_{x} \mu_{xy} \leq m_y \\
& \mu_{xy} \leq \mu_{xy}^{P,t}
\end{aligned}
$$

Adjustment phase:

$$
\mu_{xy}^{A,t+1} = \mu_{xy}^{A,t} - \left(\mu_{xy}^{P,t} - \mu_{xy}^{K,t} \right)
$$

In the proposal and disposal phase, there are Lagrange multipliers $\tau_{xy}^{\alpha} \geq 0$ and $\tau_{xy}^{\gamma} \geq 0$ which are such that the perceived utility of workers and firms are respectively

$$
\alpha_{xy} - \tau_{xy}^{\alpha} \text{ and } \gamma_{xy} - \tau_{xy}^{\gamma}
$$

and we can show that τ_{xy}^{α} is increasing and τ_{xy}^{γ} is decreasing.

Research question: Gale and Shapley with tiebreak at random vs outcome of DARUM.

Research question: Darum for one-to-many.