Bachelor Thesis
Decoding the Color Code

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November 22, 2022

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1 Introduction to the Algebra

1.1 Schroedinger and Heisenberg picture

1.1.1 Schroedinger picture

In the Schroedinger picture, we focus on the time evolution of states:

$$|\psi\rangle = |\psi\rangle(t) \tag{1}$$

In this picture we can introduce quantum circuit diagram notation, whereby:

- States progress in time along horizontal parallel lines
- Time goes from left to right
- Gates denoted X,Y,Z are the pauli matrices $\sigma_x, \sigma_y, \sigma_z$
- Gates can act on one or multiple qubits, whereby an X gate on qubit 1 in a 3-qubit system should be interpreted as:
  $(X \otimes \mathbb{I} \otimes \mathbb{I})(|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle)$

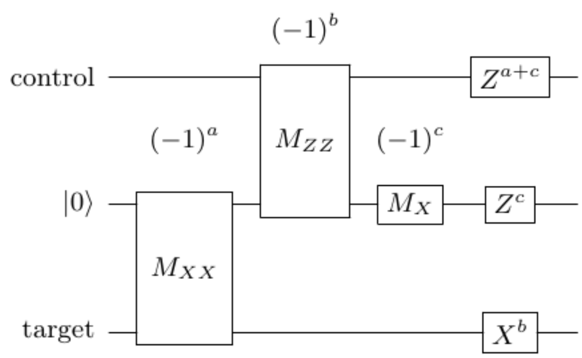

Figure 1: A Quantum Circuit, where $|0\rangle$ is the +1 eigenstate in σ_z-basis

In the quantum circuit depicted in figure 1 the input state can be written as $|\psi_{control}\rangle \otimes |0\rangle \otimes |\psi_{target}\rangle$ and the measurement in the first timestep can be expressed as $\mathbb{I} \otimes X \otimes X$.
To simplify calculation we can write $|\psi_{target}\rangle$ as $\alpha|0\rangle + \beta|1\rangle$
The input state is thus: $|\psi_{control}\rangle \otimes |0\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$.
The initial state $|\phi_{t=0}\rangle = |\psi_{control}\rangle \otimes |\psi_{ancilla}\rangle \otimes |\psi_{target}\rangle$
where
$|\psi_{control}\rangle = \alpha|0\rangle + \beta|1\rangle$
$|\psi_{ancilla}\rangle = |0\rangle$
$|\psi_{target}\rangle = \gamma|0\rangle + \delta|1\rangle$
therefore:

$$|\phi_{t=0}\rangle = \alpha\left(\gamma|000\rangle + \delta|001\rangle\right) + \beta\left(\gamma|100\rangle + \delta|101\rangle\right) \tag{2}$$

If the first measurement result is +1, the state becomes:

$$\begin{aligned}
|\phi^{+}_{t=1}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} + \mathbb{I}\otimes X\otimes X\right)|\phi_{t=0}\rangle \\
&= \alpha\left(\gamma\left(|000\rangle + |011\rangle\right) + \delta\left(|001\rangle + |010\rangle\right)\right) \\
&+ \beta\left(\gamma\left(|100\rangle + |111\rangle\right) + \delta\left(|101\rangle + |110\rangle\right)\right)
\end{aligned}$$

if the result is -1, it becomes:

$$\begin{aligned}
|\phi^{-}_{t=1}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} - \mathbb{I}\otimes X\otimes X\right)|\phi_{t=0}\rangle \\
&= \alpha\left(\gamma\left(|000\rangle - |011\rangle\right) + \delta\left(|001\rangle - |010\rangle\right)\right) \\
&+ \beta\left(\gamma\left(|100\rangle - |111\rangle\right) + \delta\left(|101\rangle - |110\rangle\right)\right)
\end{aligned}$$

In the case of the +1 Measurement $\rightarrow$ a=0:

$$\begin{aligned}
|\phi^{++}_{t=2}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} + Z\otimes Z\otimes\mathbb{I}\right)|\phi^{+}_{t=1}\rangle \\
&= (|000\rangle\langle 000| + |001\rangle\langle 001| + |110\rangle\langle 110| + |111\rangle\langle 111|)|\phi^{+}_{t=1}\rangle \\
&= \alpha\left(\gamma|000\rangle + \delta|001\rangle\right) + \beta\left(\gamma|111\rangle + \delta|110\rangle\right)
\end{aligned}$$

$$\begin{aligned}
|\phi^{+-}_{t=2}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} - Z\otimes Z\otimes\mathbb{I}\right)|\phi^{+}_{t=1}\rangle \\
&= (|010\rangle\langle 010| + |011\rangle\langle 011| + |100\rangle\langle 100| + |101\rangle\langle 101|)|\phi^{+}_{t=1}\rangle \\
&= \alpha\left(\gamma|011\rangle + \delta|010\rangle\right) + \beta\left(\gamma|100\rangle + \delta|101\rangle\right)
\end{aligned}$$

In the case of the -1 Measurement $\rightarrow$ a=1:

$$
\begin{aligned}
|\phi^{-+}_{t=2}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} + Z\otimes Z\otimes\mathbb{I}\right)|\phi^{-}_{t=1}\rangle \\
&= \alpha\left(\gamma|000\rangle + \delta|001\rangle\right) - \beta\left(\gamma|111\rangle + \delta|110\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
|\phi^{--}_{t=2}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} - Z\otimes Z\otimes\mathbb{I}\right)|\phi^{-}_{t=1}\rangle \\
&= -\alpha\left(\gamma|011\rangle + \delta|010\rangle\right) + \beta\left(\gamma|100\rangle + \delta|101\rangle\right)
\end{aligned}
$$

Now the applied measurement is $\mathbb{I}\otimes X\otimes\mathbb{I}$, which means:

$$
\begin{aligned}
|\phi^{+++}_{t=3}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} + \mathbb{I}\otimes X\otimes\mathbb{I}\right)|\phi^{++}_{t=2}\rangle \\
&= \frac{1}{2}((|010\rangle + |000\rangle)\langle 000| + (|011\rangle + |001\rangle)\langle 001| \\
&+ (|000\rangle + |010\rangle)\langle 010| + (|001\rangle + |011\rangle)\langle 011| \\
&+ (|110\rangle + |100\rangle)\langle 100| + (|111\rangle + |101\rangle)\langle 101| \\
&+ (|100\rangle + |110\rangle)\langle 110| + (|101\rangle + |111\rangle)\langle 111|)|\phi^{++}_{t=2}\rangle \\
&= \frac{1}{2}(\alpha\left(\gamma(|000\rangle + |010\rangle) + \delta(|011\rangle + |001\rangle)\right) \\
&+ \beta\left(\gamma(|101\rangle + |111\rangle) + \delta(|100\rangle + |110\rangle)\right))
\end{aligned}
$$

In this case, a,b and c would each be zero, therefore no further gate would be applied.
As intended, this state is equivalent to $CNOT_{Control\rightarrow Target}|\phi_{t=0}\rangle$.

2 Conclusion

3 Appendix