NUS Reading Seminar Summer 2023

Session 5

Clement Yung

9 Jun 2023

Kripke-Platek Set Theory

Kripke-Platek (usually abbreviated as KP) set theory is a weakened version of ZF set theory. It is closely related to descriptive set theory and higher recursion theory.

The Axiomatic system KP is the set of the following axioms:

1. **Extensionality**, **Pairing** and **Union**.
2. **Regularity**: For every formula $\phi(x)$ is a formula with y not occurring free, then:

$$\exists x\, \phi(x) \to \exists x[\phi(x) \wedge \forall y \in x\, \neg\phi(y)]$$

3. Δ_0-**Separation**: For each Δ_0-formula ϕ:

$$\forall x \exists y \forall z[z \in y \leftrightarrow (z \in x \wedge \phi(z))]$$

4. Δ_0-**Collection**: For each Δ_0-formula ϕ:

$$\forall x[(\forall y \in x\, \exists z\, \phi(y, z)) \to \exists u\, \forall y \in x\, \exists z \in u\, \phi(y, z)]$$

KPω is used to denote KP $+$ Axiom of infinity (in the usual ZF sense).

Admissible set theory thus studies models of KP. These models need not be *ω-models* - that is, while a model $\mathfrak{A}$ of KP may contain a set $x \in \mathfrak{A}$ such that:

$$\mathfrak{A} \models \text{“}(x, \in) \models \mathsf{PA}\text{”}$$

we need not have $x = \omega$ (where ω is taken in the universe). In other words, $(x, \in)$ is a *non-standard model of arithmetic*.

If $\omega \in \mathfrak{A}$, then $\mathfrak{A}$ has a standard model of arithmetic, so we say that $\mathfrak{A}$ is an *ω-model*. Note that if $\mathfrak{A} \models \mathrm{KP}$ and $\mathfrak{A}$ is well-founded, then $\mathfrak{A}$ is an ω-model - it's transitive collapse has a copy of ω.

A model being an ω-model is helpful as:

Lemma

If $\mathfrak{A}$ is an ω-model of KP, $x \in \mathfrak{A}$ and $\phi(x)$ is an arithmetical formula, then:

$$\phi(x) \iff \mathfrak{A} \models \phi(x)$$

Δ_1 Properties of KP

KP models in fact satisfy a stronger variant of separation axiom.

Proposition

KP $\vdash$ Δ_1-*separation. That is, if* φ *is a* Δ_1-*formula, then:*

$$\mathrm{KP} \vdash \forall x \exists y \forall z [z \in y \leftrightarrow (z \in x \wedge \varphi(z))]$$

Proof.

Work in a model of KP. We wish to show that for any set x, there exists some set y such that:

$$\forall z[z \in y \leftrightarrow (z \in x \wedge \varphi(z))]$$

Let ϕ, ψ be Δ_0-formulas such that for all z, $\varphi(z)$ iff $\exists w\, \phi(z, w)$ iff $\neg\exists w\, \psi(z, w)$. In other words, we have that:

$$\forall z[\exists w\, \phi(z, w) \leftrightarrow \neg\exists w\, \psi(z, w)]$$

Taking $\rightarrow$ in particular, we have that:

$$\forall z\, \exists w[\phi(z, w) \vee \psi(z, w)]$$

Proof (Cont.)

Then in particular:

$$\forall z \in x \, \exists w [\phi(z, w) \vee \psi(z, w)]$$

Applying Δ_0-collection, we obtain some u such that:

$$\forall z \in x \, \exists w \in u [\phi(z, w) \vee \psi(z, w)]$$

Now let:

$$y := \{z \in x : \exists w \in u \, \phi(z, w)\}$$

This is a well-defined set as "$\exists w \in u \, \phi(z, w)$" is Δ_0, and we have Δ_0-separation. Then this set y works. $\square$

Proposition

KP $\vdash$ Σ_1-*collection. That is, if* φ *is a* Σ_1-*formula, then:*

$$\mathrm{KP} \vdash \forall x[(\forall y \in x \, \exists z \, \phi(y,z)) \to \exists u \, \forall y \in x \, \exists z \in u \, \phi(y,z)]$$

Proof.

Suppose $\varphi(y, z)$ iff $\exists w\, \phi(y, z, w)$, where ϕ is Δ_0. Fix some set x, and suppose:

$$\forall y \in x\, \exists z\, \exists w\, \phi(y, z, w)$$

Define a formula ψ by stipulating that:

$$\psi(y, v) \iff v = \{z, w\} \wedge \phi(y, z, w)$$

This is Δ_0, as "$v = \{z, w\}$" is Δ_0 (and it makes sense as KP has the pairing axiom). Then the first statement implies that:

$$\forall y \in x\, \exists u\, \psi(y, u)$$

Proof (Cont.)

By Δ_0-Collection, there exists a set u such that:

$$\begin{aligned}
& \forall y \in x\, \exists v \in u\, \psi(y, v) \\
\implies & \forall y \in x\, \exists v \in u[v = \{z, w\} \wedge \phi(y, z, w)] \\
\implies & \forall y \in x \exists z \in \bigcup u\, \exists w\, \phi(y, z, w) \\
\implies & \forall y \in x \exists z \in \bigcup u\, \varphi(y, z)
\end{aligned}$$

so collection holds for φ. $\square$

Ordinals in KP models

If $\mathfrak{A}$ is a transitive model of KP, then $\mathfrak{A} \cap \mathbf{ORD}$ is an initial segment of **ORD**, and is hence an ordinal. However, $\mathfrak{A}$ need not even be well-founded (the ordinals in $\mathfrak{A}$ need not even be well-ordered!), so we need to adopt a different notion of "$\mathfrak{A} \cap \mathbf{ORD}$".

Notation

Let $\mathfrak{A} \models \mathrm{KP}$. Then:

$$s(\mathfrak{A}) := \sup\{\mathrm{otp}(S) : S \text{ is an initial segment of } \mathbf{ORD}^{\mathfrak{A}} \\ \wedge S \text{ is well-ordered}\}$$

Example

If $\mathfrak{A}$ is not an ω-model, then $s(\mathfrak{A}) = \omega$. This is because KP is strong enough to prove that $0, 1, 2, 3, \cdots \in \mathfrak{A}$, but $\omega \notin \mathfrak{A}$.

But what if $\mathfrak{A}$ is an ω-model?

Notation

Given a tree T and $s \in \omega^{<\omega}$, we define:

$$T/s := \{t \in \omega^{<\omega} : s^\frown t \in T\}$$

Theorem

If $\mathfrak{A}$ is an ω-model for KP *and $T \in \mathfrak{A}$ is a well-founded tree, then the height function:*

$$s \mapsto \|T/s\|$$

is in $\mathfrak{A}$.

As a consequence, we have that:

Corollary

If $\mathfrak{A}$ is an ω-model of $\mathsf{KP}\omega$, then all $x \in (\omega^\omega)^{\mathfrak{A}}$, $\omega_1^x \leq s(\mathfrak{A})$. In particular, $\omega_1^{\mathsf{CK}} \leq s(\mathfrak{A})$.

Proof.

Let $x \in (\omega^\omega)^{\mathfrak{A}}$. For each $\alpha < \omega_1^x$, let T be a well-founded tree, recursive in x, such that $\|T\| \geq \alpha$. Since $x \in \mathfrak{A}$ and recursive functions are Δ_1 functions of arithmetic, it is a Δ_0 function (of set theory), so by Δ_0-separation we have that $T \in \mathfrak{A}$. By the theorem, the height function $s \mapsto \|T/s\|$ is in $\mathfrak{A}$, and in particular $\|T\| \subseteq \mathfrak{A}$ (as range of a function is Δ_0). Therefore $\omega_1^x \leq s(\mathfrak{A})$. □

We now prove the theorem. Recall that since "T is well-founded" is Π^1_1, the well-foundedness of T is absolute across all models of sufficiently large fragment of ZF to prove that well-founded trees have a rank function. This would allow us to prove the theorem with an easy induction. Unfortunately, KP is not enough to prove such a statement, so we can't use that here.

We need a cleverer approach to prove this theorem.

Proof.

We induct on $\|T\|$, and assume that a height function exists for all subtrees T/s, where $s \neq \emptyset$. We define a formula ϕ by:

$$\phi(f, T) \iff f \text{ is a height function for } T$$

ϕ is a Δ_0 formula (Exercise). Applying the induction hypothesis, we have that:

$$\mathfrak{A} \models \forall s \in T \, \exists f [s = \emptyset \vee \phi(f, T/s)]$$

Applying Δ_0-collection, we obtain a set $x \in \mathfrak{A}$ such that:

$$\mathfrak{A} \models \forall s \in T \, \exists f \in x [s = \emptyset \vee \phi(f, T/s)]$$

Proof (Cont.)

Let $a := T \times T \times \bigcup\bigcup\bigcup x$. $a \in \mathfrak{A}$ as KP models are closed under finite Cartesian products (Exercise). Note that if $f \in x$, then $\mathrm{ran}(f) \subseteq \bigcup\bigcup\bigcup x$. By Δ_0-separation, we may define the function $F(s,t)$ by:

$$F := \{(s,t,\alpha) \in a : \mathfrak{A} \models \exists f \in x[\phi(f, T/s) \wedge f(t) = \alpha]\}$$

Note that for all s, t such that $s^\frown t \in T$, we have that:

$$F(s,t) = f(t) = \|T/(s^\frown t)\|$$

where f is the height function for the tree T/s. $\square$

Proof.

Let $b := \text{ran}(F) \cap \mathbf{ORD}$, which is a well-defined set in $\mathfrak{A}$ as range and $\mathbf{ORD}$ are Δ_0. We see that:

$$\begin{aligned}\sup b &= \sup(\text{ran}(F) \cap \mathbf{ORD}) \\ &= \sup_{s \neq \emptyset} \| T/s \| \\ &= \| T \|\end{aligned}$$

Thus, we let $\alpha_0 := \sup b = \bigcup b$. We may use Δ_0-separation again to define a function G such that:

$$G(s) := \begin{cases} \alpha_0, & \text{if } s = \emptyset \\ F((k), t), & \text{if } s = (k)^\frown t \end{cases}$$

Then G is the height function of T, as desired. $\square$

KP and constructibility

Suppose G is a function on some transitive class A. Recursion theorem asserts that there exists a function F on A such that for all $x \in A$:

$$F(x) = G(F \restriction x)$$

(See Theorem 6.5 of Jech). Using only Δ_1-separation and Σ_1-collection, we can get the following theorem of KP:

Theorem (Krivine, Σ_1-Recursion theorem)

If G is a Σ_1-function on some transitive class A, then there exists a Σ_1-function F on A such that for all $x \in A$:

$$F(x) = G(F \restriction x)$$

It turns out that Σ_1-recursion is sufficient for us to define the function:

$$\alpha \mapsto L_\alpha$$

That is, KP is sufficient to define the constructible hierarchy. This is done with the help of Gödel operations - see more at §12 of Jech.

We get the following result:

Theorem

If $\mathfrak{A}$ is a model of $\mathrm{KP}\omega$, *then:*

$$\{x \in \mathfrak{A} : \mathfrak{A} \models x \in L\}$$

is a model of $\mathrm{KP} + V = L$.

KP is also sufficient to prove the well-known *condensation lemma*:

Theorem

If $\mathfrak{A}$ *is a transitive model for* $\mathsf{KP} + V = L$*, then* $\mathfrak{A} = L_\alpha$ *for some ordinal* α*.*

A model of KP

We've seen that if $\mathfrak{A}$ is an ω-model of KP, then $s(\mathfrak{A}) \geq \omega_1^{\mathsf{CK}}$.

Theorem

There is an ω-model $\mathfrak{A}$ of KP *with $s(\mathfrak{A}) = \omega_1^{\mathsf{CK}}$.*

We shall borrow a few facts from the theory of hyperarithmetic sets. Define:

$$\mathrm{WFG} := \{\ulcorner T \urcorner : T \text{ is a recursive tree}\}$$

Fact

WFG *is a* Π^1_1*, but not* Σ^1_1*, subset of* ω*.*

Fact

If $x \subseteq \omega$ *is not* Δ^1_1*, then there exists an* ω*-model* $\mathfrak{A}$ *of* $\mathrm{KP}\omega$ *such that* $x \notin \mathfrak{A}$*.*

Lemma

Assume $\omega_1^{\mathsf{CK}} \subseteq \mathfrak{A}$. For all $\alpha < \omega_1^{\mathsf{CK}}$, there exists some $n_\alpha < \omega$ such that if $T \in \mathfrak{A}$ is a recursive tree of height α, then its height function is in $L^{\mathfrak{A}}_{\alpha+n_\alpha}$.

The assumption "$\omega_1^{\mathsf{CK}} \subseteq \mathfrak{A}$" is to ensure that the set $L^{\mathfrak{A}}_\alpha$ makes sense for all $\alpha <$ CK.

Proof.

We first note that since T is recursive, there exists a formula ϕ, Δ_1 in arithmetic (so Δ_0), such that $T = \{s \in \omega^{<\omega} : \phi(s)\}$, so $T \in L_{\omega+1}$.

We induct on α. We recap the proof that ω-models of KP contains the height function of T. We first observe that there exists some $n'_\alpha < \omega$: such that $\alpha + n'_\alpha = \sup_{\gamma<\alpha} \gamma + n_\gamma$. The induction hypothesis precisely asserts that

$$\mathfrak{A} \models \forall s \in T\, \exists f \in L_{\alpha+n'_\alpha}[s \neq \emptyset \vee \phi(f, T/s)]$$

where $\phi(f, T/s)$ is the Δ_0 formula asserting that f is the height function of T/s.

Proof (Cont.)

Let $a := T \times T \times \bigcup\bigcup\bigcup L_{\alpha+n'_\alpha}$, and by Δ_0-separation we may define the function:

$$F := \{(s, t, \xi) \in a : \mathfrak{A} \models \exists f \in L_{\alpha+n'_\alpha}[\phi(f, T/s) \wedge f(t) = \xi]\}$$

We have that $T \in L_{\alpha+n'_\alpha}$ and $\bigcup\bigcup\bigcup L_{\alpha+n'_\alpha} \in L_{\alpha+n'_\alpha+4}$, so $F \in L_{\alpha+n'_\alpha+5}$. We may then define G, the height function of T, from F and a, so $G \in L_{\alpha+n'_\alpha+6}$. Let $n_\alpha := n'_\alpha + 6$, and the induction is complete. $\square$

Proof of Theorem.

Let $\mathfrak{A}$ be an ω-model of KP such that WFG $\notin \mathfrak{A}$. We wish to show that $\omega_1^{\mathsf{CK}} \notin \mathfrak{A}$. Suppose otherwise. Recall that a tree has a height function iff it is well-founded (in the universe). We may define $L^{\mathfrak{A}}_{\omega_1^{\mathsf{CK}}} \in \mathfrak{A}$, and we see that:

$$x \in \mathrm{WFG}$$

$$\iff$$

$$x \in \omega \wedge \exists f \in L_{\omega_1^{\mathsf{CK}}+4} \exists T \in L_{\omega+1}[x = \ulcorner T \urcorner \wedge \varphi(f, T)]$$

The formula on RHS is Δ_0, so WFG $\in \mathfrak{A}$, a contradiction. $\square$

Well-founded models of KP

Here are some heads up of what we will be covering after summer school.

Suppose $\mathfrak{A} = (A, E)$ is a model of KP, not necessarily well-founded. Consider the following way to extract the well-founded part of $\mathfrak{A}$: Given each $x \in A$, define T_x to be the set:

$$\{s : s(0) = x \wedge s \text{ is a } E\text{-decreasing sequence of elements of } A\}$$

Then $(T_x, \sqsubseteq)$ is a well-defined tree. Now define:

$$B := \{x \in A : \|T_x\| < \omega\}$$

Then the set $\mathfrak{B} := (B, E)$ is well-founded (w.r.t. E).

Theorem

$\mathfrak{B}$ *is a well-founded* ω*-model of* KP.

Corollary

$L_{\omega_1^{\mathrm{CK}}}$ *is a well-founded* ω*-model of* KP.

I plan to prove these two results after summer school.

Admissible ordinals

Models of KP of the form L_α are of much interest, and are very important in the proof of Σ^1_1-AD $\to 0^\sharp$ exists.

Definition

A countable ordinal α is *admissible* if $L_\alpha \models$ KP.

Another theorem which we will be proving is the following:

Theorem

α is admissible iff $\alpha = \omega_1^x$ for some real x.