# NUS Reading Seminar Summer 2023

## Session 2

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# Axiom of Determinacy

## Definition

The Axiom of Determinacy, also written as AD, asserts that the game $G_A$ is determined for all $A \subseteq \omega^\omega$.

**Lemma**

*If* AC *holds, then $G_A$ is not determined for some $A \subseteq \omega^\omega$. Hence,* AD *is incompatible with* AC.

Given $x = (b_0, b_1, \dots)$ and a strategy $\sigma$ for Player I, we denotet

$$\sigma * x := (\sigma(\emptyset), b_0, \sigma(\sigma(\emptyset), b_0), b_1, \dots)$$

**Proof.**

We first note that each strategy is a function from $\omega^{<\omega}$ to $\omega$, so each player has at most $2^{\aleph_0}$ many strategies for games of the form $G_A$. It's easy to construct $2^{\aleph_0}$ many strategies for each player. Also, observe that the map $x \mapsto \sigma * x$ is injective, so for each $\sigma$ the set $\{\sigma * x : x \in \omega^\omega\}$ has size $2^{\aleph_0}$.

## Proof (Cont.)

Let $\{\sigma_\alpha : \alpha < 2^{\aleph_0}\}$ and $\{\tau_\alpha : \alpha < 2^{\aleph_0}\}$ enumerate all strategies. Define $X = \{x_\alpha : \alpha < 2^{\aleph_0}\}$, $Y = \{y_\alpha : \alpha < 2^{\aleph_0}\} \subseteq \omega^\omega$ as follows:

(1) Suppose $\{x_\xi : \xi < \alpha\}$ has been defined. Choose $y_\alpha \notin \{x_\xi : \xi < \alpha\}$ such that $y_\alpha = \sigma_\alpha * z$ for some $z \in \omega^\omega$.

(2) Suppose $\{y_\xi : \xi \leq \alpha\}$ has been defined. Choose $x_\alpha \notin \{y_\xi : \xi \leq \alpha\}$ such that $x_\alpha = \tau_\alpha * z$ for some $z \in \omega^\omega$.

Clearly $X$ and $Y$ are disjoint, and no strategy for either player works: If Player I chooses strategy $\sigma$, then by construction Player II can play some sequence $z \in \omega^\omega$ such that $\sigma * z \notin X$, (and similarly for Player II. □

However, AD implies that countable choice for real numbers holds.

**Lemma**

AD *implies that every countable set of non-empty subsets of* $\omega^\omega$ *has a choice function.*

**Proof.**

Let $\{X_n : n < \omega\}$ be a family of non-empty subsets of $\omega^\omega$. Define:

$$A := \{x = (a_0, b_0, a_1, b_1, \dots) \in \omega^\omega : (b_0, b_1, \dots) \notin X_{a_0}\}$$

Consider the game $G_A$. Clearly Player I does not have a winning strategy, for if Player I plays $a_0$, then Player II may choose any $(b_0, b_1, \dots) \in X_{a_0}$ and plays it. By AD, Player II has a winning strategy $\tau$. A choice function would thus be:

$$f(X_n) := \tau * (n, 0, 0, \dots)$$

$\square$

An important consequence of this fact is that:

**Corollary**

AD *implies that* $\omega_1$ *is regular.*

Note that ZF + "$\omega_1$ is singular" is indeed consistent (if ZFC is consistent).

A few concluding remarks on AD:

(1) AD is essentially a large cardinal axiom. In fact, a result of Woodin says that $\mathsf{ZF} + \mathsf{AD}$ is equiconsistent with $\mathsf{ZFC} + \omega$ many Woodin cardinals.

(2) AD is compatible with dependent choice (DC). However, I cannot find any resources on whether it's known that $\mathsf{AD} \to \mathsf{DC}$.

## Recursive Trees

Recall that a tree is a subset $T \subseteq \omega^{<\omega}$ closed under initial segments.

Recall also that if $T$ is a tree, then a rank function $f : T \to \mathbf{ORD}$ is a function such that:

$$s \sqsubseteq t \implies f(t) < f(s)$$

We have proved via hand-waving that every well-founded tree has a rank function.

## Definition

Let $T$ be a tree. The *height* of $T$, denoted by $\|T\|$, is:

$$\|T\| := \min\{f(\emptyset) : f : T \to \mathbf{ORD} \text{ is a rank function}\}$$

See Example 1.14 of *Recursive Aspects of Descriptive Set Theory*, Mansfield-Weitkamp for some examples.

### Lemma

*For all $\alpha < \omega_1$, there exists a tree $T$ such that $\|T\| = \alpha$.*

### Sketch of Proof.

This lemma is hard to prove without pictures, so I shall just include a rough explanation of how to prove this lemma. We induct on $\alpha$.

(1) If $\alpha = 0$, the tree $T = \{\emptyset\}$ works.

(2) Suppose $\alpha = \beta + 1$. Let $T$ be a tree such that $\|T\| = \beta$. Append a node above the root of the tree, and the new tree has height $\alpha$.

(3) Suppose $\alpha = \sup_{n<\omega} \alpha_n$ is a limit ordinal. Let $T_n$ be a tree with height $\alpha_n$. Start with a root with infinitely many branches. Append each $T_n$ to one of these branches. The new tree has height $\alpha$.

$\square$

**Definition**

Let $x \in \omega^\omega$. We define:

$$\omega_1^x := \sup\{\|T\| : T \text{ is a tree recursive in } x\}$$

Since $\|T\| < \omega_1$ for all $T$, and there are only countably many trees recursive in $x$, we have that $\omega_1^x < \omega_1$ as long as $\omega_1$ is regular.

Clearly if $x \equiv_{\mathrm{T}} y$, then $\omega_1^x = \omega_1^y$. Thus, if $\Gamma$ is the set of Turing degrees, then:

$$\mathsf{CK} : \Gamma \to \omega_1, \;\; \mathsf{CK}([x]) := \omega_1^x$$

is a well-defined function (as long as $\omega_1$ is regular).

# Martin's Measure

Given $x \in \omega^\omega$, recall that a *cone* is a set $C_x$ of the form:

$$C_x := \{\deg(y) : x \leq_{\mathrm{T}} y\}$$

$x$ is also called the *apex* of the cone.

## Definition

The *Martin measure* is the set:

$$\mathcal{D} := \{X \subseteq \omega_1 : \mathsf{CK}^{-1}[X] \text{ contains a cone}\}$$

$\mathcal{D}$ is clearly a filter, as the intersection of two cones remains to be a cone (if $A$ and $B$ are Turing degrees, then $C_A \cap C_B = C_{A\oplus B}$). Furthermore, using countable choice, given Turing degrees $\{A_n\}_{n<\omega}$ we may define the supremum $A := \bigoplus_{n<\omega} A_n$. Then:

$$\bigcap_{n<\omega} C_{A_n} = C_A$$

Therefore, $\mathcal{D}$ is a $\omega_1$-complete filter.

**Theorem (Martin, Solovay)**

*If* AD *holds, then $\mathcal{D}$ is a $\omega_1$-complete non-principal ultrafilter on $\omega_1$.*

## Proof.

We first use AD to show that $\mathcal{D}$ is an ultrafilter. Let $X \subseteq \omega_1$. Note that $\mathsf{CK}^{-1}[\omega_1 \setminus X] = \Gamma \setminus \mathsf{CK}^{-1}[X]$. Let $\Lambda := \mathsf{CK}^{-1}[X]$, and it suffices to show that either $\Lambda$ or $\Gamma \setminus \Lambda$ contains a cone.

Let $A_\Lambda := \{x \in \omega^\omega : [x] \in \Lambda\}$. By AD, $G_{A_\Lambda}$ is determined, so there exists a winning strategy $\sigma$ for either Player I or II. We consider the cone $C := C_{\deg(\sigma)}$.

(1) Suppose $\sigma$ is a winning strategy for I. Let $x \in \omega^\omega$ such that $\sigma \leq_{\mathrm{T}} x$. Let $y := \sigma * x$. Then:

$$x \leq_{\mathrm{T}} y \leq_{\mathrm{T}} \sigma * x \leq_{\mathrm{T}} x$$

so $[x] = [y]$. Since I wins with $\sigma$, $x \in A_\Lambda$. Therefore $[y] \in \Lambda$, so $C \subseteq \Lambda$.

(2) Similarly, if $\sigma$ is a winning strategy for I, then $C \subseteq \Gamma \setminus \Lambda$.

Thus $\mathcal{D}$ is an ultrafilter.

## Proof (Cont.)

We now show that $\mathcal{D}$ is non-principal. Suppose not, so $\{\alpha\} \in \mathcal{D}$ for some $\alpha < \omega_1$. Let $x \in \omega^\omega$ such that $C_x \subseteq \mathsf{CK}^{-1}(\alpha)$. In other words, we have that:

$$x \leq_{\mathrm{T}} y \implies \omega_1^y = \alpha$$

Let $T$ be a tree such that $\|T\| > \alpha$. Let $y \in \omega^\omega$ such that $[y] = [x] \oplus \deg(T)$. Clearly $x \leq_{\mathrm{T}} y$, so by the above we have that $\omega_1^y = \alpha$. But then $T$ is recursive in $y$, so $\omega_1^y \geq \|T\| > \alpha$, a contradiction. $\square$

A few remarks on some related results:

(1) The theorem can be proved without recursion theory. Given $x, y \in \omega^\omega$, define:

$$x \preceq y \iff x \in L[y]$$

This is a relation that behaves very similarly to $\leq_{\mathrm{T}}$. We can basically repeat the proof with $\leq_{\mathrm{T}}$ replaced by $\preceq$.

(2) AD implies that $\aleph_2$ is also measurable.

(3) AD implies that $\mathrm{cf}(\omega_n) = \omega_2$ for all $n \geq 2$. In particular, $\aleph_n$ is not measurable for $n \geq 3$.

# AD and Measurability

**Theorem**

AD *implies that:*

*(1) Every set of reals is Lebesgue measurable.*

*(2) Every set of reals has the property of Baire.*

*(3) Every uncountable set of reals contains a perfect subset.*

We shall only prove (1).

Recall the following measure-theoretic fact (which can be proven in ZF + CC):

**Fact**

*For any $A \subseteq \mathbb{R}$ and $\varepsilon > 0$, there exists an open $U \supseteq A$ such that $\mu(U) \leq \mu^*(A) + \varepsilon$.*

Taking countable intersections of such open sets, there exists some measurable $E \supseteq A$ such that every measurable subset of $E \setminus A$ is null. Therefore, it suffices to show that:

Under AD, if $S \subseteq \mathbb{R}$ is such that every measurable subset of $S$ is null, then $S$ is null.

# The Covering Game

We introduce a game that we need to apply AD to. Fix some $S \subseteq [0,1]$ and $\varepsilon > 0$ such that every measurable subset of $S$ is null. We let $K_n$ to be the set of all sets $G \subseteq \mathbb{R}$ such that:

(1) $G$ is a finite union of rational intervals.

(2) $\mu(G) \leq \frac{\varepsilon}{2^{2(n+1)}}$.

By countable choice, $K_n$ is countable. We enumerate $K_n$ by writing $K_n = \{G_k^n : n < \omega\}$.

Given $(a_0, a_1, \dots) \in \{0,1\}^\omega$, we define:

$$a := \sum_{n=0}^{\infty} \frac{a_n}{2^{n+1}}$$

The rules of the game are as follows:

(1) $a_n = 0$ or $1$ for all $n$.
(2) $a \in S$.
(3) $a \notin \bigcup_{n=0}^{\infty} G^n_{b_n}$.

Intuitively, Player I tries to play a real number $a \in S$, and Player II tries to cover $a$ by some union $\bigcup_{n=0}^{\infty} H_n$ such that $H_n \in K_n$ for all $n$.

### Lemma

*Player I does not have a winning strategy in this game.*

### Proof.

Suppose $\sigma$ is a winning strategy for I. Define $f : \omega^\omega \to \omega^\omega$ by:

$$f(b) = a = (a_0, a_1, \dots), \text{ where } \sigma * b = (a_0, b_0, a_1, b_1, \dots)$$

Clearly $f$ is continuous. We borrow the fact that the continuous image of AN open set is measurable, and so $Z := f[\omega^\omega]$ is a measurable subset of $S$.

By the hypothesis on $S$, $Z$ is null. Since null sets can be covered by arbitrarily small open sets, we may pick $G^n_{b_n} \in K_n$ such that $Z \subseteq \bigcup_{n=0}^\infty G^n_{b_n}$. If II plays $(b_0, b_1, \dots)$, then clearly I always lose whenever I follows the strategy $\sigma$, a contradiction. □

### Proof of Theorem.

By AD and the previous lemma, II has a winning strategy $\tau$. It suffices to show that $\mu^*(S) \leq \varepsilon$ for arbitrarily $\varepsilon > 0$.

For each $s = (a_0, \ldots, a_n)$ of 0 and 1, let:

$$G_s := G^n_{b_n}, \text{ where } b_n = \sigma(a_0, b_0, \ldots, b_{n-1}, a_n)$$

Since $\tau$ is a winning strategy, for any $a = (a_0, a_1, \ldots) \in S$ which I plays, we have that $a \in \bigcup_{s \sqsubseteq a} G_s$. Thus:

$$S \subseteq \bigcup_{s \in \{0,1\}^{<\omega}} G_s = \bigcup_{n=1}^{\infty} \bigcup_{s \in \{0,1\}^n} G_s$$

## Proof of Theorem (Cont.)

Now for any $n$, we have that:

$$\mu\left(\bigcup_{s\in\{0,1\}^n} G_s\right) \leq \sum_{s\in\{0,1\}^n} \mu(G_s) \leq 2^n \cdot \frac{\varepsilon}{2^{2n}} = \frac{\varepsilon}{2^n}$$

Therefore:

$$\mu^*(S) \leq \sum_{n=1}^{\infty} \mu\left(\bigcup_{s\in\{0,1\}^n} G_s\right) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

as desired. $\square$

Concluding remarks:

(1) Unsurprisingly, the "converse" to the theorem is false. We have that "$\mathsf{ZF} + \mathsf{DC} +$ Every subset of $\mathbb{R}$ is Lebesgue measurable" is equiconsistent with "$\mathsf{ZFC} + \exists$inaccessible", while we recall that $\mathsf{ZF} + \mathsf{AD}$ is equiconsistent with $\omega$ many Woodin cardinals.

(2) However, we do not know if the three properties are "independent" - for instance, it's open if "$\mathsf{ZF} + \mathsf{DC} +$ Every subset of $\mathbb{R}$ is Lebesgue measurable" implies that every subset of $\mathbb{R}$ has the perfect set property.