Exercise Sheet 3

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1 Fourier-Galerking Approximation: Exercise 1

1.1 Derivation

We are considering the variable coefficient problem

$$\frac{\partial u}{\partial t} + \sin(x)\frac{\partial u}{\partial x} = 0 \tag{1}$$

with periodic boundary condition.
We assume that $u(x,t)$ is periodic in $x \in [0, 2\pi]$ and use a truncated Fourier series to define

$$u_N = \sum_{n=0}^{N} \hat{u}_n(t)\phi_n(x) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \hat{u}_n(t)e^{inx} \tag{2}$$

where the continues expansion coefficients $\hat{u}_n$ are defined as (see Lecture 8)

$$\hat{u}_n(t) = \frac{1}{2\pi}\int_0^{2\pi} u(x,t)e^{-inx}dx. \tag{3}$$

We can now substitute u_N into the PDE (1):

$$\frac{\partial u_N}{\partial t} + \sin(x)\frac{\partial u_N}{\partial x} = 0 \tag{4}$$

Computing each term

- *Time derivative*:

$$\frac{\partial u_N}{\partial t} = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \frac{d\hat{u}_n}{dt}e^{inx} \tag{5}$$

- *Spatial derivative*:

$$\frac{\partial u_N}{\partial x} = \sin(x)\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \hat{u}_n \frac{d}{dx}\left[e^{inx}\right] = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} in\hat{u}_n \sin(x)e^{inx} \tag{6}$$

The Residual is now given as follows

$$R_N(x,t) = \frac{\partial u_N}{\partial t} - \mathcal{L}u_N = \frac{\partial u_N}{\partial t} + \sin(x)\frac{\partial u_N}{\partial x} \tag{7}$$

We enforce $(R_N, \psi_m)_w = 0, \forall m \in [-N/2, N/2]$, where $\psi_m = \frac{1}{\gamma_m}\phi_m$, so that $(\phi_m, \psi_n)_w = \delta_{mn}$.

$$(R_N, \psi_m)_w = \int_0^{2\pi}\left(\frac{\partial u_N}{\partial t} - \mathcal{L}u_N\right)\overline{\psi_m}w dx = \int_0^{2\pi}\left(\frac{\partial u_N}{\partial t} + \sin(x)\frac{\partial u_N}{\partial x}\right)\frac{1}{2\pi}e^{-imx} = 0 \tag{8}$$

Choosing $w(x) = 1$, and using orthonormal basis $\psi_m(x) = \frac{1}{2\pi}e^{imx}$, we have:

$$(\phi_n, \psi_m)_w = \int_0^{2\pi} e^{inx}\cdot\frac{1}{2\pi}e^{-imx}dx = \delta_{nm}. \tag{9}$$

We now plugin the results of (5) and (6) into (8) we get

$$(R_N, \psi_m)_w = \int_0^{2\pi}\left(\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \frac{d\hat{u}_n}{dt}e^{inx} + \sin(x)\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} in\hat{u}_n e^{inx}\right)\cdot\frac{1}{2\pi}e^{-imx}dx. \tag{10}$$

First lets simplify the first term inside the integral

$$\int_0^{2\pi} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \frac{d\hat{u}_n}{dt} e^{inx} \frac{1}{2\pi} e^{-imx} dx = \frac{1}{2\pi} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \frac{d\hat{u}_n}{dt} \int_0^{2\pi} e^{i(n-m)x} = \frac{1}{2\pi} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \frac{d\hat{u}_n}{dt} 2\pi \delta_{nm} = \frac{d\hat{u}_m}{dt} \tag{11}$$

For the second term we can rewrite sine as

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \tag{12}$$

and then substitute it into the second term

$$\begin{aligned} \int_0^{2\pi} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} in\hat{u}_n \sin(x) e^{inx} \frac{1}{2\pi} e^{-imx} dx &= \frac{1}{2\pi} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} in\hat{u}_n \int_0^{2\pi} \frac{e^{ix} - e^{-ix}}{2i} e^{i(n-m)x} dx \\ &= \frac{1}{2\pi} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} in\hat{u}_n \int_0^{2\pi} \frac{e^{i(n-m+1)x} - e^{i(n-m-1)x}}{2i} dx \end{aligned} \tag{13}$$

using the orthogonality property again, this integral becomes non-zero only when:

$$\begin{aligned} n - m + 1 = 0 \Rightarrow n = m - 1 \\ n - m - 1 = 0 \Rightarrow n = m + 1 \end{aligned} \tag{14}$$

Therefore the second term becomes

$$i(m-1)\hat{u}_{m-1} \frac{1}{2\pi} \frac{\int_0^{2\pi} e^0 dx}{2i} - i(m+1)\hat{u}_{m+1} \frac{1}{2\pi} \frac{\int_0^{2\pi} e^0 dx}{2i} = \frac{1}{2} \left[(m-1)\hat{u}_{m-1} - (m+1)\hat{u}_{m+1} \right] \tag{15}$$

Now we can combine both terms, resulting in

$$\frac{d\hat{u}_m}{dt} + \frac{1}{2} \left[(m-1)\hat{u}_{m-1} - (m+1)\hat{u}_{m+1} \right] = 0 \tag{16}$$

Rearranging this result leads to our systems of ODEs for the Fourier-Galerking approximation

$$\frac{d\hat{u}_m}{dt} = \frac{(m+1)\hat{u}_{m+1} - (m-1)\hat{u}_{m-1}}{2} \tag{17}$$

and the initial condition is given as

$$\hat{u}_m(0) = \frac{1}{2\pi} \int_0^{2\pi} g(x) e^{-imx} dx \tag{18}$$

1.2 Is $P_N u = u_N$?

If we apply the Fourier-Galerking Approximation to the equation satisfied by the exact solution u, we get

$$\frac{d\hat{u}_m}{dt} = \frac{(m+1)\hat{u}_{m+1} - (m-1)\hat{u}_{m-1}}{2} + \epsilon_T \tag{19}$$

where ϵ_T is the truncation error. In order for the $P_N u = u_N$ to hold ϵ_T has to be 0.

$$P_N \left(\frac{\partial u}{\partial t} + \sin(x) \frac{\partial u}{\partial x} \right) = \frac{\partial P_N u}{\partial t} + P_N \left(\sin(x) \frac{\partial u}{\partial x} \right) = P_N(0) = 0 \tag{20}$$

The Problem is that in general

$$P_N \left(\sin(x) \frac{\partial u}{\partial x} \right) \neq \sin(x) \frac{\partial P_N u}{\partial x} \tag{21}$$

because when we are multiplying $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$ with $\frac{\partial u}{dx} = \sum_{-\infty}^{\infty} in\hat{u}_n e^{inx}$ it creates frequencies that are outside our truncated space, created by the projection into finite space by P_N.

2 Fourier-Galerkin Approximation: Exercise 2

We are considering the variable coefficient problem

$$\frac{\partial u}{\partial t} + \sin(x)\frac{\partial u}{\partial x} = 0 \tag{22}$$

with Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0 \tag{23}$$

For our Basis function we choose the sine as our basis due to the it naturally satisfying the boundary condition

$$\sin(nx)|_{x=0} = \sin(nx)|_{x=\pi} = 0 \tag{24}$$

therefore we define for the approximation

$$u_N(x,t) = \sum_{n=1}^{N} \hat{u}_n(t)\sin(nx) \tag{25}$$

If we now follow the Galerking Approach first we substitute our approximation u_N into the PDE:

$$\frac{\partial u_N}{\partial t} + \sin(x)\frac{\partial u_N}{\partial x} = 0 \tag{26}$$

The residual is then given as

$$R_N(x,t) = \frac{\partial u_N}{\partial t} + \sin(x)\frac{\partial u_N}{\partial x} \tag{27}$$

Computing each term we get:

- *Time derivative*:

$$\frac{\partial u_N}{\partial t} = \sum_{n=1}^{N} \frac{d\hat{u}_n(t)}{dt}\sin(nx) \tag{28}$$

- *Spatial derivative*:

$$\frac{\partial u_N}{\partial x} = \sum_{n=1}^{N} \hat{u}_n(t) n\cos(nx) \tag{29}$$

Therefore the residual becomes

$$R_N(x,t) = \sum_{n=1}^{N} \frac{d\hat{u}_n(t)}{dt}\sin(nx) + \sin(x)\sum_{n=1}^{N} \hat{u}_n(t) n\cos(nx) \tag{30}$$

For the final step we want to make the residual orthogonal to the basis function. Hence for sine functions on the interval $[0,\pi]$ with weight function $w(x) = 1$, we have:

$$(\phi_n, \psi_m)_w = \int_0^{\pi} \sin(mx)\frac{2}{\pi}\sin(nx)dx = \delta_{mn} \tag{31}$$

Therefore, $\gamma_m = \frac{\pi}{2}$ and our test functions should be:

$$\psi_m = \frac{2}{\pi}\sin(mx) \tag{32}$$

We require that the residual is orthogonal to these test functions: $(R_N, \psi_m)_w = 0 \quad$ for all $m \in [1, N]$ This gives us:

$$(R_N, \psi_m)_w = \frac{2}{\pi}\int_0^{\pi}\left(\sum_{n=1}^{N} \frac{d\hat{u}_n(t)}{dt}\sin(nx) + \sin(x)\sum_{n=1}^{N} \hat{u}_n(t) n\cos(nx)\right)\sin(mx)dx = 0 \tag{33}$$

Looking at each term individually we get

- *First term* with applied orthogonality property:

$$\frac{2}{\pi}\int_0^{\pi}\sum_{n=1}^{N} \frac{d\hat{u}_n(t)}{dt}\sin(nx)\sin(mx)dx = \frac{2}{\pi}\cdot\frac{\pi}{2}\frac{d\hat{u}_m(t)}{dt} = \frac{d\hat{u}_m(t)}{dt} \tag{34}$$

- *Second term*:

$$\frac{2}{\pi}\sum_{n=1}^{N}\hat{u}_n(t)n\int_0^{\pi}\sin(x)\cos(nx)\sin(mx)dx \tag{35}$$

To evaluate the integral, we use the trigonometric identity:

$$\sin(a)\cos(b) = \frac{1}{2}[\sin(a+b)+\sin(a-b)] \tag{36}$$

in our case this results in:

$$\begin{aligned}\sin(x)\cos(nx) &= \frac{1}{2}[\sin(x+nx)+\sin(x-nx)] = \frac{1}{2}[\sin((n+1)x)+\sin((1-n)x)]\\ &= \frac{1}{2}[\sin((n+1)x)-\sin((n-1)x)]\end{aligned} \tag{37}$$

since $\sin((1-n)x) = -\sin((n-1)x)$ for $n > 1$. We can now use another identity:

$$\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b)-\cos(a+b)] \tag{38}$$

resulting in the following two integrals

$$\begin{aligned}\int_0^{\pi}\sin((n+1)x)\sin(mx)dx &= \frac{1}{2}\int_0^{\pi}[\cos((n+1-m)x)-\cos((n+1+m)x)]dx\\ \int_0^{\pi}\sin((n-1)x)\sin(mx)dx &= \frac{1}{2}\int_0^{\pi}[\cos((n-1-m)x)-\cos((n-1+m)x)]dx\end{aligned} \tag{39}$$

Evaluating these cosine intergrals we note that

$$\int_0^{\pi}\cos(kx)dx = \begin{cases}\pi & \text{if } k=0\\ \frac{\sin(k\pi)}{k}=0 & \text{if } k\neq 0\end{cases} \tag{40}$$

This means that the integrals are non-zero only when:

$$\begin{aligned}&n+1-m=0 \Rightarrow n=m-1\\ &n+1+m=0 \text{ (not possible for positive } n,m)\\ &n-1-m=0 \Rightarrow n=m+1\\ &n-1+m=0 \text{ (not possible for positive } n,m)\end{aligned} \tag{41}$$

Evaluating these conditions by using the identities introduces before:

  - When $n = m-1$:

$$\begin{aligned}\frac{2}{\pi}\hat{u}_{m-1}(t)\int_0^{\pi}\sin(x)\cos((m-1)x)\sin(mx)dx &= \frac{2}{\pi}\hat{u}_{m-1}(t)\int_0^{\pi}\sin(x)\cos((m-1)x)\sin(mx)dx\\ &= \frac{2}{\pi}\hat{u}_{m-1}(t)\frac{1}{2}\int_0^{\pi}\sin(mx)^2-(\cos(2x)-cos((2m-2)x))dx\\ &= \frac{2}{\pi}\hat{u}_{m-1}(t)\frac{1}{2}\left(\frac{\pi}{2}-0\right)\\ &= \frac{1}{2}(m-1)\hat{u}_{m-1}(t)\end{aligned} \tag{42}$$

    Note that far any non-zero integer k, $\int_0^{\pi}\cos(kx)dx = 0$. Since 2 and $2m-2$ are non zero integers for $m > 1$ therefore the term which includes the two cosine term evaluates to zero.

  - When $n = m+1$:

$$\begin{aligned}\frac{2}{\pi}\hat{u}_{m+1}(t)\int_0^{\pi}\sin(x)\cos((m+1)x)\sin(mx)dx &= \frac{2}{\pi}\hat{u}_{m+1}(t)\int_0^{\pi}\sin(x)\cos((m+1)x)\sin(mx)dx\\ &= \frac{2}{\pi}\hat{u}_{m+1}(t)\frac{1}{2}\int_0^{\pi}(\cos(2x)-\cos((2m+2)x))-\sin(mx)^2dx\\ &= \frac{2}{\pi}\hat{u}_{m+1}(t)\frac{1}{2}\left(0-\frac{\pi}{2}\right)\\ &= \frac{1}{2}(m+1)\hat{u}_{m+1}(t)\end{aligned} \tag{43}$$

    Note that the same argument can be made for this term Since 2 and $2m+2$ are non zero integers therefore the term which includes the two cosine term evaluates to zero.

If we now combine both terms we end up with following ODE system

$$\frac{d\hat{u}_m(t)}{dt} = \frac{1}{2}[(m+1)\hat{u}_{m+1}(t) - (m-1)\hat{u}_{m-1}(t)] \tag{44}$$

For the initial condition, we project the initial function $g(x)$ onto our basis:

$$\hat{u}_m(0) = \frac{2}{\pi}\int_0^\pi g(x)\sin(mx)dx \tag{45}$$

This completes the Fourier-Galerkin approximation for the given variable coefficient problem with Dirichlet boundary conditions.

3 Tau Approximation

We are considering the variable coefficient problem

$$\frac{\partial u}{\partial t} + \sin(x)\frac{\partial u}{\partial x} = 0 \tag{46}$$

with Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0. \tag{47}$$

The solution approximation is given by:

$$u_N(x,t) = \sum_{n=0}^{N+N_b} \hat{u}_n(t)\cos(nx) \tag{48}$$

where $N_b = 2$ is the number of boundary conditions.
Important to note is that the basis function $\cos(nx)$ does not on its own satisfy the boundary conditions. The residual is then given as

$$R_N(x,t) = \frac{\partial u_N}{\partial t} + \sin(x)\frac{\partial u_N}{\partial x} \tag{49}$$

Computing each term we get:

- *Time derivative*:
$$\frac{\partial u_N}{\partial t} = \sum_{n=0}^{N+2} \frac{d\hat{u}_n(t)}{dt}\cos(nx) \tag{50}$$

- *Spatial derivative*:
$$\frac{\partial u_N}{\partial x} = \sum_{n=0}^{N+2} -\hat{u}_n(t) n \sin(nx) \tag{51}$$

Substituting both terms back into the residual results in

$$R_N(x,t) = \sum_{n=0}^{N+2} \frac{d\hat{u}_n(t)}{dt}\cos(nx) - \sin(x)\sum_{n=0}^{N+2}\hat{u}_n(t) n\sin(nx) \tag{52}$$

We choose as the test function $\psi_m(x) = \frac{2}{\pi}cos(mx)$ for $m \in [0,N]$ with weight function $w(x) = 1$

$$(\phi_n, \psi_m)_w = \int_0^\pi \cos(mx)\frac{2}{\pi}\cos(nx)dx = \delta_{mn} \tag{53}$$

We require that the residual is orthogonal to these test functions: $(R_N, \psi_m)_w = 0 \quad \text{for all } m$
This gives us:

$$(R_N, \psi_m)_w = \frac{2}{\pi}\int_0^\pi \left(\sum_{n=0}^{N+2}\frac{d\hat{u}_n(t)}{dt}\cos(nx) - \sin(x)\sum_{n=0}^{N+2}\hat{u}_n(t) n \sin(nx)\right)\cos(mx)dx = 0 \tag{54}$$

Looking at each term individually

- *First term*:

$$\frac{2}{\pi}\sum_{n=0}^{N+2}\frac{d\hat{u}_n(t)}{dt}\int_0^\pi \cos(nx)\cos(mx)dx \tag{55}$$

Using the orthogonally property of cosine functions

$$\int_0^\pi \cos(nx)\cos(mx)dx = \begin{cases} \frac{\pi}{2} & \text{if } n = m \neq 0 \\ \pi & \text{if } n = m = 0 \\ 0 & \text{if } n \neq m \end{cases} \tag{56}$$

Resulting in

$$\frac{2}{\pi}\sum_{n=0}^{N+2}\frac{d\hat{u}_n(t)}{dt}\int_0^\pi \cos(nx)\cos(mx)dx = \begin{cases} \frac{d\hat{u}_m(t)}{dt} & \text{if } m \neq 0 \\ 2\frac{d\hat{u}_0(t)}{dt} & \text{if } m = 0 \end{cases} \tag{57}$$

- *Second Term*:

$$-\frac{2}{\pi}\sum_{n=0}^{N+2}\hat{u}_n(t)n\int_0^\pi \sin(x)\sin(nx)\cos(mx)dx \tag{58}$$

We can rewrite the integral as follows:

$$\int_0^\pi \sin(x)\sin(nx)\cos(mx)dx = \frac{1}{2}\int_0^\pi \cos((n-1)x) - \cos((n+1)x)\cos(mx)dx \tag{59}$$

splitting up this integral leads to

$$\begin{aligned} \int_0^\pi \cos((n-1)x)\cos(mx)dx &= \frac{1}{2}\int_0^\pi [\cos((n-1+m)x) + \cos((n-1-m)x)]dx \\ \int_0^\pi \cos((n+1)x)\cos(mx)dx &= \frac{1}{2}\int_0^\pi [\cos((n+1+m)x) + \cos((n+1-m)x)]dx \end{aligned} \tag{60}$$

For a cosine integral:

$$\int_0^\pi \cos(kx)dx = \begin{cases} \pi & \text{if } k = 0 \\ \frac{\sin(k\pi)}{k} = 0 & \text{if } k \neq 0 \end{cases} \tag{61}$$

Therefore these integrals are non-zero only when:

$$\begin{aligned} &(n-1+m) = 0 \Rightarrow n = 1-m \\ &(n-1-m) = 0 \Rightarrow n = m+1 \\ &(n+1+m) = 0 \Rightarrow n = -(m+1) \text{ (impossible for } n, m \geq 0) \\ &(n+1-m) = 0 \Rightarrow n = m-1 \end{aligned} \tag{62}$$

Therfore for $m \geq 1$ the only non-zero contributions come from

  - $n = m+1$: $\frac{m+1}{2}\hat{u}_{m+1}$
  - $n = m-1$: $\frac{m-1}{2}\hat{u}_{m-1}$

For $m = 0$ the non-zero term is

  - $n = 1$: $-\frac{1}{2}\hat{u}_1$

Combining both terms for $m \geq 1$ we end up with

$$\frac{\hat{u}_m(t)}{dt} = \frac{1}{2}\left[(m+1)\hat{u}_{m+1}(t) - (m-1)\hat{u}_{m-1}(t)\right] \tag{63}$$

and for $m = 0$

$$\frac{\hat{u}_0(t)}{dt} = \frac{1}{4}\hat{u}_1(t) \tag{64}$$

Applying the boundary conditions then results in

1. $u_N(0,t) = \sum_{n=0}^{N+2}\hat{u}_n(t)\cos(0) = \sum_{n=0}^{N+2}\hat{u}_n(t) = 0$
2. $u_N(\pi,t) = \sum_{n=0}^{N+2}\hat{u}_n(t)\cos(m\pi) = \sum_{n=0}^{N+2}(-1)^n\hat{u}_n(t) = 0$

which provide additional constraints and the initial conditions are for $m \in [0,$

$$\hat{u}_m(0) = \frac{2}{\pi}\int_0^\pi g(x)\cos(mx)dx \tag{65}$$

4 Fourier-Collocation Approximation for Burgers Equation

For this exercise we consider Burgers' equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2} \tag{66}$$

with periodic boundary conditions.
We introduce a grid with $N+1$ equidistant grid points (Odd Method)

$$x_j = \frac{2\pi j}{N+1}, \quad j = 0, \ldots, N \tag{67}$$

We use the trigonometric polynomial to approximate the solution:

$$u_N(x,t) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \tilde{u}_n(t) e^{inx} = \sum_{j=0}^{N} u_N(x_j,t) h_j(x) \tag{68}$$

where $h_j(x)$ is defined as

$$h_j(x) = \frac{1}{N} \frac{\sin\left(\frac{N+1}{2}(x - x_j)\right)}{\sin\left(\frac{x-x_j}{2}\right)} \tag{69}$$

these functions have the property $h_j(x_k) = \delta_{jk}$ at the collocation points.
The residual at each collocation point is

$$R_N(x_j,t) = \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} - \epsilon \frac{\partial^2 u}{\partial x^2} \tag{70}$$

For $N+1$ equations we require

$$R_N(x_j,t) = 0 \quad j = 0, \ldots, N \tag{71}$$

Computing each term in the residual

- *Time Derivative*:
$$\frac{\partial u_N}{\partial t}(x_j,t) = \frac{du_N(x_j,t)}{dt} \tag{72}$$

- *Second Spatial Derivative*: Using the Odd method's second order differentiation matrix $\tilde{D}^{(2)} = \tilde{D} \cdot \tilde{D}$
$$\frac{\partial^2 u_N}{\partial x^2}(x_j,t) = \sum_{k=0}^{N} \tilde{D}^{(2)}_{jk} u_N(x_k,t) \tag{73}$$

- *Nonlinear Spatial Derivative*:
$$\frac{\partial u^2}{\partial x} = \sum_{k=0}^{N} \tilde{D}_{jk} \left[u_N(x_k,t)\right]^2 \tag{74}$$

Substituting this back into the equation we can derive the system of $N+1$ ODEs

$$\frac{du_N(x_j,t)}{dt} = -\frac{1}{2}\sum_{k=0}^{N} \tilde{D}_{jk} \left[u_N(x_k,t)\right]^2 + \epsilon \sum_{k=0}^{N} \tilde{D}^{(2)}_{jk} u_N(x_k,t) \tag{75}$$

and $N+1$ initial conditions

$$u_N(x_j,0) = g(x_j) \quad \forall j = 0, \ldots, N \tag{76}$$