Work and Energy

Physics Department, Bingham University

PHY 101 Lectures by Mrs. O. V. Oyelade

Part B: Introductory Classical Mechanics

Course content

- Work and Energy
- Rotational Motion
- Bodies in Equilibrium
- Newton's Universal Gravitation
- Simple Harmonic Motion

Work and Energy

Learning Objectives

- To represent the work done by any force and evaluate the work done by varying forces
- To calculate the kinetic energy of a particle given its mass and its velocity or momentum
- To apply the work-energy theorem to find information about the motion of a particle, given the forces acting on it
- To use the work-energy theorem to find information about the forces acting on a particle, given information about its motion
- To relate the work done during a time interval to the power delivered
- To find the power expended by a force acting on a moving body

Definition of Work *W*

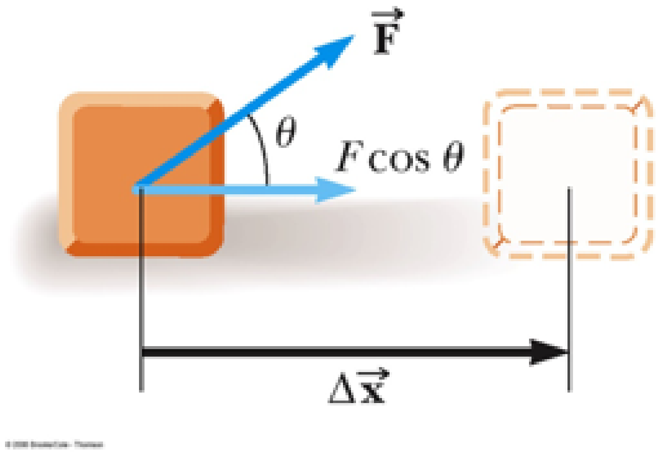
- **work** is done on an object when energy is transferred to the object.
- The work done by a force is the integral of the force with respect to displacement along the path of the displacement:

$$W_{AB} = \int_{\text{path } AB} \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{r}}$$

- The simplest work to evaluate is that done by a force that is constant in magnitude and direction.

Work *W* done by a constant force

- The work, *W*, done by a constant force on an object is defined as the product of the component of the force along the direction of displacement and the magnitude of the displacement



$$W \equiv (F\cos\theta)\Delta x$$

- Where F is the magnitude of the force, Δx is the magnitude of the object's displacement and θ is the angle between F and Δx

Definition of Work *W*

- Work is a scalar quantity
- *SI unit*
  - ➢ Newton • meter (Nm) = Joule
  - ➢ $kg \bullet m^2 / s^2 = (kg \bullet m / s^2) \bullet m = J$
- Work can be positive, negative, or zero. The sign of the work depends on the direction of the force relative to the displacement
  - Work positive: $W > 0$ if $90^\circ > \theta > 0^\circ$ i.e when component of force is in the direction of displacement
  - Work negative: $W < 0$ if $180^\circ > \theta > 90^\circ$
  - Work zero: $W = 0$ if $\theta = 90^\circ$
  - Work maximum if $\theta = 0^\circ$
  - Work minimum if $\theta = 180$

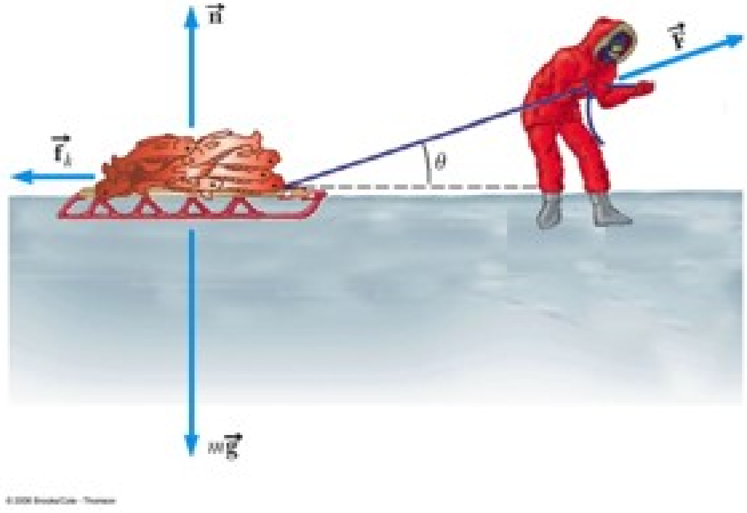
Example of when work is zero

- A man carries a bucket of water horizontally at constant velocity.
- The force does no work on the bucket
- Displacement is horizontal
- Force is vertical
- cos 90° = 0

$$W \equiv (F \cos \theta)\Delta x$$

Example

- A boy pulls a sled as shown. The total mass of the sled is 50.0 kg, and he exerts a force of 1.20×10^2 N on the sled by pulling on the rope. How much work does he do on the sled if θ = 30° and he pulls the sled 5.0 m ?



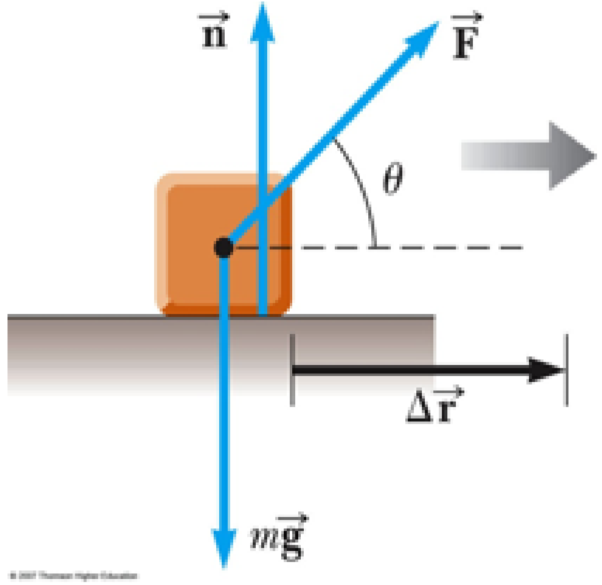
$$W = (F\cos\theta)\Delta x$$
$$= (1.20\times10^2 N)(\cos 30^\circ)(5.0m)$$
$$= 5.2\times10^2 J$$

Work Done by Multiple Forces

- If more than one force acts on an object, then the total work is equal to the algebraic sum of the work done by the individual forces
- Remember work is a scalar, so this is the algebraic sum

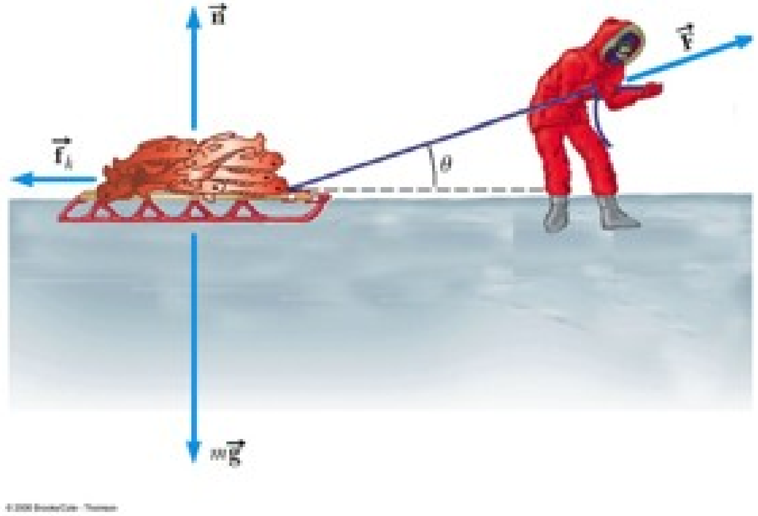
$$W_{\text{net}} = \sum W_{\text{by individual forces}}$$

$$W_{net} = W_g + W_N + W_F = (F\cos\theta)\Delta r$$



Example

- Suppose μ_k = 0.200, How much work done on the mass by friction, and the net work if θ = 30° and the man pulls the sled 5.0 m ?

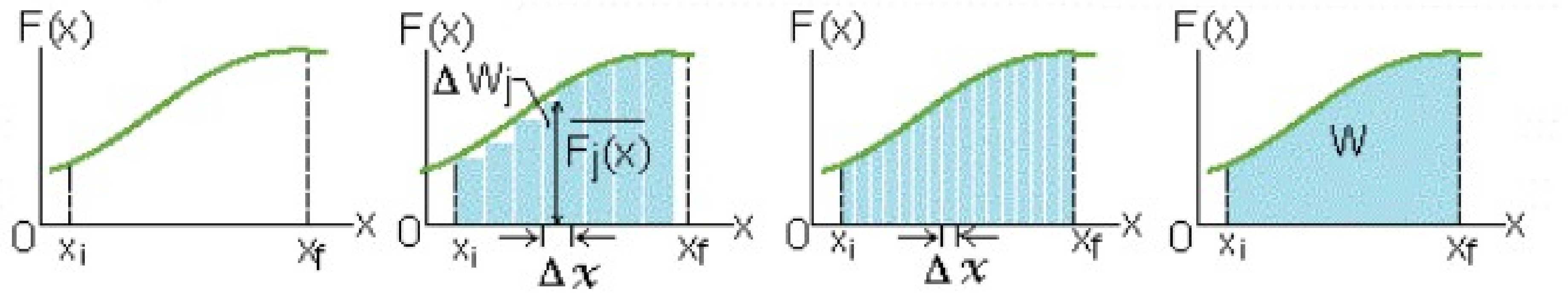


$$W_{fric} = (f_k \cos 180^\circ)\Delta x = -f_k \Delta x$$
$$= -\mu_k N \Delta x = -\mu_k (mg - F\sin\theta)\Delta x$$
$$= -(0.200)(50.0kg \cdot 9.8m/s^2$$
$$-1.2\times 10^2 N \sin 30^\circ)(5.0m)$$
$$= -4.3\times 10^2 J$$

$$F_{net,y} = N - mg + F\sin\theta = 0$$
$$N = mg - F\sin\theta$$

$$W_{net} = W_F + W_{fric} + W_N + W_g$$
$$= 5.2\times 10^2 J - 4.3\times 10^2 J + 0 + 0$$
$$= 90.0J$$

Work done by Varying Forces



$$W = \int_{x_i}^{x_f} F(x)dx$$

- The total work is a line integral, or the limit of a sum of infinitesimal amounts of work.

Work done by gravity and spring force

- The work done by gravity, over any path from *A* to *B*, is

$$W_{\text{grav}, AB} = -mg\,\hat{\mathbf{j}} \cdot (\vec{\mathbf{r}}_B - \vec{\mathbf{r}}_A) = -mg(y_B - y_A).$$

- Work done by a spring force

$$W_{\text{spring}, AB} = \int_A^B F_x\,dx = -k\int_A^B x\,dx = -k\frac{x^2}{2}\Bigg|_A^B = -\frac{1}{2}k\left(x_B^2 - x_A^2\right).$$

Examples

1. How much work is done (a) if a person exerts a constant force of 75.0 N at an angle 35° below the horizontal and pushes a lawn mower 25.0 m on level ground?

$$W = (75.0\,\text{N})(25.0\,\text{m})\cos(35.0°) = 1.54 \times 10^3\ \text{J}.$$

2. A perfectly elastic spring requires 0.54 J of work to stretch 6 cm from its equilibrium position (a) What is its spring constant *k*?

(b) How much work is required to stretch it an additional 6 cm?

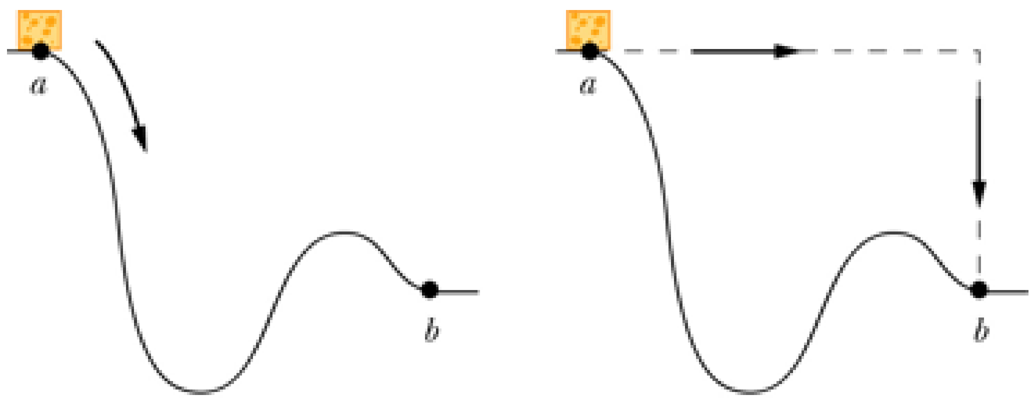
$$W = \frac{1}{2}k(x_B^2 - x_A^2).$$

a. $W = 0.54\ \text{J} = \frac{1}{2}k[(6\ \text{cm})^2 - 0]$, so $k = 3\ \text{N/cm}$.

b. $W = \frac{1}{2}(3\ \text{N/cm})[(12\ \text{cm})^2 - (6\ \text{cm})^2] = 1.62\ \text{J}$.

Energy

- Energy is a scalar quantity. It does not have a direction associated with it
- The energy approach to describing motion is particularly useful when Newton's Laws are difficult or impossible to use, for non-linear motions as shown below:



- SI unit: joule (J)

$$1 \text{ joule} = 1 \text{ J} = 1 \text{ kg m}^2/\text{s}^2$$

Kinetic Energy

- Kinetic Energy is energy associated with the state of motion of an object
- Kinetic Energy for an object moving with a speed of v is

$$KE = \frac{1}{2}mv^2$$

- Kinetic Energy for a system of particles

$$K = \sum \frac{1}{2}mv^2.$$

Kinetic Energy

- The kinetic energy, K of a particle can be expressed in terms of its mass and momentum

$$p = mv \qquad v = p/m$$

$$K = \frac{1}{2}m\left(\frac{p}{m}\right)^2 = \frac{p^2}{2m}$$

What was its mass of an object travelling at 22 km/s and releasing 4.2 x 10^{23} J of kinetic energy upon impact?

$$m = 2K/v^2 = 2(4.2 \times 10^{23}\,\mathrm{J})/(22\,\mathrm{km/s})^2 = 1.7 \times 10^{15}\ \mathrm{kg}.$$

Work-Energy Theorem

- When work is done by a net force on an object and the only change in the object is its speed, the work done is equal to the change in the object’s kinetic energy
  - Speed will increase if work is positive
  - Speed will decrease if work is negative

$$W_{net} = \int_{x_i}^{x_f} F_{net}(x)dx$$

- W_{net} is the work done by
- F_{net} the net force acting on a body.

- Work-Energy Theorem: work done is equal to the change in kinetic energy

Work-Energy Theorem

$$W_{net} = \int_{x_i}^{x_f} F_{net}\,dx$$

$$= \int_{x_i}^{x_f} m\,a\,dx = m\int_{x_i}^{x_f} \frac{dv}{dt}\,dx$$

$$= m\int_{v_i}^{v_f} \frac{dx}{dt}\,dv = m\int_{v_i}^{v_f} v\,dv$$

$$= m\int_{v_i}^{v_f} v\,dv$$

$$= m\left[\frac{v^2}{2}\right]_{v_i}^{v_f} = \tfrac{1}{2}m(v_f^2 - v_i^2)$$

$$= \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$

Example: Work-Energy Theorem

- A bullet with a mass of 2.60 g and a of 335 m/s penetrates a distance of 15.2 cm into a board. What is the average stopping force exerted by the board?
- $W_{net} = F_{net}\Delta s = \Delta KE$

$$F_{net} = \frac{\Delta KE}{\Delta s}$$

$$F_{net} = \frac{\frac{1}{2}(2.6\ x\ 10^{-3})335^2\Delta KE)}{0.152} = 960N$$

Power

- Power is defined as the rate of doing work, or the limit of the average power for time intervals approaching zero,

$$P = \frac{dW}{dt}.$$

- If the power during an interval varies with time, then the work done is the time integral of the power,

$$W = \int P dt.$$

- the SI unit for power is Watts, W: 1 J/s = 1 W
- Another common unit for expressing the power capability of everyday devices is horsepower: 1 hp = 746 W .

Fixed Axis Rotation

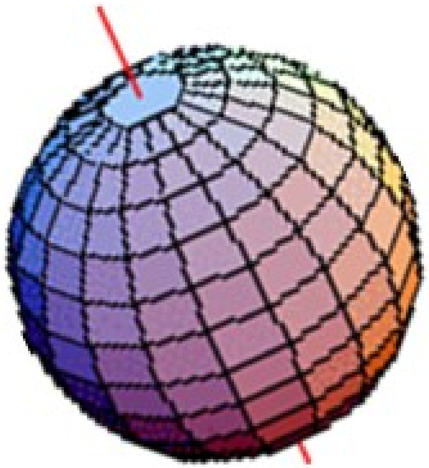
Learning Objectives

By the end of this section, you will be able to describe the physical meaning and calculate:

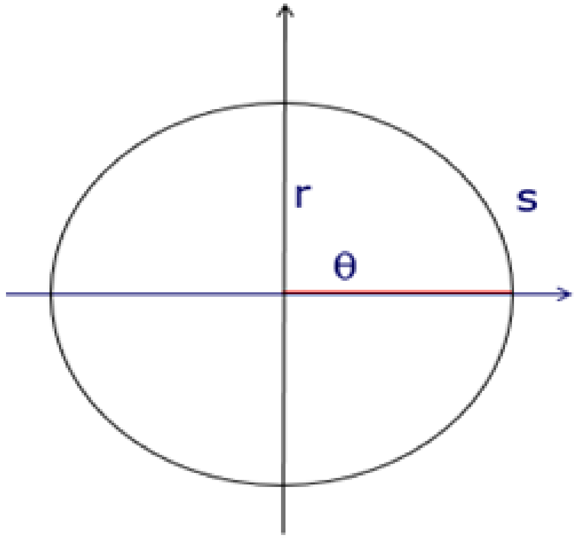
- Angular position and radian
- Angular displacement
- Angular velocity
- Angular acceleration
- Rotational motion under constant angular acceleration
- Relations between angular and linear quantities
- Circular motion

Fixed-axis rotation

- Fixed-axis rotation describes the rotation around a fixed axis of a rigid body; that is, an object that does not deform as it moves. Some examples are shown below



Angle and Radian



θ is the arc length s along a circle divided by the radius r.

θ is measured in radians (rad)

$$s = (2\pi)r$$

$$2\pi = \frac{s}{r}$$

Whenever using rotational equations, you must use angles expressed in radians

Conversions

- Comparing degrees and radians

$$2\pi(rad) = 360^\circ \qquad \pi(rad) = 180^\circ$$

- Converting from degrees to radians

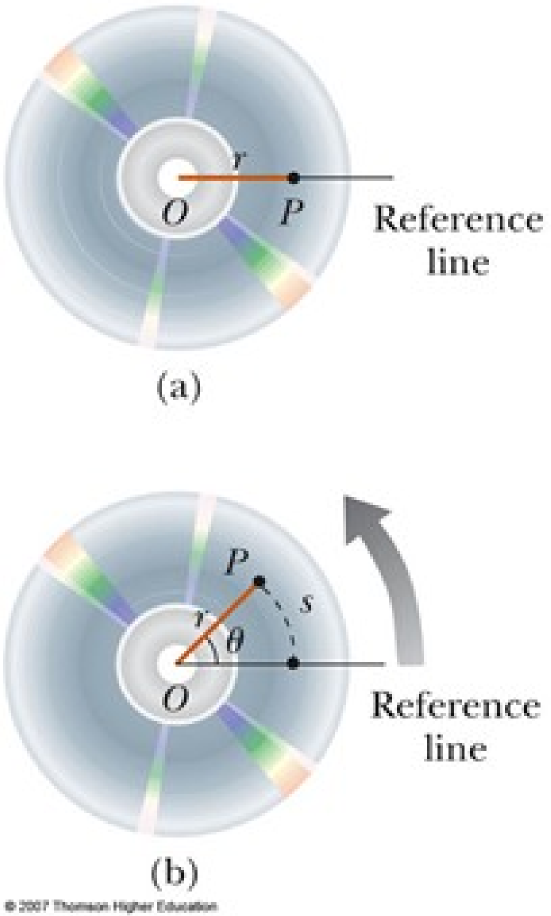
$$\theta\left(rad\right) = \frac{\pi}{180^\circ}\,\theta\left(degrees\right)$$

- Converting from radians to degrees

$$\theta(\deg rees) = \frac{180^\circ}{\pi}\theta(rad) \qquad 1\,rad = \frac{360^\circ}{2\pi} = 57.3^\circ$$

Angular Position

- Axis of rotation is the center of the disc
- Choose a fixed reference line
- Point P is at a fixed distance r from the origin
- As the particle moves, the only coordinate that changes is θ
- As the particle moves through θ, it moves though an arc length s.
- The angle θ, measured in radians, is called the angular position.

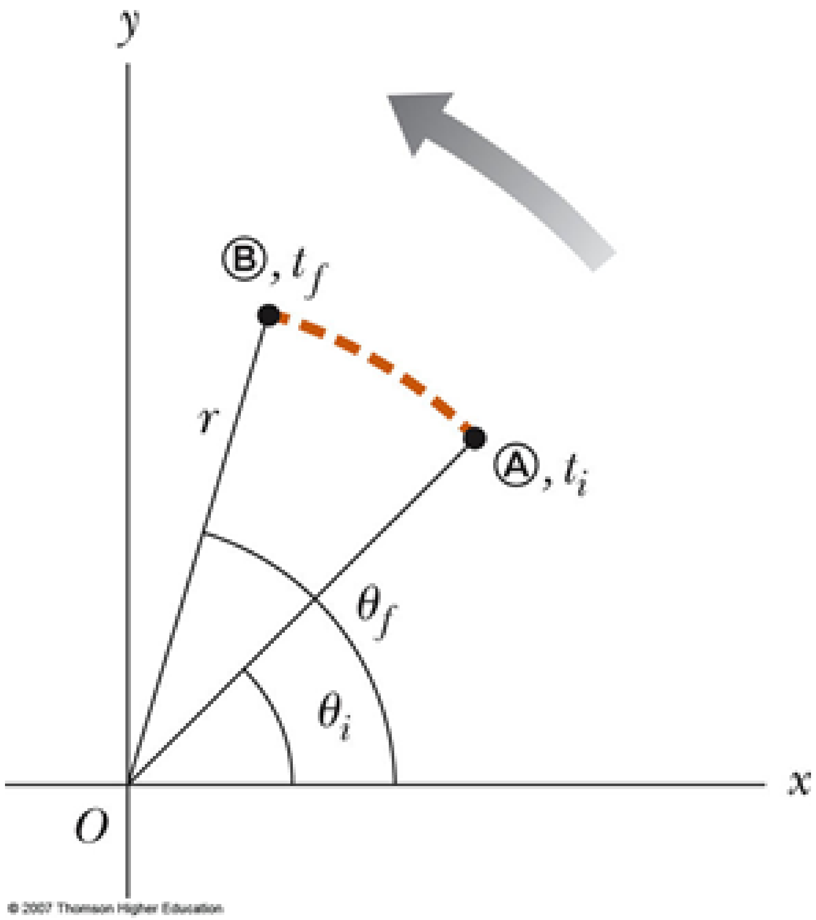


Angular Displacement

- The angular displacement is defined as the angle the object rotates through during some time interval

$$\Delta\theta = \theta_f - \theta_i$$

- SI unit: radian (rad)
- This is the angle that the reference line of length *r* sweeps out



Average and Instantaneous Angular Speed

- The *average* angular speed, ω_{avg}, of a rotating rigid object is the ratio of the angular displacement to the time interval

$$\omega_{avg} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

- The *instantaneous* angular speed is defined as the limit of the average speed as the time interval approaches zero

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

- SI unit: radian per second (rad/s)
- Angular speed positive if rotating in counterclockwise
- Angular speed will be negative if rotating in clockwise

Average Angular Acceleration

- The average angular acceleration, α, of an object is defined as the ratio of the change in the angular speed to the time it takes for the object to undergo the change:

$$\alpha_{avg} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous Angular Acceleration

- The instantaneous angular acceleration is defined as the limit of the average angular acceleration as the time goes to 0

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}$$

- SI Units of angular acceleration: rad/s^2
- Positive angular acceleration is in the counterclockwise.
  - if an object rotating counterclockwise is speeding up
  - if an object rotating clockwise is slowing down
- Negative angular acceleration is in the clockwise.
  - if an object rotating counterclockwise is slowing down
  - if an object rotating clockwise is speeding up

Rotational and Translational motion

- A number of parallels exist between the equations for rotational motion and those for linear motion

$$v_{avg} = \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t} \qquad \omega_{avg} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta \theta}{\Delta t}$$

- Under **constant angular acceleration**, we can describe the motion of the rigid object using a set of kinematic equations
  - These are similar to the kinematic equations for linear motion
  - The rotational equations have the same mathematical form as the linear equations

Kinematic equations for Rotational and Translational motion under constant acceleration

Rotational Motion About a Fixed Axis	Linear Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$

Example

A wheel rotates with a constant angular acceleration of 3.50 rad/s2. If the angular speed of the wheel is 2.00 rad/s at *ti* = 0, (a) through what angle does the wheel rotate in 2.00 s? (b) What is the angular speed at *t* = 2.00 s?

Solution

$$\theta_f = \theta_i + \omega_i t + \tfrac{1}{2}\alpha t^2$$

$$\theta_f - \theta_i = \omega_i t + \tfrac{1}{2}\alpha t^2$$

$$= (2.00\ \text{rad/s})(2.00\ \text{s}) + \tfrac{1}{2}(3.50\ \text{rad/s}^2)(2.00\ \text{s})^2$$

$$= 11.0\ \text{rad} = (11.0\ \text{rad})(57.3°/\text{rad}) = \boxed{630°}$$

$$= \frac{630°}{360°/\text{rev}} = \boxed{1.75\ \text{rev}}$$

(b) $\omega_f = \omega_i + \alpha t$

$$= 2.00\ \text{rad/s} + (3.50\ \text{rad/s}^2)(2.00\ \text{s})$$

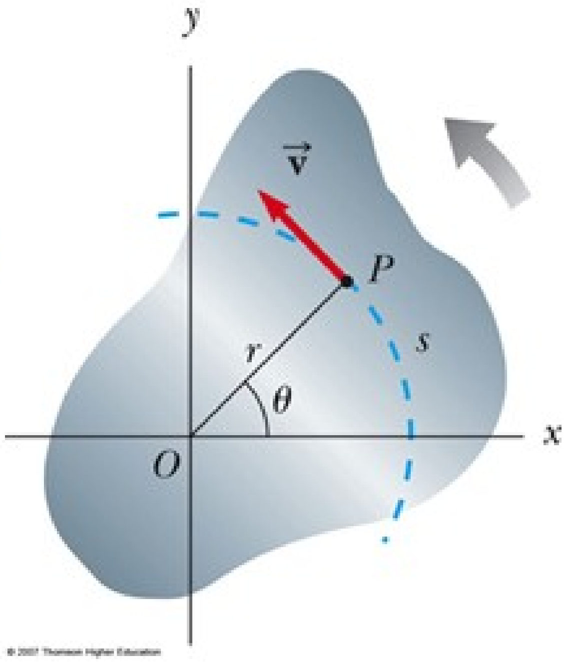
$$= \boxed{9.00\ \text{rad/s}}$$

Relationship Between Angular and Linear Quantities

- Every point on the rotating object has the same angular motion.
- Every point on the rotating object does **not** have the same linear motion.
- Displacement $s = \theta r$
- Speeds $v = \omega r$
- Accelerations $a = \alpha r$

Speed Comparison

- The linear velocity is always tangent to the circular path called the tangential velocity
- The magnitude is defined by the tangential speed



$$\Delta\theta = \frac{\Delta s}{r}$$

$$\frac{\Delta\theta}{\Delta t} = \frac{\Delta s}{r\Delta t} = \frac{1}{r}\frac{\Delta s}{\Delta t}$$

$$\omega = \frac{v}{r} \quad or \quad v = r\omega$$

Acceleration Comparison

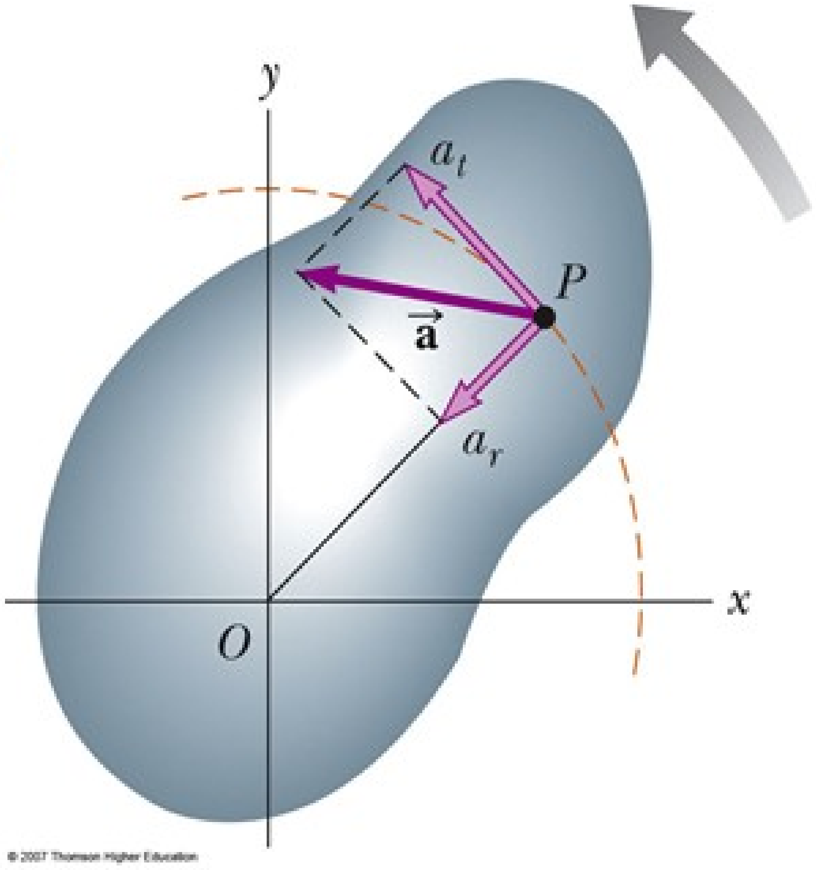
- The tangential acceleration

is the derivative of the tangential velocity

$$\Delta v = r\Delta\omega$$

$$\frac{\Delta v}{\Delta t} = r\frac{\Delta\omega}{\Delta t} = r\alpha$$

$$a_t = r\alpha$$



Speed and Acceleration

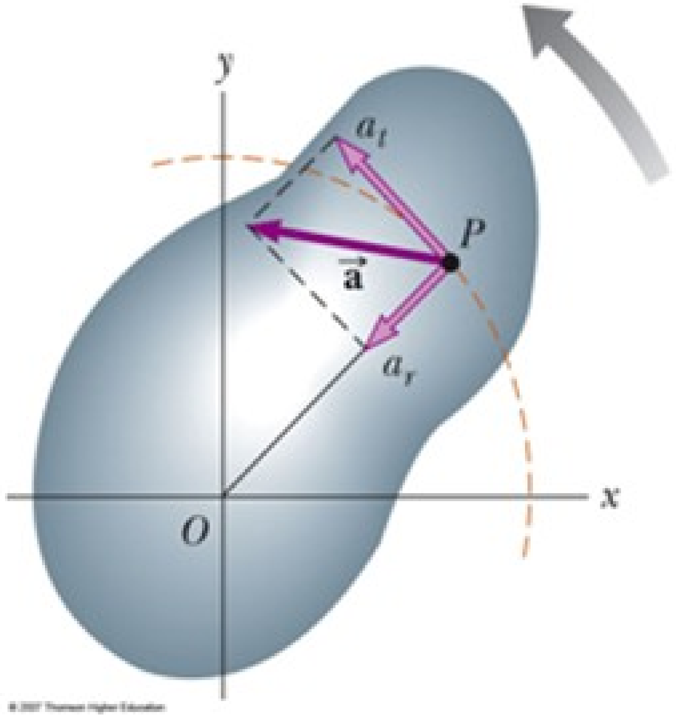
- All points on the rigid object will have the same angular speed, but not the same tangential speed
- All points on the rigid object will have the same angular acceleration, but not the same tangential acceleration
- The tangential quantities depend on r, and r is not the same for all points on the object

$$\omega = \frac{v}{r} \quad or \quad v = r\omega \qquad\qquad a_t = r\alpha$$

Centripetal Acceleration

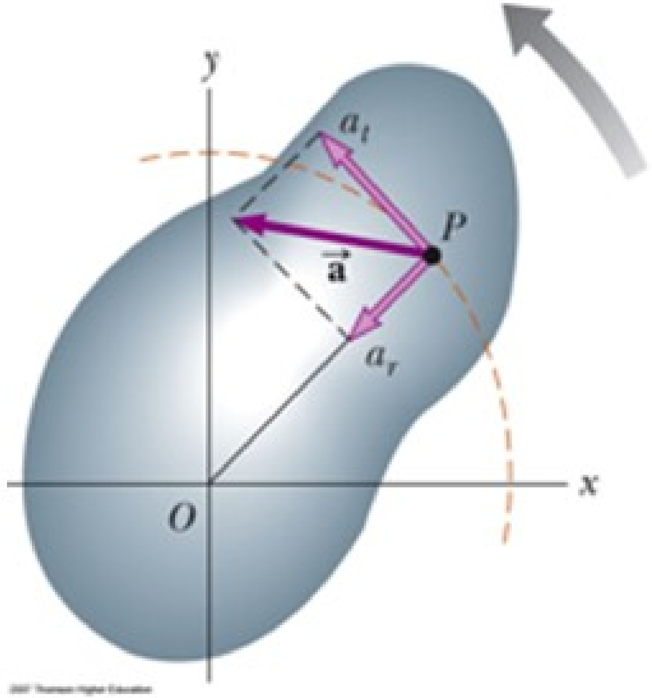
- An object traveling in a circle, even though it moves with a constant speed, will have an acceleration
- Therefore, each point on a rotating rigid object will experience a centripetal acceleration

$$a_r = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$



Resultant Acceleration

- The tangential component of the acceleration is due to changing speed
- The centripetal component of the acceleration is due to changing direction
- Total acceleration can be found from these components



$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

Example

Arm rotates as rigid body: r = 0.8 m

Constant angular acceleration $\alpha = 50$ rad/s^2

Find the tangential and radial (centripetal) components of the acceleration

$$a_{tang} = \alpha r = 50 \times 0.8 = 40\, m/s^2$$

- constant magnitude
- direction: tangent to rotary motion

$$a_{rad} = \omega^2 r = (\omega_0 + \alpha t)^2 r$$

- ω not constant

Start from rest → $\omega_0 = 0$

$$\therefore a_{rad} = (\alpha t)^2 r$$

- centripetal acceleration growing rapidly as square of time

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