Asymptotic Statistics : Random vectors 2

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Relationships between the modes of CV

Theorem: $X_n \xrightarrow[n\to+\infty]{a.s.} X \implies X_n \xrightarrow[n\to+\infty]{\mathbb{P}} X \implies X_n \xrightarrow[n\to+\infty]{\mathscr{L},d} X$.

Theorem: If $X_n \xrightarrow[n\to+\infty]{\mathscr{L},d} c$ for a constant c, then $X_n \xrightarrow[n\to+\infty]{\mathbb{P}} c$.

Theorem: If $X_n \xrightarrow[n\to+\infty]{\mathscr{L},d} X$ and $\|X_n - Y_n\| \xrightarrow[n\to+\infty]{\mathbb{P}} 0$, then $Y_n \xrightarrow[n\to+\infty]{\mathscr{L},d} X$.

How to combine convergences ?

Theorem: Let g be a measurable function and X be a random vector such that, if we denote by O the set of continuity points of g, $\mathbb{P}(X \in O) = 1$.

- If $X_n \xrightarrow[n\to+\infty]{a.s.} X$ and $Y_n \xrightarrow[n\to+\infty]{a.s.} Y$ then $(X_n, Y_n) \xrightarrow[n\to+\infty]{a.s.} (X, Y)$.
- If $X_n \xrightarrow[n\to+\infty]{\mathbb{P}} X$ and $Y_n \xrightarrow[n\to+\infty]{\mathbb{P}} Y$ then $(X_n, Y_n) \xrightarrow[n\to+\infty]{\mathbb{P}} (X, Y)$.
- If $X_n \xrightarrow[n\to+\infty]{\mathscr{L},d} X$ and $Y_n \xrightarrow[n\to+\infty]{\mathbb{P}} c$ for a constant c, then $(X_n, Y_n) \xrightarrow[n\to+\infty]{\mathscr{L},d} (X, c)$ (Slutsky).

Exercice: Show that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow[n\to+\infty]{d} Y$ does not always imply $(X_n, Y_n) \xrightarrow[n\to+\infty]{\mathscr{L},d} (X, Y)$.

Continuous mapping

Theorem: Let g be a measurable function and X be a random vector such that, if we denote by O the set of continuity points of g, $\mathbb{P}(X \in O) = 1$.

- If $X_n \xrightarrow[n\to+\infty]{a.s.} X$ then $g(X_n) \xrightarrow[n\to+\infty]{a.s.} g(X)$.
- If $X_n \xrightarrow[n\to+\infty]{\mathbb{P}} X$ then $g(X_n) \xrightarrow[n\to+\infty]{\mathbb{P}} g(X)$.
- If $X_n \xrightarrow[n\to+\infty]{\mathscr{L},d} X$ then $g(X_n) \xrightarrow[n\to+\infty]{\mathscr{L},d} g(X)$.

A first example in statistics

Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} B(p)$, a Bernoulli distribution of probability of success $p \in (0,1)$.

Exercice: Show that $\dfrac{\sqrt{n}(\bar{X}_n - p)}{\bar{X}_n(1 - \bar{X}_n)} \xrightarrow{d} N(0,1)$.

We can use that to obtain asymptotic confidence intervals or to design asymptotic tests !

Asymptotic probabilistic notations

Definition: Let $(X_n)_{n\in\mathbb{N}}$, $(R_n)_{n\in\mathbb{N}}$ be a sequence of random vectors.

- We write $X_n = o_{\mathbb{P}}(R_n)$ if there exists $Y_n \xrightarrow[n\to+\infty]{\mathbb{P}} 0$ such that $\forall n, X_n = Y_n R_n$.
- We write $X_n = O_{\mathbb{P}}(R_n)$ if there exists (Y_n) uniformly tight such that $\forall n, X_n = Y_n R_n$.

Exercice: Show that $o_{\mathbb{P}}(O_{\mathbb{P}}(1)) = o_{\mathbb{P}}(1)$.

Theorem: Let $(X_n)_{n\in\mathbb{N}}$ such that $X_n \xrightarrow[n\to+\infty]{\mathbb{P}} 0$, R be a measurable function and $q > 0$.

- If $R(h) = o_0(\|h\|^q)$, then We write $R(X_n) = o_{\mathbb{P}}(\|X_n\|^q)$.

- If $R(h) = O_0(\|h\|^q)$, then We write $R(X_n) = O_{\mathbb{P}}(\|X_n\|^q)$.