M2RI Asymptotic Statistics Lecture 2: Random Vectors 2

I. Relationships between various modes of convergence & properties

Theorem $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{\mathbb{P}} X \Rightarrow X_n \xrightarrow{d} X$

Proof:

(i) Let $\varepsilon > 0$

$$\mathbb{P}(\|X_n - X\| \geq \varepsilon) = \mathbb{E}\left(\mathbb{1}(\|X_n - X\| \geq \varepsilon)\right)$$

$$= \int_\Omega \underbrace{\mathbb{1}(\|X_n(\omega) - X(\omega)\| \geq \varepsilon)}_{} \, d\mathbb{P}(\omega)$$

$\xrightarrow[n \to +\infty]{} 0$ almost everywhere because $X_n \xrightarrow{a.s.} X$.

$\xrightarrow[n \to +\infty]{} 0$ (Dominated convergence).

(ii) Let f be Lipschitz Bounded.

$$|\mathbb{E}(f(X_n)) - \mathbb{E}(f(X))| \leq \mathbb{E}\left|\,|f(X_n) - f(X)|\,\right|$$

$$\leq \mathbb{E}\left(L\varepsilon \mathbb{1}\{\|X_n - X\| \leq \varepsilon\} + 2M\, \mathbb{1}\{\|X_n - X\| \geq \varepsilon\}\right)$$

$$\leq L\varepsilon \underbrace{\mathbb{P}(\|X_n - X\| \leq \varepsilon)}_{\downarrow \ 1} + 2M \underbrace{\mathbb{P}(\|X_n - X\| \geq \varepsilon)}_{\downarrow \ 0}$$

Theorem If $X_n \xrightarrow{\mathcal{L}} c$ for a constant c, then $X_n \xrightarrow{\mathbb{P}} c$.

Proof: Let $\varepsilon > 0$. By the portmanteau theorem,

$$\limsup_n \mathbb{P}(\|X_n - c\| \geq \varepsilon) \geq \mathbb{P}(\|c - c\| \geq \varepsilon) = 0.$$

Theorem If $X_n \xrightarrow{\mathcal{L}} X$ and $\|X_n - Y_n\| \xrightarrow{\mathbb{P}} 0$ then $Y_n \xrightarrow{\mathcal{L}} X$

Proof: Left as an exercise (we've already seen the techniques).

Theorem (How to combine convergences)

(i) If $X_n \xrightarrow{\mathcal{L}} X$ and $Y_n \xrightarrow{\mathcal{L}} c$ (constant), then (Slutsky)
$$(X_n, Y_n) \xrightarrow{\mathcal{L}} (X, c)$$

(ii) If $X_n \xrightarrow{\mathbb{P}} X$ and $Y_n \xrightarrow{\mathbb{P}} Y$, then
$$(X_n, Y_n) \xrightarrow{\mathbb{P}} (X, Y)$$

(iii) If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then
$$(X_n, Y_n) \xrightarrow{a.s.} (X, Y).$$

Proof:

(i) $Y_n \xrightarrow[n \to +\infty]{\mathcal{L}} c$ so $Y_n \xrightarrow[n \to +\infty]{\mathbb{P}} c$.

Thus, $\|(X_n, Y_n) - (X_n, c)\| \xrightarrow{\mathbb{P}} 0$

Hence, if we show that $(X_n, c) \xrightarrow{\mathcal{L}} (X, c)$, we would have won by the previous theorem.

Let f be continuous bounded.

$$\mathbb{E}(f(X_n, c)) = \mathbb{E}(f_c(X_n)) \xrightarrow{\text{because } X_n \xrightarrow{\mathcal{L}} X \text{ and } f_c \text{ cont. bound.}} \mathbb{E}(f_c(X)) = \mathbb{E}(f(X, c))$$
$$f_c(x) \equiv f(x, c)$$

(ii) We consider the norm such that $\|(X, Y)\| = \|X\| + \|Y\|$ since they are all equivalent.

Let $\varepsilon > 0$

$$\mathbb{P}(\|(X_n, Y_n) - (X, Y)\| \geq \varepsilon) \leq \mathbb{P}\left[\left(\|X_n - X\| \geq \frac{\varepsilon}{2}\right) \cup \left(\|Y_n - Y\| \geq \frac{\varepsilon}{2}\right)\right]$$
$$\leq \mathbb{P}\left(\|X_n - X\| \geq \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\|Y_n - Y\| \geq \frac{\varepsilon}{2}\right).$$

$$\downarrow n \to +\infty \qquad\qquad \downarrow n \to +\infty$$
$$0 \qquad\qquad\qquad 0$$

(iii) $(X_n, Y_n) \to (X, Y)$ i.f.f. $X_n \to X$ and $Y_n \to Y$

and any countable intersection of almost sure events is almost sure.

Theorem (Continuous Mapping) Let g be such that, if $C \equiv \{x \mid g \text{ is cont. at } x\}$,

$$P(X \in C) = 1$$

then,

(i) $X_n \xrightarrow{\mathcal{L}} X \implies g(X_n) \xrightarrow{\mathcal{L}} g(X)$

(ii) $X_n \xrightarrow{\mathbb{P}} X \implies g(X_n) \xrightarrow{\mathbb{P}} g(X)$

(iii) $X_n \xrightarrow{a.s.} X \implies g(X_n) \xrightarrow{a.s.} g(X)$

Proof (i) Let F be closed. We want to show that $\limsup \mathbb{P}(g(X_n) \in F) \leq \mathbb{P}(g(X) \in F)$,

$\forall n$, $(g(X_n) \in F) = (X_n \in g^{-1}(F))$ and $(g(X) \in F) = (X \in g^{-1}(F))$.

Furthermore, $g^{-1}(F) \subset \overline{g^{-1}(F)} \subset g^{-1}(F) \cup C^c$

So,
$$\begin{aligned}
\limsup \mathbb{P}(g(X_n) \in F) &= \limsup \mathbb{P}(X_n \in g^{-1}(F)) \\
&\leq \limsup \mathbb{P}(X_n \in \overline{g^{-1}(F)}) \\
&\leq \mathbb{P}(X \in \overline{g^{-1}(F)}) \quad \text{(Portmanteau)} \\
&\leq \mathbb{P}(X \in g^{-1}(F) \cup C^c) \\
&\leq \mathbb{P}(X \in g^{-1}(F)) + \underbrace{\mathbb{P}(X \notin C)}_{=0}
\end{aligned}$$

(ii)

(ii) Let $\varepsilon > 0$, $\delta > 0$.

$$\mathbb{P}(\|g(X_n) - g(X)\| \geq \varepsilon) = \mathbb{P}(\|g(X_n) - g(X)\| \geq \varepsilon, \|X_n - X\| \leq \delta) + \mathbb{P}(\|g(X_n) - g(X)\| \geq \varepsilon, \|X_n - X\| > \delta)$$

$$\leq \mathbb{P}(\|g(X_n) - g(X)\| \geq \varepsilon, \|X_n - X\| \leq \delta) + \underbrace{\mathbb{P}(\|X_n - X\| > \delta)}_{\xrightarrow{n \to +\infty} 0}$$

Let $B_\delta = \{x : \exists y \text{ st } \|x - y\| \leq \delta, \|g(x) - g(y)\| \geq \varepsilon\}$

then $\limsup_n \mathbb{P}(\|g(X_n) - g(X)\| \geq \varepsilon) \leq \mathbb{P}(X \in B_\delta)$

$= \mathbb{P}(X \in B_\delta \cap C)$ since $\mathbb{P}(X \in C) = 1$

$\forall x$, $\mathbb{1}\{x \in B_\delta \cap C\} \xrightarrow[\delta \to 0]{} 0$ and so, by dominated convergence,

$$\mathbb{P}(X \in B_\delta \cap C) \xrightarrow[\delta \to 0]{} 0 .$$

(iii) $X_n(\omega) \to X(\omega) \Rightarrow g(X_n(\omega)) \Rightarrow g(X(\omega))$ if g is continuous at $X(\omega)$.

Furthermore, $\mathbb{P}(X_n \to X) = 1$ and $\mathbb{P}(X \in C) = 1$

thus $\mathbb{P}(g(X_n) \to g(X)) = 1$.

II. A first example in statistics:

$X_1, \dots, X_n \overset{iid}{\sim} \mathcal{B}(p)$. Objective: What is p? ($p \neq 1$ and $p \neq 0$).

Weak law of large numbers: $\bar{X}_n \equiv \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{\mathcal{L}} p$.

Central limit theorem: $\sqrt{n}\left(\bar{X}_n - p\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, p(1-p))$

Continuous Mapping: $\dfrac{\sqrt{n}\left(\bar{X}_n - p\right)}{p(1-p)} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)$.

We have $\bar{X}_n \xrightarrow{\mathcal{L}} p$ so $\bar{X}_n \xrightarrow{\mathbb{P}} p$ (p is a constant).

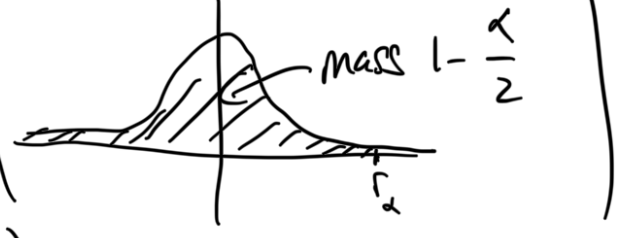
so $(\bar{X}_n, 1 - \bar{X}_n) \xrightarrow{\mathbb{P}} p$

so $\bar{X}_n(1 - \bar{X}_n) \xrightarrow{\mathbb{P}} p(1-p)$ (Continuous Mapping)

So, by Slutsky, $\left(\sqrt{n}\left(\bar{X}_n - p\right), \bar{X}_n(1-\bar{X}_n)\right) \xrightarrow{\mathcal{L}} \left(\mathcal{N}(0, p(1-p)), p(1-p)\right)$

So, by continuous mapping, $\dfrac{\sqrt{n}\left(\bar{X}_n - p\right)}{\bar{X}_n(1-\bar{X}_n)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$.

So, if r_α is the quantile of order $1 - \frac{\alpha}{2}$ of $\mathcal{N}(0,1)$



$$\mathbb{P}\left(\frac{\sqrt{n}\left(\bar{X}_n - p\right)}{\bar{X}_n(1-\bar{X}_n)} \in [-r_\alpha, r_\alpha]\right) \xrightarrow[n \to +\infty]{} 1 - \alpha$$

$$\text{i.e. } \mathbb{P}\left(p \in \underbrace{\left[\overline{X}_n \pm \frac{X_n(1-X_n)}{\sqrt{n}} r_\alpha \right]}_{\text{Confidence interval of level } \alpha.} \right) \xrightarrow[n \to +\infty]{} 1-\alpha$$

III. Asymptotic probabilistic notations.

Definition:

- $X_n = o_{\mathbb{P}}(1)$ if $\|X_n\| \xrightarrow{\mathbb{P}} 0$. $X_n = o_{\mathbb{P}}(R_n)$ if $\exists (Y_n)$ st $X_n = Y_n R_n$, $Y_n = o_{\mathbb{P}}(1)$.

- $X_n = O_{\mathbb{P}}(1)$ if (X_n) is uniformly tight. $X_n = O_{\mathbb{P}}(R_n)$ if $\exists (Y_n)$ s.t. $X_n = Y_n R_n$, $Y_n = O_{\mathbb{P}}(1)$.

Theorem: Let R be a deterministic function and $q > 0$.
Let $X_n \xrightarrow{\mathbb{P}} 0$.

- $\|R(h)\| = \underset{h \to 0}{o}(\|h\|^q) \Rightarrow \|R(X_n)\| = o_{\mathbb{P}}(\|X_n\|^q)$
- $\|R(h)\| = \underset{h \to 0}{O}(\|h\|^q) \Rightarrow \|R(X_n)\| = O_{\mathbb{P}}(\|X_n\|^q)$.

Proof: Let's define $g(h) = \dfrac{R(h)}{\|h\|^q}$ if $h \neq 0$ and $g(h) = 0$ otherwise.

Then $R(X_n) = g(X_n) \|X_n\|^q$

- In the first case, g is continuous at 0. Hence, by the continuous mapping,
  $g(X_n) \xrightarrow{\mathbb{P}} 0$. So $\|R(X_n)\| = o_{\mathbb{P}}(\|X_n\|^q)$.
- $\|R(h)\| = \underset{h \to 0}{O}(\|h\|^q) \Rightarrow \exists \delta > 0$ st $\|h\| \leq \delta \Rightarrow R(h) \leq M\|h\|^q$

so, $\limsup \mathbb{P}(\|g(X_n)\| \geq M) \leq \limsup \mathbb{P}(\|X_n\| \geq \delta) = 0$
since $X_n \xrightarrow{\mathbb{P}} 0$.

so $(g(X_n))$ is uniformly tight

Exercice: Show that $o_{\mathbb{P}}\left(O_{\mathbb{P}}(1)\right) = o_{\mathbb{P}}(1)$.

Solution: Let $X_n = O_{\mathbb{P}}(1)$ and $Y_n = o_{\mathbb{P}}(X_n)$.

By definition, $\exists\, Z_n$ st $Y_n = R_n X_n$ and $Z_n \xrightarrow{\mathbb{P}} 0$.

Let $\varepsilon > 0$. $\mathbb{P}(|Y_n| > \varepsilon) = \mathbb{P}(|R_n X_n| > \varepsilon)$.

$$= \mathbb{P}\left(|R_n X_n| > \varepsilon, |X_n| \geq M\right) + \mathbb{P}\left(|R_n X_n| > \varepsilon, |X_n| < M\right)$$

$$\leq \underbrace{\mathbb{P}(|X_n| \geq M)}_{\leq \varepsilon \text{ if } M \text{ is chosen big enough.}} + \underbrace{\mathbb{P}(|R_n| > \varepsilon/M)}_{\xrightarrow[n \to +\infty]{} 0}$$

because $R_n \xrightarrow{\mathbb{P}} 0$.