math+econ+code on equilibrium virtual whiteboard, day 3

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1 Matching with nonlinear taxation

Before (linear tax) we had
$u_x \geq \alpha_{xy} + (1-\theta) w_{xy}$
$v_y \geq \gamma_{xy} - w_{xy}$

Now we have
$u_x \geq \alpha_{xy} + N(w_{xy})$
$v_y \geq \gamma_{xy} - w_{xy}$

$n^{k+1} = n^k + \left(1-\tau^k\right)\left(w^{k+1} - w^k\right)$
$N^k = n^k - w^k\left(1-\tau^k\right)$

2 Bargaining sets over pairs

Assume if w is the net wage, then
$u = \mathcal{U}(w)$

For a given worker-firm pair, define the feasible set of utilities (or bargaining set) as
$\mathcal{F} = \{(U,V) : \exists w : U \leq \mathcal{U}(w) \text{ and } V \leq \mathcal{V}(w)\}$.
Remark 1. We are assuming free disposal here.
Remark 2. w does not have to be a wage; it can be the set of terms of a contract.

Let's see some examples. taken from Galichon, Kominers and Weber (2019).
Example 1. Transferable utiity (Becker model):
$\mathcal{U}(w) = \alpha + w$
$\mathcal{V}(w) = \gamma - w$
$\mathcal{F} = \{(U,V) : U + V \leq \alpha + \gamma\}$

Example 2. Non-transferable utility

in that case,
$\mathcal{F} = \{(U,V) : U \leq \alpha \text{ and } V \leq \gamma\}$ this is the non-transferable utility case.

Example 3. Marriage model with marital surplus and private consumption
Private consumption c^i, c^j $c^i + c^j = B$ joint budget of the household
$U = \alpha + \tau \log c^i$
$V = \gamma + \tau \log c^j$
$c^i = \exp\left(\frac{U-\alpha}{\tau}\right)$ and $c^j = \exp\left(\frac{V-\gamma}{\tau}\right)$ and thus
$\mathcal{F} = \left\{(U,V) : \exp\left(\frac{U-\alpha}{\tau}\right) + \exp\left(\frac{V-\gamma}{\tau}\right) \leq B\right\}$
When $\tau \to +\infty$, get at first order
$2 + \frac{U-\alpha}{\tau} + \frac{V-\gamma}{\tau} \leq B$
Assume $B = 2$
that is in the limit, $U + V \leq \alpha + \gamma$. hence transferable utility.
When $\tau \to 0$,
$\tau \log\left(\exp\left(\frac{U-\alpha}{\tau}\right) + \exp\left(\frac{V-\gamma}{\tau}\right)\right) \leq \tau \log B$
$\to$
$\max\{U-\alpha, V-\gamma\} \leq 0$

Example 4. Marriage with a public good
$g \in G$ is a public good that need to be jointly decided
e.g. the number of kids; buying a house
Assume that conditional on $g \in G$, the utilities are
$U = \alpha^g(w)$ increasing
$V = \gamma^g(w)$ decreasing
where w is the terms of match - say the share of private consumption that goes to the man.
We can compute the conditional bargaining set
$\mathcal{F}^g = \left\{(U,V) : U \leq \alpha^g\left((\gamma^g)^{-1}(V)\right)\right\}$
Indeed, conditional on having decided on a g, (U,V) is feasible if
there is a w such that $U \leq \alpha^g(w)$ and $V \leq \gamma^g(w)$ thus $w \leq (\gamma^g)^{-1}(V)$ and therefore
$U \leq \alpha^g\left((\gamma^g)^{-1}(V)\right)$
In this case, the bargaining set is
$\mathcal{F} = \cup_{g \in G} \mathcal{F}^g$.
$\mathcal{F} = \left\{(U,V) : U \leq \max_g \alpha^g\left((\gamma^g)^{-1}(V)\right)\right\}$
Example 5. Matching with progressive taxation.
$U \leq \alpha + N(w)$
$V \leq \gamma - w$
we have therefore
$\mathcal{F} = \{(U,V) : U \leq \alpha + N(\gamma - V)\}$
$N(w) = \min_k\left\{n^k + \left(1-\tau^k\right)\left(w - w^k\right)\right\}$
therefore
$\mathcal{F} = \left\{(U,V) : U \leq \min_k\left\{\alpha + n^k + \left(1-\tau^k\right)\left(\gamma - V - w^k\right)\right\}\right\}$

and thus
$\mathcal{F} = \cap_k \mathcal{F}^k$
where
$\mathcal{F}^k = \left\{ (U,V) : U \leq \alpha + n^k + \left(1 - \tau^k\right)\left(\gamma - V - w^k\right) \right\}$

2.1 A convenient description of the bargaining sets

Given a feasible set $\mathcal{F}$, compute the distance to the frontier of $\mathcal{F}$ along the diagonal, with a minus sign if in the interior.
Compute $D(U,V) = \min\{t \in \mathbb{R} : (U-t, V-t) \in \mathcal{F}\}$.
We have $D(U+a, V+a) = D(U,V) + a$

Example 1. TU case $\mathcal{F} = \{(U,V) : U + V \leq \Phi\}$ where $\Phi = \alpha + \gamma$
in that case $D(U,V) = \frac{U+V-\Phi}{2}$

Example 2. NTU case $\mathcal{F} = \{(U,V) : U \leq \alpha, V \leq \gamma\}$
in that case $D(U,V) = \max(U-\alpha, V-\gamma)$
$\min\{t \in \mathbb{R} : \max(U-\alpha, V-\gamma) - t \leq 0\} = \min\{t \in \mathbb{R} : t \geq \max(U-\alpha, V-\gamma)\}$

Example 3. model w marital prefs+private consumption
$\mathcal{F} = \left\{ (U,V) : \exp\left(\frac{U-\alpha}{\tau}\right) + \exp\left(\frac{V-\gamma}{\tau}\right) \leq 2 \right\}$ (wlog can take $B = 2$)
$D(U,V) = \tau \log\left(\frac{\exp\left(\frac{U-t-\alpha}{\tau}\right) + \exp\left(\frac{V-t-\gamma}{\tau}\right)}{2} \right)$.
indeed, $D(U,V) = t$ such that $\exp\left(\frac{U-t-\alpha}{\tau}\right) + \exp\left(\frac{V-t-\gamma}{\tau}\right) = 2$.

Example 4. $\mathcal{F} = \cup_g \mathcal{F}^g$
$D_{\mathcal{F}}(U,V) = \min_g D_{\mathcal{F}^g}(U,V)$

Example 5. $\mathcal{F} = \cap_k \mathcal{F}^k$
$D_{\mathcal{F}}(U,V) = \max_k D_{\mathcal{F}^k}(U,V)$

A preview of the sequel.
In Choo and Siow, the supply-demand analysis led to matching functions.
$\mu_{xy} = M_{xy}\left(\mu_{x0}, \mu_{0y}\right)$
here we are going to see that
$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \exp\left(-D_{xy}\left(-\ln \mu_{x0}, -\ln \mu_{0y}\right)\right)$.

3 The matching model

Assume w_{xy} is the wage and consider the workers' and the firms' problems
$u_x = \max_y \{\mathcal{U}_{xy}(w_{xy}), 0\}$
$v_y = \max_x \{\mathcal{V}_{xy}(w_{xy}), 0\}$

(μ, u, v, w) is an equilibrium matching if the following conditions hold

(i) population constraint
$\sum_y \mu_{xy} + \mu_{x0} = n_x$
$\sum_x \mu_{xy} + \mu_{0y} = m_y$
(ii) Stability
$u_x \geq \mathcal{U}_{xy}(w_{xy})$
$v_y \geq \mathcal{V}_{xy}(w_{xy})$
$u_x \geq 0$
$v_y \geq 0$
(iii) Complementarity
$\mu_{xy} > 0$ implies $u_x = \mathcal{U}_{xy}(w_{xy})$ and $v_y = \mathcal{V}_{xy}(w_{xy})$
$\mu_{x0} > 0$ implies $u_x = 0$
$\mu_{0y} > 0$ implies $v_y = 0$

Consider a version of the problem with random utility

$u_x = \mathbb{E}\left[\max_y \left\{\mathcal{U}_{xy}(w_{xy}) + T\varepsilon_y, T\varepsilon_0\right\}\right]$
$v_y = \mathbb{E}\left[\max_x \left\{\mathcal{V}_{xy}(w_{xy}) + T\eta_x, T\eta_0\right\}\right]$
Denote $U_{xy} = \mathcal{U}_{xy}(w_{xy})$ and $V_{xy} = \mathcal{V}_{xy}(w_{xy})$.
$u_x = \mathbb{E}\left[\max_y \left\{U_{xy} + T\varepsilon_y, T\varepsilon_0\right\}\right] = T\log\left(1 + \sum_y \exp\frac{U_{xy}}{T}\right)$
$v_y = T\log\left(1 + \sum_x \exp\frac{V_{xy}}{T}\right)$

Note that we have $(U_{xy}, V_{xy}) \in \mathcal{F}_{xy}$. Thus, we reexpress
$U_{xy} = \mathcal{U}_{xy}(w_{xy})$ and $V_{xy} = \mathcal{V}_{xy}(w_{xy})$ for some w_{xy}
as
$D_{xy}(U_{xy}, V_{xy}) = 0$ where D_{xy} is the distance function associated with $\mathcal{F}_{xy}$.
$\frac{\mu_{xy}}{n_x} = \Pr(y|x) = \frac{\exp\left(\frac{U_{xy}}{T}\right)}{1+\sum_{y'}\exp\left(\frac{U_{xy'}}{T}\right)} = \exp\left(\frac{U_{xy}-u_x}{T}\right)$
$\frac{\mu_{x0}}{n_x} = \exp\left(\frac{-u_x}{T}\right)$
$\frac{\mu_{xy}}{m_y} = \exp\left(\frac{V_{xy}-v_y}{T}\right)$
$\frac{\mu_{0y}}{m_y} = \exp\left(\frac{-v_y}{T}\right)$
Thus
$\frac{\mu_{xy}}{\mu_{x0}} = \exp\left(\frac{U_{xy}}{T}\right)$ thus $U_{xy} = T\ln\frac{\mu_{xy}}{\mu_{x0}}$
$\frac{\mu_{xy}}{\mu_{0y}} = \exp\left(\frac{V_{xy}}{T}\right)$ thus $V_{xy} = T\ln\frac{\mu_{xy}}{\mu_{0y}}$

Recall that
$D_{xy}(U_{xy}, V_{xy}) = 0$ hence
$D_{xy}\left(T\ln\frac{\mu_{xy}}{\mu_{x0}}, T\ln\frac{\mu_{xy}}{\mu_{0y}}\right) = 0$
$D_{xy}\left(T\ln\mu_{xy} - T\ln\mu_{x0}, T\ln\mu_{xy} - T\ln\mu_{0y}\right) = 0$
$T\ln\mu_{xy} + D_{xy}\left(-T\ln\mu_{x0}, -T\ln\mu_{0y}\right) = 0$

therefore $\mu_{xy} = M_{xy}\left(\mu_{x0}, \mu_{0y}\right)$, where

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \exp\left(-\frac{1}{T} D_{xy}\left(-T\ln\mu_{x0}, -T\ln\mu_{0y}\right)\right)$$

where μ_{x0} and μ_{0y} satisfy

$$\begin{aligned} \mu_{x0} + \sum_{y\in\mathcal{Y}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) &= n_x \\ \mu_{0y} + \sum_{x\in\mathcal{X}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) &= m_y \end{aligned}$$

4 Solving for the equilibrium using Gauss-Seidel

We will verify that we are in the Gross Substitutes / BGH case. Recall

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \exp\left(-\frac{1}{T} D_{xy}\left(-T\ln\mu_{x0}, -T\ln\mu_{0y}\right)\right)$$

Introduce $p_z = (p_x, p_y)$ with $p_x = -\mu_{x0}$ and $p_y = \mu_{0y}$

$$\begin{aligned} e_x(p) &= p_x - \sum_{y\in\mathcal{Y}} M_{xy}\left(-p_x, p_y\right) + n_x \\ e_y(p) &= p_y + \sum_{x\in\mathcal{X}} M_{xy}\left(-p_x, p_y\right) - m_y \end{aligned}$$

Gross substitutes hold.

Law of aggregate supply holds:

$\sum_x e_x(p) + \sum_y e_y(p) = \sum_x p_x + \sum_y p_y + \sum_x n_x - \sum_y m_y$ is increasing in all the prices.

Hence

e is inverse isotone.

Now let's see that there is a subsolution and a supersolution.

$p_z = N$, N large enough yields a supersolution, while

$p_z = -N$, N large enough yields a subsolution.

Hence there is a solution, and Gauss-Seidel converges.