Definability of mad families of vector spaces and the fullness property

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Warning!!!

For the entirety of this talk, the **zero vector** is ignored.

Almost disjoint subspaces

Let $\mathbb{F}$ be a countable field (possibly finite). Let E be a $\mathbb{F}$-vector space with a fixed Hamel basis $(e_n)_{n<\omega}$.

Definition

Let $V, W \subseteq E$ be two infinite-dimensional subspaces. We say that V, W are *almost disjoint* if $V \cap W$ is a finite-dimensional subspace of E.

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Let $\mathcal{A}$ be a family of infinite-dimensional subspaces of E. We say that $\mathcal{A}$ is *almost disjoint* if all subspaces in $\mathcal{A}$ are pairwise almost disjoint. We say that $\mathcal{A}$ is *maximal almost disjoint* (or just *mad*) if $\mathcal{A}$ is not strictly contained in another almost disjoint family of infinite-dimensional subspaces.

Block subspaces

Recall that E has a fixed Hamel basis $(e_n)_{n<\omega}$. Given a vector $x \in E$, we may write

$$x = \sum_{n<\omega} \lambda_n(x) e_n,$$

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Notation

Given two vectors x, y we write:

$$x < y \iff \max(\mathrm{supp}(x)) < \min(\mathrm{supp}(y)).$$

Definition

An infinite-dimensional subspace $V \subseteq W$ is a *block subspace* if it has a *block basis*. That is, V is spanned by the basis $(x_n)_{n<\omega}$, where:

$$x_0 < x_1 < x_2 < \cdots .$$

Note that the block basis of a block subspace is unique (up to scalar multiplication of each vector).

Fact

Every infinite-dimensional subspace of E contains an infinite-dimensional block subspace.

Consequently, if $\mathcal{A}$ is an almost disjoint family such that there is no block subspace that is almost disjoint with every element of $\mathcal{A}$, then $\mathcal{A}$ is mad.

Definition

We define the cardinal invariant:

$$\mathfrak{a}^*_{\mathrm{vec},\mathbb{F}} := \min\{|\mathcal{A}| > 1 : \mathcal{A} \text{ is an mad family of subspaces}\},$$

$$\mathfrak{a}_{\mathrm{vec},\mathbb{F}} := \min\{|\mathcal{A}| > 1 : \mathcal{A} \text{ is a mad family of block subspaces}\}.$$

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Proposition (user527492 (on MathOverflow) and Y., 2025)

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Definability of mad families

Let $E^{[\infty]}$ be the set of all block sequences (= block subspaces) of E.

Consider equipping E with the discrete topology, and $E^{\mathbb{N}}$ with the product topology. Since E is countable, $E^{\mathbb{N}}$ is Polish. Then $E^{[\infty]} \subseteq E^{\mathbb{N}}$ is a closed subspace, so the subspace topology of $E^{[\infty]}$ is also Polish.

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Current status. Mostly open, but there are some partial positive results.

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Given an almost disjoint family $\mathcal{A} \subseteq E^{[\infty]}$ of block subspaces, we define the set $\mathcal{I}(\mathcal{A})$ to be:

$$\left\{ X \subseteq E : \exists A_0, \ldots, A_{n-1} \in \mathcal{A} \text{ such that } X \setminus \bigcup_{i<n} \langle A_i \rangle \text{ is small} \right\}.$$

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$\mathcal{I}(\mathcal{A})$ is $\subseteq$-downward closed, but it is not necessarily an ideal. We also let $\mathcal{I}^+(\mathcal{A}) := \mathcal{P}(E) \setminus \mathcal{I}(\mathcal{A})$.

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2. We say that $\mathcal{A}$ is *full* if for all $Y \subseteq E$ such that Y is $\mathcal{I}^+(\mathcal{A})$-dense below some $B \in \mathcal{I}^+(\mathcal{A})$, there exists some $C^* \in \mathcal{I}^+(\mathcal{A}){\restriction}B$ such that $\langle C^* \rangle \subseteq Y$.

The fullness property should be seen as a weak form of the pigeonhole principle.

Lemma

Suppose that E is a vector space over $\mathbb{F}_2$. For any almost disjoint family $\mathcal{A} \subseteq E^{[\infty]}$, the following are equivalent:

1. *$\mathcal{A}$ is full.*
2. *For every $B \in \mathcal{I}^+(\mathcal{A})$ and every partition $Y \subseteq E$, there exists some $C \in \mathcal{I}^+(\mathcal{A}) \upharpoonright B$ such that $\langle C \rangle \subseteq Y$ or $\langle C \rangle \subseteq Y^c$.*

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1. (ZFC) Is there a full mad family?
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Theorem (Smythe, 2019)

If $\mathfrak{p} = \mathfrak{c}$, then there exists a full mad family.

Construction of full mad families

Theorem (Y., 2025)

If $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$ *or* $\mathfrak{p} = \max\{\mathfrak{b}, \mathfrak{s}\}$, *then there exists a full mad family of block subspaces.*

The construction mimics that of a completely separable mad family of subsets of ω, assuming $\mathfrak{s} \leq \mathfrak{a}$.

Completely separable mad families

Let $\mathcal{A} \subseteq [\omega]^\omega$ be an almost disjoint family (not necessarily mad). Recall that:

$$\mathcal{I}(\mathcal{A}) := \left\{ X \subseteq \omega : \exists A_0, \ldots, A_{n-1} \in \mathcal{A} \text{ such that } X \subseteq^* \bigcup_{i<n} A_i \right\}.$$

$\mathcal{I}(\mathcal{A})$ is an ideal (as finite union of finite sets is finite). We let $\mathcal{I}^+(\mathcal{A}) := \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$.

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Definition

A mad family $\mathcal{A} \subseteq [\omega]^\omega$ is *completely separable* if for every $Y \in \mathcal{I}^+(\mathcal{A})$, there exists some $A \in \mathcal{A}$ such that $A \subseteq Y$.

Theorem (Mildenberger-Raghavan-Steprāns, 2014)

If $\mathfrak{s} \leq \mathfrak{a}$, then there exists a completely separable mad family.

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Assume that $\mathcal{A}_\alpha = \{X_\beta : \beta < \alpha\}$. We shall find some X_α such that:

- X_α is almost disjoint from X_β for all $\beta < \alpha$.
- If $Y_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$, then $X_\alpha \subseteq Y_\alpha$.

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There are two crucial components of the construction that we will discuss:

1. **(Main lemma)** Using block-splitting families to split a set $X \in \mathcal{I}^+(\mathcal{A}_\alpha)$ into two $\mathcal{I}(\mathcal{A}_\alpha)$-positive pieces.
2. **(Applying assumptions)** Applying $\mathfrak{s} \leq \mathfrak{a}$ to obtain the required X_α.

Definition

1. Let $\mathcal{P} = \{P_n : n < \omega\}$ be an interval partition. We say that a set $S \in [\omega]^\omega$ *block-splits* $\mathcal{P}$ if $\{n < \omega : P_n \subseteq S\}$ and $\{n < \omega : P_n \cap S = \emptyset\}$ are both infinite.
2. A *block-splitting family* is a family $\mathcal{S} \subseteq [\omega]^\omega$ such that every interval partition is block-split by some $S \in \mathcal{S}$.

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Lemma (Kamburelis-Weglorz, 1996)

$\max\{\mathfrak{b}, \mathfrak{s}\} = \min\{|\mathcal{S}| :$ *$\mathcal{S}$ is a block-splitting family*$\}$.

Main lemma.

Lemma

Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. Let $\mathcal{A}$ be an almost disjoint family, and let $X \in \mathcal{I}^+(\mathcal{A})$. Then the following properties hold:

1. *For all α, $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$ or $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.*
2. *If $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some $Y \in \mathcal{I}^+(\mathcal{A})$ such that $Y \subseteq X \cap S_\alpha$.*
3. *If $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some $Y \in \mathcal{I}^+(\mathcal{A})$ such that $Y \subseteq X \setminus S_\alpha$.*
4. *There exists some α such that $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.*

Note that properties (1)-(3) are trivial.

Applying assumptions. Using the previous lemma (+ some non-trivial workings), we obtain $\mathcal{B}$ and Y'_α such that:

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Let:

$$\mathcal{B}' := \{Z \cap Y'_\alpha : Z \in \mathcal{B}\}.$$

If $\mathfrak{s} \leq \mathfrak{a}$, then $|\mathcal{B}'| < \max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}$, so there exists some $X_\alpha \subseteq Y'_\alpha$ that is almost disjoint from all elements of $\mathcal{B}'$. Choosing X_α completes the induction.

Full mad families

Recall the theorem of focus today:

Theorem (Y., 2025)

If $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$ *or* $\mathfrak{p} = \max\{\mathfrak{b}, \mathfrak{s}\}$, *then there exists a full mad family of block subspaces.*

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We need two enumerations:

1. $\{Y_\alpha : \alpha < \mathrm{Lim}(\mathfrak{c})\}$ enumerates all infinite subsets of E, and;
2. $\{B_\alpha : \alpha < \mathrm{Lim}(\mathfrak{c})\}$ enumerates all elements of $E^{[\infty]}$ (i.e. all block subspaces);

such that for all $Y \subseteq E$ and $B \in E^{[\infty]}$, $(Y, B) = (Y_\alpha, B_\alpha)$ for cofinally many α.

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Sketch of Proof. Let $Y \subseteq E$ be $\mathcal{I}^+(\mathcal{A})$-dense below some $B \in \mathcal{I}^+(\mathcal{A})$, i.e. for all $C \in \mathcal{I}^+(\mathcal{A})\restriction B$, there exists some $D \leq C$ such that $\langle D \rangle \subseteq Y$. We need to find some $C^* \in \mathcal{I}^+(\mathcal{A})\restriction B$ such that $\langle C^* \rangle \subseteq Y$.

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Fact

If $\mathcal{A}$ is a mad family of block subspaces, then $B \in \mathcal{I}^+(\mathcal{A})$ iff there exists an infinite $\{\beta_i : i < \omega\}$ such that A_{β_i} is compatible with B for all i.

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- there exists some $D \leq B_\alpha$, almost disjoint from $\mathcal{C}_\alpha$ such that $\langle D \rangle \subseteq Y$; True, as 1. $A'_\alpha \in \mathcal{I}^+(\mathcal{A}) \upharpoonright B_\alpha$, 2. A'_α almost disjoint from $\mathcal{C}_\alpha$, and 3. Y is $\mathcal{I}^+(\mathcal{A})$-dense below B_α, so there exists some $D \leq A'_\alpha$ such that $\langle D \rangle \subseteq Y$.

then $A'_\alpha \leq B_\alpha$ is such that $\langle A'_\alpha \rangle \subseteq Y$, and A'_α is almost disjoint from $\mathcal{A}_\alpha$. We then let $\{A_{\alpha+n} : n < \omega\}$ be any almost disjoint family such that $A_{\alpha+n} \leq A'_\alpha$ for all n, which defines $\mathcal{A}_{\alpha+\omega}$.

Let $\alpha > \beta_i$ for all i be a limit ordinal such that $(Y, B) = (Y_\alpha, B_\alpha)$.

Step α of the induction. We shall find some A'_α such that if:

- the set:

$$\mathcal{C}_\alpha := \{A_\beta \in \mathcal{A}_\alpha : A_\beta \text{ and } B_\alpha \text{ are compatible}\}$$

  is infinite, and; True, as $\{A_{\beta_i} : i < \omega\} \subseteq \mathcal{C}_\alpha$.
- there exists some $D \leq B_\alpha$, almost disjoint from $\mathcal{C}_\alpha$ such that $\langle D \rangle \subseteq Y$; True, as 1. $A'_\alpha \in \mathcal{I}^+(\mathcal{A}) \restriction B_\alpha$, 2. A'_α almost disjoint from $\mathcal{C}_\alpha$, and 3. Y is $\mathcal{I}^+(\mathcal{A})$-dense below B_α, so there exists some $D \leq A'_\alpha$ such that $\langle D \rangle \subseteq Y$.

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Main lemma. Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family, and for each α, let:

$$Z_\alpha := \langle (e_n)_{n \in S_\alpha} \rangle .$$

Main lemma. Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family, and for each α, let:

$$Z_\alpha := \langle (e_n)_{n \in S_\alpha} \rangle \,.$$

Lemma

Let $\mathcal{A} \subseteq E^{[\infty]}$ be an almost disjoint family, and let $B \in E^{[\infty]}$ be such that $\langle B \rangle \in \mathcal{I}^+(\mathcal{A})$.

1. *For all α, $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ or $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.*
2. *If $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some $C \in E^{[\infty]}$ such that $\langle C \rangle \in \mathcal{I}^+(\mathcal{A})$ and $\langle C \rangle \subseteq \langle B \rangle \cap Z_\alpha$.*
3. *If $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some $C \in E^{[\infty]}$ such that $\langle C \rangle \in \mathcal{I}^+(\mathcal{A})$ and $\langle C \rangle \subseteq \langle B \rangle \setminus Z_\alpha$.*
4. *There exists some α such that $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.*

Applying assumptions. Using the previous lemma (+ some non-trivial workings), we obtain $\mathcal{B}$ and B'_α such that:

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Applying assumptions. Using the previous lemma (+ some non-trivial workings), we obtain $\mathcal{B}$ and B'_α such that:

- $\mathcal{B}$ is an almost disjoint family such that $|\mathcal{B}| < \max\{\mathfrak{b}, \mathfrak{s}\}$.
- $B'_\alpha \leq B_\alpha$.
- If $C \leq B'_\alpha$ is almost disjoint from all elements of $\mathcal{B}$, then C is almost disjoint from A_β for all $\beta < \alpha$.

Let:

$$\mathcal{B}' := \{\langle D \rangle \cap \langle B'_\alpha \rangle : D \in \mathcal{B}\}.$$

This is an almost disjoint of subspaces of B'_α, but not necessarily block subspaces.

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1. If $\mathfrak{p} = \max\{\mathfrak{b}, \mathfrak{s}\}$, then $|\mathcal{B}'| < \mathfrak{p}$. By a Lemma of (Smythe, 2019), there is some $A'_\alpha \leq B'_\alpha$ that is almost disjoint from $\mathcal{B}'$.

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2. If $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$, then $|\mathcal{B}'| < \mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$. Thus, there exists some $A'_\alpha \leq B'_\alpha$ that is almost disjoint from all elements of $\mathcal{B}'$.

Choosing this A'_α completes the induction.

Open problems

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Problem

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Problem

Does $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$ hold in ZFC?

It's unknown there exists a completely separable mad family in ZFC.

Theorem (Shelah, 2011)

If $\mathfrak{c} < \aleph_\omega$, then there is a completely separable mad family.

Problem

If $\mathfrak{c} < \aleph_\omega$, is there a full mad family of block subspaces?

Thanks for listening!

1. Subspaces V, W are *almost disjoint* if $V \cap W$ is finite-dimensional. This gives us the cardinal invariants:
   - $\mathfrak{a}^*_{\mathrm{vec},\mathbb{F}} := \min\{|\mathcal{A}| : \mathcal{A}$ is a mad family of subspaces$\}$.
   - $\mathfrak{a}_{\mathrm{vec},\mathbb{F}} := \min\{|\mathcal{A}| : \mathcal{A}$ is a mad family of block subspaces$\}$.
2. A mad family $\mathcal{A} \subseteq E^{[\infty]}$ is *full* if $\mathcal{I}^+(\mathcal{A})$ satisfies a weak pigeonhole principle.
   - (Smythe, 2019) If $\mathfrak{p} = \mathfrak{c}$, then there exists a full mad family.
   - Is there a full mad family in ZFC? Is every mad family full?
3. (Y., 2025) If $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$ or $\mathfrak{p} = \max\{\mathfrak{b}, \mathfrak{s}\}$, then there exists a full mad family.
   - The proof is inspired by the construction of a completely separable mad family (of $[\omega]^\omega$), assuming $\mathfrak{s} \leq \mathfrak{a}$.
   - The key modifications are the main lemma involving a block-splitting family, and how the assumptions are applied.