Class notes on submodularity

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math+econ+code masterclass on equilibrium transport and matching models in economics
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1 Agenda

This class will introduce topics such as lattices, submodularity, Veinott's strong set ordering, and Topkis' theorem. This class is only meant to provide an bird's-eye view of topic.

2 Some references

Topkis' book: Topkis, Donald M. (1998). Supermodularity and Complementarity. Princeton University Press.

Veinott's lectute notes http://dl.icdst.org/pdfs/files/079ac78c59f5789ec9bbf319e5c9835c.pdf

Galichon, The unreasonable effectiveness of optimal transport in economics. https://arxiv.org/abs/2107.04700

3 An introduction

3.1 Some examples

Question: consider

$$P(q) = \arg\max_p F(p, q) \tag{1}$$

when is P increasing in q?

Example 1: supply problem. Consider the producer's problem

$$Q(p) = \arg\max_q \left\{ p^\top q - C(q) \right\}$$

then letting the indirect profit of the producer as

$$C^*(p) = \max_q \left\{ p^\top q - C(q) \right\}$$

we have by convex duality that $Q = P^{-1}$ where P given by

$$P(q) = \arg\max_p \left\{ p^\top q - C^*(p) \right\}.$$

This has the same form as in (**??**) with $F(p,q) = p^\top q - C^*(p)$.

$C^*(p) = \max_q \left\{p^\top q - C(q)\right\}$

1) FOC – q as a function of p: $p_i = \partial c(q) / \partial q_i$ that is $p = \nabla C(q)$

2) Enveloppe theorem – p as a function of q. this gives $\nabla C^*(p) = q$.

Example 2: optimal transport. Consider the problem

$$\min_{u,v} \left\{ \sum_x n_x u_x + \sum_y m_y v_y : u_x + v_y \geq \Phi_{xy} \right\}$$

then setting $p = (u_x, -v_y)$ and $q = (-n_x, m_y)$ yields

$$P(q) = \arg\max_p \left\{p^\top q - C(p)\right\}$$

where $C(p) = 0$ if $p_y - p_x \leq -\Phi_{xy}$ for all xy and $+\infty$ otherwise.

Example 3: regularized optimal transport. Same as the above with

$$C(p) = \sigma \sum_{xy} \exp\left(\frac{\Phi_{xy} + p_y - p_x}{\sigma}\right).$$

Question: what is the response of the prices p to a change in quantities q? In particular, when is p isotone in q, i.e. $q \leq q'$ implies $p \leq p'$?

3.2 Le Châtelier principle

Informal discussion. Assume differentiability. Then

$$P(q) = \{P(q)\} = \arg\max_p \{F(p,q)\},$$

and we would like to explore when $D_q P \geq 0$ by which we mean that $\partial P_i(q) / \partial q_j \geq 0$.

Write down first order conditions

$$D_p F(P(q), q) = 0 \tag{2}$$

and second order conditions ensure that $D_p^2 F(P(q), q)$, which is a symmetric matrix, is definite negative.

By differentiation of (**??**), we get

$$\left(D_{pp}^2 F\right) D_q P + D_{pq}^2 F = 0$$

and as a result

$$D_q P = -\left(D_{pp}^2 F\right)^{-1} \left(D_{pq}^2 F\right).$$

As a result, a sufficient condition for $D_q P \geq 0$ (termwise) is $-\left(D_{pp}^2 F\right)^{-1} \geq 0$ and $D_{pq}^2 F \geq 0$.

- $D^2_{pq}F \geq 0$ is called increasing differences: means that D_pF is increasing in q. This is satisfied for instance in the examples above when $F(p,q) = p^\top q - C(q)$.
- $-\left(D^2_{pp}F\right)^{-1} \geq 0$ is related in a fundamental way to **gross substitutes**.

3.3 Gross substitutes and submodularity

Assume in this paragraph that $F(p,q) = p^\top q - C^*(p)$.

Consider the supply function $Q(p) = P^{-1}(p)$.

We have $-D^2_{pp}F(p,q) = D^2C^*(p)$.

We have $Q(p) = \nabla C^*(p)$ by foc in $\arg\max_p \left\{p^\top q - C^*(p)\right\}$

Thus $D_pQ(p) = D^2C^*(p) = -D^2_{pp}F(p, Q(p))$.

Gross substitutes expresses that when the price of i increases, the supply for good $j \neq i$ decreases. (Producers “substitute” i to j as it is becoming more attractive to produce i). Therefore **gross substitutes is expressed by the property that D_pQ is a Z-matrix**. That means that

$$\frac{\partial^2 C^*(p)}{\partial p_i \partial p_j} \leq 0 \forall i \neq j.$$

Which is exactly submodularity of $C^*(p)$.

Gross substitutes is equivalent in this setting with the fact that $C^*(p)$ should be submodular – that is F is supermodular in p.

3.4 Submodularity, and Stieltjes matrices

Back to the general setting and simply assume that F is supermodular in p. We have

$$D_qP = -\left(D^2_{pp}F\right)^{-1} D^2_{pq}F.$$

We can assume that F is supermodular in p, i.e. $-\left(D^2_{pp}F\right)$ is a Z-matrix.

Can we conclude that $-\left(D^2_{pp}F\right)^{-1} \geq 0$ termwise?

In general (cf. previous lecture), A a Z-matrix does not imply $A^{-1} \geq 0$.

As we saw in the previous lecture, a Z-matrix A is such that $A^{-1} \geq 0$ if and only if it is a M-matrix, that is, if and only if it is nonreversing, that is if and only if $Au \leq 0$ and $u \geq 0$ imply $u = 0$.

Here we can rely on a bit more, which is second order conditions. We know by first order conditions that $-\left(D^2_{pp}F\right)^{-1}$ is symmetric negative positive.

Definition. A Z-matrix which symmetric positive definite is called a Stieltjes matrix.

Remark. If a function $g(p)$ is strict convex and submodular, then $D^2g(p)$ is a Stieljes matrix.

Property. A Stieltjes matrix is a M-matrix.

Proof. If A is a Stieltjes, it is a Z-matrix and thus we only need to verify that it is nonreversing. Assume $Au \leq 0$ and $u \geq 0$. Then $u^\top Au \leq 0$, but this implies $u = 0$ because A is symmetric positive definite.

As a result, $-D^2_{pp}F$ is a Stieltjes matrix, and therefore $-\left(D^2_{pp}F\right)^{-1} \geq 0$.

To summarize, we have assumed that
* increasing difference holds $\partial^2_{pq}F(p,q) \geq 0$
* F is supermodular $\partial^2_{p_ip_j}F(p,q) \geq 0$ for $i \neq j$

and we "concluded" that $\partial_p P(q) \geq 0$ termwise, that is P is isotone (ie monotone with respect to the componentwise order) with respect to q.

This is the essence of Topkis theorem.

4 Lattices, submodularity, increasing differences and Veinott's order

Consider the previous problem

$$P(q) = \arg\max_{p \in L} F(p,q)$$

and assume no differentiability whatsoever.

Here we are in R^d and $(p \vee p')_i = \max\{p_i, p'_i\}$ and $(p \wedge p')_i = \min\{p_i, p'_i\}$.

Assume that L is a sublattice of R^d, that is $p, p' \in L$ implies $p \wedge p' \in L$ and $p \vee p' \in L$.

Claim: we can define supermodularity of $g(p)$ as for all p and p',

$$g(p \wedge p') + g(p \vee p') \geq g(p) + g(p')$$

this is giving us

$$g(p \vee p') - g(p') \geq g(p) - g(p \wedge p')$$

which means that for $p = P + \epsilon e^i$ where $e^i_i = 1$ and $e^i_k = 0$ for $k \neq i$.

and $p' = P + \eta e^j$ and thus $p \vee p' = P + \epsilon e^i + \eta e^j$ and $p \wedge p' = P$. Thus above becomes

$$g\left(P + \epsilon e^i + \eta e^j\right) - g\left(P + \eta e^j\right) \geq g\left(P + \epsilon e^i\right) - g(P)$$

then letting $\epsilon \to 0$ yields

$$\partial_{p_i} g\left(P + \eta e^j\right) \geq \partial_{p_i} g(P)$$

hence

$$\partial^2_{p_ip_j} g(P) \geq 0.$$

Second, we shall define increasing differences as $q \to F(p', q) - F(p, q)$ is nondecreasing in q as soon as $p' \geq p$.

Now for the interesting problem. What does it mean that $P(q)$ should be increasing. The answer is given by Veinott's strong set order.

Definition. $P \preceq_V P'$ if given $p \in P$ and $p' \in P'$, then $p \wedge p' \in P$ and $p \vee p' \in P'$.

First thing is to verify that when P and P' are singletons, this coincides with the natural order.

Property 1. Assume $P = \{p\}$ and $P' = \{p'\}$, then $P \preceq_V P'$ if and only if $p \wedge p' = p$ and $p \vee p' = p'$.

Hence that occurs if and only if $p \leq p'$.

Is $\preceq_V$ reflexive?

Property 2. $P \preceq_V P$ if and only if P is a sublalttice of R^d that is iff $p, p' \in P$ imply $p \wedge p' \in P$ and $p \vee p' \in P$.

Property 3. Assume P and P' are sublattices of R^d.

If $P \preceq_V P'$ then $\inf P \leq \inf P'$ and $\sup P \leq \sup P'$.

5 Topkis' theorem

Assume L is a lattice, $F(p, q)$ is

supermodular in p and

has increasing differences in (p, q).

Then $q \to \arg\max_{p \in L(q)} F(p, q)$ is increasing in Veinott's order.

We need to show that if $q \leq q'$ and if $p \in \arg\max_p F(p, q)$ and $p' \in \arg\max_p F(p, q')$

then $p \wedge p' \in \arg\max_p F(p, q)$ and $p \vee p' \in \arg\max_p F(p, q')$.

Let's show this.

$p \in \arg\max_p F(p, q)$ implies $F(p, q) \geq F(p \wedge p', q)$

$p' \in \arg\max_p F(p, q')$ implies $F(p', q') \geq F(p \vee p', q')$

Take the second one; we get

$0 \geq F(p \vee p', q') - F(p', q') \geq F(p \vee p', q) - F(p', q)$ because of increasing differences

This leads to

$F(p', q) \geq F(p \vee p', q)$

add with the first one above, this yield

$F(p', q) + F(p, q) \geq F(p \wedge p', q) + F(p \vee p', q)$

Now because of supermodularity, this inequality can only hold as an equality.

Hence all the inequalities written above are in fact equalities

Hence

$F(p,q) = F(p \wedge p', q)$
$F(p', q') = F(p \vee p', q')$
and QED.