Roadmap

Day 1: study theoretical and computational methods for equilibrium systems (with gross substitutes)

Day 2: Matching models with fully transferable utility

what is given up by one side of the market is fully appropriated by the other side
this is connected with optimal transport

Day 3: Matching models with impfectly transferable utility

what is given up by one side of the market is NOT fully appropriated by the other side
models from labor economics (matching with taxes)
or from familiy economics (collective models of bargaining)

Day 4: Matching models with non transferable utility

* school choice
* social housing
* taxi markets

Day 5: price equilibrium on networks

1 Day 1: equilibrium with gross substitutes

Our goal = provide a common framework for these matching models. Reformulate as a general equilibrium problem: $|Z| = n$ goods, Z is the set of goods, and

$$Q : R^n \to R^n$$

such that is p_z if the price of good $z \in Z$, then $Q_z(p)$ is the excess supply for good z (=supply-demand), and the equilibrium problem consists in finding

$p \in R^n$ such that

$$Q(p) = 0.$$

More specifically, we shall assume that Q has the *gross substitute property.*

Assume we increase price of good z. What is this going to do to the excess supply for good z'? some producers will shift from producing z' to producing z, and therefore $Q_{z'}(p)$ will decrease.

In other words, when it is derivable, Q has the gross substitute property when

$$\frac{\partial Q_z}{\partial p_{z'}}(p) \leq 0 \text{ for } z \neq z'$$

while it natural to assume

$$\frac{\partial Q_z}{\partial p_z}(p) \geq 0 \text{ for all } z.$$

Example: surge pricing in an uber-like environment. We have partitioned the city in a finite number of locations (say, blocks).

$x \in X$= location of the driver

$y \in Y$=location of the passenger

Assume $z \in Z$ is the pickup location.

Assume that for a drive at x, the cost of picking up at z is c_{xz}

if the price of the ride at z is p_z the utility if the driver is $p_z - c_{xz} + \sigma\varepsilon_z$, where the vector (ε_z) is random.

if the driver does not pickup anyone, the utility is normalized to ε_0.

Assume that $(\varepsilon_z) \sim$ *Gumbel* and is iid. Then the probability that a driver at x will demand a ride z is

$$\frac{\exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z'} \exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)}$$

Now assume that there are n_x drivers in area x, and therefore the supply for rides z is

$$S_z(p) = \sum_{x \in X} n_x \frac{\exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z' \in Z} \exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)}$$

Let's study the properties of $S(p)$. Do we have gross substitutes?

$$S_z(p) = \sum_{x \in X} n_x \frac{1}{\exp\left(\frac{-p_z + c_{xz}}{\sigma}\right) + \sum_{z' \neq z} \exp\left(\frac{p_{z'} - p_z + c_{xz} - c_{xz'}}{\sigma}\right) + 1}$$

and we immediately see that $S_z(p)$ is decreasing w.r.t. $p_{z'}$ (for $z' \neq z$), and increasing with respect to p_z.

Now. let's focus on demand. This is the same as before, except for the fact that utility of a passenger at y seeking a ride in a cell z is now

$$u_{yz} - p_z + \eta_z$$

where η is iid Gumbel. The induced demand is

$$
\begin{aligned}
D_z(p) &= \sum_{y\in Y} m_y \frac{\exp(u_{yz}-p_z)}{1+\sum_{z'}\exp(u_{yz'}-p_{z'})} \\
&= \sum_{y\in Y} m_y \frac{1}{\exp(-u_{yz}+p_z)+\sum_{z'\neq z}\exp(u_{yz'}-u_{yz}+p_z-p_{z'})+1}
\end{aligned}
$$

we see that $-D_z(p)$ has the Gross substitute property, and therefore

$$Q_z(p) = S_z(p) - D_z(p)$$

also has the gross substitute property.

Question = how do we compute p such that $Q(p)=0$.

Tatonnement = raise prices where there is excess demand ($Q_z(p)<0$) and decrease prices where there excess supply.

Essentially 2 types of methods.

1) optimization-based methods: reformulate as an optimization problem. In order to do that, try to obtain Q as a gradient– ie, is there a potential function $V: R^n \to R$ such that V is convex, and

$$Q_z(p) = \frac{\partial V}{\partial p_z}(p).$$

In that case, $Q(p)=0$ is equivalent to the fact that p is a minimizer of V.

In that case, we can minimize V using e.g. gradient descent, that is

$$
\begin{aligned}
p^{t+1} &= p^t - \epsilon \frac{\partial V}{\partial p_z}(p^t) \\
&= p^t - \epsilon Q_z(p^t)
\end{aligned}
$$

In our example, we had

$$
\begin{aligned}
Q_z(p) &= \sum_{x\in X} n_x \frac{\exp\left(\frac{p_z-c_{xz}}{\sigma}\right)}{1+\sum_{z'\in Z}\exp\left(\frac{p_{z'}-c_{xz'}}{\sigma}\right)} \\
&\quad -\sum_{y\in Y} m_y \frac{\exp(u_{yz}-p_z)}{1+\sum_{z'}\exp(u_{yz'}-p_{z'})}
\end{aligned}
$$

We have that $Q_z(p) = \partial V(p)/\partial p_z$, where

$$
\begin{aligned}
V(p) &= \sigma \sum_{x\in X} n_x \log\left(1+\sum_{z'\in Z}\exp\left(\frac{p_{z'}-c_{xz'}}{\sigma}\right)\right) \\
&\quad + \sum_{y\in Y} m_y \log\left(1+\sum_{z'}\exp(u_{yz'}-p_{z'})\right)
\end{aligned}
$$

2) substitutes-based methods. Consider the system of equations

$$Q_z\left(p\right)=0$$

Assume that we can start with a supersolution – i.e. a vector p^0 such that

$$Q_z\left(p^0\right)\geq 0 \text{ for all } z$$

Then we can consider the myopic market-clearing algorithm as follows: assume p^t has been computed and compute p^{t+1} using

$$Q_z\left(p_z^{t+1},p_{-z}^t\right)=0$$

We can show that under suitable assumptions (involving gross substitutes), this algorithm converges to the market-clearing price.

Recall we started from a supersolution p^0. We can show that for each $t\geq 0$:
* $p_z^t\geq p_z^{t+1}$
* p^{t+1} is a supersolution, that is $Q_z\left(p^{t+1}\right)\geq 0$ for all z
By the induction hypothesis,
$Q_z\left(p^t\right)\geq 0$ that is
$Q_z\left(p_z^t,p_{-z}^t\right)\geq 0$
$Q_z\left(p_z^{t+1},p_{-z}^t\right)=0$
because Q_z is increasing in p_z, this implies necessarily $p_z^t\geq p_z^{t+1}$.
Let's now show that p^{t+1} is a supersolution. We have
$Q_z\left(p_z^{t+1},p_{-z}^t\right)=0$
We have $p^t\geq p^{t+1}$ and $Q_z\left(p\right)$ is nonincreasing in p_{-z}, therefore
$Q_z\left(p_z^{t+1},p_{-z}^{t+1}\right)\geq Q_z\left(p_z^{t+1},p_{-z}^t\right)=0$

Now:
* if p^t remains bounded below, then it converges to p^*.
* if Q is continuous and p^t converges, then we can take the limit $t\to+\infty$ in

$$Q_z\left(p_z^{t+1},p_{-z}^t\right)=0$$

and we have that

$$Q_z\left(p^*\right)=0.$$

After the break, we need to show that p^t remains bounded below under reasonable assumptions. We will show assumptions under which Q is inverse isotone, i.e.

$$Q\left(p\right)\leq Q\left(p'\right)\implies p\leq p'$$

which is a fundamental property of Q which is related to M-maps.

Theorem (Berry, Gandhi and Haile, Econometrica 2013). Assume that:
(i) Q satisfies weak gross substitutes ie $Q_z\left(p\right)$ is weakly decreasing in $p_{z'}$ for $z\neq z$.

(ii) Law of aggregate supply holds, ie $\sum_z Q_z(p)$ is weakly increasing in each p_z, or in other words:

$$Q_0(p) = -\sum_z Q_z(p) \text{ is weakly decreasing in each } p_z$$

(iii) Connected strong substitutes holds: or each z, there is a path from z to 0 $z_1 = z$, $z_2 z_2 ... z_{n-1}$, $z_n = 0$ such that $Q_{z_{k+1}}(p)$ is strictly decreasing with respect to p_{z_k}.

Then Q is inverse isotone, i.e. $Q(p) \leq Q(p')$ implies $p \leq p'$.

Example where (iii) does not hold. Assume $Z = B_1 \cup B_2$ with $B_1 \cap B_2 = \emptyset$, and assume that $Q_z(p)$ only depends on the prices in B_1 for $z \in B_1$ and $Q_z(p)$ only depends on the prices in B_2 for $z \in B_2$.

Example. Consider $Q(p) = Qp$ where

* Q has gross substitutes which means that $Q_{ij} \leq 0$ for $i \neq j$, and

* Q is row-diagonally dominant .

$$Q_{ii} > \sum_{j \neq i} |Q_{ij}|$$

Then I can show that the assumptions in the BGH theorem are met.

(i) and (ii) are obvious. (iii) is obtained by the fact that $Q_0(p)$ is strictly decreasing in p_z.

Example 3. Back to

$$\begin{aligned} Q_z(p) &= \sum_{x \in X} n_x \frac{\exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z' \in Z} \exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)} \\ &\quad - \sum_{y \in Y} m_y \frac{\exp(u_{yz} - p_z)}{1 + \sum_{z'} \exp(u_{yz'} - p_{z'})} \end{aligned}$$

We have show (i) gross substitutes.

Show (ii). We have

$$\begin{aligned} \sum_z Q_z(p) &= \sum_{x \in X} n_x \sum_{z \in Z} \frac{\exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z' \in Z} \exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)} \\ &\quad - \sum_{y \in Y} m_y \sum_{z \in Z} \frac{\exp(u_{yz} - p_z)}{1 + \sum_{z'} \exp(u_{yz'} - p_{z'})} \end{aligned}$$

that is

$$\begin{aligned} \sum_z Q_z(p) &= \sum_{x \in X} n_x \left(1 - \frac{1}{1 + \sum_{z' \in Z} \exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)}\right) \\ &\quad - \sum_{y \in Y} m_y \left(1 - \frac{1}{1 + \sum_{z'} \exp(u_{yz'} - p_{z'})}\right) \end{aligned}$$

therefore $\sum_z Q_z(p)$ is strictly increasing in each of the p_z.

Two different implications of Q inverse isotone.

Implication 1: existence of a solution under the assumption that there exists both a sub- and a super-solution.

If Q is inverse isotone, then assume that in addition to what we have assumed, there is a subsolution $\underline{p}$. Then for anysupersolution p, we can show that $\underline{p} \leq p$. Indeed, $Q(\underline{p}) \leq 0 \leq Q(p)$ implies by inverse isotonicity that $\underline{p} \leq p$. Therefore the sequence constructed above, which is a sequence of supersolutions, is bounded below by $\underline{p}$.

Implication 2: uniqueness of a solution.

If Q is inverse isotone, then the equilibrium price p^* is unique. Indeed, assume $Q(p) = Q(p')$. Then $Q(p) \leq Q(p')$ and $Q(p') \leq Q(p)$ and by inverse isotonicity applied twice, we have $p \leq p'$ and $p' \leq p$, which implies $p = p'$.

Let's take an example

$$\begin{aligned} Q_1(p) &= 2p_1 - p_2 \\ Q_2(p) &= -2p_1 + 3p_2 \end{aligned}$$

$p^0 = (1,1)$. Thus $Q(p^0) = (1,1) \geq 0$ thus p^0 is a supersolution.

$$\begin{aligned} 2p_1^1 - p_2^0 &= 0 \\ -2p_1^0 + 3p_2^1 &= 0 \end{aligned}$$

thus I get

$$\begin{aligned} p_1^1 &= p_2^0/2 = 1/2 \\ p_2^1 &= 2p_1^0/3 = 2/3 \end{aligned}$$

Let's now see a tiny variant of this model, where the supply is now

$$S_z(p) = \frac{\exp\left(\frac{\alpha_{xz} p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z'} \exp\left(\frac{\alpha_{xz'} p_{z'} - c_{xz'}}{\sigma}\right)}$$

(where α_{xz} differs from a constant) and D_z is the same as before.

I am now claiming that there is NO function V such that

$$Q_z(p) = \frac{\partial V}{\partial p_z}(p).$$

Why? because assume such V existed. Then one would have

$$\frac{\partial Q_z}{\partial p_{z'}} = \frac{\partial^2 V}{\partial p_z \partial p_{z'}}(p)$$

which should be symmetric. Indeed,

$$
\begin{aligned}
& \frac{\partial}{\partial p_{z'}} \frac{\exp\left(\frac{\alpha_{xz} p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z'} \exp\left(\frac{\alpha_{xz'} p_{z'} - c_{xz'}}{\sigma}\right)} \\
= & \frac{\partial}{\partial p_{z'}} \frac{1}{\exp\left(\frac{-\alpha_{xz} p_z + c_{xz}}{\sigma}\right) + \sum_{z' \neq z} \exp\left(\frac{\alpha_{xz'} p_{z'} - \alpha_{xz} p_z + c_{xz} - c_{xz'}}{\sigma}\right) + 1} \\
= & \alpha_{xz'} * \text{sth symmetric}
\end{aligned}
$$

2 Day 2: matching with transferable utility