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LECTURE II: MTH 103

TOPIC: Concept of limit of a function

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Let us recapitulate briefly what we discussed last week in Lecture I

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Notation of functions

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$$\{x \epsilon R : a \leq x \leq b\}\ [a, b]\ closed\ interval$$

$$\{x \epsilon R : a < x < b\}\ (a, b)\ open\ interval$$

$$f(x) \rightarrow function\ of\ x$$

$$g(t) \rightarrow function\ of\ t$$

A function is a relation or mapping that assigns every $x £ X$ $a\ unique\ y £ Y$. i.e for $\forall\ x £ X$ there is $y £ Y$. A function can also be defined as a relation between two real variables $y\ and\ x$. thus we called x the independent variable and y the dependent. A Function arises where ever one quantity depends on another.

Examples: The area A of a circle depends on the radius of the circle. $A = \pi r^2$

A function is a relation or mapping that assigns every i.e. for there is A function can also be defined as a relation between two real variables. thus we called x the independent variable and y the dependent. A Function arises where ever one quantity depends on another.

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LIMIT

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Suppose f(x) is a given function of x. then, if we can make f(x) as near as we want to a given number L by choosing x sufficiently near to a number a, then L is said to be the limit of f(x) as x$\rightarrow$ a.

This means that the limit of f(x) as x approaches or tend to a i.e. x

A more precise definition of limit of a function is as follows: if f(x) approaches a limit L as x, then for any number it must be possible to find a number δ>0

This means that the limit of f(x) as x approaches or tend to a i.e. $x \to a$ *is L*

A more precise definition of limit of a function is as follows: if f(x) approaches a limit L as $x \to a$ then for any number K > 0 (*however small*) it must be possible to find a number δ>0

Such that $|f(x) - L| < K$ when $|x - a| < \boldsymbol{\delta}$ or $0 < |x - a| < \boldsymbol{\delta}$ --- (1)

It must be noted that the value of $\boldsymbol{\delta}$ depends on the value of K e.g.

Suppose $f(x) = 1 - \frac{1}{x+2}$ and assuming that $\lim_{x \to 1} f(x) = \frac{2}{3}$

Provided $\exists\, \boldsymbol{\delta} : \forall K > 0$

METHOD OF EVALUATING LIMITS

The methods by which limits of a function can be evaluated is as follows:

1. By direct substitution
2. By factorization
3. By rationalization etc.

It should be noted that in general when dealing with algebraic functions, and when x approaches a finite value, to find the limit, first reduce the given function to its lowest term using any of the methods above, then insert in the result the value that x approaches as long as the indeterminate quantity is not obtained it follows that the result is the required value.

It should be noted that in general when dealing with algebraic functions, and when x approaches a finite value, to find the limit, first reduce the given function to its lowest term using any of the methods above, then insert in the result the value that x approaches as long as the indeterminate quantity $^0/_0$ $(or\ ^\infty/_\infty)$ is not obtained it follows that the result is the required value.

Examples:

Find the values of the following limits:

$$(1)\ \lim_{x\to 1}\{x^2+3x-1\} \quad (2)\ \lim_{x\to -2}\left\{\frac{x^3+8}{x+2}\right\}$$

$$(3)\ \lim_{x\to 0}\left\{\frac{\sqrt{1-x}-\sqrt{1+x}}{x}\right\} \quad (4)\ \lim_{x\to \infty}\left\{\frac{(1+x)(2+x)x}{x^3+x}\right\}$$

Examples:

Find the values of the following limits:

(1) (2) (3) (4)

Solution: (1)

given that $\lim_{x \to 1} x^2 + 3x - 1$

This can be solved using direct substitution i.e. assume that $x = 1$

$$\lim_{x \to 1} x^2 + 3x - 1 = 1^2 + 3(1) - 1$$

$$= 1 + 3 - 1 = 3$$

Solution: (1)

given that

This can be solved using direct substitution i.e. assume that x = 1

Solution: (2)

$$\lim_{x\to -2}\left\{\frac{x^3+8}{x+2}\right\}$$ for direct substitution we have

Solution: (2)

for direct substitution we have

$$\frac{(-2)^3+8}{-2+2}=\frac{-8+8}{-2+2}=\frac{0}{0}$$ which is undefined

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Next we use factorization, i.e.

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$$\lim_{x\to -2}\left\{\frac{x^3+8}{x+2}\right\}=\lim\left\{\frac{x^3+2^3}{x+2}\right\}=\lim_{x\to -2}\left\{\frac{(x+2)(x^2-2x+4)}{x+2}\right\}$$

$$=\lim_{x\to -2}\left\{\frac{(x+2)(x^2-2x+4)}{x+2}\right\}$$

$$=$$

$$=\lim_{x\to -2}\{x^2-2x+4\}$$

$$=$$

$$=(-2)^2-2(-2)+4$$

$$=4+4+4=12$$

Solution: (3)

$$\lim_{x\to 0}\left\{\frac{\sqrt{1-x}-\sqrt{1+x}}{x}\right\}\ \textit{direct substitution gives}\ {}^{0}/_{0}$$

We then use the rationalization

Solution: (3)

We then use the rationalization

i.e

$$\text{i.e}\ \lim_{x\to 0}\left\{\frac{\sqrt{1-x}-\sqrt{1+x}}{x}\right\}=\lim_{x\to 0}\left\{\frac{\sqrt{1-x}-\sqrt{1+x}}{x}\cdot\frac{\sqrt{1-x}+\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}\right\}$$

$$=\lim_{x\to 0}\left\{\frac{\left(\sqrt{1-x}-\sqrt{1+x}\right)\left(\sqrt{1-x}+\sqrt{1+x}\right)}{x\left(\sqrt{1-x}+\sqrt{1+x}\right)}\right\}$$

$$=\lim_{x\to 0}\left\{\frac{(1-x)-(1+x)}{x\left(\sqrt{1-x}+\sqrt{1+x}\right)}\right\}, =\lim_{x\to 0}\left\{\frac{-2x}{x\left(\sqrt{1-x}+\sqrt{1+x}\right)}\right\},$$

$$=\lim_{x\to 0}\left\{\frac{-2}{\left(\sqrt{1-x}+\sqrt{1+x}\right)}\right\}=\frac{-2}{\sqrt{1-0}+\sqrt{1+0}}$$

$$=\frac{-2}{+2}=-1$$

$$\therefore\lim_{x\to 0}\left\{\frac{\sqrt{1-x}-\sqrt{1+x}}{x}\right\}=-1$$

Solution: (4)

$$\lim_{x\to\infty}\left\{\frac{(1+x)(2+x)x}{x^3+x}\right\}$$

Solution: (4)

This is a special example when $x \to \infty$, the numerator and denominator are divided by x^n, where n is the highest power of x present on expansion of both numerator and denominator.

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$$\therefore \lim_{x\to\infty}\left\{\frac{(1+x)(2+x)x}{x^3+x}\right\} = \lim_{x\to\infty}\left\{\frac{(2+3x+x^2)x}{x(x^2+1)}\right\} = \lim_{x\to\infty}\left\{\frac{2+3x+x^2}{x^2+1}\right\}$$

Now divide through by x^2, so that we have;

Now divide through by , so that we have;

$$= \lim_{x\to\infty}\left\{\frac{\frac{2}{x^2}+\frac{3x}{x^2}+\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}}\right\} = \lim_{x\to\infty}\left\{\frac{\frac{2}{x^2}+\frac{3}{x}+1}{1+\frac{1}{x^2}}\right\} = \frac{\frac{2}{\infty^2}+\frac{3}{\infty}+1}{1+\frac{1}{\infty^2}}$$

But $\frac{1}{x} \to 0 \; as \; x \to \infty$

But

$$\therefore \frac{\frac{2}{\infty^2}+\frac{3}{\infty}+1}{1+\frac{1}{\infty^2}} = \frac{0+0+1}{1+0} = \frac{1}{1} = 1$$

Thus, $\lim_{x\to\infty}\left\{\frac{(1+x)(2+x)x}{x(x^2+1)}\right\} = 1$

Thus,

EXERCISE

Find the values of the following limits:

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Find the values of the following limits:

(1) $\lim\limits_{x \to 2} \frac{3-x}{2+x}$ (2) $\lim\limits_{x \to 3} \frac{x^2-9}{x-3}$

(1) (2)

(3) $\lim\limits_{x \to 2} \frac{x^3-8}{x^2-4}$ (4) $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

(3) (4)

(5) $\lim\limits_{x \to -1} \left\{\frac{(1-x^2)}{1+x}\right\}$ (6) $\lim\limits_{x \to \infty} \frac{(2x-1)(x+4)(x-2)}{x^3}$

(5) (6)

(7) (8)

(7) $\lim\limits_{x \to \infty} \frac{5^{x+1}+7^{x+1}}{5^x - 7^x}$ (8) $\lim\limits_{n \to \infty} \frac{6n^3+17n+1}{n^3+2n^2+4}$

(9) (10)

(9) $\lim\limits_{x \to \infty} \frac{7.10^n-5.10^{2n}}{2.10^{n-1}+3.10^{2n-1}}$ (10) $\lim\limits_{h \to 0} \frac{h}{\sqrt{4+h}-2}$

BASIC THEOREMS ON LIMITS

THEOREM 1: (Sum Law)

The limits of the sum of a finite number of functions is equal to the sum of their limits i.e let f(x) and g(x) be defined on the interval (a,b) then $\lim_{x\to a}\{f(x)+g(x)\}$

$$= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

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THEOREM 1: (Sum Law)

The limits of the sum of a finite number of functions is equal to the sum of their limits i.e let f(x) and g(x) be defined on the interval (a,b) then = +

Proof of theorem 1

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let $\lim\limits_{x \to a} f(x) = L_1$ and $\lim\limits_{x \to a} g(x) = L_2$.

let = and .

From the first definition of limit, it can be

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seen that $\operatorname{Lim}\limits_{x \to a} f(x) = L_1$ as $\mathrm{x} \to a$,

seen that = as x

and $\lim\limits_{x \to a} g(x) = L_2$

and =

$\therefore f(x) + g(x) \to L_1 + L_2 \; as \;\; x \to a$ then

+ then

$$\lim_{x \to a}\{f(x) + g(x)\} = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

= +

THEOREM 2(Difference Law)

The limits of the difference of a finite number of functions is equal to the difference of their limits i.e let f(x) and g(x) be defined on the interval (a,b) then

$$\lim_{x \to a}\{f(x) - g(x)\} = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

THEOREM 2(Difference Law)

The limits of the difference of a finite number of functions is equal to the difference of their limits i.e let f(x) and g(x) be defined on the interval (a,b) then

Proof of theorem 2

Proof of theorem 2

let $\lim\limits_{x\to a} f(x) = L_1$ and $\lim\limits_{x\to a} g(x) = L_2$.

let = and = .

From the first definition of limit,

From the first definition of limit,

it can be seen that $\lim\limits_{x\to a} f(x) = L_1$ as x $\to a$,

it can be seen that = as x

and $\lim\limits_{x\to a} g(x) = L_2$

and

$\therefore f(x) + g(x) \to L_1 - L_2$ $as\ \ x \to a$ then

+ then

$$\lim_{x\to a}\{f(x) - g(x)\} = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

THEOREM 3: (Product Law)

The limits of the product of a finite number of functions is equal to the product of their limits i.e let f(x) and g(x) be defined on the interval (a,b) then $\lim_{x\to a}\{f(x) * g(x)\} =$

$$\lim_{x\to a} f(x)\ .\ \lim_{x\to a} g(x)$$

THEOREM 3: (Product Law)

The limits of the product of a finite number of functions is equal to the product of their limits i.e let f(x) and g(x) be defined on the interval (a,b) then =

Proof of theorem 3

Proof of theorem 3

let $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$.

let = and = .

From the first definition of limit,

From the first definition of limit,

it can be seen that $\lim_{x\to a} f(x) = L_1$ as $x \to a$,

it can be seen that = as x

and $\lim_{x\to a} g(x) = L_2$

and =

$\therefore f(x) * g(x) \to L_1 . L_2 \; as \; x \to a$ then

* then

$$\lim_{x\to a}\{f(x).g(x)\} = \lim_{x\to a} f(x) . \lim_{x\to a} g(x)$$

= .

THEOREM 4: (Quotient Law)

The limits of the quotient of a finite number of functions is equal to the quotient of their limits i.e let f(x) and g(x) be defined on the interval (a,b) then

$$\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

if

$$\lim_{x\to a} g(x) \neq 0$$

THEOREM 4: (Quotient Law)

The limits of the quotient of a finite number of functions is equal to the quotient of their limits i.e let f(x) and g(x) be defined on the interval (a,b)

then if

THEOREM 5: (Constant Multiple Law)

The constant multiple of a limit is equal to the constant multiple the function of the limit. i.e let f(x) and g(x) be defined on the interval (a,b) then, $\lim_{x \to a}\{Cf(x)\} = C\lim_{x \to a} f(x)$

$open$ int$erval$

EXERCISE

Prove theorem 4 and 5