Class notes on one-to-many-matching

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math+econ+code masterclass on equilibrium transport and matching models in economics
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1 Agenda

We will touch upon:

- One-to-many matchings
- Lovasz extension
- Submodular minimization

This class is only meant to provide an bird's-eye view of topic.

2 Some references

- Vohra's book. Mechanism Design A Linear Programming Approach, Chapter 5.
  https://www.cambridge.org/core/books/mechanism-design/26EA10EC04EB418E699CCBD5B09A55
- Bach's monograph
  https://www.di.ens.fr/~fbach/2200000039-Bach-Vol6-MAL-039.pdf
- Fujushige's monograph:
  https://www.elsevier.com/books/submodular-functions-and-optimization/fujishige/978-0-444-52086-9
- Paes Leme's survey paper:
  https://www.renatoppl.com/papers/gs-survey-aug-14.pdf

3 One-to-many matchings

Assume firm $y \in Y$ hires set of workers $B \subseteq X$. Economic output of bundle B for firm y is $\Phi_y(B)$. (Endogenous) wage of worker x is u_x. As a result, firm y's problem is

$$\Phi_y^*(u) = \max_{B\subseteq X}\left\{\Phi_y(B) - \sum_{x\in B} u_x\right\}$$

and letting B_y be the optimal choice of y, we see that

$$\frac{\partial \Phi_y^*(u)}{\partial u_x} = -1\{x \in B_y\}$$

and $\frac{\partial^2 \Phi_y^*(u)}{\partial u_x \partial u_{x'}} = -\frac{\partial 1\{x\in B_y\}}{\partial u_{x'}}$.

Goss substitutes means that when worker x' becomes more expensive, the firm tends to demand "more" of worker $x \neq x'$, that is $\frac{\partial 1\{x\in B_y\}}{\partial u_{x'}} \geq 0$, or in other words, $\frac{\partial^2 \Phi_y^*(u)}{\partial u_x \partial u_{x'}} \leq 0$, that is $\Phi_y^*(u)$ is submodular.

As we have seen, $\Phi_y^*(u)$ submodular **implies** that $\Phi_y(B)$ is submodular i.e.

$$\Phi_y(B) + \Phi_y(B') \geq \Phi_y(B \cap B') + \Phi_y(B \cup B')$$

The converse does not hold.

3.1 Walrasian equilibrium

The reference here is Vohra.

Solve the following problem

$$\min_{(u_x)}\left\{\sum_{x\in X} u_x + \sum_{y\in Y} \Phi_y^*(u)\right\}$$

this is a good candidate to determine equilibrium wages u, because by first order conditions

$$1 + \sum_{y\in Y} \frac{\partial \Phi_y^*(u)}{\partial u_x} = 0$$

Now, assuming differentiability of $\Phi_y^*(u)$ at u, we get $\frac{\partial \Phi_y^*(u)}{\partial u_x} = -1\{x \in B_y\}$ where B_y is the bundle of workers demanded by y, so

$$1 = \sum_{y\in Y} 1\{x \in B_y\}, \forall x \in X.$$

4 About minimization and maximization of submodular functions

4.1 Minimization of a submodular function

The reference here is Bach.

When $\Phi_y(B)$ is submodular, we may view it as a the restriction of a convex function $\phi_y(b)$ restricted to $b_x \in \{0,1\}$. Such a convex function $\phi_y(b)$ is called the Lovasz extension of $\Phi_y(B)$, or the Choquet integral.

We define

$$
\begin{aligned}
\phi_y(b) &= \max_q \left\{ q^\top b : \sum_{x\in B} q_z \leq \Phi_y(B), \sum_{z\in X} q_x = \Phi_y(X) \right\} \\
&= \max_q \min_{\substack{\lambda_B \geq 0 \\ \lambda_X}} \left\{ q^\top b + \sum_{B\subseteq X} \lambda_B \left(\Phi_y(B) - q(B) \right) \right\} \\
&= \min_{\substack{\lambda_B \geq 0 \\ \lambda_X}} \sum_{B\subseteq X} \lambda_B \Phi_y(B) + \max_q \sum_{x\in X} q_x \left(b_z - \sum_{x\in B} \lambda_B \right) \\
&= \min_{\substack{\lambda_B \geq 0 \\ \lambda_X}} \sum_{B\subseteq X} \lambda_B \Phi_y(B) : b_x = \sum_{B\subseteq X} \lambda_B 1\{x\in B\}
\end{aligned}
$$

and we have that $\phi_y(1_B) = \Phi_y(B)$. Next, we can actually maximize $\phi_y(b)$ over $b \in [0.1]^X$ which wi

4.2 Maximization of a submodular function

Exact minimization of a submodular function is a difficult problem, unless the function has the stronger property of gross substitute.

4.3 Maximization of a GS function

Our strategy forward:

- replace $\max_{B\subseteq X} \left\{ \Phi_y(B) - \sum_{x\in B} u_x \right\}$ by $\max_{b\in[0,1]^X} \left\{ \phi_y(b) - b^\top u \right\}$ [Lovasz extension]. [Minimization of submodular functions][greedy algorithm]
- look for equilibrium wage

  $$\min_u \sum_{y\in Y} \max_{B\subseteq X} \left\{ \Phi_y(B) - \sum_{x\in B} u_x \right\} + \sum_{x\in X} u_x$$

  indeed, first order conditions yield $\sum_{y\in Y} 1\{x \in B_y\} = 1$

We have
$\Phi_y^*(u) \geq \Phi_y(B) - \sum_{x\in B} u_x$
thus
$\Phi_y^*(u) + \sum_{x\in B} u_x \geq \Phi_y(B)$
that is
$\Phi_y(B) \leq \min_B \left\{\Phi_y^*(u) + \sum_{x\in B} u_x\right\}$

The question is, do we have equality?
well, $\Phi_y(B)$ is submodular, hence we can consider its Lovasz extension (bach p 20)
We have by definition $\phi_y(1_B) = \Phi_y(B)$.
Because $\Phi_y(B)$ is submodular (Bach p 27) we have

$$\begin{aligned}
\phi_y(b) &= \max_q \left\{q^\top b : q(B) \leq \Phi_y(B), q(Z) = \Phi_y(B)\right\} \\
&= \max_q \min_{\substack{\lambda_B \geq 0 \\ \lambda_Z}} \left\{q^\top b + \sum_{B\subseteq Z} \lambda_B (\Phi_y(B) - q(B))\right\} \\
&= \min_{\substack{\lambda_B \geq 0 \\ \lambda_Z}} \sum_{B\subseteq Z} \lambda_B \Phi_y(B) + \max_q \sum_z q_z \left(b_z - \sum_{z\in B} \lambda_B\right) \\
&= \min_{\substack{\lambda_B \geq 0 \\ \lambda_Z}} \sum_{B\subseteq Z} \lambda_B \Phi_y(B) : b_z = \sum_{B\subseteq Z} \lambda_B 1\{z \in B\}
\end{aligned}$$

Thus we can consider $\phi_y^*(q) = \max_b \{qb - \phi_y(b)\} = \max_b \min_q$