Whitboard for math+econ+code
Alfred Galichon, June 2022

Roadmap

Day 1: study theoretical and computational methods for equilibrium systems (with gross substitutes)

Day 2: Matching models with fully transferable utility

what is given up by one side of the market is fully appropriated by the other side

this is connected with optimal transport

Day 3: Matching models with impfectly transferable utility

what is given up by one side of the market is NOT fully appropriated by the other side

models from labor economics (matching with taxes)

or from familiy economics (collective models of bargaining)

Day 4: Matching models with non transferable utility

* school choice
    * social housing
    * taxi markets

Day 5: price equilibrium on networks

1 Day 1: equilibrium with gross substitutes

Our goal = provide a common framework for these matching models. Reformulate as a general equilibrium problem: $|Z| = n$ goods, Z is the set of goods, and

$$Q : R^n \to R^n$$

such that is p_z if the price of good $z \in Z$, then $Q_z(p)$ is the excess supply for good z (=supply-demand), and the equilibrium problem consists in finding $p \in R^n$ such that

$$Q(p) = 0.$$

More specifically, we shall assume that Q has the *gross substitute property.*

Assume we increase price of good z. What is this going to do to the excess supply for good z'? some producers will shift from producing z' to producing z, and therefore $Q_{z'}(p)$ will decrease.

In other words, when it is derivable, Q has the gross substitute property when

$$\frac{\partial Q_z}{\partial p_{z'}}(p) \leq 0 \text{ for } z \neq z'$$

while it natural to assume

$$\frac{\partial Q_z}{\partial p_z}(p) \geq 0 \text{ for all } z.$$

Example: surge pricing in an uber-like environment. We have partitioned the city in a finite number of locations (say, blocks).

$x \in X=$ location of the driver

$y \in Y=$location of the passenger

Assume $z \in Z$ is the pickup location.

Assume that for a drive at x, the cost of picking up at z is c_{xz}

if the price of the ride at z is p_z the utility if the driver is $p_z - c_{xz} + \sigma\varepsilon_z$, where the vector (ε_z) is random.

if the driver does not pickup anyone, the utility is normalized to ε_0.

Assume that $(\varepsilon_z) \sim$ *Gumbel* and is iid. Then the probability that a driver at x will demand a ride z is

$$\frac{\exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z'} \exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)}$$

Now assume that there are n_x drivers in area x, and therefore the supply for rides z is

$$S_z(p) = \sum_{x \in X} n_x \frac{\exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z' \in Z} \exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)}$$

Let's study the properties of $S(p)$. Do we have gross substitutes?

$$S_z(p) = \sum_{x \in X} n_x \frac{1}{\exp\left(\frac{-p_z + c_{xz}}{\sigma}\right) + \sum_{z' \neq z} \exp\left(\frac{p_{z'} - p_z + c_{xz} - c_{xz'}}{\sigma}\right) + 1}$$

and we immediately see that $S_z(p)$ is decreasing w.r.t. $p_{z'}$ (for $z' \neq z$), and increasing with respect to p_z.

Now. let's focus on demand. This is the same as before, except for the fact that utility of a passenger at y seeking a ride in a cell z is now

$$u_{yz} - p_z + \eta_z$$

where η is iid Gumbel. The induced demand is

$$\begin{aligned} D_z(p) &= \sum_{y \in Y} m_y \frac{\exp(u_{yz} - p_z)}{1 + \sum_{z'} \exp(u_{yz'} - p_{z'})} \\ &= \sum_{y \in Y} m_y \frac{1}{\exp(-u_{yz} + p_z) + \sum_{z' \neq z} \exp(u_{yz'} - u_{yz} + p_z - p_{z'}) + 1} \end{aligned}$$

we see that $-D_z(p)$ has the Gross substitute property, and therefore

$$Q_z(p) = S_z(p) - D_z(p)$$

also has the gross substitute property. Note that

$$\log \left\{ 1 + \sum_{z'} \exp(u_{yz'} - p_{z'}) \right\}$$

is the expected indirect utility of a passenger living at y.

Question = how do we compute p such that $Q(p) = 0$.

Tatonnement = raise prices where there is excess demand ($Q_z(p) < 0$) and decrease prices where there excess supply.

Essentially 2 types of methods.

1) optimization-based methods: reformulate as an optimization problem. In order to do that, try to obtain Q as a gradient– ie, is there a potential function $V : R^n \to R$ such that V is convex, and

$$Q_z(p) = \frac{\partial V}{\partial p_z}(p).$$

In that case, $Q(p) = 0$ is equivalent to the fact that p is a minimizer of V.

In that case, we can minimize V using e.g. gradient descent, that is

$$\begin{aligned} p^{t+1} &= p^t - \epsilon \frac{\partial V}{\partial p_z}(p^t) \\ &= p^t - \epsilon Q_z(p^t) \end{aligned}$$

In our example, we had

$$
\begin{aligned}
Q_z(p) &= \sum_{x\in X} n_x \frac{\exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{1+\sum_{z'\in Z}\exp\left(\frac{p_{z'}-c_{xz'}}{\sigma}\right)} \\
&\quad -\sum_{y\in Y} m_y \frac{\exp\left(u_{yz}-p_z\right)}{1+\sum_{z'}\exp\left(u_{yz'}-p_{z'}\right)}
\end{aligned}
$$

We have that $Q_z(p) = \partial V(p)/\partial p_z$, where

$$
\begin{aligned}
V(p) &= \sigma \sum_{x\in X} n_x \log\left(1+\sum_{z'\in Z}\exp\left(\frac{p_{z'}-c_{xz'}}{\sigma}\right)\right) \\
&\quad +\sum_{y\in Y} m_y \log\left(1+\sum_{z'}\exp\left(u_{yz'}-p_{z'}\right)\right)
\end{aligned}
$$

2) substitutes-based methods. Consider the system of equations

$$Q_z(p) = 0$$

Assume that we can start with a supersolution – i.e. a vector p^0 such that

$$Q_z\left(p^0\right) \geq 0 \text{ for all } z$$

Then we can consider the myopic market-clearing algorithm as follows: assume p^t has been computed and compute p^{t+1} using

$$Q_z\left(p_z^{t+1}, p_{-z}^t\right) = 0$$

We can show that under suitable assumptions (involving gross substitutes), this algorithm converges to the market-clearing price.

Recall we started from a supersolution p^0. We can show that for each $t \geq 0$:
* $p_z^t \geq p_z^{t+1}$
* p^{t+1} is a supersolution, that is $Q_z\left(p^{t+1}\right) \geq 0$ for all z
By the induction hypothesis,
$Q_z\left(p^t\right) \geq 0$ that is
$Q_z\left(p_z^t, p_{-z}^t\right) \geq 0$
$Q_z\left(p_z^{t+1}, p_{-z}^t\right) = 0$
because Q_z is increasing in p_z, this implies necessarily $p_z^t \geq p_z^{t+1}$.
Let's now show that p^{t+1} is a supersolution. We have
$Q_z\left(p_z^{t+1}, p_{-z}^t\right) = 0$
We have $p^t \geq p^{t+1}$ and $Q_z(p)$ is nonincreasing in p_{-z}, therefore
$Q_z\left(p_z^{t+1}, p_{-z}^{t+1}\right) \geq Q_z\left(p_z^{t+1}, p_{-z}^t\right) = 0$

Now:

* if p^t remains bounded below, then it converges to p^*.
* if Q is continuous and p^t converges, then we can take the limit $t \to +\infty$ in

$$Q_z\left(p_z^{t+1}, p_{-z}^t\right) = 0$$

and we have that

$$Q_z\left(p^*\right) = 0.$$

After the break, we need to show that p^t remains bounded below under reasonable assumptions. We will show assumptions under which Q is inverse isotone, i.e.

$$Q\left(p\right) \leq Q\left(p'\right) \implies p \leq p'$$

which is a fundamental property of Q which is related to M-maps.

Theorem (Berry, Gandhi and Haile, Econometrica 2013). Assume that:
(i) Q satisfies weak gross substitutes ie $Q_z\left(p\right)$ is weakly decreasing in $p_{z'}$ for $z \neq z$.
(ii) Law of aggregate supply holds, ie $\sum_z Q_z\left(p\right)$ is weakly increasing in each p_z, or in other words:

$$Q_0\left(p\right) = -\sum_z Q_z\left(p\right) \text{ is weakly decreasing in each } p_z$$

(iii) Connected strong substitutes holds: or each z, there is a path from z to 0 $z_1 = z$, $z_2 z_2 ... z_{n-1}$, $z_n = 0$ such that $Q_{z_{k+1}}\left(p\right)$ is strictly decreasing with respect to p_{z_k}.
Then Q is inverse isotone, i.e. $Q\left(p\right) \leq Q\left(p'\right)$ implies $p \leq p'$.

Example where (iii) does not hold. Assume $Z = B_1 \cup B_2$ with $B_1 \cap B_2 = \emptyset$, and assume that $Q_z\left(p\right)$ only depends on the prices in B_1 for $z \in B_1$ and $Q_z\left(p\right)$ only depends on the prices in B_2 for $z \in B_2$.

Example. Consider $Q\left(p\right) = Qp$ where
* Q has gross substitutes which means that $Q_{ij} \leq 0$ for $i \neq j$, and
* Q is row-diagonally dominant .

$$Q_{ii} > \sum_{j \neq i} \left|Q_{ij}\right|$$

Then I can show that the assumptions in the BGH theorem are met.
(i) and (ii) are obvious. (iii) is obtained by the fact that $Q_0\left(p\right)$ is strictly decreasing in p_z.

Example 3. Back to

$$\begin{aligned} Q_z\left(p\right) &= \sum_{x \in X} n_x \frac{\exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z' \in Z} \exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)} \\ &\quad - \sum_{y \in Y} m_y \frac{\exp\left(u_{yz} - p_z\right)}{1 + \sum_{z'} \exp\left(u_{yz'} - p_{z'}\right)} \end{aligned}$$

We have show (i) gross substitutes.

Show (ii). We have

$$\begin{aligned}\sum_z Q_z(p) &= \sum_{x\in X} n_x \sum_{z\in Z} \frac{\exp\left(\frac{p_z-c_{xz}}{\sigma}\right)}{1+\sum_{z'\in Z}\exp\left(\frac{p_{z'}-c_{xz'}}{\sigma}\right)} \\ &\quad -\sum_{y\in Y} m_y \sum_{z\in Z} \frac{\exp(u_{yz}-p_z)}{1+\sum_{z'}\exp(u_{yz'}-p_{z'})}\end{aligned}$$

that is

$$\begin{aligned}\sum_z Q_z(p) &= \sum_{x\in X} n_x \left(1-\frac{1}{1+\sum_{z'\in Z}\exp\left(\frac{p_{z'}-c_{xz'}}{\sigma}\right)}\right) \\ &\quad -\sum_{y\in Y} m_y \left(1-\frac{1}{1+\sum_{z'}\exp(u_{yz'}-p_{z'})}\right)\end{aligned}$$

therefore $\sum_z Q_z(p)$ is strictly increasing in each of the p_z.

Two different implications of Q inverse isotone.

Implication 1: existence of a solution under the assumption that there exists both a sub- and a super-solution.

If Q is inverse isotone, then assume that in addition to what we have assumed, there is a subsolution $\underline{p}$. Then for anysupersolution p, we can show that $\underline{p}\leq p$. Indeed, $Q\left(\underline{p}\right)\leq 0\leq Q(p)$ implies by inverse isotonicity that $\underline{p}\leq p$. Therefore the sequence constructed above, which is a sequence of supersolutions, is bounded below by $\underline{p}$.

Implication 2: uniqueness of a solution.

If Q is inverse isotone, then the equilibrium price p^* is unique. Indeed, assume $Q(p)=Q(p')$. Then $Q(p)\leq Q(p')$ and $Q(p')\leq Q(p)$ and by inverse isotonicity applied twice, we have $p\leq p'$ and $p'\leq p$, which implies $p=p'$.

Let's take an example

$$\begin{aligned}Q_1(p) &= 2p_1-p_2 \\ Q_2(p) &= -2p_1+3p_2\end{aligned}$$

$p^0=(1,1)$. Thus $Q\left(p^0\right)=(1,1)\geq 0$ thus p^0 is a supersolution.

$$\begin{aligned}2p_1^1-p_2^0 &= 0 \\ -2p_1^0+3p_2^1 &= 0\end{aligned}$$

thus I get

$$\begin{aligned}p_1^1 &= p_2^0/2=1/2 \\ p_2^1 &= 2p_1^0/3=2/3\end{aligned}$$

Let's now see a tiny variant of this model, where the supply is now

$$S_z(p) = \frac{\exp\left(\frac{\alpha_{xz} p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z'} \exp\left(\frac{\alpha_{xz'} p_{z'} - c_{xz'}}{\sigma}\right)}$$

(where α_{xz} differs from a constant) and D_z is the same as before.

I am now claiming that there is NO function V such that

$$Q_z(p) = \frac{\partial V}{\partial p_z}(p).$$

Why? because assume such V existed. Then one would have

$$\frac{\partial Q_z}{\partial p_{z'}} = \frac{\partial^2 V}{\partial p_z \partial p_{z'}}(p)$$

which should be symmetric. Indeed,

$$\begin{aligned}
& \frac{\partial}{\partial p_{z'}} \frac{\exp\left(\frac{\alpha_{xz} p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z'} \exp\left(\frac{\alpha_{xz'} p_{z'} - c_{xz'}}{\sigma}\right)} \\
= & \frac{\partial}{\partial p_{z'}} \frac{1}{\exp\left(\frac{-\alpha_{xz} p_z + c_{xz}}{\sigma}\right) + \sum_{z' \neq z} \exp\left(\frac{\alpha_{xz'} p_{z'} - \alpha_{xz} p_z + c_{xz} - c_{xz'}}{\sigma}\right) + 1} \\
= & \alpha_{xz'} * \text{sth symmetric}
\end{aligned}$$

2 Day 2: matching with transferable utility

Plan for today:

1) Regularized optimal transport
2) Sinkhorn's algorithn
3) Microfoundation of regularized optimal transport
4) Numerical computation

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2.1 Regularized optimal transport

Workers 'type $x \in X$, where X is finite, and there are n_x workers of type x.

Firms' type $y \in Y$, where Y is finite, and there are m_y workers of type y.

Assume that the total number of workers = total number of firms

$$\sum_x n_x = \sum_y m_y = 1$$

(where we have rescaled the numbers into probabilities wlog).

Central planner's problem.

The central planner determines who matches with whom: determine how many matches of type xy we shall form for each x and each y. Denote μ_{xy} this frequency. We have

$$\sum_{y \in Y} \mu_{xy} = n_x \text{ and } \sum_{x \in X} \mu_{xy} = m_y.$$

Assume workers and firms are matched in a random way, in which case $\mu_{xy} = n_x m_y$.

Assume that Φ_{xy} is the economic output (in monetary terms) of a match xy. Then the total economic output out of a matching (μ_{xy}) is

$$\sum_{xy} \mu_{xy} \Phi_{xy}$$

and the central planner's problem is

$$\begin{aligned} \max_{\mu_{xy} \geq 0} \quad & \sum_{xy} \mu_{xy} \Phi_{xy} \\ s.t. \quad & \sum_{y \in Y} \mu_{xy} = n_x \\ & \sum_{x \in X} \mu_{xy} = m_y. \end{aligned}$$

which is called the (unregularized) optimal transport problem.

x location of soil n_x volume of soil at x

y location of soil to fill out m_y volume at y

c_{xy} is the cost of transporting a unit of volume of soil from x to y

In this case

$$\begin{aligned} \min_{\mu_{xy} \geq 0} \quad & \sum_{xy} \mu_{xy} c_{xy} \\ s.t. \quad & \sum_{y \in Y} \mu_{xy} = n_x \\ & \sum_{x \in X} \mu_{xy} = m_y. \end{aligned}$$

Consider now the regularized version of this problem: for $\sigma > 0$, consider

$$\begin{aligned} \max_{\mu_{xy} \geq 0} \quad & \left\{ \sum_{xy} \mu_{xy} \Phi_{xy} - \sigma \sum_{xy} \mu_{xy} \ln \mu_{xy} \right\} \\ s.t. \quad & \sum_{y \in Y} \mu_{xy} = n_x \ \ [u_x] \\ & \sum_{x \in X} \mu_{xy} = m_y \ \ [v_y] \end{aligned}$$

We have

$$\max_{\mu_{xy}\geq 0}\left\{\begin{array}{c}\sum_{xy}\mu_{xy}\Phi_{xy}-\sigma\sum_{xy}\mu_{xy}\ln\mu_{xy}\\ +\sum_{x}\min_{u_x}u_x\left(n_x-\sum_{y\in Y}\mu_{xy}\right)\\ +\sum_{y}\min_{v_y}v_y\left(m_y-\sum_{x\in X}\mu_{xy}\right)\end{array}\right\}$$

thus

$$\max_{\mu_{xy}\geq 0}\min_{u_x,v_y}\sum_x n_x u_x+\sum_y m_y v_y+\sum_{xy}\mu_{xy}\left(\Phi_{xy}-u_x-v_y-\sigma\ln\mu_{xy}\right)$$

we have

$$\begin{aligned}&\min_{u_x,v_y}\max_{\mu_{xy}\geq 0}\sum_x n_x u_x+\sum_y m_y v_y+\sum_{xy}\mu_{xy}\left(\Phi_{xy}-u_x-v_y-\sigma\ln\mu_{xy}\right)\\ =\ &\min_{u_x,v_y}\left\{\sum_x n_x u_x+\sum_y m_y v_y+\max_{\mu_{xy}\geq 0}\sum_{xy}\mu_{xy}\left(\Phi_{xy}-u_x-v_y-\sigma\ln\mu_{xy}\right)\right\}\end{aligned}$$

Optimality condition with respect to μ in the inside maximization problem yields

$$\mu_{xy}=\exp\left(\frac{\Phi_{xy}-u_x-v_y-\sigma}{\sigma}\right)$$

and therefore, the problem becomes

$$\min_{u_x,v_y}\left\{\sum_x n_x u_x+\sum_y m_y v_y+\sum_{xy}\sigma\exp\left(\frac{\Phi_{xy}-u_x-v_y-\sigma}{\sigma}\right)-\sigma\right\}$$

Change of signs

$$\begin{aligned}p_x&=u_x\\ p_y&=-v_y\end{aligned}$$

and the problem becomes

$$\min_p\left\{\sum_x n_x p_x-\sum_y m_y p_y+\sum_{xy}\sigma\exp\left(\frac{\Phi_{xy}-p_x+p_y-\sigma}{\sigma}\right)-\sigma\right\}$$

FOC with respect to p_x :

$$n_x=\sum_y\exp\left(\frac{\Phi_{xy}-p_x+p_y-\sigma}{\sigma}\right)$$

with respect to p_y:

$$m_y=\sum_x\exp\left(\frac{\Phi_{xy}-p_x+p_y-\sigma}{\sigma}\right)$$

Remark. Consider μ^σ the solution to

$$\begin{aligned} V(\sigma) = \max_{\mu_{xy}\geq 0} & \left\{ \sum_{xy} \mu_{xy}\Phi_{xy} - \sigma \sum_{xy} \mu_{xy} \ln \mu_{xy} \right\} \\ s.t. & \sum_{y\in Y} \mu_{xy} = n_x \ \ [u_x] \\ & \sum_{x\in X} \mu_{xy} = m_y \ \ [v_y] \end{aligned}$$

Theorem. When $\sigma \to 0$, μ^σ converges to the minimizer of

$$\sum_{xy} \mu_{xy} \ln \mu_{xy}$$

subject to

$$\begin{aligned} \sum_{xy} \mu_{xy}\Phi_{xy} &= V(0) \\ \sum_{y\in Y} \mu_{xy} &= n_x \\ \sum_{x\in X} \mu_{xy} &= m_y \end{aligned}$$

Yesterday, we saw methods to solve systems of the form

$$Q(p) = 0$$

where Q has gross substitutes and was inverse isotone – or equivalently, Q is a M-function. Here we have

$$\begin{aligned} Q_x(p) &:= n_x - \sum_y \exp\left(\frac{\Phi_{xy} - p_x + p_y - \sigma}{\sigma}\right) = 0 \\ Q_y(p) &:= \sum_x \exp\left(\frac{\Phi_{xy} - p_x + p_y - \sigma}{\sigma}\right) - m_y = 0 \end{aligned}$$

Can Q be injective? $Q(p + c1) = Q(p)$ – Q is not injective. We have

$$\sum_x Q_x(p) + \sum_y Q_y(p) = 0$$

Fix $0 \in X$ a particular element of X and set $p_0 = 0$. I can view Q as a function of $R^{X\backslash\{0\}\cup Y} \to R^{X\backslash\{0\}\cup Y}$, and I can show that Q is inverse isotone – use Berry Gandhi and Haile. Hence I have uniquenss of the equilibrium prices (that is up to normalization) and convergence of Jacobi.

What is Gauss-Seidel algorithm?

$$
\begin{aligned}
Q_x\left(p_x^{t+1}, p_y^t\right) &= 0 \\
Q_y\left(p_y^{t+1}, p_x^{t+1}\right) &= 0
\end{aligned}
$$

Well, the first equation yields

$$\sum_y \exp\left(\frac{\Phi_{xy} - p_x^{t+1} + p_y^t - \sigma}{\sigma}\right) = n_x$$

that is

$$\exp\left(\frac{-p_x^{t+1}}{\sigma}\right) = \frac{n_x}{\sum_y \exp\left(\frac{\Phi_{xy} + p_y^t - \sigma}{\sigma}\right)}$$

and for the other one,

$$\exp\left(\frac{p_y^{t+1}}{\sigma}\right) = \frac{m_y}{\sum_x \exp\left(\frac{\Phi_{xy} - p_x^{t+1} - \sigma}{\sigma}\right)}.$$

Remark. Link with the model in D1.
Model in D1 (without the outside option) was

$$Q_z(p) = \sum_{x\in X} n_x \frac{\exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{\sum_{z'\in Z}\exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)} - \sum_{y\in Y} m_y \frac{\exp\left(u_{yz} - p_z\right)}{\sum_{z'}\exp\left(u_{yz'} - p_{z'}\right)}$$

Assume $u_{yz} = -\infty$ if $y \neq z$ and $u_{yz} = 0$ else. The previous equation becomes

$$Q_z(p) = \sum_{x\in X} n_x \frac{\exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{\sum_{z'\in Z}\exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)} - m_z$$

Let's introduce U_x such that

$$\exp\left(\frac{U_x}{\sigma}\right) = \frac{\sum_{z'\in Z}\exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)}{n_x}$$

we have

$$\sum_{x\in X}\exp\left(\frac{p_z - U_x - c_{xz}}{\sigma}\right) = m_z$$

and we have

$$\sum_{z\in Z}\exp\left(\frac{p_z - U_x - c_{xz}}{\sigma}\right) = \frac{n_x \sum_z \exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{\sum_{z'\in Z}\exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)} = n_x.$$

$$U_x = \sigma\log\left(\sum_{z'\in Z}\exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)\right) - \sigma\log n_x$$

2.2 Microfoundation of the matching model

Worker x's problem: choose a firm y such that

$$\max_{y \in Y} \{\alpha_{xy} + w_{xy} + \sigma \varepsilon_y\}$$

where α_{xy} is the monetary valuation of the job y's amenities. w_{xy} is the wage of worker x working for firm y; determined at equilibrium. ε_y is the random utilitiy, assumed logit (Gumbel distributed). As a consequence, the average indirect utility of worker x is

$$u_x = \sigma \log \left(\sum_y \exp \left(\frac{\alpha_{xy} + w_{xy}}{\sigma} \right) \right)$$

and the probability that worker x picks firm y is

$$\frac{\mu_{xy}}{n_x} = \frac{\exp\left(\frac{\alpha_{xy}+w_{xy}}{\sigma}\right)}{\sum_{y'} \exp\left(\frac{\alpha_{xy'}+w'_{xy}}{\sigma}\right)} = \frac{\exp\left(\frac{\alpha_{xy}+w_{xy}}{\sigma}\right)}{\exp\left(\frac{u_x}{\sigma}\right)} = \exp\left(\frac{\alpha_{xy} + w_{xy} - u_x}{\sigma}\right).$$

Firm's side:

$$\max_{x \in X} \{\gamma_{xy} - w_{xy} + \sigma \eta_x\}$$

and

$$v_y = \sigma \log \left(\sum_x \exp \left(\frac{\gamma_{xy} - w_{xy}}{\sigma} \right) \right)$$

and the proba that firm y picks worker x is

$$\frac{\mu_{xy}}{m_y} = \exp\left(\frac{\gamma_{xy} - w_{xy} - v_y}{\sigma}\right)$$

Let's eliminate the wages by multiplying term by term:

$$\frac{\mu_{xy}}{n_x}\frac{\mu_{xy}}{m_y} = \exp\left(\frac{\alpha_{xy} + w_{xy} - u_x}{\sigma}\right) \exp\left(\frac{\gamma_{xy} - w_{xy} - v_y}{\sigma}\right)$$

that is

$$\frac{\mu_{xy}^2}{n_x m_y} = \exp\left(\frac{\alpha_{xy} + \gamma_{xy} - u_x - v_y}{\sigma}\right)$$

Let define $\Phi_{xy} = \alpha_{xy} + \gamma_{xy}$ as the match surplus out of a match xy, and

$$\mu_{xy} = \exp\left(\frac{\Phi_{xy} - (u_x - \sigma \ln n_x) - (v_y - \sigma \ln m_y)}{2\sigma}\right)$$

define

$$a_x = u_x - \sigma \ln n_x \text{ and } b_y = v_y - \sigma \ln m_y$$

we have

$$\mu_{xy} = \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2\sigma}\right)$$

and a_x and b_y are determined so that

$$\begin{aligned}\sum_y \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2\sigma}\right) &= n_x \\ \sum_x \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2\sigma}\right) &= m_y\end{aligned}$$

3 Day 3: matching with imperfectly transferable utility

Today:
* general theory (~1 h)
* application to employer-employee matching with nonlinear taxes (~30 min)
* application to family economics and collective models (with Simon Weber)

Start from the same setting as yesterday, but with the addition of taxes. Assume w_{xy} is now the gross wage. The expected indirect utility of a worker of type x is

$$u_x = E\left[\max_{y \in Y}\left\{\alpha_{xy} + N\left(w_{xy}\right) + \sigma\varepsilon_y, \sigma\varepsilon_0\right\}\right]$$

where $N\left(w\right)$ is the net wage assuming the gross wage is w. Typically, N is
* increasing
* concave
* piecewise affine

Hence we can represent $N\left(w\right)$ as

$$N\left(w\right) = \min_{k=1,...,K}\left(1 - \tau_k\right)\left(w - w_k\right)$$

where τ_k is the kth tax bracket: $\tau_1 = 0 < \tau_2 < ... < \tau_K$.

As yesterday α_{xy} is the monetary valuation of the job y's amenities. w_{xy} is the wage of worker x working for firm y; determined at equilibrium. ε_y is the random utilitiy, assumed logit (Gumbel distributed). As a consequence, the average indirect utility of worker x is

$$u_x = \sigma \log\left(1 + \sum_y \exp\left(\frac{\alpha_{xy} + N\left(w_{xy}\right)}{\sigma}\right)\right)$$

and the probability that worker x picks firm y is

$$\frac{\mu_{xy}}{n_x} = \frac{\exp\left(\frac{\alpha_{xy} + N\left(w_{xy}\right)}{\sigma}\right)}{1 + \sum_{y'}\exp\left(\frac{\alpha_{xy'} + N\left(w'_{xy}\right)}{\sigma}\right)} = \frac{\exp\left(\frac{\alpha_{xy} + N\left(w_{xy}\right)}{\sigma}\right)}{\exp\left(\frac{u_x}{\sigma}\right)} = \exp\left(\frac{\alpha_{xy} + N\left(w_{xy}\right) - u_x}{\sigma}\right).$$

Firm's side:

$$v_y = E\left[\max_{x \in X}\left\{\gamma_{xy} - w_{xy} + \sigma\eta_x, \sigma\eta_0\right\}\right]$$

and the probability that firm y picks worker x is

$$\frac{\mu_{xy}}{m_y} = \exp\left(\frac{\gamma_{xy} - w_{xy} - v_y}{\sigma}\right).$$

Yesterday we had:
* after-transfer systematic utility of worker $U_{xy} = \alpha_{xy} + w_{xy}$
* after-transfer systematic utility of firm $V_{xy} = \gamma_{xy} - w_{xy}$
it was a transferable utility model, where $U_{xy} + V_{xy} = \alpha_{xy} + \gamma_{xy} = \Phi_{xy}$

Today, we will consider models in which $(U_{xy}, V_{xy}) \in F_{xy}$ belong to a feasible set F_{xy} – we shall make assumptions on F_{xy}.

Let's assume free disposal: if $(u, v) \in F_{xy}$ then $(u', v') \in F_{xy}$ as soon as $u' \leq u$ and $v' \leq v$.

Example. Nonlinear taxes

In this case $U_{xy} \leq \alpha_{xy} + N(w_{xy})$ and $V_{xy} \leq \gamma_{xy} - w_{xy}$. We have

$$(U_{xy}, V_{xy}) \in F_{xy} \text{ if and only } N(\gamma_{xy} - V_{xy}) \geq U_{xy} - \alpha_{xy}$$

that is in this case

$$F_{xy} = \left\{(U_{xy}, V_{xy}) : N(\gamma_{xy} - V_{xy}) \geq U_{xy} - \alpha_{xy}\right\}.$$

Describing F_{xy} by the distance-to-frontier function. Introduce

$$D_{xy}(u, v) = \min_{t \in \mathbb{R}}\left\{t : (u - t, v - t) \in F_{xy}\right\}$$

Therefore $D_{xy}(u, v) > 0$ iff $(u, v) \notin F_{xy}$.
$D_{xy}(u, v) \leq 0$ iff $(u, v) \in F_{xy}$
$D_{xy}(u, v) < 0$ iff (u, v) is not on the frontier of F_{xy}.
Example 1. Transferable utility case.
$F_{xy} = \{(U, V) : U + V = \Phi_{xy}\}$ we have

$$D_{xy}(u, v) = \frac{u + v - \Phi_{xy}}{2}$$

indeed, the minimum t such that $(u - t, v - t) \in F_{xy}$ is such that $(u - t) + (v - t) = \Phi_{xy}$ that is $u + v - 2t = \Phi_{xy}$ and thus $t = \frac{u+v-\Phi_{xy}}{2}$.

Example 2. Nonlinear taxation. We have

$$F_{xy} = \left\{(u, v) : N(\gamma_{xy} - v) \geq u - \alpha_{xy}\right\}.$$

where

$$N(w) = \min_{k=1,\dots,K} (1-\tau_k)(w - w_k)$$

We have

$$\begin{aligned} F_{xy} &= \left\{ (u,v) : \min_{k=1,\dots,K} (1-\tau_k)(\gamma_{xy} - v - w_k) \geq u - \alpha_{xy} \right\} \\ &= \cap_k \left\{ (u,v) : (1-\tau_k)(\gamma_{xy} - v - w_k) \geq u - \alpha_{xy} \right\}. \end{aligned}$$

Therefore we need to understand how to compute the distance to the intersection of elementary sets.

$$D_{F_1 \cap F_2} = \max \{ D_{F_1}, D_{F_2} \}$$

and

$$D_{F_1 \cup F_2} = \min \{ D_{F_1}, D_{F_2} \}.$$

Back to our taxation problem

$$D_{F_{xy}}(u,v) = \max_k D^k_{xy}(u,v)$$

where

$$D^k_{xy}(u,v) = \frac{u - \alpha_{xy} + (1-\tau_k)(v - \gamma_{xy} + w_k)}{2-\tau_k}.$$

Therefore

$$D_{xy} = \max_{k=1..K} \left\{ \frac{u - \alpha_{xy} + (1-\tau_k)(v - \gamma_{xy} + w_k)}{2-\tau_k} \right\}.$$

Note that D is additive

$$D(u+t, v+t) = t + D(u,v).$$

Back to the matching model. We had

$$\begin{aligned} u_x &= E\left[\max_{y \in Y} \{ \alpha_{xy} + N(w_{xy}) + \sigma\varepsilon_y, \sigma\varepsilon_0 \} \right] \\ &= E\left[\max_{y \in Y} \{ U_{xy} + \sigma\varepsilon_y, \sigma\varepsilon_0 \} \right] \\ &= \log\left(1 + \sum_y \exp \frac{U_{xy}}{\sigma} \right) \end{aligned}$$

and

$$\begin{aligned} v_y &= E\left[\max_{x \in X} \{ \gamma_{xy} - w_{xy} + \sigma\eta_x, \sigma\eta_0 \} \right] \\ &= E\left[\max_{x \in X} \{ V_{xy} + \sigma\eta_x, \sigma\eta_0 \} \right] \\ &= \log\left(1 + \sum_x \exp \frac{V_{xy}}{\sigma} \right) \end{aligned}$$

and

$$\begin{aligned}
\frac{\mu_{xy}}{n_x} &= \exp\left(\frac{U_{xy} - u_x}{\sigma}\right) \\
\frac{\mu_{xy}}{m_y} &= \exp\left(\frac{V_{xy} - v_y}{\sigma}\right)
\end{aligned}$$

therefore

$$\begin{aligned}
U_{xy} &= u_x + \sigma \ln \frac{\mu_{xy}}{n_x} \\
V_{xy} &= v_y + \sigma \ln \frac{\mu_{xy}}{m_y}
\end{aligned}$$

and we have

$$D_{xy}\left(U_{xy}, V_{xy}\right) = 0$$

Replacing, we get

$$D_{xy}\left(u_x + \sigma \ln \frac{\mu_{xy}}{n_x}, v_y + \sigma \ln \frac{\mu_{xy}}{m_y}\right) = 0$$

Recalling that

$$D\left(u + t, v + t\right) = t + D\left(u, v\right),$$

this yields

$$\sigma \ln \mu_{xy} + D_{xy}\left(u_x - \sigma \ln n_x, v_y - \sigma \ln m_y\right) = 0$$

that is

$$\begin{aligned}
\mu_{xy} &= \exp\left(-\frac{D_{xy}\left(u_x - \sigma \ln n_x, v_y - \sigma \ln m_y\right)}{\sigma}\right), \\
\mu_{x0} &= \exp\left(-\frac{u_x - \sigma \ln n_x}{\sigma}\right) \\
\mu_{0y} &= \exp\left(-\frac{v_y - \sigma \ln m_y}{\sigma}\right)
\end{aligned}$$

and denoting $a_x = u_x - \sigma \ln n_x$ and $b_y = v_y - \sigma \ln m_y$ we obtain

$$\begin{aligned}
\mu_{xy} &= \exp\left(-\frac{D_{xy}\left(a_x, b_y\right)}{\sigma}\right) \\
\mu_{x0} &= \exp\left(-\frac{a_x}{\sigma}\right) \\
\mu_{0y} &= \exp\left(-\frac{b_y}{\sigma}\right)
\end{aligned}$$

Recall

$$\frac{\mu_{x0}}{n_x} = \frac{\exp\left(0\right)}{\exp\left(0\right) + \sum_y \exp \frac{U_{xy}}{\sigma}} = \frac{1}{\exp\left(\frac{u_x}{\sigma}\right)} = \exp\left(-\frac{u_x}{\sigma}\right)$$

Now we just need to solve for (a_x, b_y), which we do using

$$\begin{aligned}
\sum_y \exp\left(-\frac{D_{xy}(a_x, b_y)}{\sigma}\right) + \exp\left(-\frac{a_x}{\sigma}\right) &= n_x \\
\sum_x \exp\left(-\frac{D_{xy}(a_x, b_y)}{\sigma}\right) + \exp\left(-\frac{b_y}{\sigma}\right) &= m_y
\end{aligned}$$

Introduce $p_x = a_x$, $p_y = -b_y$ and

$$\begin{aligned}
Q_x(p) &= -\sum_y \exp\left(-\frac{D_{xy}(p_x, -p_y)}{\sigma}\right) + \exp\left(-\frac{p_x}{\sigma}\right) + n_x \\
Q_y(p) &= \sum_x \exp\left(-\frac{D_{xy}(p_x, -p_y)}{\sigma}\right) + \exp\left(\frac{p_y}{\sigma}\right) - m_y
\end{aligned}$$

If this were to a gradient, one would have $Q_x(p) = \partial V(p)/\partial p_x$ and thus

$$\frac{\partial Q_x(p)}{\partial p_y} = \frac{\partial^2 V}{\partial p_x \partial p_y} = \frac{\partial Q_y(p)}{\partial p_x}$$

But we have

$$\begin{aligned}
\frac{\partial Q_x(p)}{\partial p_y} &= \exp\left(-\frac{D_{xy}(p_x, -p_y)}{\sigma}\right) \frac{\partial_{p_y} D_{xy}(p_x, -p_y)}{\sigma} \\
\frac{\partial Q_y(p)}{\partial p_x} &= \exp\left(-\frac{D_{xy}(p_x, -p_y)}{\sigma}\right) \frac{\partial_{p_x} D_{xy}(p_x, -p_y)}{\sigma}
\end{aligned}$$

Some comments.
Assume 2 players bargain over a utility set which is

$$F_{xy} = \{(u, v) \text{ feasible}\}$$

Assume that if x does not participate, can generate in autharky 0
Assume that if y does not participate, can generate in autharky 0

Nash solution: $u = v, (u, v) \in F_{xy}$
Pareton solution: take differential bargaining weights λ_x and ν_y, and the solution would satisfy

$$\max\{\lambda_x u + \nu_y v : (u, v) \in F_{xy}\}$$

(assuming F_{xy} is convex)

In the matching context that we say yesterday (with $\sigma = 1$), U_{xy} and V_{xy} was determined in a unique way. Recall that we had

$$\mu_{xy} = \exp(-D_{xy}(a_x, b_y))$$

and (a_x, b_y) was determined in a unique way (because we characterized it as a solution to $Q(p) = 0$).

and we had

$$\frac{\mu_{xy}}{n_x} = \exp(U_{xy} - u_x)$$

thus

$$\mu_{xy} = \exp(U_{xy} - a_x)$$

thus

$$\begin{aligned} U_{xy} &= a_x + \ln \mu_{xy} \\ V_{xy} &= b_y + \ln \mu_{xy}. \end{aligned}$$

Remark. Indirect utility of x is

$$\begin{aligned} u_x &= E\left[\max_y \{U_{xy} + \varepsilon_y\}\right] \\ v_y &= E\left[\max_x \{V_{xy} + \eta_y\}\right] \end{aligned}$$

this contrasts with a situation where J is the set of individual firms and

$$u_i = \max_{j \in J} \{U_{ij} + \varepsilon_{ij}\}$$

even if $U_{ij} = 0$, we have $u_i = \max_{j \in J} \{\varepsilon_{ij}\}$ thus $E[u_i] = \log \sum_{j=1}^{J} \exp 0 = \log J$.

Dagsvik (IER 2000) and Menzel (Econometrica 2015) have increasing returns to scale.

4 Day 4

Adachi (Ec Letters 2000) – reformulation of stable matchings in the NTU case.

Individual workers $i \in I$ and firms $j \in J$.

Assume that if i and j match, then:

i gets α_{ij}

j gets γ_{ij}

(right now, we are NOT assuming free disposal).

If i is unmatched, then gets α_{i0}

If j is unmatched, then gets γ_{0j}

Introduce the payoff that i gets at equilibrium u_i and j gets at equilibrium v_j.

Adachi's idea: define i's opportunity pool = set of firms that are willing to match with i. This is

$$\{j \in J : \gamma_{ij} \geq v_j\}$$

at equilibrium,

$$u_i = \max\left\{\max_{j\in J}\left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\}, \alpha_{i0}\right\}$$

by symmetry, the same goes for j, and we have

$$v_j = \max\left\{\max_{i\in I}\left\{\gamma_{ij} : \alpha_{ij} \geq u_i\right\}, \gamma_{0j}\right\}.$$

Assume that there are strict preferences, ie $\alpha_{ij} \neq \alpha_{ij'}$ for all $j \neq j' \in J \cup \{0\}$ and $\gamma_{ij} \neq \gamma_{i'j}$ for all $i \neq i' \in I \cup \{0\}$.

$\mu_{ij} \in \{0, 1\}$ is the indicator that i and j are matched, and μ_{i0} is the indicator that i is unmatched and μ_{0j} is the indicator that j unmatched. μ is feasible if and only

$$\begin{aligned} \sum_{j\in J\cup\{0\}} \mu_{ij} &= 1 \\ \sum_{i\in I\cup\{0\}} \mu_{ij} &= 1 \end{aligned}$$

In a match μ, i gets utility $U_i^\mu = \sum_{j\in J\cup\{0\}} \mu_{ij}\alpha_{ij}$ and $V_j^\mu = \sum_{i\in I\cup\{0\}} \mu_{ij}\gamma_{ij}$.
A stable matching is a feasible matching μ such that
(1) there is no blocking pair ij, i.e. no pair i and j such that

$$\alpha_{ij} > U_i^\mu \text{ and } \gamma_{ij} > V_j^\mu$$

(2) there is no i such that $\alpha_{i0} > U_i^\mu$
(3) there is no j such that $\gamma_{0j} > V_j^\mu$.

Remark 1. No ties makes that we could have written

$$\alpha_{ij} > U_i^\mu \text{ and } \gamma_{ij} > V_j^\mu$$

with one of these inequalities being weak.

Remark 2. Ordinal or cardinal. This notion is an ordinal notion ,meaning that the cardinal value of α and γ is unimportant – just the rank ordering over the j and the $\acute{\imath}$'s is.

Adachi's theorem. If μ is stable matching, then $u = U^\mu$ and $v = V^\mu$ satisfy Adachi's equations

$$\begin{aligned} u_i &= \max\left\{\max_{j\in J}\left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\}, \alpha_{i0}\right\} \\ v_j &= \max\left\{\max_{i\in I}\left\{\gamma_{ij} : \alpha_{ij} \geq u_i\right\}, \gamma_{0j}\right\}. \end{aligned}$$

Conversely, if (u, v) satisfy Adachi's equations, then defining μ_{ij} as

$$\mu_{ij} = 1\{\alpha_{ij} = u_i\}$$

then one can show that $\mu_{ij} = 1\{\gamma_{ij} = v_j\}$ and that μ is a stable matching.

Now:
- reformulate these equations as $Q(p) = 0$
- investigate various coordinate algorithms to solve $Q(p) = 0$.
[- version of the model with random utility.]

Define

$$p_i = -u_i \text{ and } p_j = v_j$$

and

$$\begin{aligned} Q_i(p) &= p_i + \max\left\{\max_{j \in J}\{\alpha_{ij} : \gamma_{ij} \geq p_j\}, \alpha_{i0}\right\} \\ Q_j(p) &= p_j - \max\left\{\max_{i \in I}\{\gamma_{ij} : \alpha_{ij} \geq -p_i\}, \gamma_{0j}\right\} \end{aligned}$$

Remark. Q has gross substitutes but it is not inverse isotone. That's OK – we know we cannot have uniqueness.

Gauss-Seidel's algorithm.

Initialize the p^0's to have supersolution. $Q_k(p) \geq 0$. Take $p_i^0 = \max_{j \in J_0}\{-\alpha_{ij}\}$ and $p_j^0 = \max_{i \in I_0}\{\gamma_{ij}\}$, then

$$Q(p^0) \geq 0$$

This means that the proposing side is j.

We have

$$\begin{aligned} p_i^{t+1} &= -\max\left\{\max_{j \in J}\{\alpha_{ij} : \gamma_{ij} \geq p_j^t\}, \alpha_{i0}\right\} \\ p_j^{t+1} &= \max\left\{\max_{i \in I}\{\gamma_{ij} : \alpha_{ij} \geq -p_i^{t+1}\}, \gamma_{0j}\right\} \end{aligned}$$

which is Adachi's algorithm. Alternatively, we could have taken

$$\begin{aligned} p_j^{t+1} &= \max\left\{\max_{i \in I}\{\gamma_{ij} : \alpha_{ij} \geq -p_i^t\}, \gamma_{0j}\right\} \\ p_i^{t+1} &= -\max\left\{\max_{j \in J}\{\alpha_{ij} : \gamma_{ij} \geq p_j^{t+1}\}, \alpha_{i0}\right\} \end{aligned}$$

Remark – Tarski's fixed point theorem. The operator $T(p)$ defined by

$$\begin{aligned} T_i(p) &= -\max\left\{\max_{j \in J}\{\alpha_{ij} : \gamma_{ij} \geq p_j\}, \alpha_{i0}\right\} \\ T_j(p) &= \max\left\{\max_{i \in I}\{\gamma_{ij} : \alpha_{ij} \geq -p_i\}, \gamma_{0j}\right\} \end{aligned}$$

is an order-preserving operator. We know by Tarski's fixed point theorem that the set of fixed points of this operator is a lattice. Hence the set of p such that $Q(p) = 0$ is a lattice.

BE CAREFUL! P is a lattice does not mean that if $p, p' \in P$ then p'' defined $p''_i = \max\{p_i, p'_i\}$ belongs to P.

P is a lattice means that $p \vee p' \in P$, but $(p \vee p')_i$ in general differs from $\max\{p_i, p'_i\}$.

Take $P = \{(0,0), (1,2), (2,1), (3,3)\}$ is a lattice but $(1,2) \vee (2,1) = (3,3)$.