NUS Reading Seminar Summer 2023

Session 1

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13 May 2023

Large Cardinals

Definition

A cardinal κ is *regular* if $\mathrm{cf}(\kappa) = \kappa$.

Definition

A cardinal κ is:

(1) *weakly inaccessible* if κ is a regular limit cardinal.

(2) *strongly inaccessible* if κ is a regular *strong limit* cardinal, i.e. $\lambda < \kappa \to 2^\lambda < \kappa$.

Recall that the *Von Neumann universe* V is constructed recursivley as follows:

$V_0 := \emptyset$.

$V_{\alpha+1} := \mathcal{P}(V_\alpha)$.

If α limit, $V_\alpha := \bigcup_{\beta<\alpha} V_\beta$.

$V := \bigcup_{\alpha \in \mathbf{ORD}} V_\alpha$.

Theorem

If κ is strongly inaccessible, then V_κ is a model of ZFC*.*

By Gödel's incompleteness theorem, the existence of strongly inaccessible cardinals is unprovable in ZFC. This makes strongly inaccessible cardinals a form of *large cardinal*.

Vaguely speaking, a large cardinal is a cardinal with combinatorial properties so strong that its existence is unprovable in ZFC.

Measurable Cardinals

Definition

Let $\mathcal{U}$ be an ultrafilter, and let κ be a cardinal. $\mathcal{U}$ is *κ-complete* if it is closed under λ-intersections for all $\lambda < \kappa$. In other words, for all $\lambda < \kappa$ and $\{X_\alpha : \alpha < \lambda\} \subseteq \mathcal{U}$, $\bigcap_{\alpha<\lambda} X_\alpha \in \mathcal{U}$.

Definition

A cardinal κ is *measurable* if there exists a κ-complete non-principal ultrafilter $\mathcal{U}$ on κ.

A non-principal ultrafilter is also called a *measure*, as a non-principal ultrafilter $\mathcal{U}$ on κ induces a non-trivial measure μ on κ by:

$$\mu(X) := \begin{cases} 1, & \text{if } X \in \mathcal{U} \\ 0, & \text{if } X \notin \mathcal{U} \end{cases}$$

See §10 of Jech for more about the relationship between measurable cardinals and the measure problem.

Lemma

If κ is measurable, it is strongly inaccessible.

Proof.

Regular: Suppose $\mathrm{cf}(\kappa) = \lambda < \kappa$. Let $\{\kappa_\alpha : \alpha < \lambda\}$ be cofinal in κ. Since $\mathcal{U}$ is κ-complete non-principal, $\kappa \setminus \kappa_\alpha \in \mathcal{U}$ for all α. Then $\emptyset = \bigcap_{\alpha<\lambda}(\kappa \setminus \kappa_\alpha) \in \mathcal{U}$, a contradiction.

Strong Limit: Suppose $\lambda < \kappa$ and $2^\lambda \geq \kappa$. Let S be a set of functions $f : \lambda \to \{0, 1\}$ with $|S| = \kappa$. For each $\alpha < \lambda$, let X_α be the set $\{f \in S : f(\alpha) = 0\}$ or $\{f \in S : f(\alpha) = 1\}$ that is in $\mathcal{U}$, and let $\varepsilon_\alpha \in \{0, 1\}$ be the respective value. Then $X := \bigcap_{\alpha<\lambda} X_\alpha \in \mathcal{U}$. But X only contains the function $f(\alpha) = \varepsilon_\alpha$. $\square$

How "strong" a large cardinal is is measured by its *consistency strength*.

Since measurable cardinals are strongly inaccessible, we have:

$$\begin{gathered}\mathrm{Con}(\mathrm{ZFC} + \exists \text{ a measurable cardinal}) \\ \downarrow \\ \mathrm{Con}(\mathrm{ZFC} + \exists \text{ a strongly inaccessible cardinal})\end{gathered}$$

So a measurable cardinal has a higher consistency strength than a strongly inaccessible cardinal.

Constructible Universe

Definition

Let M be a set. A set X is *definable over* $(M, \in)$ if there exists a formula φ and $a_1, \ldots, a_n \in M$ such that:

$$X = \{x \in M : (M, \in) \models \varphi[x, a_1, \ldots, a_n]\}$$

Definition

The *definable power set*, $\mathrm{def}(M)$, is defined as:

$$\mathrm{def}(M) := \{X \subseteq M : X \text{ is definable over } (M, \in)\}$$

Von Neumann Universe, V:

$V_0 := \emptyset$.

$V_{\alpha+1} := \mathcal{P}(V_\alpha)$.

If α limit, $V_\alpha := \bigcup_{\beta<\alpha} V_\beta$.

$V := \bigcup_{\alpha \in \mathbf{ORD}} V_\alpha$.

Constructible Universe, L:

$L_0 := \emptyset$.

$L_{\alpha+1} := \mathrm{def}(L_\alpha)$.

If α limit, $L_\alpha := \bigcup_{\beta<\alpha} L_\beta$.

$L := \bigcup_{\alpha \in \mathbf{ORD}} L_\alpha$.

Definition

A set x is *constructible* if $x \in L$.

Theorem

L is a model of ZFC.

Definition

The statement "$V = L$" abbreviates the statement "every set is constructible".

Theorem

(1) $L \models$ "$V = L$".

(2) $L \models$ GCH. Thus, if $V \not\models$ GCH, $V \neq L$.

Projective Hierarchy

Let ω^ω denote the space of countable sequences of natural numbers. These sequences are also called *reals*.

Definition

Let $A \subseteq \omega^\omega$.

(1) A is Σ^1_1 if there exists a recursive relation R such that:

$$x \in A \iff \exists y \in \omega^\omega \, \forall n \in \omega \, R(x \restriction n, y \restriction n)$$

(2) A is $\Sigma^1_1(a)$, where $a \in \omega^\omega$, if there exists a recursive relation R such that:

$$x \in A \iff \exists y \in \omega^\omega \, \forall n \in \omega \, R(x \restriction n, y \restriction n, a \restriction n)$$

Definition

Let $A \subseteq \omega^\omega$.

(1) A is Π^1_n (in a) if $\omega^\omega \setminus A$ is Σ^1_n.

(2) A is Σ^1_{n+1} (in a) if there exists some Π^1_n (in a) set $B \subseteq \omega^\omega \times \omega^\omega$ such that:

$$x \in A \iff \exists y \in \omega^\omega \, (x, y) \in B$$

(3) A is Δ^1_n if A is both Σ^1_n and Π^1_n.

Definition

$$\mathbf{\Sigma}^1_n := \bigcup_{a \in \omega^\omega} \Sigma^1_1(a), \;\; \mathbf{\Pi}^1_n := \bigcup_{a \in \omega^\omega} \Pi^1_1(a)$$

Replacing ω^ω with $\prod_{i<n} \omega^\omega$ for some n, we may instead consider a hierarchy of relations instead of sets of reals.

Lemma

(1) If A, B are $\Sigma^1_n(a)$, then so are $\exists x\, A$, $A \wedge B$, $A \vee B$.

(2) If A, B are $\Pi^1_n(a)$, then so are $\forall x\, A$, $A \wedge B$, $A \vee B$.

(3) If A is $\Sigma^1_n(a)$, then $\neg A$ is $\Pi^1_n(a)$. If A is $\Pi^1_n(a)$, then $\neg A$ is $\Sigma^1_n(a)$.

See also Lemma 25.2, Jech.

Π^1_1 Normal Form

The normal form theorem allows us to express Π^1_1 in terms of trees. This expression is very useful in proving absoluteness results about sets in the projective hierarchy.

By a *tree* we refer to a subset $T \subseteq \omega^{<\omega}$, ordered by initial segment $\sqsubseteq$, that is closed under initial segments - i.e. if $t \in T$, then $t{\restriction}n \in T$ for all $n \leq |t|$.

We also use $[T]$ to denote the set of branches of T, i.e.:

$$[T] := \{x \in \omega^\omega : x{\restriction}n \in T \text{ for all } n\}$$

Let Seq_r denote the set of r-tuples $(s_1, \ldots, s_r)$, where $s_i \in \omega^{<\omega}$, such that $|s_1| = \cdots = |s_r|$.

Definition

An *(r-dimensional) sequential tree* is a subset $T \subseteq \mathrm{Seq}_r$ that is closed under initial segments - i.e. if $(s_1, \ldots, s_r) \in T$, then for all $n \leq |s_i|$, $(s_1 \restriction n, \ldots, s_r \restriction n) \in T$.

Definition

A sequential tree is *well-founded* if it has no infinite branch. That is. the set:

$$[T] := \{(x_1, \ldots, x_r) \in (\omega^\omega)^r : \forall n \, (x_1 \restriction n, \ldots, x_r \restriction n) \in T\}$$

is empty.

If T is an $(r+1)$-dimensional sequential tree, then:

$$T(x) := \{(s_1, \ldots, s_r) \in \mathrm{Seq}_r : (x \restriction |s_i|, s_1, \ldots, s_r) \in T\}$$

Definition

A sequential tree T is *recursive* if the map $x \mapsto T(x)$ is recursive.

Theorem (Normal Form for Π^1_1 Sets)

Let $A \subseteq \omega^\omega$. Then A is Π^1_1 iff there exists a recursive $T \subseteq \mathrm{Seq}_2$ such that:

$$x \in A \iff T(x) \text{ is well-founded}$$

This follows from the observation that a recursive relation R defining a Σ^1_1 set is a subset of Seq_2. However, R need not be closed under initial segments, so a small modification is necessary for one direction.

Proof.

It suffices to show that if $A \subseteq \omega^\omega$ is Σ^1_1, then there exists a recursive $T \subseteq \mathrm{Seq}_2$ such that:

$$x \in A \iff T(x) \text{ is ill-founded}$$

$\underline{\Longleftarrow:}$ If T is one such sequential tree, then:

$$\begin{aligned} x \in A &\iff T(x) \text{ is ill-founded} \\ &\iff \exists y \in \omega^\omega \, y \in [T(x)] \\ &\iff \exists y \in \omega^\omega \, \forall n \, T(x{\restriction}n, y{\restriction}n) \end{aligned}$$

Since T is recursive, A is Σ^1_1.

Proof (Cont.)

$\underline{\Longrightarrow:}$ Suppose R is a recursive relation and:

$$x \in A \iff \exists y \in \omega^\omega \, \forall n \, R(x \restriction n, y \restriction n)$$

Define a sequential tree $T \subseteq \mathrm{Seq}_2$ by:

$$T := \{(s, t) \in \mathrm{Seq}_2 : \forall n \leq |s| \, R(s \restriction n, t \restriction n)\}$$

Clearly T is indeed closed under initial segments. Then:

$$\begin{aligned} x \in A &\iff (x, y) \in [T] \text{ for some } y \in \omega^\omega \\ &\iff y \in [T(x)] \text{ for some } y \in \omega^\omega \\ &\iff [T(x)] \text{ is ill-founded} \end{aligned}$$

$\square$

As mentioned earlier, the power of the normal forms stems from its ability to prove absoluteness results.

Theorem (Mostowki's Absoluteness)

If P is a $\boldsymbol{\Sigma}^1_1$ property, then P is absolute for every transitive model that is adequate for P.

Proof.

Suppose P is $\Sigma^1_1(a)$. Here "adequate" means that the model $(M, \in)$ satisfies a sufficiently large enough fragment of ZFC for well-founded trees to have a rank function, and that $a \in M$.
Let $T \in M$ such that $P = \{x \in \omega^\omega : T(x) \text{ is ill-founded}\}$. Fix some $x \in \omega^\omega$, and we wish to show that $M \models P(x)$ iff $P(x)$.

(1) If $M \models (T(x) \text{ is ill-founded })$, then $M \models \exists y \in \omega^\omega \ \in [T(x)]$. This is a Σ_1 formula, which is upward-absolute.

(2) If $M \models (T(x) \text{ is well-founded })$, then $M \models \exists f : T(x) \to \mathbf{ORD}$ such that $s \sqsubseteq t \to f(t) < f(s)$. This is again a Σ_1 formula, which is upward-absolute.

□

Stronger absoluteness results can be proven using deeper set theory which uses normal form.

Theorem (Shoenfield's Absoluteness)

If $\mathbb{P}$ is a $\Sigma^1_2(a)$ property, then it is absolute for all inner models M of $\mathrm{ZF} + \mathrm{DC}$ such that $a \in M$.

Infinite Games

We consider games played by two players, in which each player take turn picking some element of ω. The players take turns infinitely many times.

To formalise this: Let $A \subseteq \omega^\omega$. We use G_A to denote the following two-player game:

> Player I starts by picking $a_0 \in \omega$.
>
> Player II then picks $b_0 \in \omega$.
>
> Player I then picks $a_1 \in \omega$.
>
> Player II then picks $b_1 \in \omega$.
>
> ...

Player I wins iff by the end of the game, $(a_0, b_0, a_1, b_1, \dots) \in A$.

A *winning strategy* for Player I (Player II) is a strategy σ such that as long as Player I (Player II) follows this strategy, Player I (Player II) is guaranteed to win the game.

More precisely, a winning strategy for Player I is a function σ such that:

$$\sigma(\emptyset) = a_0$$
$$\sigma(a_0, b_0) = a_1$$
$$\sigma(a_0, b_0, a_1, b_1) = a_2$$
$$\vdots$$

Note that the number a_1 depends on a_0, b_0, a_2 depends on a_0, b_0, a_1, b_1 etc. Then $(a_0, b_0, a_1, b_1, \dots) \in A$ for any sequence $(b_0, b_1, \dots)$.

Example

Let:

$$A := \left\{ x \in \omega^\omega : \left\{ \frac{1}{x(n)+1} \right\}_{n<\omega} \text{ converges to some real number} \right\}$$

Then Player II has a winning strategy as follows:

(1) If Player I plays $a_n = 0$, then Player II plays $b_n = 1$.

(2) If Player I plays $a_n > 0$, then Player II plays $b_n = 0$.

The sequence $\frac{1}{a_0+1}, \frac{1}{b_0+1}, \frac{1}{a_1+1}, \frac{1}{b_1+1}$ has infinitely many 1s, so for it to converge it must be eventually 1, which is impossible by Player II's strategy.

We say that a game is *determined* if either Player I or II has a strategy.

Theorem

All open games are determined. That is, if $A \subseteq \omega^\omega$ is open, then G_A is determined.

Note that we are equipping ω^ω with the usual topology $\{O(s) : s \in \omega^{<\omega}\}$, where:

$$O(s) := \{x \in \omega^\omega : s \sqsubseteq x\}$$

Proof.

Let $A \subseteq \omega^\omega$ be open. If Player I has a winning strategy, then we're done, so assume otherwise. Player II shall play as follows:

(1) Player I starts by picking any $a_0 \in \omega$.

(2) Since Player I has no winning strategy, Player II has not lost, so, in particular, there exists some $b_0 \in \omega$ such that Player II has not lost after the play (a_0, b_0).

(3) Player I plays any $a_1 \in \omega$.

(4) Player II chooses some $b_1 \in \omega$ such that Player II has not lost after the play (a_0, b_0, a_1, b_1).

(5) $\ldots$.

Suppose $x := (a_0, b_0, a_1, b_1, \ldots) \in A$. Since A is open, there exists some $s = (a_0, b_0, \ldots, a_n) \sqsubseteq x$ such that $O(s) \subseteq A$. But that would mean that II has already lost by the time Player I plays a_n. $\square$

In fact, more is true:

Theorem (Martin)

All Borel games are determined.

For now, we do not have plans to prove this.

Borel determinacy is the “strongest” ZFC determinacy theorem available. The next natural determinacy would be analytic determinacy, which is equivalent to the existence of $0^\sharp$. A measurable cardinal implies the existence of $0^\sharp$. Thus, Con(ZFC $+ \exists$ a measurable cardinal) $\rightarrow$ Con(ZFC$+$ Analytic determinacy).