Exercise Sheet 1

Dennys Huber

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1 Exercise 1

A $2m$-order local interpolating polynomial to $u(x)$ in the neighborhood of x_j given as

$$u(x) = \sum_{k=-m}^{m} u_{j+k} L_{j+k}(x) \tag{1}$$

where the grid function $u_{j+k} = u(x_{j+k})$ and the *Lagrange interpolation polynomial* is

$$L_{j+k} = \prod_{l=-m, l\neq k}^{m} \frac{x - x_{j+l}}{x_{j+k} - x_{j+l}}. \tag{2}$$

A 6th order accurate central finite difference approximation is derived as follows:

$$u(x) = \sum_{k=-3}^{3} u_{j+k} L_{j+k}(x) \tag{3}$$

Because only the Lagrange interpolation polynomial depends on x we can find the derivative of $u(x)$ by differentiating L_{j+k} which can be found using the logarithmic differentiation method, resulting in:

$$\begin{aligned} \ln(L_{j+k}(x)) &= \sum_{l=-m, l\neq k}^{m} \ln\left(\frac{x - x_{j+l}}{x_{j+k} - x_{j+l}}\right) \\ &= \sum_{l=-m, l\neq k}^{m} [\ln(x - x_{j+l}) - \ln(x_{j+k} - x_{j+l})] \end{aligned} \tag{4}$$

Differentiating both sides with respect to x

$$\begin{aligned} \frac{1}{L_{j+k}(x)} \frac{d}{dx} L_{j+k}(x) &= \sum_{l=-m, l\neq k}^{m} \frac{d}{dx} [\ln(x - x_{j+l}) - \ln(x_{j+k} - x_{j+l})] \\ &= \sum_{l=-m, l\neq k}^{m} \frac{d}{dx} \ln(x - x_{j+l}) \\ &= \sum_{l=-m, l\neq k}^{m} \frac{1}{x - x_{j+l}} \end{aligned} \tag{5}$$

Solving for the derivative of $L_{j+k}(x)$

$$\frac{d}{dx} L_{j+k}(x) = L_{j+k}(x) \sum_{l=-m, l\neq k}^{m} \frac{1}{x - x_{j+l}} \tag{6}$$

Resulting in the derivative of $u(x)$ being:

$$\begin{aligned} \frac{d}{dx} u(x) &= \sum_{k=-3}^{3} u_{j+k} \frac{d}{dx} L_{j+k}(x) \\ &= \sum_{k=-3}^{3} u_{j+k} \left[L_{j+k}(x) \sum_{l=-3, l\neq k}^{3} \frac{1}{x - x_{j+l}} \right] \end{aligned} \tag{7}$$

Now if we set $x = x_j$ and plug into (7) we need to examine what happens for different k. In order to do so we can rewrite (7) as

$$\frac{d}{dx}u(x_j) = \sum_{k=-3}^{3} u_{j+k}c_k \tag{8}$$

For $k = 0$ we know that $L_j(x_j) = 1$ by definition of the Lagrange polynomials, resulting in the following coefficient

$$\begin{aligned} c_0 &= \left(\frac{1}{x_j - x_{j-3}} + \frac{1}{x_j - x_{j-2}} + \frac{1}{x_j - x_{j-1}} + \frac{1}{x_j - x_{j+1}} + \frac{1}{x_j - x_{j+2}} + \frac{1}{x_j - x_{j+3}}\right) \\ c_0 &= \left(\frac{1}{-3\Delta x} + \frac{1}{-2\Delta x} + \frac{1}{-\Delta x} + \frac{1}{\Delta x} + \frac{1}{2\Delta x} + \frac{1}{3\Delta x}\right) = 0 \end{aligned} \tag{9}$$

For the case where $k \neq 0$, the lagrange polynomial is $L_{j+k}(x_j) = 0$, because in the product used for the langrange polynomial will be a factor of $(x_j - x_j) = 0$. However in the product of $L_{j+k}(x)\sum_{l=-3, l\neq k}^{3} \frac{1}{x - x_{j+l}}$, we have a division by zero when $l = 0$, this creates the indetermined form $0 \cdot \infty$.
Therefore we compute the limit:

$$\begin{aligned} \lim_{x\to x_j} c_k &= \lim_{x\to x_j} \left[L_{j+k}(x) \sum_{l=-3, l\neq k}^{3} \frac{1}{x - x_{j+l}} \right] \\ &= \lim_{x\to x_j} \left[L_{j+k}(x) \left(\frac{1}{x - x_j} + \sum_{l=-3, l\neq k, l\neq 0}^{3} \frac{1}{x - x_{j+l}} \right) \right] \\ &= \lim_{x\to x_j} \left[L_{j+k}(x) \frac{1}{x - x_j} + \underbrace{L_{j+k}(x) \sum_{l=-3, l\neq k, l\neq 0}^{3} \frac{1}{x - x_{j+l}}}_{\to 0} \right] \end{aligned} \tag{10}$$

the second term goes to zero due to $L_{j+k}(x_j) = 0$, but for the first term we can write out the langrange polynomial and take out the term for $l = 0$ in the product resulting in

$$\begin{aligned} \lim_{x\to x_j} c_k &= \lim_{x\to x_j} \frac{1}{x - x_j} \frac{x - x_j}{x_{j+k} - x_j} \prod_{l=-3, l\neq k, l\neq 0} \frac{x - x_{j+l}}{x_{j+k} - x_{j+l}} \\ &= \lim_{x\to x_j} \frac{1}{x_{j+k} - x_j} \cdot \prod_{l=-3, l\neq k, l\neq 0} \frac{x - x_{j+l}}{x_{j+k} - x_{j+l}} \end{aligned} \tag{11}$$

Evaluating this at $x = x_j$ leads to a non zero term for $k \neq 0$.

$$c_k = \frac{1}{x_{j+k} - x_j} \prod_{l=-3, l\neq k, l\neq 0} \frac{x_j - x_{j+l}}{x_{j+k} - x_{j+l}} \tag{12}$$

Using the result from (9) and (12) leads to the following coefficients:
For $k = -3$:

$$\begin{aligned} c_{-3} &= \frac{1}{x_{j-3} - x_j} \prod_{l\neq 0, l\neq -3} \frac{x_j - x_{j+l}}{x_{j-3} - x_{j+l}} \\ &= \frac{1}{-3\Delta x} \cdot \frac{(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j-3} - x_{j-2})(x_{j-3} - x_{j-1})(x_{j-3} - x_{j+1})(x_{j-3} - x_{j+2})(x_{j-3} - x_{j+3})} \\ &= \frac{1}{-3\Delta x} \cdot \frac{(2\Delta x)(\Delta x)(-\Delta x)(-2\Delta x)(-3\Delta x)}{(-\Delta x)(-2\Delta x)(-4\Delta x)(-5\Delta x)(-6\Delta x)} \\ &= \frac{1}{-3\Delta x} \cdot \frac{-12\Delta x^5}{-240\Delta x^5} = \frac{-1}{60\Delta x} \end{aligned} \tag{13}$$

For $k = -2$:

$$\begin{aligned} c_{-2} &= \frac{1}{x_{j-2} - x_j} \prod_{l \neq 0, l \neq -2} \frac{x_j - x_{j+l}}{x_{j-2} - x_{j+l}} \\ &= \frac{1}{-2\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j-2} - x_{j-3})(x_{j-2} - x_{j-1})(x_{j-2} - x_{j+1})(x_{j-2} - x_{j+2})(x_{j-2} - x_{j+3})} \\ &= \frac{1}{-2\Delta x} \cdot \frac{(3\Delta x)(\Delta x)(-\Delta x)(-2\Delta x)(-3\Delta x)}{(\Delta x)(-\Delta x)(-3\Delta x)(-4\Delta x)(-5\Delta x)} \\ &= \frac{1}{-2\Delta x} \cdot \frac{-18\Delta x^5}{60\Delta x^5} = \frac{3}{20\Delta x} \end{aligned} \tag{14}$$

For $k = -1$:

$$\begin{aligned} c_{-1} &= \frac{1}{x_{j-1} - x_j} \prod_{l \neq 0, l \neq -1} \frac{x_j - x_{j+l}}{x_{j-1} - x_{j+l}} \\ &= \frac{1}{-\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j+1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j-1} - x_{j-3})(x_{j-1} - x_{j-2})(x_{j-1} - x_{j+1})(x_{j-1} - x_{j+2})(x_{j-1} - x_{j+3})} \\ &= \frac{1}{-\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(-\Delta x)(-2\Delta x)(-3\Delta x)}{(2\Delta x)(\Delta x)(-2\Delta x)(-3\Delta x)(-4\Delta x)} \\ &= \frac{1}{-\Delta x} \cdot \frac{-36\Delta x^5}{-48\Delta x^5} = \frac{-3}{4\Delta x} \end{aligned} \tag{15}$$

For $k = 0$ the coefficient was computed in (9) and is $c_0 = 0$
For $k = 1$:

$$\begin{aligned} c_{1} &= \frac{1}{x_{j+1} - x_j} \prod_{l \neq 0, l \neq 1} \frac{x_j - x_{j+l}}{x_{j+1} - x_{j+l}} \\ &= \frac{1}{\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+2})(x_j - x_{j+3})}{(x_{j+1} - x_{j-3})(x_{j+1} - x_{j-2})(x_{j+1} - x_{j-1})(x_{j+1} - x_{j+2})(x_{j+1} - x_{j+3})} \\ &= \frac{1}{\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(\Delta x)(-2\Delta x)(-3\Delta x)}{(4\Delta x)(3\Delta x)(2\Delta x)(-\Delta x)(-2\Delta x)} \\ &= \frac{1}{\Delta x} \cdot \frac{36\Delta x^5}{48\Delta x^5} = \frac{3}{4\Delta x} \end{aligned} \tag{16}$$

For $k = 2$:

$$\begin{aligned} c_{2} &= \frac{1}{x_{j+2} - x_j} \prod_{l \neq 0, l \neq 2} \frac{x_j - x_{j+l}}{x_{j+2} - x_{j+l}} \\ &= \frac{1}{2\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+3})}{(x_{j+2} - x_{j-3})(x_{j+2} - x_{j-2})(x_{j+2} - x_{j-1})(x_{j+2} - x_{j+1})(x_{j+2} - x_{j+3})} \\ &= \frac{1}{2\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(\Delta x)(-\Delta x)(-3\Delta x)}{(5\Delta x)(4\Delta x)(3\Delta x)(\Delta x)(-\Delta x)} \\ &= \frac{1}{2\Delta x} \cdot \frac{18\Delta x^5}{-60\Delta x^5} = \frac{-3}{20\Delta x} \end{aligned} \tag{17}$$

For $k = 3$:

$$\begin{aligned} c_{3} &= \frac{1}{x_{j+3} - x_j} \prod_{l \neq 0, l \neq 3} \frac{x_j - x_{j+l}}{x_{j+3} - x_{j+l}} \\ &= \frac{1}{3\Delta x} \cdot \frac{(x_j - x_{j-3})(x_j - x_{j-2})(x_j - x_{j-1})(x_j - x_{j+1})(x_j - x_{j+2})}{(x_{j+3} - x_{j-3})(x_{j+3} - x_{j-2})(x_{j+3} - x_{j-1})(x_{j+3} - x_{j+1})(x_{j+3} - x_{j+2})} \\ &= \frac{1}{3\Delta x} \cdot \frac{(3\Delta x)(2\Delta x)(\Delta x)(-\Delta x)(-2\Delta x)}{(6\Delta x)(5\Delta x)(4\Delta x)(2\Delta x)(\Delta x)} \\ &= \frac{1}{3\Delta x} \cdot \frac{12\Delta x^5}{240\Delta x^5} = \frac{1}{60\Delta x} \end{aligned} \tag{18}$$

Finally we can put it all together and 6th order accurate central finite difference approximation is given as:

$$\begin{aligned} \left.\frac{du}{dx}\right|_{x=x_j} &= u_{j-3}\frac{-1}{60\Delta x} + u_{j-2}\frac{3}{20\Delta x} + u_{j-1}\frac{-3}{4\Delta x} + u_{j+1}\frac{3}{4\Delta x} + u_{j+2}\frac{-3}{20\Delta x} + u_{j+3}\frac{1}{60\Delta x} \\ &= \frac{-u_{j-3} + 9u_{j-2} - 45u_{j-1} + 45u_{j+1} - 9u_{j+2} + u_{j+3}}{60\Delta x} \end{aligned} \tag{19}$$