Class notes on gross substitutes

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math+econ+code masterclass on equilibrium transport and matching models in economics
Special lecture 4, November 12, 2021

1 Agenda

The following topics will be (briefly) touched upon: **polymatroids** and **exchangeability**, Lovasz extensions, **discrete convex analysis, L / L# / M / M# convexity** as well as unified gross substitutes. This class is only meant to provide an bird's-eye view of topic.

2 Some references

- Murota's monograph
  https://www.amazon.com/Discrete-Analysis-Monographs-Mathematics-Applications/dp/0898715407
- Murota's overview
  https://www.comp.tmu.ac.jp/kzmurota/paper/HIMSummerSchool15Murota.pdf
- Gul and Stacchetti papers:
  https://www.sciencedirect.com/science/article/abs/pii/S0022053199925310
  https://www.sciencedirect.com/science/article/abs/pii/S0022053199925802
- Bach's monograph:
  https://www.di.ens.fr/~fbach/2200000039-Bach-Vol6-MAL-039.pdf
- 2 working papers of mine (available upon request):
  Galichon, Hsieh and Sylvestre. "Monotone comparative statics for submodular functions, with an application to aggregated deferred acceptance."
  Galichon, Samuelson and Vernet. "Unified Gross Substitutes".

3 Reminders on gross substitutes

Recall the example from last time, which was the supplier's problem

$$Q(p) = \arg\max_q \left\{ p^\top q - C(q) \right\}$$

where $C(q)$ is the cost of producing q, which is assumed convex. Consider the indirect utility

$$C^*(p) = \max_q \left\{ p^\top q - C(q) \right\}$$

and by duality

$$C(q) = \max_p \left\{ p^\top q - C^*(p) \right\}$$

we have $Q(p) = \partial C^*(p)$. Recall that gross substitutes is equivalent with submodularity of C^*. Indeed, assuming differentiability, $Q_z(p) = \partial C^*(p) / \partial p_z$, and

$$\frac{\partial Q_z(p)}{\partial p_{z'}} = \frac{\partial^2 C^*(p)}{\partial p_z \partial p_{z'}}$$

thus gross substutues mean that $\frac{\partial^2 C^*(p)}{\partial p_z \partial p_{z'}} \leq 0$ for $z \neq z'$.

This lecture is all about gross substitutes, which here means $C^*(p)$ submodular [and convex].

Assume $f(p)$ is submodular and convex. Define $f^*(q) = \max_p \left\{ p^\top q - f(p) \right\}$. Denote

$$\begin{aligned} Q(p) &= \arg\max_q \left\{ p^\top q - f^*(q) \right\} = \partial f(p) \\ Q^{-1}(q) &= \arg\max_p \left\{ p^\top q - f(p) \right\} = \partial f^*(q) \end{aligned}$$

What can I say about f^*?

* it's convex

* it's supermodular. Heuristically. Show that

$$\frac{\partial^2 f^*(q)}{\partial q_z \partial q_{z'}} \geq 0 \text{ for } z \neq z'$$

well,

$$\frac{\partial^2 f^*(q)}{\partial q_z \partial q_{z'}} = \frac{\partial}{\partial q_z} \frac{\partial f^*(q)}{\partial q_{z'}}$$

and $\frac{\partial f^*(q)}{\partial q_{z'}} = Q^{-1}(q) = \arg\max_p \left\{ p^\top q - f(p) \right\}$. Now we can apply Topkis to $(p, q) \to p^\top q - f(p)$.

This function satisfies increasing differences; and it is supermodular in p. Hence $\arg\max_p \left\{ p^\top q - f(p) \right\}$ is nondecreasing in q. Hence $\frac{\partial f^*(q)}{\partial q_{z'}}$ is a nondecreasing function of q. Hence its derivative wrt any q_z is ≥ 0. Thus

$$\frac{\partial^2 f^*(q)}{\partial q_z \partial q_{z'}} = \frac{\partial}{\partial q_z} \frac{\partial f^*(q)}{\partial q_{z'}} \geq 0.$$

* Proof that if f is submodular and convex, then f^* is supermodular and convex.

$f^*(q \vee q') + f^*(q \wedge q') \geq f^*(q) + f^*(q')$

$f^*(q) = \max \{pq - f(p)\} = pq - f(p)$ for some p

$f^{*}(q') = \max\{pq' - f(p)\} = p'q' - f(p')$ for some p'

$pq - f(p) + p'q' - f(p') \leq pq + p'q' - (f(p) + f(p')) \leq pq + p'q' - (f(p \vee p') + f(p \wedge p'))$
But
$pq + p'q' \leq (p \wedge p')(q \wedge p') + (p' \vee p')(q' \vee q')$
hence
$pq - f(p) + p'q' - f(p') \leq (p \wedge p')(q \wedge p') + (p' \vee p')(q' \vee q') - (f(p \wedge p') + f(p \vee p'))$
$\leq f^{*}(q \vee q') + f^{*}(q \wedge q')$

Question: does the converse hold. Ie does $g(q)$ convex and supermodular imply that $g^{*}(p) = \max_q \{pq - g(q)\}$ is submodular.

Yes when the dimension of p and q is 2.

In fact, no as soon as dimension ≥ 3.

Submodularity of $f(p)$ means that $D^2 f(p)$ is a Z-matrix. Assuming f is strictly convex, $D^2 f(p)$ is strictly nonreversing, thus $D^2 f(p)$ is an M-matrix. We have that it follows that

$$\left(D^2 f(p)\right)^{-1}$$

has nonnegative entries. But

$$\left(D^2 f(p)\right)^{-1} = D^2 f^{*}(q)$$

where $q = \nabla f(p)$.

We can do this by blocks. We could have

$$D^2 g(q) = \begin{pmatrix} D_1 & P^{\top} \\ P & D_2 \end{pmatrix}$$

where P has nonnegative entries and D_1 and D_2 are diagonal matrices with positive diagonal terms.

Because of the particular structure., I can change the signs of q and write $\tilde{q} = (q_X, -q_Y)$

Define $\tilde{g}(\tilde{q}) = g(\tilde{q}_X, -\tilde{q}_Y)$. Compute the Hessian

$$D^2 \tilde{g}(\tilde{q}) = \begin{pmatrix} D_1 & -P^{\top} \\ -P & D_2 \end{pmatrix}$$

thus now $D^2 \tilde{g}(\tilde{q})$ is a Z-matrix, and $\tilde{g}(\tilde{q})$ is submodular.

$\mu_{xy} = \exp(\Phi_{xy} - u_x - v_y)$
$\sum_y \exp(\Phi_{xy} - u_x - v_y) = n_x$

4 Discrete convexity, exchangeability and polymatroids

1. L-Convexity

We shall assume that the domain of C is contained in $\Delta_a = \left\{q : \sum_{z\in Z} q_z = a\right\}$, eg $a = 1$. That is

$$C(q) = +\infty \text{ if } q \notin \Delta_a$$

and therefore

$$C^*(p) = \max\{pq - C(q)\}$$

we have

$$C^*(p + t1_Z) = C^*(p) + ta.$$

indeed

$$\begin{aligned} C^*(p + t1_Z) &= \max_q \left\{(p + t1)^\top q - C(q)\right\} = \max_q \left\{ta + p^\top q - C(q)\right\} \\ &= C^*(p) + ta. \end{aligned}$$

A $f(p)$ function is L-convex iff
f is convex
$f(p + t) = f(p) + ta$ for some a.
f is submodular

2. M-convexity
A function $g(q)$ is M-convex iff
g is convex and g^* is L-convex.

In particular,
* the domain of g is Δ_a
* g supermodular.
But this is not sufficient!

Theorem (Murota). A function $g(q)$ is M-convex iff
g is convex
The domain of g is contained in $\Delta_a = \left\{q : \sum_{z\in Z} q_z = a\right\}$
Exchageability is satisfied:
given $q, q' \in \Delta_a$, for any $\delta_1 : 0 \leq \delta^1 \leq (q' - q)^+$, $\exists \delta^2 : 0 \leq \delta^2 \leq (q - q')^+$ with $\sum_z \delta_z^1 = \sum_z \delta_z^2$ s.t.
$g\left(q + \delta^1 - \delta^2\right) + g\left(q' + \delta^2 - \delta^1\right) \leq g(q) + g(q')$.

3. M#- and L#- convexity
g is M#-convex iff $\tilde{g}(q_0, q_1, ..., q_z) = g(q_1, ..., q_z)$ if $q_0 + \sum_z q_z = 0$ is M-convex.
$\tilde{f}(\pi, p_1, .., p_n) = \max_q \left\{(p - \pi 1)^\top q - g(q_1, ..., q_z)\right\} = g^*(p_1 - \pi, ..., p_Z - \pi)$
Thus f is L#-convex iff $(\pi, p_1, ..., p_Z) \to f(p_1 - \pi, ..., p_Z - \pi)$ is L-convex.

4. Base polymatroid /base polyhedron

Given a closed convex set Q, Q is a base polymatroid when it support function

$$\iota_Q^* (b) = \sup_{q \in Q} q^\top b$$

is a L-function.

Equivalently, when its indicator function ι_Q is a M-function, where the indicator function is defined by $\iota_Q (q) = 0$ if $q \in Q$, $+\infty$ if $q \notin Q$

As a result, a base polymatroid can be expressed as $Q = \left\{ q : \sum_{z \in B} q_z \leq h (B) , \forall B \subseteq Z, \sum_{z \in Z} q_z = a \right\}$ where h is a submodular function.

Equivalently, when Q is contained in $\Delta_a = \left\{ q : \sum_{z \in Z} q_z = a \right\}$ and when Q satisfies an exchangeability axiom:

given $q, q' \in Q$, for $\delta_1 : 0 \leq \delta^1 \leq (q' - q)^+$, $\exists \delta^2 : 0 \leq \delta^2 \leq (q - q')^+$ s.t. $q + \delta^1 - \delta^2 \in Q$ and $q' + \delta^2 - \delta^1 \in Q$.

One of our leading questions is how does this translate on the properties C. When

$$Q (p) = \arg\max_q \left\{ p^\top q - C (q) \right\}$$

Here, we are going to assume that the domain of C is Δ_1. We want to show that $Q (p)$ is a base polymatroid, hence that there is a submodular function $h (B)$ such that

$$Q (p) = \left\{ q \in \Delta_1 : \sum_{z \in B} q_z \leq h (B) \right\}$$

We have

$$\begin{aligned} Q (p) &= \arg\max_q \left\{ p^\top q - C (q) \right\} = \left\{ q \in \Delta_1 : p \in \arg\max_p \left\{ pq - C^* (p) \right\} \right\} \\ &= \left\{ q \in \Delta_1 : \frac{d}{dt} \left\{ C^* (p + tb) - (p + tb)^\top q \right\} |_{0^+} \geq 0 \forall b \right\} \\ &= \left\{ q \in \Delta_1 : q^\top b \leq \frac{d}{dt} \left\{ C^* (p + tb) \right\} |_{0^+} \forall b \right\} \end{aligned}$$

define $\tilde{h} (b) = \frac{d}{dt} \left\{ C^* (p + tb) \right\} |_{0^+}$. We have

$$Q (p) = \left\{ q \in \Delta_1 : \sum_z q_z b_z \leq \tilde{h} (b) \right\}$$

Take $b_z = 1 \{ z \in B \}$ that is $b = 1_B$, and let's define $h (B) = \tilde{h} (1_B)$. Then we see that

$$Q (p) \subseteq \left\{ q \in \Delta_1 : \sum_{z \in B} q_z \leq h (B) \right\}$$

But we would like to show that the converse holds

h submodular means $h\left(B\cap B'\right)+h\left(B\cup B'\right)\leq h\left(B\right)+h\left(B'\right)$
this is equivalent with $\tilde{h}\left(1_{B}\wedge1_{B'}\right)+\tilde{h}\left(1_{B}\vee1_{B'}\right)\leq\tilde{h}\left(1_{B}\right)+\tilde{h}\left(1_{B'}\right)$.
$1_{B}\wedge1_{B'}=1_{B\cap B'}$ and $1_{B}\vee1_{B'}=1_{B\cup B'}$
.

A remark Typically we consider problems of the sort

$$Q\left(p\right)=q$$

with gross substitutes. We expect $Q\left(p\right)$ to be a base polymatroid, and $Q^{-1}\left(q\right)$ to be a lattice. Hence base polymatroids is somehow the "dual structure" to lattice structure. Prices live in lattices, quantities live in (base) polymatroids.