

math+econ+code on equilibrium virtual whiteboard, day 2

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$$\begin{aligned} u_x &= \max_y \{ \alpha_{xy} + w_{xy}, 0 \} \\ v_y &= \max_x \{ \gamma_{xy} - w_{xy}, 0 \} \end{aligned}$$

Assume we have determined (u, v) .

$u_x \geq \alpha_{xy} + w_{xy}$ hence $w_{xy} \leq u_x - \alpha_{xy}$

$v_y \geq \gamma_{xy} - w_{xy}$ hence $w_{xy} \geq \gamma_{xy} - v_y$

therefore any w such that

$$\mathbb{w}_{xy} := \gamma_{xy} - v_y \leq w_{xy} \leq \bar{w}_{xy} := u_x - \alpha_{xy}$$

Indeed, if $\mu_{xy} > 0$, then $\bar{w}_{xy} = \mathbb{w}_{xy}$

if $\mu_{xy} = 0$ then in general $\bar{w}_{xy} \geq \mathbb{w}_{xy}$.

1 The Choo-Siow model as an optimization problem

Choo-Siow model

$$\begin{aligned} n_x &= f_x(a, b) := \exp\left(-\frac{a_x}{T}\right) + \sum_y \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) \\ m_y &= f_y(a, b) := \exp\left(-\frac{b_y}{T}\right) + \sum_x \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) \end{aligned}$$

Write $M_{x0} = \exp\left(-\frac{a_x}{T}\right)$ and $M_{xy} = \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right)$

Write down the Jacobian of this system

$$\begin{aligned} &\begin{pmatrix} \frac{\partial f_x}{\partial a_{x'}} & \frac{\partial f_x}{\partial b_{y'}} \\ \frac{\partial f_y}{\partial a_{x'}} & \frac{\partial f_y}{\partial b_{y'}} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{T} \text{diag}\left(\left(M_{x0} + \frac{1}{2} \sum_y M_{xy}\right)_x\right) & \left(-\frac{1}{2T} M_{xy'}\right)_{xy'} \\ \left(-\frac{1}{2T} M_{x'y}\right)_{x'y} & -\frac{1}{T} \text{diag}\left(\left(M_{0y} + \frac{1}{2} \sum_x M_{xy}\right)_y\right) \end{pmatrix} \end{aligned}$$

The Jacobian is symmetric.
Consider

$$\min_{a,b} F(a, b)$$

where

$$F(a, b) = \left\{ \begin{array}{l} \sum_x n_x a_x + \sum_y m_y b_y \\ + T \sum_x \exp\left(-\frac{a_x}{T}\right) + T \sum_y \exp\left(-\frac{b_y}{T}\right) \\ + 2T \sum_{xy} \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) \end{array} \right\}$$

and we have $f_x(a, b) = \partial F(a, b) / \partial a_x$ and $f_y(a, b) = \partial F(a, b) / \partial b_y$.

We can reformulate Choo-Siow's equations as

$$\begin{aligned} \partial F(a, b) / \partial a_x &= 0 \\ \partial F(a, b) / \partial b_y &= 0 \end{aligned}$$

1.1 The Choo-Siow model as an equilibrium problem with gross substitutes

Recall that an equilibrium pb w gross substitutes can be expressed as

$$Q_z(p) = 0$$

where the Jacobian of Q is positive on the diagonal, and off-diagonal nonpositive.

$$\frac{\partial Q_z}{\partial p_{z'}} \leq 0 \text{ for } z \neq z'$$

ie the Jacobian of Q is a Z-matrix.

Here, the Jacobian of the Choo-Siow equations is the Hessian of F , which is

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

where H_{11} and H_{22} diagonal negative on the diagonal

and all the entries of $H_{12} = H_{21}^\top$ are negative.

$$Q_x(p) = M_{x0}(-p_x) + \sum_y M_{xy}(-p_x, p_y) - n_x$$

$$Q_y(p) = -M_{0y}(p_y) - \sum_x M_{xy}(-p_x, p_y) + m_y$$

$$p_z = (p_x, p_y)$$

$$p_x = -a_x$$

$$p_y = b_y$$

$$Q_x(p) = f_x(-(p_x), (p_y)) - n_x$$

$$Q_y(p) = m_y - f_y(-(p_x), (p_y))$$

Coordinate update algorithm for this system.

Blockwise Jacobi / Gauss-Seidel:

Start with an initial value $p^0 = (-a_x^0, b_y^0)$

Solve for a_x^{t+1} such that

$$M_{x0}(a_x^{t+1}) + \sum_y M_{xy}(a_x^t, b_y^t) = n_x$$

Solve b_y^{t+1} such that

$$M_{0y}(b_y^{t+1}) + \sum_x M_{xy}(a_x^{t+1}, b_y^{t+1}) = m_y$$

Note that this can interpreted as a coordinate descent algorithm

$$a^{t+1} = \arg \min_a F(a, b^t)$$

$$b^{t+1} = \arg \min_b F(a^{t+1}, b)$$

Back to the problem:

$$\exp\left(-\frac{a_x^{t+1}}{T}\right) + \sum_y \exp\left(\frac{\Phi_{xy}}{2T}\right) \exp\left(-\frac{a_x^{t+1}}{2T}\right) \exp\left(-\frac{b_y^t}{2T}\right) = n_x$$

and

$$\exp\left(-\frac{b_y^{t+1}}{T}\right) + \sum_x \exp\left(\frac{\Phi_{xy}}{2T}\right) \exp\left(-\frac{a_x^{t+1}}{2T}\right) \exp\left(-\frac{b_y^{t+1}}{2T}\right) = n_x$$

Let's introduce $K_{xy} = \exp\left(\frac{\Phi_{xy}}{2T}\right)$, and denote

$$A_x^t = \exp\left(-\frac{a_x^t}{2T}\right)$$

$$B_y^t = \exp\left(-\frac{b_y^t}{2T}\right)$$

and rewrite the system as

$$(A_x^{t+1})^2 + \sum_y K_{xy} A_x^{t+1} B_y^t = n_x$$

and

$$(B_y^{t+1})^2 + \sum_x K_{xy} A_x^{t+1} B_y^{t+1} = n_x$$

eg let's solve the first one

$$A_x^2 + 2A_x \frac{\sum_y K_{xy} B_y}{2} + \left(\frac{\sum_y K_{xy} B_y}{2}\right)^2 = n_x + \left(\frac{\sum_y K_{xy} B_y}{2}\right)^2$$

$$\left(A_x + \frac{\sum_y K_{xy} B_y}{2}\right)^2 = n_x + \left(\frac{\sum_y K_{xy} B_y}{2}\right)^2$$

$$A_x = -\frac{\sum_y K_{xy} B_y}{2} + \sqrt{n_x + \left(\frac{\sum_y K_{xy} B_y}{2}\right)^2}$$

Let's check that BGH is satisfied

$$Q_x(p) = M_{x0}(-p_x) + \sum_y M_{xy}(-p_x, p_y) - n_x$$

$$Q_y(p) = -M_{0y}(p_y) - \sum_x M_{xy}(-p_x, p_y) + m_y$$

$$\sum_x Q_x(p) + \sum_y Q_y(p) = \sum_x M_{x0}(-p_x) - \sum_y M_{0y}(p_y) + cte$$

Thus the law of aggregate supply holds.

Thus the system is inverse isotone.

2 Connecting with yesterday's model

x = driver

y = passengers

z = locations

$U_{xz} + P_z$ is the utility of driver picking at z

$V_{zy} - P_z$ is the utility of passenger y picked up at z

Chiappori, McCann and Nesheim (ET 2011)

One can show that at equilibrium if x picks y , then it is in location z such that

$$z \in \arg \max_z \{U_{xz} + V_{zy}\}$$

Further, yesterday's problem can be reformulated as a matching problem between passengers and drivers where the matching surplus is

$$\Phi_{xy} = \max_z \{U_{xz} + V_{zy}\}.$$

Conversely, from the matching model to the hedonic model, assume $Y = Z$ and assume that $V_{zy} = 0$ if $z = y$ and $V_{zy} = -\infty$ if $z \neq y$. In that case,

$$\Phi_{xy} = \max_z \{U_{xz} + V_{zy}\} = U_{xy}.$$