On the Relationship between Arithmetic and Geometric Returns

Dimitry Mindlin, ASA, MAAA, PhD
President

CDI Advisors LLC
dmindlin@cdiadvisors.com

August 14, 2011

Arithmetic and geometric averages are important and somewhat controversial measurements of the past and future investment returns. Numerous publications have discussed the pros and cons of these measurements as well as relationships between them. Yet, the controversy surrounding arithmetic and geometric averages appears to persist.

Vital decisions for pension plans, for example, are often based on estimates of future investment returns. It is imperative to utilize appropriate measurements of returns and apply them properly in a forward-looking manner. In particular, it is important to distinguish arithmetic and geometric averages for asset classes and portfolios as well as specify the relationships between these averages.

A popular formula presented in several publications stipulates that the geometric average is approximately equal to the arithmetic average minus half of the variance. However, a proper justification for this formula and the assessment of the quality of this approximation are hard to find. Moreover, this popular formula may significantly underestimate the geometric return in practical applications.

Recognizing the need for clarity in this area and the desirability of alternative solutions, this paper presents three additional formulas for approximate calculations of geometric averages and provides simple quantitative explanations for all four formulas. The results of these formulas are compared to historic geometric averages and to each other. The paper shows in particular, that the three other formulas are often superior to the popular one.

This author hopes that this paper would be useful to practitioners in clarifying the relationship between arithmetic and geometric averages as well as their pros and cons.

Arithmetic and geometric averages are some of the most commonly utilized measurements of investment returns.[1] Despite their extensive utilization, however, there has been a great deal of controversy and confusion surrounding these measurements. A number of publications have attempted to clarify the issues related to arithmetic and geometric averages, but the controversy and confusion appear to persist.[2]

According to de La Grandville [1998], "*A number of serious, widely held errors and misconceptions about the long-term expected rate of return need to be dispelled.*" One of these "misconceptions" is related to the calculation of the geometric return of a portfolio. According to several publications, *"the geometric average is approximately equal to the arithmetic average minus half of the variance."* Despite the popularity of this formula, few publications attempt to justify this approximation and gauge its quality.

As demonstrated in this paper, this formula is the result of a couple of relatively crude approximations. More troubling, this formula tends to underestimate the geometric average. This tendency, in particular, should be of concern to pension plans that employ geometric portfolio returns to determine their discount rates.

Calculations of geometric returns can have a significant impact on asset allocation decisions. A pension plan, for example, may select the lowest risk portfolio with the geometric return equal to the plan's discount rate (by itself, an idea of questionable utility). A calculation that underestimates the geometric return would force the plan to needlessly increase the riskiness of the portfolio in order to hit the "target" return. In other words, the plan would take additional risk solely due to questionable math.

Unlike geometric averages, arithmetic averages are relatively easy to use. In particular, the arithmetic return for a portfolio is equal to the weighted average of the arithmetic returns of underlying asset classes. This "rule," however, does not work for geometric returns – a weighted average of asset classes' geometric returns is *not* equal to the geometric return of the corresponding portfolio. Therefore, there is a need to "convert" arithmetic portfolio returns to the geometric ones, and vice versa.

Attempting to establish a better understanding of the relationship between arithmetic and geometric averages, this paper

- provides a simple quantitative explanation for the abovementioned popular formula;
- presents three more formulas that connect arithmetic and geometric returns;
- develops connections between all four formulas;
- demonstrates that the popular formula tends to produce sub-optimal results;
- identifies the formula that should be expected to produce better results.

Geometric and Arithmetic Averages: Return Series

For a series of returns, this section develops four formulas that connect arithmetic and geometric averages.

The *arithmetic* average A of the series of returns $r_1, \ldots, r_n$ is defined simply as the average value of the series:

$$A = \frac{1}{n} \sum_{k=1}^{n} r_k \tag{1}$$

One of the main advantages of the arithmetic average is it is an unbiased estimate of the return. One of the main disadvantages of the arithmetic average is the probability of achieving the arithmetic average return may be unsatisfactory. In other words, as a prediction of future returns, the arithmetic average may be too optimistic. Another disadvantage of the arithmetic return is its "incompatibility" with the starting and ending asset values. Specifically, the starting asset value multiplied by the compounded arithmetic return factor $(1+A)^n$ is greater than the ending asset value.[3]

The concept of *geometric* average is specifically designed to correct this problem. If A_0 and A_n are the starting and ending asset values correspondingly, then, by definition,

$$A_0 \left(1+r_1\right)\ldots\left(1+r_n\right)=A_n \tag{2}$$

The geometric average G is defined as the rate of return that connects the starting and ending asset values if assumed in all periods. Namely, the starting asset value multiplied by the compounded return factor $\left(1+G\right)^n$ is equal to the ending asset value:

$$A_0 \left(1+G\right)^n = A_n \tag{3}$$

Combining (2) and (3), we get a standard "textbook" definition of the geometric average G:

$$G = -1 + \prod_{k=1}^{n}\left(1+r_k\right)^{1/n} \tag{4}$$

Let us try to determine how the arithmetic and geometric averages relate to each other. Firstly, it is well-known that the arithmetic average is always greater or equal to the arithmetic average:[4]

$$A \geq G \tag{5}$$

Following a long-established tradition, only the first two moments of the underlying variables will be used in developing relationships between A and G. Therefore, the relationships between A and G considered in this paper also involve variance V.

Let us present four formulas that connect arithmetic and geometric returns and specify the required approximations to derive each formula.[5]

Formula # 1 (A1)

Let us make the following two approximations on the right side of (4).

1. For all k, replace each factor $\left(1+r_k\right)^{1/n}$ by its Maclaurin series expansion up to the second degree.
2. In the resulting product, ignore all summands of the third degree and higher.

See the Appendix for more details. After these approximations, the right side of (4) becomes $A - V/2$, where V is the sample variance defined as[6]

$$V = \frac{1}{n}\sum_{k=1}^{n}\left(r_k - A\right)^2 \tag{6}$$

Therefore, we get the following relationship (denoted as (A1) throughout this paper):

$$G \approx A - V/2 \tag{A1}$$

Relationship (A1) is the popular formula discussed above; it is well-known among practitioners.[7]

Formula # 2 (A2)

Note that (4) implies

$$\left(1+G\right)^2 = \prod_{k=1}^{n} \left(1+r_k\right)^{2/n} \tag{7}$$

Let us make the following two approximations on the right side of (7).

1. For all k, replace each factor $\left(1+r_k\right)^{2/n}$ by its Maclaurin series expansion up to the second degree.
2. In the resulting product, ignore all summands of the third degree and higher.

After these approximations, the right side of (7) becomes $\left(1+A\right)^2 - V$, where V is defined in (6). Therefore, we get the following relationship (denoted as (A2) throughout this paper):

$$\left(1+G\right)^2 \approx \left(1+A\right)^2 - V \tag{A2}$$

Relationship (A2) is not as well-known as (A1) among practitioners, even though it has been known for a long time.[8] Interestingly, formula (A2) is exact when the return series has just two points (see the Appendix for more details).

Formula # 3 (A3)

Note that (4) implies

$$\ln\left(1+G\right) = \frac{1}{n} \sum_{k=1}^{n} \ln\left(1+r_k\right) \tag{8}$$

On the right side of (8), let us replace each summand $\ln\left(1+r_k\right)$ by its Taylor series expansion around A up to the second degree. See the Appendix for more details.

After this approximation, the right side of (8) becomes $\ln\left(1+A\right) - \frac{1}{2}V\left(1+A\right)^{-2}$. Therefore, we get the following relationship (denoted as (A3) throughout this paper):

$$\ln\left(1+G\right) \approx \ln\left(1+A\right) - \frac{1}{2}V\left(1+A\right)^{-2} \qquad \text{(A3)}$$

or, equivalently,

$$1+G \approx \left(1+A\right)\exp\left(-\frac{1}{2}V\left(1+A\right)^{-2}\right) \qquad \text{(A3)}$$

In this author's experience, relationship (A3) is little known among practitioners, even though it has been presented in some publications.[9]

Formula # 4 (A4)

In (A3), using approximation $\ln\left(1+x\right) \approx x$, let us replace $V\left(1+A\right)^{-2}$ with $\ln\left(1+V\left(1+A\right)^{-2}\right)$. As a result, we get the following relationship (denoted as (A4) throughout this paper):

$$1+G \approx \left(1+A\right)\left(1+V\left(1+A\right)^{-2}\right)^{-1/2} \qquad \text{(A4)}$$

As demonstrated in the next section, this relationship is exact when arithmetic and geometric averages (means) are defined for a lognormal distribution.

It should be noted that there is a sequence of approximations and simplifications that turn (A3) into (A4), as presented above, then turn (A4) into (A2), and then turn (A2) into (A1) (see the Appendix for more details). It is also worth noticing that the geometric average estimate (A4) is always greater than (A3), which in turn is always greater than (A2).[10] Loosely speaking,

$$(A2) < (A3) < (A4)$$

Interestingly, the geometric average estimate (A2) is not necessarily greater than (A1), although this is true for most practical examples.[11] See the Appendix for more details.

To recap, formulas (A1) – (A4), which work for any return sample, establish approximate relationships between the geometric and arithmetic averages and the variance. These formulas are based on Taylor series expansions up to the second degree.

Geometric and Arithmetic Means: Return Distributions

The previous section developed the relationships between the arithmetic and geometric averages defined for a series of returns. This section, in contrast, develops similar results when the distribution of return is given. To avoid confusion with the previous section, this section defines arithmetic and geometric *means* (rather than *averages*), which are denoted as E and M correspondingly (as opposed to averages A and G in the previous section).

In this case, the *arithmetic* mean E of return R is defined as the expected value of R:[12]

$$E = E(R)$$

The *geometric* mean *M* of return *R* is defined as follows:

$$M = \exp\left(E\left(\ln\left(1+R\right)\right)\right) - 1 \tag{9}$$

The primary motivation for these definitions comes from the fact that the arithmetic and geometric means are the limits of appropriately selected series of arithmetic and geometric averages, as demonstrated below.

Specifically, let us define arithmetic averages A_n and geometric averages G_n for a series of independent identically distributed returns $\{r_k\}$:

$$A_n = \frac{1}{n}\sum_{k=1}^{n} r_k \tag{10}$$

$$G_n = -1 + \prod_{k=1}^{n}\left(1+r_k\right)^{1/n} \tag{11}$$

According to the Law of Large Numbers (LLN), A_n converges to *E*. Also, from (11) we have

$$\ln\left(1+G_n\right) = \frac{1}{n}\sum_{k=1}^{n}\ln\left(1+r_k\right) \tag{12}$$

Again, according to LLN, $\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+r_k\right)$ and, therefore, $\ln\left(1+G_n\right)$ converge to the expected value $E\left(\ln\left(1+R\right)\right)$. Consequently, G_n converges to $\exp\left(E\left(\ln\left(1+R\right)\right)\right) - 1$, which is equal to *M*.

To recap, A_n converges to *E* and G_n converges to *M* when *n* tends to infinity. As discussed in the previous section, relationships (A1) – (A4) are true for A_n and G_n, where sample variance V_n is defined similar to (6):

$$V_n = \frac{1}{n}\sum_{k=1}^{n}\left(r_k - A_n\right)^2 \tag{13}$$

Since series V_n converges to the variance of returns *V* when *n* tends to infinity, relationships (A1) – (A4) are true for *E* and *M* as well:

$$M \approx E - V/2 \tag{A1}$$

$$\left(1+M\right)^2 \approx \left(1+E\right)^2 - V \tag{A2}$$

$$1+M \approx \left(1+E\right)\exp\left(-\frac{1}{2}V\left(1+E\right)^{-2}\right) \tag{A3}$$

$$1+M \approx \left(1+E\right)\left(1+V\left(1+E\right)^{-2}\right)^{-1/2} \tag{A4}$$

It should be emphasized that, as a general principle, one should avoid approximations whenever direct calculations are possible.[13] As demonstrated below, (A4) represents the exact relationship between the arithmetic and geometric means under common assumptions.

Let us assume that the return factor $1+R$ is lognormally distributed, which means $\ln\left(1+R\right)$ is normally distributed with parameters μ and σ. Under this assumption, the following formulas are well-known:[14]

$$1+E = \exp\left(\mu+\frac{1}{2}\sigma^2\right) \tag{14}$$

$$1+M = \exp\left(\mu\right) \tag{15}$$

$$V = \exp\left(2\mu+\sigma^2\right)\left(\exp\left(\sigma^2\right)-1\right) \tag{16}$$

It easily follows from (14)-(16) that the geometric mean is calculated as (A4):

$$1+M = \left(1+E\right)\left(1+V\left(1+E\right)^{-2}\right)^{-1/2} \tag{A4}$$

Thus, the relationship (A4) is exact under the lognormal assumption.

If there is a need to calculate the arithmetic mean when the geometric mean and the variance are given, then, from (A4), the arithmetic mean is calculated as follows:

$$1+E = \left(1+M\right)\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4V}{\left(1+M\right)^2}}} \tag{17}$$

Which formula among $(A1)-(A4)$ should work better? The utilization of independent identically distributed lognormal return factors may be a reasonable forward-looking assumption.

Therefore, formula (A4) may be the right choice *for forward-looking analysis. A priori*, however, this is not necessarily the case for historical data. The next section explores this issue.

Historical Arithmetic and Geometric Averages

This section presents the arithmetic and geometric averages for historical data and analyzes the quality of the approximations discussed in prior sections. The section analyzes three sets of historical data: equity real rates of return (*Exhibit 1*),[15] equity premium relative to bills (*Exhibit 2*), and equity premium relative to bonds (*Exhibit 3*) from 1900 to 2005. Each dataset contains the arithmetic averages, geometric averages and standard deviations calculated exactly. For each dataset, we calculate four approximations of the geometric averages (A1) – (A4) and compare the approximations to the actual values.

Exhibit 1

Equity Real Rates of Return, 1900–2005

	Data			Geometric Average Approximation					
	Arithmetic Average	Standard Deviation	Geometric Average	A1	A2	A3	A4	Best	Worst
Australia	9.21%	17.64%	7.70%	7.65%	7.78%	7.79%	7.81%	A1	A4
Belgium	4.58%	22.10%	2.40%	2.14%	2.22%	2.27%	2.32%	A4	A1
Canada	7.56%	16.77%	6.24%	6.15%	6.24%	6.26%	6.28%	A2	A1
Denmark	6.91%	20.26%	5.25%	4.86%	4.97%	5.01%	5.04%	A4	A1
France	6.08%	23.16%	3.60%	3.40%	3.52%	3.58%	3.64%	A3	A1
Germany*	8.21%	32.53%	3.09%	2.92%	3.20%	3.43%	3.63%	A2	A4
Ireland	7.02%	22.10%	4.79%	4.58%	4.71%	4.76%	4.81%	A4	A1
Italy	6.49%	29.07%	2.46%	2.26%	2.45%	2.60%	2.73%	A2	A4
Japan	9.26%	30.05%	4.51%	4.74%	5.05%	5.20%	5.35%	A1	A4
Netherlands	7.22%	21.29%	5.26%	4.95%	5.09%	5.13%	5.17%	A4	A1
Norway	7.08%	26.96%	4.28%	3.45%	3.63%	3.74%	3.84%	A4	A1
South Africa	9.46%	22.57%	7.25%	6.91%	7.11%	7.16%	7.20%	A4	A1
Spain	5.90%	21.88%	3.74%	3.51%	3.62%	3.66%	3.71%	A4	A1
Sweden	10.07%	22.62%	7.80%	7.51%	7.72%	7.77%	7.82%	A4	A1
Switzerland	6.28%	19.73%	4.48%	4.33%	4.43%	4.46%	4.49%	A4	A1
U.K.	7.36%	19.96%	5.50%	5.37%	5.49%	5.52%	5.55%	A2	A1
U.S.	8.50%	20.19%	6.52%	6.46%	6.60%	6.64%	6.67%	A1	A4
World	7.16%	17.23%	5.75%	5.68%	5.77%	5.78%	5.80%	A2	A1
World ex-U.S.	7.02%	19.79%	5.23%	5.06%	5.17%	5.21%	5.24%	A4	A1

* excludes 1922–1923
Source: Dimson, Marsh, and Staunton (2006).

Exhibit 2

Equity Premium Relative to Bills, 1900–2005

	Data			Geometric Average Approximation					
	Arithmetic Average	Standard Deviation	Geometric Average	A1	A2	A3	A4	Best	Worst
Australia	8.49%	17.00%	7.08%	7.05%	7.15%	7.17%	7.18%	A1	A4
Belgium	4.99%	23.06%	2.80%	2.33%	2.43%	2.49%	2.55%	A4	A1
Canada	5.88%	16.71%	4.54%	4.48%	4.55%	4.57%	4.59%	A2	A1
Denmark	4.51%	19.85%	2.87%	2.54%	2.61%	2.64%	2.67%	A4	A1
France	9.27%	24.19%	6.79%	6.34%	6.56%	6.62%	6.69%	A4	A1
Germany*	9.07%	33.49%	3.83%	3.46%	3.80%	4.05%	4.27%	A2	A4
Ireland	5.98%	20.33%	4.09%	3.91%	4.01%	4.05%	4.08%	A4	A1
Italy	10.46%	32.09%	6.55%	5.31%	5.70%	5.90%	6.07%	A4	A1
Japan	9.84%	27.82%	6.67%	5.97%	6.26%	6.37%	6.48%	A4	A1
Netherlands	6.61%	22.36%	4.55%	4.11%	4.24%	4.29%	4.34%	A4	A1
Norway	5.70%	25.90%	3.07%	2.35%	2.48%	2.57%	2.66%	A4	A1
South Africa	8.25%	22.09%	6.20%	5.81%	5.97%	6.02%	6.06%	A4	A1
Spain	5.46%	21.45%	3.40%	3.16%	3.26%	3.30%	3.34%	A4	A1
Sweden	7.98%	22.09%	5.73%	5.54%	5.70%	5.74%	5.79%	A3	A1
Switzerland	5.29%	18.79%	3.63%	3.52%	3.60%	3.63%	3.65%	A3	A1
U.K.	6.14%	19.84%	4.43%	4.17%	4.27%	4.30%	4.33%	A4	A1
U.S.	7.41%	19.64%	5.51%	5.48%	5.60%	5.63%	5.66%	A1	A4
World	5.93%	19.33%	4.23%	4.06%	4.15%	4.18%	4.21%	A4	A1
World ex-U.S.	6.07%	16.65%	4.74%	4.68%	4.76%	4.77%	4.79%	A2	A1

* excludes 1922–1923
Source: Dimson, Marsh, and Staunton (2006).

Exhibit 3

Equity Premium Relative to Bonds, 1900–2005

	Data			Geometric Average Approximation					
	Arithmetic Average	Standard Deviation	Geometric Average	A1	A2	A3	A4	Best	Worst
Australia	7.81%	18.80%	6.22%	6.04%	6.16%	6.18%	6.21%	A4	A1
Belgium	4.37%	20.10%	2.57%	2.35%	2.42%	2.45%	2.49%	A4	A1
Canada	5.67%	17.95%	4.15%	4.06%	4.13%	4.16%	4.18%	A3	A1
Denmark	3.27%	16.18%	2.07%	1.96%	1.99%	2.01%	2.03%	A4	A1
France	6.03%	22.29%	3.86%	3.55%	3.66%	3.71%	3.76%	A4	A1
Germany*	8.35%	27.41%	5.28%	4.59%	4.83%	4.94%	5.04%	A4	A1
Ireland	5.18%	18.37%	3.62%	3.49%	3.56%	3.59%	3.61%	A4	A1
Italy	7.68%	29.73%	4.30%	3.26%	3.49%	3.65%	3.80%	A4	A1
Japan	9.98%	33.06%	5.91%	4.52%	4.89%	5.12%	5.32%	A4	A1
Netherlands	5.95%	21.63%	3.86%	3.61%	3.72%	3.76%	3.81%	A4	A1
Norway	5.26%	27.43%	2.55%	1.50%	1.62%	1.75%	1.86%	A4	A1
South Africa	7.03%	19.32%	5.35%	5.16%	5.27%	5.30%	5.33%	A4	A1
Spain	4.21%	20.20%	2.32%	2.17%	2.23%	2.27%	2.31%	A4	A1
Sweden	7.51%	22.34%	5.21%	5.01%	5.16%	5.21%	5.26%	A3	A1
Switzerland	3.28%	17.52%	1.80%	1.75%	1.78%	1.80%	1.83%	A3	A1
U.K.	5.29%	16.60%	4.06%	3.91%	3.97%	3.99%	4.01%	A4	A1
U.S.	6.49%	20.16%	4.52%	4.46%	4.56%	4.60%	4.63%	A2	A4
World	5.18%	15.19%	4.10%	4.03%	4.08%	4.09%	4.10%	A4	A1
World ex-U.S.	5.15%	14.96%	4.04%	4.03%	4.08%	4.09%	4.10%	A1	A4

* excludes 1922–1923
Source: Dimson, Marsh, and Staunton (2006).

Exhibits 1-3 contain data for 17 countries plus two totals – 19 data series overall. For each data series, we measure the distance between approximations (A1) – (A4) of the geometric average and the actual geometric average. The approximation that is closest to actual value is ranked the best; the farthest is ranked the worst. For example, looking at the data for Australia in *Exhibit 3*, (A1) is 18 bps away from the actual value (6.04% vs. 6.22%), (A2) is 6 bps away from the actual value (6.16% vs. 6.22%), (A3) is 4 bp away from the actual value (6.18% vs. 6.22%), and (A4) is 1 bp away from the actual value (6.21% vs. 6.22%). Therefore, (A4) is ranked the best and (A1) is ranked the worst.

For each exhibit and each approximation, we count the number of data series for which the approximation is the best and the worst. These counts are presented in *Exhibit 4*.

Exhibit 4

Approximation Rankings

	A1		A2		A3		A4	
	Best	Worst	Best	Worst	Best	Worst	Best	Worst
Equity Premium Relative to Bonds (*Exhibit 1*)	3	14	5	0	1	0	10	5
Equity Premium Relative to Bills (*Exhibit 2*)	2	16	3	0	2	0	12	3
Equity Real Rates of Return (*Exhibit 3*)	1	17	1	0	3	0	14	2
Total #	**6**	**47**	**9**	**0**	**6**	**0**	**36**	**10**
Total %	**11%**	**82%**	**16%**	**0%**	**11%**	**0%**	**63%**	**18%**

Exhibit 5

A4 Compared to A1-A3

	A4 is better than A1	A4 is better than A2	A4 is better than A3
Equity Premium Relative to Bonds (*Exhibit 1*)	14	11	10
Equity Premium Relative to Bills (*Exhibit 2*)	16	13	12
Equity Real Rates of Return (*Exhibit 3*)	17	14	14
Total #	**47**	**38**	**36**
Total %	**82%**	**67%**	**63%**

Overall, (A4) largely looks better than (A1) – (A3), as it is the best approximations in 63% cases (see *Exhibit 4*). (A1) largely looks worse than (A2) – (A4), as it is the worst approximation in 82% cases (see *Exhibit 4*). (A2) and (A3) are mostly in-between, and they are never the worst. The results within *Exhibits 1-3* are consistent with this conclusion.

Exhibit 5 contains the results of direct comparisons of (A4) to (A1) – (A3) for each data series. (A4) works better than (A1) in 82% cases, better than (A2) in 67% cases, and better than (A3) in 63% cases. The results within *Exhibits 1-3* are consistent with this conclusion. While the results of (A4) are not vastly superior, they do demonstrate a clear pattern. Another clear pattern is the tendency of (A1) to underestimate the geometric return. It happens in 56 out of 57 data series, and, sometimes, by a significant margin.

Yet, (A1) should not be dismissed easily, not so fast, at least. (A1) provides the best match for the U.S. data in *Exhibits 1* and *2*; it is a close second in *Exhibit 3*, in which it is also the best match for the "World ex. U.S." data series.

The results of (A1) – (A4) can occasionally be far apart, especially for high volatility portfolios. Let us consider the following example. *Exhibit 6* shows the data for the U.S. stocks divided into "large" and "small" stocks (as defined in the source). For the large stocks, (A1) is the best and (A4) is the worst approximation. For the small stocks, the opposite is true – (A4) is the best and (A1) is the worst approximation. But (A1) is not just the worst approximation among the four – it is astounding 162 bps lower than the actual value. (A2) is 100 bps closer, but still disappointing 62 bps below the actual value. (A3) is another 37 bps closer, but still 25 bps below the actual value. (A4) is the only one that provides a decent approximation.

Exhibit 6

U.S. Large and Small Stocks

Large Stocks

	Data	Geometric Average Approximation			
		A1	A2	A3	A4
Arithmetic Average	12.49%				
Geometric Average	10.51%	10.43%	10.64%	10.67%	10.70%
Standard Deviation	20.30%				

Small Stocks

	Data	Geometric Average Approximation			
		A1	A2	A3	A4
Arithmetic Average	18.29%				
Geometric Average	12.19%	10.57%	11.57%	11.94%	12.26%
Standard Deviation	39.28%				

Source: Bodie [2004], Table 5.3, p. 141.

Conclusion

This paper analyzes the following four relationships between arithmetic and geometric averages (means) that work for any return sample:

$$G \approx A - V/2 \tag{A1}$$

$$(1+G)^2 \approx (1+A)^2 - V \tag{A2}$$

$$1+G \approx (1+A)\exp\left(-\frac{1}{2}V(1+A)^{-2}\right) \tag{A3}$$

$$1+G \approx (1+A)\left(1+V(1+A)^{-2}\right)^{-1/2} \tag{A4}$$

When the return factor is lognormally distributed (a common forward-looking assumption), the relationship (A4) is exact:

$$1+M = (1+E)\left(1+V(1+E)^{-2}\right)^{-1/2} \tag{A4}$$

In this case, there is no need for approximations.

Relationship (A1) is the simplest, popularized in many publications, but usually sub-optimal and tends to underestimate the geometric return. Relationships (A2) – (A4) are slightly more complicated, but, in most cases, should be expected to produce better results than (A1).

Overall, (A4) looks like a winner – it works better in both backward- and forward looking settings. Still, (A1) – (A3) should not be dismissed summarily, and more research is needed to determine the conditions under which a particular formula may work better. For a practitioner, it may be a good idea to compare the results of all four formulas. There may be significant disparities among these approximations, especially for high volatility portfolios.

Both arithmetic averages and geometric averages are required for a clear understanding of investment returns. This author hopes that this paper would be useful to practitioners in clarifying the relationships between these averages as well as their pros and cons.

APPENDIX: The Development of Formulas (A1) – (A3) and Transitions from (A4) to (A1)

This Appendix contains the technical details of the development of formulas (A1) – (A4). The *arithmetic* average A of a series of returns $r_1, \ldots, r_n$ is defined as the average value of the series:

$$A = \frac{1}{n}\sum_{k=1}^{n} r_k \tag{1}$$

The *geometric* average G of a series of returns $r_1, \ldots, r_n$ is defined as follows:

$$G = -1 + \prod_{k=1}^{n}(1+r_k)^{1/n} \tag{4}$$

Sample variance V is defined as

$$V = \frac{1}{n}\sum_{k=1}^{n}(r_k - A)^2 \tag{6}$$

Formula # 1 (A1)

Let us take the Maclaurin series expansion for the function $f(x) = (1+x)^{1/n}$ up to the second degree and ignore the remainder:

$$(1+x)^{1/n} \approx 1 + \frac{1}{n}x + \frac{1-n}{2n^2}x^2 \tag{18}$$

Substituting (18) into (4) and ignoring summands of the third degree and higher, we get (A1):

$$G \approx -1 + \prod_{k=1}^{n}\left(1 + \frac{1}{n}r_k + \frac{1-n}{2n^2}r_k^2\right) \approx$$
$$\approx \frac{1}{n}\sum_{k=1}^{n} r_k + \frac{1}{n^2}\sum_{k\neq l}^{n} r_k r_l + \frac{1-n}{2n^2}\sum_{k=1}^{n} r_k^2 = \tag{19}$$
$$= A - V/2$$

Formula # 2 (A2)

From the definition of G, we get

$$(1+G)^2 = \prod_{k=1}^{n}(1+r_k)^{2/n} \tag{20}$$

Let us take the Maclaurin series expansion for the function $f(x) = (1+x)^{2/n}$ up to the second degree and ignore the remainder:

$$(1+x)^{2/n} \approx 1 + \frac{2}{n}x + \frac{2-n}{2n^2}x^2 \tag{21}$$

Substituting (21) into (20) and ignoring summands of the third degree and higher, we get (A2):

$$(1+G)^2 \approx \prod_{k=1}^{n}\left(1 + \frac{2}{n}x_k + \frac{2-n}{2n^2}x_k^2\right) \approx$$

$$\approx 1 + \frac{2}{n}\sum_{i=1}^{n}x_k + \frac{4}{n^2}\sum_{k \neq l}x_k x_l + \frac{2-n}{n^2}\sum_{k=1}^{n}x_k^2 = \tag{22}$$

$$= (1+A)^2 - V$$

Formula # 3 (A3)

Let us take the Taylor series expansion for the function $f(x) = \ln(1+x)$ around point A up to the second degree and ignore the remainder:

$$\ln(1+x) \approx \ln(1+A) + \frac{x-A}{1+A} - \frac{(x-A)^2}{2(1+A)^2} \tag{23}$$

Using (23) on the right side of (8), we get (A3):

$$\ln(1+G) = \frac{1}{n}\sum_{k=1}^{n}\ln(1+r_k) \approx$$

$$\approx \ln(1+A) + \frac{1}{n(1+A)}\sum_{k=1}^{n}(r_k - A) - \frac{1}{2(1+A)^2}\frac{1}{n}\sum_{k=1}^{n}(r_k - A)^2 = \tag{A3}$$

$$= \ln(1+A) - \frac{V}{2(1+A)^2}$$

Now, below are the sequences of approximations and simplifications that turn (A3) into (A4), then turn (A4) into (A2), and then turn (A2) into (A1) as well as the proof that

$$(A2) < (A3) < (A4)$$

(A3) – (A4) Transition

The transition from (A3) to (A4) is achieved via replacing $V(1+A)^{-2}$ with $\ln\left(1+V(1+A)^{-2}\right)$ (using approximation $\ln(1+x) \approx x$). Noting that $1+x \leq \exp x$, we get

$$(1+A)\exp\left(-\frac{1}{2}V(1+A)^{-2}\right) \leq (1+A)\left(1+V(1+A)^{-2}\right)^{-1/2}$$

which means the geometric average estimate (A4) is no less than (A3).

(A4) – (A2) Transition

The transition from (A4) to (A2) is achieved via replacing $\left(1+V(1+A)^{-2}\right)^{-1}$ with $1-V(1+A)^{-2}$ (using approximation $(1+x)^{-1} \approx 1-x$). Noting that $(1+x)^{-1} \geq 1-x$, we get

$$(1+A)\left(1+V(1+A)^{-2}\right)^{-1/2} \geq (1+A)\left(1-V(1+A)^{-2}\right)^{1/2} = \left((1+A)^2 - V\right)^{1/2}$$

which means the geometric average estimate (A3) is no less than (A2).

(A2) – (A1) Transition

The transition from (A2) to (A1) is achieved via replacing $(1+G)^2$ and $(1+A)^2$ with $1+2A$ and $1+2G$ correspondingly (using approximation $(1+x)^2 \approx 1+2x$).

The geometric average estimate (A1) is not necessarily less than (A2), although this is true for all data series in *Exhibits 1- 3* and most practical applications. For example, if $r_1 = -99\%$ and $r_2 = 100\%$, then the geometric average estimate (A1) is equal to -49%, and the geometric average estimate (A2) is equal to -86% (which is equal to the actual geometric average of this return series, see below).

Finally, formula (A2) is exact when the return series contains just two points, due to the following.

$$(1+G)^2 = (1+r_1)(1+r_2) = 1 + r_1 + r_2 + \frac{1}{2}(r_1+r_2)^2 - \frac{1}{2}(r_1^2 + r_2^2) =$$

$$= 1 + r_1 + r_2 + \frac{1}{4}(r_1+r_2)^2 - \left(\frac{1}{2}(r_1^2+r_2^2) - \frac{1}{4}(r_1+r_2)^2\right) =$$

$$= (1+A)^2 - V$$

REFERENCES

Bodie, Z., Kane, A., Marcus, A.J. [1999]. *Investments*, McGraw-Hill, 4th Ed., 1999.

Bodie, Z., Kane, A., Marcus, A.J. [2004]. *Essentials of Investments*, McGraw-Hill, 2nd Ed., 2004.

Booth, D.G., Fama, E.F. [1992]. *Diversification Returns and Asset Contributions*, Financial Analysts Journal, May/June 1992, p. 26–32.

de La Grandville, O. [1998]. *The Long-Term Expected Rate of Return: Setting It Right*, Financial Analysts Journal, November/December 1998, p. 75–80.

Dimson, E., Marsh, P., Staunton, M. [2006]. *The Worldwide Equity Premium: A Smaller Puzzle*, London Business School, April 2006, http://papers.ssrn.com/sol3/papers.cfm?abstract_id=891620.

de La Grandville, O., Pakes, A.G., Tricot, C. [2002]. *Random Rates of Growth and Return: Introducing the Expo-Normal Distribution*, Applied Stochastic Models in Business and Industry, 2002; 18:23–51.

Hughson, E., Stutzer, M., Yung, C. [2006]. *The Misuse of Expected Returns*, Financial Analysts Journal, November/December 2006, p. 88-96.

Jacquier, E., Kane, A, Marcus, A. [2003]. *Geometric Mean or Arithmetic Mean: A Reconsideration*, Financial Analysts Journal, November/December 2003, p. 46-53.

Jean, W.H., Helms, W.P. {1983]. *Geometric Mean Approximations*, Journal of Financial and Quantitative Analysis, Vol. 18, No. 3, September, 1983, p. 287-293.

Jordan, B. D., Miller, T. W. [2008] *Fundamentals of Investments*, McGraw-Hill, 4th Ed., 2008.

Klugman, S.A., Panjer, H.H., Willmot, G.E. [1998] *Loss Distributions*, Wiley, 1998.

Latane, H.A. [1959]. *Criteria for Choice among Risky Ventures*, Journal of Political Economy, Vol. 67, April 1959, p. 144-155.

MacBeth, J.D. [1995]. *What's the Long-Term Expected Return to Your Portfolio?* Financial Analysts Journal, September/October 1995, p. 6–8.

Maginn, J.L., Tuttle, D.L., McLeavey, D.W., Pinto, J.E. [2007]. *Managing Investment Portfolios*, Wiley, 3rd Ed., 2007.

Markowitz, H.M. [1991]. *Portfolio Selection*, 2nd Ed., Blackwell Publishing, 1991.

Siegel, J. J. [2008] *Stocks for the Long Run*, McGraw-Hill, 4th Ed., 2008, p. 22.

Pinto, J. E., Henry, E., Robinson, T. R., Stowe, J. D. *Equity Asset Valuation*, Wiley, 2nd Ed., 2010.

Important Information

---

[1] *Arithmetic* and *geometric* averages are two of the three classical *Pythagorean means*. The third one is the *harmonic* average.

[2] For example, see MacBeth [1995], de La Grandville [1998], Jacquier [2003], Hughson [2006].

[3] That is, obviously, if returns $r_1, \ldots, r_n$ are not all the same, as assumed in this section. If $r_1 = r_2 = \ldots = r_n$, then the problem is trivial as the arithmetic and geometric averages are equal.

[4] This fact is a corollary of the *Jencen's inequality*.

[5] For the purposes of this section, the concerns about the quality of these approximations are set aside.

[6] For the purposes of this paper, the concerns that the sample variance as defined in (7) is not an unbiased estimate are set aside.

[7] For example, see Bodie [1999], p. 751, Jordan [2008], p. 25, Pinto [2010], p. 49.

[8] According to Jean, Helms [1983], formula (A2) was originally proposed in Latane [1959].

[9] For example, see Markowitz [1991], p. 122, Jean, Helms [1983], Booth, Fama [1992].

[10] Reminder: in this section, it is assumed that returns $r_1, \ldots, r_n$ are not all the same (see endnote 3).

[11] The geometric average estimate (A1) is less than (A2) when $A > V/4$, which is usually the case.

[12] We assume that the first and the second moments of the return distribution are finite.

[13] de La Grandville [1998] and de La Grandville [2002] contain a similar message.

[14] For example, see Klugman [1998], p. 582.

[15] This data is also presented in Maginn [2007], Exhibit 7-2, p. 410.