



CMP418: Algorithm and Complexity Analysis (3 units)

Lecture 3: Brute Force and Exhaustive Search

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Chapter 3 Outline

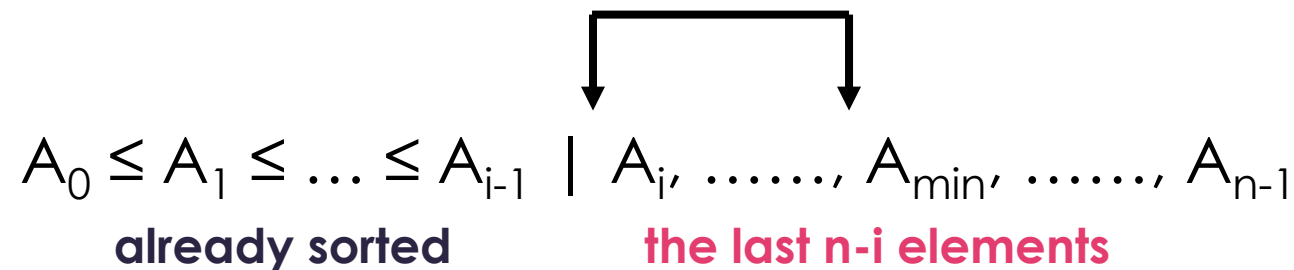
- ▶ Selection Sort and Bubble Sort
- ▶ Sequential Search and Brute-Force String Matching
- ▶ Closest-Pair and Convex-Hull Problems by Brute Force
- ▶ Exhaustive Search
- ▶ Depth-First Search and Breadth-First Search

What is Brute Force Approach (BFA)?

- ▶ The first algorithm design Approach
- ▶ A straightforward approach to solving problem,
- ▶ Usually based on problem statement and definitions of the concepts involved
- ▶ “**Force**” comes from using computer power not intellectual power
- ▶ In short, “**brute force**” means “Just do it!”
- ▶ It is the only general approach that always works
- ▶ Seldom gives efficient solution, but one can easily improve the brute force version.
- ▶ Serves as a yardstick to compare with more efficient solutions

Brute Force Case Studies – Selection Sort

- ▶ We start selection sort by scanning the entire given list to find its smallest element and Exchange it with the first element,
 - ▶ putting the smallest element in its final position in the sorted list.
- ▶ Then we scan the list, starting with the second element, to find the smallest among the last $n - 1$ elements and exchange it with the second element,
 - ▶ putting the second smallest element in its final position.
- ▶ On the i -th pass (i goes from 0 to $n-2$) the algorithm searches for the smallest item among the last $n-i$ elements and swaps it with A_i



Brute Force Case Study – Selection Sort Algorithm

ALGORITHM SelectionSort(A[0,..n-1])

for i <- 0 **to** n-2 **do**

min <- i

for j <- i+1 **to** n-1 **do**

if A[j] < A[min]

min <- j

swap A[i] and A[min]

		89	45	68	90	29	34	17
17		45	68	90	29	34	89	
17	29		68	90	45	34	89	
17	29	34	45		90	68	89	
17	29	34	45	68		90	89	
17	29	34	45	68	89		90	

$$\begin{aligned}
 C(n) &= \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 \\
 &= \sum_{i=0}^{n-2} [(n-1) - (i+1) + 1] \\
 &= \frac{(n-1)n}{2}
 \end{aligned}$$

$C(n) \in \Theta(n^2)$

of key swaps $\in \Theta(n)$

Brute Force Case Study – Bubble Sort

- ▶ To compare adjacent elements of the list and exchange them if they are out of order.
- ▶ By doing it repeatedly, we end up “**bubbling up**” the largest element to the last position on the list.
- ▶ The next pass bubbles up the second largest element, and so on, until after $n - 1$ passes the list is sorted.
- ▶ Pass i ($0 \leq i \leq n - 2$) of bubble sort can be represented by the following diagram:

$$A_0, \dots, A_j \overset{?}{<-->} A_{j+1}, \dots, A_{n-i-1} \mid A_{n-i} \leq \dots \leq A_{n-1}$$

On their final positions

Brute Force Case Study – Bubble Sort...

ALGORITHM BubbleSort($A[0..n-1]$)

for $i \leftarrow 0$ to $n-2$ **do**

for $j \leftarrow 0$ to $n-2-i$ **do**

if $A[j+1] < A[j]$

 swap $A[j]$ and $A[j+1]$

$$\begin{aligned} C(n) &= \sum_{i=0}^{n-2} \sum_{j=0}^{n-2-i} 1 \\ &= \sum_{i=0}^{n-2} [(n-2-i) - (0) + 1] \end{aligned}$$

$$C(n) \in \Theta(n^2)$$

Could you improve it?

What about 89, 45, 68, 90, 29, 34, 17 ?

Brute Force Case Study – Bubble Sort...

What about sorting 89, 45, 68, 90, 29, 34, 17 ?

Pass 1

89, 45, 68, 90, 29, 34, 17

45, **89**, 68, 90, 29, 34, 17

45, 68, **89**, 90, 29, 34, 17

45, 68, 89, 90, 29, 34, 17 No Swap

45, 68, 89, 29, **90**, 34, 17

45, 68, 89, 29, 34, **90**, 17

45, 68, 89, 29, 34, 17, **90**

$n-2-i, i=0$

Pass 2

45, 68, 89, 29, 34, 17, **90**

45, 68, 89, 29, 34, 17, **90** No Swap

45, 68, 89, 29, 34, 17, **90** No Swap

45, 68, 29, **89**, 34, 17, **90**

45, 68, 29, 34, **89**, 17, **90**

45, 68, 29, 34, 17, **89**, **90**

$n-2-i, i=1$

Brute Force Case Studies – Bubble Sort...

What about sorting 89, 45, 68, 90, 29, 34, 17 ?

Pass 3

45, 68, 29, 34, 17, **89, 90**

45, 68, 29, 34, 17, **89, 90** No Swap

45, 29, **68**, 34, 17, **89, 90**

45, 29, 34, **68**, 17, **89, 90**

45, 29, 34, 17, **68**, **89, 90**

n-2-l, i=2

Pass 4

45, 29, 34, 17, **68, 89, 90**

29, **45**, 34, 17, **68, 89, 90**

29, 34, **45**, 17, **68, 89, 90**

29, 34, 17, **45**, **68, 89, 90**

n-2-l, i=3

Brute Force Case Studies – Bubble Sort...

What about sorting 89, 45, 68, 90, 29, 34, 17 ?

Pass 5

29, 34, 17, 45, 68, 89, 90
29, 34, 17, 45, 68, 89, 90 No Swap
29, 17, 34, 45, 68, 89, 90

$n-2-i, i=4$

Pass 6

29, 17, 34, 45, 68, 89, 90
17, 29, 45, 68, 68, 89, 90 Sorted!

$n-2-i, i=5$

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Brute Force Case Study – Sequential Search

- ▶ The algorithm simply compares successive elements of a given list with a given search key until either a match is encountered (successful search)
- ▶ Or the list is exhausted without finding a match (unsuccessful search).
- ▶ A simple extra trick is often employed in implementing sequential search: if we append the search key to the end of

Brute Force Case Studies – Sequential Search...

ALGORITHM SequentialSearch(A[0..n-1], K)

//Output: index of the first element in A, whose

//value is equal to K or -1 if no such element is found

i ← 0

while i < n **and** A[i] ≠ K **do**

 i ← i+1

if i < n

return i

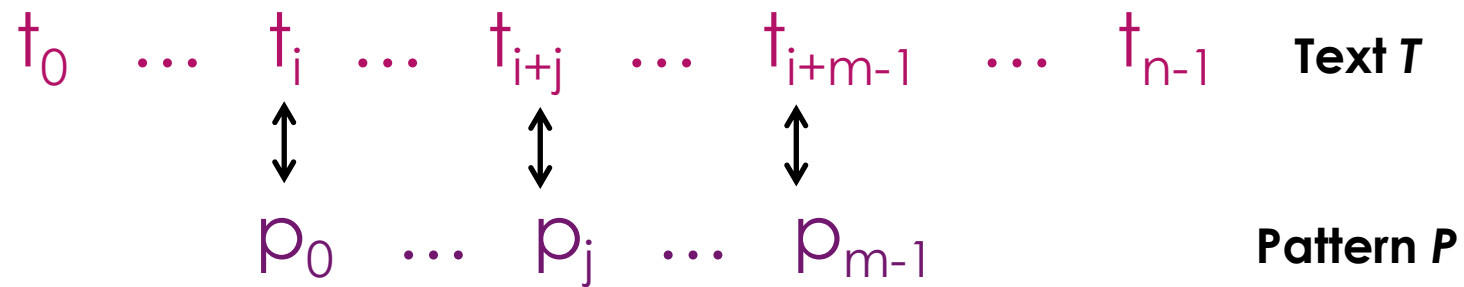
else

return -1

$$C_{\text{worst}}(n) = n$$

Brute Force Case Studies – String Matching

- ▶ Given a string of n characters called the **text** and a string of m characters ($m \leq n$) called the **pattern**,
- ▶ We find a substring of the text that matches the pattern.
- ▶ To put it more precisely, we want to find the index of the leftmost character of the first matching substring in the text



p_0 should be tested with up to t_j

Brute Force Case Study – String Matching...

ALGORITHM BruteForceStringMatching($T[0..n-1]$, $P[0..m-1]$)

for $i \leftarrow 0$ **to** $n-m$ **do**

$j \leftarrow 0$

while $j < m$ **and** $P[j] = T[i+j]$ **do**

$j \leftarrow j+1$

if $j = m$

return i

return -1

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$$C_{\text{worst}}(n, m) = m(n-m+1) \in O(nm)$$

$$C_{\text{avg}}(n, m) \in \Theta(n)$$

Brute Force Case Study – String Matching...

- ▶ Length of text = n
- ▶ Length of pattern = m
- ▶ Maximum number of comparison = $m(n - m + 1)$

Exercise: How many comparisons (both successful and unsuccessful) will be made by the brute force algorithm in searching for each of the following patterns in the binary tree of **one thousand zeros**

Pattern 1 = 00001

Pattern 2 = 01010

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Brute Force Case Studies – Closest-Pair

- ▶ Find the two closest points in a set of n points
- ▶ Points can be airplanes (most probable collision candidates), database records, DNA sequences, etc.
- ▶ Cluster analysis: pick two points, if they are close enough they are in the same cluster, pick another point,
- ▶ Euclidean distance, $d(p_i, p_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$
- ▶ Brute-force: compute distance between each pair of disjoint points and find a pair with the smallest distance $d(p_i, p_j) = d(p_j, p_i)$, so we consider only $d(p_i, p_j)$ for $i < j$

Brute Force Case Study – Closest-Pair...

ALGORITHM BruteForceClosestPair(P)

//Input: A list P of n ($n \geq 2$) points $p_1(x_1, y_1)$,

// $p_2(x_2, y_2)$, ..., $p_n(x_n, y_n)$

//Output: distance between closest pair

$d \leftarrow \infty$

for $i \leftarrow 1$ **to** $n-1$ **do**

for $j \leftarrow i+1$ **to** n **do**

$d \leftarrow \min(d, \text{sqrt}((x_i - x_j)^2 + (y_i - y_j)^2))$

return d

$$\begin{aligned}
 C(n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n 2 \\
 &= 2 \sum_{i=1}^{n-1} (n-i) \\
 &= 2[(n-1) + (n-2) + \dots + 1] \\
 &= (n-1)n \in \Theta(n^2)
 \end{aligned}$$

Chapter 3 Outline

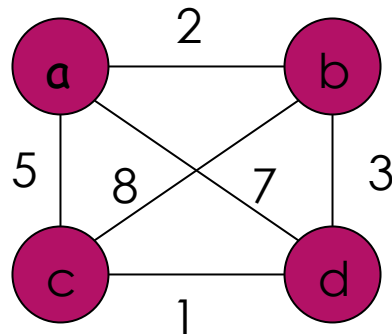
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Brute Force Case Study – Exhaustive Search

- ▶ Traveling Salesman Problem (TSP)
 - ▶ Find the shortest tour through a given set of n cities that visits each city exactly once before returning to the city where it started
 - ▶ Can be conveniently modeled by a weighted graph; vertices are cities and edge weights are distances
 - ▶ Same as finding “Hamiltonian Circuit”: find a cycle that passes through all vertices exactly once

Brute Force Case Studies

Exhaustive Search: Travelling Sales Man (TSM)



Consider only when b precedes c

$\frac{1}{2}(n-1)!$ permutations

a → b → c → d → a

$$2+8+1+7 = 18$$

a → b → d → c → a

$$2+3+1+5 = 11 \leftarrow \text{optimal}$$

a → c → b → d → a

$$5+8+3+7 = 23$$

a → c → d → b → a

$$5+1+3+2 = 11 \leftarrow \text{optimal}$$

a → d → b → c → a

$$7+3+8+5 = 23$$

a → d → c → b → a

$$7+1+8+2 = 18$$

Brute Force Case Studies

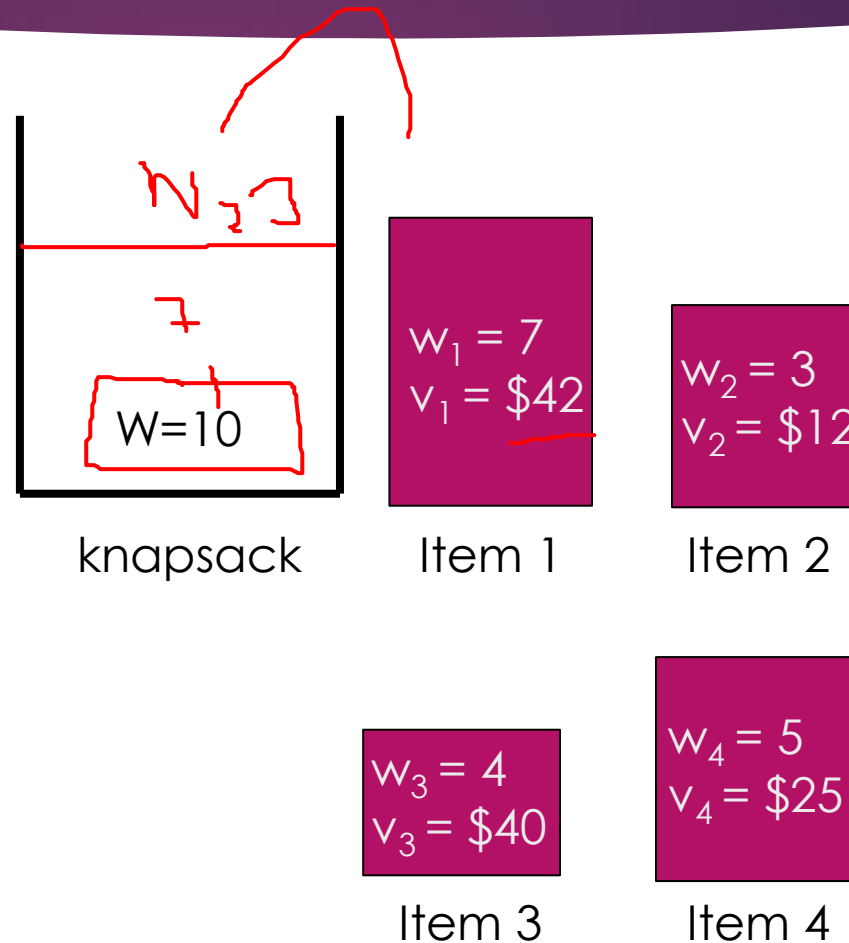
Exhaustive Search: Knapsack Problem (KP)

- ▶ Given n items of weights w_1, w_2, \dots, w_n and values v_1, v_2, \dots, v_n and a knapsack of capacity W , find the most valuable subset of the items that fit into the knapsack
- ▶ A transport plane has to deliver the most valuable set of items to a remote location without exceeding its capacity
- ▶ How do you solve this?
- ▶ Brute force: Generate all possible subsets of the n items, compute total weight of each subset to identify feasible subsets, and find the subset of the largest value

Brute Force Case Studies

Exhaustive Search: KP1

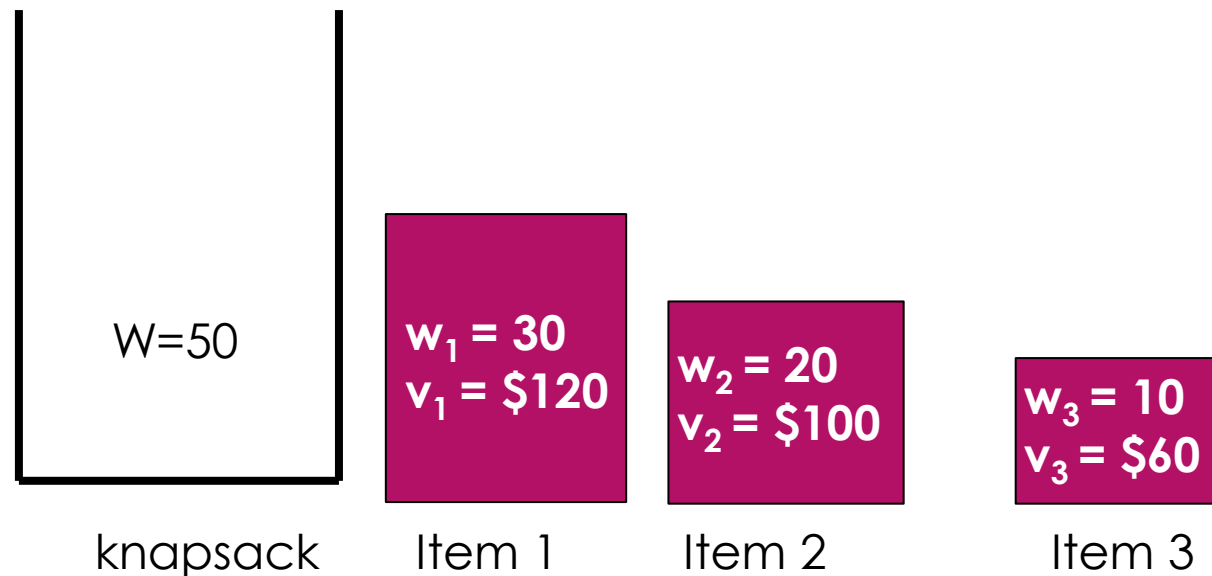
subset	weight	value
\emptyset	0	\$0
{1}	7	\$42 ✓
{2}	3	\$12
{3}	4	\$40
{4}	5	\$25
{1,2}	10	\$54 ✓
{1,3}	11	!feasible
{1,4}	12	!feasible
{2,3}	7	\$52
{2,4}	8	\$37
{3,4}	9	<u>\$65</u>
{1,2,3}	14	!feasible
{1,2,4}	15	!feasible
{1,3,4}	16	!feasible
{2,3,4}	12	!feasible
{1,2,3,4}	19	!feasible



Works for this example! 😊

Brute Force Case Studies

Exhaustive Search: KP2



Item 1: \$4/unit
Item 2: \$5/unit
Item 3: \$6/unit

$$\{\text{Item3, Item2}\} = \$60 + \$100 = \$160$$

$$\{\text{Item3, Item1}\} = \$60 + \$120 = \$180$$

$$\{\text{Item2, Item1}\} = \$100 + \$120 = \$220$$

Brute Force Case Study - Exhaustive Search

- ▶ A brute force approach to combinatorial problems (which require generation of permutations, or subsets)
- ▶ Generate every element of problem domain
- ▶ Select feasible ones (the ones that satisfy constraints)
- ▶ Find the desired one (the one that optimizes some objective function)
- ▶ For both TSM and KP, exhaustive search gives exponential time complexity.
- ▶ These are NP-hard problems, no known polynomial-time algorithm

Brute Force Case Study

Exhaustive Search: Assignment Problem (AP)

- ▶ There are n people who need to be assigned to execute n jobs, one person per job.
- ▶ $C[i, j]$: cost that would accrue if i -th person is assigned to j -th job.
- ▶ Find an assignment with the minimum total cost.
- ▶ **Generate all permutations of $\langle 1, 2, 3, 4 \rangle$, and compute total cost, find the smallest cost**

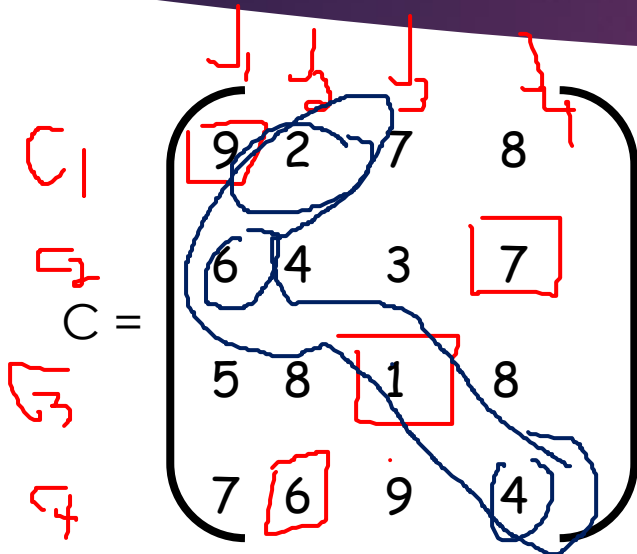
Cost matrix

	<i>F</i> Job 1	<i>B</i> Job 2	<i>P</i> Job 3	<i>R</i> Job 4
Person 1	9 <i>m</i>	<u>2</u> <i>m</i>	7	8
Person 2	6 <i>m</i>	4 <i>m</i>	<u>3</u>	7
Person 3	<u>5</u> <i>m</i>	8 <i>m</i>	<u>1</u>	8
Person 4	7	6 <i>m</i>	<u>9</u>	<u>4</u>

$$4 + 5 + 6 + 4 = 20$$

Brute Force Case Studies

Exhaustive Search: AP



Complexity is $\Omega(n!)$

Efficient algorithm exists
Called the "Hungarian
method".

$$1) \langle 1, 2, 3, 4 \rangle \text{ cost} = 9+4+1+4 = 18$$

$$2) \langle 1, 2, 4, 3 \rangle \text{ cost} = 9+4+8+9 = 30$$

$$3) \langle 1, 3, 2, 4 \rangle \text{ cost} = 9+3+8+4 = 24$$

$$4) \langle 1, 3, 4, 2 \rangle \text{ cost} = 9+3+8+6 = 26$$

$$5) \langle 1, 4, 2, 3 \rangle \text{ cost} = 9+7+8+9 = 33$$

$$6) \langle 1, 4, 3, 2 \rangle \text{ cost} = 9+7+1+6 = 23$$

$$7) \langle 2, 1, 3, 4 \rangle \text{ cost} = 2+6+1+4 = 13$$

$$8) \langle 2, 1, 4, 3 \rangle \text{ cost} = 2+6+8+9 = 25$$

$$9) \langle 2, 3, 1, 4 \rangle \text{ cost} = 2+3+5+4 = 14$$

$$10) \langle 2, 3, 4, 1 \rangle \text{ cost} = 2+3+8+7 = 20$$

$$11) \langle 2, 4, 1, 3 \rangle \text{ cost} = 2+7+5+9 = 23$$

$$12) \langle 2, 4, 3, 1 \rangle \text{ cost} = 2+7+1+7 = 17$$

$$13) \langle 3, 1, 2, 4 \rangle \text{ cost} = 7+6+8+4 = 25$$

$$13) \langle 3, 1, 4, 2 \rangle \text{ cost} = 7+6+8+6 = 27$$

$$15) \langle 3, 2, 1, 4 \rangle \text{ cost} = 7+4+1+4 = 16$$

$$16) \langle 3, 2, 4, 1 \rangle \text{ cost} = 7+4+8+7 = 26$$

$$17) \langle 3, 4, 1, 2 \rangle \text{ cost} = 7+7+5+6 = 25$$

$$18) \langle 3, 4, 2, 1 \rangle \text{ cost} = 7+7+8+7 = 29$$

$$19) \langle 4, 1, 2, 3 \rangle \text{ cost} = 8+6+8+9 = 31$$

$$20) \langle 4, 1, 3, 2 \rangle \text{ cost} = 8+6+1+6 = 21$$

$$21) \langle 4, 2, 1, 3 \rangle \text{ cost} = 8+4+5+9 = 26$$

$$22) \langle 4, 2, 3, 1 \rangle \text{ cost} = 8+4+1+7 = 20$$

$$23) \langle 4, 3, 1, 2 \rangle \text{ cost} = 8+3+1+6 = 18$$

$$24) \langle 4, 3, 2, 1 \rangle \text{ cost} = 8+3+8+7 = 26$$

Worst Assignment = 5

Best Assignment = 7

Complexity is $\Omega(4!) = 24$

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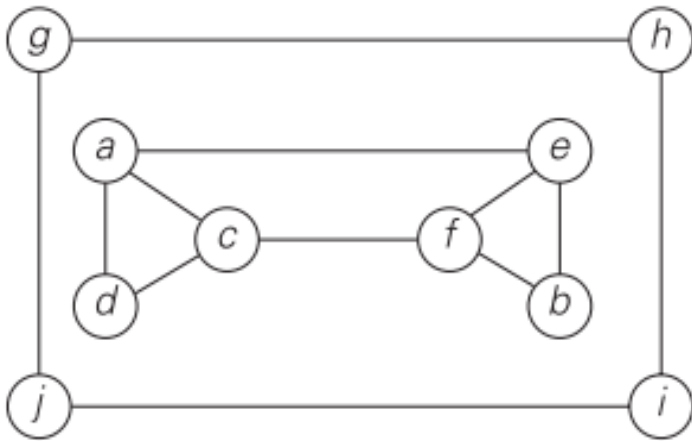
Graph Traversals

- ▶ Problem domain can be a graph
- ▶ Exhaustive search needs to visit each vertex and do something at it
- ▶ Two important graph-traversal algorithms: depth-first search, breadth first search

Graph Traversals: Depth-First Search (DFS)

- ▶ Start at a vertex, mark it as visited
- ▶ Go to one of unvisited neighbors, mark it visited, and so on.
- ▶ At dead end, backs up one edge to the vertex it came from and tries to visit unvisited vertices from there.
- ▶ Eventually halts after backing up to the starting vertex, by then one connected component has been traversed.
- ▶ If unvisited vertices remain, dfs starts at one of them.

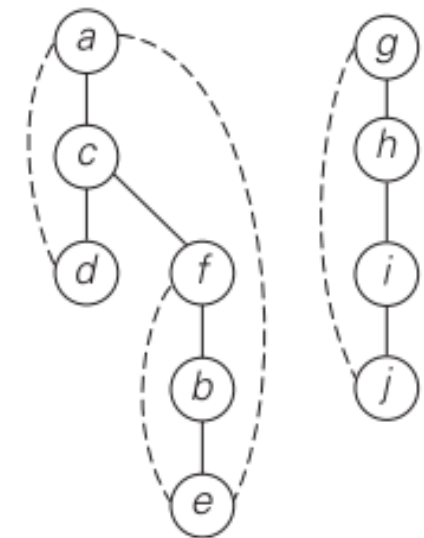
Graph Traversals: DFS



Graph

	$e_{6,2}$	
	$b_{5,3}$	$j_{10,7}$
$d_{3,1}$	$f_{4,4}$	$i_{9,8}$
$c_{2,5}$		$h_{8,9}$
$a_{1,6}$		$g_{7,10}$

Traversal's stack: Traversal's stack (the first subscript number indicates the order in which a vertex is visited, i.e., pushed onto the stack; the second one indicates the order in which it becomes a dead-end, i.e., popped off the stack).



DFS **forest** with the tree and back edges shown with solid and dashed lines, respectively.

Graph Traversals: DFS

ALGORITHM: DFS(G)

// Input: Graph= $\langle V, E \rangle$

mark each vertex in V with 0 as a mark of being “unvisited”

count $\leftarrow 0$

for each vertex v in V **do**

if v is marked with 0

 dfs(v)

dfs(v)

count \leftarrow count+1; mark v with count

for each vertex w in V adjacent to v **do**

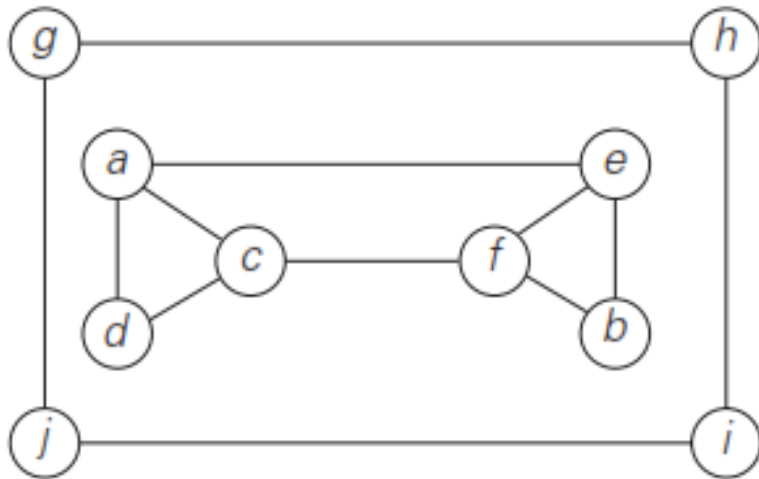
if w is marked with 0

 dfs(w)

Graph Traversals: BFS

- ▶ Start at a vertex, mark it visited
- ▶ Go to each of its neighbors and put them in a list
- ▶ Delete the starting vertex.
- ▶ Start at one of the remaining in the list, mark it visited, and so on...

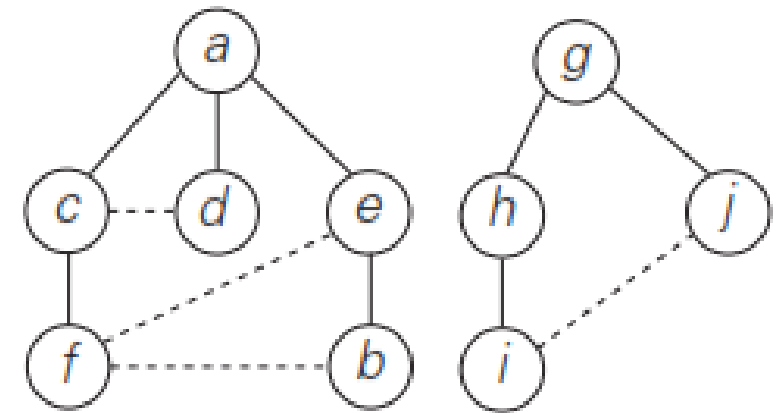
Graph Traversals: BFS



Graph

$a_1 c_2 d_3 e_4 f_5 b_6$
 $g_7 h_8 j_9 i_{10}$

Traversal queue, with the numbers indicating the order in which the vertices are visited, i.e., added to (and removed from) the queue.



BFS forest with the tree and cross edges shown with solid and dotted lines, respectively.

Graph Traversals: BFS

ALGORITHM BFS(G)

mark each vertex in V with 0 as a mark of being “unvisited”

count \leftarrow 0

for each vertex v in V **do**

if v is marked with 0

 bfs(v)

bfs(v)

Count \leftarrow count+1; mark v with count and initialize a queue with v

while the queue is not empty **do**

for each vertex w in V adjacent to the front vertex **do**

if w is marked with 0

 count \leftarrow count+1; mark w with count

 add w to queue

 remove the front vertex from the queue

Summary of DFS and BFS

	DFS	BFS
Data structure	Stack	Queue
Edge types	Tree and back edges	Tree and cross edges
Applications	Connectivity, Acyclicity, Articulation points	Connectivity, Acyclicity, Minimum-edge paths
Efficiency for adjacency matrix	$\Theta(V ^2)$	$\Theta(V ^2)$
Efficiency for adjacency list	$\Theta(V + E)$	$\Theta(V + E)$

Summary

- ▶ *BFA is a straightforward approach to solving a problem, usually directly based on the **problem statement** and **definitions** of the concepts involved.*
- ▶ *The principal strengths of the BFA are **wide applicability** and **simplicity**; its principal weakness is the **subpar efficiency** of most brute-force algorithms.*
- ▶ *Exhaustive search is a brute-force approach to combinatorial problems.*
- ▶ *The TSM, the KP, and the AP are typical examples of problems that can be solved, at least theoretically, by exhaustive-search algorithms (ESA).*
- ▶ *DFS and BFS are two principal graph-traversal algorithms.*

Thank You