

Bachelor Thesis  
Decoding the Color Code

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# 1 Introduction to the Algebra

## 1.1 Schroedinger and Heisenberg picture

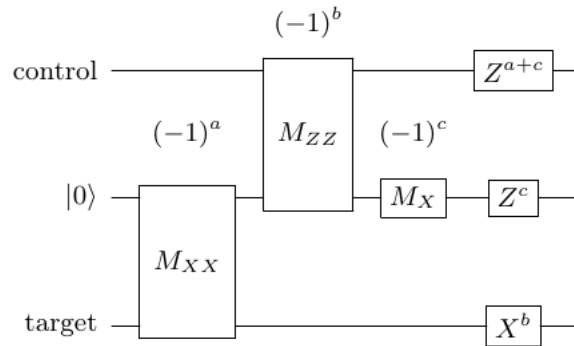
### 1.1.1 Schroedinger picture

In the Schroedinger picture, we focus on the time evolution of states:

$$|\psi\rangle = |\psi\rangle(t) \quad (1)$$

In this picture we can introduce quantum circuit diagram notation, whereby:

- States progress in time along horizontal parallel lines
- Time goes from left to right
- Gates denoted X,Y,Z are the pauli matrices  $\sigma_x, \sigma_y, \sigma_z$
- Gates can act on one or multiple qubits, whereby an X gate on qubit 1 in a 3-qubit system should be interpreted as:  
 $(X \otimes \mathbb{I} \otimes \mathbb{I})(|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle)$



**Figure 1:** A Quantum Circuit, where  $|0\rangle$  is the +1 eigenstate in  $\sigma_z$ -basis

As can be seen explicitly calculated in the familiar Schroedinger picture in Appendix 3.1, the circuit from figure 1 implements a CNOT-gate from the control qubit to the target qubit.

We will now analyze this circuit in the Heisenberg picture, finding that it results in an equal output.

### 1.1.2 Heisenberg Picture

In this picture, we focus on the time evolution of operators instead of states:

$$A = A(t) \tag{2}$$

By considering specifically the operators to which the input state space is part of those operators eigenstatespace, we can compute the output of any circuit:

$$Circuit(|\phi\rangle) = Circuit(A)|\phi\rangle \tag{3}$$

if  $|\phi\rangle$  is an eigenstate of  $A$ .

We call an operator/gate, to which an input state is an eigenvector the “Stabilizer” of that input state.

The “Stabilizer group” is a group of such operators, comprising the Stabilizers that stabilize all allowed input states

So e.g. the input state in figure 1 is only stabilized by  $\mathbb{I} \otimes Z \otimes \mathbb{I}$  (and trivially  $\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I}$ ).

## 2 Conclusion

## 3 Appendix

### 3.1 Calculation 1

In the quantum circuit depicted in figure 1 the input state can be written as  $|\psi_{control}\rangle \otimes |0\rangle \otimes |\psi_{target}\rangle$  and the measurement in the first timestep can be expressed as  $\mathbb{I} \otimes X \otimes X$ .

The initial state  $|\phi_{t=0}\rangle = |\psi_{control}\rangle \otimes |\psi_{ancilla}\rangle \otimes |\psi_{target}\rangle$  where

$$|\psi_{control}\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi_{ancilla}\rangle = |0\rangle$$

$$|\psi_{target}\rangle = \gamma|0\rangle + \delta|1\rangle$$

therefore:

$$|\phi_{t=0}\rangle = \alpha (\gamma|000\rangle + \delta|001\rangle) + \beta (\gamma|100\rangle + \delta|101\rangle) \quad (4)$$

If the first measurement result is +1, the state becomes:

$$\begin{aligned} |\phi_{t=1}^+\rangle &= \frac{1}{2} (\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes X \otimes X) |\phi_{t=0}\rangle \\ &= \alpha (\gamma (|000\rangle + |011\rangle) + \delta (|001\rangle + |010\rangle)) \\ &\quad + \beta (\gamma (|100\rangle + |111\rangle) + \delta (|101\rangle + |110\rangle)) \end{aligned}$$

if the result is -1, it becomes:

$$\begin{aligned} |\phi_{t=1}^-\rangle &= \frac{1}{2} (\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes X \otimes X) |\phi_{t=0}\rangle \\ &= \alpha (\gamma (|000\rangle - |011\rangle) + \delta (|001\rangle - |010\rangle)) \\ &\quad + \beta (\gamma (|100\rangle - |111\rangle) + \delta (|101\rangle - |110\rangle)) \end{aligned}$$

In the case of the +1 Measurement  $\rightarrow a=0$ :

$$\begin{aligned} |\phi_{t=2}^{++}\rangle &= \frac{1}{2} (\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} + Z \otimes Z \otimes \mathbb{I}) |\phi_{t=1}^+\rangle \\ &= (|000\rangle\langle 000| + |001\rangle\langle 001| + |110\rangle\langle 110| + |111\rangle\langle 111|) |\phi_{t=1}^+\rangle \\ &= \alpha (\gamma|000\rangle + \delta|001\rangle) + \beta (\gamma|111\rangle + \delta|110\rangle) \end{aligned}$$

$$\begin{aligned} |\phi_{t=2}^{+-}\rangle &= \frac{1}{2} (\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} - Z \otimes Z \otimes \mathbb{I}) |\phi_{t=1}^+\rangle \\ &= (|010\rangle\langle 010| + |011\rangle\langle 011| + |100\rangle\langle 100| + |101\rangle\langle 101|) |\phi_{t=1}^+\rangle \\ &= \alpha (\gamma|011\rangle + \delta|010\rangle) + \beta (\gamma|100\rangle + \delta|101\rangle) \end{aligned}$$

In the case of the -1 Measurement  $\rightarrow a=1$ :

$$\begin{aligned} |\phi_{t=2}^{-+}\rangle &= \frac{1}{2} (\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} + Z \otimes Z \otimes \mathbb{I}) |\phi_{t=1}^{-}\rangle \\ &= \alpha (\gamma|000\rangle + \delta|001\rangle) - \beta (\gamma|111\rangle + \delta|110\rangle) \end{aligned}$$

$$\begin{aligned} |\phi_{t=2}^{--}\rangle &= \frac{1}{2} (\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} - Z \otimes Z \otimes \mathbb{I}) |\phi_{t=1}^{-}\rangle \\ &= -\alpha (\gamma|011\rangle + \delta|010\rangle) + \beta (\gamma|100\rangle + \delta|101\rangle) \end{aligned}$$

Now the applied measurement is  $\mathbb{I} \otimes X \otimes \mathbb{I}$ , which means:

$$\begin{aligned} |\phi_{t=3}^{+++}\rangle &= \frac{1}{2} (\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes X \otimes \mathbb{I}) |\phi_{t=2}^{++}\rangle \\ &= \frac{1}{2} ((|010\rangle + |000\rangle)\langle 000| + (|011\rangle + |001\rangle)\langle 001| \\ &\quad + (|000\rangle + |010\rangle)\langle 010| + (|001\rangle + |011\rangle)\langle 011| \\ &\quad + (|110\rangle + |100\rangle)\langle 100| + (|111\rangle + |101\rangle)\langle 101| \\ &\quad + (|100\rangle + |110\rangle)\langle 110| + (|101\rangle + |111\rangle)\langle 111|) |\phi_{t=2}^{++}\rangle \\ &= \frac{1}{2} (\alpha (\gamma(|000\rangle + |010\rangle) + \delta(|011\rangle + |001\rangle)) \\ &\quad + \beta (\gamma(|101\rangle + |111\rangle) + \delta(|100\rangle + |110\rangle))) \end{aligned}$$

In this case, a,b and c would each be zero, therefore no further gate would be applied.

As intended, this state is equivalent to  $CNOT_{|\psi_{Control}\rangle \rightarrow |\psi_{Target}\rangle} |\phi_{t=0}\rangle$ .

If the last measurement result is -1:

$$\begin{aligned} |\phi_{t=3}^{++-}\rangle &= \frac{1}{2} (\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes X \otimes \mathbb{I}) |\phi_{t=2}^{++}\rangle \\ &= \frac{1}{2} ((|010\rangle + |000\rangle)\langle 000| + (|001\rangle - |011\rangle)\langle 001| \\ &\quad + (|010\rangle - |000\rangle)\langle 010| + (|011\rangle - |001\rangle)\langle 011| \\ &\quad + (|100\rangle - |110\rangle)\langle 100| + (|101\rangle - |111\rangle)\langle 101| \\ &\quad + (|110\rangle - |100\rangle)\langle 110| + (|111\rangle - |101\rangle)\langle 111|) |\phi_{t=2}^{++}\rangle \\ &= \frac{1}{2} (\alpha (TODOTODOTODOTODOTODO\gamma(|000\rangle + |010\rangle) + \delta(|011\rangle + |001\rangle)) \\ &\quad + \beta (\gamma(|101\rangle + |111\rangle) + \delta(|100\rangle + |110\rangle))) \end{aligned}$$

Notably, each measurement sequence has a differing resulting ancilla state, however we do not care since ancillas are meant to be discarded. For now, the other 7 final computation steps are left as an exercise to the reader, however I probably will still finish that.