Bachelor Thesis
Decoding the Color Code

Clemens Schumann,
Advised by Peter-Jan Derks

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1 Introduction to the Algebra

1.1 Schroedinger and Heisenberg picture

1.1.1 Schroedinger picture

In the Schroedinger picture, we focus on the time evolution of states:

$$|\psi\rangle = |\psi\rangle(t) \tag{1}$$

In this picture we can introduce quantum circuit diagram notation, whereby:

- States progress in time along horizontal parallel lines
- Time goes from left to right
- Gates denoted X,Y,Z are the pauli matrices $\sigma_x, \sigma_y, \sigma_z$
- Gates can act on one or multiple qubits, whereby an X gate on qubit 1 in a 3-qubit system should be interpreted as:
  $(X \otimes \mathbb{I} \otimes \mathbb{I})(|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle)$

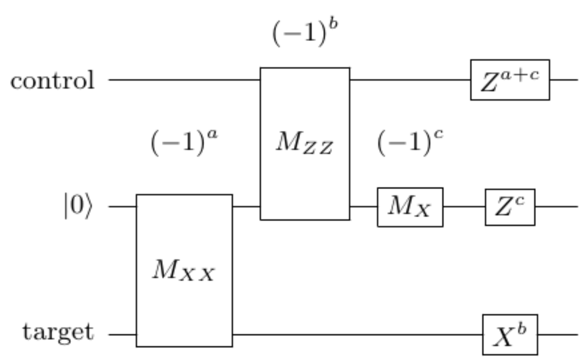

Figure 1: A Quantum Circuit, where $|0\rangle$ is the +1 eigenstate in σ_z-basis

As can be seen explicitly calculated in the familiar Schroedinger picture in Appendix 3.1, the circuit from figure 1 implements a CNOT-gate from the control qubit to the target qubit.

We will now analyze this circuit in the Heisenberg picture, finding that it results in an equivalent result.

1.1.2 Heisenberg Picture

In this picture, we focus on the time evolution of operators instead of states:

$$A = A(t) \tag{2}$$

By considering specifically the operators to which the input state space is part of those operators eigenstatespace, we can compute the output of any circuit:

$$Circuit(|\phi\rangle) = Circuit(A)|\phi\rangle \tag{3}$$

if $|\phi\rangle$ is an eigenstate of A.

2 Conclusion

3 Appendix

3.1 Calculation 1

In the quantum circuit depicted in figure 1 the input state can be written as $|\psi_{control}\rangle \otimes |0\rangle \otimes |\psi_{target}\rangle$ and the measurement in the first timestep can be expressed as $\mathbb{I} \otimes X \otimes X$.

The initial state $|\phi_{t=0}\rangle = |\psi_{control}\rangle \otimes |\psi_{ancilla}\rangle \otimes |\psi_{target}\rangle$
where
$|\psi_{control}\rangle = \alpha|0\rangle + \beta|1\rangle$
$|\psi_{ancilla}\rangle = |0\rangle$
$|\psi_{target}\rangle = \gamma|0\rangle + \delta|1\rangle$
therefore:

$$|\phi_{t=0}\rangle = \alpha\left(\gamma|000\rangle + \delta|001\rangle\right) + \beta\left(\gamma|100\rangle + \delta|101\rangle\right) \tag{4}$$

If the first measurement result is +1, the state becomes:

$$\begin{aligned}|\phi^{+}_{t=1}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} + \mathbb{I}\otimes X\otimes X\right)|\phi_{t=0}\rangle \\ &= \alpha\left(\gamma\left(|000\rangle + |011\rangle\right) + \delta\left(|001\rangle + |010\rangle\right)\right) \\ &+ \beta\left(\gamma\left(|100\rangle + |111\rangle\right) + \delta\left(|101\rangle + |110\rangle\right)\right)\end{aligned}$$

if the result is -1, it becomes:

$$\begin{aligned}|\phi^{-}_{t=1}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} - \mathbb{I}\otimes X\otimes X\right)|\phi_{t=0}\rangle \\ &= \alpha\left(\gamma\left(|000\rangle - |011\rangle\right) + \delta\left(|001\rangle - |010\rangle\right)\right) \\ &+ \beta\left(\gamma\left(|100\rangle - |111\rangle\right) + \delta\left(|101\rangle - |110\rangle\right)\right)\end{aligned}$$

In the case of the +1 Measurement $\rightarrow$ a=0:

$$\begin{aligned}|\phi^{++}_{t=2}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} + Z\otimes Z\otimes\mathbb{I}\right)|\phi^{+}_{t=1}\rangle \\ &= (|000\rangle\langle 000| + |001\rangle\langle 001| + |110\rangle\langle 110| + |111\rangle\langle 111|)|\phi^{+}_{t=1}\rangle \\ &= \alpha\left(\gamma|000\rangle + \delta|001\rangle\right) + \beta\left(\gamma|111\rangle + \delta|110\rangle\right)\end{aligned}$$

$$\begin{aligned}|\phi^{+-}_{t=2}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} - Z\otimes Z\otimes\mathbb{I}\right)|\phi^{+}_{t=1}\rangle \\ &= (|010\rangle\langle 010| + |011\rangle\langle 011| + |100\rangle\langle 100| + |101\rangle\langle 101|)|\phi^{+}_{t=1}\rangle \\ &= \alpha\left(\gamma|011\rangle + \delta|010\rangle\right) + \beta\left(\gamma|100\rangle + \delta|101\rangle\right)\end{aligned}$$

In the case of the -1 Measurement $\rightarrow$ a=1:

$$
\begin{aligned}
|\phi_{t=2}^{-+}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} + Z\otimes Z\otimes\mathbb{I}\right)|\phi_{t=1}^{-}\rangle \\
&= \alpha\left(\gamma|000\rangle + \delta|001\rangle\right) - \beta\left(\gamma|111\rangle + \delta|110\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
|\phi_{t=2}^{--}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} - Z\otimes Z\otimes\mathbb{I}\right)|\phi_{t=1}^{-}\rangle \\
&= -\alpha\left(\gamma|011\rangle + \delta|010\rangle\right) + \beta\left(\gamma|100\rangle + \delta|101\rangle\right)
\end{aligned}
$$

Now the applied measurement is $\mathbb{I}\otimes X\otimes\mathbb{I}$, which means:

$$
\begin{aligned}
|\phi_{t=3}^{+++}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} + \mathbb{I}\otimes X\otimes\mathbb{I}\right)|\phi_{t=2}^{++}\rangle \\
&= \frac{1}{2}((|010\rangle + |000\rangle)\langle 000| + (|011\rangle + |001\rangle)\langle 001| \\
&+ (|000\rangle + |010\rangle)\langle 010| + (|001\rangle + |011\rangle)\langle 011| \\
&+ (|110\rangle + |100\rangle)\langle 100| + (|111\rangle + |101\rangle)\langle 101| \\
&+ (|100\rangle + |110\rangle)\langle 110| + (|101\rangle + |111\rangle)\langle 111|)|\phi_{t=2}^{++}\rangle \\
&= \frac{1}{2}(\alpha\left(\gamma(|000\rangle + |010\rangle) + \delta(|011\rangle + |001\rangle)\right) \\
&+ \beta\left(\gamma(|101\rangle + |111\rangle) + \delta(|100\rangle + |110\rangle)\right))
\end{aligned}
$$

In this case, a,b and c would each be zero, therefore no further gate would be applied.
As intended, this state is equivalent to $CNOT_{|\psi_{Control}\rangle\rightarrow|\psi_{Target}\rangle}|\phi_{t=0}\rangle$.