Exercise Sheet 1

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Exercise 1

A $2m$-order local interpolating polynomial to $u(x)$ in the neighborhood of x_j given as

$$u(x) = \sum_{k=-m}^{m} u_{j+k} L_{j+k}(x) \tag{1}$$

where the grid function $u_{j+k} = u(x_{j+k})$ and the *Lagrange interpolation polynomial* is

$$L_{j+k} = \prod_{l=-m, l\neq k}^{m} \frac{x - x_{j+l}}{x_{j+k} - x_{j+l}}. \tag{2}$$

A 6th order accurate central finite difference approximation is derived as follows:

$$u(x) = \sum_{k=-3}^{3} u_{j+k} L_{j+k}(x) \tag{3}$$

Because only the Lagrange interpolation polynomial depends on x we can find the derivative of $u(x)$ by differentiating L_{j+k} which can be found using the logarithmic differentiation method, resulting in:

$$\begin{aligned} \ln(L_{j+k}(x)) &= \sum_{l=-m, l\neq k}^{m} \ln\left(\frac{x - x_{j+l}}{x_{j+k} - x_{j+l}}\right) \\ &= \sum_{l=-m, l\neq k}^{m} \left[\ln(x - x_{j+l}) - \ln(x_{j+k} - x_{j+l})\right] \end{aligned} \tag{4}$$

Differentiating both sides with respect to x

$$\begin{aligned} \frac{1}{L_{j+k}(x)} \frac{d}{dx} L_{j+k}(x) &= \sum_{l=-m, l\neq k}^{m} \frac{d}{dx} \left[\ln(x - x_{j+l}) - \ln(x_{j+k} - x_{j+l})\right] \\ &= \sum_{l=-m, l\neq k}^{m} \frac{d}{dx} \ln(x - x_{j+l}) \\ &= \sum_{l=-m, l\neq k}^{m} \frac{1}{x - x_{j+l}} \end{aligned} \tag{5}$$

Solving for the derivative of $L_{j+k}(x)$

$$\frac{d}{dx} L_{j+k}(x) = L_{j+k}(x) \sum_{l=-m, l\neq k}^{m} \frac{1}{x - x_{j+l}} \tag{6}$$

Resulting in the derivative of $u(x)$ being:

$$\begin{aligned} \frac{d}{dx} u(x) &= \sum_{k=-3}^{3} u_{j+k} \frac{d}{dx} L_{j+k}(x) \\ &= \sum_{k=-3}^{3} u_{j+k} \left[L_{j+k}(x) \sum_{l=-3, l\neq k}^{3} \frac{1}{x - x_{j+l}} \right] \end{aligned} \tag{7}$$

Now if we set $x = x_j$ and plug into (7) we need to examine what happens for different k. In order to do so we can rewrite (7) as

$$\frac{d}{dx}u(x_j) = \sum_{k=-3}^{3} u_{j+k}c_k \tag{8}$$

For $k = 0$ we know that $L_j(x_j) = 1$ by definition of the Lagrange polynomials, resulting in the following coefficient

$$\begin{aligned} c_0 &= \left(\frac{1}{x_j - x_{j-3}} + \frac{1}{x_j - x_{j-2}} + \frac{1}{x_j - x_{j-1}} + \frac{1}{x_j - x_{j+1}} + \frac{1}{x_j - x_{j+2}} + \frac{1}{x_j - x_{j+3}}\right) \\ c_0 &= \left(\frac{1}{-3\Delta x} + \frac{1}{-2\Delta x} + \frac{1}{-\Delta x} + \frac{1}{\Delta x} + \frac{1}{2\Delta x} + \frac{1}{3\Delta x}\right) = 0 \end{aligned} \tag{9}$$

For the case where $k \neq 0$, the lagrange polynomial is $L_{j+k}(x_j) = 0$, because in the product used for the langrange polynomial will be a factor of $(x_j - x_j) = 0$. However in the product of $L_{j+k}(x)\sum_{l=-3,l\neq k}^{3} \frac{1}{x - x_{j+l}}$, we have a division by zero when $l = 0$, this creates the indetermined form $0 \cdot \infty$.
Therefore we compute the limit:

$$\begin{aligned} \lim_{x\to x_j} c_k &= \lim_{x\to x_j}\left[L_{j+k}(x)\sum_{l=-3,l\neq k}^{3}\frac{1}{x - x_{j+l}}\right] \\ &= \lim_{x\to x_j}\left[L_{j+k}(x)\left(\frac{1}{x - x_j} + \sum_{l=-3,l\neq k,l\neq 0}^{3}\frac{1}{x - x_{j+l}}\right)\right] \\ &= \lim_{x\to x_j}\left[L_{j+k}(x)\frac{1}{x - x_j} + \underbrace{L_{j+k}(x)\sum_{l=-3,l\neq k,l\neq 0}^{3}\frac{1}{x - x_{j+l}}}_{\to 0}\right] \end{aligned} \tag{10}$$

the second term goes to zero due to $L_{j+k}(x_j) = 0$, but for the first term we can write out the langrange polynomial and take out the term for $l = 0$ in the product resulting in

$$\begin{aligned} \lim_{x\to x_j} c_k &= \lim_{x\to x_j} \frac{1}{x - x_j}\frac{x - x_j}{x_{j+k} - x_j}\prod_{l=-3,l\neq k,l\neq 0}^{3}\frac{x - x_{j+l}}{x_{j+k} - x_{j+l}} \\ &= \lim_{x\to x_j} \frac{1}{x_{j+k} - x_j} \cdot \prod_{l=-3,l\neq k,l\neq 0}^{3}\frac{x - x_{j+l}}{x_{j+k} - x_{j+l}} \end{aligned} \tag{11}$$

Evaluating this at $x = x_j$ leads to a non zero term for $k \neq 0$.

$$c_k = \frac{1}{x_{j+k} - x_j}\prod_{l=-3,l\neq k,l\neq 0}^{3}\frac{x_j - x_{j+l}}{x_{j+k} - x_{j+l}} \tag{12}$$

Using the result from (9) and (12) leads to the following coefficients:
For $k = -3$:

$$\begin{aligned} c_{-3} &= \frac{1}{x_{j-3} - x_j}\prod_{l=-3,l\neq 0,l\neq -3}^{3}\frac{x_j - x_{j+l}}{x_{j-3} - x_{j+l}} \\ &= \frac{1}{-3\Delta x} \cdot \frac{(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j-3} - x_{j-2})(x_{j-3} - x_{j-1})(x_{j-3} - x_{j+1})(x_{j-3} - x_{j+2})(x_{j-3} - x_{j+3})} \\ &= \frac{1}{-3\Delta x} \cdot \frac{(2\Delta x)(\Delta x)(-\Delta x)(-2\Delta x)(-3\Delta x)}{(-\Delta x)(-2\Delta x)(-4\Delta x)(-5\Delta x)(-6\Delta x)} \\ &= \frac{1}{-3\Delta x} \cdot \frac{-12\Delta x^5}{-240\Delta x^5} = \frac{-1}{60\Delta x} \end{aligned} \tag{13}$$

For $k = -2$:

$$\begin{aligned}
c_{-2} &= \frac{1}{x_{j-2} - x_j} \prod_{l=-3 l\neq 0, l\neq -2}^{3} \frac{x_j - x_{j+l}}{x_{j-2} - x_{j+l}} \\
&= \frac{1}{-2\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j-2} - x_{j-3})(x_{j-2} - x_{j-1})(x_{j-2} - x_{j+1})(x_{j-2} - x_{j+2})(x_{j-2} - x_{j+3})} \\
&= \frac{1}{-2\Delta x} \cdot \frac{(3\Delta x)(\Delta x)(-\Delta x)(-2\Delta x)(-3\Delta x)}{(\Delta x)(-\Delta x)(-3\Delta x)(-4\Delta x)(-5\Delta x)} \\
&= \frac{1}{-2\Delta x} \cdot \frac{-18\Delta x^5}{60\Delta x^5} = \frac{3}{20\Delta x}
\end{aligned} \tag{14}$$

For $k = -1$:

$$\begin{aligned}
c_{-1} &= \frac{1}{x_{j-1} - x_j} \prod_{l=-3, l\neq 0, l\neq -1}^{3} \frac{x_j - x_{j+l}}{x_{j-1} - x_{j+l}} \\
&= \frac{1}{-\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j+1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j-1} - x_{j-3})(x_{j-1} - x_{j-2})(x_{j-1} - x_{j+1})(x_{j-1} - x_{j+2})(x_{j-1} - x_{j+3})} \\
&= \frac{1}{-\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(-\Delta x)(-2\Delta x)(-3\Delta x)}{(2\Delta x)(\Delta x)(-2\Delta x)(-3\Delta x)(-4\Delta x)} \\
&= \frac{1}{-\Delta x} \cdot \frac{-36\Delta x^5}{-48\Delta x^5} = \frac{-3}{4\Delta x}
\end{aligned} \tag{15}$$

For $k = 0$ the coefficient was computed in (9) and is $c_0 = 0$
For $k = 1$:

$$\begin{aligned}
c_1 &= \frac{1}{x_{j+1} - x_j} \prod_{l=-3, l\neq 0, l\neq 1}^{3} \frac{x_j - x_{j+l}}{x_{j+1} - x_{j+l}} \\
&= \frac{1}{\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j+1} - x_{j-3})(x_{j+1} - x_{j-2})(x_{j+1} - x_{j-1})(x_{j+1} - x_{j+2})(x_{j+1} - x_{j+3})} \\
&= \frac{1}{\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(\Delta x)(-2\Delta x)(-3\Delta x)}{(4\Delta x)(3\Delta x)(2\Delta x)(-\Delta x)(-2\Delta x)} \\
&= \frac{1}{\Delta x} \cdot \frac{36\Delta x^5}{48\Delta x^5} = \frac{3}{4\Delta x}
\end{aligned} \tag{16}$$

For $k = 2$:

$$\begin{aligned}
c_2 &= \frac{1}{x_{j+2} - x_j} \prod_{l=-3, l\neq 0, l\neq 2}^{3} \frac{x_j - x_{j+l}}{x_{j+2} - x_{j+l}} \\
&= \frac{1}{2\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+3})}{(x_{j+2} - x_{j-3})(x_{j+2} - x_{j-2})(x_{j+2} - x_{j-1})(x_{j+2} - x_{j+1})(x_{j+2} - x_{j+3})} \\
&= \frac{1}{2\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(\Delta x)(-\Delta x)(-3\Delta x)}{(5\Delta x)(4\Delta x)(3\Delta x)(\Delta x)(-\Delta x)} \\
&= \frac{1}{2\Delta x} \cdot \frac{18\Delta x^5}{-60\Delta x^5} = \frac{-3}{20\Delta x}
\end{aligned} \tag{17}$$

For $k = 3$:

$$\begin{aligned}
c_3 &= \frac{1}{x_{j+3} - x_j} \prod_{l=-3, l\neq 0, l\neq 3}^{3} \frac{x_j - x_{j+l}}{x_{j+3} - x_{j+l}} \\
&= \frac{1}{3\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+2})}{(x_{j+3} - x_{j-3})(x_{j+3} - x_{j-2})(x_{j+3} - x_{j-1})(x_{j+3} - x_{j+1})(x_{j+3} - x_{j+2})} \\
&= \frac{1}{3\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(\Delta x)(-\Delta x)(-2\Delta x)}{(6\Delta x)(5\Delta x)(4\Delta x)(2\Delta x)(\Delta x)} \\
&= \frac{1}{3\Delta x} \cdot \frac{12\Delta x^5}{240\Delta x^5} = \frac{1}{60\Delta x}
\end{aligned} \tag{18}$$

Finally we can put it all together and 6th order accurate central finite difference approximation is given as:

$$\begin{aligned}
\left.\frac{du}{dx}\right|_{x=x_j} &= u_{j-3}\frac{-1}{60\Delta x} + u_{j-2}\frac{3}{20\Delta x} + u_{j-1}\frac{-3}{4\Delta x} + u_{j+1}\frac{3}{4\Delta x} + u_{j+2}\frac{-3}{20\Delta x} + u_{j+3}\frac{1}{60\Delta x} \\
&= \frac{-u_{j-3} + 9u_{j-2} - 45u_{j-1} + 45u_{j+1} - 9u_{j+2} + u_{j+3}}{60\Delta x}
\end{aligned} \tag{19}$$

Exercise 2

Wave Speed

We assume that the solution to the finite difference scheme is rightward traveling wave

$$v(x,t) = e^{ik(x-c_6 t)} \tag{20}$$

We plug (20) into our finite difference scheme (19)

$$\begin{aligned}-ikc_6 e^{ik(x_j-c_6t)} = -\frac{c}{60\Delta x}[&-e^{ik(x_{j-3}-c_6t)} + 9e^{ik(x_{j-2}-c_6t)} - 45e^{ik(x_{j-1}-c_6t)} \\ &+ 45e^{ik(x_{j+1}-c_6t)} - 9e^{ik(x_{j+2}-c_6t)} + e^{ik(x_{j+3}-c_6t)}]\end{aligned} \tag{21}$$

Now divided both side by $e^{ik(x_j-c_6t)}$

$$\begin{aligned}-ikc_6 &= -\frac{c}{60\Delta x}[-e^{ik(x_{j-3}-x_j)} + 9e^{ik(x_{j-2}-x_j)} - 45e^{ik(x_{j-1}-x_j)} \\ &\quad + 45e^{ik(x_{j+1}-x_j)} - 9e^{ik(x_{j+2}-x_j)} + e^{ik(x_{j+3}-x_j)}] \\ &= -\frac{c}{60\Delta x}\left[-e^{-3ik\Delta x} + 9e^{-2ik\Delta x} - 45e^{-ik\Delta x} + 45e^{ik\Delta x} - 9e^{2ik\Delta x} + e^{3ik\Delta x}\right]\end{aligned} \tag{22}$$

We can now use the following formula $e^{ik\phi} - e^{-ik\phi} = 2i\sin(k\phi)$ to simplify the equation even further

$$\begin{aligned}-ikc_6 &= -\frac{c}{60\Delta x}\left[2i\sin(3k\Delta x) - 18i\sin(2k\Delta x) + 90i\sin(k\Delta x)\right] \\ &= -\frac{c}{30\Delta x}i\left[\sin(3k\Delta x) - 9\sin(2k\Delta x) + 45\sin(k\Delta x)\right]\end{aligned} \tag{23}$$

Dividing by $-ik$ we end up with the wave speed for the 6th order approximation

$$c_6(k) = c\frac{45\sin(k\Delta x) - 9\sin(2k\Delta x) + \sin(3k\Delta x)}{30k\Delta x} \tag{24}$$

Phase Error

For the $2m$-order scheme we can define the phase error as

$$e_m(k) = \left|\frac{u(x,t) - v(x,t)}{u(x,t)}\right| = \left|1 - e^{ik(c-c_m)t}\right| \approx kt\left|c - c_m(k)\right| \tag{25}$$

Now using our result in (24) we can define the phase error for a 6th order scheme as follows

$$e_6(k,t) = kct\left|1 - \frac{45\sin(k\Delta x) - 9\sin(2k\Delta x) + \sin(3k\Delta x)}{30k\Delta x}\right| \tag{26}$$

To measure the accuracy of the 6th scheme, let's introduce the number of grid points per waive length

$$p = \frac{\lambda}{\Delta x} = \frac{2\pi}{k\Delta x} \tag{27}$$

and the number of time the solution returns to itself, due to the periodicity

$$\nu = \frac{ct}{\lambda} \tag{28}$$

Rewriting the phase error (26) in terms of p for k and ν for t results in

$$\begin{aligned}e_6(p,\nu) &= \frac{2\pi}{(\lambda/\Delta x)\Delta x}c\frac{\nu\lambda}{c}\left|1 - \frac{45\sin(2\pi p^{-1}) - 9\sin(2\cdot 2\pi p^{-1}) + \sin(3\cdot 2\pi p^{-1})}{(30\cdot 2\pi p^{-1})}\right| \\ &= 2\pi\nu\left|1 - \frac{45\sin(2\pi p^{-1}) - 9\sin(4\pi p^{-1}) + \sin(6\pi p^{-1})}{(60\pi p^{-1})}\right|\end{aligned} \tag{29}$$

If we now perform a leading-order approximation ($p \to \infty$) this means the terms including p^{-1} become small we can use the Taylor series expansion for sin, which is given by the following formula To perform a leading-order approximation as $p \to \infty$, we use the Taylor series expansion for sine:

$$\sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \tag{30}$$

e now expand each sine term in the numerator: For $\sin(2\pi p^{-1})$:

$$\sin(2\pi p^{-1}) = 2\pi p^{-1} - \frac{(2\pi p^{-1})^3}{3!} + \frac{(2\pi p^{-1})^5}{5!} - \frac{(2\pi p^{-1})^7}{7!} + \mathcal{O}(p^{-9}) \tag{30}$$

For $\sin(4\pi p^{-1})$:

$$\sin(4\pi p^{-1}) = 4\pi p^{-1} - \frac{(4\pi p^{-1})^3}{3!} + \frac{(4\pi p^{-1})^5}{5!} - \frac{(4\pi p^{-1})^7}{7!} + \mathcal{O}(p^{-9}) \tag{31}$$

For $\sin(6\pi p^{-1})$:

$$\sin(6\pi p^{-1}) = 6\pi p^{-1} - \frac{(6\pi p^{-1})^3}{3!} + \frac{(6\pi p^{-1})^5}{5!} - \frac{(6\pi p^{-1})^7}{7!} + \mathcal{O}(p^{-9}) \tag{32}$$

Now we calcualte the numerator for each term of each order.
For the first order terms:

$$45\left(2\pi p^{-1}\right) - 9\left(4\pi p^{-1}\right) + \left(6\pi p^{-1}\right) = (90 - 36 + 6)\pi p^{-1} = 60\pi p^{-1} \tag{33}$$

For the third order terms:

$$45\left(8\pi^3 p^{-3}\right) - 9\left(64\pi^3 p^{-3}\right) + \left(216\pi^3 p^{-3}\right) = (360 - 576 + 216)\pi^3 p^{-3} = 0\pi^3 p^{-3} = 0 \tag{34}$$

For the fifth order terms:

$$45\left(32\pi^5 p^{-5}\right) - 9\left(1024\pi^5 p^{-5}\right) + \left(7776\pi^5 p^{-5}\right) = (1440 - 9216 + 7776)\pi^5 p^{-5} = 0\pi^5 p^{-5} = 0 \tag{35}$$

and seventh order terms:

$$\begin{aligned}
&45\left(-\frac{128\pi^7}{5040}p^{-7}\right) - 9\left(-\frac{16384\pi^7}{5040}p^{-7}\right) + \left(-\frac{279936\pi^7}{5040}p^{-7}\right)\\
&= -\frac{45\cdot 128\pi^7}{5040}p^{-7} + \frac{9\cdot 16384\pi^7}{5040}p^{-7} - \frac{279936\pi^7}{5040}p^{-7}\\
&= -\frac{5760\pi^7}{5040}p^{-7} + \frac{147456\pi^7}{5040}p^{-7} - \frac{279936\pi^7}{5040}p^{-7}\\
&= \frac{-5760 + 147456 - 279936}{5040}\pi^7 p^{-7}\\
&= \frac{-138240}{5040}\pi^7 p^{-7}\\
&= \frac{-192}{7}\pi^7 p^{-7}
\end{aligned} \tag{36}$$

Now, substituting back into our original expression:

$$\begin{aligned}
e_6(p,\nu) &= 2\pi\nu\left|1 - \frac{60\pi p{-1}}{60\pi p{-1}} + \frac{\frac{-192}{7}\pi^7 p^{-7}}{(60\pi p^{-1})}\right|\\
&= 2\pi\nu\left|\frac{192}{420}\pi^6 p^{-6}\right|\\
&= 2\pi\nu\left|\frac{16}{35}\pi^6 p^{-6}\right|
\end{aligned} \tag{37}$$

We can remove the absolute brackets due to p being a large postive number, resulting in the final step:

$$e_3(p,\nu) = 2\pi\nu\frac{16}{35}\pi^6 p^{-6} = \frac{\pi\nu}{70}\left(\frac{2\pi}{p}\right)^6 \tag{38}$$

Let's assume we can accept an error ϵ_p after ν periods of evolution. We can now derive $p_3(\epsilon_p, \nu)$ the number of points per wavelength required to ensure the phase error is bounded by ϵ_p

$$\begin{aligned} e_3(p,\nu) &= \frac{\pi\nu}{70}\left(\frac{2\pi}{p}\right)^6 \\ \frac{70e_3(p,\nu)}{\pi\nu} &= \left(\frac{2\pi}{p}\right)^6 \\ \frac{70e_3(p,\nu)}{\pi\nu} &= \left(\frac{2\pi}{p}\right)^6 \\ \sqrt[6]{\frac{70e_3(p,\nu)}{\pi\nu}} &= \frac{2\pi}{p} \\ \sqrt[6]{\frac{70e_3(p,\nu)}{\pi\nu}} &= \frac{2\pi}{p} \\ p &= 2\pi\sqrt[6]{\frac{\pi\nu}{70e_3(p,\nu)}} \end{aligned} \tag{39}$$

Hence the bound is given as

$$p_3(\epsilon_p,\nu) \geq 2\pi\sqrt[6]{\frac{\pi\nu}{70\epsilon_p}} \tag{40}$$

If we now compare the 2nd, 4th, and 6th order schemes using $\epsilon_p = 0.1$ and $\epsilon_p = 0.01$ in Table 1, we can observe that for $\epsilon_p = 0.1$, increasing the order doesn't yield much gain, except when integration times are very long (high values of ν). For $\epsilon_p = 0.01$, we observe that higher order schemes become more beneficial even at lower integration times. Although the difference between the 2nd order and 4th order schemes is more significant at this error level, if we further decrease the error tolerance and hence increase the required accuracy, the 6th order scheme will become even more beneficial. This trend becomes particularly pronounced for very small error tolerances (e.g., $\epsilon_p = 10^{-5}$). Our analysis shows that the computational efficiency advantage of higher order schemes increases with stricter accuracy requirements. In conclusion, the 6th order scheme is recommended when either higher accuracy is required or when simulating over long periods of time that require capturing many oscillations of the solution.

ϵ_p	2nd Order Scheme (p_1)	4th Order Scheme (p_2)	6th Order Scheme (p_3)
0.1	$p_1 \geq 20\sqrt[2]{\nu}$	$p_2 \geq 7\sqrt[4]{\nu}$	$p_3 \geq 5.5\sqrt[6]{\nu}$
0.01	$p_1 \geq 64\sqrt[2]{\nu}$	$p_2 \geq 13\sqrt[4]{\nu}$	$p_3 \geq 8\sqrt[6]{\nu}$

Table 1: Comparison of numerical scheme requirements for different error tolerances (ϵ_p)

Exercise 3

To evaluate the accuracy of spectral methods for differentiation, we tested the Fourier differentiation matrix for odd number of grid points on a function with a known analytical derivative. The Fourier differentiation matrix $\tilde{D}$ is given as:

$$\tilde{D}_{ji} = \begin{cases} \frac{(-1)^{j+i}}{2}\left[\sin\left(\frac{(j-i)\pi}{N+1}\right)\right]^{-1} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \tag{41}$$

Please note note that this particular matrix is designed for an odd number of grid points. In the course slides, the domain is discretized as $x \in [0, 2\pi]$ with $x_j = \frac{2\pi j}{N+1}$ for$j \in [0, \ldots, N]$. These $N+1$ equidistant grid points result in a grid with an odd number of points when N is even.

The accuracy of this differentiation matrix was evaluated on the following function:

$$u(x) = \exp(k \sin x) \tag{42}$$

defined on the interval $x \in [0, 2\pi]$, where k is a parameter that controls the oscillatory behavior of the function. The corresponding derivative is given by

$$u'(x) = k\cos x \cdot \exp(k \sin x) \tag{43}$$

and used as the analytical version of the function and used to compute the relative error. Table 2 shows the minimum number of grid points needed to achieve the maximum relative error of 10^{-5}

k	Minimum N	Max Relative Error
2	22	$3.7751 \cdot 10^{-6}$
4	32	$5.6209 \cdot 10^{-6}$
6	42	$4.1225 \cdot 10^{-6}$
8	52	$2.4255 \cdot 10^{-6}$
10	60	$7.8919 \cdot 10^{-6}$
12	70	$3.8702 \cdot 10^{-6}$

Table 2: Minimum N required to achieve a maximum relative error below the threshold of 10^{-5}.

The results show that the number of grid points required to achieve the specified accuracy increases approximately linearly with k. This can be explained by the fact that larger values of k lead to more oscillatory functions. To accurately capture these higher frequencies, a finer grid resolution is necessary.

I implemented this exercise in C++ (more details can be found in the README.md file). For higher values of k, computing the relative error reaches the limits of floating-point precision due to the extremely small magnitude of the error. Depending on the machine and operating system, slight variations in results can be expected. To mitigate this issue, I used the Boost library and introduced a specialized double type with 50 bits of precision.