OPTIMAL SOLVING (Using Gauss Elimination)

1) $x_1 + 2x_2 - x_3 = 3$

$2x_1 + 5x_2 - 4x_3 = 5$

$5x_1 + 4x_2 + 2x_3 = 12$

SOLUTION

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & -4 & 5 \\ 5 & 4 & 2 & 12 \end{array}\right]$$

$R_2 = R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 5 & 4 & 2 & 12 \end{bmatrix}$$

$R_3 = R_3 - 5R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & -6 & 7 & -3 \end{bmatrix}$$

$R_3 = R_3 + 6R_2$

We use plus because subtraction won't give us $a_{32} = 0$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -5 & -9 \end{bmatrix}$$

$R_3 = -\frac{1}{5}R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 9/5 \end{bmatrix}$$

$z = 9/5$ //

• Find y

$0 + y - 2(9/5) = -1$

$y - \frac{18}{5} = -1$

$y = -1 + \frac{18}{5}$

$y = \frac{13}{5}$ //

• Find x

$1x + 2(13/5) - 1(9/5) = 3$

$x + 17/5 = 3$

$x = 3 - 17/5$

$x = -\frac{2}{5}$

$x = -2/5$ ✓

$y = 13/5$ ✓

$Z = 9/5$ ✓

DONE

OR (Continue Reducing)

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 9/5 \end{bmatrix}$$

$R_1 = R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 9/5 \end{bmatrix}$$

$R_1 = R_1 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -2/5 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 9/5 \end{bmatrix}$$

$R_2 = R_2 + 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -2/5 \\ 0 & 1 & 0 & 13/5 \\ 0 & 0 & 1 & 9/5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$ with all ZEROS ✓

DONE

$x = -2/5$ ✓

$y = 13/5$ ✓

$z = 9/5$ ✓

TWO WAYS OF FINDING THE VECTORS

OPTIMAL SOLVING (Using Gauss Elimination)

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SOLUTION

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$y = \frac{13}{5}$ //

- Find x

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$x + 17/5 = 3$

$x = 3 - 17/5$

$x = -\frac{2}{5}$

$x = -2/5$ ✓

$y = 13/5$ ✓

$z = 9/5$ ✓

DONE

OR (Continue Reducing)

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 9/5 \end{bmatrix}$$

$R_1 = R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 9/5 \end{bmatrix}$$

$R_1 = R_1 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -2/5 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 9/5 \end{bmatrix}$$

$R_2 = R_2 + 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -2/5 \\ 0 & 1 & 0 & 13/5 \\ 0 & 0 & 1 & 9/5 \end{bmatrix}$$

$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ with all ZEROS ✓

DONE

$x = -2/5$ ✓

$y = 13/5$ ✓

$z = 9/5$ ✓

TWO WAYS OF FINDING THE VECTORS

2) $r_1 + 2r_2 + 2r_3 = 14$

$3r_1 + r_2 + 2r_3 = 11$

$2r_1 + 3r_2 + r_3 = 11$

SOLUTION

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 14 \\ 3 & 1 & 2 & 11 \\ 2 & 3 & 1 & 11 \end{array}\right]$$

$R_2 = R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 2 & 14 \\ 0 & -5 & -4 & -31 \\ 2 & 3 & 1 & 11 \end{bmatrix}$$

$R_3 = R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 2 & 14 \\ 0 & -5 & -4 & -31 \\ 0 & -1 & -3 & -17 \end{bmatrix}$$

$R_2 = -\frac{1}{5} R_2$

$$\begin{bmatrix} 1 & 2 & 2 & 14 \\ 0 & 1 & 4/5 & 31/5 \\ 0 & -1 & -3 & -17 \end{bmatrix}$$

$R_3 = R_3 + 1R_2$

$$\begin{bmatrix} 1 & 2 & 2 & 14 \\ 0 & 1 & 4/5 & 31/5 \\ 0 & 0 & -\frac{11}{5} & -\frac{54}{5} \end{bmatrix}$$

$R_3 = -\frac{5}{11} R_3$

$$\begin{bmatrix} 1 & 2 & 2 & 14 \\ 0 & 1 & 4/5 & 31/5 \\ 0 & 0 & 1 & \frac{594}{25} \end{bmatrix}$$

- $z = 594/25$

~~Find y~~

$\cancel{x + 2\left(\frac{-1601}{125}\right) + 2\left(\frac{594}{25}\right) = 14}$

$\cancel{x + \frac{2738}{125} = 14}$

$\cancel{x = 14 - \frac{2738}{125} = -\frac{988}{125}}$

- Find y

$0 + y + 4/5\left(\frac{594}{25}\right) = 31/5$

$y + \frac{2376}{125} = 31/5$

$y = 31/5 - \frac{2376}{125}$

$y = \frac{-1601}{125}$

- Find x

$x + 2\left(\frac{-1601}{125}\right) + 2\left(\frac{594}{25}\right) = 14$

$x + \frac{2738}{125} = 14$

$x = 14 - \frac{2738}{125}$

$x = -\frac{988}{125}$

$x = -\frac{988}{125}$ ✓

$y = \frac{-1601}{125}$ ✓

$z = \frac{594}{25}$

3) $C_1 + 2C_2 + 3C_3 = 5$

$3C_1 - C_2 + 2C_3 = 8$

$4C_1 - 6C_2 - 4C_3 = -2$

SOLUTION

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 3 & -1 & 2 & 8 \\ 4 & -6 & -4 & -2 \end{array}\right]$$

$R_2 = R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -7 & -7 & -7 \\ 4 & -6 & -4 & -2 \end{bmatrix}$$

$R_3 = R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -7 & -7 & -7 \\ 0 & -14 & -16 & -22 \end{bmatrix}$$

$R_2 = -\frac{1}{7}R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & -14 & -16 & -22 \end{bmatrix}$$

$R_3 = R_3 + 14R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -8 \end{bmatrix}$$

$R_3 = -\frac{1}{2}R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

- $Z = 4$ //
- Find y (or C_2)

$0 + y + 1(4) = 1$

$y + 4 = 1$

$y = 1 - 4$

$y = -3$ //

- Find x (or C_1)

$x + 2(-3) + 3(4) = 5$

$x - 6 + 12 = 5$

$x + 6 = 5$

$x = 5 - 6$

$x = -1$ //

$x = -1$ ✓

$y = -3$ ✓

$Z = 4$ ✓

4)
$$w + 2x - y + z = 6$$
$$-w + x + 2y - z = 3$$
$$2w - x + 2y + 2z = 14$$
$$w + x - y + 2z = 8$$

SOLUTION

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 6 \\ -1 & 1 & 2 & -1 & 3 \\ 2 & -1 & 2 & 2 & 14 \\ 1 & 1 & -1 & 2 & 8 \end{bmatrix}$$

R/2/F

$R_2 = R_2 + 1R_1$
$$\begin{bmatrix} 1 & 2 & -1 & 1 & 6 \\ 0 & 3 & 1 & 0 & 9 \\ 2 & -1 & 2 & 2 & 14 \\ 1 & 1 & -1 & 2 & 8 \end{bmatrix}$$

$R_3 = R_3 - 2R_1$
$$\begin{bmatrix} 1 & 2 & -1 & 1 & 6 \\ 0 & 3 & 1 & 0 & 9 \\ 0 & -3 & 4 & 0 & 2 \\ 1 & 1 & -1 & 2 & 8 \end{bmatrix}$$

$R_4 = R_4 - 1R_1$
$$\begin{bmatrix} 1 & 2 & -1 & 1 & 6 \\ 0 & 3 & 1 & 0 & 9 \\ 0 & -3 & 4 & 0 & 2 \\ 0 & -1 & 0 & 1 & 2 \end{bmatrix}$$

$R_2 = \frac{1}{3}R_2$
$$\begin{bmatrix} 1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 1/3 & 0 & 3 \\ 0 & -3 & 4 & 0 & 2 \\ 0 & -1 & 0 & 1 & 2 \end{bmatrix}$$

$R_3 = R_3 + 3R_2$
$$\begin{bmatrix} 1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 1/3 & 0 & 3 \\ 0 & 0 & 5 & 0 & 11 \\ 0 & -1 & 0 & 1 & 2 \end{bmatrix}$$

$R_4 = R_4 + 1R_2$
$$\begin{bmatrix} 1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 1/3 & 0 & 3 \\ 0 & 0 & 5 & 0 & 11 \\ 0 & 0 & -1/3 & 1 & 5 \end{bmatrix}$$

$R_3 = \frac{1}{5}R_3$
$$\begin{bmatrix} 1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 1/3 & 0 & 3 \\ 0 & 0 & 1 & 0 & 11/5 \\ 0 & 0 & -1/3 & 1 & 5 \end{bmatrix}$$

$R_4 = R_4 + \frac{1}{3}R_3$
$$\begin{bmatrix} 1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 1/3 & 0 & 3 \\ 0 & 0 & 1 & 0 & 11/5 \\ 0 & 0 & 0 & 1 & \frac{86}{15} \end{bmatrix}$$

- $z = \frac{86}{15}$
- Find y

$$0 + 0 + 1y + 0\left(\frac{86}{15}\right) = 11/5$$
$$y = 11/5$$

- Find x

$$0 + x + 1/3(11/5) + 0\left(\frac{86}{15}\right) = 3$$
$$x + 11/15 = 3$$
$$x = 3 - 11/15 = 34/15$$

- Find w

$$w + 2(34/15) - 1(11/5) + 1(86/15) = 6$$
$$w + \frac{121}{15} = 6$$
$$w = 6 - \frac{121}{15} = \frac{-31}{15}$$

$w = -31/15$ ✓
$x = 34/15$ ✓
$y = 11/5$
$z = 86/15$