math+econ+code on equilibrium virtual whiteboard, day 1

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Excess supply function

$$e_z^\sigma(p) = \sum_i \frac{\exp\left(\frac{u_{iz}(p_z)}{\sigma}\right)}{\sum_{z'\in Z_0}\exp\left(\frac{u_{iz'}(p_{z'})}{\sigma}\right)} - \sum_j \frac{\exp\left(\frac{-c_{jz}(p_z)}{\sigma}\right)}{\sum_{z'\in Z_0}\exp\left(\frac{-c_{jz'}(p_{z'})}{\sigma}\right)}$$

where

$$Z_0 = Z \cup \{0\}$$

Properties of e^σ :

Gross substitutes: $e_z^\sigma(p)$ is increasing in p_z and decreasing in p_x $(x \neq z)$. Indeed

$$\begin{aligned} e_z^\sigma(p) &= \sum_i \frac{1}{\sum_{z'\in Z_0}\exp\left(\frac{u_{iz'}(p_{z'})}{\sigma} - \frac{u_{iz}(p_z)}{\sigma}\right)} \\ &\quad - \sum_j \frac{1}{\sum_{z'\in Z_0}\exp\left(\frac{c_{jz}(p_z)}{\sigma} - \frac{c_{jz'}(p_{z'})}{\sigma}\right)} \end{aligned}$$

We want to solve

$$e_z^\sigma(p) = 0 \ \forall z \in Z.$$

Coordinate update algorithms. (Jacobi and Gauss-Seidel).

Idea (Jacobi) = given a guess p_z^t of the solution, update p_z^{t+1} such that

$$e_z^\sigma\left(p_z^{t+1}; p_{-z}^t\right) = 0$$

where p_{-z}^t means all the other entries of p but the zth entry.

Great if you have access to parallel computing. For instance if $|Z| = 3$

$$\begin{aligned} e_1^\sigma\left(p_{z_1}^{t+1}, p_{z_2}^t, p_{z_3}^t\right) &= 0 \\ e_2^\sigma\left(p_{z_1}^t, p_{z_2}^{t+1}, p_{z_3}^t\right) &= 0 \\ e_3^\sigma\left(p_{z_1}^t, p_{z_2}^t, p_{z_3}^{t+1}\right) &= 0 \end{aligned}$$

Serial version = Gauss-Seidel. Assume $|Z| = 3$

$$\begin{aligned}
e_1^\sigma \left(p_{z_1}^{t+1}, p_{z_2}^{t}, p_{z_3}^{t}\right) &= 0 \\
e_2^\sigma \left(p_{z_1}^{t+1}, p_{z_2}^{t+1}, p_{z_3}^{t}\right) &= 0 \\
e_3^\sigma \left(p_{z_1}^{t+1}, p_{z_2}^{t+1}, p_{z_3}^{t+1}\right) &= 0
\end{aligned}$$

Reference: James Ortega and Werner Rheinboldt (1970). Iterative Solution of Nonlinear Equations in Several Variables. SIAM.

0.1 Coordinate update function

Define the coordinate update function $cu_z\left(p_{-z}\right)$ as the solution p_z' to

$$e_z\left(p_z', p_{-z}\right) = 0$$

We can describe Jacobi as
$p_z^{t+1} = cu_z\left(p_{-z}^t\right)$ for each z.

1 Convergence of Jacobi

Jacobi can be written as

$$p_z^{t+1} = cu_z\left(p^t\right)$$

Note: we are looking for p^* such that $e\left(p^*\right) = 0$. That is, we are looking for p^* such that $p^* = cu\left(p^*\right)$.

Property 1. cu_z is increasing, in the sense that

$$p \leq p' \implies cu\left(p\right) \leq cu\left(p'\right)$$

Indeed,

$$e_z\left(cu_z\left(p_{-z}\right); p_{-z}\right) = 0$$

take derivative wrt p_x where $x \neq z$ and get

$$\frac{\partial e_z}{\partial p_z}\frac{\partial cu_z}{\partial p_x}\left(p_{-z}\right) + \frac{\partial e_z}{\partial p_x} = 0$$

hence

$$\frac{\partial cu_z}{\partial p_x}\left(p_{-z}\right) = -\frac{\frac{\partial e_z}{\partial p_x}}{\frac{\partial e_z}{\partial p_z}}$$

Now we know that $\frac{\partial e_z}{\partial p_z} > 0$ and $\frac{\partial e_z}{\partial p_x} \leq 0$ (gross substitutes).

Therefore, the coordinate update is a monotone increasing function.

If p^* exists, and if p^0 is such that $p^0 \leq p^*$ for all z, then

$$p^1 = cu\left(p^0\right) \leq cu\left(p^*\right) = p^*$$

thus by induction all the Jacobi sequence p^t will be less that p^*.

Assume that $p^0 \leq p^1$. Then $cu\left(p^0\right) = p^1 \leq cu\left(p^1\right) = p^2$ and by induction p^t will be increasing.

Assuming e_z is continuous in p_z, this implies $p^t \to p^*$.

Theorem (Berry, Gandhi, Haile, weak version). Assuming e_z is increasing in p_z, decreasing in p_x for $x \neq z$, and that $\sum_{z \in \mathcal{Z}} e_z\left(p_z\right)$ is increasing in each of the p_x, $x \in \mathcal{Z}$ (law of aggregate supply). The the map $p \to e\left(p\right)$ is inverse isotone, meaning that for any two price vectors p and p', then

$$e_z\left(p\right) \leq e_z\left(p'\right) \ \ \forall z \in Z$$

implies $p_z \leq p'_z$ for all $z \in Z$.

Why is this useful?

for our purposes, assume that p is a subsolution, i.e.

$$e_z\left(p\right) \leq 0 = e_z\left(p^*\right) \ \ \forall z$$

then 1) $p \leq p^*$.

2) $e_z\left(p\right) \leq 0 = e_z\left(cu_z\left(p\right); p_{-z}\right)$

therefore $p_z \leq cu_z\left(p\right)$.

As a result, if we start from a subsolution, then the Jacobi sequence

$$p^{t+1} = cu\left(p^t\right)$$

is converging to p^*.

We start from $e\left(p^0\right) \leq 0 = e\left(p^*\right)$ (subsolution). By inverse isotonicity, it follows

$$p^0 \leq p^*$$

as a result because cu is isotone (=monotone=order preserving)

$$p^t = cu^t\left(p^0\right) \leq cu^t\left(p^*\right)$$

but as p^* is a fixed point of cu, it follows that

$$p^t \leq p^*.$$

Second, if $p^0 \leq p^*$, then $p^0 \leq cu\left(p^0\right) = p^1$ and as a result, by applying cu^t again

$$p^t \leq p^{t+1}$$

Therefore p^t_z increases and is bounded above. As a result it converges to $\bar{p}_z$.

$$e_z\left(cu\left(p^t_z\right), p^t_{-z}\right) = 0$$

therefore

$$e_z\left(\bar{p}_z, \bar{p}_{-z}\right) = 0.$$

$cu\left(p_z^t\right) \geq p_z^t$

Apply this to the supply function above

$$e_z^\sigma(p) = \sum_i \frac{\exp\left(\frac{u_{iz}(p_z)}{\sigma}\right)}{\sum_{z' \in Z_0} \exp\left(\frac{u_{iz'}(p_{z'})}{\sigma}\right)} - \sum_j \frac{\exp\left(\frac{-c_{jz}(p_z)}{\sigma}\right)}{\sum_{z' \in Z_0} \exp\left(\frac{-c_{jz'}(p_{z'})}{\sigma}\right)}$$

We shall:
1. Show that the assumptions in Berry Gandhi and Haile are met.
2. There is a subsolution.
3. There is a solution p^* – here, we shall show that there is a supersolution.

1. BGH assumptions.
Gross substitutes hold.
Let's show that the law of aggregate supply holds, ie

$$\sum_{z \in Z} e_z(p)$$

is increasing in all the p_z's.

$$\begin{aligned} \sum_{z \in Z} e_z^\sigma(p) &= \sum_i \left(1 - \frac{1}{\sum_{z' \in Z_0} \exp\left(\frac{u_{iz'}(p_{z'})}{\sigma}\right)}\right) \\ &\quad - \sum_j \left(1 - \frac{1}{\sum_{z' \in Z_0} \exp\left(\frac{-c_{jz'}(p_{z'})}{\sigma}\right)}\right) \end{aligned}$$

thus

$$\begin{aligned} \sum_{z \in Z} e_z^\sigma(p) &= |I| - |J| + \sum_j \frac{1}{\sum_{z' \in Z_0} \exp\left(\frac{-c_{jz'}(p_{z'})}{\sigma}\right)} \\ &\quad - \sum_i \frac{1}{\sum_{z' \in Z_0} \exp\left(\frac{u_{iz'}(p_{z'})}{\sigma}\right)} \end{aligned}$$

which shows that

$$\sum_{z \in Z} e_z^\sigma(p)$$

is increasing in each p_z and therefore law of aggregate supply holds, and therefore BGH assumptions are met.

2. There is a subsolution p^{sub}. Simply take $p_z = -N$ for N large enough.

3. There is a supersolutuion $p^{\sup}$. Simply take $p_z = +N$ for N large enough.
As a result $e\left(p^{sub}\right) \leq e\left(p^{\sup}\right)$

as a result, the Jacobi sequence that starts from p^{sub} is increasing and bounded above by $p^{\sup}$, and therefore converges to the equilibrium price p^*. Similarly he Jacobi sequence that starts from p^{sup} is decreasing and converges to p^*.

1.1 Convergence of Jacobi from any starting point

Now assume BGH assumptions are met.

Assume a solution p exists $e\left(p\right) = q$ for any q (therefore e is invertible)

Call p^* the solution to $e\left(p^*\right) = 0$.
Then Jacobi converges from any starting point.

Consider any initial price vector p^0, and consider the Jacobi sequence that starts from p^0. ie $p^{t+1} = cu\left(p^t\right)$.

$p^{lower} = e^{-1}\left(e\left(p^0\right) \wedge 0\right)$
$p^{upper} = e^{-1}\left(e\left(p^0\right) \vee 0\right)$
We have by definition
$e\left(p^{lower}\right) = e\left(p^0\right) \wedge 0 \leq 0$ is a subsolution and
$e\left(p^{lower}\right) = e\left(p^0\right) \wedge 0 \leq e\left(p^0\right)$ therefore
$p^{lower} \leq p^0$.

Similarly, p^{upper} is a supersolution which is greater than p^0.
$p^{lower} \leq p^0 \leq p^{upper}$
thus
$cu^t\left(p^{lower}\right) \leq cu^t\left(p^0\right) \leq cu^t\left(p^{upper}\right)$
both upper and lower bounds converge to p^*, therefore
$cu^t\left(p^0\right)$ converges to p^*.

Notation:
$a \vee b$ is the vector such that $\left(a \vee b\right)_z = \max\left(a_z, b_z\right)$
$a \wedge b$ is the vector such that $\left(a \wedge b\right)_z = \min\left(a_z, b_z\right)$