

Exercise Sheet 1

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1 Exercise 1

A $2m$ -order local interpolating polynomial to $u(x)$ in the neighborhood of x_j given as

$$u(x) = \sum_{k=-m}^m u_{j+k} L_{j+k}(x) \quad (1)$$

where the grid function $u_{j+k} = u(x_{j+k})$ and the *Lagrange interpolation polynomial* is

$$L_{j+k} = \prod_{l=-m, l \neq k}^m \frac{x - x_{j+l}}{x_{j+k} - x_{j+l}}. \quad (2)$$

A 6th order accurate central finite difference approximation is derived as follows:

$$u(x) = \sum_{k=-3}^3 u_{j+k} L_{j+k}(x) \quad (3)$$

Because only the Lagrange interpolation polynomial depends on x we can find the derivative of $u(x)$ by differentiating L_{j+k} which can be found using the logarithmic differentiation method, resulting in:

$$\begin{aligned} \ln(L_{j+k}(x)) &= \sum_{l=-m, l \neq k}^m \ln\left(\frac{x - x_{j+l}}{x_{j+k} - x_{j+l}}\right) \\ &= \sum_{l=-m, l \neq k}^m [\ln(x - x_{j+l}) - \ln(x_{j+k} - x_{j+l})] \end{aligned} \quad (4)$$

Differentiating both sides with respect to x

$$\begin{aligned} \frac{1}{L_{j+k}(x)} \frac{d}{dx} L_{j+k}(x) &= \sum_{l=-m, l \neq k}^m \frac{d}{dx} [\ln(x - x_{j+l}) - \ln(x_{j+k} - x_{j+l})] \\ &= \sum_{l=-m, l \neq k}^m \frac{d}{dx} \ln(x - x_{j+l}) \\ &= \sum_{l=-m, l \neq k}^m \frac{1}{x - x_{j+l}} \end{aligned} \quad (5)$$

Solving for the derivative of $L_{j+k}(x)$

$$\frac{d}{dx} L_{j+k}(x) = L_{j+k}(x) \sum_{l=-m, l \neq k}^m \frac{1}{x - x_{j+l}} \quad (6)$$

Resulting in the derivative of $u(x)$ being:

$$\begin{aligned} \frac{d}{dx} u(x) &= \sum_{k=-3}^3 u_{j+k} \frac{d}{dx} L_{j+k}(x) \\ &= \sum_{k=-3}^3 u_{j+k} \left[L_{j+k}(x) \sum_{l=-3, l \neq k}^3 \frac{1}{x - x_{j+l}} \right] \end{aligned} \quad (7)$$

Now if we set $x = x_j$ and plug into (7) we need to examine what happens for different k . In order to do so we can rewrite (7) as

$$\frac{d}{dx}u(x_j) = \sum_{k=-3}^3 u_{j+k} c_k \quad (8)$$

For $k = 0$ we know that $L_j(x_j) = 1$ by definition of the Lagrange polynomials, resulting in the following coefficient

$$\begin{aligned} c_0 &= \left(\frac{1}{x_j - x_{j-3}} + \frac{1}{x_j - x_{j-2}} + \frac{1}{x_j - x_{j-1}} + \frac{1}{x_j - x_{j+1}} + \frac{1}{x_j - x_{j+2}} + \frac{1}{x_j - x_{j+3}} \right) \\ c_0 &= \left(\frac{1}{-3\Delta x} + \frac{1}{-2\Delta x} + \frac{1}{-\Delta x} + \frac{1}{\Delta x} + \frac{1}{2\Delta x} + \frac{1}{3\Delta x} \right) = 0 \end{aligned} \quad (9)$$

For the case where $k \neq 0$, the lagrange polynomial is $L_{j+k}(x_j) = 0$, because in the product used for the langrange polynomial will be a factor of $(x_j - x_j) = 0$. However in the product of $L_{j+k}(x) \sum_{l=-3, l \neq k}^3 \frac{1}{x - x_{j+l}}$, we have a division by zero when $l = 0$, this creates the indetermined form $0 \cdot \infty$.

Therefore we compute the limit:

$$\begin{aligned} \lim_{x \rightarrow x_j} c_k &= \lim_{x \rightarrow x_j} \left[L_{j+k}(x) \sum_{l=-3, l \neq k}^3 \frac{1}{x - x_{j+l}} \right] \\ &= \lim_{x \rightarrow x_j} \left[L_{j+k}(x) \left(\frac{1}{x - x_j} + \sum_{l=-3, l \neq k, l \neq 0}^3 \frac{1}{x - x_{j+l}} \right) \right] \\ &= \lim_{x \rightarrow x_j} \left[L_{j+k}(x) \frac{1}{x - x_j} + \underbrace{L_{j+k}(x) \sum_{l=-3, l \neq k, l \neq 0}^3 \frac{1}{x - x_{j+l}}}_{\rightarrow 0} \right] \end{aligned} \quad (10)$$

the second term goes to zero due to $L_{j+k}(x_j) = 0$, but for the first term we can write out the langrange polynomial and take out the term for $l = 0$ in the product resulting in

$$\begin{aligned} \lim_{x \rightarrow x_j} c_k &= \lim_{x \rightarrow x_j} \frac{1}{x - x_j} \frac{x - x_j}{x_{j+k} - x_j} \prod_{l=-3, l \neq k, l \neq 0} \frac{x - x_{j+l}}{x_{j+k} - x_{j+l}} \\ &= \lim_{x \rightarrow x_j} \frac{1}{x_{j+k} - x_j} \cdot \prod_{l=-3, l \neq k, l \neq 0} \frac{x - x_{j+l}}{x_{j+k} - x_{j+l}} \end{aligned} \quad (11)$$

Evaluating this at $x = x_j$ leads to a non zero term for $k \neq 0$.

$$c_k = \frac{1}{x_{j+k} - x_j} \prod_{l=-3, l \neq k, l \neq 0} \frac{x_j - x_{j+l}}{x_{j+k} - x_{j+l}} \quad (12)$$

Using the result from (9) and (12) leads to the following coefficients:

For $k = -3$:

$$\begin{aligned} c_{-3} &= \frac{1}{x_{j-3} - x_j} \prod_{l \neq 0, l \neq -3} \frac{x_j - x_{j+l}}{x_{j-3} - x_{j+l}} \\ &= \frac{1}{-3\Delta x} \cdot \frac{(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j-3} - x_{j-2})(x_{j-3} - x_{j-1})(x_{j-3} - x_{j+1})(x_{j-3} - x_{j+2})(x_{j-3} - x_{j+3})} \\ &= \frac{1}{-3\Delta x} \cdot \frac{(2\Delta x)(\Delta x)(-\Delta x)(-2\Delta x)(-3\Delta x)}{(-\Delta x)(-2\Delta x)(-4\Delta x)(-5\Delta x)(-6\Delta x)} \\ &= \frac{1}{-3\Delta x} \cdot \frac{-12\Delta x^5}{-240\Delta x^5} = \frac{-1}{60\Delta x} \end{aligned} \quad (13)$$

For $k = -2$:

$$\begin{aligned}
c_{-2} &= \frac{1}{x_{j-2} - x_j} \prod_{l \neq 0, l \neq -2} \frac{x_j - x_{j+l}}{x_{j-2} - x_{j+l}} \\
&= \frac{1}{-2\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j-2} - x_{j-3})(x_{j-2} - x_{j-1})(x_{j-2} - x_{j+1})(x_{j-2} - x_{j+2})(x_{j-2} - x_{j+3})} \\
&= \frac{1}{-2\Delta x} \cdot \frac{(3\Delta x)(\Delta x)(-\Delta x)(-2\Delta x)(-3\Delta x)}{(\Delta x)(-\Delta x)(-3\Delta x)(-4\Delta x)(-5\Delta x)} \\
&= \frac{1}{-2\Delta x} \cdot \frac{-18\Delta x^5}{60\Delta x^5} = \frac{3}{20\Delta x}
\end{aligned} \tag{14}$$

For $k = -1$:

$$\begin{aligned}
c_{-1} &= \frac{1}{x_{j-1} - x_j} \prod_{l \neq 0, l \neq -1} \frac{x_j - x_{j+l}}{x_{j-1} - x_{j+l}} \\
&= \frac{1}{-\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j+1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j-1} - x_{j-3})(x_{j-1} - x_{j-2})(x_{j-1} - x_{j+1})(x_{j-1} - x_{j+2})(x_{j-1} - x_{j+3})} \\
&= \frac{1}{-\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(-\Delta x)(-2\Delta x)(-3\Delta x)}{(2\Delta x)(\Delta x)(-2\Delta x)(-3\Delta x)(-4\Delta x)} \\
&= \frac{1}{-\Delta x} \cdot \frac{-36\Delta x^5}{-48\Delta x^5} = \frac{-3}{4\Delta x}
\end{aligned} \tag{15}$$

For $k = 0$ the coefficient was computed in (9) and is $c_0 = 0$

For $k = 1$:

$$\begin{aligned}
c_1 &= \frac{1}{x_{j+1} - x_j} \prod_{l \neq 0, l \neq 1} \frac{x_j - x_{j+l}}{x_{j+1} - x_{j+l}} \\
&= \frac{1}{\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j+1} - x_{j-3})(x_{j+1} - x_{j-2})(x_{j+1} - x_{j-1})(x_{j+1} - x_{j+2})(x_{j+1} - x_{j+3})} \\
&= \frac{1}{\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(\Delta x)(-2\Delta x)(-3\Delta x)}{(4\Delta x)(3\Delta x)(2\Delta x)(-\Delta x)(-2\Delta x)} \\
&= \frac{1}{\Delta x} \cdot \frac{36\Delta x^5}{48\Delta x^5} = \frac{3}{4\Delta x}
\end{aligned} \tag{16}$$

For $k = 2$:

$$\begin{aligned}
c_2 &= \frac{1}{x_{j+2} - x_j} \prod_{l \neq 0, l \neq 2} \frac{x_j - x_{j+l}}{x_{j+2} - x_{j+l}} \\
&= \frac{1}{2\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+3})}{(x_{j+2} - x_{j-3})(x_{j+2} - x_{j-2})(x_{j+2} - x_{j-1})(x_{j+2} - x_{j+1})(x_{j+2} - x_{j+3})} \\
&= \frac{1}{2\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(\Delta x)(-\Delta x)(-3\Delta x)}{(5\Delta x)(4\Delta x)(3\Delta x)(\Delta x)(-\Delta x)} \\
&= \frac{1}{2\Delta x} \cdot \frac{18\Delta x^5}{-60\Delta x^5} = \frac{-3}{20\Delta x}
\end{aligned} \tag{17}$$

For $k = 3$:

$$\begin{aligned}
c_3 &= \frac{1}{x_{j+3} - x_j} \prod_{l \neq 0, l \neq 3} \frac{x_j - x_{j+l}}{x_{j+3} - x_{j+l}} \\
&= \frac{1}{3\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+2})}{(x_{j+3} - x_{j-3})(x_{j+3} - x_{j-2})(x_{j+3} - x_{j-1})(x_{j+3} - x_{j+1})(x_{j+3} - x_{j+2})} \\
&= \frac{1}{3\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(\Delta x)(-\Delta x)(-2\Delta x)}{(6\Delta x)(5\Delta x)(4\Delta x)(2\Delta x)(\Delta x)} \\
&= \frac{1}{3\Delta x} \cdot \frac{12\Delta x^5}{240\Delta x^5} = \frac{1}{60\Delta x}
\end{aligned} \tag{18}$$

Finally we can put it all together and 6th order accurate central finite difference approximation is given as:

$$\begin{aligned}
\left. \frac{du}{dx} \right|_{x=x_j} &= u_{j-3} \frac{-1}{60\Delta x} + u_{j-2} \frac{3}{20\Delta x} + u_{j-1} \frac{-3}{4\Delta x} + u_{j+1} \frac{3}{4\Delta x} + u_{j+2} \frac{-3}{20\Delta x} + u_{j+3} \frac{1}{60\Delta x} \\
&= \frac{-u_{j-3} + 9u_{j-2} - 45u_{j-1} + 45u_{j+1} - 9u_{j+2} + u_{j+3}}{60\Delta x}
\end{aligned} \tag{19}$$

Exercise 2

Wave Speed

We assume that the solution to the finite difference scheme is rightward traveling wave

$$v(x, t) = e^{ik(x - c_6 t)} \quad (20)$$

We plug (20) into our finite difference scheme (19)

$$\begin{aligned} -ikc_6 e^{ik(x_j - c_6 t)} &= -\frac{c}{60\Delta x} [-e^{ik(x_{j-3} - c_6 t)} + 9e^{ik(x_{j-2} - c_6 t)} - 45e^{ik(x_{j-1} - c_6 t)} \\ &\quad + 45e^{ik(x_{j+1} - c_6 t)} - 9e^{ik(x_{j+2} - c_6 t)} + e^{ik(x_{j+3} - c_6 t)}] \end{aligned} \quad (21)$$

Now divided both side by $e^{ik(x_j - c_6 t)}$

$$\begin{aligned} -ikc_6 &= -\frac{c}{60\Delta x} [-e^{ik(x_{j-3} - x_j)} + 9e^{ik(x_{j-2} - x_j)} - 45e^{ik(x_{j-1} - x_j)} \\ &\quad + 45e^{ik(x_{j+1} - x_j)} - 9e^{ik(x_{j+2} - x_j)} + e^{ik(x_{j+3} - x_j)}] \\ &= -\frac{c}{60\Delta x} [-e^{-3ik\Delta x} + 9e^{-2ik\Delta x} - 45e^{-ik\Delta x} + 45e^{ik\Delta x} - 9e^{2ik\Delta x} + e^{3ik\Delta x}] \end{aligned} \quad (22)$$

We can now use the following formula $e^{ik\phi} - e^{-ik\phi} = 2i \sin(k\phi)$ to simplify the equation even further

$$\begin{aligned} -ikc_6 &= -\frac{c}{60\Delta x} [2i \sin(3k\Delta x) - 18i \sin(2k\Delta x) + 90i \sin(k\Delta x)] \\ &= -\frac{c}{30\Delta x} i [\sin(3k\Delta x) - 9 \sin(2k\Delta x) + 45 \sin(k\Delta x)] \end{aligned} \quad (23)$$

Dividing by $-ik$ we end up with the wave speed for the 6th order approximation

$$c_6(k) = c \frac{45 \sin(k\Delta x) - 9 \sin(2k\Delta x) + \sin(3k\Delta x)}{30k\Delta x} \quad (24)$$

Phase Error

For the $2m$ -order scheme we can define the phase error as

$$e_m(k) = \left| \frac{u(x, t) - v(x, t)}{u(x, t)} \right| = \left| 1 - e^{ik(c - c_m)t} \right| \approx kt |c - c_m(k)| \quad (25)$$

Now using our result in (24) we can define the phase error for a 6th order scheme as follows

$$e_6(k, t) = kt \left| 1 - \frac{45 \sin(k\Delta x) - 9 \sin(2k\Delta x) + \sin(3k\Delta x)}{30k\Delta x} \right| \quad (26)$$

To measure the accuracy of the 6th scheme, let's introduce the number of grid points per wave length

$$p = \frac{\lambda}{\Delta x} = \frac{2\pi}{k\Delta x} \quad (27)$$

and the number of time the solution returns to itself, due to the periodicity

$$\nu = \frac{ct}{\lambda} \quad (28)$$

Rewriting the phase error (26) in terms of p for k and ν for t results in

$$\begin{aligned} e_6(p, \nu) &= \frac{2\pi}{(\lambda/\Delta x)\Delta x} c \frac{\nu\lambda}{c} \left| 1 - \frac{45 \sin(2\pi p^{-1}) - 9 \sin(2 \cdot 2\pi p^{-1}) + \sin(3 \cdot 2\pi p^{-1})}{(30 \cdot 2\pi p^{-1})} \right| \\ &= 2\pi\nu \left| 1 - \frac{45 \sin(2\pi p^{-1}) - 9 \sin(4\pi p^{-1}) + \sin(6\pi p^{-1})}{(60\pi p^{-1})} \right| \end{aligned} \quad (29)$$

If we now perform a leading-order approximation ($p \rightarrow \infty$) this means the terms including p^{-1} become small we can use the Taylor series expansion for \sin , which is given by the following formula

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad (30)$$