Weak A2 spaces, the Kastanas game and strategically Ramsey sets

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Ramsey Theory

In general, Ramsey theory addresses the following types of questions:

Let X be a set, and let $Y_0, \ldots, Y_{n-1}$ be a partition of X. Does there exist some $i < n$ such that Y_i contains some substructure of interest?

Example.

1. X is infinite.
2. Structure = infinite set.

Fact (Pigeonhole principle)

Let X be an infinite set. If $Y_0, \ldots, Y_{n-1}$ is a partition of X, then there exists some $i < n$ such that Y_i is infinite.

Example.

1. $X = [\mathbb{N}]^n$.
2. Structure = homogeneous set, i.e. $[H]^n$ for some infinite $H \subseteq \omega$.

Theorem (Ramsey)

If $Y_0, \ldots, Y_{n-1}$ is a partition of $[\mathbb{N}]^n$, then there exists some $i < n$ and an infinite $H \subseteq \omega$ such that $[H]^n \subseteq Y_i$.

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$$x = \sum_{n<\omega} \lambda_n(x) e_n,$$

where only finitely many λ_n's are non-zero. We may then write:

$$\mathrm{supp}(x) := \{n < \omega : \lambda_n(x) \neq 0\}.$$

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$$\mathrm{supp}(x) := \{n < \omega : \lambda_n(x) \neq 0\}.$$

Example

If $x = 2e_3 - 6e_{17} + 5e_{58}$, then $\mathrm{supp}(x) = \{3, 17, 58\}$.

Notation

Given two vectors x, y we write:

$$x < y \iff \max(\mathrm{supp}(x)) < \min(\mathrm{supp}(y)).$$

Example

If:

1. $x = 2e_3 - 6e_{17} + 5e_{58}$,
2. $y = 5e_{67} + 990e_{133} - 155e_{236}$,
3. $z = -32e_{43} + 5e_{665}$,

then $x < y$ but $x \not< z$.

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An infinite-dimensional subspace $V \subseteq W$ is a *block subspace* if $V = \mathrm{span}\{x_n : n < \omega\}$ for some infinite block sequence $(x_n)_{n<\omega}$. Note that $\{x_n\}_{n<\omega}$ is a (unique) basis of V.

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Fact

Every infinite-dimensional subspace of E contains an infinite-dimensional block subspace.

Now consider the following setting:

1. $X = E \setminus \{0\}$, the set of non-zero vectors.
2. Structure = infinite-dimensional block subspaces (without 0).

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Theorem (Hindman)

Suppose that $|\mathbb{F}| = 2$. If $Y_0, \ldots, Y_{n-1}$ is a partition of $E \setminus \{0\}$, then there exists some $i < n$ and some infinite-dimensional block subspace V such that $V \setminus \{0\} \subseteq Y_i$.

This theorem fails if $|\mathbb{F}| > 2$. We define the set Y as:

$$\begin{aligned} &\{x \in E \setminus \{0\} : x = e_n + y \text{ for some } e_n < y\}, \\ = &\{x \in E \setminus \{0\} : x = e_{n_0} + \lambda n_1 e_{n_1} + \cdots + \lambda_{n_k} e_{n_k} \text{ and } n_0 < \cdots < n_k\}. \end{aligned}$$

Then Y, Y^c partitions $E \setminus \{0\}$, but neither Y nor Y^c contains an infinite-dimensional subspace.

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Then Y, Y^c partitions $E \setminus \{0\}$, but neither Y nor Y^c contains an infinite-dimensional subspace.

Example

Let $A = (e_0 + e_1, e_2 + e_3, \ldots)$. Then $e_0 + e_1 \in Y$, but $2e_0 + 2e_1 \in Y^c$, so $\mathrm{span}(A) \setminus \{0\}$ is not a subset of either Y or Y^c.

Infinite-dimensional Ramsey theory

Infinite-dimensional Ramsey theory addresses a similar type of question, but instead, we partition $X^{\mathbb{N}}$ or a closed subest $\mathcal{R}$ of $X^{\mathbb{N}}$. Here we equip X with the discrete topology, and $X^{\mathbb{N}}$ with the product topology.

More precisely:

Let X be a set. Let $\mathcal{X}_0, \ldots, \mathcal{X}_{n-1}$ be a partition of $\mathcal{R}$, a closed subset of $X^{\mathbb{N}}$. Does there exist some $i < n$ such that $\mathcal{X}_i$ contains some substructure of interest?

Example.

1. $X = \mathbb{N}$.
2. $\mathcal{R} = [\mathbb{N}]^\infty$, which may be identified as the set of strictly increasing sequences in $\mathbb{N}^\mathbb{N}$.
3. Structure = Ellentuck neighbourhood of infinite subset.

Let $[\mathbb{N}]^{<\infty} \restriction A := \{a \in [\mathbb{N}]^{<\infty} : a \subseteq A\}$.

Notation

Given $A \in [\mathbb{N}]^{\infty}$ and $a \in [\mathbb{N}]^{<\infty} \restriction A$, we write:

$$[a, A] := \{B \in [\mathbb{N}]^{\infty} : a \sqsubseteq B \text{ and } B \subseteq A\}$$

where $a \sqsubseteq B$ means that $B \cap \max(a) = a$.

Each $[a, A]$ is also called an *Ellentuck neighbourhood*.

Definition

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is *Ramsey* if for all $A \in [\mathbb{N}]^\infty$ and $a \in [\mathbb{N}]^{<\infty} \restriction A$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

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Theorem (Galvin-Prikry)

If $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is Borel, then $\mathcal{X}$ is Ramsey.

Example. Let E a vector space over a countable field $\mathbb{F}$ of dimension $\aleph_0$, with Hamel basis $(e_n)_{n<\omega}$.

1. $X = E$.
2. $\mathcal{R} = E^{[\infty]}$, the set of all infinite block sequences of vectors ($\Leftrightarrow$ infinite-dimensional block subspaces).
3. Structure $=$ Ellentuck neighbourhood of infinite-dimensional block subspaces.

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3. Structure = Ellentuck neighbourhood of infinite-dimensional block subspaces.

Notation

If $A = (x_n)_{n<\omega}$ and $B = (y_n)_{n<\omega}$ are two elements of $E^{[\infty]}$, then we write:

$$B \leq A \iff \mathrm{span}(B) \subseteq \mathrm{span}(A).$$

Notation

Given $A = (x_n)_{n<\omega} \in E^{[\infty]}$, let $E^{[<\infty]}\restriction A$ be the set of finite block subspaces of A. In other words, the set of $(y_m)_{m<N}$ such that $\mathrm{span}\{y_0, \ldots, y_{N-1}\} \subseteq \mathrm{span}(A)$.

Notation

Given $A \in E^{[\infty]}$ and $a \in E^{[<\infty]}\restriction A$, we write:

$$[a, A] := \{B \in E^{[\infty]} : a \sqsubseteq B \text{ and } B \leq A\}$$

where $a \sqsubseteq B$ means that a is an initial segment of B.

Definition

A set $\mathcal{X} \subseteq E^{[\infty]}$ is *Ramsey* if for all $A \in E^{[\infty]}$ and $a \in E^{[<\infty]} \restriction A$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

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A set $\mathcal{X} \subseteq E^{[\infty]}$ is *Ramsey* if for all $A \in E^{[\infty]}$ and $a \in E^{[<\infty]} \restriction A$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

Theorem (Infinite-dimensional Hindman's theorem)

Suppose that $|\mathbb{F}| = 2$. If $\mathcal{X} \subseteq E^{[\infty]}$ is Borel, then it is Ramsey.

Again, this theorem fails for $|\mathbb{F}| > 2$ - there exists a clopen subset $\mathcal{X}$ of $E^{[\infty]}$ which is not Ramsey.

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Question. Can this pattern be captured and made into an abstract framework?

Topological Ramsey theory

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2. $\leq$ is a quasi-order on $\mathcal{R}$.
3. $\mathcal{AR}$ is a non-empty set, representing the set of finite increasing sequences.
   - For $\mathcal{R} = [\mathbb{N}]^{\infty}$, $\mathcal{AR} = [\mathbb{N}]^{<\infty}$.
   - For $\mathcal{R} = E^{[\infty]}$, $\mathcal{AR} = E^{[<\infty]}$.

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4. $r : \mathcal{R} \times \omega \to \mathcal{AR}$ is a function, with $r_n(-) := r(-, n)$, is a restriction map that takes the first n elements of the sequence.
    - If $A = \{x_0, x_1, \dots\} \in [\mathbb{N}]^{\infty}$, then $r_n(A) = \{x_0, \dots, x_{n-1}\}$.
    - If $A = (x_0, x_1, \dots) \in E^{[\infty]}$, then $r_n(A) = (x_0, \dots, x_{n-1})$.

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Notation

Given a triple $(\mathcal{R}, \leq, r)$, for $A \in \mathcal{R}$ and $a \in \mathcal{AR}$:

$$a \sqsubseteq A \iff a = r_n(A) \text{ for some } n \in \mathbb{N}.$$

Notation

If $A \in \mathcal{R}$ and $a \in \mathcal{AR}$,

$$[a, A] := \{B \in \mathcal{AR} : a \sqsubseteq B \text{ and } B \leq A\}.$$

Axiom (A1, Sequencing)

(1) $r_0(A) = \emptyset$ for all $A \in \mathcal{AR}$.

(2) $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n.

(3) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

Axiom (**A2**, Finitisation)

There is a quasi-ordering $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that:

(1) $\{b \in \mathcal{AR} : b \leq_{\text{fin}} a\}$ is finite for all $b \in \mathcal{AR}$.

(2) $A \leq B$ iff $\forall n \exists m [r_n(A) \leq_{\text{fin}} r_m(B)]$.

(3) $\forall a, b \in \mathcal{AR}[a \sqsubseteq b \wedge b \leq_{\text{fin}} c \rightarrow \exists d \sqsubseteq c[a \leq_{\text{fin}} d]]$.

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1. $([\mathbb{N}]^\infty, \subseteq, r)$ satisfies **A2**, as for all $a \in [\mathbb{N}]^{<\infty}$, $\{b : b \subseteq a\}$ is finite.
2. $(E^{[\infty]}, \leq, r)$ satisfies **A2** if $|\mathbb{F}| < \infty$, as if $a = (x_i)_{i<n} \in E^{[<\infty]}$, then there are finitely many subspaces of a.
3. $(E^{[\infty]}, \leq, r)$ does not satisfy **A2** if $|\mathbb{F}| = \infty$, as if $a = (x_0, x_1)$, then $\operatorname{span}\{\lambda x_0 + x_1\}$ is a (block) subspace for any $\lambda \in \mathbb{F}$.

Axiom (**A3**, Amalgamation)

The depth function defined by, for $B \in \mathcal{R}$ and $a \in \mathcal{AR}$:

$$\operatorname{depth}_B(a) := \begin{cases} \min\{n < \omega : a \leq_{\text{fin}} r_n(B)\}, & \text{if such } n \text{ exists} \\ \infty, & \text{otherwise} \end{cases}$$

satisfies the following:

(1) If $\operatorname{depth}_B(a) < \infty$, then for all $A \in [\operatorname{depth}_B(a), B]$, $[a, A] \neq \emptyset$.

(2) $A \leq B$ and $[a, A] \neq \emptyset$ imply that there exists $A' \in [\operatorname{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

We let $\mathcal{AR}_n$ be the image of the map $r_n(-)$, i.e. the set of all finite approximations of length n.

Axiom (A4, Pigeonhole)

If $\mathrm{depth}_B(a) < \infty$ and if $\mathcal{O} \subseteq \mathcal{AR}_{\mathrm{lh}(a)+1}$, then there exists $A \in [\mathrm{depth}_B(a), B]$ such that $r_{\mathrm{lh}(a)+1}[a, A] \subseteq \mathcal{O}$ or $r_{\mathrm{lh}(a)+1}[a, A] \subseteq \mathcal{O}^c$.

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1. $([\mathbb{N}]^\infty, \subseteq, r)$ satisfies **A4**, due to the pigeonhole principle.
2. $(E^{[\infty]}, \leq, r)$ satisfies **A4** if $|\mathbb{F}| = 2$, due to Hindman's theorem.

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1. $([\mathbb{N}]^\infty, \subseteq, r)$ satisfies **A4**, due to the pigeonhole principle.
2. $(E^{[\infty]}, \leq, r)$ satisfies **A4** if $|\mathbb{F}| = 2$, due to Hindman's theorem.
3. $(E^{[\infty]}, \leq, r)$ does not satisfy **A4** if $|\mathbb{F}| > 2$, as Hindman's theorem fails for $|\mathbb{F}| > 2$.

Observe that every $A \in \mathcal{R}$ can be uniquely identified as the sequence $(r_n(A))_{n<\omega}$, an element of $\mathcal{AR}^{\mathbb{N}}$. Therefore, we may identify $\mathcal{R}$ as a subset of $\mathcal{AR}^{\mathbb{N}}$.

Recall that in infinite-dimensional Ramsey theory, we require $\mathcal{R}$ to be a closed subset of $X^{\mathbb{N}}$. We make a similar requirement here.

Definition

$(\mathcal{R}, \leq, r)$ is a *closed triple* if for all $\sqsubseteq$-increasing sequence $(a_n)_{n<\omega}$ of elements in $\mathcal{AR}$ such that $\mathrm{lh}(a_n) = n$, there exists some $A \in \mathcal{R}$ such that $r_n(A) = a_n$ for all n.

In other words, $\mathcal{R}$ is a metrically closed subset of $\mathcal{AR}^{\mathbb{N}}$.

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Definition

Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space. A set $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey* if for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

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Theorem (Todorčević)

Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space. If $\mathcal{AR}$ is countable and $\mathcal{X} \subseteq \mathcal{R}$ is Borel, then $\mathcal{X}$ is Ramsey.

Weak **A2** spaces

Although countable vector spaces fail to satisfy the axioms **A1-A4** for $|\mathbb{F}| > 2$, a rich Ramsey theory of countable vector spaces has been developed in the past 20 years with lots of similarities to topological Ramsey theory.

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Question. Is there an overarching framework that encompasses topological Ramsey theory and the Ramsey theory of countable vector spaces?

Axiom (wA2, Weak Finitisation)

There is a quasi-ordering $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that:

(w1) $\{b \in \mathcal{AR} : b \leq_{\text{fin}} a\}$ is countable for all $b \in \mathcal{AR}$.

(2) $A \leq B$ iff $\forall n \exists m [r_n(A) \leq_{\text{fin}} r_m(B)]$.

(3) $\forall a, b \in \mathcal{AR} [a \sqsubseteq b \wedge b \leq_{\text{fin}} c \to \exists d \sqsubseteq c [a \leq_{\text{fin}} d]]$.

Definition

A triple $(\mathcal{R}, \leq, r)$ is a *weak* **A2** *space*, or just **wA2**-*space*, if it is a closed triple satisfying **A1**, **wA2**, **A3**.

Thus, topological Ramsey spaces and countable vector spaces are examples of **wA2**-spaces.

Abstract Kastanas Game

We discuss one application of **wA2**-spaces by introducing the abstract Kastanas game. Unless stated otherwise, we assume that $(\mathcal{R}, \leq, r)$ is a **wA2**-space.

Definition (Kastanas, Cano-Di Prisco)

Let $A \in \mathcal{R}$ and $a \in \mathcal{AR}\restriction A$. The *Kastanas game* played below $[a, A]$, denoted as $K[a, A]$, is:

$$
\begin{array}{c|ccccc}
\textbf{I} & A_0 \in [a, A] & & A_1 \in [a_1, B_0] & & \cdots \\
\hline
\textbf{II} & & \begin{array}{c} a_1 \in r_{\mathrm{lh}(a)+1}[a, A_0] \\ B_0 \in r_{\mathrm{lh}(a_1)+1}[a_1, A_0] \end{array} & & \begin{array}{c} a_2 \in [a_1, A_1] \\ B_1 \in [a_2, A_1] \end{array} &
\end{array}
$$

The outcome of this game is $\lim_{n\to\infty} a_n$, i.e. the unique $B \in \mathcal{R}$ such that $r_{\mathrm{lh}(a)+n}(B) = a_n$ for all n.

Definition

We say that **I** (similarly **II**) has a strategy in $K[a, A]$ to *reach* $\mathcal{X} \subseteq \mathcal{R}$ if it has a strategy in $K[a, A]$ to ensure the outcome is in $\mathcal{X}$.

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Definition

A set $\mathcal{X} \subseteq \mathcal{R}$ is *Kastanas Ramsey* if for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B]$ to reach $\mathcal{X}^c$.
2. **II** has a strategy in $K[a, B]$ to reach $\mathcal{X}$.

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A set $\mathcal{X} \subseteq \mathcal{R}$ is *Kastanas Ramsey* if for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B]$ to reach $\mathcal{X}^c$.
   (Definition of Ramsey: $[a, B] \subseteq \mathcal{X}^c$.)
2. **II** has a strategy in $K[a, B]$ to reach $\mathcal{X}$.
   (Definition of Ramsey: $[a, B] \subseteq \mathcal{X}$.)

Theorem (Kastanas)

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is Ramsey iff Kastanas Ramsey.

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Question. Can we generalise this fact to topological Ramsey spaces?

Theorem (Y.)

If $(\mathcal{R}, \leq, r)$ is a closed triple satisfying **A1**-**A4**, *then $\mathcal{X} \subseteq \mathcal{R}$ is Ramsey iff it is Kastanas Ramsey.*

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If $(\mathcal{R}, \leq, r)$ is a closed triple satisfying **A1**-**A4**, *then $\mathcal{X} \subseteq \mathcal{R}$ is Ramsey iff it is Kastanas Ramsey.*

1. By the Borel determinacy of Polish spaces, we can conclude that if $\mathcal{AR}$ is countable, then every Borel subset of $\mathcal{R}$ is Kastanas Ramsey.
2. Since Kastanas Ramsey $\iff$ Ramsey, we get Todorčević's theorem that every Borel subset of $\mathcal{R}$ is Ramsey.

What about analytic sets?

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Theorem (Mathias-Silver)

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Theorem (Mathias-Silver)

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Theorem (Todorčević)

Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space, and assume that $\mathcal{AR}$ is countable. Then every analytic subset of $\mathcal{R}$ is Ramsey.

Since analytic determinacy is not a theorem of ZFC, it's not clear that the equivalence between Kastanas Ramsey sets and Ramsey sets implies both theorems.

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Good news. We can use the equivalence to prove both theorems.

To simplify things, we shall demonstrate this for $[\mathbb{N}]^\infty$.

Goal. Provide a proof of the Mathias-Silver theorem in the following steps:

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Goal. Provide a proof of the Mathias-Silver theorem in the following steps:

1. Define a version of the Kastanas game (and Kastanas Ramsey sets) on $[\mathbb{N}]^\infty \times 2^\infty$. By the Borel determinacy for Polish spaces, all Borel subsets of $[\mathbb{N}]^\infty \times 2^\infty$ are Kastanas Ramsey.

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Goal. Provide a proof of the Mathias-Silver theorem in the following steps:

1. Define a version of the Kastanas game (and Kastanas Ramsey sets) on $[\mathbb{N}]^\infty \times 2^\infty$. By the Borel determinacy for Polish spaces, all Borel subsets of $[\mathbb{N}]^\infty \times 2^\infty$ are Kastanas Ramsey.
2. Show that Kastanas Ramsey sets are closed under projections. Therefore, analytic subsets of $[\mathbb{N}]^\infty$ are Kastanas Ramsey.

To simplify things, we shall demonstrate this for $[\mathbb{N}]^\infty$.

Goal. Provide a proof of the Mathias-Silver theorem in the following steps:

1. Define a version of the Kastanas game (and Kastanas Ramsey sets) on $[\mathbb{N}]^\infty \times 2^\infty$. By the Borel determinacy for Polish spaces, all Borel subsets of $[\mathbb{N}]^\infty \times 2^\infty$ are Kastanas Ramsey.
2. Show that Kastanas Ramsey sets are closed under projections. Therefore, analytic subsets of $[\mathbb{N}]^\infty$ are Kastanas Ramsey.
3. By Kastanas' theorem, analytic subsets of $[\mathbb{N}]^\infty$ are Ramsey.

Kastanas game on $[\mathbb{N}]^\infty \times 2^\infty$

Definition

Let $A \in [\mathbb{N}]^\infty$, and let $a \in [\mathbb{N}]^{<\infty}$ and $p \in 2^{|a|}$. The *Kastanas game* played below $[a, A, p]$, denoted as $K[a, A, p]$, is:

I	$A_0 = A$	$A_1 \subseteq B_0$	$\cdots$
II	$x_0 \in A_0$	$x_1 \in A_1$	$\cdots$
	$\varepsilon_0 \in \{0,1\}$	$\varepsilon_1 \in \{0,1\}$	$\cdots$
	$B_0 \subseteq A_0$	$B_1 \subseteq A_1$	$\cdots$

where:

- $\max(a) < x_0 < x_1 < \cdots$.
- A_n, B_n are infinite subsets of $\mathbb{N}$.

The outcome of the game is
$(a \cup \{x_0, x_1, \dots\}, p^\frown(\varepsilon_0, \varepsilon_1, \dots)) \in [a, A] \times 2^\infty$.

Definition

We say that **I** (similarly **II**) has a strategy in $K[a, A, p]$ to *reach* $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ if it has a strategy in $K[a, A, p]$ to ensure the outcome is in $\mathcal{C}$.

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Definition

A set $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ is *Kastanas Ramsey* if for all $A \in [\mathbb{N}]^\infty$, $a \in [\mathbb{N}]^{<\infty} \restriction A$ and $p \in 2^{|a|}$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B, p]$ to reach $\mathcal{C}^c$.
2. **II** has a strategy in $K[a, B, p]$ to reach $\mathcal{C}$.

Let $\pi_0 : [\mathbb{N}]^\infty \times 2^\infty \to [\mathbb{N}]^\infty$ be the projection to the first coordinate.

Theorem

If $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ is Kastanas Ramsey, then $\pi_0[\mathcal{C}] \subseteq [\mathbb{N}]^\infty$ is Kastanas Ramsey.

We split the proof of the theorem into two lemmas.

Lemma

Let $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ be a subset. Let $A \in [\mathbb{N}]^\infty$, $a \in [\mathbb{N}]^{<\infty}\restriction A$. If **II** *has a strategy in $K[a, A, p]$ to reach $\mathcal{C}$ for some $p \in 2^{\mathrm{lh}(a)}$, then* **II** *has a strategy in $K[a, A]$ to reach $\pi_0[\mathcal{C}]$.*

We split the proof of the theorem into two lemmas.

Lemma

Let $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ be a subset. Let $A \in [\mathbb{N}]^\infty$, $a \in [\mathbb{N}]^{<\infty}\restriction A$. If **II** *has a strategy in $K[a, A, p]$ to reach $\mathcal{C}$ for some $p \in 2^{\mathrm{lh}(a)}$, then* **II** *has a strategy in $K[a, A]$ to reach $\pi_0[\mathcal{C}]$.*

Proof.

The strategy by **II** in the game $K[a, A, p]$ to reach $\mathcal{C}$, with the ε_n's ignored, is a strategy for **II** in $K[a, A]$ to reach $\pi_0[\mathcal{C}]$. $\square$

Lemma

Let $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ be a subset. Let $A \in [\mathbb{N}]^\infty$, $a \in [\mathbb{N}]^{<\infty} \restriction A$. If for all $p \in 2^{\mathrm{lh}(a)}$, there exists some $C \in [a, A]$ such that $\mathbf{I}$ has a strategy in $K[a, C, p]$ to reach $\mathcal{C}^c$, then there exists some $B \in [a, A]$ such that $\mathbf{I}$ has a strategy in $K[a, B]$ to reach $\pi_0[\mathcal{C}]^c$.

Since $\pi_0[\mathcal{C}^c] \neq \pi_0[\mathcal{C}]^c$ in general, the same naive argument doesn't work here.

In the interest of time, we shall prove this lemma only for $a = \emptyset$.

Proof of the second Lemma

Let $B \in [A]^\infty$ and σ be a strategy for **I** in $K[\emptyset, B, \emptyset]$ (in $[\mathbb{N}]^\infty \times 2^\infty$) to reach $\mathcal{C}^c$. How do we define a strategy τ for **I** in $K[\emptyset, B]$ (in $[\mathbb{N}]^\infty$) to reach $\pi_0[\mathcal{C}]^c$?

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- Say that the outcome of a complete run in $K[\emptyset, B]$ (in $[\mathbb{N}]^\infty$), following τ, is $D = \{x_0, x_1, \dots\}$.

Proof of the second Lemma

Let $B \in [A]^\infty$ and σ be a strategy for **I** in $K[\emptyset, B, \emptyset]$ (in $[\mathbb{N}]^\infty \times 2^\infty$) to reach $\mathcal{C}^c$. How do we define a strategy τ for **I** in $K[\emptyset, B]$ (in $[\mathbb{N}]^\infty$) to reach $\pi_0[\mathcal{C}]^c$?

- Say that the outcome of a complete run in $K[\emptyset, B]$ (in $[\mathbb{N}]^\infty$), following τ, is $D = \{x_0, x_1, \dots\}$.
- $D \in \pi_0[\mathcal{C}]^c$ iff for all $x \in 2^\infty$, $(D, x) \in \mathcal{C}^c$.

Proof of the second Lemma

Let $B \in [A]^\infty$ and σ be a strategy for **I** in $K[\emptyset, B, \emptyset]$ (in $[\mathbb{N}]^\infty \times 2^\infty$) to reach $\mathcal{C}^c$. How do we define a strategy τ for **I** in $K[\emptyset, B]$ (in $[\mathbb{N}]^\infty$) to reach $\pi_0[\mathcal{C}]^c$?

- Say that the outcome of a complete run in $K[\emptyset, B]$ (in $[\mathbb{N}]^\infty$), following τ, is $D = \{x_0, x_1, \ldots\}$.
- $D \in \pi_0[\mathcal{C}]^c$ iff for all $x \in 2^\infty$, $(D, x) \in \mathcal{C}^c$.
- **Goal.** Design τ such that, for any outcome D and any $x \in 2^\infty$ (in $[\mathbb{N}]^\infty$), there is a simulation of the game in $K[\emptyset, B, \emptyset]$ (in $[\mathbb{N}]^\infty \times 2^\infty$) following σ, such that the outcome is (D, x). By our choice of σ, $(D, x) \in \mathcal{C}^c$.

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	

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$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$
	$B_0 \subseteq A_0$

$K[\emptyset, B]$, defining τ for **I**:

$$
\begin{array}{c|l}
\mathbf{I} & A_0 = B \\
\hline
\mathbf{II} & \qquad\qquad x_0 \in A_0 \\
 & \qquad\qquad B_0 \subseteq A_0
\end{array}
$$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

$$
\begin{array}{c|l}
\mathbf{I} & A_0 = B \\
\hline
\mathbf{II} & \\
 & \\
 &
\end{array}
$$

or

$$
\begin{array}{c|l}
\mathbf{I} & A_0 = B \\
\hline
\mathbf{II} & \\
 & \\
 &
\end{array}
$$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$
	$B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$
II	$x_0 \in A_0$
	$\varepsilon_0 = 0$

or

I	$A_0 = B$
II	$x_0 \in A_0$
	$\varepsilon_0 = 1$

$K[\emptyset, B]$, defining τ for **I**:

$$\begin{array}{c|l}
\textbf{I} & A_0 = B \\ \hline
\textbf{II} & \qquad\qquad x_0 \in A_0 \\
 & \qquad\qquad B_0 \subseteq A_0
\end{array}$$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

$$\begin{array}{c|l}
\textbf{I} & A_0 = B \\ \hline
\textbf{II} & \qquad\qquad x_0 \in A_0 \\
 & \qquad\qquad \color{red}{\varepsilon_0 = 0} \\
 & \qquad\qquad B_0 \subseteq A_0
\end{array}$$

or

$$\begin{array}{c|l}
\textbf{I} & A_0 = B \\ \hline
\textbf{II} & \qquad\qquad x_0 \in A_0 \\
 & \qquad\qquad \color{red}{\varepsilon_0 = 1} \\
 &
\end{array}$$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	

or

I	$A_0 = B$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$

$K[\emptyset, B]$, defining τ for **I**:

$$\begin{array}{c|l}
\mathbf{I} & A_0 = B \\
\hline
\mathbf{II} & \qquad x_0 \in A_0 \\
 & \qquad B_0 \subseteq A_0
\end{array}$$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

$$\begin{array}{c|ll}
\mathbf{I} & A_0 = B & \color{blue}{A_1^0 := \sigma(x_0, 0, B_0)} \\
\hline
\mathbf{II} & \qquad x_0 \in A_0 & \\
 & \qquad \color{red}{\varepsilon_0 = 0} & \\
 & \qquad B_0 \subseteq A_0 &
\end{array}$$

or

$$\begin{array}{c|l}
\mathbf{I} & A_0 = B \\
\hline
\mathbf{II} & \qquad x_0 \in A_0 \\
 & \qquad \color{red}{\varepsilon_0 = 1} \\
 & \qquad \color{blue}{A_1^0 \subseteq A_0}
\end{array}$$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$ $\varepsilon_0 := 0$ $B_0 \subseteq A_0$	

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$ $A_1^0 \subseteq A_0$	

$K[\emptyset, B]$, defining τ for **I**:

$$
\begin{array}{c|ll}
\mathbf{I} & A_0 = B & \qquad \tau(x_0, B_0) := A_1^1 \\
\hline
\mathbf{II} & \qquad x_0 \in A_0 & \\
 & \qquad B_0 \subseteq A_0 &
\end{array}
$$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

$$
\begin{array}{c|ll}
\mathbf{I} & A_0 = B & \qquad A_1^0 := \sigma(x_0, 0, B_0) \\
\hline
\mathbf{II} & \qquad x_0 \in A_0 & \\
 & \qquad \varepsilon_0 = 0 & \\
 & \qquad B_0 \subseteq A_0 &
\end{array}
$$

or

$$
\begin{array}{c|ll}
\mathbf{I} & A_0 = B & \qquad A_1^1 := \sigma(x_0, 1, A_1^0) \\
\hline
\mathbf{II} & \qquad x_0 \in A_0 & \\
 & \qquad \varepsilon_0 = 1 & \\
 & \qquad A_1^0 \subseteq A_0 &
\end{array}
$$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$	$x_1 \in A_1$ $B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$ $A_1^0 \subseteq A_0$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$	$x_1 \in A_1$ $B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 0)$:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 0$

or

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 1$

$K[\emptyset, B]$, defining τ for **I**:

$$
\begin{array}{c|lll}
\mathbf{I} & A_0 = B & \color{brown}{\tau(x_0, B_0) := A_1^1} & \\
\hline
\mathbf{II} & & x_0 \in A_0 & x_1 \in A_1 \\
 & & B_0 \subseteq A_0 & B_1 \subseteq A_1
\end{array}
$$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $\color{blue}{(\varepsilon_0 = 0)}$:

$$
\begin{array}{c|llll}
\mathbf{I} & A_0 = B & \color{blue}{A_1^0 := \sigma(x_0, 0, B_0)} & & \color{cyan}{A_2^0 := \sigma(x_0, 0, B_0, x_1, 0, B_1)} \\
\hline
\mathbf{II} & & x_0 \in A_0 & x_1 \in A_1^1 & \\
 & & \color{red}{\varepsilon_0 = 0} & \color{red}{\varepsilon_1 = 0} & \\
 & & B_0 \subseteq A_0 & B_1 \subseteq A_1^1 &
\end{array}
$$

or

$$
\begin{array}{c|lll}
\mathbf{I} & A_0 = B & \color{blue}{A_1^0 := \sigma(x_0, 0, B_0)} & \\
\hline
\mathbf{II} & & x_0 \in A_0 & x_1 \in A_1^1 \\
 & & \color{red}{\varepsilon_0 = 0} & \color{red}{\varepsilon_1 = 1} \\
 & & B_0 \subseteq A_0 &
\end{array}
$$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$	$x_1 \in A_1$ $B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 0)$:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$	$A_2^0 := \sigma(x_0, 0, B_0, x_1, 0, B_1)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 0$ $B_1 \subseteq A_1^1$	

or

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$	$A_2^1 := \sigma(x_0, 0, B_0, x_1, 1, A_2^0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 1$ $A_2^0 \subseteq A_1^1$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$	$x_1 \in A_1$
	$B_0 \subseteq A_0$	$B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 1)$:

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$
II	$x_0 \in A_0$	$x_1 \in A_1^1$
	$\varepsilon_0 = 1$	$\varepsilon_1 = 0$
	$A_1^0 \subseteq A_0$	$A_2^1 \subseteq A_1^1$

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$
II	$x_0 \in A_0$	$x_1 \in A_1^1$
	$\varepsilon_0 = 1$	$\varepsilon_1 = 1$
	$A_1^0 \subseteq A_0$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$	
II	$x_0 \in A_0$		$x_1 \in A_1$
	$B_0 \subseteq A_0$		$B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 1)$:

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$		$A_2^2 := \sigma(x_0, 1, B_0, x_1, 0, A_2^1)$
II	$x_0 \in A_0$		$x_1 \in A_1^1$	
	$\varepsilon_0 = 1$		$\varepsilon_1 = 0$	
	$A_1^0 \subseteq A_0$		$A_2^1 \subseteq A_1^1$	

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$	
II	$x_0 \in A_0$		$x_1 \in A_1^1$
	$\varepsilon_0 = 1$		$\varepsilon_1 = 1$
	$A_1^0 \subseteq A_0$		

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$	$x_1 \in A_1$
	$B_0 \subseteq A_0$	$B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 1)$:

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$	$A_2^2 := \sigma(x_0, 1, B_0, x_1, 0, A_2^1)$
II	$x_0 \in A_0$	$x_1 \in A_1^1$	
	$\varepsilon_0 = 1$	$\varepsilon_1 = 0$	
	$A_1^0 \subseteq A_0$	$A_2^1 \subseteq A_1^1$	

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$	$A_2^3 := \sigma(x_0, 1, A_1^0, x_1, 1, A_2^2)$
II	$x_0 \in A_0$	$x_1 \in A_1^1$	
	$\varepsilon_0 = 1$	$\varepsilon_1 = 1$	
	$A_1^0 \subseteq A_0$	$A_2^2 \subseteq A_1^1$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$		$\tau(x_0, B_0) := A^1_1$		$\tau(x_0, B_0, x_1, B_1) := A^3_2$
II		$x_0 \in A_0$		$x_1 \in A_1$	
		$B_0 \subseteq A_0$		$B_1 \subseteq A_1$	

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 1)$:

I	$A_0 = B$		$A^1_1 := \sigma(x_0, 1, A^0_1)$		$A^2_2 := \sigma(x_0, 1, B_0, x_1, 0, A^1_2)$
II		$x_0 \in A_0$		$x_1 \in A^1_1$	
		$\varepsilon_0 = 1$		$\varepsilon_1 = 0$	
		$A^0_1 \subseteq A_0$		$A^1_2 \subseteq A^1_1$	

or

I	$A_0 = B$		$A^1_1 := \sigma(x_0, 1, A^0_1)$		$A^3_2 := \sigma(x_0, 1, A^0_1, x_1, 1, A^2_2)$
II		$x_0 \in A_0$		$x_1 \in A^1_1$	
		$\varepsilon_0 = 1$		$\varepsilon_1 = 1$	
		$A^0_1 \subseteq A_0$		$A^2_2 \subseteq A^1_1$	

Generalising to **wA2**-spaces

Let $(\mathcal{R}, \leq, r)$ be a **wA2**-space. In a way similar to how we go from $[\mathbb{N}]^\infty$ to $[\mathbb{N}]^\infty \times 2^\infty$, we may consider going from $\mathcal{R}$ to $\mathcal{R} \times 2^\infty$.

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More precisely, we shall construct the triple $(\mathcal{R} \times 2^\infty, \preceq, r)$ in the following manner:

1. $(A, u) \preceq (B, v) \iff A \leq B$.
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Note that $\preceq$ is not a partial order.

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Lemma

*Let $(\mathcal{R}, \leq, r)$ be a **wA2**-space. Then the closed triple $(\mathcal{R} \times 2^\infty, \preceq, r)$ defined above is a **wA2**-space which does not satisfy **A4**.*

This means that $([\mathbb{N}]^\infty \times 2^\infty, \preceq, r)$ is a **wA2**-space, so we may consider the abstract Kastanas game on $([\mathbb{N}]^\infty \times 2^\infty, \preceq, r)$.

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Fact

The abstract Kastanas game on $([\mathbb{N}]^\infty \times 2^\infty, \preceq, r)$ is precisely the "modified" Kastanas game that we presented earlier.

Theorem (Y.)

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space. If $\mathcal{C} \subseteq \mathcal{R} \times 2^{\infty}$ is Kastanas Ramsey, then $\pi_0[\mathcal{C}] \subseteq \mathcal{R}$ is Kastanas Ramsey.*

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Corollary (Y.)

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space, and assume that $\mathcal{AR}$ is countable. Then every analytic subset of $\mathcal{R}$ is Kastanas Ramsey.*

Strategically Ramsey sets

Todorčević's theorem asserts that if $(\mathcal{R}, \leq, r)$ is a closed triple satisfying **A1**-**A4**, and $\mathcal{AR}$ is countable, then every analytic subset of $\mathcal{R}$ is Kastanas Ramsey. What about countable vector spaces?

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Theorem (Rosendal)

Every analytic subset of $E^{[\infty]}$ is strategically Ramsey.

Proposition

A subset $\mathcal{X} \subseteq E^{[\infty]}$ is Kastanas Ramsey iff it is strategically Ramsey.

Thanks for listening!

1. The Ramsey theorem for $([\mathbb{N}]^{\infty}, \subseteq, r)$ (pigeonhole principle) and $(E^{[\infty]}, \leq, r)$ when $|\mathbb{F}| = 2$ are both true.
2. Todorčević developed topological Ramsey theory to provide a general framework to prove these results.
3. $(E^{[\infty]}, \leq, r)$ for $|\mathbb{F}| > 2$ is not a topological Ramsey space, but still contains a rich Ramsey theory. **wA2**-space proposes an extension of topological Ramsey theory to such spaces.
4. We defined the abstract Kastanas game for **wA2**-spaces and Kastanas Ramsey sets. For topological Ramsey spaces, Kastanas Ramsey sets are precisely Ramsey sets.
5. By considering $(\mathcal{R} \times 2^{\infty}, \preceq, r)$, we showed that every analytic subset of $\mathcal{R}$ is Kastanas Ramsey. This implies that every analytic subset of $E^{[\infty]}$ is strategically Ramsey.