

## MAT337 Introduction to Real Analysis - Fall 2025

### Week 8 Tutorial (Addendum)

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There was a small issue with the solution to Problem 5.1.D I gave in the tutorial. I used the fact that for all  $t > -1$ :

$$\frac{t}{1+t} \leq \log(1+t) \leq t.$$

Letting  $t = xy$ , we have that at a neighbourhood around  $(0, y_0)$ :

$$\frac{xy}{1+xy} \leq \log(1+xy) \leq xy.$$

Then I claimed that this implies:

$$\frac{y}{1+xy} \leq \frac{\log(1+xy)}{x} \leq y.$$

This is only true when  $x > 0$ , so we need to address the case when  $x < 0$ . The correct argument can be found below. I have also included a proof of the inequality involving the logarithm for completeness.

### Problem 5.1.D.

Prove that  $f$  is continuous at  $(0, y_0)$ , where  $f$  is defined on  $\mathbb{R}^2$  by:

$$f(x, y) = \begin{cases} (1+xy)^{\frac{1}{x}}, & \text{if } x \neq 0, \\ e^y, & \text{if } x = 0. \end{cases}$$

#### Solution

The key observation is that:

$$\lim_{(x,y) \rightarrow (0,y_0)} (1+xy)^{\frac{1}{x}} = \lim_{x \rightarrow 0} e^{\frac{\log(1+xy)}{x}} = e^{\lim_{(x,y) \rightarrow (0,y_0)} \frac{\log(1+xy)}{x}}.$$

So it suffices to show that:

$$\lim_{(x,y) \rightarrow (0,y_0)} \frac{\log(1+xy)}{x} = y_0.$$

**Exercise.** Show that for all  $t > -1$ :

$$\frac{t}{1+t} \leq \log(1+t) \leq t.$$

*Proof.* We first show that  $\log(1+t) \leq t$  for all  $t > -1$ . Let  $g(t) = \log(1+t)$  and  $f(t) = t$ . Then  $g'(t) = \frac{1}{1+t}$  and  $f'(t) = 1$ . Note that  $g(0) = 0 = f(0)$ .

- (1) Since  $g'(t) < f'(t)$  for all  $t > 0$  and  $g(0) = f(0)$ , we have that  $g(t) < f(t)$  for all  $t > 0$ .
- (2) Since  $g'(t) > f'(t)$  for all  $-1 < t < 0$  and  $g(0) = f(0)$ , we have that  $g(t) < f(t)$  for all  $t > 0$ .

Note that if  $t > -1$ , then  $-\frac{t}{1+t} > -1$ . Therefore, applying the same inequality with  $t$  substituted as  $\frac{t}{1+t}$ , we get:

$$\begin{aligned} \log\left(1 - \frac{t}{1+t}\right) &\leq -\frac{t}{1+t} \implies \log\left(\frac{1}{1+t}\right) \leq -\frac{t}{1+t} \\ &\implies -\log(1+t) \leq -\frac{t}{1+t} \\ &\implies \frac{t}{1+t} \leq \log(1+t). \end{aligned}$$

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Letting  $t = xy$ , we have that at a neighbourhood around  $(0, y_0)$ :

$$\frac{xy}{1+xy} \leq \log(1+xy) \leq xy.$$

Therefore:

- (1) If  $x > 0$ , then:

$$\frac{y}{1+xy} \leq \frac{\log(1+xy)}{x} \leq y.$$

- (2) If  $x < 0$ , then:

$$y \leq \frac{\log(1+xy)}{x} \leq \frac{y}{1+xy}.$$

This means that, for any  $x$ :

$$\min \left\{ y, \frac{y}{1+xy} \right\} \leq \frac{\log(1+xy)}{x} \leq \max \left\{ y, \frac{y}{1+xy} \right\}.$$

Observe that  $\lim_{(x,y) \rightarrow (0,y_0)} y = y_0$  and  $\lim_{(x,y) \rightarrow (0,y_0)} \frac{y}{1+xy} = y_0$ . Since  $\max$  and  $\min$  are continuous functions (see Exercise 5.3.J, it's a homework problem!):

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,y_0)} \min \left\{ y, \frac{y}{1+xy} \right\} &= y_0. \\ \lim_{(x,y) \rightarrow (0,y_0)} \max \left\{ y, \frac{y}{1+xy} \right\} &= y_0. \end{aligned}$$

By the squeeze theorem,  $\lim_{(x,y) \rightarrow (0,y_0)} \frac{\log(1+xy)}{x} = y_0$ .