Exercise Sheet 1

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Exercise 1

A $2m$-order local interpolating polynomial to $u(x)$ in the neighborhood of x_j given as

$$u(x) = \sum_{k=-m}^{m} u_{j+k} L_{j+k}(x) \tag{1}$$

where the grid function $u_{j+k} = u(x_{j+k})$ and the *Lagrange interpolation polynomial* is

$$L_{j+k} = \prod_{l=-m, l\neq k}^{m} \frac{x - x_{j+l}}{x_{j+k} - x_{j+l}}. \tag{2}$$

A 6th order accurate central finite difference approximation is derived as follows:

$$u(x) = \sum_{k=-3}^{3} u_{j+k} L_{j+k}(x) \tag{3}$$

Because only the Lagrange interpolation polynomial depends on x we can find the derivative of $u(x)$ by differentiating L_{j+k} which can be found using the logarithmic differentiation method, resulting in:

$$\begin{aligned} \ln(L_{j+k}(x)) &= \sum_{l=-m, l\neq k}^{m} \ln\left(\frac{x - x_{j+l}}{x_{j+k} - x_{j+l}}\right) \\ &= \sum_{l=-m, l\neq k}^{m} \left[\ln(x - x_{j+l}) - \ln(x_{j+k} - x_{j+l})\right] \end{aligned} \tag{4}$$

Differentiating both sides with respect to x

$$\begin{aligned} \frac{1}{L_{j+k}(x)} \frac{d}{dx} L_{j+k}(x) &= \sum_{l=-m, l\neq k}^{m} \frac{d}{dx} \left[\ln(x - x_{j+l}) - \ln(x_{j+k} - x_{j+l})\right] \\ &= \sum_{l=-m, l\neq k}^{m} \frac{d}{dx} \ln(x - x_{j+l}) \\ &= \sum_{l=-m, l\neq k}^{m} \frac{1}{x - x_{j+l}} \end{aligned} \tag{5}$$

Solving for the derivative of $L_{j+k}(x)$

$$\frac{d}{dx} L_{j+k}(x) = L_{j+k}(x) \sum_{l=-m, l\neq k}^{m} \frac{1}{x - x_{j+l}} \tag{6}$$

Resulting in the derivative of $u(x)$ being:

$$\begin{aligned} \frac{d}{dx} u(x) &= \sum_{k=-3}^{3} u_{j+k} \frac{d}{dx} L_{j+k}(x) \\ &= \sum_{k=-3}^{3} u_{j+k} \left[L_{j+k}(x) \sum_{l=-3, l\neq k}^{3} \frac{1}{x - x_{j+l}} \right] \end{aligned} \tag{7}$$

Now if we set $x = x_j$ and plug into (7) we need to examine what happens for different k. In order to do so we can rewrite (7) as

$$\frac{d}{dx}u(x_j) = \sum_{k=-3}^{3} u_{j+k}c_k \tag{8}$$

For $k = 0$ we know that $L_j(x_j) = 1$ by definition of the Lagrange polynomials, resulting in the following coefficient

$$\begin{aligned} c_0 &= \left(\frac{1}{x_j - x_{j-3}} + \frac{1}{x_j - x_{j-2}} + \frac{1}{x_j - x_{j-1}} + \frac{1}{x_j - x_{j+1}} + \frac{1}{x_j - x_{j+2}} + \frac{1}{x_j - x_{j+3}}\right) \\ c_0 &= \left(\frac{1}{-3\Delta x} + \frac{1}{-2\Delta x} + \frac{1}{-\Delta x} + \frac{1}{\Delta x} + \frac{1}{2\Delta x} + \frac{1}{3\Delta x}\right) = 0 \end{aligned} \tag{9}$$

For the case where $k \neq 0$, the lagrange polynomial is $L_{j+k}(x_j) = 0$, because in the product used for the langrange polynomial will be a factor of $(x_j - x_j) = 0$. However in the product of $L_{j+k}(x)\sum_{l=-3,l\neq k}^{3} \frac{1}{x-x_{j+l}}$, we have a division by zero when $l = 0$, this creates the indetermined form $0 \cdot \infty$.
Therefore we compute the limit:

$$\begin{aligned} \lim_{x \to x_j} c_k &= \lim_{x \to x_j} \left[L_{j+k}(x) \sum_{l=-3,l\neq k}^{3} \frac{1}{x - x_{j+l}} \right] \\ &= \lim_{x \to x_j} \left[L_{j+k}(x) \left(\frac{1}{x - x_j} + \sum_{l=-3,l\neq k,l\neq 0}^{3} \frac{1}{x - x_{j+l}} \right) \right] \\ &= \lim_{x \to x_j} \left[L_{j+k}(x)\frac{1}{x - x_j} + \underbrace{L_{j+k}(x) \sum_{l=-3,l\neq k,l\neq 0}^{3} \frac{1}{x - x_{j+l}}}_{\to 0} \right] \end{aligned} \tag{10}$$

the second term goes to zero due to $L_{j+k}(x_j) = 0$, but for the first term we can write out the langrange polynomial and take out the term for $l = 0$ in the product resulting in

$$\begin{aligned} \lim_{x \to x_j} c_k &= \lim_{x \to x_j} \frac{1}{x - x_j} \frac{x - x_j}{x_{j+k} - x_j} \prod_{l=-3,l\neq k,l\neq 0}^{3} \frac{x - x_{j+l}}{x_{j+k} - x_{j+l}} \\ &= \lim_{x \to x_j} \frac{1}{x_{j+k} - x_j} \cdot \prod_{l=-3,l\neq k,l\neq 0}^{3} \frac{x - x_{j+l}}{x_{j+k} - x_{j+l}} \end{aligned} \tag{11}$$

Evaluating this at $x = x_j$ leads to a non zero term for $k \neq 0$.

$$c_k = \frac{1}{x_{j+k} - x_j} \prod_{l=-3,l\neq k,l\neq 0}^{3} \frac{x_j - x_{j+l}}{x_{j+k} - x_{j+l}} \tag{12}$$

Using the result from (9) and (12) leads to the following coefficients:
For $k = -3$:

$$\begin{aligned} c_{-3} &= \frac{1}{x_{j-3} - x_j} \prod_{l=-3,l\neq 0,l\neq -3}^{3} \frac{x_j - x_{j+l}}{x_{j-3} - x_{j+l}} \\ &= \frac{1}{-3\Delta x} \cdot \frac{(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j-3} - x_{j-2})(x_{j-3} - x_{j-1})(x_{j-3} - x_{j+1})(x_{j-3} - x_{j+2})(x_{j-3} - x_{j+3})} \\ &= \frac{1}{-3\Delta x} \cdot \frac{(2\Delta x)(\Delta x)(-\Delta x)(-2\Delta x)(-3\Delta x)}{(-\Delta x)(-2\Delta x)(-4\Delta x)(-5\Delta x)(-6\Delta x)} \\ &= \frac{1}{-3\Delta x} \cdot \frac{-12\Delta x^5}{-240\Delta x^5} = \frac{-1}{60\Delta x} \end{aligned} \tag{13}$$

For $k = -2$:

$$\begin{aligned}
c_{-2} &= \frac{1}{x_{j-2} - x_j} \prod_{l=-3 l\neq 0, l\neq -2}^{3} \frac{x_j - x_{j+l}}{x_{j-2} - x_{j+l}} \\
&= \frac{1}{-2\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j-2} - x_{j-3})(x_{j-2} - x_{j-1})(x_{j-2} - x_{j+1})(x_{j-2} - x_{j+2})(x_{j-2} - x_{j+3})} \\
&= \frac{1}{-2\Delta x} \cdot \frac{(3\Delta x)(\Delta x)(-\Delta x)(-2\Delta x)(-3\Delta x)}{(\Delta x)(-\Delta x)(-3\Delta x)(-4\Delta x)(-5\Delta x)} \\
&= \frac{1}{-2\Delta x} \cdot \frac{-18\Delta x^5}{60\Delta x^5} = \frac{3}{20\Delta x}
\end{aligned} \tag{14}$$

For $k = -1$:

$$\begin{aligned}
c_{-1} &= \frac{1}{x_{j-1} - x_j} \prod_{l=-3, l\neq 0, l\neq -1}^{3} \frac{x_j - x_{j+l}}{x_{j-1} - x_{j+l}} \\
&= \frac{1}{-\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j+1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j-1} - x_{j-3})(x_{j-1} - x_{j-2})(x_{j-1} - x_{j+1})(x_{j-1} - x_{j+2})(x_{j-1} - x_{j+3})} \\
&= \frac{1}{-\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(-\Delta x)(-2\Delta x)(-3\Delta x)}{(2\Delta x)(\Delta x)(-2\Delta x)(-3\Delta x)(-4\Delta x)} \\
&= \frac{1}{-\Delta x} \cdot \frac{-36\Delta x^5}{-48\Delta x^5} = \frac{-3}{4\Delta x}
\end{aligned} \tag{15}$$

For $k = 0$ the coefficient was computed in (9) and is $c_0 = 0$
For $k = 1$:

$$\begin{aligned}
c_{1} &= \frac{1}{x_{j+1} - x_j} \prod_{l=-3, l\neq 0, l\neq 1}^{3} \frac{x_j - x_{j+l}}{x_{j+1} - x_{j+l}} \\
&= \frac{1}{\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j+1} - x_{j-3})(x_{j+1} - x_{j-2})(x_{j+1} - x_{j-1})(x_{j+1} - x_{j+2})(x_{j+1} - x_{j+3})} \\
&= \frac{1}{\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(\Delta x)(-2\Delta x)(-3\Delta x)}{(4\Delta x)(3\Delta x)(2\Delta x)(-\Delta x)(-2\Delta x)} \\
&= \frac{1}{\Delta x} \cdot \frac{36\Delta x^5}{48\Delta x^5} = \frac{3}{4\Delta x}
\end{aligned} \tag{16}$$

For $k = 2$:

$$\begin{aligned}
c_{2} &= \frac{1}{x_{j+2} - x_j} \prod_{l=-3, l\neq 0, l\neq 2}^{3} \frac{x_j - x_{j+l}}{x_{j+2} - x_{j+l}} \\
&= \frac{1}{2\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+3})}{(x_{j+2} - x_{j-3})(x_{j+2} - x_{j-2})(x_{j+2} - x_{j-1})(x_{j+2} - x_{j+1})(x_{j+2} - x_{j+3})} \\
&= \frac{1}{2\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(\Delta x)(-\Delta x)(-3\Delta x)}{(5\Delta x)(4\Delta x)(3\Delta x)(\Delta x)(-\Delta x)} \\
&= \frac{1}{2\Delta x} \cdot \frac{18\Delta x^5}{-60\Delta x^5} = \frac{-3}{20\Delta x}
\end{aligned} \tag{17}$$

For $k = 3$:

$$\begin{aligned}
c_{3} &= \frac{1}{x_{j+3} - x_j} \prod_{l=-3, l\neq 0, l\neq 3}^{3} \frac{x_j - x_{j+l}}{x_{j+3} - x_{j+l}} \\
&= \frac{1}{3\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+2})}{(x_{j+3} - x_{j-3})(x_{j+3} - x_{j-2})(x_{j+3} - x_{j-1})(x_{j+3} - x_{j+1})(x_{j+3} - x_{j+2})} \\
&= \frac{1}{3\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(\Delta x)(-\Delta x)(-2\Delta x)}{(6\Delta x)(5\Delta x)(4\Delta x)(2\Delta x)(\Delta x)} \\
&= \frac{1}{3\Delta x} \cdot \frac{12\Delta x^5}{240\Delta x^5} = \frac{1}{60\Delta x}
\end{aligned} \tag{18}$$

Finally we can put it all together and 6th order accurate central finite difference approximation is given as:

$$\begin{aligned}
\left.\frac{du}{dx}\right|_{x=x_j} &= u_{j-3}\frac{-1}{60\Delta x} + u_{j-2}\frac{3}{20\Delta x} + u_{j-1}\frac{-3}{4\Delta x} + u_{j+1}\frac{3}{4\Delta x} + u_{j+2}\frac{-3}{20\Delta x} + u_{j+3}\frac{1}{60\Delta x} \\
&= \frac{-u_{j-3} + 9u_{j-2} - 45u_{j-1} + 45u_{j+1} - 9u_{j+2} + u_{j+3}}{60\Delta x}
\end{aligned} \tag{19}$$

Exercise 2

Wave Speed

We assume that the solution to the finite difference scheme is rightward traveling wave

$$v(x,t) = e^{ik(x-c_6 t)} \tag{20}$$

We plug (20) into our finite difference scheme (19)

$$\begin{aligned} -ikc_6 e^{ik(x_j-c_6t)} = -\frac{c}{60\Delta x}[&-e^{ik(x_{j-3}-c_6t)} + 9e^{ik(x_{j-2}-c_6t)} - 45e^{ik(x_{j-1}-c_6t)} \\ &+ 45e^{ik(x_{j+1}-c_6t)} - 9e^{ik(x_{j+2}-c_6t)} + e^{ik(x_{j+3}-c_6t)}] \end{aligned} \tag{21}$$

Now divided both side by $e^{ik(x_j-c_6t)}$

$$\begin{aligned} -ikc_6 &= -\frac{c}{60\Delta x}[-e^{ik(x_{j-3}-x_j)} + 9e^{ik(x_{j-2}-x_j)} - 45e^{ik(x_{j-1}-x_j)} \\ &\quad + 45e^{ik(x_{j+1}-x_j)} - 9e^{ik(x_{j+2}-x_j)} + e^{ik(x_{j+3}-x_j)}] \\ &= -\frac{c}{60\Delta x}\left[-e^{-3ik\Delta x} + 9e^{-2ik\Delta x} - 45e^{-ik\Delta x} + 45e^{ik\Delta x} - 9e^{2ik\Delta x} + e^{3ik\Delta x}\right] \end{aligned} \tag{22}$$

We can now use the following formula $e^{ik\phi} - e^{-ik\phi} = 2i\sin(k\phi)$ to simplify the equation even further

$$\begin{aligned} -ikc_6 &= -\frac{c}{60\Delta x}\left[2i\sin(3k\Delta x) - 18i\sin(2k\Delta x) + 90i\sin(k\Delta x)\right] \\ &= -\frac{c}{30\Delta x} i\left[\sin(3k\Delta x) - 9\sin(2k\Delta x) + 45\sin(k\Delta x)\right] \end{aligned} \tag{23}$$

Dividing by $-ik$ we end up with the wave speed for the 6th order approximation

$$c_6(k) = c\frac{45\sin(k\Delta x) - 9\sin(2k\Delta x) + \sin(3k\Delta x)}{30k\Delta x} \tag{24}$$

Phase Error

For the $2m$-order scheme we can define the phase error as

$$e_m(k) = \left|\frac{u(x,t) - v(x,t)}{u(x,t)}\right| = \left|1 - e^{ik(c-c_m)t}\right| \approx kt\,|c - c_m(k)| \tag{25}$$

Now using our result in (24) we can define the phase error for a 6th order scheme as follows

$$e_6(k,t) = kct\left|1 - \frac{45\sin(k\Delta x) - 9\sin(2k\Delta x) + \sin(3k\Delta x)}{30k\Delta x}\right| \tag{26}$$

To measure the accuracy of the 6th scheme, let's introduce the number of grid points per waive length

$$p = \frac{\lambda}{\Delta x} = \frac{2\pi}{k\Delta x} \tag{27}$$

and the number of time the solution returns to itself, due to the periodicity

$$\nu = \frac{ct}{\lambda} \tag{28}$$

Rewriting the phase error (26) in terms of p for k and ν for t results in

$$\begin{aligned} e_6(p,\nu) &= \frac{2\pi}{(\lambda/\Delta x)\Delta x} c\frac{\nu\lambda}{c}\left|1 - \frac{45\sin(2\pi p^{-1}) - 9\sin(2\cdot 2\pi p^{-1}) + \sin(3\cdot 2\pi p^{-1})}{(30\cdot 2\pi p^{-1})}\right| \\ &= 2\pi\nu\left|1 - \frac{45\sin(2\pi p^{-1}) - 9\sin(4\pi p^{-1}) + \sin(6\pi p^{-1})}{(60\pi p^{-1})}\right| \end{aligned} \tag{29}$$

If we now perform a leading-order approximation ($p \to \infty$) this means the terms including p^{-1} become small we can use the Taylor series expansion for sin, which is given by the following formula To perform a leading-order approximation as $p \to \infty$, we use the Taylor series expansion for sine:

$$\sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \tag{30}$$

e now expand each sine term in the numerator: For $\sin(2\pi p^{-1})$:

$$\sin(2\pi p^{-1}) = 2\pi p^{-1} - \frac{(2\pi p^{-1})^3}{3!} + \frac{(2\pi p^{-1})^5}{5!} - \frac{(2\pi p^{-1})^7}{7!} + \mathcal{O}(p^{-9}) \tag{30}$$

For $\sin(4\pi p^{-1})$:

$$\sin(4\pi p^{-1}) = 4\pi p^{-1} - \frac{(4\pi p^{-1})^3}{3!} + \frac{(4\pi p^{-1})^5}{5!} - \frac{(4\pi p^{-1})^7}{7!} + \mathcal{O}(p^{-9}) \tag{31}$$

For $\sin(6\pi p^{-1})$:

$$\sin(6\pi p^{-1}) = 6\pi p^{-1} - \frac{(6\pi p^{-1})^3}{3!} + \frac{(6\pi p^{-1})^5}{5!} - \frac{(6\pi p^{-1})^7}{7!} + \mathcal{O}(p^{-9}) \tag{32}$$

Now we calcualte the numerator for each term of each order.
For the first order terms:

$$45\left(2\pi p^{-1}\right) - 9\left(4\pi p^{-1}\right) + \left(6\pi p^{-1}\right) = (90 - 36 + 6)\pi p^{-1} = 60\pi p^{-1} \tag{33}$$

For the third order terms:

$$45\left(8\pi^3 p^{-3}\right) - 9\left(64\pi^3 p^{-3}\right) + \left(216\pi^3 p^{-3}\right) = (360 - 576 + 216)\pi^3 p^{-3} = 0\pi^3 p^{-3} = 0 \tag{34}$$

For the fifth order terms:

$$45\left(32\pi^5 p^{-5}\right) - 9\left(1024\pi^5 p^{-5}\right) + \left(7776\pi^5 p^{-5}\right) = (1440 - 9216 + 7776)\pi^5 p^{-5} = 0\pi^3 p^{-3} = 0 \tag{35}$$

and seventh order terms:

$$\begin{aligned}
&45\left(-\frac{128\pi^7}{5040}p^{-7}\right) - 9\left(-\frac{16384\pi^7}{5040}p^{-7}\right) + \left(-\frac{279936\pi^7}{5040}p^{-7}\right)\\
&= -\frac{45\cdot 128\pi^7}{5040}p^{-7} + \frac{9\cdot 16384\pi^7}{5040}p^{-7} - \frac{279936\pi^7}{5040}p^{-7}\\
&= -\frac{5760\pi^7}{5040}p^{-7} + \frac{147456\pi^7}{5040}p^{-7} - \frac{279936\pi^7}{5040}p^{-7}\\
&= \frac{-5760 + 147456 - 279936}{5040}\pi^7 p^{-7}\\
&= \frac{-138240}{5040}\pi^7 p^{-7}\\
&= \frac{-192}{7}\pi^7 p^{-7}
\end{aligned} \tag{36}$$

Now, substituting back into our original expression:

$$\begin{aligned}
e_6(p,\nu) &= 2\pi\nu\left|1 - \frac{60\pi p^-1}{60\pi p^-1} + \frac{\frac{-192}{7}\pi^7 p^{-7}}{(60\pi p^{-1})}\right|\\
&= 2\pi\nu\left|\frac{192}{420}\pi^6 p^{-6}\right|\\
&= 2\pi\nu\left|\frac{16}{35}\pi^6 p^{-6}\right|
\end{aligned} \tag{37}$$

We can remove the absolute brackets due to p being a large postive number, resulting in the final step:

$$e_3(p,\nu) = 2\pi\nu\frac{16}{35}\pi^6 p^{-6} = \frac{\pi\nu}{70}\left(\frac{2\pi}{p}\right)^6 \tag{38}$$

We can use now the obtained equation to derive the number of points per wavelength required to ensure that the phase error is bounded by ϵ_3.

$$\begin{aligned}
e_3(p,\nu) &= \frac{\pi\nu}{70}\left(\frac{2\pi}{p}\right)^6 \\
\frac{70e_3(p,\nu)}{\pi\nu} &= \left(\frac{2\pi}{p}\right)^6 \\
\frac{70e_3(p,\nu)}{\pi\nu} &= \left(\frac{2\pi}{p}\right)^6 \\
\sqrt[6]{\frac{70e_3(p,\nu)}{\pi\nu}} &= \frac{2\pi}{p} \\
\sqrt[6]{\frac{70e_3(p,\nu)}{\pi\nu}} &= \frac{2\pi}{p} \\
p &= 2\pi\sqrt[6]{\frac{\pi\nu}{70e_3(p,\nu)}}
\end{aligned} \tag{39}$$

Hence the bound is given as

$$p_3(\epsilon_p,\nu) \geq 2\pi\sqrt[6]{\frac{\pi\nu}{70\epsilon_p}} \tag{40}$$

ϵ_p	2nd Order Scheme (p_1)	4th Order Scheme (p_2)	6th Order Scheme (p_3)
0.1	$p_1 \geq 20\sqrt[2]{\nu}$	$p_2 \geq 7\sqrt[4]{\nu}$	$p_3 \geq 5.5\sqrt[6]{\nu}$
0.01	$p_1 \geq 64\sqrt[2]{\nu}$	$p_2 \geq 13\sqrt[4]{\nu}$	$p_3 \geq 8\sqrt[6]{\nu}$

Table 1: Comparison of numerical scheme requirements for different error tolerances (ϵ_p)

Exercise 3

k	Minimum N	Max Relative Error
2	22	$3.78 \cdot 10^{-6}$
4	32	$5.62 \cdot 10^{-6}$
6	42	$4.12 \cdot 10^{-6}$
8	52	$2.44 \cdot 10^{-6}$
10	60	$6.44 \cdot 10^{-6}$
12	80	$1.67 \cdot 10^{-5}$

Table 2: Relationship between k values, minimum N requirements, and maximum relative errors