Diagonalisation Forcing with Infinite Block Sequences

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Abstract

This thesis consists of two components. The first component (Chapter 2 and 3) is expository - in Chapter 2, we provide a detailed description on the classical construction of Solovay's model. In Chapter 3, we introduce the notion of topological Ramsey spaces, and provide a proof that the well-known space of infinite block sequences $\mathbf{FIN}_k^{[\infty]}$ carries this structure.

The second component (Chapter 4 and 5) consists of original research - in Chapter 4, we describe the new notion of abstract selective coideal introduced in [DMN15], and develop a modified version of diagonalisation forcing (see [NN16]) for $\mathbf{FIN}_k^{[\infty]}$. We apply the forcing notion to show that, assuming the existence of an inaccessible cardinal and several combinatorial hypotheses for $\mathbf{FIN}_k^{[\infty]}$, it holds in Solovay's model that every $\mathcal{X} \subseteq \mathbf{FIN}_k^{[\infty]}$ is $\mathcal{H}$-Ramsey for all selective coideals $\mathcal{H}$. We conclude the thesis by exploring further the ultrafilter hypothesis in Chapter 5.

Contents

1 Introduction and Historical Notes

This thesis studies some Ramsey-theoretic behaviour of subsets of the topological Ramsey space $\mathbf{FIN}_k^{[\infty]}$ in Solovay's model.

Robert M. Solovay first introduced Solovay's Model in his famous paper *A model of set-theory in which every set of reals is Lebesgue measurable* [Sol70], to answer the long-standing question "Is Axiom of Choice AC necessary to construct a non-Lebesgue measurable set?". This model satisfies $\mathsf{ZF}+\mathsf{DC}+$ "All subsets of $\mathbb{R}$ are Lebesgue measurable" (where DC denotes the Axiom of Dependent Choice, which is still sufficient to develop most of real analysis [Ber42]).

Since then, the set-theoretic properties of Solovay's model have been extensively studied by mathematicians. In 1977, Mathias proved that every subset of $[\omega]^\omega$ is Ramsey in his celebrated paper *Happy Families* [Mat77]. The forcing notion introduced in his paper is now widely known as Mathias forcing. He also showed that, assuming the existence of a Mahlo cardinal, Solovay's model contains no happy families (selective coideals). Mathias then asked if one may lower the strength to that of just an inaccessible.

This was originally answered positively by Törnquist [Tör15], but he did it in a purely combinatorial manner. Neeman and Norwood then introduced diagonalisation forcing in [NN16] to provide a second positive answer using the notion of happy families (selective coideals) introduced by Mathias in his original paper. Neeman-Norwood's paper also provided a strengthening of a theorem in [Tör15].

Meanwhile, Todorčević published his renowned *Introduction to Ramsey Spaces* [Tod10] in 2005. This book introduces the notion of topological Ramsey spaces, which possess important Ramsey-theoretic properties. One of such space is the space of infinite block sequences $\mathbf{FIN}_k^{[\infty]}$ - the set of blocks $\mathbf{FIN}_k$ was first introduced by Gowers [Gow92] in his groundbreaking *Lipschitz functions on classical spaces*, which resolved a long-standing open problem in Banach space theory using a Ramsey-theoretic argument on sequences of blocks. Gowers showed a close relation between $\mathbf{FIN}_k$ and the positive part of the unitary sphere in the Banach space c_0. Since then, the Ramsey theory behind $\mathbf{FIN}_k^{[\infty]}$ (the space of infinite block sequences) has been a subject of much interest, such as [CDM18].

Di Prisco-Mijares-Nieto have also been working on treating local Ramsey theory abstractly. Local Ramsey theory originally served to study Ramsey spaces of the form $(\mathbb{N}^{[\infty]}, \mathcal{H}, \subseteq, r)$, and properties of this space were studied when $\mathcal{H}$ is assumed to be a semiselective/selective coideal. Mijares kick-started the development of abstract local Ramsey theory in [Mij07], where he generalised Ramsey spaces to sets other than $\mathbb{N}^{[\infty]}$. He also introduced a notion of selective ultrafilters for topological Ramsey spaces. His work has since then been furthered in the paper [DMN15], where the idea of coideals and semiselectivity/selectivity have been extended to topological Ramsey spaces as well. The tools they introduced allowed them to develop a Mathias-like forcing for Ramsey spaces and leave many possibilities of other forcing notions to be generalised to Ramsey spaces.

The structure of this thesis is as follows: In Chapter 2, we provide a construction of Solovay's model $\mathbf{L}(\mathbb{R})$. In Chapter 3, we will introduce topological Ramsey spaces (following [Tod10]), and describe the topological Ramsey space property of $\mathbf{FIN}_k^{[\infty]}$ as an example. In Chapter 4, we introduce an abstract notion of coideals and ultrafilters for topological Ramsey spaces as introduced in [Mij07] and [DMN15], use these extended notions to develop our diagonalisation-like forcing for $\mathbf{FIN}_k^{[\infty]}$. In Chapter 5 we examine the ultrafilter hypothesis on $\mathbf{FIN}_k^{[\infty]}$.

2 | Solovay's Model

In this chapter we provide a construction of the Solovay's model $\mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$, where $\mathcal{G}$ is the generic filter of the Lévy collapse $\mathrm{Col}(\omega, \kappa)$ of an inaccessible cardinal κ (where $\mathrm{Col}(\omega, \kappa)$ collapses cardinals $< \kappa$), along with proofs of some basic properties of this model. The reference texts here are [Jec03], [Kan03] and [Ung15].

2.1 The Lévy Collapse

The Lévy collapse was developed by Azriel Lévy shortly after Cohen introduced forcing to prove the independence of continuum hypothesis. Lévy collapse is one of the most well-known notion used to collapse cardinals, partially due to the homogeneity property that it enjoys (Lemma 2.1.6).

2.1.1 Basic Properties of Lévy Collapse

Heuristically, the Lévy Collapse is a generalisation of the original Cohen forcing. Instead of collapsing a single cardinal, the Lévy collapse simultaneously collapses a set of ordinals into a single cardinal.

Definition 2.1.1. Let $S \subseteq \mathbf{ORD}$, and let λ be a regular cardinal. The **Lévy Collapse**, $\mathrm{Col}(\lambda, S)$, is the forcing notion that consists of conditions p such that:

(1) p is a function.

(2) $\mathrm{dom}(p) \subseteq S \times \lambda$.

(3) $|\mathrm{dom}(p)| < \lambda$.

(4) For all $(\alpha, \xi) \in \mathrm{dom}(p)$, we have $p(\alpha, \xi) < \alpha$.

along with a partial order $\leq$ defined by:

$$p \leq q \iff p \supseteq q$$

Intuitively, the Lévy collapse simultaneously adds surjections $\lambda \to \alpha$ for every $\alpha \in S$. In particular, $\mathrm{Col}(\omega, \{\omega\})$ is isomorphic to Cohen's original forcing.

Notation 2.1.2. Let $\mathcal{G}$ be a generic filter over $\mathrm{Col}(\lambda, \kappa)$. For $\lambda < \delta < \mu < \kappa$, write:

(1) $\mathcal{G}{\restriction}\mu := \mathcal{G} \cap \mathrm{Col}(\lambda, \mu)$.

(2) $\mathcal{G}{\restriction}[\delta, \mu) := \mathcal{G} \cap \mathrm{Col}(\delta, \mu)$.

Lemma 2.1.3.

(1) Suppose $S = X \sqcup Y$. Set $\mathbb{P}_0 := \mathrm{Col}(\lambda, X)$ and $\mathbb{P}_1 := \mathrm{Col}(\lambda, Y)$. Then $\mathcal{G}$ is $\mathrm{Col}(\lambda, S)$-generic iff:

$$\mathcal{G} = \{p \cup q : p \in \mathcal{G}_0 \wedge q \in \mathcal{G}_1\}$$

for some $\mathcal{G}_0, \mathcal{G}_1$, where $\mathcal{G}_0$ is $\mathbb{P}_0$-generic, $\mathrm{Col}(\lambda, Y)^{\mathbf{V}[\mathcal{G}_0]} = \mathbb{P}_1$, and $\mathcal{G}_1$ is $\mathbb{P}_1$-generic over $\mathbf{V}[\mathcal{G}_0]$.

(2) If κ is inaccessible, then $\mathrm{Col}(\lambda, \kappa)$ is κ-cc.

(3) If $\mathrm{Col}(\lambda, \kappa)$ is κ-cc, then forcing with it preserves cardinals $\leq \lambda$ and $\geq \kappa$.

(4) Suppose κ is regular, $\mathrm{Col}(\lambda, \kappa)$ is κ-cc, and $\mathcal{G}$ is $\mathrm{Col}(\lambda, \kappa)$-generic. Let $X \in \mathbf{V}$ with $|X| < \kappa$, and let $Y \in \mathbf{V}[\mathcal{G}]$ such that $Y \subseteq X$. Then there is a $\beta < \kappa$ such that $X \in \mathbf{V}[\mathcal{G}{\restriction}\beta]$.

Proof.

(1) For each $p \in \mathrm{Col}(\lambda, S)$, the map/correspondence given by:

$$p \mapsto (p{\restriction}(X \times \lambda), p{\restriction}(Y \times \lambda))$$

yields an obvious isomorphism $\mathrm{Col}(\lambda, S) \cong \mathbb{P}_0 \times \mathbb{P}_1$. To see that $\mathrm{Col}(\lambda, Y)^{\mathbf{V}[\mathcal{G}_0]} = \mathbb{P}_1$, we note that clearly $\mathbb{P}_0$ is λ-closed, so no new sequences of ordinals of length $< \lambda$ are added. Now apply the Product Lemma (Proposition 10.9, [Kan03]).

(2) We borrow the Δ-System Lemma (Proposition 10.4, [Kan03]). Let $\mathcal{A} = \{p_\beta : \beta < \kappa\}$ be a set of conditions of size κ, and let $\mathcal{A}' := \{\mathrm{dom}(p_\beta) : \beta < \kappa\}$. Since κ is inaccessible, the hypothesis of Δ-System Lemma is satisfied, so applying it to $\mathcal{A}'$ yields a set $\mathcal{B} \subseteq \mathcal{A}'$ such that $|\mathcal{B}| = \kappa$ and a root $r \subseteq \lambda \times \kappa$ such that $x \cap y = r$ for distinct $x, y \in \mathcal{B}$. Now since $p(\alpha, \xi) < \alpha$ for all $p \in \mathcal{B}$, the possible values which $p{\restriction}r$ can take is at most $|\kappa \times \lambda|^{<\kappa}$, and since κ is inaccessible, $|\kappa \times \lambda|^\kappa = \kappa$. Thus, there must exist $\mathcal{C} \in [\mathcal{B}]^\kappa$ such that all elements in $\mathcal{C}$ is pairwise compatible. In particular, $\mathcal{A}$ is not an antichain.

(3) This follows from λ-closedness.

(4) Let $\dot{Y} \in \mathbf{V}$ be a name of Y (so $\dot{Y}_\mathcal{G} = Y$). For each $x \in X$, let $\mathcal{A}_x \subseteq \mathrm{Col}(\lambda, \kappa)$ be a maximal antichain such that for all $p \in \mathcal{A}_x$, p decides the statement "$\hat{x} \in \dot{Y}$". Since $\mathrm{Col}(\lambda, \kappa)$ is κ-cc, $|\mathcal{A}_x| < \kappa$ for all $x \in X$. Now let:

$$\mathcal{A} := \bigcup_{x \in X} \mathcal{A}_x$$

Since $|X| < \kappa$ and κ is regular, $|\mathcal{A}| < \kappa$. Thus, there exists some $\lambda < \beta < \kappa$ such that $\mathcal{A} \subseteq \mathrm{Col}(\lambda, \beta)$. Then we see that:

$$Y = \{x \in X : \exists p \in \mathcal{G}{\restriction}\beta [p \Vdash \hat{x} \in \dot{Y}]\}$$

so $Y \in \mathbf{V}[\mathcal{G}{\restriction}\beta]$.

□

Thus, if $\mathrm{Col}(\lambda, \kappa)$, with $\kappa > \lambda$ is κ-cc, it collapses κ to λ^+. That is, it collapses exactly the cardinals $\lambda < \alpha < \kappa$.

2.1.2 Weak Homogeneity

We now addresses two important properties of the Lévy Collapse. The first property is defined as follows:

Definition 2.1.4. For a forcing poset $\mathbb{P}$, we say that $\mathbb{P}$ is **weakly homogeneous** iff for every $p, q \in \mathbb{P}$, there exists an automorphism π such that $\pi(p)$ and q are compatible.

Lemma 2.1.5. *If $\mathbb{P}$ is weakly homogeneous, then for any formula φ of n free variables and $x_1, \ldots, x_n \in \mathbf{V}$, we have either $\Vdash_\mathbb{P} \varphi(\hat{x}_1, \ldots, \hat{x}_n)$ or $\Vdash_\mathbb{P} \neg\varphi(\hat{x}_1, \ldots, \hat{x}_n)$.*

Lemma 2.1.6. $\mathrm{Col}(\lambda, \kappa)$ *is weakly homogeneous.*

Proof. Let $p, q \in \mathrm{Col}(\lambda, \kappa)$. Since $|\operatorname{dom}(q)| < \lambda$, there exists a set X such that $|X| = |\operatorname{dom}(p)|$ and $X \cap \operatorname{dom}(q) = \emptyset$. Let $f_p : X \to \operatorname{dom}(p)$ be any bijection, and define an automorphism π by stipulating that for any $p \in \mathbb{P}$, $\pi(p) : X \to \kappa$ is the partial function defined by $\pi(p)(\alpha, \xi) := p(f_p(\alpha, \xi))$. Then $\operatorname{dom}(\pi(p)) \cap \operatorname{dom}(q) = \emptyset$, so $\pi(p)$ and q are compatible. $\square$

We remark that, in fact, $\mathrm{Col}(\lambda, \kappa)$ is **homogeneous**, i.e. for any $p, q \in \mathbb{P}$ we may produce an automorphism such that $\pi(p) = q$ (Corollary 26.13, [Jec03]).

2.1.3 The Factor Lemma

In the presence of multiple generic filters in which one is generic over a Lévy collapse, the Factor Lemma allows us to "factor" out the filter which is generic over the Lévy collapse from the rest.

Lemma 2.1.7. *Let $\mathbb{P}$ be a separative forcing poset. Suppose for some uncountable ordinal α, we have $|\mathbb{P}| \leq |\alpha|$, and:*

$$\Vdash_{\mathbb{P}} \exists f[f : \omega \to \alpha \text{ is surjective}]$$

Then there is an injective, dense embedding of a dense subset of $\mathrm{Col}(\omega, \{\alpha\})$ *into* $\mathbb{P}$.

Proof. Let $\nu = |\alpha|$. Since $\nu > \omega$, the failure of ν-cc implies that there is an antichain of size ν below every $p \in \mathbb{P}$. Let $\dot{g}$ be a term such that $\Vdash_{\mathbb{P}} \dot{g} : \omega \to \dot{\mathcal{G}}$ is surjective, where $\mathcal{G}$ is $\mathbb{P}$-generic (which exists by the maximal principle). It's easy to see that:

$$\mathcal{D} := \{p \in \mathbb{P} : \exists n \in \omega \, \exists r \in \mathcal{G}[p \Vdash \dot{g}(n) = \hat{r}]\}$$

is dense in $\mathbb{P}$. Now define:

$$\mathcal{D}_\alpha := \{p \in \mathrm{Col}(\omega, \{\alpha\}) : \exists n \in \omega[p : \{\alpha\} \times n \to \alpha]\}$$

Clearly $\mathcal{D}_\alpha$ is dense in $\mathrm{Col}(\omega, \{\alpha\})$. Thus, it suffices to define an embedding $\pi : \mathcal{D}_\alpha \to \mathcal{D}$. We do this recursively:

- $\pi(\emptyset) = \mathbb{1}_{\mathbb{P}}$.
- Suppose $\pi(p)$ is defined, where $p \in \mathcal{D}_\alpha$ with $\operatorname{dom}(p) = \{\alpha\} \times n$. Let $\left\langle a^p_\xi : \xi < \alpha \right\rangle$ be a maximal antichain below $\pi(p)$ of size ν. By taking stronger conditions if necessary (while preserving denseness of $\mathcal{D}_\alpha$), assume for each ξ there exists an r such that $a^p_\xi \Vdash (\hat{r} \in \dot{\mathcal{G}} \wedge \dot{g}(n) = \hat{r})$. Now define:

$$\pi(p \cup \{((\alpha, n), \xi)\}) := a^p_\xi$$

Then clearly $\pi : \mathcal{D}_\alpha \to \pi[\mathcal{D}_\alpha]$ is an isomorphism. To see that $\pi[\mathcal{D}_\alpha]$ is dense in $\mathcal{D}$, let $r \in \mathcal{D}$ be arbitrary. The separative property of $\mathbb{P}$ ensures that $r \Vdash \hat{r} \in \dot{\mathcal{G}}$. Thus, let $s \in \mathcal{D}$ be such that $s \Vdash \dot{g}(n) = \hat{r}$ for some n. By the construction, one may produce a p such that $\pi(p) \Vdash (\hat{r} \in \dot{\mathcal{G}} \wedge \dot{g}(n) = \hat{r})$. In particular, $\pi(p) \Vdash \hat{r} \in \dot{\mathcal{G}}$ implies that $\pi(p) \leq r$, due to the separative property. □

Lemma 2.1.8 (Factor Lemma). *Suppose $\kappa > \omega$ is regular and $\mathcal{G}$ is* $\mathrm{Col}(\kappa, \omega)$*-generic. Let $X \in \mathbf{V}$ with $|X| < \kappa$. Then for any $Y \in \mathbf{V}[\mathcal{G}]$ such that $Y \subseteq X$, there exists a $\mathcal{G}^*$, which is* $\mathrm{Col}(\omega, \kappa)$*-generic over $\mathbf{V}[Y]$, such that $\mathbf{V}[\mathcal{G}] = \mathbf{V}[Y][\mathcal{G}^*]$.*

Proof. By Lemma 2.1.3(4), we obtain a $\beta < \kappa$ such that $Y \in \mathbf{V}[\mathcal{G}\restriction\beta]$. By the intermediate model property (Proposition 10.10, [Kan03]), we have that $\mathbf{V}[\mathcal{G}\restriction\beta] = \mathbf{V}[Y][\mathcal{H}_0]$ for some separative poset $\mathbb{P} \in \mathbf{V}[Y]$, and $\mathbb{P}$-generic filter $\mathcal{H}_0$ over $\mathbf{V}[Y]$.

Let $\mathbb{Q} := \mathbb{P} \times \mathrm{Col}(\omega, \{\beta\})$. Then $|\mathbb{Q}| \leq |\mathrm{Col}(\omega, \beta + 1)| \leq |\beta|$, and the $\mathrm{Col}(\omega, \{\beta\})$ component in $\mathbb{Q}$ ensures that $\Vdash_{\mathbb{Q}} \exists f [f : \omega \to \beta \text{ is surjective}]$. By Lemma 2.1.7, $\mathbb{Q}$ and $\mathrm{Col}(\omega, \{\beta\})$ are forcing equivalent, so there exists a $\mathrm{Col}(\omega, \{\beta\})$-generic $\mathcal{H}_1$ such that $\mathbf{V}[Y][\mathcal{H}_0][\mathcal{G}\restriction\{\beta\}] = \mathbf{V}[Y][\mathcal{H}_1]$. On the other hand, we can apply Lemma 2.1.7 again to $\mathrm{Col}(\omega, \beta + 1)$ directly to obtain a $\mathcal{H}_2$ that corresponds to this $\mathcal{H}_1$, i.e. $\mathbf{V}[Y][\mathcal{H}_2] = \mathbf{V}[Y][\mathcal{H}_1]$. Thus:

$$\begin{aligned}
\mathbf{V}[\mathcal{G}] &= \mathbf{V}[\mathcal{G}\restriction\beta][\mathcal{G}\restriction\{\beta\}][\mathcal{G}\restriction[\beta + 1, \kappa)] \\
&= \mathbf{V}[Y][\mathcal{H}_0][\mathcal{G}\restriction\{\beta\}][\mathcal{G}\restriction[\beta + 1, \kappa)] \\
&= \mathbf{V}[Y][\mathcal{H}_1][\mathcal{G}\restriction[\beta + 1, \kappa)]
\end{aligned}$$

Now let $\mathcal{G}^* = \mathcal{H}_1 \times \mathcal{G}\restriction[\beta + 1, \kappa)$, and the theorem follows. □

2.2 Borel Sets and Reals

Let $\mathcal{N}$ denote the Baire space ${}^\omega\omega$. Every Borel set of reals can be obtained by taking a countable sequence of intervals, countable unions and complements. This section illustrates how one can code this sequence with a real $c \in \mathcal{N}$, so there is a correspondence between a code c and its associated Borel set A_c.

2.2.1 Borel Codes

Let $\{I_k : k \in \{1, 2, \dots\}\}$ be a recursive enumeration of open intervals with rational endpoints (i.e. the sequence of the pairs of endpoints is recursive). Such an enumeration is possible as there exists an explicit bijective pairing function $\Gamma : \omega \times \omega \to \omega$ (e.g. the Cantor pairing function).

Notation 2.2.1. For each $c \in \mathcal{N}$, define two functions $u, v_i : \mathcal{N} \to \mathcal{N}$ (where $i \in \omega$) as follows:

- For $n \in \omega$, $u(c)(n) := c(n+1)$.
- For $n \in \omega$, $v_i(c)(n) := c(\Gamma(i,n)+1)$.

It's easy to see that u, v_i are all recursive functions.

Definition 2.2.2. For $0 < \alpha < \omega_1$, we define sets $\varSigma_n, \varPi_n \subseteq \mathcal{N}$ recursively as follows:

(1) $\varSigma_1 := \{c \in \mathcal{N} : c(0) > 1\}$.

(2) $\varPi_\alpha := \left(\bigcup_{\beta<\alpha} \varSigma_\beta \cup \varPi_\beta\right) \cup \varPi^*_\alpha$, where:

$$\varPi^*_\alpha := \{c \in \mathcal{N} : c(0) = 0 \wedge u(c) \in \varSigma_\alpha\}$$

(3) $\varSigma_\alpha := \left(\bigcup_{\beta<\alpha} \varSigma_\beta \cup \varPi_\beta\right) \cup \varSigma^*_\alpha$, where:

$$\varSigma^*_\alpha := \left\{c \in \mathcal{N} : c(0) = 1 \wedge \forall i \left[v_i(c) \in \bigcup_{\beta<\alpha} \varSigma_\beta \cup \varPi_\beta\right]\right\}$$

If $c \in \varSigma_\alpha$ (if $c \in \varPi_\alpha$), we call c a $\mathbf{\Sigma}^0_\alpha$**-code** (a $\mathbf{\Pi}^0_\alpha$**-code**).

Definition 2.2.3. For every $c \in \mathbf{BC}$, we define a Borel set A_c as follows (and say that c **codes** A_c):

$$A_c := \begin{cases} \bigcup_{c(n)=1} I_n, & \text{if } c \in \varSigma_1 \\ \mathbb{R} \setminus A_{u(c)}, & \text{if } c \in \varPi_\alpha \text{ and } c(0) = 0 \\ \bigcup_{i=0}^{\infty} A_{v_i(c)}, & \text{if } c \in \varSigma_\alpha \text{ and } c(0) = 1 \end{cases}$$

Definition 2.2.4. Define the **lightface hierarchy** as follows:

(1) A set $A \subseteq \mathcal{N}$ is Σ^1_1 if there exists a recursive set $R \subseteq \bigcup_{n=0}^{\infty}(\omega^n \times \omega^n)$ such that for all $x \in \mathcal{N}$:

$$x \in A \iff \exists y \in \mathcal{N} \forall n \in \omega R(x\restriction n, y\restriction n)$$

(2) $A \subseteq \mathcal{N}$ if Π^1_n (in a) if the complement of A is in Σ^1_n (in a).

(3) $A \subseteq \mathcal{N}$ is Σ^1_{n+1} (in a) if it is the projection of a Π^1_n (in a) subset of $\mathcal{N}^2$.

(4) $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$

Definition 2.2.5. Define the **Borel (boldface) hierarchy** as follows:

- $\mathbf{\Sigma}^0_1$ is the collection of all open subsets of $\mathcal{N}$.
- $\mathbf{\Pi}^0_1$ is the collection of all closed subsets of $\mathcal{N}$.
- $\mathbf{\Sigma}^0_\alpha$ is the collection of all sets $A = \bigcup_{i=0}^\infty A_i$ such that $A_i \in \mathbf{\Pi}^0_\beta$ for some $\beta < \alpha$.
- $\mathbf{\Pi}^0_\alpha$ is the collection of all complements of sets in $\mathbf{\Sigma}^0_\alpha$.

Note that if $\alpha < \beta$, then $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta \cap \mathbf{\Sigma}^0_\beta$ (Page 140, [Jec03]).

The following lemma should provide the intuition for this definition, as it gives a nice correspondence between the sets $\mathit{\Sigma}_\alpha$ ($\mathit{\Pi}_\alpha$) and the Borel sets $\mathbf{\Sigma}^0_\alpha$ ($\mathbf{\Pi}^0_\alpha$).

Lemma 2.2.6. *Let $\alpha \in \omega_1 \setminus \{0\}$.*

(1) For every $c \in \mathit{\Sigma}_\alpha$ ($c \in \mathit{\Pi}_\alpha$), we have that $A_c \in \mathbf{\Sigma}^0_\alpha$ ($A_c \in \mathbf{\Pi}^0_\alpha$).

(2) For every $B \in \mathbf{\Sigma}^0_\alpha$ ($B \in \mathbf{\Pi}^0_\alpha$), there exists a $c \in \mathit{\Sigma}_\alpha$ ($c \in \mathit{\Pi}_\alpha$) such that $B = A_c$.

Proof. For both directions of α, We induct on $\alpha > 0$.

(1) For $\alpha = 1$, clearly $A_c = \bigcup_{c(n)=1} I_n$ is open if $c \in \mathit{\Sigma}_1$, so $A_c \in \mathbf{\Sigma}^0_1$. If $c \in \mathit{\Pi}_1$, then $u(c) \in \mathit{\Sigma}_1$, so $A_{u(c)}$ is open. Hence $A_c = \mathbb{R} \setminus A_{u(c)}$ is closed, so once again $A_c \in \mathbf{\Pi}^0_1$.

For $\alpha > 1$, let $c \in \mathit{\Sigma}_\alpha$. If $c \in \bigcup_{\beta<\alpha}(\mathit{\Sigma}_\beta \cup \mathit{\Pi}_\beta)$, then $A_c \in \bigcup_{\beta<\alpha}(\mathbf{\Sigma}^0_\beta \cup \mathbf{\Pi}^0_\beta) \subseteq \mathbf{\Sigma}^0_\alpha$. Otherwise, $v_i(c) \in \bigcup_{\beta<\alpha}(\mathit{\Sigma}_\beta \cup \mathit{\Pi}_\beta)$ for each i, so $A_{v_i(c)} \in \mathbf{\Sigma}^0_\alpha$. Therefore, $A_c = \bigcup_{i=0}^\infty A_{v_i(c)} \in \mathbf{\Sigma}^0_\alpha$ as well.

For the case $c \in \mathit{\Pi}_\alpha$, we can repeat the same argument for $\alpha = 1$ to obtain $A_c \in \mathbf{\Pi}^0_\alpha$.

(2) For $\alpha = 1$, first suppose $B \in \mathbf{\Sigma}^0_1$ is open. Then clearly $B = \bigcup_{c(n)=1} I_n$ for some suitable sequence $c \in \mathcal{N}$ with $c \in \mathit{\Sigma}_1$ (simply set $c(0) > 1$). If $B \in \mathbf{\Pi}^0_1$ is closed, then suppose $\mathbb{R} \setminus B = \bigcup\{I(n) : c(n) = 1\}$ with $c \in \mathit{\Sigma}_1$. Define $c' \in \mathcal{N}$ by:

$$c'(n) := \begin{cases} 0, & \text{if } n = 0 \\ c(n-1), & \text{if } n > 0 \end{cases}$$

Then clearly $c'(0)$ and $u(c') = c \in \mathit{\Sigma}_1$, so $A_{c'} = \mathbb{R} \setminus A_{u(c)} = B$, and $c' \in \mathit{\Pi}_1$.

For $\alpha > 1$, first suppose $B \in \mathbf{\Sigma}^0_\alpha$. Then $B = \bigcup_{i=0}^\infty B_i$, where $B_i \in \mathbf{\Pi}^0_{\beta_i}$ for some $\beta_i < \alpha$. By induction hypothesis, there exists $c_i \in \mathit{\Pi}_{\beta_i}$ such that $B_i = A_{c_i}$ for all i. Now define a new $c \in \mathcal{N}$ as follows:

$$c(n) := \begin{cases} 1, & \text{if } n = 0 \\ c_i(m), & \text{if } n = \Gamma(i,m) + 1 \\ 0, & \text{otherwise} \end{cases}$$

Clearly $c(0) = 1$ and $v_i(c) \in \bigcup_{\beta<\alpha}(\mathit{\Sigma}_\beta \cup \mathit{\Pi}_\alpha)$, so $c \in \mathit{\Sigma}_\alpha$. Then $B_i = A_{v_i(c)}$, so $B = \bigcup_{i=0}^\infty A_{v_i(c)} = A_c$, as desired.

For $\alpha > 1$ and $B \in \mathbf{\Pi}^0_\alpha$, the argument is the same as that of $\alpha = 1$.

□

Thus, $\{A_c : c \in \mathbf{BC}\}$ is the collection of all Borel sets.

Lemma 2.2.7. *The set* $\mathbf{BC}$ *of all Borel codes is* Π^1_1.

Proof. Define a relation E on $\mathcal{N}$ as follows:

$$x\,E\,y \iff (y(0) = 0 \wedge x = u(y)) \vee (y(0) = 1 \wedge \exists i[x = v_i(y)]) \tag{2.2.1}$$

Since u and v_i are recursive, E is easily seen to be Σ_1. Now we note that:

(1) If $y \in \mathit{\Sigma}_1$, then y is clearly E-minimal (i.e. $\nexists x[x\,E\,y]$), as $y(0)$ is neither 0 or 1. The converse also holds.

(2) If $y \in \mathit{\Pi}_\alpha$ and $x\,E\,y$, then $x \in \mathit{\Sigma}_\alpha$ by definition of $\mathit{\Pi}_\alpha$.

(3) If $y \in \mathit{\Sigma}_\alpha$ $(\alpha > 1)$ and $x\,E\,y$, then by definition again we have $x \in \bigcup_{\beta<\alpha}(\mathit{\Sigma}_\beta \cup \mathit{\Pi}_\beta)$.

Claim. $y \in \mathbf{BC}$ *iff* E *is well-founded below* y.

This implies that $\mathbf{BC}$ is Π^1_1, as we have:

$$y \in \mathbf{BC} \iff \neg\,(\exists z_0, z_1, \dots \in \mathcal{N}\,\forall n[z_0 = y \wedge z_{n+1}\,E\,z_n])$$

Proof of Claim. $\underline{\implies:}$ If $y \in \mathbf{BC}$, then $y \in \mathit{\Sigma}_\alpha \cup \mathit{\Pi}_\alpha$ for some $\alpha < \omega_1$. If $\cdots E\,z_1\,E\,z_0\,E\,y$, then we have $z_n \in \mathit{\Sigma}_{\alpha_n} \cup \mathit{\Pi}_{\alpha_n}$. This implies that the sequence $\langle \alpha_n : n < \omega \rangle$ is strictly decreasing, a contradiction.

$\underline{\impliedby:}$ If E is well-founded below y, we may define a rank function for E on $\mathrm{ext}_E(y)$. Since E-minimal elements are in $\mathit{\Sigma}^1_1$ (as explained above), we see by induction that every $x \in \mathrm{ext}_E(y)$ is a Borel code. Hence, y itself is a Borel code. ■

□

Lemma 2.2.8. *The properties $A_c \subseteq A_d$, $A_c = A_d$, $A_c = \emptyset$ are Π^1_1 properties of Borel codes.*

Proof. This lemma follows from the following claim:

Claim. *There exist properties $P, Q \subseteq \mathbb{R} \times \mathcal{N}$ such that:*

- *P is Π^1_1.*
- *Q is Σ^1_1.*
- *For every $c \in \mathbf{BC}$:*

$$a \in A_c \iff (a, c) \in P \iff (a, d) \in Q$$

Then we have that:

$$\begin{aligned} A_c \subseteq A_d &\iff c, d \in \mathbf{BC} \wedge \forall a[(a, c) \in Q \to (a, d) \in P] \\ A_c = A_d &\iff c, d \in \mathbf{BC} \wedge A_c \subseteq A_d \wedge A_d \subseteq A_c \\ A_c = \emptyset &\iff c \in \mathbf{BC} \wedge \forall a[(a, c) \notin Q] \end{aligned}$$

Proof. Let E be the relation defined in (2.2.1). Let T be the E-transitive closure of $\text{ext}_E(x) \cup \{x\}$. That is, T is the smallest set which:

$$x \in T \wedge (y \in T \wedge z \, E \, y \to z \in T) \tag{TC}$$

Then T is countable, as each node branches out at most ω many times, but it has finite height (as Borel sets are constructed from open/closed sets via finite number of operations).

Fix $a \in \mathbb{R}$, and let $h_a : T \to \{0, 1\}$ be a function defined as follows - for $y \in T$, we have:

- If $y(0) > 1$, then $h_a(y) = 1$ iff for some n, $y(n) = 1$ and $a \in I_n$.
- If $y(0) = 0$, then $h_a(y) = 1$ iff $h_a(u(y)) = 0$.
- If $y(0) = 1$, then $h_a(y) = 1$ iff for some i, $h_a(v_i(y)) = 1$.

We say that h_a satisfies the "h-property".

Now there is a unique smallest countable set $T \subseteq \mathcal{N}$ which has the property (TC), and a unique function h (the uniqueness can be proved via induction) with the above property such that:

$$h_a(y) = 1 \iff a \in A_y$$

Using the existence and uniqueness properties, we see that:

$$\begin{aligned}
(a, x) \in P &\iff \forall T[T \subseteq \mathcal{N} \wedge |T| = \aleph_0 \wedge \forall h : T \to \{0,1\}[(\text{TC}) \wedge (h\text{-property}) \to h(x) = 1]] \\
(a, x) \in Q &\iff \exists T[T \subseteq \mathcal{N} \wedge |T| = \aleph_0 \wedge \exists h : T \to \{0,1\}[(\text{TC}) \wedge (h\text{-property}) \to h(x) = 1]]
\end{aligned}$$

It's immediately clear that if $c \in \mathbf{BC}$, then:

$$a \in A_c \iff (a, c) \in P \iff (a, c) \in Q$$

It's not difficult to check that P and Q are indeed Π^1_1 and Σ^1_1 respectively. ■

□

With this lemma, we have the following result to conclude the section:

Lemma 2.2.9. *The following properties (of codes) are absolute for all transitive models M of $\mathsf{ZF} + \mathsf{DC}$:*

$$A_e = A_c \cup A_d, \quad A_e = A_c \cap A_d,$$
$$A_e = \mathbb{R} \setminus A_c, \quad A_e = A_c \bigtriangleup A_d, \quad A_e = \bigcup_{n=0}^{\infty} A_{c_n}$$

assuming that the codes c, d, e are in M, and so is the sequence $(c_n)_{n\in\omega}$. We say that the operations above on Borel sets with codes in M are absolute for M.

We note that Mostowski's Absoluteness Theorem (Theorem 25.4, [Jec03]) asserts that Σ^1_1 (and hence Π^1_1) properties of Borel codes are absolute for transitive models of $\mathsf{ZF} + \mathsf{DC}$. Since $\mathbf{BC}$ is Π^1_1, we thus have for models $M \subseteq \mathbf{V}$ of $\mathsf{ZF} + \mathsf{DC}$:

$$c \text{ is a Borel code} \iff M \models c \text{ is a Borel code}$$

Furthermore, since $a \in A_c$ is Π^1_1, and $A_c \subseteq A_d$, $A_c = d$ and $A_c = \emptyset$ are also Π^1_1, we have that they are all absolute, with $A_c^M = A_c \cap M$.

Proof. We first prove for the last operation $\bigcup_{n=0}^{\infty}$.

$A_e = \bigcup_{n=0}^{\infty} A_{c_n}$ is absolute: Let $(c_n)_{n\in\omega}$ be a sequence of Borel codes in M, and let $c \in \mathcal{N}$ such that $c(0) = 1$ and $v_i(c) = c_i$ for all $i \in \omega$ (the explicit construction can be found in the proof of Lemma 2.2.6). Note that clearly c is a Borel code, $c \in M$ (the construction for c can be done completely in M), and c codes the Borel set $\bigcup_{n=0}^{\infty} A_{c_n}$ in both M and $\mathbf{V}$. Thus, we have that for all Borel codes $e \in M$:

$$A_e^M = \bigcup_{n=0}^{\infty} A_{c_n}^M \iff A_e^M = A_c^M \iff A_e = A_c \iff A_e = \bigcup_{n=0}^{\infty} A_{c_n}$$

as $A_e = A_c$ is absolute for M. Thus, $A_e = \bigcup_{n=0}^{\infty} A_{c_n}$ is absolute.

$\mathbb{R} \setminus A_c$ is absolute: Similar to the above argument, we can construct a Borel code d (within M) such that $c = u(d)$. Then using the absoluteness of equality again, we have that $\mathbb{R}\setminus A_c$ is absolute.

$A_e = A_c \cup A_d$, $A_e = A_c \cap A_d$, $A_e = A_c \triangle A_d$ are absolute: Since they can be defined from the above two operations, they are also absolute. $\square$

2.2.2 Random Reals

With Borel codes discussed, we may now proceed to talk about the relationship between Borel sets and random reals.

Let $\mathcal{B}$ be the σ-algebra of non-empty Borel sets of reals, and let μ denote the Lebesgue measure. Define an equivalence relation $\sim$ on $\mathcal{B}$ such that for $A, B \in \mathcal{B}$, we have $A \sim B$ iff $\mu(A \triangle B) = 0$. Then denote $\mathbb{P} := \mathcal{B}/\sim$, along with a partial order $\leq$ on $\mathbb{P}$ defined by:

$$[A] \leq [B] \iff \mu(A \setminus B) = 0$$

Lemma 2.2.10. *Let $\{A_{c_n}\}_{n\in\omega}$ be a countable set of Borel sets. Let $\mathcal{G}$ be a generic filter over $\mathbb{P}$. Then:*

$$[A_{c_n}] \in \mathcal{G} \text{ for some } n \iff \left[\bigcup_{n=0}^{\infty} A_{c_n}\right] \in \mathcal{G}$$

Proof. $\Longrightarrow$ is immediate. For $\Longleftarrow$, denote $A := \bigcup_{n=0}^{\infty} A_{c_n}$. Let:

$$\mathcal{D}_n := \{[B] \in \mathbb{P} : [B] \leq [A_{c_n}]\}$$
$$\mathcal{D}' := \{[B] \in \mathbb{P} : [B] \leq [\mathbb{R} \setminus A]\}$$

It's easy to see that $\mathcal{D} := \mathcal{D}' \cup \bigcup_{n=0}^{\infty} \mathcal{D}_n$ is dense open. Since $\mathcal{G}$ is generic, we have that $\mathcal{G} \cap \mathcal{D} \neq \emptyset$. Now $\mathcal{G} \cap \mathcal{D}' = \emptyset$ as $\mathcal{G}$ is a filter and $[A] \in \mathcal{G}$, so we must have $\mathcal{G} \cap \mathcal{D}_n \neq \emptyset$ for some n. Then $A_{c_n} \in \mathcal{G}$. $\square$

Given a transitive model M of $\mathsf{ZF}+\mathsf{DC}$, let $\mathcal{B}_M$ denote the collection of all Borel sets in M, along with the partial order $\mathbb{P}_M := \mathcal{B}_M/\sim$. We have seen that for all $B \in \mathcal{B}_M$, there exists $B^* \in \mathcal{B}$ such that $B = B^* \cap M$.

Lemma 2.2.11. *(A_c is null)^M iff A_c is null.*

Proof. We note that the statements $c \in \Sigma_1$ and $c \in \Pi_1$ are absolute, as M contains the whole Baire space $\mathcal{N}$ so the construction of the Borel hierarchy is identical for both.

We first prove the following two claims:

Claim. *If c is a $\mathbf{\Sigma}^0_1$-code or $\mathbf{\Pi}^0_1$-code, then $\mu^M(A_c^M) = \mu(A_c)$.*

Proof. Let $(k_n)_{n\in\omega}$ be a sequence of all natural numbers k such that $c(k) = 1$, so $A_c = \bigcup_{n=0}^{\infty} I_{k_n}$, a union of open intervals with rational endpoints. Write:

$$X_n := I_{k_n} \setminus \bigcup_{i=0}^{n-1} I_{k_i}$$

Then $A_c = \bigsqcup_{n=0}^{\infty} X_n$, and $\mu(A_c) = \sum_{n=0}^{\infty} \mu(X_n)$. Now each of X_n is a finite union of intervals, and for each interval I_k of the form (a,b), $(a,b]$, $[a,b)$ or $[a,b]$, we have $I_k^M = I_k$, and:

$$\mu^M(I_k) = \mu(I_k) = b - a$$

Therefore $\mu^M(X_n^M) = \mu(X_n)$ for all n, which gives $\mu^M(A_c^M) = \mu(A_c)$.

A similar argument shows that if $c \in M$ is a $\mathbf{\Pi}^0_1$-code, then $\mu^M(A_c^M) = \mu(A_c)$. ■

We now prove the lemma. If (A_c is null)^M, then M satisfies:

$$\exists e[e \in \Sigma_1 \wedge A_e \supseteq A_c \wedge \forall n[\mu(A_e) \leq 1/n]]$$

Since whatever is inside [...] is upward absolute (by the absoluteness of $\mu(A_e)$ in the first claim), A_c is null in $\mathbf{V}$.

On the other hand, suppose (A_c is not null)^M. Recall that every set of positive measure contains a closed subset of positive measure. Thus, M satisfies:

$$\exists e[e \in \Pi_1 \wedge A_e \subseteq A_c \wedge \mu(A_e) > 0]$$

Again, whatever is inside [...] is upward absolute, so A_c is not null in $\mathbf{V}$. □

Lemma 2.2.12. *If $\mathcal{G}$ is a $\mathbb{P}_M$-generic filter over M, then there is a unique real number $x_{\mathcal{G}}$ such that for all $B \in \mathcal{B}$:*

$$x_{\mathcal{G}} \in \iff [B]_m \in \mathcal{G} \tag{R}$$

*Furthermore, the formula (R) determines $\mathcal{G}$ and hence $M[\mathcal{G}] = M[x_{\mathcal{G}}]$. Such a real is called a **random real**.*

Proof. We wish to find a unique x that satisfy:

$$x \in B^* \iff [B] \in \mathcal{G} \ \ \text{(for all } B \in \mathcal{B}) \tag{G}$$

We call such an x a **generic real**.

Uniqueness of x: If $x < y$ are two generic reals, then pick any rational $r \in (x, y)$, and we have that $(-\infty, r)$ and $[r, \infty)$ are two disjoint Borel sets such that x lies in the first and y lies in the second. Yet, only one of such interval lies in $\mathcal{G}$.

Existence of x: Let:

$$x := \sup\{r \in \mathbb{Q}^M : [(r, \infty)] \in \mathcal{G}\}$$

where here $\mathbb{Q}$ denotes the set of rational numbers, and $\mathbb{Q}^M := \mathbb{Q} \cap M$. Since the set of bounded Borel subsets of $\mathbb{R}$ is dense in $\mathbb{P}_M$, the genericity of $\mathcal{G}$ guarantees us a bounded set in $\mathcal{G}$, so there must be at least one unbounded set not in $\mathcal{G}$. Therefore, the supremum exists (in $\mathbf{V}$).

To finish the proof, we may induct on Borel codes $c \in \mathbf{BC}^M$ and show that for all Borel codes c:

$$x \in A_c^* \iff [A_c] \in \mathcal{G}$$

We note that as $\mathcal{G}$ is a filter, given $[A], [B] \in \mathcal{G}$ we have $[A \cap B] \in \mathcal{G}$.

Step 1.1 - $c \in \Sigma_1 \cap M$, $c(n) = 1$ for exactly one n: Then c codes the interval $I_n = A_c$. Let $I_n = (p, q)$. Then:

$$\begin{aligned} x \in A_c^* &\iff p < x < q \\ &\iff p < \sup\{r : [(r, \infty)] \in \mathcal{G}\} < q \\ &\iff [(p, \infty)] \in \mathcal{G} \wedge [(q, \infty) \notin \mathcal{G}] \\ &\iff [(p, \infty)] \in \mathcal{G} \wedge [\mathbb{R} \setminus (q, \infty) \in \mathcal{G}] \\ &\iff [(p, q)] \in \mathcal{G} \\ &\iff [A_c] \in \mathcal{G} \end{aligned}$$

Step 1.2 - $c \in \Sigma_1 \cap M$, general case: Suppose $A_c = \bigcup_{n=0}^{\infty} I_{k_n}$, where $\{k_n : n \in \omega\} = \{k : c(k) = 1\}$. Then:

$$\begin{aligned} x \in A_c^* &\iff x \in \bigcup_{n=0}^{\infty} I_{k_n}^* \\ &\iff \exists n[x \in I_{k_n}^*] \\ &\iff \exists n[[I_{k_n}] \in \mathcal{G}] && \text{by Step 1.1} \\ &\iff \left[\bigcup_{n=0}^{\infty} I_{k_n}\right] \in \mathcal{G} && \text{by Lemma 2.2.10} \\ &\iff [A_c] \in \mathcal{G} \end{aligned}$$

Step 2 - $c \in \Pi_\alpha \cap M$, $\alpha < \omega_1^M$: Assume that $x \in A_d^* \iff [A_d] \in \mathcal{G}$ for $d \in \Sigma_\alpha \cap M$. By definition, we have $u(c) \in \Sigma_\alpha \cap M$, and $A_c = \mathbb{R} \setminus A_{u(c)}$. Then:

$$x \in A_c^* \iff x \notin A_{u(c)}^* \iff [A_{u(c)}] \notin G \iff [A_c] \in \mathcal{G}$$

Step 3 - $c \in \Sigma_\alpha \cap M$, $1 < \alpha < \omega_1^M$: The proof is similar to that of $c \in \Sigma_1$, in which we write $A_c = \bigcup_{n=0}^{\infty} A_{v_i(c)}$ instead of $\bigcup_{n=0}^{\infty} I_{k_n}$.

The induction is complete. □

Lemma 2.2.13 (Characterisation of Random Reals)**.** *A real number is random over M iff it does not belong to any null Borel sets coded in M.*

Thus, if $\mathcal{R}_M$ the sets of all random reals random over M, we have:

$$\mathcal{R}_M = \mathbb{R}^* \setminus \bigcup\{A_c^* : c \in \mathbf{BC}^M \wedge A_c^* \text{ is null}\}$$

Proof. For $\implies$, we simply let $\mathcal{G}$ be the corresponding generic filter for $x = x_{\mathcal{G}}$ in Lemma 2.2.12, and it's easy to check that it works. For $\impliedby$, also motivated by Lemma 2.2.12, the natural choice of $\mathcal{G}$ would be:

$$\mathcal{G} := \{[A_c] : x \in A_c^*\}$$

We shall show that $\mathcal{G}$ is indeed generic over M. We start by observing that if $[A_c] = [A_d]$, then $A_c \triangle A_d$ is null, so by Lemma 2.2.11 we have that $A_c^* \triangle A_d^*$ is null, so $x \in A_c^*$ iff $x \in A_d^*$.

It's easy to see that $\mathcal{G}$ is a filter. For genericity, recall that it suffices to show that $\mathcal{G}$ intersects every maximal antichain in $\mathbb{P}$. Now since $\mathbb{P}$ is ccc, every maximal antichain is of the form $\{[A_{c_n}] : n < \omega\}$, so $\mathbb{R} \setminus \bigcup_{n=1}^{\infty} A_{c_n}^*$ is null. By hypothesis, $x \notin \mathbb{R} \setminus \bigcup_{n=1}^{\infty} A_{c_n}^*$, so $x \in \bigcup_{n=1}^{\infty} A_{c_n}^*$, hence $x \in A_{c_n}^*$ for some n. This implies that $A_{c_n}^* \in \mathcal{G}$, as desired. □

2.3 Solovay's Theorem on $\mathbf{L}(\mathbb{R})$

With the setup on Borel Sets and codes and the Lévy collapse ready, we're finally ready to show Solovay's theorem.

Theorem 2.3.1 (Solovay)**.** *Let κ be an inaccessible cardinal. Let $\mathcal{G}$ be a filter that is $\mathrm{Col}(\omega, \kappa)$-generic over $\mathbf{V}$. Then $\mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ is a model of $\mathsf{ZF} + \mathsf{DC}$ in which all subsets of $\mathbb{R}$ are Lebesgue measurable.*

We shall abbreviate $\mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ with $\mathbf{L}(\mathbb{R})$.

Proof. We will split the proof into two components: The first component proves that $\mathbf{L}(\mathbb{R}) \models \mathsf{DC}$, while the second proves that all subsets of $\mathbb{R}$ are Lebesgue measurable.

Step 1 - $\mathbf{L}(\mathbb{R})$ satisfies DC: We begin with a lemma.

Lemma 2.3.2. *There exists a surjection $F : \mathbf{ORD} \times \mathbb{R} \to \mathbf{L}(\mathbb{R})$.*

Proof. We first note that for all $X \in \mathbf{L}(\mathbb{R})$, there exists an ordinal α such that:

$$X = \{x \in L_\alpha(\mathbb{R}) : L_\alpha(\mathbb{R}) \models \phi(x, \vec{v})\}$$

for some formula ϕ and parameters $\vec{v} = (v_1, \ldots, v_n)$, where $v_i \in L_\alpha(\mathbb{R})$ for all i. For each v_i, we once again have that for some $\beta < \alpha$:

$$v_i = \{x \in L_\beta(\mathbb{R}) : L_\beta(\mathbb{R}) \models \psi(x, \vec{u})\}$$

for some formula ψ and parameters $\vec{u}$ in $L_\beta(\mathbb{R})$. We may repeat this until it terminates (as $\mathbf{ORD}$ is well-ordered), and we have that each $X \in \mathbf{L}(\mathbb{R})$ can be expressed in terms of a finite set of ordinals, reals and formulas. The required surjection F can thus be done by coding such finite sets using an ordinal and a real. ■

Let $R \in \mathbf{L}(\mathbb{R})$ be a relation on a set A satisfying such that for all $x \in A$, there exists $y \in A$ such that $x\,R\,y$. Working in $\mathbf{V}[\mathcal{G}]$, we define $f : \omega \to \mathbf{L}(\mathbb{R})$, $(x_n)_{n\in\omega}$ and $(\alpha_n)_{n\in\omega}$ as follows:

- Fix any $f(0) \in A$. Let:

  $$\alpha_0 := \min\{\alpha \in \mathbf{ORD} : \exists x \in \mathbb{R}[F(\alpha, x) = f(0)]\}$$

  and let $x_0 \in \mathbb{R}$ such that $F(\alpha_0, x_0) = f(0)$.

- Let:

  $$\alpha_{n+1} := \min\{\alpha \in \mathbf{ORD} : \exists x \in \mathbb{R}[f(n)\, R\, F(\alpha, x)]\}$$

  and let x_{n+1} such that $f(n)\, R\, F(\alpha_{n+1}, x_{n+1})$. Define $f(n+1) := F(\alpha_{n+1}, x_{n+1})$.

We see that f is definable from A, $(\alpha_n)_{n\in\omega}$ and $(x_n)_{n\in\omega}$. $(x_n)_{n\in\omega}$ can be coded as a real, and since $A \in \mathbf{L}(\mathbb{R})$, it is also real-definable. Finally, the construction of f actually does not require $(\alpha_n)_{n\in\omega}$, as we may construction F without explicitly building the sequence $(\alpha_n)_{n\in\omega}$. Thus, $f \in \mathbf{L}(\mathbb{R})$, and $(f(n))_{n\in\omega}$ is the choice sequence we need.

Step 2 - All subsets of $\mathbb{R}$ in $\mathbf{L}(\mathbb{R})$ are Lebesgue measurable: We first introduce a definition.

Definition 2.3.3. Let $S \in M[\mathcal{G}]$ be a subset of $\mathbb{R}$, where M is a transitive model of ZFC. Then S is **Solovay** over M if there is a formula $\varphi(x)$, with parameters in M, such that for all reals x:

$$x \in S \iff M[x] \models \varphi(x)$$

Proposition 2.3.4. *Let $r \in \mathbf{V}[\mathcal{G}] \cap \mathbb{R}$.*

(1) Let $X \in \mathbf{L}(\mathbb{R})$ be a subset of $\mathbb{R}$ that is definable from r. Then X is Solovay over $\mathbf{V}[r]$.

(2) Let S be a Solovay set of reals over $\mathbf{V}[r]$. Then there exists a Borel sets A such that $S \cap \mathcal{R}_{\mathbf{V}[r]} = A \cap \mathcal{R}_{\mathbf{V}[r]}$.

(3) The set of all reals that are not random over $\mathbf{V}[r]$ is null.

Thus, the X in (1) is in fact Lebesgue measurable.

Proof.

(1) We first prove that:

Claim. *Given a formula φ, there is a formula $\tilde{\varphi}$ such that for every $x \in \mathbf{V}[\mathcal{G}] \cap \mathbb{R}$:*

$$\mathbf{V}[\mathcal{G}] \models \varphi(x) \iff \mathbf{V}[x] \models \tilde{\varphi}(x)$$

Proof. By the definability of the forcing in the ground model, we may let:

$$\tilde{\varphi}(x) \equiv \ulcorner \Vdash_{\mathrm{Col}(\omega,\kappa)} \varphi(\hat{x}) \urcorner$$

Fix $x \in \mathbf{V}[\mathcal{G}] \cap \mathbb{R}$. The Factor Lemma provides us a $\mathbb{P}$-generic filter $\mathcal{G}^*$ over $\mathbf{V}[x]$ such that $\mathbf{V}[\mathcal{G}] = \mathbf{V}[x][\mathcal{G}^*]$. Then, by the weak homogeneity of the Lévy collapse:

$$\mathbf{V}[\mathcal{G}] \models \varphi(x) \iff \mathbf{V}[x][\mathcal{G}^*] \models \varphi(x) \iff \mathbf{V}[x] \models \tilde{\varphi}(x)$$

■

We may apply the argument similarly for formulas $\varphi(x, y)$ with two variables to get:

$$\mathbf{V}[\mathcal{G}] \models \varphi(x, y) \iff \mathbf{V}[x, y] \models \tilde{\varphi}(x, y)$$

Thus, if $X = \{x : \varphi(x, r)\} \in \mathbf{V}[\mathcal{G}]$, then:

$$x \in X \iff \mathbf{V}[\mathcal{G}] \models \varphi(x, r) \iff \mathbf{V}[r][x] \models \tilde{\varphi}(x, r)$$

so X is Solovay over $\mathbf{V}[r]$.

(2) Recall that $\mathbb{P} = \mathcal{B}/\sim$. Let φ be the formula such that $X = \{x : \mathbf{V}[x] \models \varphi(x)\}$. Let:

$$\mathcal{A} := \{[B] \in \mathbb{P} : [B] \Vdash \varphi(\hat{x})\}$$

Let $A := \bigcup_{[B]\in\mathcal{A}} B$. Since $\mathbb{P}$ is ccc, we have that $[A] \in \mathbb{P}$. We write the random real which corresponds to the generic filter $\mathcal{G}$ as $x_{\mathcal{G}}$. Then:

$$\begin{aligned} x_{\mathcal{G}} \in S &\iff \mathbf{V}[x] \models \varphi(x) \\ &\iff \mathbf{V}[\mathcal{G}] \models \varphi(x) \\ &\iff \exists [A_c] \in \mathcal{G}[[A_c] \Vdash \varphi(\hat{x})] \\ &\iff [A] \in \mathcal{G} \\ &\iff x_{\mathcal{G}} \in A \end{aligned}$$

where the last equivalence follows from Lemma 2.2.13.

(3) By the remark proceeding Lemma 2.2.13, we have:

$$\mathcal{R}_{\mathbf{V}[r]} = \mathbb{R}^* \setminus \bigcup \{A_c^* : c \in \mathbf{BC}^{\mathbf{V}[r]} \wedge A_c^* \text{ is null}\}$$

Recall from Lemma 2.1.3(4) that there exists a $\delta < \kappa$ such that $r \in \mathbf{V}[\mathcal{G} \restriction \delta]$, and this δ can be chosen such that $\delta > 2^{\aleph_0}$. Clearly $|\mathbb{P}| < \kappa$, so κ remains inaccessible in $\mathbf{V}[r]$ (Lemma 10.14, [Kan03]). Thus, $\mathbf{V}[\mathcal{G}]$ thinks that ${}^{\omega}\omega \cap \mathbf{V}[r]$ is countable. This means that $\mathbb{R}^* \setminus \mathcal{R}_{\mathbf{V}[r]}$ is a countable union of null sets, which remains null.

Thus for the set X in (1), if A is the Borel set as in (2), then we have $X \triangle A \subseteq \mathbb{R} \setminus \mathcal{R}_M$, which is null by (3). X is then a null set away from a Borel set, so it is hence Lebesgue measurable. $\blacksquare$

Since statements such as "X is Lebesgue measurable" and "X is null" are absolute, $\mathbf{L}(\mathbb{R}) \models$ All subsets of $\mathbb{R}$ are Lebesgue measurable. $\square$

We remark that $\mathbf{L}(\mathbb{R})$ also satisfies:

- Every subset of $\mathbb{R}$ has the property of Baire. The proof for this can be done by replacing $\mathbb{P}$ with $\mathcal{B}/\sim^*$ the *Cohen algebra*, i.e. we have:

$$A \sim^* B \iff A \triangle B \text{ is meagre}$$

  The corresponding generic real for the Cohen algebra is called a **Cohen real**. The steps to prove that all subsets have the property of Baire is almost identical to that of all subsets are null (except for the proof that the statement "A_c is meagre" is absolute, but it is not difficult).

- Every subset of $\mathbb{R}$ has the perfect set property, i.e. every set is either countable or contains a non-empty perfect set.

A natural question to ask at this point is if the hypothesis of the existence of an inaccessible cardinal is necessary for the statement "All subsets of $\mathbb{R}$ is Lebesgue measurable". This was answered positively by Shelah in 1984 (see [She84]).

3 | Topological Ramsey Spaces

We provide an exposition of topological Ramsey spaces, along with the well-known example $\mathbf{FIN}_k^{[\infty]}$. The main reference is Chapter 4 and 5 of [Tod10], and we shall borrow all the notations there.

3.1 Axioms of Topological Ramsey Spaces

Intuitively, a topological Ramsey space is a space of infinite sequences satisfying certain axioms. They come in a form of a triple $(\mathcal{R}, \leq, r)$, where $\leq$ is a reflexive and transitive relation in $\mathcal{R}$, and r behaves like a map that restricts infinite sequences to finite sequences.

More precisely, given a set $\mathcal{R}$, the map r is a map:

$$r : \mathcal{R} \times \omega \to \mathcal{AR}$$

In which we define, for $A \in \mathcal{R}$ and $n \in \omega$:

$$r_n(A) := r(A, n)$$

Intuitively one may think of the map r_n as restricting an infinite sequence A to its first n terms, and $\mathcal{AR}$ is the space of all finite approximations of infinite sequences in $\mathcal{R}$. We define $\mathcal{AR}_n := \operatorname{ran}(r_n)$, which one may think of as the space of sequences of length n.

The canonical example of a topological Ramsey space is $(\mathcal{R}, \leq, r) = ([\omega]^\omega, \subseteq, r)$, where $r : [\omega]^\omega \times \omega \to [\omega]^{<\omega}$ is defined as $r(X, n) =$ the set of n smallest elements of X. Thus, we may introduce the four axioms of a topological Ramsey space with this example in mind.

Axiom A1 (Sequencing). For any $A, B \in \mathcal{R}$:

(1) $r_0(A) = r_0(B)$.

(2) If $A \neq B$, then $r_n(A) \neq r_n(B)$ for some n.

(3) If $r_n(A) = r_m(B)$, then $n = m$ and $r_k(A) = r_k(B)$ for all $k \leq m$.

This axiom allows us to identify elements in $\mathcal{R}$ as sequences of finite approximations. In other words, for $A \in \mathcal{R}$ the map:

$$A \mapsto (r_n(A))_{n\in\omega}$$

is a bijection between $\mathcal{R}$ and $\prod_{n\in\omega} \mathcal{AR}_n$. Similarly, we may identify finite sequences $a \in \mathcal{AR}$ as $(r_n(A))_{n<m}$ for some $m \in \omega$. Axiom **A1**(3) asserts that this m is unique, and we call it the **length** of a (written as $\mathrm{lh}(a)$). Furthermore, given $a, b \in \mathcal{R}$, we say that b is an **end-extension** of a if there exists an $A \in \mathcal{R}$ such that $a = r_n(A)$ and $b = r_m(A)$, and $n \leq m$ (written $a \sqsubseteq b$).

Axiom A2 (Finitisation). There exists a reflexive and transitive order $\leq_{\mathrm{fin}}$ on $\mathcal{AR}$ such that:

(1) For all $a \in \mathcal{AR}$, the set $\{b : b \leq_{\mathrm{fin}} a\}$ is finite.

(2) For $A, B \in \mathcal{R}$, we have:

$$A \leq B \iff \forall n \exists m [r_n(A) \leq_{\mathrm{fin}} r_m(B)]$$

(3) $\forall a, b \in \mathcal{AR}[a \sqsubseteq b \wedge b \leq_{\mathrm{fin}} c \to \exists d \sqsubseteq c[a \leq_{\mathrm{fin}} d]]$.

In the example of $([\omega]^\omega, \subseteq, r)$, we have $\leq_{\mathrm{fin}} = \leq = \subseteq$. Axiom **A1** and **A2** allow us to define the **basic sets**: Given $a \in \mathcal{AR}$, $A \in \mathcal{R}$ and $n \in \omega$, define:

$$[a] := \{B \in \mathcal{R} : r_{\mathrm{lh}(a)}(B) = a\}$$
$$[a, A] := \{B \in [a] : B \leq A\}$$
$$[n, A] := [r_n(A), A]$$

The sets of the form $[a, A]$ generates a basis of a topology on $\mathcal{R}$, which we call the **natural topology**, or **Ellentuck topology**. This topology extends the metrisable topology generated by open sets of the form $[a]$ for $a \in \mathcal{AR}$ (the metric here refers to the first difference metric defined on the Tychonoff cube $\mathcal{AR}^\omega$).

Before stating the next axiom, we first introduce a new notation. Given $a \in \mathcal{AR}$ and $A \in \mathcal{R}$, define:

$$\mathrm{depth}_A(a) := \begin{cases} \min\{n : a \leq_{\mathrm{fin}} r_n(A)\}, & \text{if } \exists n[a \leq_{\mathrm{fin}} r_n(A)] \\ \infty, & \text{otherwise} \end{cases}$$

Axiom A3 (Amalgamation). Let $a \in \mathcal{AR}$ and $A, B \in \mathcal{R}$.

(1) If $\mathrm{depth}_B(a) < \infty$, then $[a, A] \neq \emptyset$ for all $A \in [\mathrm{depth}_B(a), B]$.

(2) If $A \leq B$ and $[a, A] \neq \emptyset$, then there exists $A' \in [\mathrm{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

Note that the assertion $\emptyset \neq [a, A']$ in **A3**(2) follows from **A3**(2). It's easy to see that, as an immediate consequence:

Lemma 3.1.1 (Lemma 4.20, [Tod10]). *Suppose* $(\mathcal{R}, \leq, r)$ *satisfies axioms **A1-A3**. If* $a \sqsubseteq b$ *and* $[b, A] \neq \emptyset$*, then* $[a, A] \neq \emptyset$ *and* $\mathrm{depth}_A(a) \leq \mathrm{depth}_Y(b) < \infty$*.*

The final axiom is the most important and least trivial axiom of all, which asserts:

Axiom A4 (Pigeonhole). Let $a \in \mathcal{AR}$ and $A \in \mathcal{R}$. If $\mathrm{depth}_A(a) < \infty$ and $\mathcal{O} \subseteq \mathcal{AR}_{\mathrm{lh}(a)+1}$, then there exists $B \in [\mathrm{depth}_A(a), A]$ such that either $r_{\mathrm{lh}(a)+1}[a, B] \subseteq \mathcal{O}$ or $r_{\mathrm{lh}(a)+1}[a, B] \subseteq \mathcal{O}^c$.

These axioms alone, however, do not define a topological Ramsey space, but with an additional assumption they form a sufficient condition for $(\mathcal{R}, \leq, r)$ to be a topological Ramsey space.

Definition 3.1.2. A subset $\mathcal{X} \subseteq \mathcal{R}$ is **Ramsey** if for every $\emptyset \neq [a, A]$, there exists $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$. If $[a, B] \subseteq \mathcal{X}^c$ always, we say that $\mathcal{X}$ is **Ramsey null**.

Recall that a set $\mathcal{X}$ has the property of Baire (or, simply, is Baire) if $\mathcal{X} = \mathcal{O} \triangle \mathcal{M}$ with respect to a topology, where $\mathcal{O}$ is open with respect to the topology and $\mathcal{M}$ is meagre with respect to the topology.

Definition 3.1.3 (Definition 5.3, [Tod10]). A triple a **topological Ramsey space** if every Baire subset of $\mathcal{R}$ (w.r.t. the Ellentuck topology) is Ramsey, and if every meagre subset of $\mathcal{R}$ is Ramsey null.

Theorem 3.1.4 (Abstract Ellentuck Theorem 5.4, [Tod10]). *Suppose* $(\mathcal{R}, \leq, r)$ *satisfies axioms **A1-A4**, and* $\mathcal{R}$ *is closed when identified as a subset of* $\mathcal{AR}^\omega$ *under the metrisable topology. Then* $(\mathcal{R}, \leq, r)$ *is a topological Ramsey space.*

As a result, we may conclude that $([\omega]^\omega, \subseteq, r)$ is indeed a topological Ramsey space.

3.2 Infinite Block Sequences $\mathbf{FIN}_k^{[\infty]}$

In this section we provide an additional example of a topological Ramsey space, the space of infinite block sequences $(\mathbf{FIN}_k^{[\infty]}, \leq, r)$.

3.2.1 Preliminaries for FIN_k

Definition 3.2.1. The set $\mathbf{FIN}_k$ consists of functions $p : \omega \to \omega$ such that:

(1) $\mathrm{supp}(p) := \{n \in \omega : p(n) \neq 0\}$ is finite.

(2) $\mathrm{ran}(p) \leq k$.

(3) $p(n) = k$ for some $n \in \mathrm{supp}(p)$.

An element of $\mathbf{FIN}_k$ is also called a **block**.

Notation 3.2.2. For $p, q \in \mathbf{FIN}_k$, we write:

$$p < q \iff \max(\mathrm{supp}(p)) < \min(\mathrm{supp}(q))$$

Definition 3.2.3. Given $\alpha \leq \omega$, a **block sequence (of length α)** is a sequence $(p_n)_{n<\alpha}$, with $p_n \in \mathbf{FIN}_k$, such that $p_n < p_{n+1}$ for all $n+1 < \alpha$. We denote $\mathbf{FIN}_k^{[\alpha]}$ to be the set of all infinite block sequences of length α, and $\mathbf{FIN}_k^{[\infty]} := \mathbf{FIN}_k^{[\omega]}$.

We also write $\mathbf{FIN}_k^{[<\infty]} := \bigcup_{n=1}^{\infty} \mathbf{FIN}_k^{[n]}$. We may define the obvious $r : \mathbf{FIN}_k^{[\infty]} \to \mathbf{FIN}_k^{[<\infty]}$ by letting:

$$r_m\left((p_n)_{n<\omega}\right) := (p_n)_{n<m}$$

Blocks may also form a *partial semigroup* under two operations.

Notation 3.2.4. Given $p \in \mathbf{FIN}_i$ and $q \in \mathbf{FIN}_j$, we define a partial binary operation $+$, in which whenever $\mathrm{supp}(p) \cap \mathrm{supp}(q) = \emptyset$, to be:

$$(p+q)(n) := \begin{cases} p(n), & \text{if } n \in \mathrm{supp}(p) \\ q(n), & \text{if } n \in \mathrm{supp}(q) \\ 0, & \text{otherwise} \end{cases}$$

Note that clearly $p + q \in \mathbf{FIN}_{\max\{i,j\}}$.

Definition 3.2.5. We define the **tetris operation** $T : \mathbf{FIN}_k \to \mathbf{FIN}_{k-1}$ by:

$$T(p)(n) := \max\{p(n) - 1, 0\}$$

Thus, given $P = (p_n)_{n<\alpha}$ and $Q = (q_n)_{n<\beta}$, we may define $Q \leq P$ by stipulating that for all n, there exists $i_0 < \cdots < i_m$ and $0 \leq j_0, \ldots, j_m \leq k$ with $\min\{j_i : i < m\} = 0$, such that:

$$q_n = T^{j_0}(p_{i_0}) + \cdots + T^{j_m}(p_{i_m})$$

Note that q_n is well-defined, as $p_{i_0} < \cdots < p_{i_m}$, so their supports are pairwise disjoint. The *partial semigroup generated by* $P \in \mathbf{FIN}_k^{[\infty]}$ we have:

$$\langle P \rangle_k := \{p \in \mathbf{FIN}_k : (p) \leq P\}$$

We also write $\langle s \rangle_k$ similarly for $s \in \mathbf{FIN}_k^{[<\infty]}$. If $s = (s_0, \ldots, s_{n-1})$, we also write:

$$\langle s_0, \ldots, s_{n-1} \rangle_k := \langle s \rangle_k$$

3.2.2 Topological Ramsey Space $\mathbf{FIN}_k^{[\infty]}$

The main theorem of this section is as follows:

Theorem 3.2.6 (Gowers, Theorem 5.21, [Tod10])**.** *$(\mathbf{FIN}_k^{[\infty]}, \leq, r)$ is a topological Ramsey space.*

We note that since Axioms **A1-A3** are trivially satisfied by $\mathbf{FIN}_k^{[\infty]}$, and clearly $\mathbf{FIN}_k^{[\infty]}$ is a closed subset of $\mathbf{FIN}_k^{[<\infty]}$ (under the metrisable topology). Thus, proving Theorem 3.2.6 boils down to proving that $\mathbf{FIN}_k^{[\infty]}$ satisfies Axiom **A4**. It is in fact sufficient proving the following statement:

Theorem 3.2.7 (Gower's $\mathbf{FIN}_k$ Theorem)**.** *Let $r \in \omega$, and let $f : \mathbf{FIN}_k \to r$ be a finite coloring. Then for all $Q \in \mathbf{FIN}_k^{[\infty]}$, there exists $P \leq Q$ such that $f \restriction \langle P \rangle_k$ is constant.*

The proof that this is sufficient follows from Lemma 4.2.2, with $\mathcal{H} = \mathbf{FIN}_k^{[\infty]}$.

Lemma 3.2.8. *It suffices to prove Theorem 3.2.7 without the $P \leq Q$ restriction.*

Proof. For a fixed $Q = (q_n)_{n\in\omega} \in \mathbf{FIN}_k^{[\infty]}$, consider the map $F : \mathbf{FIN}_k \to \langle Q \rangle_k$ defined by:

$$F(p) := \sum_{n \in \mathrm{supp}(p)} T^{k-p(n)}(q_n)$$

We may then extend this F to $\mathbf{FIN}_k^{[\infty]}$, by:

$$F((p_n))_{n\in\omega} := (F(p_n))_{n\in\omega}$$

For $p, q \in \mathbf{FIN}_k$, the function F has the following properties:

- F is bijective.
- $p < q$ iff $F(p) < F(q)$. Thus, $F(P)$ is well-defined for $P \in \mathbf{FIN}_k^{[\infty]}$.
- If $\mathrm{supp}(p) \cap \mathrm{supp}(q) \neq \emptyset$, then $F(p+q) = F(p) + F(q)$.
- $T(F(p)) = F(T(p))$.
- $P \leq R$ iff $F(P) \leq F(R)$.

Thus, for any finite coloring $f : \mathbf{FIN}_k \to r$, consider the coloring $g := f \circ F$. If Theorem 3.2.7 without the $P \leq Q$ restriction holds, we obtain a $P \in \mathbf{FIN}_k^{[\infty]}$ such that $g \restriction \langle P \rangle_k$ is constant. Then $f \restriction \langle F(P) \rangle_k$ is constant, and $F(P) \leq Q$. □

Idempotent Set-Ultrafilters

Recall that a **set-ultrafilter** U (we reserve the term "ultrafilter" for Chapter 4) on a set X is a subset of X such that:

- $X \in \mathsf{U}$, $\emptyset \notin \mathsf{U}$.
- If $Y \in \mathsf{U}$ and $Y \subseteq Z$, then $Z \in \mathsf{U}$.
- If $Y, Z \in \mathsf{U}$ then $Y \cap Z \in \mathsf{U}$.

A set-ultrafilter is **cofinite** if it contains all cofinite subsets of X.

Let βX denote the set of all set-ultrafilters of X, and let $\gamma X \subseteq \beta X$ denote the set of all cofinite set-ultrafilters. We may equip βX with the topology generated by the basic open sets of the form:

$$\overline{Y} := \{\mathsf{U} \in \beta X : Y \in \mathsf{U}\}$$

where Y iterates across all subsets of X. Under this topology, βX is the Stone-Čech compactification of X, so in particular it is compact.

Since a set-ultrafilter is cofinite iff it is non-principal, γX is a closed subspace of βX. This is because for any principal set-ultrafilter U, we have $\{x\} \in \mathsf{U}$ for some $x \in X$, so $\overline{\{x\}} = \{\mathsf{U}\} \subseteq \beta X \setminus \gamma X$. Thus, under the subspace topology, we have that γX is compact.

We now work with $\gamma\mathbf{FIN}_k$. We note that the partial operation $+$ in $\mathbf{FIN}_k$ (Notation 3.2.4) may be extended to $\gamma\mathbf{FIN}_k$ by stipulating that, for $\mathsf{U}, \mathsf{V} \in \gamma\mathbf{FIN}_k$:

$$X \in \mathsf{U} + \mathsf{V} \iff \mathsf{U} p \mathsf{V} q [p + q \in X]$$

where $\mathsf{U}x\,\phi(x)$ is an abbreviation for $\exists X \in \mathsf{U}\,\forall x \in X\,\phi(x)$ for all formulas ϕ. We may also extend the tetris operation (Definition 3.2.5) by stipulating that:

$$X \in T(\mathsf{U}) \iff T^{-1}[X] \in \mathsf{U}$$

I.e. $T(\mathsf{U})$ is the **Rudin-Keisler image** of U under T. It's easy to check that $T : \gamma\mathbf{FIN}_k \to \gamma\mathbf{FIN}_{k-1}$ is a continuous onto homomorphism of semigroups, so in particular it preserves $+$ (i.e. $T(\mathsf{U}+\mathsf{V}) = T(\mathsf{U}) + T(\mathsf{V})$.

Lemma 3.2.9 (Ellis)**.** *Every compact semigroup has an idempotent element.*

This proof is a short application of Zorn's Lemma (see Lemma 2.1, [Tod10]). Note that it was discovered in a recent work by Mauro-Eleftherios [NT17] that the full AC is not required here - the Ultrafilter Lemma would suffice.

In particular, we have that there exists an idempotent set-ultrafilter on any $\mathbf{FIN}_k$.

3.2.3 Proof of Gower's $\mathbf{FIN}_k$ Theorem

Lemma 3.2.10. *For $k > 0$, there exist idempotent set-ultrafilters U_k on $\mathbf{FIN}_k$ such that for $i < j$, we have:*

(1) $\mathsf{U}_i + \mathsf{U}_j = \mathsf{U}_j + \mathsf{U}_i = \mathsf{U}_j$.

(2) $\mathsf{U}_i = T^{(j-i)}(\mathsf{U}_j)$.

Proof. We define U_k inductively. Let U_1 be any idempotent set-ultrafilter on $\mathbf{FIN}_1$. Now suppose U_i has been defined for $1 \leq i < k$. Consider the set:

$$T^{-1}(\mathsf{U}_{k-1}) + \mathsf{U}_{k-1} := \{\mathsf{V} + \mathsf{U}_{k-1} : \mathsf{V} \in T^{-1}(\mathsf{U}_{k-1})\}$$

Since T is continuous, $T^{-1}(\mathsf{U}_{k-1}) \neq \emptyset$ is closed, and thus so is $T^{-1}(\mathsf{U}_{k-1}) + \mathsf{U}_{k-1}$. Furthermore, observe that if $\mathsf{V}, \mathsf{W} \in T^{-1}(\mathsf{U}_{k-1})$ we have:

$$\begin{aligned} T(\mathsf{V} + \mathsf{U}_{k-1} + \mathsf{W}) &= T(\mathsf{V}) + T(\mathsf{U}_{k-1}) + T(\mathsf{W}) \\ &= \mathsf{U}_{k-1} + \mathsf{U}_{k-2} + \mathsf{U}_{k-1} \\ &= \mathsf{U}_{k-1} + \mathsf{U}_{k-1} && \text{by (1) of induction hypothesis} \\ &= \mathsf{U}_{k-1} && \text{by idempotence of } \mathsf{U}_{k-1} \end{aligned}$$

(if $k = 2$, we take U_{k-2} as the identity 0) Thus, $T^{-1}(\mathsf{U}_{k-1}) + \mathsf{U}_{k-1}$ is closed under addition and forms a compact subsemigroup of $\gamma\mathbf{FIN}_k$. By Ellis Lemma 3.2.9, there exists an idempotent $\mathsf{W} = \mathsf{V} + \mathsf{U}_{k-1} \in T^{-1}(\mathsf{U}_{k-1}) + \mathsf{U}_{k-1}$. Let:

$$\mathsf{U}_k := \mathsf{U}_{k-1} + \mathsf{V} + \mathsf{U}_{k-1}$$

Then clearly $\mathsf{U}_k + \mathsf{U}_{k-1} = \mathsf{U}_k$ by idempotence of U_{k-1}, and:

$$\begin{aligned}\mathsf{U}_k + \mathsf{U}_k &= \mathsf{U}_{k-1} + \mathsf{V} + \mathsf{U}_{k-1} + \mathsf{U}_{k-1} + \mathsf{V} + \mathsf{U}_{k-1} \\ &= \mathsf{U}_{k-1} + \mathsf{V} + \mathsf{U}_{k-1} + \mathsf{V} + \mathsf{U}_{k-1} \\ &= \mathsf{U}_{k-1} + \mathsf{V} + \mathsf{U}_{k-1} && \text{as } \mathsf{W} = \mathsf{V} + \mathsf{U}_{k-1} \text{ is idempotent} \\ &= \mathsf{U}_k\end{aligned}$$

which completes the induction. □

We are now ready to prove Gower's $\mathbf{FIN}_k$ theorem.

Proof of Theorem 3.2.7. For each $k > 0$, let U_k be the set-ultrafilter given in the proof of Lemma 3.2.10. Let $f : \mathbf{FIN}_k \to r$ be a finite coloring, so we have that $X := f^{-1}[\{i\}] \in \mathsf{U}_k$ for some $i < r$. We shall inductively construct an infinite block sequence $P = (p_n)_{n\in\omega}$, along with $X_0^l \supseteq X_1^l \supseteq \cdots$, a decreasing sequence of elements of U_l, such that:

(1) $X_0^k = X$.

(2) $p_n \in X_n^k$, and $T^{(k-l)}[X_n^k] = X_n^l$.

(3) For $1 \le i, j \le k$, we have:

$$\mathsf{U}_k p[T^{(k-i)}(p_n) + T^{(k-j)}(p) \in X_n^{\max\{i,j\}}]$$

Base Step: We let $X_0^l := T^{(k-l)}[X]$. We see that since:

$$X_0^j \in \mathsf{U}_j = \mathsf{U}_i + \mathsf{U}_j = T^{(k-i)}(\mathsf{U}_k) + T^{(k-j)}(\mathsf{U}_k)$$

we have that:

$$\mathsf{U}_k p \, \mathsf{U}_k q[T^{(k-i)}(p) + T^{(k-j)}(q) \in X_0^j]$$

Thus, for any $i \ne j$ there exists a $Y_0^{i,j} \in \mathsf{U}_k$ such that for all $q_{i,j} \in Y_0^{i,j}$:

$$\mathsf{U}_k q\left[T^{(k-i)}(q_{i,j}) + T^{(k-j)}(q) \in X_0^{\max\{i,j\}}\right]$$

Now let $Y_0 := X_0^k \cap \bigcap_{1\le i,j\le k, i\ne j} Y_0^{i,j} \in \mathsf{U}_k$. Then we may pick any $p_0 \in Y_0$, and we have that criteria (1)-(3) are all satisfied.

Induction Step: Assume p_i and X_i^l have all been constructed for $i < n$ and $1 \le l \le k$, satisfying (1)-(3). For $1 \le i, j \le k$ and $m < n$, let:

$$Z_m^{i,j} := \left\{p \in \mathbf{FIN}_k : T^{(k-i)}(p_m) + T^{(k-j)}(p) \in X_m^{\max\{i,j\}}\right\}$$

By induction hypothesis with (3), we have that $Z_m^{i,j} \in \mathsf{U}_k$. Now let:

$$X_n^k := X_{n-1} \cap \bigcap_{\substack{1 \leq i,j \leq k \\ m < n}} Z_m^{i,j} X_n^l := T^{(k-l)}[X_n^k]$$

Clearly $X_n^l \in \mathsf{U}_l$ for $1 \leq l \leq k$. By the same reasoning as the base case, we may let $Y_n^{i,j} \in \mathsf{U}_k$ such that for all $q_{i,j} \in Y_n^{i,j}$:

$$\mathsf{U}_k q \left[T^{(k-i)}(q_{i,j}) + T^{(k-j)}(q) \in X_n^{\max\{i,j\}}\right]$$

Now let $Y_n = X_n^k \cap \bigcap_{1 \leq i,j \leq k, i \neq j} Y_n^{i,j} \in \mathsf{U}_k$. Since U_k is cofinite, pick any $p_n \in Y_n$ such that $p_n > p_{n-1}$. Then p_n is chosen such that (1)-(3) of the induction are satisfied. Now let $P := (p_n)_{n \in \omega}$.

Verify that $\langle P \rangle_k \subseteq X$: We finish the proof by showing that P is the desired infinite block sequence. It suffices to show that for $n_0 < \cdots < n_m$, $1 \leq l_0, \ldots, l_m \leq k$ and $q \in X_{n_p}^{l_p}$, we have:

$$r := T^{(k-l_0)}(p_{n_0}) + \cdots + T^{(k-l_{m-1})}(p_{n_{m-1}}) + q \in X_{n_0}^{\max\{l_0,\ldots,l_m\}}$$

This implies the theorem, as if $l = \max\{l_0, \ldots, l_p\}$, then $T^{(k-l)}(r') \in T^{(k-l)}[X]$ for all r' such that $T(r') = r$, and therefore $r' \in X$ (and clearly $T(r')$ will have such a form for all $r' \in \langle P \rangle_k$).

We induct on m. The case $m = 0$ follows immediately from property (3) and the choice of p_{n_0}. For $m > 0$, let:

$$q' := T^{(k-l_1)}(p_{n_1}) + \cdots + T^{(k-l_{m-1})}(p_{n_{m-1}}) + q$$

By inductive hypothesis, we have that $q' \in X_{n_1}^{l'}$, where $l' := \max\{l_1, \ldots, l_p\}$. Let $q^* \in X_{n_1}^k$ such that $q' = T^{(k-l')}(q^*)$. Since $q^* \in X_{n_0+1}^k$, we have that $y^* \in Z_{n_0}^{l_0,l'}$ (as in the induction step), so:

$$T^{(k-l_0)}(p_{n_0}) + T^{(k-l')}(q^*) \in X_{n_0}^l$$

which completes the proof. □

There were recent efforts in eliminating the need of set-ultrafilters in the proof of Gower's $\mathbf{FIN}_k$ theorem. For instance, a proof the finitised version of Theorem 3.2.7 using only Peano Arithmetic PA was discovered by Ojeda-Aristizabal (see [Oje17]).

4 | Diagonalisation Forcing for $\mathbf{FIN}_k^{[\infty]}$

The following is a theorem of Neeman-Norwood:

Theorem 4.0.1 (Theorem 16, [NN16])**.** *Let κ be an inaccessible cardinal. Let $\mathcal{G}$ be $\mathrm{Col}(\omega, \kappa)$-generic over $\mathbf{V}$. If $\mathcal{X} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ is a subset of $[\omega]^\omega$, and $\mathcal{H} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ is a selective coideal, then $\mathcal{X}$ is $\mathcal{H}$-Ramsey.*

In this chapter we shall show that, using the abstract notions of coideals introduced in [Mij07], one may prove an analogous result for $\mathbf{FIN}_k^{[\infty]}$, assuming some combinatorial hypotheses.

4.1 Coideals and Ultrafilters

Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space.

Notation 4.1.1. Fix $A \in \mathcal{R}$ and $a \in \mathcal{AR}$ such that $a \leq_{\mathrm{fin}} A$.

(1) Given $\mathcal{X} \subseteq \mathcal{R}$, write $\mathcal{X}\restriction[a, A] := \{B \in \mathcal{X} : a \sqsubseteq B \wedge B \leq A\}$.

(2) Given $\mathcal{X} \subseteq \mathcal{AR}$, write $\mathcal{X}\restriction[a, A] := \{b \in \mathcal{X} : a \sqsubseteq b \wedge \exists n[b \leq_{\mathrm{fin}} r_n(A)]\}$.

In both cases, we abbreviate $\mathcal{X}\restriction A := \mathcal{X}\restriction[r_0(A), A]$.

Definition 4.1.2 ([Mij07; DMN15])**.** A set $\mathcal{H} \subseteq \mathcal{R}$ is a **coideal** if it satisfies the following:

(1) (Upward closure) If $A \in \mathcal{H}$ and $A \leq B$, then $B \in \mathcal{H}$.

(2) (**A3** mod $\mathcal{H}$) For all $A \in \mathcal{H}$ and $a \in \mathcal{AR}\restriction A$, we have that:

  (a) $[a, B] \cap \mathcal{H} \neq \emptyset$ for all $B \in [\mathrm{depth}_A(a), A] \cap \mathcal{H}$.

(b) If $B \in \mathcal{H}\restriction A$ and $[a, B] \neq \emptyset$, then there exists $A' \in [\mathrm{depth}_A(a), A] \cap \mathcal{H}$ such that $\emptyset \neq [a, A'] \subseteq [a, B]$.

(3) (**A4** mod $\mathcal{H}$) Fix $A \in \mathcal{H}$ and $a \in \mathcal{AR}\restriction A$. For any $\mathcal{O} \subseteq \mathcal{AR}_{\mathrm{lh}(a)+1}$, there exists $B \in [\mathrm{depth}_A(a), A] \cap \mathcal{H}$ such that $r_{\mathrm{lh}(a)+1}[a, B] \subseteq \mathcal{O}$ or $r_{\mathrm{lh}(a)+1}[a, B] \subseteq \mathcal{O}^c$.

Definition 4.1.3 ([Mij07; DMN15])**.** A set $\mathcal{F} \subseteq \mathcal{R}$ is a **filter** if it satisfies the following:

(1) (Upward closure) If $A \in \mathcal{F}$ and $A \leq B$, then $B \in \mathcal{F}$.

(2) For all $A, B \in \mathcal{F}$, there exists $C \in \mathcal{F}$ such that $C \leq A$ and $C \leq B$.

The filter is said to be **maximal** if $\mathcal{F}'$ is another filter and $\mathcal{F} \subseteq \mathcal{F}'$, then $\mathcal{F} = \mathcal{F}'$. A maximal filter that is also a coideal is called an **ultrafilter**.

In other words, a coideal (ultrafilter) $\mathcal{H}$ is an upward-closed subset of $\mathcal{R}$ (maximal filter of $\mathcal{R}$) such that $(\mathcal{R}, \mathcal{H}, \leq, r)$ satisfies Axioms **A1-A4** of abstract Ramsey spaces (see Chapter 4, [Tod10]). In [DMN15], Di Prisco-Mijares-Nieto studied the conditions of $\mathcal{H}$ required to make $(\mathcal{R}, \mathcal{H}, \leq, r)$ an abstract Ramsey space.

4.1.1 Combinatorial Hypothses of Coideals/Ultrafilters

Unlike coideals/ultrafilter on ω, the combinatorial properties of coideals/ultrafilters are less explored. We introduce four important combinatorial hypotheses which would be relevant to our discussion later.

Question (Combinatoiral Hypotheses of Coideals/Ultrafilters)**.**

(A) (Ultrafilter Hypothesis) Does every coideal of $\mathcal{R}$ contain an ultrafilter?

(B) (Maximal Filter Hypothesis) Is every maximal filter $\mathcal{U}$ satisfying **A3** mod $\mathcal{U}$ an ultrafilter?

(C) (Sub-Coideal Hypothesis) If $N \subseteq M$ are two models of ZFC, and $M \models \mathcal{H}$ is a coideal, is it true that $N \models \mathcal{H} \cap N$ is a coideal?

(D) (Sub-Ultrafilter Hypothesis) If $N \subseteq M$ are two models of ZFC, and $M \models \mathcal{H}$ is an ultrafilter, is it true that $N \models \mathcal{U} \cap N$ is an ultrafilter?

For convenience we may abbreviate the hypotheses names by their corresponding alphabets (e.g. Hypothesis (B) refers to maximal filter hypothesis). It is remarked without proof in [DMN15] that the hypothesis (B) holds. Regardless, it is consistent with ZFC that there exists an ultrafilter (see Theorem 4.4.5)

Hypotheses (A) and (B) in the case of $\mathcal{R} = \mathbf{FIN}_k^{[\infty]}$ will be explored more in depth in Chapter 5.

4.2 Coideals of $\mathbf{FIN}_k^{[\infty]}$

From this section onwards we focus on $\mathcal{R} = \mathbf{FIN}_k^{[\infty]}$.

Definition 4.2.1. A set $\mathcal{X} \subseteq \mathbf{FIN}_k^{[\infty]}$ is **cofinite** if for all $P \in \mathcal{X}$ and $Q \in \mathbf{FIN}_k^{[\infty]}$ such that $P \triangle Q$ is finite, then we have $Q \in \mathcal{X}$ as well.

In $\mathbf{FIN}_k^{[\infty]}$, cofinite coideals enjoy a nice characterisation which makes them easier to handle.

Lemma 4.2.2. *Let $\mathcal{H} \subseteq \mathbf{FIN}_k^{[\infty]}$. Then $\mathcal{H}$ is a cofinite coideal iff it satisfies:*

(1) $\mathcal{H}$ is cofinite.

(2) $\mathcal{H}$ is upward-closed.

*(3) **(Weak A4 mod $\mathcal{H}$)** For any $P \in \mathcal{H}$ and $\mathsf{X} \subseteq \mathbf{FIN}_k$, there exists $Q \in \mathcal{H}$ with $Q \leq P$ such that either $\mathbf{FIN}_k \restriction Q \subseteq \mathsf{X}$ or $\mathbf{FIN}_k \restriction Q \subseteq \mathsf{X}^c$.*

Proof. It's clear that all cofinite coideals satisfy properties (1)-(3). Suppose $\mathcal{H}$ satisfies (1)-(3). Cofiniteness implies that $\mathcal{H}$ satisfies **A3** mod $\mathcal{H}$ trivially. We shall show that $\mathcal{H}$ satisfies **A4** mod $\mathcal{H}$.

Fix $P \in \mathcal{H}$ and $s \in \mathbf{FIN}_k^{[n]} \restriction P$ for some n. Let $\mathcal{O} \subseteq \mathbf{FIN}_k^{[n+1]}$. Define:

$$\mathcal{O}' := \{t_n : t = (t_0, \dots, t_n) \in \mathcal{O} \wedge (t_0. \dots, t_{n-1}) = s\} \subseteq \mathbf{FIN}_k$$

Let $m := \mathrm{depth}_P(s)$. Now $\mathcal{H}$ is cofinite, so $P' := P \setminus r_m(P) \in \mathcal{H}$. By weak **A4** mod $\mathcal{H}$, there exists $Q' \leq P'$ such that $\mathbf{FIN}_k \restriction Q' \subseteq \mathcal{O}'$ or $\mathbf{FIN}_k \restriction Q' \subseteq (\mathcal{O}')^c$.

Since $r_m(P) < P'$ and $Q' \leq P'$, we have $r_m(P) < Q'$, so $Q := r_m(P)^\frown Q'$ is well-defined. By cofiniteness of $\mathcal{H}$, $Q \in [\mathrm{depth}_P(s), P] \cap \mathcal{H}$. It's easy to check that then $r_{n+1}[s, Q] \subseteq \mathcal{O}$ or $r_{n+1}[s, Q] \subseteq \mathcal{O}^c$ respectively. □

4.3 Diagonalisation Forcing $\mathbb{P}_{\mathcal{U},A}$

4.3.1 Selective Coideals of $\mathrm{FIN}_k^{[\infty]}$

We define a notion of selectivity for coideals of $\mathbf{FIN}_k^{[\infty]}$.

Definition 4.3.1 (Selectivity)**.** Let $\mathcal{H}$ be a coideal and let $A \in \mathcal{H}$.

(1) A sequence $\mathcal{A} = (A_n)_{n\in\omega} \subseteq \mathcal{H}$ is called a **decreasing sequence** if $A_{n+1} \leq A_n$ for all n.

(2) An infinite block sequence $P = (p_n)_{n\in\omega}$ is said to be a **diagonalisation** of $\mathcal{A}$ (or **diagonalises** $\mathcal{A}$) **within** A if $p_{n+1} \in \mathbf{FIN}_k \restriction A_{\mathrm{depth}_A(p_n)}$ for all n.

(3) $\mathcal{H}$ is said to be **selective** if for all $A \in \mathcal{H}$ and for all decreasing families $\mathcal{A} \subseteq \mathcal{H}$ with $A_0 \leq A$, there exists a $P \in \mathcal{H}\restriction A$ that diagonalises $\mathcal{A}$ within A.

Note that this definition of selectivity differs from that defined in [Mij07], [DMN15] and [CDM18].

Lemma 4.3.2. *Let $A \in \mathbf{FIN}_k^{[\infty]}$, and let $\mathcal{A}$ be a decreasing family such that $A_0 \leq A$. If $P \in \mathbf{FIN}_k^{[\infty]}$ diagonalises $\mathcal{A}$ within A and $Q \leq P$, then Q also diagonalises $\mathcal{A}$ within A.*

Proof. Let $P = (p_n)_{n\in\omega}$ and $Q = (q_n)_{n\in\omega}$. Write:

$$q_n = T^{(j_0)}(p_{i_0}) + \cdots T^{(j_m)}(p_{i_m})$$
$$q_{n+1} = T^{(j'_0)}(p_{i'_0}) + \cdots T^{(j'_{m'})}(p_{i'_{m'}})$$

where $0 \leq j_0, \ldots, j_m < k$, $\min_i j_i = 0$, $0 \leq j'_0, \ldots, j'_{m'} < k$, $\min_i j'_i = 0$ and $i_m < i'_0$. We see that $\mathrm{depth}_A(q_n) = \mathrm{depth}_A(p_{i_m})$, and since P diagonalises $\mathcal{A}$, we have that $p_{i'_0}, \ldots, p_{i'_{m'}} \in \mathbf{FIN}_k \restriction A_{\mathrm{depth}_A(p_{i_m})}$. This implies that $q_{n+1} \in \mathbf{FIN}_k \restriction A_{\mathrm{depth}_A(q_n)}$, as desired. □

Definition 4.3.3. An infinite block sequence $P = (p_n)_{n\in\omega}$ is said to be a **almost diagonalisation** of $\mathcal{A}$ (or **almost diagonalises** $\mathcal{A}$) within A if $(p_n)_{n>N}$ diagonalises $\mathcal{A}$ within A for some N.

By a similar idea, we obtain:

Lemma 4.3.4. *Let $\mathcal{A}$ be a decreasing family and let $P, Q \in \mathbf{FIN}_k^{[\infty]}$. If P almost diagonalises $\mathcal{A}$ within A and $Q \leq P$, then Q also almost diagonalises $\mathcal{A}$ within A.*

4.3.2 Definition and Motivation

Fix an ultrafilter $\mathcal{U}$, and let $A \in \mathcal{U}$. Given $s \in \mathbf{FIN}_k^{[<\infty]}$, we write $s = (s_0, \dots, s_{\mathrm{lh}(s)-1})$. From now, all (almost) diagonalisations are understood to be done within A.

Definition 4.3.5. The forcing poset $\mathbb{P}_{\mathcal{U},A}$ consists of pairs $\langle s, \mathcal{A} \rangle$ such that:

(1) $s \in \mathbf{FIN}_k^{[<\infty]} \restriction A$.

(2) $\mathcal{A} = (A_n)_{n\in\omega} \subseteq \mathcal{U}$ is a decreasing sequence.

We define a partial order $\leq$ on $\mathbb{P}_{\mathcal{U},A}$ such that $\langle t, \mathcal{B} \rangle \leq s, \mathcal{A}$, where $\mathcal{B} = (B_n)_{n\in\omega}$, iff:

(1) $s \sqsubseteq t$.

(2) $B_n \leq A_n$ for every $n \geq \mathrm{lh}(b)$.

(3) For $\mathrm{lh}(s) \leq n < \mathrm{lh}(t) - 1$, $t_{n+1} \in A_{\mathrm{depth}_A(t_n)}$.

Notation 4.3.6. Let $p, q \in \mathbb{P}_{\mathcal{U},A}$.

(1) If $p = \langle s, \mathcal{A} \rangle$, we write $\mathrm{stem}(p) := s$.

(2) $q \leq_0 p$ off $q \leq p$ and $\mathrm{stem}(q) = \mathrm{stem}(p)$.

(3) Given two decreasing sequences $\mathcal{A} = (A_n)_{n\in\omega}$ and $\mathcal{B} = (B_n)_{n\in\omega}$, we write $\mathcal{A} \leq \mathcal{B}$ iff $A_n \leq B_n$ for all n.

We may then define:

$$[\![s, \mathcal{A}]\!] := \left\{ t \in \mathbf{FIN}_k^{[<\infty]} : t \sqsubseteq s \vee \left(s \sqsubseteq t \wedge t_{n+1} \in \mathbf{FIN}_k \restriction A_{\mathrm{depth}_A(t_n)} \text{ for } n \in [\mathrm{lh}(s), \mathrm{lh}(t) - 1) \right) \right\}$$

We also write $[a, \mathcal{A}]$ as the set of branches of the set $[\![a, \mathcal{A}]\!]$. We call these branches **almost diagonalisations of $\mathcal{A}$ extending** a. It's easy to see that $[a, \mathcal{A}] \neq \emptyset$ for all $\langle a, \mathcal{A} \rangle \in \mathbb{P}_{\mathcal{U},A}$.

Definition 4.3.7. An element $G \in \mathbf{FIN}_k^{[\infty]}$ is $\mathbb{P}_{\mathcal{U},A}$**-generic** over a model $\mathbf{V}$ if for every dense open subset $\mathcal{D} \in \mathbf{V}$ of $\mathbb{P}_{\mathcal{U},A}$, there exists $\langle a, \mathcal{A} \rangle \in \mathcal{D}$ such that $G \in [a, \mathcal{A}]$.

4.3.3 Properties of $\mathbb{P}_{\mathcal{U},A}$

Lemma 4.3.8. *Let $\mathcal{O} \subseteq \mathcal{R}$ be an open set under the Ellentuck topology. Then there is a condition $\langle \emptyset, \mathcal{A} \rangle \in \mathbb{P}_{\mathcal{U},A}$ such that either $[\emptyset, \mathcal{A}] \subseteq \mathcal{O}$ or $[\emptyset, \mathcal{A}] \subseteq \mathcal{O}^c$.*

Note that $[\emptyset, \mathcal{A}]$ is nothing but the set of all diagonalisations of $\mathcal{A}$.

Proof. Player I and I shall play a game of the following form:

$$\begin{array}{c|ccccccc} \mathrm{I} & A_0 & \geq & A_1 & \geq & A_2 & \geq & \cdots \\ \hline \mathrm{II} & & p_0 \in \mathbf{FIN}_k \restriction A_0 & & p_1 \in \mathbf{FIN}_k \restriction A_1 & & p_2 \in \mathbf{FIN}_k \restriction A_2 & \cdots \end{array}$$

More precisely, the rules require the to, for every $n \in \omega$:

(1) $A_n \in \mathcal{U}$.

(2) $A_n \geq A_{n+1}$.

(3) $p_n < p_{n+1}$.

(4) $p_{n+1} \in \mathbf{FIN}_k \restriction A_n$.

Write $\mathcal{A} := (A_n)_{n\in\omega}$. At the end of the game, Player I wins iff $(p_n)_{n\in\omega} \in \mathcal{O}$. The determinacy of open games asserts that one of the two players has a winning strategy σ. We consider each case separately.

- Suppose that σ is a winning strategy for Player I. For each n, let $Q_n \in \mathcal{U}$ be such that $Q_n \leq \sigma(s)$ for all $s \in \mathbf{FIN}_k^{[<\infty]}$ which $\mathrm{depth}_A(s) \leq n$. This is possible as there are only finitely many such s.

  Let $\mathcal{C} := (Q_n)_{n\in\omega}$ so $\mathcal{C}$ is a decreasing sequence. We see that $[\emptyset, \mathcal{C}] \subseteq \mathcal{O}$ - indeed, suppose $P = (p_n)_{n\in\omega}$ diagonalises $\mathcal{C}$, and consider the game above with $A_0 = Q_0$ and $A_n = Q_{\mathrm{depth}_A(p_n)}$. Since $Q_{\mathrm{depth}_A(p_n)} \leq \sigma(r_n(P))$ by the definitions of Q_n's, this sequence of play by the players is consistent with Player I's winning strategy, so $P \in \mathcal{O}$.

- Suppose that, on the other hand, σ is a winning strategy for Player II.

  Claim. *Fix $Q_0, \ldots, Q_{n-1} \in \mathcal{U}$ with $Q_0 \geq \cdots \geq Q_{n-1}$. Then there exists a $P \in \mathcal{U}$ such that::*

$$\forall p \in \mathbf{FIN}_k \restriction P \, \exists Q_n \in \mathcal{U} \restriction Q_{n-1} [s = \sigma(\langle Q_0, \ldots, Q_{n-1}, Q_n \rangle)]$$

Proof. Let:

$$\mathsf{X} := \{p \in \mathbf{FIN}_k : \exists Q_n \in \mathcal{U}[p = \sigma(\langle Q_0, \dots, Q_{n-1}, Q_n\rangle)]\}$$

Since $Q_{n-1} \in \mathcal{U}$, by **A4** mod $\mathcal{U}$ we have a $P \leq Q_{n-1} \cap \mathcal{U}$ such that $\mathbf{FIN}_k \restriction P \subseteq \mathsf{X}$ or $\mathbf{FIN}_k \restriction P \subseteq \mathsf{X}^c$. We see that the latter case is not possible - otherwise, if Player I plays $A_0 = Q_0, \dots, A_{n-1} = Q_{n-1}$, $A_n = P$, and $t_n := \sigma(\langle Q_0, \dots, Q_{n-1}, P\rangle)$, then $t_n \in \mathsf{X}^c$ so:

$$\forall Q_n \in \mathcal{U}[t_n \neq \sigma(\langle Q_0, \dots, Q_{n-1}, Q_n\rangle)]$$

This contradicts that $t_n = \sigma(\langle Q_0, \dots, Q_{n-1}, P\rangle)$. ■

Now for $s \in \mathbf{FIN}_k^{[<\infty]}$, define $R_s \in \mathbf{FIN}_k^{[\infty]}$ inductively as follows: We let $R_\emptyset \in \mathcal{U}$ be any infinite block sequence. Suppose $\mathrm{lh}(s) = n$, and suppose R_t is defined for all $\mathrm{lh}(t) < n$. By the claim, define $R_s \in \mathcal{U}$ such that:

$$\forall p \in \mathbf{FIN}_k \restriction B_s \, \exists Q_n \in \mathcal{U}[q = \sigma(\left\langle R_\emptyset, R_{r_1(s)}, \dots, R_{r_{n-1}(s)}, Q_n \right\rangle)]$$

We may also choose R_s such that $R_s \leq R_{r_m(s)}$ for $m \leq n$. Now define a sequence $\mathcal{D} = (S_n)_{n\in\omega}$ such that $S_0 = R_\emptyset$, and $S_{n+1} \leq S_n$ such that $S_{n+1} \leq R_s$ for all $\mathrm{depth}_A(s) \leq n$. We shall show that $[\emptyset, \mathcal{D}] \subseteq \mathcal{O}^c$, as desired.

Let $P = (p_n)_{n\in\omega}$ be a diagonalisation of $\mathcal{D}$. Consider playing the game above with player II playing P, and player I playing $A_n = S_{\mathrm{depth}_A(p_n)}$. Since P diagonalises $\mathcal{D}$, this run obeys the rules of the game. Furthermore, $S_{\mathrm{depth}_A(p_n)} \leq R_{p_n}$, so p_{n+1} is always chosen such that it is consistent with the strategy σ of Player II. Thus, $P \in \mathcal{O}^c$.

□

Definition 4.3.9. We say that a condition $\langle s, \mathcal{A}\rangle$ **captures** a dense set $\mathcal{D} \subseteq \mathbb{P}_{\mathcal{U},A}$ if for every $P \in [s, \mathcal{A}]$, there exists an n such that $r_n(P) \in [\![s, \mathcal{A}]\!]$ and $\langle r_n(P), \mathcal{A}\rangle \in \mathcal{D}$.

Informally, a condition captures $\mathcal{D}$ if one can enter $\mathcal{D}$ by extending s (the stem of the condition) in an way that diagonalises $\mathcal{A}$ in a finite number of steps.

We also note that if $\langle s, \mathcal{A}\rangle$ captures $\mathcal{D}$, then $\langle s, \mathcal{A}\rangle$ still capture D in any outer model, as long as the relation $\sqsubseteq$ is absolute across all models. Indeed, by replacing the Baire space $\mathcal{N}$ with $\mathbf{FIN}_k^{[\infty]}$, we can perform descriptive set theory on $\mathbf{FIN}_k^{[\infty]}$ and repeat the proof of Mistowski's Π^1_1-absoluteness (see Chapter 25, [Jec03] for a proof) for $\mathbf{FIN}_k^{[\infty]}$. Thus, if we let:

$$\mathcal{V} := \{Q \in \mathbf{FIN}_k^{[\infty]} : \exists t \sqsubseteq Q[\langle t, \mathcal{B}\rangle \in \mathcal{D}]\}$$

then this set is open in the metrisable topology, and hence in the forcing topology. Then:

$$\langle s, \mathcal{A}\rangle \text{ captures } \mathcal{D} \iff \forall Q \in [s, \mathcal{A}][Q \in \mathcal{V}]$$

By Π^1_1 absoluteness, we have that $\langle s, \mathcal{A}\rangle$ also captures $\mathcal{D}$ in any outer model.

Lemma 4.3.10. *Let* $p \in \mathbb{P}_{\mathcal{U},A}$.

(1) For every dense open set $\mathcal{D} \subseteq \mathbb{P}_{\mathcal{U},A}$*, there is a condition* $q \leq_0 p$ *that captures* $\mathcal{D}$.

(2) In fact, more is true - for every countable family of dense open sets $(\mathcal{D}_n)_{n\in\omega}$*, there is a condition* $q \leq_0 p$ *that captures all of* $\mathcal{D}_n$.

Proof. For both statements we assume for convenience that p is the empty condition - the proof for the more general result is similar.

(1) For each $t \in \mathbf{FIN}_k^{[<\infty]}$, let $\mathcal{A}^t \subseteq \mathcal{U}$ be a decreasing sequence such that $\langle t, \mathcal{A}^t\rangle \in \mathcal{D}$ (if no such $\mathcal{A}^t$ exists, set $\mathcal{A}^t$ as your favourite decreasing sequence). Let $A_n^\infty \in \mathcal{U}$ be such that $A_n^\infty \leq A_n^t$ for all $\text{depth}_A(t) \leq n$. Write $\mathcal{A}^\infty := (A_n^\infty)_{n\in\omega}$.

Observe that if $\langle s, \mathcal{B}\rangle \in \mathcal{D}$ for some $s \in \mathbf{FIN}_k^{[<\infty]}$ and some decreasing sequence $\mathcal{B} \subseteq \mathcal{U}$, then we may pick $\mathcal{B}$ such that $A_n^\infty \leq B_n$ for all $n \geq \text{depth}_A(s)$. Since $\mathcal{D}$ is open, we have that $\langle s, \mathcal{A}^\infty\rangle \in \mathcal{D}$. Now consider the set:

$$\mathcal{O} := \{Q \in \mathbf{FIN}_k^{[\infty]} : \exists t \sqsubseteq Q[\langle t, \mathcal{A}^\infty\rangle \in \mathcal{D}]\}$$

Clearly $\mathcal{O}$ is open. By Lemma 4.3.8, there exists an $\mathcal{A}^* \leq \mathcal{A}^\infty$ such that $[\emptyset, \mathcal{A}^*] \subseteq \mathcal{O}$ or $[\emptyset, \mathcal{A}^*] \subseteq \mathcal{O}^c$. Now if $[\emptyset, \mathcal{A}^*] \subseteq \mathcal{O}$, then by definition of $\mathcal{O}$ and the fact that $\mathcal{D}$ is open, we have $\langle \emptyset, \mathcal{A}^*\rangle$. It remains to show that the latter case is not possible.

Suppose $[\emptyset, \mathcal{A}^*] \subseteq \mathcal{O}^c$. Since $\mathcal{D}$ is dense, there exists $\langle t, \mathcal{B}\rangle \leq \langle \emptyset, \mathcal{A}^*\rangle$ such that $\langle t, \mathcal{B}\rangle \in \mathcal{D}$. As discussed above, we have that $\langle t, \mathcal{A}^*\rangle \in \mathcal{D}$, and since $\mathcal{A}^* \leq \mathcal{A}^\infty$ and $\mathcal{D}$ is open, $\langle t, \mathcal{A}^*\rangle \in \mathcal{D}$ (note that we're not asserting that $\mathcal{B} \leq \mathcal{A}^*$). Therefore, any almost diagonalisation of $\mathcal{A}^*$ extending t is in $\mathcal{O}$, so $\emptyset \neq [t, \mathcal{A}^*] \subseteq \mathcal{O}$. This contradicts that $[t, \mathcal{A}^*] \subseteq [\emptyset, \mathcal{A}^*] \subseteq \mathcal{O}$ (note that $[t, \mathcal{A}^*] \subseteq [\emptyset, \mathcal{A}^*]$ as $\langle t, \mathcal{A}^*\rangle \leq \langle \emptyset, \mathcal{A}^*\rangle$).

(2) For each $t \in \mathbf{FIN}_k^{[<\infty]}$, let $\mathcal{A}^t \subseteq \mathcal{U}$ be a decreasing sequence such that $\langle t, \mathcal{A}^t\rangle \in \mathcal{D}_i$ for all $i \leq \text{depth}_A(t)$ (again, if no such t $\mathcal{A}^t$ exists, set $\mathcal{A}^t$ as your favourite decreasing sequence). Then define $\mathcal{A}^\infty$ similarly as in (1) above. (1) asserts that for each n, we may find a $\mathcal{A}^{*n}$ such that $\langle \emptyset, \mathcal{A}^{*n}\rangle \leq \langle \emptyset, \mathcal{A}^\infty\rangle$ and $\langle \emptyset, \mathcal{A}^{*n}\rangle$ captures $\mathcal{D}_n$.

Claim. *For any* $Q_0, \ldots, Q_{n-1} \in \mathcal{U}$*, we may choose* $\mathcal{A}^{*n}$ *above such that* $A_i^{*n} = Q_i$ *for* $i < n$.

Proof. Let $s \in \mathbf{FIN}_k^{[<\infty]}$, with $\mathrm{lh}(s)$, such that s diagonalises $\langle Q_0, \dots, Q_{n-1}\rangle$. We repeat the construction of $\mathcal{A}^{*n}$ in (1) below the condition $\langle a, \mathcal{A}^\infty\rangle$. Let:

$$\mathcal{O}_n := \{Q \in \mathbf{FIN}_k^{[\infty]} : \exists t \sqsubseteq Q[\langle t, \mathcal{A}^\infty\rangle \in \mathcal{D}_n]\}$$

By the same argument we still have that $[a, \mathcal{A}^{*n}] \subseteq \mathcal{O}_n^c$ is not possible, so $[a, \mathcal{A}^{*n}] \subseteq \mathcal{O}_n$. Now define $A_i^{*n} \in \mathcal{U}$ such that:

- If $i < n$, let $A_i^{*n} := Q_i$.
- If $i \geq n$, let $A_i^{*n} \in \mathcal{U}$ be such that $A_i^{*n} \leq A_i^t$ for $\mathrm{depth}_A(t) \leq i$.

This implies that for any $Q \in [\emptyset, \mathcal{A}^{*n}]$, if $t \sqsubseteq Q$ and $\mathrm{depth}_A(b) \geq n$, then $Q \in [b, \mathcal{A}^{*n}] \subseteq \mathcal{O}$, as desired. ■

In particular, we obtain $\mathcal{A}^{*n}$ such that for all $i < n$, $A_i^{*n} = A_i^{*(n-1)}$. Set A_n^* such that $A_n^* \leq A_n^{*m}$ for $m \leq n$. Set $\mathcal{A} := (\mathcal{A}_n^*)_{n\in\omega}$. Then $\langle \emptyset, \mathcal{A}^*\rangle$ captures $\mathcal{D}_i$ for all i by the last statement of the proof of the claim, as desired.

□

Lemma 4.3.11 (Prikry Property)**.** *If* $p = \langle s, \mathcal{A}\rangle \in \mathbb{P}_{\mathcal{U},A}$ *is a condition and* σ *is a sentence in the forcing language, then there is a condition* $p^* \leq_0 p$ *that forces either* σ *or* $\neq \sigma$*.*

Proof. Again, assume that $s = \emptyset$, and the proof for the general case is similar. Let $\mathcal{D}$ be the open dense set of conditions that decide σ:

$$\mathcal{D} := \{q \in \mathbb{P}_{\mathcal{U},A} : q \Vdash \sigma \text{ or } q \Vdash \neg\sigma\}$$

By Lemma 4.3.10, there is a condition $\langle \emptyset, \mathcal{A}^*\rangle \leq \langle \emptyset, \mathcal{A}\rangle$ that captures $\mathcal{D}$. Now define:

$$\mathcal{X} := \{Q \in \mathbf{FIN}_k^{[\infty]} : \exists t \sqsubseteq Q[\langle t, \mathcal{A}^*\rangle \Vdash \sigma]\}$$
$$\mathcal{Y} := \{Q \in \mathbf{FIN}_k^{[\infty]} : \exists t \sqsubseteq Q[\langle t, \mathcal{A}^*\rangle \Vdash \neg\sigma]\}$$

Notice that both $\mathcal{X}$ and $\mathcal{Y}$ are open subsets of $\mathbf{FIN}_k^{[\infty]}$ and $[\emptyset, \mathcal{A}^*] \subseteq \mathcal{X} \sqcup \mathcal{Y}$ (by the definitions of $\mathcal{X}$ and $\mathcal{Y}$ and of capturing). Applying Lemma 4.3.8 again, we have a condition $\langle \emptyset, \mathcal{A}^{**}\rangle \leq \langle \emptyset, \mathcal{A}^*\rangle$ such that $[\emptyset, \mathcal{A}^{**}] \subseteq \mathcal{X}$ or $[\emptyset, \mathcal{A}^{**}] \subseteq \mathcal{Y}$ (Lemma 4.3.8 can be easily tweaked to such that for $\mathcal{O}' \subseteq \mathcal{O}$ both open, either $[\emptyset, \mathcal{A}] \subseteq \mathcal{O}'$ or $[\emptyset, \mathcal{A}] \subseteq \mathcal{O} \setminus \mathcal{O}'$).

We shall show that $\langle \emptyset, \mathcal{A}^{**}\rangle \Vdash \sigma$ in the case when $[\emptyset, \mathcal{A}^{**}] \subseteq \mathcal{X}$, and the other case is similar. Suppose not, so let $\langle t, \mathcal{B}\rangle \leq \langle \emptyset, \mathcal{A}^{**}\rangle$ that forces $\neg\sigma$. Choose any $B \in [t, \mathcal{B}] \subseteq [\emptyset, \mathcal{A}^{**}] \subseteq \mathcal{X}$, and let $t' \sqsubseteq B$ such that $\langle t', \mathcal{A}^*\rangle \Vdash \sigma$. This then contradicts the assumption that $\langle t, \mathcal{B}\rangle \Vdash \neg\sigma$ and $\langle t', \mathcal{A}\rangle$ and $\langle t, \mathcal{B}\rangle$ are compatible conditions (since one of t and t' is an initial segment of another). □

Lemma 4.3.12. *Let $\mathbf{V} \subseteq \mathbf{W}$ be models of set theory. If $\mathbb{P}_{\mathcal{U},A} \in \mathbf{V}$ is a diagonalisation forcing notion (so $\mathcal{U}, A \in \mathbf{V}$), then for any $G \in \mathbf{FIN}_k^{[\infty]} \cap \mathbf{W}$, the following are equivalent:*

(1) $\mathcal{G}$ is $\mathbb{P}_{\mathcal{U},A}$-generic over $\mathbf{V}$.

(2) For every decreasing sequence $\mathcal{A} \subseteq \mathcal{U} \cap \mathbf{V}$, $\mathcal{G}$ almost diagonalises $\mathcal{A}$.

Proof. $\underline{(1) \implies (2)}$: Let $G \in \mathbf{FIN}_k^{[\infty]}$ be $\mathbb{P}_{\mathcal{U},A}$-generic over $\mathbf{V}$. Fix a decreasing sequence $\mathcal{A} \subseteq \mathcal{U}$ that belongs to the ground model $\mathbf{V}$. The set $\mathcal{D}$ of conditions $\langle t, \mathcal{B} \rangle$ such that $\mathcal{B} \leq \mathcal{A}$ is a dense set in $\mathbf{V}$, as for any $\langle u, \mathcal{C} \rangle$ we may let $B_n \in \mathcal{U}$ such that $B_n \leq A_n$ and $B_n \leq C_n$, and we have that $\langle u, \mathcal{B} \rangle \in \mathcal{D}$. Since $\mathcal{G}$ is generic, we have that $G \in [t, \mathcal{B}]$ for some $\mathcal{B} \leq \mathcal{A}$. In other words, we have that $g_{n+1} \in \mathbf{FIN}_k \restriction A_{\mathrm{depth}_A(g_n)}$ for $n \geq \mathrm{lh}(t)$, so $\mathcal{G}$ almost diagonalises $\mathcal{A}$.

$\underline{(2) \implies (1)}$: Suppose $\mathcal{G}$ almost diagonalises every decreasing sequence in $\mathcal{U} \cap \mathbf{V}$. Let $\mathcal{D} \in \mathbf{V}$ be an arbitrary open dense subset of $\mathbb{P}_{\mathcal{U},A}$.

Notation 4.3.13. Fix $p = \langle s, \mathcal{A} \rangle \in \mathbb{P}_{\mathcal{U},A}$. Given $a \in \mathbf{FIN}_k^{[<\infty]}$ such that $a < s$, define $\mathrm{copy}_a(p) := \langle a^\frown s, \mathcal{A} \rangle$. If $\mathcal{D} \subseteq \mathbb{P}_{\mathcal{U},A}$, then:

$$\mathrm{copy}_a(\mathcal{D}) := \{p \in \mathbb{P}_{\mathcal{U},A} : a < \mathrm{stem}(p)\, \mathrm{copy}_a(p) \in \mathcal{D}\} \cup \{p \in \mathbb{P}_{\mathcal{U},A} : a \not< \mathrm{stem}(p)\}$$

By Lemma 4.3.10(2), there exists a decreasing sequence $\mathcal{A} \subseteq \mathcal{U}$ such that $\langle \emptyset, \mathcal{A} \rangle$ that captures $\mathrm{copy}_a(\mathcal{D})$ for every $a \in \mathcal{AR} \restriction A$. By hypothesis, $\mathcal{G}$ diagonalises $\mathcal{A}$ beyond some n, so there exists $k > n$ such that $\langle r_k(G) \setminus r_n(G), \mathcal{A} \rangle \in \mathrm{copy}_{r_n(G)}(\mathcal{D})$. Since $r_n(G) < r_k(G) \setminus r_n(G)$, we have that $\langle r_k(G), \mathcal{A} \rangle \in \mathcal{D}$. Then $G \in [r_k(G), \mathcal{A}]$, so $\mathcal{G}$ is generic, as desired. Note that the absoluteness of capturing from $\mathbf{V}$ to $\mathbf{W}$ is used here, as $\mathcal{G}$ does not belong to $\mathbf{V}$. □

Corollary 4.3.14 (Mathias Property)**.** *If $\mathcal{G}$ is $\mathbb{P}_{\mathcal{U},A}$-generic over $\mathbf{V}$ and $G' \leq G$, then G' is also $\mathbb{P}_{\mathcal{U},A}$-generic.*

Proof. This follows from Lemma 4.3.2 (for almost diagonalisation) and Lemma 4.3.12. □

4.4 Diagonalisation Forcing in $\mathbf{L}(\mathbb{R})$

We now use diagonalisation forcing to prove the analogous version of Theorem 4.0.1 for $\mathbf{FIN}_k^{[\infty]}$, assuming various combinatorial hypotheses. We fix a cofinite coideal $\mathcal{H} \subseteq \mathbf{FIN}_k^{[\infty]}$.

Definition 4.4.1. A set $\mathcal{X} \subseteq \mathbf{FIN}_k^{[\infty]}$ is **$\mathcal{H}$-Ramsey** if for every $P \in \mathcal{H}$, there exists $Q \in \mathcal{H}\restriction P$ such that $\mathbf{FIN}_k^{[\infty]}\restriction Q \subseteq \mathcal{X}$ or $\mathbf{FIN}_k^{[\infty]}\restriction Q \subseteq \mathcal{X}^c$.

Let κ be an an inaccessible cardinal, and let $\mathcal{G}$ be generic over $\mathbf{V}$ for the Lévy collapse $\mathrm{Col}(\omega, \kappa)$. Let $\mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ denote the Solovay's model.

4.4.1 With Hypotheses (A) and (C)

If we try to mimic the proof in [NN16] as much as possible, hypotheses (A) and (C) are required.

Theorem 4.4.2. *Suppose $\mathbf{FIN}_k^{[\infty]}$ satisfies hypotheses (A) and (C). If $\mathcal{X} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ is a subset of $\mathbf{FIN}_k^{[\infty]}$ and $\mathcal{H} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ is a selective coideal, then $\mathcal{X}$ is $\mathcal{H}$-Ramsey.*

We note that notions such as "$\mathcal{H}$-Ramsey" and "selective coideals" are absolute between $\mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ and $\mathbf{V}[\mathcal{G}]$, so we may work entirely in $\mathbf{V}[\mathcal{G}]$.

Proof of Theorem 4.4.2. Let $\mathcal{H} \in \mathbf{L}(\mathbb{R})$ be a selective coideal and let $A \in \mathcal{H}$. Let $\mathcal{X} \in \mathbf{L}(\mathbb{R})$ be a subset of $\mathbf{FIN}_k^{[\infty]}$. The definability of $\mathcal{X}$ gives us a formula $\phi = \phi(x, y)$, along with a real $r \in \mathbf{V}[\mathcal{G}]$ such that:

$$A \in \mathcal{X} \iff \mathbf{V}[\mathcal{G}] \models \phi(\dot{A}, r)$$

Since κ is inaccessible and $\mathcal{G}$ is generic over $\mathbf{V}$ (w.r.t. the Lévy collapse), κ-cc-ness of $\mathrm{Col}(\omega, \kappa)$ gives an ordinal $\beta < \kappa$ such that $A, r \in \mathbf{V}[\mathcal{G}\restriction\beta]$. By the definability of forcing, we have a natural formula $\phi^* = \phi^*(x, y)$ satisfying the equivalence (in $\mathbf{V}[\mathcal{G}\restriction\beta]$):

$$\phi^*(A, y) \iff \emptyset \Vdash_{\mathrm{Col}(\omega,\kappa)} \phi(A, \hat{y})$$

Furthermore, let:

$$\overline{\mathcal{X}} := \{A \in \mathcal{R} \cap \mathbf{V}[\mathcal{G}\restriction\beta] : \mathbf{V}[\mathcal{G}\restriction\beta] \models \phi^*(A, r)\}$$

Notice that $\overline{\mathcal{X}} \in \mathbf{V}[\mathcal{G}\restriction\beta]$, again by the definability of the forcing relation. By the homogeneity of the collapse, we have that $\overline{\mathcal{X}} = \mathcal{X} \cap \mathbf{V}[\mathcal{G}\restriction\beta]$, as:

$$\begin{aligned} A \in \mathcal{X} &\iff \mathbf{V}[\mathcal{G}] \models \phi(A, r) \\ &\iff \emptyset \Vdash_{\mathrm{Col}(\omega,\kappa)} \phi(A, r) \\ &\iff A \in \overline{\mathcal{X}} \end{aligned}$$

By similar arguments, $\overline{\mathcal{H}} := \mathcal{H} \cap \mathbf{V}[\mathcal{G}\restriction\beta]$ belongs to $\mathbf{V}[\mathcal{G}\restriction\beta]$. Assuming the sub-coideal hypothesis, $\overline{\mathcal{H}}$ remains a coideal, and by the ultrafilter hypothesis we may let $\mathcal{U} \subseteq \overline{\mathcal{H}}$ be an ultrafilter.

Claim. *Fix a poset* $\mathbb{P} \in \mathbf{V}[\mathcal{G}\restriction\beta]$*. Let* $G \in \mathbf{FIN}_k^{[\infty]} \cap \mathbf{V}[\mathcal{G}]$ *be* $\mathbb{P}$*-generic over* $\mathbf{V}[\mathcal{G}\restriction\beta]$*. Then* $G \in \mathcal{X}$ *iff* $\mathbf{V}[\mathcal{G}\restriction\beta][G] \models \phi^*(G, r)$*.*

Proof. Work in $\mathbf{V}[\mathcal{G}]$. First note that since $\mathrm{Col}(\omega, \kappa) * \dot{\mathbb{P}}$ is a poset in $\mathbf{V}$ of size $< \kappa$, so by Factor Lemma 2.1.8 there exists a filter $\mathcal{G}^*$ that is $\mathrm{Col}(\omega, \kappa)$-generic over $\mathbf{V}[\mathcal{G}\restriction\beta][G]$ such that:

$$\mathbf{V}[\mathcal{G}\restriction\beta][G][\mathcal{G}^*] = \mathbf{V}[\mathcal{G}]$$

$\underline{\implies:}$ Suppose $G \in \mathcal{X}$, so $\mathbf{V}[\mathcal{G}] \models \phi(G, r)$. By the homogeneity of the Lévy collapse, we have that, in $\mathbf{V}[\mathcal{G}\restriction\beta][G]$, the empty condition forces $\phi(\dot{\mathcal{G}}, \hat{r})$. That is:

$$\mathbf{V}[\mathcal{G}\restriction\beta][G] \models \phi^*(G, r)$$

$\underline{\impliedby:}$ If $\mathbf{V}[\mathcal{G}\restriction\beta][G] \models \phi^*(G, r)$, then $\mathbf{V}[\mathcal{G}] = \mathbf{V}[\mathcal{G}\restriction\beta][G][\mathcal{G}']$ satisfies $\phi(G, r)$ (as the empty condition forces it), so $G \in \mathcal{X}$. ∎

Claim. *In* $\mathbf{V}[\mathcal{G}]$*, for every* $\langle \emptyset, \mathcal{A} \rangle \in \mathbb{P}_{\mathcal{U},A}$*, there is a* $G \in \mathcal{H} \cap [a, A]$ *that is* $\mathbb{P}_{\mathcal{U},A}$*-generic below* $\langle \emptyset, \mathcal{A} \rangle$ *over* $\mathbf{V}[\mathcal{G}\restriction\beta]$*.*

Proof. Work in $\mathbf{V}[\mathcal{G}]$. By Lemma 4.3.12, it suffices to show that $\mathcal{G}$ almost-diagonalises every decreasing sequence in $\mathcal{U}$ that belongs to $\mathbf{V}[\mathcal{G}\restriction\beta]$. We note that we may choose $\beta > 2^{\aleph_0}$, in which then $|\mathbf{FIN}_k^{[\infty]} \cap \mathbf{V}[\mathcal{G}\restriction\beta]|^{\mathbf{V}[\mathcal{G}]} = \aleph_0$. This implies that $\mathbf{V}[\mathcal{G}]$ thinks there are countably many decreasing sequences in $\mathbf{V}[\mathcal{G}\restriction\beta]$. Let $\{\mathcal{A}_n : n \in \omega\}$, with $\mathcal{A}_n = (A_m^{(n)})_{m\in\omega}$, denote the set of all decreasing sequences in $\mathbf{V}[\mathcal{G}\restriction\beta]$. Define $\mathcal{B}$ by stipulating that $B_n \leq A_n$ and $B_n \leq A_m^{(n)}$ for $m \leq n$. It's easy to see that almost diagonalising $\mathcal{B}$ would almost diagonalise all $\mathcal{A}_n$'s. Moreover, $\langle a, \mathcal{B} \rangle \leq \langle a, \mathcal{A} \rangle$.

Since $\mathcal{H}$ is a selective coideal, there exists a $B \in \mathcal{H}\restriction A$ (in $\mathbf{V}[\mathcal{G}]$) that diagonalises $\mathcal{B}$, as desired. ∎

Using the Prikry property of $\mathbb{P}_{\mathcal{U},A}$, we obtain a condition $\langle \emptyset, \mathcal{A} \rangle \in \mathbf{V}[\mathcal{G}\restriction\beta]$ that forces either $\phi^*(G, \hat{r})$ or its negation.

Claim. *The element* $G \in [\emptyset, \mathcal{A}] \cap \mathcal{H}$ *provided in the second claim will witness that* $\mathcal{X}$ *is* $\mathcal{H}$*-Ramsey.*

Proof. The argument is symmetric in $\mathcal{X}$ and $\mathcal{X}^c$, so assume WLOG that $G \in \mathcal{X}$. By the first claim, we have that $\langle \emptyset, \mathcal{A} \rangle \Vdash \phi^*(G, \hat{r})$ and not its negation. Working in $\mathbf{V}[\mathcal{G}]$, the Mathias property of $\mathbb{P}_{\mathcal{U},A}$ ensures that G' is also $\mathbb{P}_{\mathcal{U},A}$-generic over $\mathbf{V}[\mathcal{G}\restriction\beta]$ for any $G' \leq G$. Since $\mathcal{G}$ diagonalises $\mathcal{A}$, so does G'. In other words, $G' \in [\emptyset, \mathcal{A}]$, so:

$$\mathbf{V}[\mathcal{G}\restriction\beta][G'] \models \phi^*(G', r)$$

since $\langle \emptyset, \mathcal{A} \rangle$ forces it. Now apply the first claim to conclude that $G' \in \mathcal{X}$, hence $\mathbf{FIN}_k^{[\infty]} \upharpoonright G \subseteq \mathcal{X}$. ■

Thus, $\mathcal{X}$ is $\mathcal{H}$-Ramsey. □

4.4.2 With Hypothesis (C) and (D)

Alternatively, we may avoid hypothesis (A) by forcing ultrafilters inside selective coideals, and use hypothesis (D) instead to obtain an ultrafilter inside $\overline{\mathcal{H}}$ in the proof.

Notation 4.4.3. For $P, Q \in \mathbf{FIN}_k^{[\infty]}$, we write $P \leq^* Q$ if there exists a $P' \in \mathbf{FIN}_k^{[\infty]}$, with $P \triangle P'$ finite, such that $P' \leq Q$.

Lemma 4.4.4. *A maximal filter is cofinite.*

Proof. Note that a filter $\mathcal{F}$ is not maximal iff there exists a $P \in \mathbf{FIN}_k^{[\infty]}$ such that for all $Q \in \mathcal{F}$, there exists $R \in \mathbf{FIN}_k^{[\infty]}$ such that $R \leq P$ and $R \leq Q$, as in the case we may define:

$$\mathcal{F}' := \left\{ S \in \mathbf{FIN}_k^{[\infty]} : \exists A \in \mathbf{FIN}_k^{[\infty]} \, \exists Q \in \mathcal{F} [A \leq Q \wedge A \leq P \wedge A \leq S] \right\}$$

then we have that $\mathcal{F} \subseteq \mathcal{F}'$ and $P \in \mathcal{F}' \setminus \mathcal{F}$.

Let $P = (p_n)_{n \in \omega} \in \mathcal{F}$ and $P' \in \mathbf{FIN}_k^{[\infty]}$ such that $P \triangle P'$ is finite. By upward closure of $\mathcal{F}$, we may assume WLOG that $P' \subseteq P$. Let $Q \in \mathcal{F}$. Since $P \in \mathcal{F}$ and $\mathcal{F}$ is a filter, there exists an $R \in \mathcal{F}$ with $R \leq P$ and $R \leq Q$.

Write $R = (r_n)_{n \in \omega}$. Since $R \leq P$, for each r_n we may write:

$$r_n = T^{j(n,0)}(p_{i(n,0)}) + \cdots + T^{j(n,m(n))}(p_{i(n,m(n))})$$

Let:

$$X := \{ n \in \omega : p_n \in P \setminus P' \}$$

Note that X is finite, so let N be large enough so that $i(N, 0) > \max X$. Now let $S = (s_n)_{n \in \omega}$ be such that $s_n = r_{n+N}$. Then clearly $S \leq R \leq Q$ and $S \leq P'$. □

Theorem 4.4.5. *If $\mathcal{H}$ is a selective coideal, then forcing with $(\mathcal{H}, \leq^*)$ adds no new elements of $\mathbf{FIN}_k^{[\infty]}$. Furthermore, if $\mathcal{U}$ is a $(\mathcal{H}, \leq^*)$-generic filter over $\mathbf{V}$, then $\mathcal{U}$ is a cofinite ultrafilter in $\mathbf{V}[\mathcal{U}]$.*

Proof. Observe that if $\mathcal{A} = (A_n)_{n\in\omega} \subseteq \mathcal{H}$ is a decreasing sequence, and P diagonalises $\mathcal{A}$, then $P \leq^* A_n$ for all n. Thus, $(\mathcal{H}, \leq^*)$ is σ-closed, and is hence σ-distributive. Therefore, forcing with $(\mathcal{H}, \leq^*)$ adds no new infinite block sequences.

We shall show that $\mathcal{U}$ is a cofinite ultrafilter. $\mathcal{U}$ is a maximal filter by genericity, so by Lemma 4.4.4 $\mathcal{U}$ is cofinite. It remains to show that weak **A4** mod $\mathcal{U}$ holds.

Let $\mathsf{X} \subseteq \mathbf{FIN}_k$, and fix $P \in \mathcal{U}$. Let:

$$\mathcal{D}_{\mathsf{X}} := \{Q \in \mathbf{FIN}_k^{[\infty]} : \langle Q\rangle_k \subseteq \mathsf{X} \vee \langle Q\rangle_k \subseteq \mathsf{X}^c\}$$

Clearly $\mathcal{D}_{\mathsf{X}}$ is open. Furthermore, since $\mathcal{H}$ satisfies **A4** mod $\mathcal{H}$, we have that for all $P' \in \mathcal{H}$, there exists $Q \in \mathcal{H}{\restriction}P'$ such that $Q \in \mathcal{D}_{\mathsf{X}}$. In other words, $\mathcal{D}_{\mathsf{X}}$ is dense in $\mathcal{H}$. By genericity, $\mathcal{U} \cap \mathcal{D}_{\mathsf{X}} \neq \emptyset$.

Let $Q \in \mathcal{U} \cap \mathcal{D}_{\mathsf{X}}$, and let $R \in \mathcal{U}$ such that $R \leq Q$ and $R \leq P$. Then $\langle R\rangle_k \subseteq \mathsf{X}$ or $\langle R\rangle_k \subseteq \mathsf{X}^c$, as desired. □

In particular, this shows that it is consistent with ZFC that there exists an ultrafilter of $\mathbf{FIN}_k^{[\infty]}$.

We are now ready to reprove Theorem 4.4.2 with hypothesis (A) replaced by hypothesis (D).

Theorem 4.4.6. *Suppose* $\mathbf{FIN}_k^{[\infty]}$ *satisfies hypotheses (C) and (D). If* $\mathcal{X} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ *is a subset of* $\mathbf{FIN}_k^{[\infty]}$ *and* $\mathcal{H} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ *is a selective coideal, then* $\mathcal{X}$ *is* $\mathcal{H}$*-Ramsey.*

Proof. Fix a selective coideal $\mathcal{H} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$, so again we have a formula $\phi = \phi(x, y)$, along with a real $r \in \mathbf{V}[\mathcal{G}]$:

$$Q \in \mathcal{X} \iff \mathbf{V}[\mathcal{G}] \models \phi(\dot{Q}, r)$$

We may let ϕ^* be the same formula as in the proof of Theorem 4.4.2. Consider the forcing notion $(\mathcal{H}, \leq^*)$. By Theorem 4.4.5 it adds an ultafilter $\mathcal{U} \subseteq \mathcal{H}$ in $\mathbf{V}[\mathcal{G}][\mathcal{U}]$, and no new infinite block sequences are added. Note that we have $\mathbf{V}[\mathcal{G}{\restriction}\beta][\mathcal{U}][\mathcal{G}{\restriction}[\beta, \kappa)] = \mathbf{V}[\mathcal{G}][\mathcal{U}]$.

Let $P \in \mathcal{H}$. Let $\beta < \kappa$ such that $\mathbf{V}[\mathcal{G}{\restriction}\kappa]$ contains r and P. Since $\mathcal{X} \in \mathbf{L}(r, \alpha)^{\mathbf{V}[\mathcal{G}]} \subseteq \mathbf{L}(r, \alpha)^{V[\mathcal{G}][\mathcal{U}]}$, and that $\mathcal{U}$ adds no new elements of $\mathbf{FIN}_k^{[\infty]}$, there exists a formula $\psi = \psi(A, x)$ such that:

$$\mathbf{V}[\mathcal{G}] \models \phi(Q, r) \iff \mathbf{V}[\mathcal{G}][\mathcal{U}] \models \psi(Q, r)$$

This gives another natural formula ψ^*, satisfying the equivalence (in $\mathbf{V}[G{\restriction}\beta][\mathcal{U}]$):

$$\psi^*(Q, y) \iff \emptyset \Vdash_{\mathrm{Col}(\omega, <\kappa)} \psi(\dot{Q}, \hat{y})$$

Now let $\overline{\mathcal{H}} = \mathcal{H} \cap \mathbf{V}[\mathcal{G}\restriction\beta]$, $\overline{\mathcal{U}} = \mathcal{U} \cap \mathbf{V}[\mathcal{G}\restriction\beta]$. By hypotheses (C) and (D), $\overline{\mathcal{H}}$ remains a coideal and $\overline{\mathcal{U}}$ remains an ultrafilter, with $\overline{\mathcal{U}} \subseteq \overline{\mathcal{H}}$. We may now follow the same steps as in the proof of Theorem 4.4.2 with diagonalisation forcing $\mathbb{P}_{\overline{\mathcal{U}},A}$, and the fact that:

- $\mathbf{V}[\mathcal{G}] \models \mathcal{H}$ is a selective coideal $\iff$ $\mathbf{V}[\mathcal{G}][\mathcal{U}] \models \mathcal{H}$ is a selective coideal.
- $\mathbf{V}[\mathcal{G}] \models \mathcal{X}$ is $\mathcal{H}$-Ramsey $\iff$ $\mathbf{V}[\mathcal{G}][\mathcal{U}] \models \mathcal{X}$ is $\mathcal{H}$-Ramsey.

Therefore, the theorem follows. □

4.4.3 κ is Mahlo

If we begin the proof of Theorem 4.4.2 by assuming further that κ is Mahlo, then we may choose $\beta < \kappa$ such that $\overline{\mathcal{H}}$ is a selective coideal (following closely the workings of [Mat77]). We can then add an ultrafilter $\mathcal{U} \subseteq \overline{\mathcal{H}}$ instead of $\mathcal{H}$.

Theorem 4.4.7. *Suppose κ is Mahlo. If $\mathcal{X} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ is a subset of $\mathbf{FIN}_k^{[\infty]}$ and $\mathcal{H} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ is a selective coideal, then $\mathcal{X}$ is $\mathcal{H}$-Ramsey.*

Proof. Once again, fix a selective coideal $\mathcal{H} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]}$ along with formulas ϕ and ϕ^*. Since κ is Mahlo, there exists an inaccessible $\beta < \kappa$ such that $\overline{\mathcal{H}} := \mathcal{H} \cap \mathbf{V}[\mathcal{G}\restriction\beta]$ remains a selective coideal. By forcing with $(\overline{\mathcal{H}}, \leq^*)$ (in $\mathbf{V}[\mathcal{G}\restriction\beta]$), we add an ultrafilter $\mathcal{U} \subseteq \overline{\mathcal{H}}$. By considering the model $\mathbf{V}[\mathcal{G}][\mathcal{U}]$, we may follow the proof of Theorem 4.4.6 and the result follows. □

4.5 Mathias' Theorem and $\mathbf{FIN}_k^{[\infty]}$

Call a set $\mathcal{X} \subseteq \mathbf{FIN}_k^{[\infty]}$ **Ramsey** if it is $\mathbf{FIN}_k^{[\infty]}$-Ramsey. Since $\overline{\mathbf{FIN}_k^{[\infty]}} = \mathbf{FIN}_k^{[\infty]} \cap \mathbf{V}[\mathcal{G}\restriction\beta]$, it is trivially a selective coideal in $\mathbf{V}[\mathcal{G}\restriction\beta]$. Without the need that κ is Mahlo or any combinatorial hypotheses, we may add an ultrafilter $\mathcal{U} \subseteq \overline{\mathbf{FIN}_k^{[\infty]}}$. This gives:

Theorem 4.5.1. $\mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]} \models$ *Every subset of $\mathbf{FIN}_k^{[\infty]}$ is Ramsey.*

We thus have the $\mathbf{FIN}_k^{[\infty]}$ analog of Mathias' result, which asserts that $\mathbf{L}(\mathbb{R})^{\mathbf{V}[\mathcal{G}]} \models$ Every subset of $[\omega]^\omega$ is Ramsey.

5 | Ultrafilter Hypothesis for $\mathrm{FIN}_k^{[\infty]}$

In this chapter we discuss the ultrafilter hypothesis (and maximal filter hypothesis) for the topological Ramsey space of infinite block sequences $\mathbf{FIN}_k^{[\infty]}$. We describe a correspondence between coideals/filters of $\mathbf{FIN}_k^{[\infty]}$ (as in Definition 4.1.2 and 4.1.3) and coideals/filters of $\mathbf{FIN}_k$ (as sets). This correspondence gives a possible approach to obtain a positive answer to the ultrafilter hypothesis for $\mathbf{FIN}_k^{[\infty]}$.

5.1 The Correspondence Theorem

In this section we shall describe a correspondence between coideals/filters of $\mathbf{FIN}_k^{[\infty]}$ and set-coideals/set-filters of $\mathbf{FIN}_k$. Given an infinite block sequence $P \in \mathbf{FIN}_k^{[\infty]}$, we abuse notation by occasionally treating $P = (p_n)_{n\in\omega}$ as the set $\{p_n\}_{n\in\omega}$.

We call "usual" notions of a coideal H a **set-coideal**. That is, a set-coideal over a set X, $\mathsf{H} \subseteq \mathcal{P}(X)$, satisfies:

(1) $X \in \mathsf{H}$, $\emptyset \notin \mathsf{H}$.

(2) (Upward Closure) If $Y \in \mathsf{H}$ and $Y \subseteq Z$ then $Z \in \mathsf{H}$.

(3) (Pigeonhole) If $Y \cup Z \in \mathsf{H}$ then $Y \in \mathsf{H}$ or $Z \in \mathsf{H}$.

Similarly, the "usual" notions of an ultrafilter U shall be called a **set-ultrafilter**.

Definition 5.1.1.

(1) Let $\mathsf{H} \subseteq \mathcal{P}(\mathbf{FIN}_k)$ be a set-coideal. An **ordered base** for H is a set $\mathcal{B} \subseteq \mathbf{FIN}_k^{[\infty]}$ such that:

- $\forall P \in \mathcal{B}\, [\langle P\rangle_k \in \mathsf{H}]$.

- $\forall \mathcal{O} \in \mathsf{H}\, \exists P \in \mathcal{B}\, [\langle P \rangle_k \subseteq \mathcal{O}]$.

(2) A set-coideal $\mathsf{H} \subseteq \mathcal{P}(\mathbf{FIN}_k)$ is **ordered** of it has an ordered base. If the base is cofinite, we say that H is **cofinitely ordered**.

Given a set-coideal $\mathsf{H} \subseteq \mathcal{P}(\mathbf{FIN}_k)$, we define:

$$\mathsf{H}^{[\infty]} := \left\{ P \in \mathbf{FIN}_k^{[\infty]} : \langle P \rangle_k \in \mathsf{H} \right\}$$

Lemma 5.1.2. *If* H *is an cofintely ordered set-coideal, then* $\mathsf{H}^{[\infty]}$ *is a cofinite coideal.*

Proof. We check the axioms.

(1) (Upward Closure) Let $P \in \mathsf{H}^{[\infty]}$ and $P \leq Q$. Then $\langle P \rangle_k \subseteq \langle Q \rangle_k$, so by upward closure of set-coideals $\langle Q \rangle_k \in \mathsf{H}$. Hence $Q \in \mathsf{H}^{[\infty]}$.

(2) (Weak **A4** mod $\mathsf{H}^{[\infty]}$) Let $\mathcal{O} \subseteq \mathbf{FIN}_k$ and $P \in \mathsf{H}^{[\infty]}$. By the pigeonhole property of H, we have $\langle P \rangle_k \cap \mathcal{O} \in \mathsf{H}_n$ or $\langle P \rangle_k \restriction P \cap \mathcal{O}^c \in \mathsf{H}$. Assume WLOG the first case. Since H is ordered there exists a $Q \in \mathcal{B}$ such that $\langle Q \rangle_k \subseteq \langle P \rangle_k \cap \mathcal{O}$. This implies that $\langle Q \rangle_k \subseteq \langle P \rangle_k$, so $Q \leq P$ and $\langle Q \rangle_k \subseteq \mathcal{O}$, as desired.

(3) (Cofiniteness) Let $P \in \mathsf{H}^{[\infty]}$, and let $P' \in \mathbf{FIN}_k^{[\infty]}$ such that $P \triangle P'$ is finite. By upward closure, we may assume WLOG that $P' \subseteq P$. Now $\langle P \rangle_k \in \mathsf{H}$, so we have $\langle Q \rangle_k \subseteq \langle P \rangle_k$ for some $Q \in \mathcal{B}$. Since $\mathcal{B}$ is cofinite, let $Q' \in \mathcal{B}$ such that $Q' \subseteq P' \cap Q$. Then $\langle Q' \rangle_k \subseteq \langle P' \rangle_k$, so $P' \in \mathsf{H}^{[\infty]}$.

$\square$

Conversely, suppose we have a cofinite coideal $\mathcal{H} \subseteq \mathbf{FIN}_k^{[\infty]}$. We let:

$$\mathcal{H}^* := \{ \mathcal{O} \subseteq \mathbf{FIN}_k : \exists P \in \mathcal{H}[\langle P \rangle_k \subseteq \mathcal{O}] \}$$

Lemma 5.1.3. *Given any cofinite coideal* $\mathcal{H}$*,* $\mathcal{H}^*$ *is a cofinitely ordered set-coideal.*

Proof. It is clear that we have $\mathbf{FIN}_k \in \mathcal{H}^*$, $\emptyset \notin \mathcal{H}^*$ (as $\langle P \rangle_k \neq \emptyset$ for all P), and that they are upward closed. To see the pigeonhole property, suppose $\mathcal{O}_1 \sqcup \mathcal{O}_2 \in \mathcal{H}^*$ (it suffices to show the case for $\mathcal{O}_1$ and $\mathcal{O}_2$ disjoint). Then $\langle P \rangle_k \subseteq \mathcal{O}_1 \sqcup \mathcal{O}_2$ for some $P \in \mathcal{H}$, and by weak **A4** mod $\mathcal{H}$ there exists a $Q \in \mathcal{H}$ with $Q \leq P$ such that $\langle Q \rangle_k \subseteq \mathcal{O}_1$ or $\langle Q \rangle_k \subseteq \mathcal{O}_1^c$. Then $\mathcal{O}_1 \in \mathcal{H}^*$ in the first case and $\mathcal{O}_2 \in \mathcal{H}^*$ in the second. Finally, $\mathcal{H}^*$ is clearly cofinitely ordered by $\mathcal{H}$ itself. $\square$

Lemma 5.1.4.

(1) Let H *and* H' *be two cofinitely ordered set-coideals. Then* $\mathsf{H}^{[\infty]} \subseteq (\mathsf{H}')^{[\infty]}$ *iff* $\mathsf{H} \subseteq \mathsf{H}'$.

(2) Let $\mathcal{H}$ *and* $\mathcal{H}'$ *be two cofinite coideals. Then* $\mathcal{H}^* \subseteq (\mathcal{H}')^*$ *iff* $\mathcal{H} \subseteq \mathcal{H}'$.

Proof. For both statements, $\Longleftarrow$ is clear by definition, so we only prove $\Longrightarrow$.

(1) Let $\mathcal{O} \in \mathsf{H}$, so there exists $P \in \mathcal{B}$ (where $\mathcal{B}$ is the base for H) such that $\langle P \rangle_k \subseteq \mathcal{O}$. Then $P \in \mathsf{H}^{[\infty]} \subseteq (\mathsf{H}')^{[\infty]}$, so $\langle P \rangle_k \in \mathsf{H}'$ as well. Hence $\mathcal{O} \in \mathsf{H}'$ by upward closure.

(2) Let $P \in \mathcal{H}$, so $\langle P \rangle_k \in \mathcal{H}^* \subseteq (\mathcal{H}')^*$. Then $\langle Q \rangle_k \subseteq \langle P \rangle_k$ for some $Q \in \mathcal{H}'$, which implies that $Q \leq P$. Then $P \in \mathcal{H}'$ by upward closure.

□

We may also repeat Definition 5.1.1 and define these two dual notions analgously for cofinite filters/cofinitely ordered set-filters.

Lemma 5.1.5.

(1) If F *is a cofinitely ordered set-filter, then* $\mathsf{F}^{[\infty]}$ *is a cofinite filter.*

(2) If $\mathcal{F}$ *is a cofinite filter, then* $\mathcal{F}^*$ *is a cofinitely ordered set-filter.*

Proof. The proof of upward closure and cofiniteness are the same as that of coideals/set-coideals.

(1) Let $P, Q \in \mathsf{F}^{[\infty]}$. Then $\langle P \rangle_k, \langle Q \rangle_k \in \mathsf{F}$, and since F is ordered, there exists some $R \in \mathbf{FIN}_k^{[\infty]}$, with $\langle R \rangle_k \in \mathsf{F}$, such that $\langle R \rangle_k \subseteq \langle P \rangle_k \cap \langle Q \rangle_k$. This implies that $R \leq P$ and $R \leq Q$, and $R \in \mathsf{F}^{[\infty]}$ by definition.

(2) Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{F}^*$. Then $\langle P \rangle_k \subseteq \mathcal{O}_1$ and $\langle Q \rangle_k \subseteq \mathcal{O}_2$ for some $P, Q \in \mathcal{F}$. Since $\mathcal{F}$ is a filter, take some $R \in \mathcal{F}$ such that $R \leq P$ and $R \leq Q$. Then $\langle R \rangle_k \subseteq \langle P \rangle_k \cap \langle Q \rangle_k \subseteq \mathcal{O}_1 \cap \mathcal{O}_2$, so $\mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{F}^*$.

□

Lemma 5.1.6.

(1) If H *is a cofinitely ordered set-coideal (or set-filter), then* $\mathsf{H} = (\mathsf{H}^{[\infty]})^*$.

(2) If $\mathcal{H}$ *is a cofinite coideal (or a filter), then* $\mathcal{H} = (\mathcal{H}^*)^{[\infty]}$.

Proof. We prove for coideals/set-coideals, and the proof for filters/set-filters are the same (we just need upward closure and orderedness).

(1) Let $\mathcal{O} \in \mathsf{H}$, so $\langle P \rangle_k \subseteq \mathcal{O}$ for some $P \in \mathsf{H}^{[\infty]}$ (by definition of orderedness). Then $\langle P \rangle_k \in (\mathsf{H}^{[\infty]})^*$, so $\mathcal{O} \in (\mathsf{H}^{[\infty]})^*$ by upward closure.

On the other hand, let $\mathcal{O} \in (\mathsf{H}^{[\infty]})^*$, so there exists $P \in \mathsf{H}^{[\infty]}$ such that $\langle P \rangle_k \subseteq \mathcal{O}$. By definition of $\mathsf{H}^{[\infty]}$ we have that $\langle P \rangle_k \in \mathsf{H}$, so $\mathcal{O} \in \mathsf{H}$ by upward closure.

(2) Let $P \in \mathcal{H}$, so $\langle P \rangle_k \in \mathcal{H}^*$, which implies that $P \in (\mathcal{H}^*)^{[\infty]}$ by definition.

On the other hand, let $P \in (\mathcal{H}^*)^{[\infty]}$, so $P \in \langle P \rangle_k \in \mathcal{H}^*$. This means that there exists $Q \in \mathcal{H}$ such that $\langle Q \rangle_k \subseteq \langle P \rangle_k$. This implies that $Q \leq P$, so $P \in \mathcal{H}$.

□

These lemmas combine to give the correspondence theorem.

Theorem 5.1.7 (Correspondence Theorem for $\mathbf{FIN}_k^{[\infty]}$)**.** *There exists a correspondence between cofinite coideals (filters) of* $\mathbf{FIN}_k^{[\infty]}$ *and cofinitely ordered set-coideals (set-filters) of* $\mathbf{FIN}_k$ *which respects inclusion.*

The correspondence theorem holds for any topological Ramsey space $\mathcal{R}$ that satisfies the following: Given $A, B \in \mathcal{R}$, we have $A \leq B$ iff $\mathcal{AR}_1{\restriction}A \subseteq \mathcal{AR}_1{\restriction}B$.

5.1.1 Maximally Ordered Set-Filters

The correspondence of ultrafilters and set-filters follows immediately from the previous lemmas.

Definition 5.1.8. An ordered set-filter $\mathsf{F} \subseteq \mathcal{P}(\mathbf{FIN}_k)$ is **maximally-ordered** if whenever F' is another ordered filter and $\mathsf{F} \subseteq \mathsf{F}'$, we have $\mathsf{F} = \mathsf{F}'$.

Lemma 5.1.9.

(1) If F *is a cofinitely maximally ordered set-filter, then* $\mathsf{F}^{[\infty]}$ *is a maximal cofinite filter.*

(2) If $\mathcal{F}$ *is a maximal cofinite filter, then* $\mathcal{F}^*$ *is a maximal cofinitely ordered set-filter.*

Proof. This follows from the correspondence theorem for filters. □

Lemma 5.1.10.

(1) If U *is a maximally cofinitely ordered set-ultrafilter, then* $\mathsf{U}^{[\infty]}$ *is a ultrafilter.*

(2) If $\mathcal{U}$ *is an ultrafilter, then* $\mathcal{U}^*$ *is a maximally cofinitely ordered set-ultrafilter.*

Proof. This follows from Lemma 4.4.4, Lemma 5.1.3, Lemma 5.1.2 and Lemma 5.1.9. □

This allows us to rephrase the combinatorial hypotheses in terms of set-coideals/set-filters:

Corollary 5.1.11. *The following are equivalent:*

(1) Ultrafilter hypothesis for $\mathbf{FIN}_k^{[\infty]}$.

(2) Every cofinitely ordered set-coideal contains a cofinitely ordered set-ultrafilter.

Corollary 5.1.12. *The following are equivalent:*

(1) Maximal filter hypothesis for $\mathbf{FIN}_k^{[\infty]}$.

(2) Every cofinitely maximally ordered set-filter is a set-ultrafilter.

Question. Are the combinatorial hypotheses for $\mathbf{FIN}_k^{[\infty]}$ necessary for Theorem 4.4.2 to hold?

References

[Ber42] P. Bernays. "A system of axiomatic set theory. Part III. Infinity and enumerability. Analysis". In: *Journal of Symbolic Logic* (1942). DOI: 10.2307/2266303.

[Sol70] R. M. Solovay. "A model of set-theory in which every set of reals is Lebesgue measurable". In: *Annals of Mathematics*. Second Series (1970). DOI: 10.2307/1970696.

[Mat77] A. R. D. Mathias. "Happy families". In: *Ann. Math. Logic* (1977).

[She84] S. Shelah. "Can you take Solovay's inaccessible away?" In: *Israel Journal of Mathematics* 48.1 (Dec. 1984), pp. 1–47. ISSN: 1565-8511. DOI: 10.1007/BF02760522.

[Gow92] W.T. Gowers. "Lipschitz functions on classical spaces". In: *European Journal of Combinatorics* 13.3 (1992), pp. 141–151. ISSN: 0195-6698. DOI: https://doi.org/10.1016/0195-[]6698(92)90020-[]Z.

[Jec03] T. Jech. *Set Theory*. 3rd ed. Springer Monograph in Mathematics, 2003. ISBN: 3-540-44085-2.

[Kan03] A. Kanamori. *The Higher Infinite: Large Cardinals in Set Theory from their Beginnings*. 2nd ed. Springer Monograph in Mathematics, 2003. ISBN: 978-3-54000-384-7.

[Mij07] J. G. Mijares. "A notion of selective ultrafilter corresponding to topological Ramsey spaces". In: *Mathematical logic quarterly* 53 (3 2007).

[Tod10] S. Todorcevic. *Introduction to Ramsey Spaces*. Vol. 174. Annals of Mathematics Studies. Princeton University Press, 2010.

[DMN15] C. Di-Prisco, J. G. Mijares, and J. Nieto. "Local Ramsey theory. An abstract approach". In: (2015). arXiv: 1506.03488 [math.CO].

[Tör15] Asger Törnquist. "Definability and almost disjoint families". In: *Advances in Mathematics* 330 (Mar. 2015). DOI: 10.1016/j.aim.2018.03.005.

[Ung15] S. Unger. *A brief account of Solovay's model*. 2015.

[NN16] I. Neeman and Z. Norwood. “Happy and mad families in $L(\mathbb{R})$”. In: *Journal of Symbolic Logic* (2016). DOI: 10.1017/jsl.2017.85.

[NT17] Mauro Nasso and Eleftherios Tachtsis. “Idempotent ultrafilters without Zorn’s Lemma”. In: *Proceedings of the American Mathematical Society* 146 (Jan. 2017). DOI: 10.1090/proc/13719.

[Oje17] Diana Ojeda-Aristizabal. “Finite forms of Gowers’ theorem on the oscillation stability of c_0”. In: *Combinatorica* 37 (2017), pp. 143–155.

[CDM18] D. Calderon, C. Di-Prisco, and J. Mijares. “Ramsey subsets of the space of infinite block sequences of vectors”. In: (Oct. 2018).