2003 National Higher Education Entrance Examination
(national version)

Mathematics
(Science, engineering, agriculture and medicine)

2003/06/07 15:00 - 17:00
120 minutes

This exam is composed by **Part I** (multiple choice) and **Part II** (non-multiple choice). Please return both this exam and answer sheet after the exam period is finished.

Part I (60 points)

1. MULTIPLE CHOICES

This section consists of 12 questions. Each question is worth 5 points and there are 60 points available. Only one option is correct for each question.

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1. Given that $x \in (-\frac{\pi}{2}, 0)$, $\cos x = \frac{4}{5}$, then $\tan 2x$ is
   - A. $\frac{7}{24}$
   - B. $-\frac{7}{24}$
   - C. $\frac{24}{7}$
   - D. $-\frac{24}{7}$
2. The directrix of the conic section $\rho = \frac{8\sin\theta}{\cos^2\theta}$ is
   - A. $\rho\cos\theta = -2$
   - B. $\rho\cos\theta = 2$
   - C. $\rho\sin\theta = 2$
   - D. $\rho\sin\theta = -2$
3. Set the function $f(x) = \begin{cases} 2^{-x} - 1, & x \le 0 \\ x^{\frac{1}{2}}, & x > 0 \end{cases}$. If $f(x_0) > 1$, then the range of x_0 is
   - A. $(-1, 1)$
   - B. $(-1, +\infty)$
   - C. $(-\infty, -2) \cup (0, +\infty)$
   - D. $(-\infty, -1) \cup (1, +\infty)$
4. The maximum of the function $y = 2\sin x(\sin x + \cos x)$ is
   - A. $1 + \sqrt{2}$
   - B. $\sqrt{2} - 1$
   - C. $\sqrt{2}$
   - D. 2
5. Given a circle C: $(x-a)^2 + (y-2)^2 = 4\,(a > 0)$ and a line l: $x - y + 3 = 0$, when the chore determined by intersecting C by l has length $2\sqrt{3}$, a should be
   - A. $\sqrt{2}$
   - B. $2 - \sqrt{2}$
   - C. $\sqrt{2} - 1$
   - D. $\sqrt{2} + 1$
6. Given a cone with bottom radius R and height $3R$, for all its inscribed cylinders, the maximum surface area is
   - A. $2\pi R^2$

B. $\frac{9}{4}\pi R^2$

C. $\frac{8}{3}\pi R^2$

D. $\frac{3}{2}\pi R^2$

7. Given that the four roots of the equation $\left(x^2-2x+m\right)\left(x^2-2x+n\right)=0$ form an arithmetic sequence with the first term $\frac{1}{4}$, then $|m-n|$ is

A. 1

B. $\frac{3}{4}$

C. $\frac{1}{2}$

D. $\frac{3}{8}$

8. Suppose a hyperbola is centered at the origin and one of its focuses is $F(\sqrt{7},0)$. A line $y=x-1$ intersects the hyperbola at points M and N. The x-axis of the midpoint of MN is $-\frac{2}{3}$. Then the equation for this hyperbola is

A. $\frac{x^2}{3}-\frac{y^2}{4}=1$

B. $\frac{x^2}{4}-\frac{y^2}{3}=1$

C. $\frac{x^2}{5}-\frac{y^2}{2}=1$

D. $\frac{x^2}{2}-\frac{y^2}{5}=1$

9. The inverse, $f^{-1}(x)$, of the function $f(x)=\sin x, x\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ is

A. $-\arcsin x, x\in[-1,1]$

B. $-\pi-\arcsin x, x\in[-1,1]$

C. $\pi+\arcsin x, x\in[-1,1]$

D. $\pi-\arcsin x, x\in[-1,1]$

10. A rectangle has four vertexes $A(0,0)$, $B(2,0)$, $C(2,1)$ and $D(0,1)$. A point mass, from P_0, the midpoint of AB, along θ, the angle between itself and AB, goes to the point P_1 on BC. It is then reflected to CD, DA and AB on points P_2, P_3 and P_4 (the angle of incident equals to that of reflection). Let the coordinate of P_4 be $(x_4,0)$. If $1<x_4<2$, then the range of $\tan\theta$ is

A. $\left(\frac{1}{3},1\right)$

B. $\left(\frac{1}{3},\frac{2}{3}\right)$

C. $\left(\frac{2}{5},\frac{1}{2}\right)$

D. $\left(\frac{2}{5},\frac{2}{3}\right)$

11. $\lim\limits_{n\to\infty}\dfrac{C_2^2+C_3^2+C_4^2+\cdots+C_n^2}{n\left(C_2^1+C_3^1+C_4^1+\cdots+C_n^1\right)}=$

A. 3

B. $\frac{1}{3}$

C. $\frac{1}{6}$

D. 6

12. A tetrahedron has all of its edge length equal to $\sqrt{2}$. The four vertexes are all on the same sphere surface, then the surface area of the sphere is

A. 3π

B. 4π

C. $3\sqrt{3}\pi$

D. 6π

Part II (90 points)

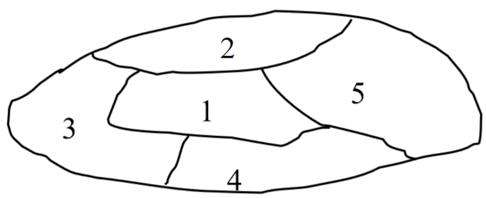
2. SHORT ANSWER

This section consists of 4 questions. Each question is worth 4 points and there are 16 points available. Fill in the answers in the blank area.

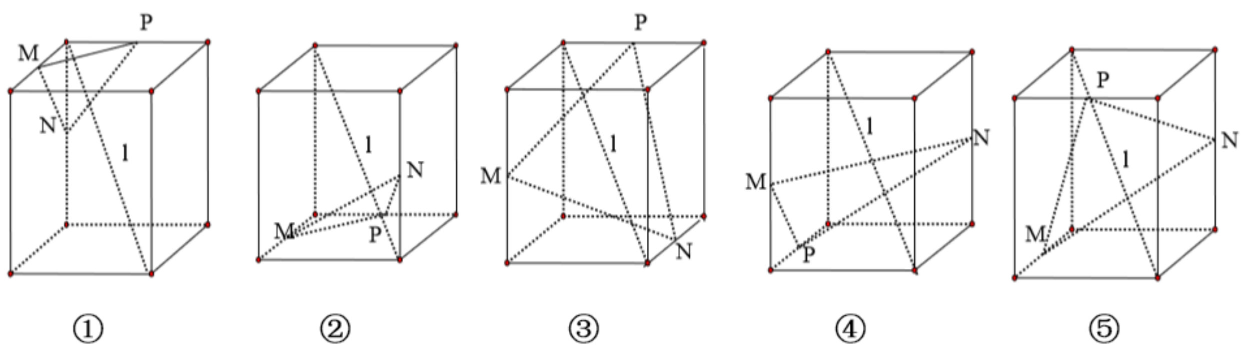
13. Find the term containing x^9 in the expansion of $\left(x^2 - \frac{1}{2x}\right)^9$: ____________

14. The range of x that makes the inequality $\log_2(-x) < x + 1$ hold is: ____________

15. A region is divided into 5 districts. We colour the map and demand that neighbours do not share the same colour. If we have 4 colours, then there are ____________ different colouring schemes. (answer in numbers)



16. For the following 5 cubes, l is a diagonal and points M, N, P are at the centers of the edges. The diagrams that have the relation $l \perp$ surface MNP are ____________. (write down all labels)



3. LONG ANSWER

This section consists of 6 questions and there are 74 points available. Candidates should write down their descriptions, proofs or steps.

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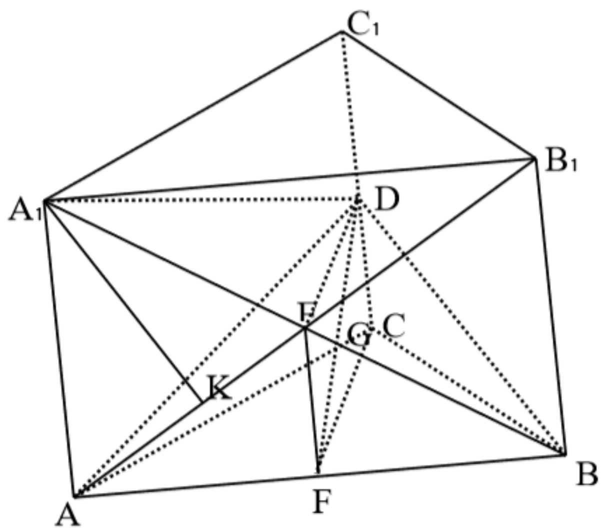
17. (12 points) Given that the argument of the complex number z is $60°$ and that $|z-1|$ is the middle term of the geometric terms $|z|$ and $|z-2|$, find $|z|$.

18. (12 points) In the below triangular prism $ABC - A_1B_1C_1$, the base is a isosceles right triangle, $\angle ACB = 90°$, the edge $AA_1 = 2$, D and E are the midpoints of CC_1 and A_1B, respectively, and the projection of point E onto surface ABD is the centroid G of $\triangle ABC$.



(I) Find the angle between A_1B and surface ABD. (express the answer in inverse trigonometric values)

(II) Find the distance between point A_1 and surface AED.

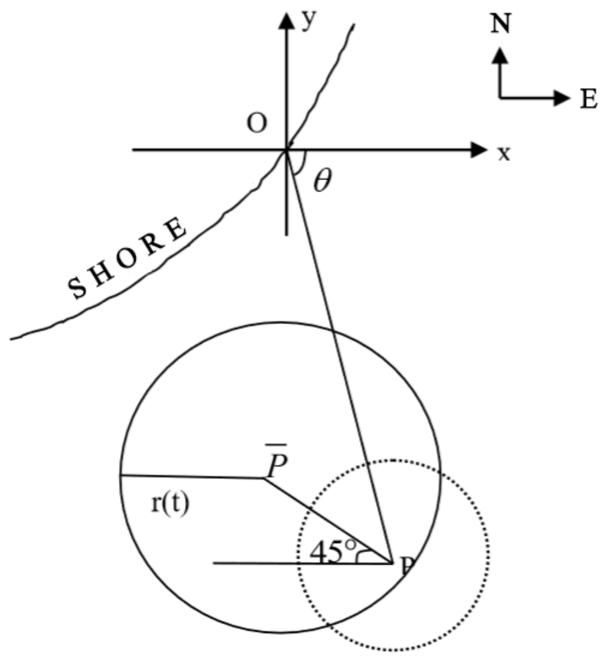
19. (12 points) Given that $c > 0$, let

P:= the function $y = c^x$ is monotone decreasing in $\mathbb{R}$.

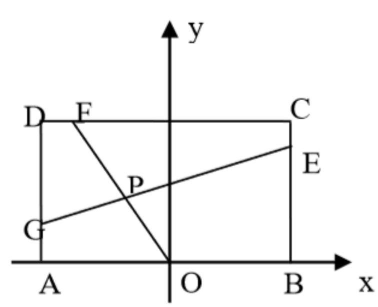
Q:= the inequality $x + |x - 2c| > 1$ has $\mathbb{R}$ as its solution.

If one and only one of P and Q is correct, find the range of c.

20. (12 points) A typhoon is spotted on the sea near to a coastal town. According to the observation, the eye of the typhoon is located at east $\theta = \arccos \frac{\sqrt{2}}{10}$ to the south, 300 km away, at the point P, relative to the town and is moving west 45° to the north at a speed of 20 km/h. The range of the typhoon is a circle and has a radius of 60 km which is expanding at a speed of 10 km/h. After how many hours will the town be affected by the typhoon?



21. (14 points) Let $a > 0$, in the rectangle $ABCD$, $AB = 4$, $BC = 4a$, O is the midpoint of AB, points E, F, G are moving on BC, CD, DA, respectively, also $\frac{BE}{BC} = \frac{CF}{CD} = \frac{DC}{DA}$, and P is the intersection of GE and OF. Do there exist two fixed points such that the sum of their distances to P is a constant? If there are, find the coordinates and such constants of these two points, otherwise explain why they do not exist.



22. (12 points) (I) Suppose that $\{a_n\}$ is a sequence formed by rearranging all numbers in the set $\{2^s + 2^t \mid 0 \leq s < t \text{ and } s, t \in \mathbb{Z}\}$ in increasing order, i.e. $a_1 = 3, a_2 = 5, a_3 = 6, a_4 = 9, a_5 = 10, a_6 = 12, \ldots$

Now write $\{a_n\}$ as a triangular number array based on: 1. the smaller is on the top and the bigger on the bottom 2. the smaller is on the left and the bigger on the right.

$$
\begin{array}{c}
3 \\
5 \quad 6 \\
9 \quad 10 \quad 12 \\
— \quad — \quad — \quad — \\
\cdots\cdots\cdots
\end{array}
$$

(1) Write down the numbers on the fourth and fifth rows of this array.

(2) Find a_{100}.

(II) (This item is a bonus question. If it is answered correctly, 4 points are added to the exam, but the total score should not exceed 150) Let $\{b_n\}$ be the sequence formed by rearranging all numbers in the set $\{2^r + 2^s + 2^t \mid 0 \leq r < s < t \text{ and } r, s, t \in \mathbb{Z}\}$ in increasing order. Given that $b_k = 1160$, find k.