math+econ+code on equilibrium
virtual whiteboard, day 5

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1 Network topology

A directed network is defined as follows:

Consider a set of nodes Z

and a set of arcs $A \subseteq Z \times Z$ so if $xy \in A$ then interpret x as the origin, and y as the destination.

A flow μ is the vector of mass along the arcs, hence $\mu \in \mathbb{R}_{+}^{A}$.

μ is determined at equilibrium

Balance of mass:

Let q_z=quantity consumed locally at z, is assumed exogenous.

($q_z > 0$ means actually consumed, $q_z < 0$ means produced.)

$q_z = \sum_{x:xz \in A} \mu_{xz} - \sum_{y:zy \in A} \mu_{zy} = \left(\nabla^{\top} \mu\right)_z = \sum_a \nabla_{a,z} \mu_a$

where $\nabla_{a,z} = 1$ if z is endpoint of a, -1 if z is startpoint of a, and 0 otherwise.

Therefore balance of mass expresses:

$q = \nabla^{\top} \mu$.

Sum mass balance over all z

$\sum_z q_z = \sum_{xz:xz \in A} \mu_{xz} - \sum_{zy:zy \in A} \mu_{zy} = 0$.

Necessarily, $\sum_z q_z = 0$

Assume $p \in R^Z$ is the vector of prices (determined at equilibrium) such that p_z is the price of the commodity at z.

Trading equilibrium.

On arc $a = xy$, there is an infinite number of traders who can perform a carry trade.

Purchase at x, carry to y, sell at y.

Assume that the profit / rent function associated with this trade is

$R_{xy}(p)$ per unit of mass shipped which is
- increasing with p_y
- decreasing with p_x
- independent on the other entries of p.

Examples:
$R_{xy}(p) = p_y - p_x - c_{xy}$ (transferable utility case).
$R_{xy}(p) = (1 - \tau_{xy}) p_y - (1 + \theta_{xy}) p_x - c_{xy}$
$R_{xy}(p) = f_{xy}(p_y - p_x - c_{xy})$ (more general taxes on surplus)
[Later we will explore settings where
$p_y - p_x - c_{xy}(\mu_{xy})$
or even $R_{xy}(p; \mu_{xy})$
in link with congestion]

Perfect competition => Absence of rents
$R_a(p) \leq 0 \ \forall a$

Free entry
$\mu_a > 0 \implies R_a(p) = 0$

To summarize: (μ, p) is the solution to an *equilibrium flow problem* if the following conditions hold:
(i) balance of mass:
$q = \nabla^\top \mu$
(ii) Absence of rent:
$R_a(p) \leq 0 \ \forall a$
(iii) Free entry:
$\mu_a > 0 \implies R_a(p) = 0.$

$q = \nabla^\top \mu.$ ==> $q = N(\mu)$ $q_z = T \log \sum_a \exp\left(\frac{\nabla_{az}\mu_a}{T}\right)$
iceberg costs $q_z = \sum_x (1-\theta)\mu_{xz} - \sum_y (1+\tau)\mu_{zy}$
Can

$q_z = \sum_x \lambda_z \mu_{xz} - \sum_y \lambda_z \mu_{zy}$
$\frac{q_z}{\lambda_z} = \sum_x \mu_{xz} - \sum_y \mu_{zy}$

Brandon's version – more general
(i) balance of mass:
$q = N(\mu)$
(ii) Absence of rent:
$R_a(p) \leq 0 \ \forall a$
(iii) Free entry:
$\mu_a > 0 \implies R_a(p) = 0.$

Application:

1. Matching with TU/ ITU without unassigned agents
In this case $Z = X \cup Y$ and $A = X \times Y$
$x \in X$, $n_x = -q_x > 0$
$y \in Y$, $m_y = q_y > 0$
$\sum n_x = \sum m_y$

$R_{xy}(p) \leq 0$ translates into absence of a blocking pair $D_{xy}(u, v) \geq 0$
thus set $D_{xy}(u, v) = -R_{xy}(p)$

$u_x = p_x$
$v_y = -p_y$

TU case $R_{xy}(p) = p_y - p_x - c_{xy}$

$D_{xy}(u, v) = -R_{xy}(u_x, -v_y) = c_{xy} + u_x + v_y$
and $R_{xy}(p) \leq 0$ rewrites as $D_{xy}(u, v) \geq 0$ that is
$u_x + v_y \geq -c_{xy} =: \Phi_{xy}$
2. Matching with TU/ ITU with unassigned agents
$\sum_y \mu_{xy} + \mu_{x0} = n_x$
$\sum_x \mu_{xy} + \mu_{0y} = m_y$
$Z = X \cup Y \cup \{0\}$ $A = (X \times Y) \cup \{X \times \{0\}\} \cup (\{0\} \times Y)$

$D_{xy}(u, v) = -R_{xy}(u_x, -v_y)$
$R_{x0}(p_x, p_0) = R_{x0}(u_x, p_0) = -u_x + p_0$
so that $R_{x0}(p) \leq 0$ writes as $u_x \geq p_0$ and we shall normalize $p_0 = 0$.
Similarly
$R_{0y}(p_0, p_y) = R_{0y}(p_0, -v_y) = -v_y + p_0$
3. Hedonic models (day 1)
$x \in X$ (producers): there are n_x producers of type x
$y \in Y$ (consumers): there are m_y consumers of type y
$z \in Z$ (quality space)
μ_{xz}=the number of producers producing in quality z
μ_{zy}=the number of comsumers consuming in quality z
$\Pi_{xz}(p_z)$= profit of producer x producing z; increasing in p_z
$C_{zy}(p_z)$=cost of the consumer consuming z; increasing in p_z
u_x =indirect profit of consumer x
c_y=indirect cost of consumer y

Hedonic equilibrium:
total production in z=total consumption in z
$\sum_x \mu_{xz} = \sum_y \mu_{zy}$
$n_x = \sum_z \mu_{xz} + \mu_{x0}$
$m_y = \sum_z \mu_{zy} + \mu_{0y}$

$u_x \geq \Pi_{xz}(p_z)$ with equality if $\mu_{xz} > 0$
$u_x \geq 0$ with equality if $\mu_{x0} > 0$

$c_y \leq C_{yz}(p_z)$ with equality if $\mu_{yz} > 0$.
$c_y \leq 0$ with equality if $\mu_{0y} > 0$

2 Regularized equilibrium transport problem

(a) balance of mass:
$q = \nabla^\top \mu$
(b) Write down the ansatz
$\mu_a = \exp(R_a(p)/T)$

$R_a(p) = T \ln \mu_a$

Rewrite the problem as a function of p.
We get

$$Q(p) = q$$

where $Q(p) = \nabla^\top \exp(R(p)/T)$
that is $Q_z(p) = \sum_{x:xz\in A} \exp\left(\frac{R_{xz}(p)}{T}\right) - \sum_{y:zy\in A} \exp\left(\frac{R_{zy}(p)}{T}\right)$.

Let's take a look at
(a) balance of mass:
$q = N(\mu)$
(b) ansatz
$\mu_a = \exp(R_a(p))$

Then $Q(p) = N(\exp R(p))$
We have

$$\frac{\partial Q_z}{\partial p_{z'}} = \sum_a \frac{\partial N_z}{\partial \mu_a} \exp(R_a(p)) \frac{\partial R_a(p)}{\partial p_{z'}}$$

Gross substitutes. The sign $\frac{\partial R_a(p)}{\partial p_{z'}}$ is the same as the sign of $\nabla_{a,z'}$.
Claim: if the sign of $\frac{\partial N_z}{\partial \mu_a}$ is the same as $\nabla_{a,z}$, then Gross substitutes will hold.
Indeed $sign\left(\frac{\partial N_z}{\partial \mu_a} \exp(R_a(p)) \frac{\partial R_a(p)}{\partial p_{z'}}\right) = \nabla_{a,z}\nabla_{a,z'}$.
But if $z = z'$ this is > 0, and if $z \neq z'$, this is ≤ 0.
$N_z(\mu)$ should be increasing in μ_{xz}'s, and decreasing in the μ_{zy}'s.

$1^\top \nabla^\top \mu = 0$
$\varphi(N(\mu)) = 0$.

Claim. When $R(p) = p_y - p_x - c_{xy}$ (this is the TU case), we have that the problem is the solution to
$\max_q \sum_z p_z q_z - \sum_a T \exp\left(\frac{\nabla p - c}{T}\right)$

indeed, by first order conditions, one has
$q_z = \sum_{xz \in A} \exp\left(\frac{p_z - p_x - c_{xz}}{T}\right) - \sum_{y:zy \in A} \exp\left(\frac{p_y - p_z - c_{zy}}{T}\right)$

Open problems.
1. General conditions on R and on the network so that

$$\nabla^\top \exp\left(R\left(p\right)\right) = q$$

has a solution.
2. Relatedly, conditions on the problem to have a subsolution.
3. Considering the problem

$$\nabla^\top \exp\left(\frac{R\left(p\right)}{T}\right) = q$$

and assuming it has a solution p^T, under which conditions can we show that the solution

$$\mu^T = \exp\left(\frac{R\left(p^T\right)}{T}\right)$$

converges as $T \to 0$ to a solution to the unregularized problem?
4. Which solution to the unregularized problem is picked up?

For 4, in the optimization case, this is quite easy to answer. Indeed, in that case, p^T is a solution to

$$\max_q \left\{ \sum_z p_z q_z - \sum_a T \exp\left(\frac{\nabla p - c}{T}\right) \right\} = \min_{\mu : \nabla^\top \mu = q} \left\{ \sum_a \mu_a c_a + T \sum \mu_a \ln \mu_a \right\}$$

therefore $\mu^T = \exp\left(\frac{\nabla p^T - c}{T}\right)$ will converge to a μ that minimimizes

$$\min_{\mu : \nabla^\top \mu = q} \left\{ \sum \mu_a \ln \mu_a \right\}$$

among the solution of

$$\min_{\mu : \nabla^\top \mu = q} \left\{ \sum_a \mu_a c_a \right\}.$$

$Q\left(p\right) = q$ where q is fixed.

$Q\left(q\right) = D\left(p\right)$ where D is an excess demand ie D_z is decreasing in p_z and weakly increasing in p_{-z}.

$Q\left(q\right) - D\left(p\right) = 0$
in particular, could assume that $D_z\left(p\right) = d_z\left(p_z\right)$ where $d_z\left(.\right)$ decreasing.