‘MATH+ECON+CODE’ MASTERCLASS ON EQUILIBRIUM TRANSPORT AND MATCHING MODELS IN ECONOMICS

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Day 1: competitive equilibrium with gross substitutes
Mathematical results

Learning objectives

- Gross substitutes
- Supermodularity
- Z-, P-, M- maps
- Inverse isotonicity and uniqueness of equilibrium
- Coordinate update algorithm

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Section 1

Motivating example

- Consider now the same model as before, but with nonlinearities
- The equilibrium price of good z is p_z. Consumer i's utility if purchasing good z is
$$u_{xz}\left(p_z\right)+\varepsilon_{iz}$$
where $u_{xz}\left(.\right)$ is decreasing and continuous, and ε_{i0} if no good is purchased, where $\left(\varepsilon_{iz}\right)_{z\in\mathcal{Z}_0}\sim$ i.i.d. Gumbel.

- The demand for good z is now

$$d_z(p) = \sum_{x \in \mathcal{X}} n_x \frac{\exp(u_{xz}(p_z))}{1 + \sum_{z' \in \mathcal{Z}} \exp(u_{xz'}(p_{z'}))}$$

- In the partial equilibrium, one takes the supply, i.e. the distribution of the goods $(l_z)_{z \in \mathcal{Z}_0}$ as given, and thus the equilibrium prices are determined by

$$d(p) = l.$$

- Consider promoters $j \in \mathcal{J}$ such that the observable type of j is $y_j \in \mathcal{Y}$ (institutional characteristics). There are m_y producers of type y. The surplus for a producer j of type y to produce quality z is

$$s_{yz}\left(p_z\right) + \eta_{jz}$$

where $s_{yz}\left(.\right)$ is increasing and continuous, and η_{0j} if no good is purchased, where $\left(\eta_{jz}\right)_{z \in \mathcal{Z}_0} \sim$ i.i.d. Gumbel.

- The supply for good z is

$$s_z\left(p\right) = \sum_{y \in \mathcal{Y}} m_y \frac{\exp\left(s_{yz}\left(p_z\right)\right)}{1 + \sum_{z' \in \mathcal{Z}} \exp\left(s_{yz'}\left(p_{z'}\right)\right)}$$

- As before, prices adjust demand and supply, and the equations are thus $d\left(p\right) = s\left(p\right)$, that us
$$e\left(p\right) = 0 \tag{1}$$
where $e : \mathbb{R}^{\mathcal{Z}} \rightarrow \mathbb{R}^{\mathcal{Z}}$ is the excess supply function, defined by $e\left(p\right) = s\left(p\right) - d\left(p\right)$.
- However, the equilibrium problem (1) can no longer be interpreted as an equilibrium problem. Indeed, Je is not symmetric, positive definite–as a matter of fact, it is not even symmetric.
- Yet, note that the map e has an interesting property: $\forall z \neq z'$, $e_z\left(p\right)$ is a nonincreasing function of $p_{z'}$, that is the off-diagonal elements of Je are negative.

Section 2

Gross substitutes

- Instead of using the property – no longer true – that JE is symmetric positive definite, we will therefore use another property: the property that the elements of JE are nonpositive.
- This notion captures the gross substitutes property: increasing the price of alternative z (weakly) increases the demand for the other alternatives, from a *substitution effect*.
  - Example: if the price of meat goes up, the demand for fish increases (fish and meat are substitutes).
  - Counterexample: if the price of gasoline goes up, the demand for automobile decreases (automobiles and gasoline are complements).

- Consider an increase of the prices of all alternatives BUT the default one. Then the only thing that can be said about the excess supplies is that the excess supply of the zero alternative has decreased. In particular, there is no reason that $p_z \leq p'_z \forall z \in \mathcal{Z}$ should imply $e_z(p) \leq e_z(p') \, \forall z \in \mathcal{Z}$. Indeed, there may be non-trivial substitution effects between the nonzero alternatives.
- However, we shall see that the converse holds true: if $e_z(p) \leq e_z(p') \, \forall z \in \mathcal{Z}$ then $p_z \leq p'_z \forall z \in \mathcal{Z}$. This fundamental result appeared in the economics literature in a paper by Berry, Gandhi and Haile (2013), with antecendents in the numerical analysis literature, see theorem 4.6 in More (1972). It will imply uniqueness of equilibrium prices, as well as a constructive method to determine those.

- Hedonic model example. Compute the Jacobian of d. Its off-diagonal term is given by $z \neq t$

$$\frac{\partial d_z(p)}{\partial p_t} = -\sum_{x \in \mathcal{X}} n_x \frac{\exp(u_{xz}(p_z)) \exp(u_{xt}(p_t))}{\left(1 + \sum_{z' \in \mathcal{Z}} \exp(u_{xz'}(p_{z'}))\right)^2} u'_{xt}(p_t)$$

  which is not symmetric unless $u'_{xz}(.)$ is a constant for all x and z.
- Therefore, there is no hope to view equilibrium equations as optimality conditions.

Let A be a $n \times n$ matrix.

- **Definition**. A is called a Z-matrix if its off-diagonal elements are nonpositive, that is if $\forall i \neq j, A_{ij} \leq 0$.
- If $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable, then f is submodular if and only if $D^2 f$ is a Z-matrix.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map.

- **Definition**. f is called a Z-map if $f_i(x)$ nonincreasing with respect to x_i.
- If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, then f is a Z-map if and only if Df is a Z-matrix.

Section 3

Uniqueness of equilibrium prices

Theorem (Berry, Gandhi and Haile, 2013). Consider $e : \mathbb{R}^{\mathcal{Z}} \rightarrow \mathbb{R}^{\mathcal{Z}}$ and define

$$e_0(p) = -\sum_{z \in \mathcal{Z}} e_z(p)$$

and denote$\mathcal{Z}_0 = \mathcal{Z} \cup \{0\}$. Assume that:
(i) the weak gross substitutes condition holds:

$$\forall z \in \mathcal{Z}_0, z' \in \mathcal{Z} \setminus \{z\}, \ e_z(p) \text{ is nonincreasing in } p_{z'}$$

(where $e_0(p) = -\sum_{z \in \mathcal{Z}} e_z(p)$), and
(ii) the connected strong substitutes condition holds: $\forall z \in \mathcal{Z}_0$, there is a sequence $z_1 = z, ..., z_K = 0$ such that

$$\forall k \in \{1, ..., K-1\}, \ e_{z_{k+1}}(p) \text{ is strictly decreasing in } p_{z_k}.$$

Then e is inverse isotone: $e(p) \leq e(p')$ implies $p \leq p'$.

Corollary. Under the same assumptions, e is injective: $e(p) = e(p')$ implies $p = p'$.

Proof. Assume $e_z(p) \leq e_z(p')$.
(a) One has under assumption (i) that for $z \in z \in \mathcal{Z}_0^{\leq}$,
$e_z\left(p_{\mathcal{Z}_0^{\leq}}, p_{\mathcal{Z}_0^{>}}\right) \leq e_z\left(p_{\mathcal{Z}_0^{\leq}}, p'_{\mathcal{Z}_0^{>}}\right)$, hence

$$\forall z \in \mathcal{Z}_0^{\leq},\ e_z(p) \leq e_z\left(p \wedge p'\right) \tag{2}$$

(b) Next, we have under assumption (i) and (ii) that for
$z \in \mathcal{Z}_0^{>},\ e_z\left(p'_{\mathcal{Z}_0^{\leq}}, p'_{\mathcal{Z}_0^{>}}\right) \leq e_z\left(p_{\mathcal{Z}_0^{\leq}}, p'_{\mathcal{Z}_0^{>}}\right)$, with *at least one strict equality* if $\mathcal{Z}_0^{>}$ is nonempty. Thus

$$\forall z \in \mathcal{Z}_0^{>},\ e_z(p) \leq e_z\left(p'\right) \leq e_z\left(p \wedge p'\right) \tag{3}$$

with *at least one strict equality* if $\mathcal{Z}_0^{>}$ is nonempty.
(c) Summing (2) and (3) yields

$$\mathcal{Z}_0^{>} \neq \emptyset \implies 0 = \sum_{z \in \mathcal{Z}_0} e_z(p) < \sum_{z \in \mathcal{Z}_0} e_z\left(p \wedge p'\right) = 0$$

which implies that $\mathcal{Z}_0^{>} = \emptyset$. Hence

$$p_z \leq p'_z\ \forall z \in \mathcal{Z}_0,\ QED.$$

- **Definition**. A is called a P-matrix if for all $x \neq 0$, $\max_{i \in \{1,...,n\}} \sum_{j=1}^{n} A_{ij} x_i x_j > 0$.
  - Note that if one replaces the outer maximization by a sum, then one would get the definition of a positive definite matrix.
  - A positive definite matrix is a P-matrix.
- Alternatively, A is a P-matrix if for all $y = Ax$ and $y' = Ax'$ such that $x \neq x'$, we have

$$\max_{i \in \{1,...,d\}} \left\{ \left(x_i' - x_i \right) \left(y_i - y_i \right) \right\} > 0$$

  which means that A cannot revert all the elementwise inequalities between x and x'.
  - Interpretation: cannot have prices in the reverse order as excess supplies across two markets.

- $f : R^n \rightarrow R^n$ is a P-map if for all $y = f(x)$ and $y' = f(x')$ such that $x \neq x'$, we have

$$\max_{i \in \{1,\dots,d\}} \left\{ \left(x_i' - x_i \right) \left(y_i - y_i \right) \right\} > 0.$$

- A linear map $f(x) = Ax$ is a P-map if and only if A is a P-matrix.

- **Definition**. A is called an M-matrix if it is both a Z- and a P-matrix.
- **Definition**. A is called a Stieltjes matrix if it is both a Z-matrix and a symmetric positive definite matrix.
  - A Stieltjes matrix is an M-matrix.

- **Definition**. f is called an M-matrix if it is both a Z- and a P-map.
- A linear map $f(x) = Ax$ is an m-map if and only if A is a M-matrix.

- **Theorem (Moré-Rheinboldt)**. Let f be an Z-map. Then f is a P-map if and only if f is inverse isotone, i.e. $f\left(x\right) \leq f\left(x'\right)$ implies $x \leq x'$.
- Note that this theorem implies in particular that if A is a Z-matrix, then it is a P-map if and only if it is invertible and its inverse has nonnegative entries.

Proof. (a) Proof of the direct implication. Assume f is an M-map. Then f it is both a Z-map and a P-map. Let $x \neq x'$ be two vectors in $\mathbb{R}^n$ such that $f(x) \leq f(x')$; we would like to show that $x \leq x'$. To do this, let $I = \{1, ..., n\}$, $I^{>} = \{i \in I : x_i > x_i'\}$ and assume by contradiction that $I^{>}$ is nonempty. One has $x \leq x \vee x'$ and for $i \in I^{>}$, one has $x_i = (x \vee x')_i$. Because f is off-diagonal antitone, one has that for $i \in I^{>}$, $f_i(x) = f(x_{i^>}, x_{I^{\leq}}) \geq f\left(x_{I^>}, x'_{I^{\leq}}\right) = f_i(x \vee x')$, thus

$$f_i\left(x'\right) \geq f_i(x) \geq f_i\left(x \vee x'\right) \ \forall i \in I^{>}.$$

If $x_i' < (x \vee x')_i$, then $i \in I^{>}$, and thus $f_i(x') \leq f_i(x \vee x')$. Hence

$$\left(x_i' - \left(x \vee x'\right)_i\right)\left(f_i\left(x'\right) - f_i\left(x \vee x'\right)\right) \leq 0 \ \forall i \in I,$$

which contradicts the fact that f is a P-function.

(b) Proof of the reverse implication. Assume that f is both a Z-function and inverse isotone. By contradiction, assume that it is not a P-function, so that there are two vectors $x \neq x'$ in $\mathbb{R}^n$ such that

$$\left(x_i - x_i'\right)\left(f_i\left(x\right) - f_i\left(x'\right)\right) \leq 0 \ \forall i \in I. \tag{4}$$

Let $I^{\geq} = \{i : x_i \geq x_i'\}$, and $I^{<} = \{i : x_i < x_i'\}$. One has $i \in I^{\geq}$ implies $x_i = \left(x \vee x'\right)_i$, thus because f is a Z-function,

$$\forall i \in I^{\geq} : f_i\left(x\right) = f\left(x_{B^{\geq}}, x_{B^{<}}\right) \geq f\left(x_{B^{\geq}}, x'_{B^{<}}\right) = f_i\left(x \vee x'\right).$$

Similarly, $f_i\left(x'\right) \geq f_i\left(x \vee x'\right)$ holds for all $i \in I^{<}$, but $i \in I^{<}$ implies $x_i < x_i'$, hence by inequality (4), $f_i\left(x\right) - f_i\left(x'\right) \geq 0$, and it follows that

$$\forall i \in I^{<} : f_i\left(x\right) \geq f_i\left(x \vee x'\right),$$

therefore $f\left(x\right) \geq f\left(x \vee x'\right)$. By the inverse isotonicity of f, this implies $x \geq x \vee x'$, thus $x \geq x'$. As x and x play symmetric roles, exchanging them in the proof yields $x' \geq x$, and hence $x = x$, a contradiction. Hence f is a P-function.

Section 4

Coordinate update algorithm

Coordinate Updates (CU) algorithm

- Instead of gradient descent, the natural algorithm is coordinate updates a.k.a. Jacobi algorithm – see e.g. Berry (1994):
- Algorithm
  - At step 0. Take an initial price vector $\left(p_z^0\right)$.
  - At step k, set p_z^{t+1} such that

$$e_z\left(p_z^{t+1}, p_{-z}^t\right) = 0$$

  - Repeat until p^{t+1} is close enough to p^t.
- Note that when $e\left(p\right) = \nabla W\left(p\right)$, this is *simply coordinate descent*, i.e. set p_z^{t+1} to solve

$$\min_{p_z} W\left(p_z, p_{-z}^t\right) = 0.$$

  However, beyond this case we are not descending, hence the expression *coordinate update*.

- Contrast Jacobi with another popular coordinate update method, Gauss-Seidel iterations. Recall that in Jacobi, at step t, set p^{t+1} such that

$$\begin{cases} e_{z_1}\left(p_{z_1}^{t+1}, p_{z_2}^{t}, p_{z_3}^{t} ..., p_{z_d}^{t}\right) = 0 \\ e_{z_2}\left(p_{z_1}^{t}, p_{z_2}^{t+1}, p_{z_3}^{t}, ..., p_{z_d}^{t}\right) = 0 \\ e_{z_3}\left(p_{z_1}^{t}, p_{z_2}^{t}, p_{z_3}^{t+1}, ..., p_{z_d}^{t}\right) = 0 \\ ... \\ e_{z_d}\left(p_{z_1}^{t}, p_{z_2}^{t}, p_{z_3}^{t}, ..., p_{z_d}^{t+1}\right) = 0 \end{cases}$$

- In contrast, in the Gauss-Seidel algorithm: at step t, one sets p^{t+1} such that

$$\begin{cases} e_{z_1}\left(p_{z_1}^{t+1}, p_{z_2}^{t}, p_{z_3}^{t} ..., p_{z_d}^{t}\right) = 0 \\ e_{z_2}\left(p_{z_1}^{t+1}, p_{z_2}^{t+1}, p_{z_3}^{t}, ..., p_{z_d}^{t}\right) = 0 \\ e_{z_3}\left(p_{z_1}^{t+1}, p_{z_2}^{t+1}, p_{z_3}^{t+1}, ..., p_{z_d}^{t}\right) = 0 \\ ... \\ e_{z_d}\left(p_{z_1}^{t+1}, p_{z_2}^{t+1}, p_{z_3}^{t+1}, ..., p_{z_d}^{t+1}\right) = 0 \end{cases}$$

- Advantage of Gauss-Seidel: uses all the information available up to time t. Advantage of Jacobi: can be implemented in parallel (see Keith's upcoming talk).

- Each core is in charge clearing the market for one good (or a group of goods). Thus at time t, one pushes the price vector p^t to all the cores, and core j has to solve the scalar equation in $p_{z_j}^{t+1}$

$$e_{z_j}\left(p_{z_1}^{t+1}, p_{z_2}^{t}, p_{z_3}^{t}, ..., p_{z_j}^{t+1}, ..., p_{z_d}^{t}\right) = 0.$$

- In synchronous parallelization, at each step we need to wait for each of the cores to return their prices. If one core is slower for some reason, this can considerably slow down the whole system.
- Therefore, one can consider asynchronous parallelization where cores return their prices once they are done, and are updated with the current set of prices without waiting for the other cores to be finished. Cf. afternoon talk on asynchronous parallelization.

Theorem (Convergence of CU, part 1). Let e be a continuous map such that (i) $p_z \to e_z(p)$ is increasing, and (ii) $p_{z'} \to e_z(p)$ is weakly decreasing, and that (iii) 0 is in the range of $e_z(., p_{-z})$ for any vector p_{-z}.
Consider p_z^0 such that $e_z\left(p^0\right) \geq 0$ for all $z \in \mathcal{Z}$, and compute p_z^t using the CU algorithm. Then for each z, the sequence p_z^t is decreasing. Further, if each p_z^t remains bounded below, then p^t converges to a p^* solution of $e(p^*) = 0$.
Proof. Show by induction that $e_z\left(p_z^t, p_{-z}^t\right) \geq 0$ and $p_{-z}^t \geq p_{-z}^{t+1}$.
Because $p_{-z}^t \geq p_{-z}^{t+1}$, one has by assumption (ii)
$0 = e_z\left(p_z^{t+1}, p_{-z}^t\right) \leq e_z\left(p_z^{t+1}, p_{-z}^{t+1}\right)$. Further, one has by assumption (iii) that there exists p_z^{t+2} such that $e_z\left(p_z^{t+2}, p_{-z}^{t+1}\right) = 0$. By assumption (i), this implies that $p_z^{t+2} \leq p_z^{t+1}$, which shows that the induction hypothesis extends.
We have shown that the sequence is decreasing. If it converges, then let us call p^* its limit; we have $\lim e_z\left(p_z^{t+1}, p_{-z}^t\right) = e_z(p^*) = 0$.

Theorem (Convergence of CU, part 2). Let e be a continuous map such that (i) $p_z \to e_z(p)$ is increasing, and (ii) $p_{z'} \to e_z(p)$ is weakly decreasing, and that (iii) 0 is in the range of $e_z(., p_{-z})$ for any vector p_{-z}. Assume further that (iv) $p \to e(p)$ is inverse isotone.
(a) Then there exists p^* such that $e(p^*) = 0$ if and only if there exist p' and p'' such that $e(p') \geq 0 \geq e(p'')$.
(b) Assume further that e is surjective. Then the CU algorithm converges no matter what the starting point p^0 is.
Proof. (a) The direct implication is obvious; simply take $p' = p'' = p$. For the converse, note that the CU algorithm starting at $p^0 = p'$ satisfies $e(p^t) \geq 0$ for all t by the virtues of the previous theorem. By inverse isotonicity, this implies $p^t \geq p''$, hence p^t converges toward the solution p^* of $e(p^*) = 0$.
(b) Let $\underline{p} = e^{-1}\left(e\left(p^0\right) \wedge 0\right)$ and $\overline{p} = e^{-1}\left(e\left(p^0\right) \vee 0\right)$, and let $\underline{p}^t$ and $\overline{p}^t$ be the two CU sequences starting from $\underline{p}$ and $\overline{p}$, respectively. By inverse isotonicity of e, one has $\underline{p} \leq p^* \leq \overline{p}$, so $\underline{p}^t$ and $\overline{p}^t$ converge monotonically to U^*. Further, still by inverse isotonicity of e, one has $\underline{p} \leq p^0 \leq \overline{p}$ thus $\underline{p}^t \leq p^t \leq \overline{p}^t$, so $p^t \to p^*$.

Section 5

Appendix A: Gross substitutes in noncooperative games

- $\mathcal{Z}$ is the set of players. Player z plays p_z. Player z's cost is $C_z(p)$, which is assumed to be strictly submodular in p and differentiable and strictly convex in p_z. (Note: the literature usually described the game in terms of profit functions, which therefore need to be supermodular and concave instead).
- A Nash equilibrium p^* is such that $p_z^* = \arg\min_{p_z} C_z\left(p_z, p_{-z}^*\right)$.
- Setting $e_z(p) = \partial C_z(p)/\partial p_z$. Note that $e_z(p)$ is increasing in p_z and decreasing in p_{-z}. By first order conditions, an equilibrium price satisfies

$$e(p) = 0$$

which is exactly a general equilibrium problem with gross substitutes.

Section 6

Appendix B: Gross substitutes in general equilibrium

- Cf. MGW 17.B.2. Consider an exchange economy where $\mathcal{Z}_0 = \mathcal{Z} \cup \{0\}$ is the set of goods. The excess demand of a consumer j is $\omega_k - X_k\left(p, \omega_k^\intercal p\right)$, where X_k is the Marshallian (aka Walrasian) demand function of consumer k, and ω_k is consumer k's endowment. Excess supply is then given by

$$Z\left(p\right) = \sum_k \left(\omega_k - X_k\left(p, \omega_k^\intercal p\right)\right).$$

- Typical properties of Z are as follows:
  - Positive homogeneity of degree 0: $Z\left(\lambda p\right) = Z\left(p\right)$ for $\lambda \geq 0$.
  - Walras' law: $p^\intercal Z\left(p\right) = 0$.
- The Gross Substitutes property can additionally hold: $p_z \rightarrow Z_y\left(p_z, p_{-z}\right)$ is decreasing for $y \neq z$. See MWG, sect. 17.F, where there are examples of individual utility functions that yield such a property.

- How can we map with our setting? Define $e_z\left(p\right) = p_z Z_z\left(p\right)$.
- One sees that the function $e_z\left(p\right)$ verifies

$$\frac{\partial e_z\left(p\right)}{\partial p_y} < 0 \text{ for } y \neq z.$$

  Therefore we may view the condition $\sum_{z \in \mathcal{Z}_0} e_z\left(p\right)$ as Walras' law.
- Further, one may normalize $p_0 = 1$ in order to take care of the positive homogeneity.