

Definability of mad families of vector spaces and the fullness property

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Warning!!!

For the entirety of this talk, the **zero vector** is ignored.

Almost disjoint subspaces

Let \mathbb{F} be a countable field (possibly finite). Let E be a \mathbb{F} -vector space with a fixed Hamel basis $(e_n)_{n < \omega}$.

Definition

Let $V, W \subseteq E$ be two infinite-dimensional subspaces. We say that V, W are *almost disjoint* if $V \cap W$ is a finite-dimensional subspace of E .

Definition

Let \mathcal{A} be a family of infinite-dimensional subspaces of E . We say that \mathcal{A} is *almost disjoint* if all subspaces in \mathcal{A} are pairwise almost disjoint. We say that \mathcal{A} is *maximal almost disjoint* (or just *mad*) if \mathcal{A} is not strictly contained in another almost disjoint family of infinite-dimensional subspaces.

Block subspaces

Recall that E has a fixed Hamel basis $(e_n)_{n < \omega}$. Given a vector $x \in E$, we may write

$$x = \sum_{n < \omega} \lambda_n(x) e_n,$$

where only finitely many λ_n 's are non-zero. We may then write:

$$\text{supp}(x) := \{n < \omega : \lambda_n(x) \neq 0\}.$$

Notation

Given two vectors x, y we write:

$$x < y \iff \max(\text{supp}(x)) < \min(\text{supp}(y)).$$

Definition

An infinite-dimensional subspace $V \subseteq W$ is a *block subspace* if it has a *block basis*. That is, V is spanned by the basis $(x_n)_{n < \omega}$, where:

$$x_0 < x_1 < x_2 < \cdots .$$

Note that the block basis of a block subspace is unique (up to scalar multiplication of each vector).

Fact

Every infinite-dimensional subspace of E contains an infinite-dimensional block subspace.

Consequently, if \mathcal{A} is an almost disjoint family such that there is no block subspace that is almost disjoint with every element of \mathcal{A} , then \mathcal{A} is mad.

Definition

We define the cardinal invariant:

$$\mathfrak{a}_{\text{vec},\mathbb{F}}^* := \min\{|\mathcal{A}| > 1 : \mathcal{A} \text{ is a mad family of subspaces}\},$$

$$\mathfrak{a}_{\text{vec},\mathbb{F}} := \min\{|\mathcal{A}| > 1 : \mathcal{A} \text{ is a mad family of block subspaces}\}.$$

It's clear that $\mathfrak{a}_{\text{vec},\mathbb{F}}^* \leq \mathfrak{a}_{\text{vec},\mathbb{F}}$.

Proposition (Smythe, 2019)

$$\mathfrak{a}_{\text{vec},\mathbb{F}} \geq \aleph_1.$$

Proposition (user527492 (on MathOverflow) and Y., 2025)

$$\mathfrak{a}_{\text{vec},\mathbb{F}}^* \geq \aleph_1.$$

Definability of mad families

Let $E^{[\infty]}$ be the set of all block sequences (= block subspaces) of E .

Consider equipping E with the discrete topology, and $E^{\mathbb{N}}$ with the product topology. Since E is countable, $E^{\mathbb{N}}$ is Polish. Then $E^{[\infty]} \subseteq E^{\mathbb{N}}$ is a closed subspace, so the subspace topology of $E^{[\infty]}$ is also Polish.

Problem (Smythe, 2019)

Is there no analytic mad family $\mathcal{A} \subseteq E^{[\infty]}$ of block subspaces?

Current status. Mostly open, but there are some partial positive results.

Definition

A subset $X \subseteq E$ is *small* if there is no subspace $V \subseteq E$ such that $V \subseteq X$.

Note that finite union of small subsets need not be small.

Given an almost disjoint family $\mathcal{A} \subseteq E^{[\infty]}$ of block subspaces, we define the set $\mathcal{I}(\mathcal{A})$ to be:

$$\left\{ X \subseteq E : \exists A_0, \dots, A_{n-1} \in \mathcal{A} \text{ such that } X \setminus \bigcup_{i < n} \langle A_i \rangle \text{ is small} \right\}.$$

$\mathcal{I}(\mathcal{A})$ is \subseteq -downward closed, but it is not necessarily an ideal. We also let $\mathcal{I}^+(\mathcal{A}) := \mathcal{P}(E) \setminus \mathcal{I}(\mathcal{A})$.

Fullness

Definition

Let $\mathcal{A} \subseteq E^{[\infty]}$ be an almost disjoint family of block subspaces.

1. A subset $Y \subseteq E$ is $\mathcal{I}^+(\mathcal{A})$ -dense below some $B \in \mathcal{I}^+(\mathcal{A})$ if for all $C \in \mathcal{I}^+(\mathcal{A}) \restriction B$ (i.e. $C \in \mathcal{I}^+(\mathcal{A})$ and $C \leq B$), there exists some $D \leq C$ such that $\langle D \rangle \subseteq Y$.
2. We say that \mathcal{A} is *full* if for all $Y \subseteq E$ such that Y is $\mathcal{I}^+(\mathcal{A})$ -dense below some $B \in \mathcal{I}^+(\mathcal{A})$, there exists some $C^* \in \mathcal{I}^+(\mathcal{A}) \restriction B$ such that $\langle C^* \rangle \subseteq Y$.

The fullness property should be seen as a weak form of the pigeonhole principle.

Lemma

Suppose that E is a vector space over \mathbb{F}_2 . For any almost disjoint family $\mathcal{A} \subseteq E^{[\infty]}$, the following are equivalent:

1. *\mathcal{A} is full.*
2. *For every $B \in \mathcal{I}^+(\mathcal{A})$ and every partition $Y \subseteq E$, there exists some $C \in \mathcal{I}^+(\mathcal{A}) \restriction B$ such that $\langle C \rangle \subseteq Y$ or $\langle C \rangle \subseteq Y^c$.*

Theorem (Smythe, 2019)

If \mathcal{A} is a full mad family of subspaces, then \mathcal{A} is not analytic.

Problem (Smythe, 2019)

1. (ZFC) Is there a full mad family?
2. (ZFC) Is every mad family full?

Theorem (Smythe, 2019)

If $\mathfrak{p} = \mathfrak{c}$, then there exists a full mad family.

Construction of full mad families

Theorem (Y., 2025)

If $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}_{\text{vec}, \mathbb{F}}^$ or $\mathfrak{p} = \max\{\mathfrak{b}, \mathfrak{s}\}$, then there exists a full mad family of block subspaces.*

The construction mimics that of a completely separable mad family of subsets of ω , assuming $\mathfrak{s} \leq \mathfrak{a}$.

Completely separable mad families

Let $\mathcal{A} \subseteq [\omega]^\omega$ be an almost disjoint family (not necessarily mad). Recall that:

$$\mathcal{I}(\mathcal{A}) := \left\{ X \subseteq \omega : \exists A_0, \dots, A_{n-1} \in \mathcal{A} \text{ such that } X \subseteq^* \bigcup_{i < n} A_i \right\}.$$

$\mathcal{I}(\mathcal{A})$ is an ideal (as finite union of finite sets is finite). We let $\mathcal{I}^+(\mathcal{A}) := \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$.

Definition

A mad family $\mathcal{A} \subseteq [\omega]^\omega$ is *completely separable* if for every $Y \in \mathcal{I}^+(\mathcal{A})$, there exists some $A \in \mathcal{A}$ such that $A \subseteq Y$.

Theorem (Mildenberger-Raghavan-Steprāns, 2014)

If $\mathfrak{s} \leq \mathfrak{a}$, then there exists a completely separable mad family.

Start of proof. We construct the completely separable mad family $\mathcal{A} = \{X_\alpha : \alpha < \mathfrak{c}\}$ inductively.

Let $\{Y_\alpha : \alpha < \mathfrak{c}\}$ enumerate all infinite subsets of ω such that for each infinite $Y \subseteq \omega$, $Y = Y_\alpha$ for cofinally many α .

Assume that $\mathcal{A}_\alpha = \{X_\beta : \beta < \alpha\}$. We shall find some X_α such that:

- X_α is almost disjoint from X_β for all $\beta < \alpha$.
- If $Y_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$, then $X_\alpha \subseteq Y_\alpha$.

There are two crucial components of the construction that we will discuss:

1. **(Main lemma)** Using block-splitting families to split a set $X \in \mathcal{I}^+(\mathcal{A}_\alpha)$ into two $\mathcal{I}(\mathcal{A}_\alpha)$ -positive pieces.
2. **(Applying assumptions)** Applying $\mathfrak{s} \leq \mathfrak{a}$ to obtain the required X_α .

Definition

1. Let $\mathcal{P} = \{P_n : n < \omega\}$ be an interval partition. We say that a set $S \in [\omega]^\omega$ *block-splits* \mathcal{P} if $\{n < \omega : P_n \subseteq S\}$ and $\{n < \omega : P_n \cap S = \emptyset\}$ are both infinite.
2. A *block-splitting family* is a family $\mathcal{S} \subseteq [\omega]^\omega$ such that every interval partition is block-split by some $S \in \mathcal{S}$.

Lemma (Kamburelis-Weglorz, 1996)

$\max\{\mathfrak{b}, \mathfrak{s}\} = \min\{|\mathcal{S}| : \mathcal{S} \text{ is a block-splitting family}\}.$

Main lemma.

Lemma

Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. Let \mathcal{A} be an almost disjoint family, and let $X \in \mathcal{I}^+(\mathcal{A})$. Then the following properties hold:

- 1. For all α , $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$ or $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.*
- 2. If $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some $Y \in \mathcal{I}^+(\mathcal{A})$ such that $Y \subseteq X \cap S_\alpha$.*
- 3. If $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some $Y \in \mathcal{I}^+(\mathcal{A})$ such that $Y \subseteq X \setminus S_\alpha$.*
- 4. There exists some α such that $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.*

Note that properties (1)-(3) are trivial.

Applying assumptions. Using the previous lemma (+ some non-trivial workings), we obtain \mathcal{B} and Y'_α such that:

- \mathcal{B} is an almost disjoint family such that $|\mathcal{B}| < \max\{\mathfrak{b}, \mathfrak{s}\}$.
- $Y'_\alpha \subseteq Y_\alpha$.
- If $X \subseteq Y'_\alpha$ is almost disjoint from all elements of \mathcal{B} , then X is almost disjoint from X_β for all $\beta < \alpha$.

Let:

$$\mathcal{B}' := \{Z \cap Y'_\alpha : Z \in \mathcal{B}\}.$$

If $\mathfrak{s} \leq \mathfrak{a}$, then $|\mathcal{B}'| < \max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}$, so there exists some $X_\alpha \subseteq Y'_\alpha$ that is almost disjoint from all elements of \mathcal{B}' . Choosing X_α completes the induction.

Full mad families

Recall the theorem of focus today:

Theorem (Y., 2025)

If $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}_{\text{vec}, \mathbb{R}}^$ or $\mathfrak{p} = \max\{\mathfrak{b}, \mathfrak{s}\}$, then there exists a full mad family of block subspaces.*

Start of proof. We construct the full mad family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ of block subspaces inductively.

We need two enumerations:

1. $\{Y_\alpha : \alpha < \text{Lim}(\mathfrak{c})\}$ enumerates all infinite subsets of E , and;
2. $\{B_\alpha : \alpha < \text{Lim}(\mathfrak{c})\}$ enumerates all elements of $E^{[\infty]}$ (i.e. all block subspaces);

such that for all $Y \subseteq E$ and $B \in E^{[\infty]}$, $(Y, B) = (Y_\alpha, B_\alpha)$ for cofinally many α .

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$, where α is a limit. We shall find some A'_α such that if:

- the set:

$$\mathcal{C}_\alpha := \{A_\beta \in \mathcal{A}_\alpha : A_\beta \text{ and } B_\alpha \text{ are compatible}\}$$

is infinite, and;

- there exists some $D \leq B_\alpha$, almost disjoint from \mathcal{C}_α such that $\langle D \rangle \subseteq Y$;

then $A'_\alpha \leq B_\alpha$ is such that $\langle A'_\alpha \rangle \subseteq Y$, and A'_α is almost disjoint from \mathcal{A}_α . We then let $\{A_{\alpha+n} : n < \omega\}$ be any almost disjoint family such that $A_{\alpha+n} \leq A'_\alpha$ for all n , which defines $\mathcal{A}_{\alpha+\omega}$.

Claim

This construction yields a full mad family.

Sketch of Proof. Let $Y \subseteq E$ be $\mathcal{I}^+(\mathcal{A})$ -dense below some $B \in \mathcal{I}^+(\mathcal{A})$, i.e. for all $C \in \mathcal{I}^+(\mathcal{A}) \restriction B$, there exists some $D \leq C$ such that $\langle D \rangle \subseteq Y$. We need to find some $C^* \in \mathcal{I}^+(\mathcal{A}) \restriction B$ such that $\langle C^* \rangle \subseteq Y$.

Fact

If \mathcal{A} is a mad family of block subspaces, then $B \in \mathcal{I}^+(\mathcal{A})$ iff there exists an infinite $\{\beta_i : i < \omega\}$ such that A_{β_i} is compatible with B for all i .

Let $\alpha > \beta_i$ for all i be a limit ordinal such that $(Y, B) = (Y_\alpha, B_\alpha)$.

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Step α of the induction. We shall find some A'_α such that if:

- the set:

$$\mathcal{C}_\alpha := \{A_\beta \in \mathcal{A}_\alpha : A_\beta \text{ and } B_\alpha \text{ are compatible}\}$$

is infinite, and; True, as $\{A_{\beta_i} : i < \omega\} \subseteq \mathcal{C}_\alpha$.

- there exists some $D \leq B_\alpha$, almost disjoint from \mathcal{C}_α such that $\langle D \rangle \subseteq Y$; True, as 1. $A'_\alpha \in \mathcal{I}^+(\mathcal{A}) \restriction B_\alpha$, 2. A'_α almost disjoint from \mathcal{C}_α , and 3. Y is $\mathcal{I}^+(\mathcal{A})$ -dense below B_α , so there exists some $D \leq A'_\alpha$ such that $\langle D \rangle \subseteq Y$.

then $A'_\alpha \leq B_\alpha$ is such that $\langle A'_\alpha \rangle \subseteq Y$, and A'_α is almost disjoint from \mathcal{A}_α . We then let $\{A_{\alpha+n} : n < \omega\}$ be any almost disjoint family such that $A_{\alpha+n} \leq A'_\alpha$ for all n , which defines $\mathcal{A}_{\alpha+\omega}$.

Main lemma. Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family, and for each α , let:

$$Z_\alpha := \langle (e_n)_{n \in S_\alpha} \rangle.$$

Lemma

Let $\mathcal{A} \subseteq E^{[\infty]}$ be an almost disjoint family, and let $B \in E^{[\infty]}$ be such that $\langle B \rangle \in \mathcal{I}^+(\mathcal{A})$.

1. For all α , $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ or $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.
2. If $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some $C \in E^{[\infty]}$ such that $\langle C \rangle \in \mathcal{I}^+(\mathcal{A})$ and $\langle C \rangle \subseteq \langle B \rangle \cap Z_\alpha$.
3. If $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some $C \in E^{[\infty]}$ such that $\langle C \rangle \in \mathcal{I}^+(\mathcal{A})$ and $\langle C \rangle \subseteq \langle B \rangle \setminus Z_\alpha$.
4. There exists some α such that $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Applying assumptions. Using the previous lemma (+ some non-trivial workings), we obtain \mathcal{B} and B'_α such that:

- \mathcal{B} is an almost disjoint family such that $|\mathcal{B}| < \max\{\mathfrak{b}, \mathfrak{s}\}$.
- $B'_\alpha \leq B_\alpha$.
- If $C \leq B'_\alpha$ is almost disjoint from all elements of \mathcal{B} , then C is almost disjoint from A_β for all $\beta < \alpha$.

Let:

$$\mathcal{B}' := \{\langle D \rangle \cap \langle B'_\alpha \rangle : D \in \mathcal{B}\}.$$

This is an almost disjoint of subspaces of B'_α , but not necessarily block subspaces.

1. If $p = \max\{b, s\}$, then $|\mathcal{B}'| < p$. By a Lemma of (Smythe, 2019), there is some $A'_\alpha \leq B'_\alpha$ that is almost disjoint from \mathcal{B}' .
2. If $\max\{b, s\} \leq \alpha_{\text{vec}, \mathbb{F}}^*$, then $|\mathcal{B}'| < \alpha_{\text{vec}, \mathbb{F}}^*$. Thus, there exists some $A'_\alpha \leq B'_\alpha$ that is almost disjoint from all elements of \mathcal{B}' .

Choosing this A'_α completes the induction.

Open problems

It's still unknown if one may construct a full mad family of block subspaces in ZFC.

Problem

Does $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}_{\text{vec}, \mathbb{F}}^*$ hold in ZFC?

It's unknown there exists a completely separable mad family in ZFC.

Theorem (Shelah, 2011)

If $\mathfrak{c} < \aleph_\omega$, then there is a completely separable mad family.

Problem

If $\mathfrak{c} < \aleph_\omega$, is there a full mad family of block subspaces?

Thanks for listening!

1. Subspaces V, W are *almost disjoint* if $V \cap W$ is finite-dimensional. This gives us the cardinal invariants:
 - $\mathfrak{a}_{\text{vec}, \mathbb{F}}^* := \min\{|\mathcal{A}| : \mathcal{A} \text{ is a mad family of subspaces}\}.$
 - $\mathfrak{a}_{\text{vec}, \mathbb{F}} := \min\{|\mathcal{A}| : \mathcal{A} \text{ is a mad family of block subspaces}\}.$
2. A mad family $\mathcal{A} \subseteq E^{[\infty]}$ is *full* if $\mathcal{I}^+(\mathcal{A})$ satisfies a weak pigeonhole principle.
 - (Smythe, 2019) If $\mathfrak{p} = \mathfrak{c}$, then there exists a full mad family.
 - Is there a full mad family in ZFC? Is every mad family full?
3. (Y., 2025) If $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}_{\text{vec}, \mathbb{F}}^*$ or $\mathfrak{p} = \max\{\mathfrak{b}, \mathfrak{s}\}$, then there exists a full mad family.
 - The proof is inspired by the construction of a completely separable mad family (of $[\omega]^\omega$), assuming $\mathfrak{s} \leq \mathfrak{a}$.
 - The key modifications are the main lemma involving a block-splitting family, and how the assumptions are applied.