$u_x = \max_y \{\alpha_{xy} + w_{xy}, 0\}$
$v_y = \max_x \{\gamma_{xy} - w_{xy}, 0\}$

Assume we have determined (u, v).
$u_x \geq \alpha_{xy} + w_{xy}$ hence $w_{xy} \leq u_x - \alpha_{xy}$
$v_y \geq \gamma_{xy} - w_{xy}$ hence $w_{xy} \geq \gamma_{xy} - v_y$
therefore any w such that

$$\underline{w}_{xy} := \gamma_{xy} - v_y \leq w_{xy} \leq \bar{w}_{xy} := u_x - \alpha_{xy}$$

Indeed, if $\mu_{xy} > 0$, then $\bar{w}_{xy} = \underline{w}_{xy}$
if $\mu_{xy} = 0$ then in general $\bar{w}_{xy} \geq \underline{w}_{xy}$.

1 The Choo-Siow model as an optimization problem

Choo-Siow model

$$\begin{aligned} n_x &= f_x(a,b) := \exp\left(-\frac{a_x}{T}\right) + \sum_y \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) \\ m_y &= f_y(a,b) := \exp\left(-\frac{b_y}{T}\right) + \sum_x \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) \end{aligned}$$

Write $M_{x0} = \exp\left(-\frac{a_x}{T}\right)$ and $M_{xy} = \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right)$
Write down the Jacobian of this system

$$\begin{pmatrix} \frac{\partial f_x}{\partial a_{x'}} & \frac{\partial f_x}{\partial b_{y'}} \\ \frac{\partial f_y}{\partial a_{x'}} & \frac{\partial f_y}{\partial b_{y'}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{T} diag\left(\left(M_{x0} + \frac{1}{2}\sum_y M_{xy}\right)_x\right) & \left(-\frac{1}{2T} M_{xy'}\right)_{xy'} \\ \left(-\frac{1}{2T} M_{x'y}\right)_{x'y} & -\frac{1}{T} diag\left(\left(M_{0y} + \frac{1}{2}\sum_x M_{xy}\right)_y\right) \end{pmatrix}$$

The Jacobian is symmetric.
Consider

$$\min_{a,b} F(a,b)$$

where

$$F(a,b) = \left\{ \begin{array}{c} \sum_x n_x a_x + \sum_y m_y b_y \\ +T\sum_x \exp\left(-\frac{a_x}{T}\right) + T\sum_y \exp\left(-\frac{b_y}{T}\right) \\ +2T\sum_{xy} \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) \end{array} \right\}$$

and we have $f_x(a,b) = \partial F(a,b)/\partial a_x$ and $f_y(a,b) = \partial F(a,b)/\partial b_y$.

We can reformulate Choo-Siow's equations as

$$\begin{aligned} \partial F(a,b)/\partial a_x &= 0 \\ \partial F(a,b)/\partial b_y &= 0 \end{aligned}$$

1.1 The Choo-Siow model as an equilibrium problem with gross substitutes

Recall that an equilbrium pb w gross substitutes can be expressed as

$$Q_z(p) = 0$$

where the Jacobian of Q is positive on the diagonal, and off-diagonal nonpositive.

$$\frac{\partial Q_z}{\partial p_{z'}} \leq 0 \text{ for } z \neq z'$$

ie the Jacobian of Q is a Z-matrix.

Here, the Jacobian of the Choo-Siow equations is the Hessian of F, which is

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

where H_{11} and H_{22} diagonal negative on the diagonal

and all the entries of $H_{12} = H_{21}^{\top}$ are negative.
$Q_x(p) = M_{x0}(-p_x) + \sum_y M_{xy}(-p_x, p_y) - n_x$
$Q_y(p) = -M_{0y}(p_y) - \sum_x M_{xy}(-p_x, p_y) + m_y$
$p_z = (p_x, p_y)$
$p_x = -a_x$
$p_y = b_y$
$Q_x(p) = f_x(-(p_x), (p_y)) - n_x$
$Q_y(p) = m_y - f_y(-(p_x), (p_y))$

Coordinate update algorithm for this system.
Blockwise Jacobi / Gauss-Seidel:
Start with an initial value $p^0 = \left(-a_x^0, b_y^0\right)$
Solve for a_x^{t+1} such that
$M_{x0}\left(a_x^{t+1}\right) + \sum_y M_{xy}\left(a_x^t, b_y^t\right) = n_x$
Solve b_y^{t+1} such that
$M_{0y}\left(b_y^{t+1}\right) + \sum_x M_{xy}\left(a_x^{t+1}, b_y^{t+1}\right) = m_y$

Note that this can interpreted as a coordinate descent algorithm
$a^{t+1} = \arg\min_a F(a, b^t)$
$b^{t+1} = \arg\min_b F\left(a^{t+1}, b\right)$

Back to the problem:
$\exp\left(-\frac{a_x^{t+1}}{T}\right) + \sum_y \exp\left(\frac{\Phi_{xy}}{2T}\right) \exp\left(-\frac{a_x^{t+1}}{2T}\right) \exp\left(-\frac{b_y^t}{2T}\right) = n_x$

and
$\exp\left(-\frac{b_y^{t+1}}{T}\right)+\sum_x \exp\left(\frac{\Phi_{xy}}{2T}\right)\exp\left(-\frac{a_x^{t+1}}{2T}\right)\exp\left(-\frac{b_y^{t+1}}{2T}\right)=n_x$
Let's introduce $K_{xy}=\exp\left(\frac{\Phi_{xy}}{2T}\right)$, and denote
$A_x^t=\exp\left(-\frac{a_x^{t+1}}{2T}\right)$
$B_y^t=\exp\left(-\frac{b_y^t}{2T}\right)$
and rewrite the system as
$\left(A_x^{t+1}\right)^2+\sum_y K_{xy}A_x^{t+1}B_y^t=n_x$
and
$\left(B_y^{t+1}\right)^2+\sum_y K_{xy}A_x^{t+1}B_y^{t+1}=n_x$

eg let's solve the first one
$A_x^2+2A_x\frac{\sum_y K_{xy}B_y}{2}+\left(\frac{\sum_y K_{xy}B_y}{2}\right)^2=n_x+\left(\frac{\sum_y K_{xy}B_y}{2}\right)^2$
$\left(A_x+\frac{\sum_y K_{xy}B_y}{2}\right)^2=n_x+\left(\frac{\sum_y K_{xy}B_y}{2}\right)^2$
$A_x=-\frac{\sum_y K_{xy}B_y}{2}+\sqrt{n_x+\left(\frac{\sum_y K_{xy}B_y}{2}\right)^2}$
Let's check that BGH is satisfied
$Q_x\left(p\right)=M_{x0}\left(-p_x\right)+\sum_y M_{xy}\left(-p_x,p_y\right)-n_x$
$Q_y\left(p\right)=-M_{0y}\left(p_y\right)-\sum_x M_{xy}\left(-p_x,p_y\right)+m_y$
$\sum_x Q_x\left(p\right)+\sum_y Q_y\left(p\right)=\sum_x M_{x0}\left(-p_x\right)-\sum_y M_{0y}\left(p_y\right)+cte$
Thus the law of aggrete supply holds.
Thus the system is inverse isotone.

2 Connecting with yesterday's model

$x=$ driver
$y=$ passengers
$z=$ locations
$U_{xz}+P_z$ is the utility of driver picking at z
$V_{zy}-P_z$ is the utility of passenger y picked up at z
Chiappori, McCann and Nesheim (ET 2011)

One can show that at equilibirum if x picks y, then it is in location z such that
$z\in\arg\max_z\left\{U_{xz}+V_{zy}\right\}$

Further, yesterday's problem can be reformulated as a matching problem between passengers and drivers where the matching surplus is

$$\Phi_{xy}=\max_z\left\{U_{xz}+V_{zy}\right\}.$$

Conversely, from the matching model to the hedonic model, assume $Y=Z$

and assume that $V_{zy} = 0$ if $z = y$ and $V_{zy} = -\infty$ if $z \neq y$. In that case,

$$\Phi_{xy} = \max_{z} \{U_{xz} + V_{zy}\} = U_{xy}.$$