Exercise Sheet 2

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1 Odd vs. Even Method

In this exercise, we compared the accuracy of two spectral differentiation methods, the Odd and Even Fourier methods. The function $u(x) = \exp(k\sin(x))$ and its derivative were used to compare the two techniques. The Table below summarizes the minimum number of grid points N required to achieve the maximum error of 10^{-5} and the corresponding maximum relative errors for increasing values of the parameter k.

k	ODD Method		EVEN Method	
	Minimum N	Max Relative Error	Minimum N	Max Relative Error
2	22	3.78×10^{-6}	20	3.13×10^{-6}
4	32	5.62×10^{-6}	32	3.02×10^{-7}
6	42	4.12×10^{-6}	40	1.19×10^{-6}
8	52	2.43×10^{-6}	48	3.37×10^{-6}
10	60	7.89×10^{-6}	56	8.22×10^{-6}
12	70	3.87×10^{-6}	68	5.79×10^{-7}

Table 1: Comparison of ODD and EVEN Fourier Differentiation Methods for $u(x) = \exp(k\sin(x))$

The numerical evidence show that the Even method consistently outperforms the Odd method in terms of accuracy, by requiring less grid points to achieve a similar relative error. As an example for $k = 4$ the Even method achieves a significant lower maximum relative error. Furthermore the Even method needs most of the time 2 to 4 less grid points to fall below the required error threshold.
In conclusion, the EVEN Fourier method proves to be more reliable across and is preferable when high accuracy is required with fewer grid points.

2 Even Method for Various Types of Derivatives

In this exercise we compute the derivative using the Even Method over the interval $[0, 2\pi]$ of the following functions

- **Case 1**: $f(x) = \cos(10x)$
- **Case 2**: $f(x) = \cos(x/2)$
- **Case 3**: $f(x) = x$

Using the pointwise error L_∞ and the global error L_2 for increasing values of N we observe the following behavior in each of the cases.

2.1 Case 1

For this case of function $f(x) = \cos(10x)$, we observe in Table 2 that

- Initially, the errors increase from $N = 8$ to $N = 16$.
- Then the error drops drastically to the order of 10^{-14} around $N = 32$, which coincides with the expected behavior of a Spectral Fourier method.
- After N=32, the errors are slowly increasing again to the order of 10^{-11} at $N = 2048$.

This behavior suggests, that the function has a high-frequency components (factor of 10) and at $N = 32$, we achieve an optimal sampling that effectively captures these oscillations. The subsequent increase in error, is likely caused by accumulating round-off errors, rather than discretization errors. Furthermore it is important to not that the function is 2π periodic and builds a continuous wave using the periodic boundary conditions.

N	L_∞ Error	L_2 Error	L_∞ Ratio	L_2 Ratio
8	8.00×10^0	1.60×10^1	—	—
16	1.60×10^1	4.53×10^1	0.50	0.35
32	8.62×10^{-14}	1.36×10^{-13}	1.86×10^{14}	3.33×10^{14}
64	1.29×10^{-13}	3.23×10^{-13}	0.67	0.42
128	1.87×10^{-13}	6.76×10^{-13}	0.69	0.48
256	3.52×10^{-13}	1.67×10^{-12}	0.53	0.40
512	4.80×10^{-12}	9.44×10^{-12}	0.07	0.18
1024	3.80×10^{-12}	1.96×10^{-11}	1.26	0.48
2048	5.89×10^{-11}	9.69×10^{-11}	0.06	0.20

Table 2: Convergence Analysis for $f(x) = \cos(10x)$ using Even Fourier Differentiation

2.2 Case 2

In this case of function $f(x) = \cos(x/2)$, we observe in Table 3 that

- The errors consistently increases as N increases.
- The convergence rate for L_∞ and L_2 is always around 0.5.
- No convergence like in Case 1 is observed.

This is an unexpected behavior for a spectral method for smooth functions. The constant error growth indicates the method is not suitable for this low-frequency function. Importantly the function values at $\cos(0) = 1$ and $\cos(\frac{2\pi}{2}) = 0$ are different, creating a discontinuity when using periodic boundary conditions. This mismatch prevents the convergences and explains the poor performance.

N	L_∞ Error	L_2 Error	L_∞ Ratio	L_2 Ratio
8	1.75×10^0	2.65×10^0	—	—
16	3.52×10^0	5.65×10^0	0.50	0.47
32	7.06×10^0	1.17×10^1	0.50	0.49
64	1.41×10^1	2.37×10^1	0.50	0.49
128	2.82×10^1	4.76×10^1	0.50	0.50
256	5.65×10^1	9.56×10^1	0.50	0.50
512	1.13×10^2	1.92×10^2	0.50	0.50
1024	2.26×10^2	3.83×10^2	0.50	0.50
2048	4.52×10^2	7.67×10^2	0.50	0.50

Table 3: Convergence Analysis for $f(x) = \cos(x/2)$ using Even Fourier Differentiation

2.3 Case 3

In the final case for $f(x) = x$ we can observe

- The ratio also stabilizes at around 0.50.
- The errors is increasing and is even larger than in Case 2.

The linear function behaves similarly to Case 2 with fixed convergence ratios and consistently increasing errors. For this case there's even a more significant mismatch at the bondary (0 to 2π). This function fundamentally contradicts the periodic assumption used by the Fourier methods, which results in large errors, which grow with increasing resolution.

N	L_∞ Error	L_2 Error	L_∞ Ratio	L_2 Ratio
8	5.44×10^0	8.40×10^0	—	—
16	1.10×10^1	1.78×10^1	0.49	0.47
32	2.22×10^1	3.66×10^1	0.50	0.49
64	4.43×10^1	7.43×10^1	0.50	0.49
128	8.87×10^1	1.50×10^2	0.50	0.50
256	1.77×10^2	3.00×10^2	0.50	0.50
512	3.55×10^2	6.02×10^2	0.50	0.50
1024	7.10×10^2	1.20×10^3	0.50	0.50
2048	1.42×10^3	2.41×10^3	0.50	0.50

Table 4: Convergence Analysis for $f(x) = x$ using Even Fourier Differentiation

2.4 Comparison

The fundamental difference between the cases can be explained by the following three key points:

- **Function Characteristics**
  - **Case 1**: A periodic, high-frequency function.
  - **Case 2**: A low-frequency function with period 4π, and not matching $[0, 2\pi]$
  - **Case 3**: A non periodic function on $[0, 2\pi]$
- **Spectral Properties**
  - Fourier methods work very well for periodic functions
  - Only for Case 1 the Fourier method converged.
  - The other cases experience the Gibbs phenomena at the boundaries.
- **Covergence Rate**
  - **Case 1**: Spectral convergence until machine precision, then experiencing round-of errors.
  - **Case 2 & 3**: First-order convergence with constant ratio 0.5 and is unable to accurately represent the function even for higher N.

This leads to the key insight that Even fourier distributions work best for functions that are periodic on the domain. For function that don't match this such as seen in Case 2 and 3, the method introduces errors that grow with the resolution.

3 Scalar Hyperbolic Problem

This exercise is dedicated on solving a scalar hyperbolic problem given as

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= -2\pi \frac{\partial u(x,t)}{\partial x} \\ u(0,t) &= u(2\pi, t) \\ u(x,0) &= \exp[\sin(x)] \end{aligned} \tag{1}$$

where $u(x,t) \in C^\infty[0, 2\pi]$ is assumed periodic. This represents a wave propagating at velocity 2π in the positive x-direction, while keeping its shape. This problem is in particular appealing due to the existence of a known analytical solution

$$u(x,t) = \exp[\sin(x - 2\pi t)], \tag{2}$$

making it ideal to benchmark different numerical differentiation methods. For the time integration, the fourth-order Runga-Kutta method was implemented, which provides high accuracy for temporal discretization. Thus being able to focus on the analysis of the following three different spatial discretization schemes:

- Second-order centered finite difference approximation

$$\left.\frac{du}{dx}\right|_{x_j} = \frac{u_{j+1} - u_{j-1}}{2\Delta x}. \tag{3}$$

- Fourth-order centered finite difference approximation

$$\left.\frac{du}{dx}\right|_{x_j} = \frac{u_{j-2} - 8u_{j-1} + 8u_{j+1} - u_{j+2}}{12\Delta x}. \tag{4}$$

- Global infinite-order approximation using the Odd Fourier method.

$$\left.\frac{du}{dx}\right|_{x_j} = \sum_{i=0}^{N} \tilde{D}_{ji} u_i \tag{5}$$

By looking into the L_∞ error convergence rates and their long-term behaviors, we can evaluate their efficiency, accuracy, and stability characteristics for scalar hyperbolic problems.

3.1 Convergence Analysis

The L_∞ error data in Table 5 and the corresponding convergence rates in Table 6 for the problem at $t = \pi$ reveals distinct coverngence behaviors.

- **Second-Order Scheme**
  - Starts with a rather high error and steadily converges as N increases.
  - Achieves a final error ($N = 2048$) of 1.26×10^{-4} .
  - The stabilization of the convergence rate happenst at $N \geq 128$ and ≈ 2.0, perfectly reflecting the theoretical second order accuracy.
- **Fourth-Order Scheme**
  - Has an initial error which is lower than the second order scheme.
  - Also shows significantly faster error reduction rate.
  - Reaching a final error value ($N = 2048$) of 1.45×10^{-9}.
  - Convergences rate stabilizes at ≈ 4.0 for $N \geq 512$, reflecting the theoretical fourth-order accuracy.
- **Fourier Scheme**
  - Shows dramatically lower errors even at low resolution compared to the other methods.
  - Achieves extremely rapid initial convergence with a rate of 15.15 between N=8 and N=16.
  - Reaches machine precision at approximately N=16 with an error of 2.28×10^{-8}.
  - After reaching machine precision, error remains constant around 6.3×10^{-9} for higher resolutions up to N=512.
  - Truncation to zero error at N=1024 and N=2048 indicates numerical precision limits.

N	Second Order	Fourth Order	Fourier
8	1.73×10^{0}	6.38×10^{-1}	8.31×10^{-4}
16	1.06×10^{0}	2.02×10^{-1}	2.28×10^{-8}
32	4.42×10^{-1}	2.13×10^{-2}	6.20×10^{-9}
64	1.31×10^{-1}	1.41×10^{-3}	6.34×10^{-9}
128	3.26×10^{-2}	9.16×10^{-5}	6.36×10^{-9}
256	8.06×10^{-3}	5.83×10^{-6}	6.36×10^{-9}
512	2.01×10^{-3}	3.74×10^{-7}	6.36×10^{-9}
1024	5.03×10^{-4}	2.94×10^{-8}	0.00×10^{0}
2048	1.26×10^{-4}	7.80×10^{-9}	0.00×10^{0}

Table 5: L_∞-Error for the Scalar Hyperbolic Problem at $t = \pi$

N	Second Order	Fourth Order	Fourier
16	0.71	1.66	15.15
32	1.26	3.24	1.88
64	1.75	3.92	-0.03
128	2.01	3.95	-0.00
256	2.01	3.97	0.00
512	2.00	3.96	-0.00
1024	2.00	3.67	inf
2048	2.00	1.91	nan

Table 6: Convergence Rates for the Scalar Hyperbolic Problem

In conclusion, the observed convergence rates match the theoretical expectation for the finite difference schemes, while the Fourier method demonstrates spectacular convergence characteristics. It shows an extremely rapid initial error reduction (rate of 15.15), quickly reaching machine precision at merely N=16, after which the error remains constant due to floating-point limitations.

In order to achieve the same error as the second-order scheme at $N = 2048$ (1.26×10^{-4}), the fourth-order scheme requires only approximately $N = 128$ (9.16×10^{-5}) grid points, while the Fourier scheme reaches far better accuracy with just $N = 8$ (8.31×10^{-4}) and dramatically better at $N = 16$ (2.28×10^{-8}). This demonstrates the exceptional efficiency of the spectral method, which achieves accuracy orders of magnitude better with far fewer grid points. These results demonstrate that the Fourier method should be the clear choice for problems with smooth, periodic solutions, where it can provide machine-precision accuracy with minimal computational resources.

One final important note is how I addressed the challenge of π being irrational, which means that any fixed time step dt would likely not divide it evenly, making it difficult to reach $t_{final} = \pi$. I solved this issue, by setting $dt = 0.001$ for all methods and then instead of taking π as my t_{final}, I used:

$$t'_{final} = dt \cdot \lfloor t_{final}/dt \rfloor \tag{6}$$

3.2 Longterm Convergence

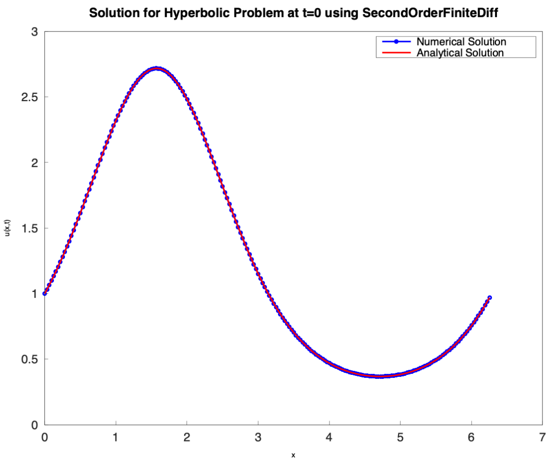
In this final exercise we analyze the longtime convergence behavior of the second-order ($N = 200$) and spectral Fourier method ($N = 10$), when solving the scalar hyperbolic problem, which is depicted in Figure 1. When looking at the time steps $t = 0, 100, 200$ we can observe the following:

- **Initial conditions** ($t = 0$)
  - **Second-Order Method**: This being the initial condition the analytical and numerical solution are identical.
  - **Spectral Fourier Method**: Similar to the second-order method the analytical and numerical are identical.
- **Medium term** ($t = 100$)
  - **Second-Order Method**: Beginning to show phase and amplitude errors. The numerical solution has small oscillations are visible near steep gradients, furthermore the solution is slightly shifted to the right.
  - **Spectral Fourier Method**: Even with just 10 grid points, the numerical solution maintains the identical shape to the analytical solution.
- **Long term** ($t = 200$)
  - **Second-Order Method**: Shows substantial degradation with the numerical solution now being noticeably out of phase. Around $x = 5$ we can observe weird oscillations, particular in the region around $x = 5$, which indicates numerical instability.
  - **Spectral Fourier Method**: Continues of being identical to the analytical solution.

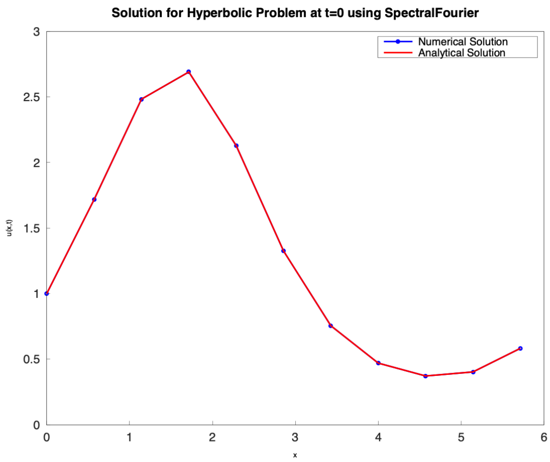
These results lead to the following four key observations:

1. **Stability**: Second Order method over time starts to accumulate phase errors and develops instabilities, while the spectral Fourier method maintains an exact solution.

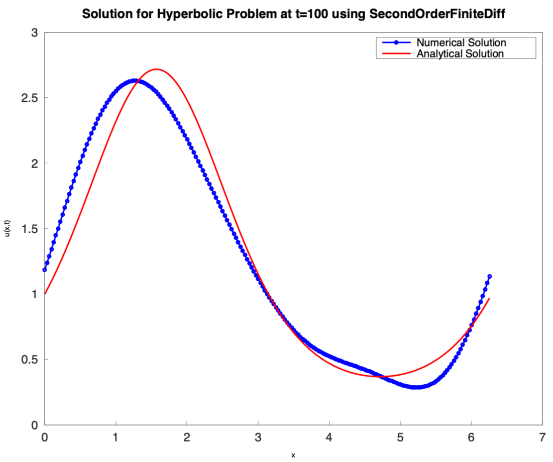
2. **Efficiency**: The spectral method is significantly more efficient by representing very accurate numerical solution with 20 fold less grid points, compared to the second order.
3. **Dispersion**: The second order method showed typical dispersion characteristics, where different components propagate at different speeds, leading to phase errors. The spectral method didn't show any signs of dispersion characteristics.
4. **Resolution**: This comparison showed how dramatically better the Fourier method (global) can capture propagation dynamics with fewer grid points.



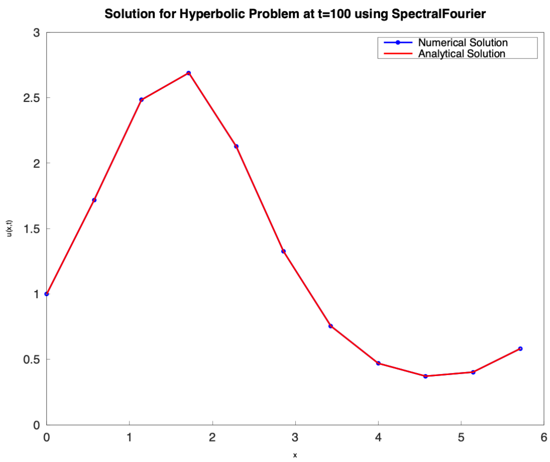
(a) Second Order at $t = 0$



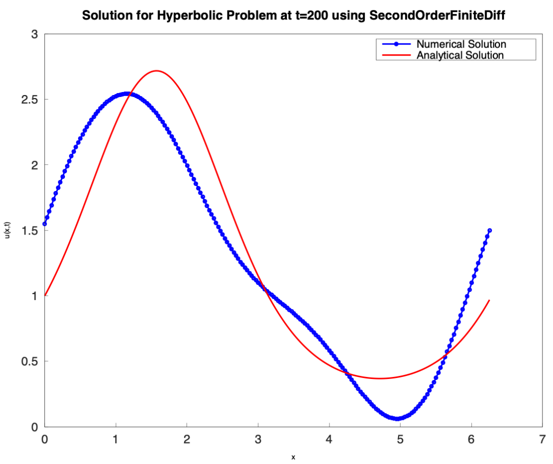
(b) Spectral Fourier at $t = 0$



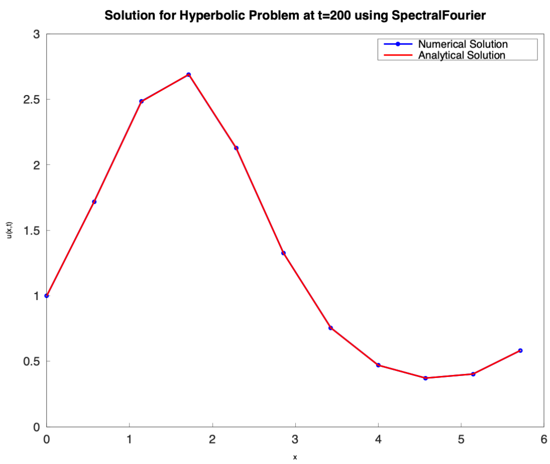
(c) Second Order at $t = 100$



(d) Spectral Fourier at $t = 100$



(e) Second Order at $t = 200$



(f) Spectral Fourier at $t = 200$

Figure 1: **Comparison of Second Order Finite Difference and Spectral Fourier**