Bachelor Thesis
Decoding the Color Code

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1 Introduction to the Algebra

1.1 Schroedinger and Heisenberg picture

1.1.1 Schroedinger picture

In the Schroedinger picture, we focus on the time evolution of states:

$$|\psi\rangle = |\psi\rangle(t) \tag{1}$$

In this picture we can introduce quantum circuit diagram notation, whereby:

- States progress in time along horizontal parallel lines
- Time goes from left to right
- Gates denoted X,Y,Z are the pauli matrices $\sigma_x, \sigma_y, \sigma_z$
- Gates can act on one or multiple qubits, whereby an X gate on qubit 1 in a 3-qubit system should be interpreted as: $(X \otimes \mathbb{I} \otimes \mathbb{I})(|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle)$

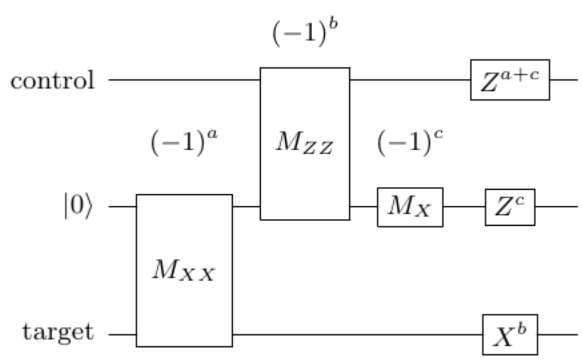

Figure 1: A Quantum Circuit, where $|0\rangle$ is the +1 eigenstate in σ_z-basis

In the quantum circuit depicted in figure 1 the input state can be written as $|\psi_{control}\rangle \otimes |0\rangle \otimes |\psi_{target}\rangle$ and the measurement in the first timestep can be expressed as $\mathbb{I} \otimes X \otimes X$.
To simplify calculation we can write $|\psi_{target}\rangle$ as $\alpha|+\rangle + \beta|-\rangle$, where $|+\rangle, |-\rangle$ are the +1 and -1 Eigenstates of the σ_x-matrix.
The input state is thus: $|\psi_{control}\rangle \otimes |0\rangle \otimes (\alpha|+\rangle + \beta|-\rangle)$.
Upon the first measurement, if the measurement result on ancilla $|0\rangle$ is +1, the state becomes:

$$|\phi^{+}_{t=1}\rangle = |\psi_{control}\rangle \otimes |+\rangle \otimes \alpha|+\rangle \tag{2}$$

In this case, a = 0. If the measurement result is -1, the state becomes:

$$|\phi^{-}_{t=1}\rangle = |\psi_{control}\rangle \otimes |-\rangle \otimes \beta|-\rangle \tag{3}$$

In this case, a = 1.
We now write $|\psi_{control}\rangle$ as $\gamma|0\rangle + \delta|1\rangle$ and $|\pm\rangle$ as $\frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$.
Upon the second measurement then, if the measurement result on the control qubit is +1, and the first ancilla measurement was also +1, the state becomes:

$$|\phi^{++}_{t=2}\rangle = \gamma|0\rangle \otimes |0\rangle \otimes \alpha|+\rangle \tag{4}$$

Similarly:

$$|\phi^{+-}_{t=2}\rangle = -\delta|1\rangle \otimes -|1\rangle \otimes \alpha|+\rangle \tag{5}$$

$$|\phi^{-+}_{t=2}\rangle = \gamma|0\rangle \otimes |0\rangle \otimes \beta|-\rangle \tag{6}$$

$$|\phi^{--}_{t=2}\rangle = -\delta|1\rangle \otimes |1\rangle \otimes \beta|-\rangle \tag{7}$$

In X basis, the state of the ancilla will be:

$$|\psi^{++}_{A}\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, |\psi^{+-}_{A}\rangle = \frac{|-\rangle - |+\rangle}{\sqrt{2}} \tag{8}$$

$$|\psi^{-+}_{A}\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, |\psi^{--}_{A}\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}} \tag{9}$$

Therefore, measuring X on the ancilla at t=3 will yield -1 or +1 both with probability $\frac{1}{2}$ in the ++ and the -+ case, always -1 in the +- case and always +1 in the – case.
So then the target qubit will be resolved in the following way:

1. $Z^{1+1}|\phi^{--}_{t=4}\rangle = -\delta|1\rangle \neq |\phi_{t=0}\rangle$???

Meanwhile, in the Heisenberg picture we focus on the time-evolution of Operators:

$$H = H(t) \tag{10}$$

By specifically looking at the time evolution of those operators to which the states in the system's input state space are eigenstates, we can figure out a systems output state space via:

$$Circuit(\psi) = Circuit(H)\psi \tag{11}$$

2 Conclusion

3 Appendix