Bachelor Thesis
Decoding the Color Code

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Contents

1 Introduction to the Algebra

In the following we will give an overview of the utilized algebra in quantum information theory.
To this end we must first return to the fundamentals of quantum mechanics.

1.1 Schroedinger and Heisenberg picture

1.1.1 Schroedinger picture

In the Schroedinger picture, we focus on the time evolution of states:

$$|\psi\rangle = |\psi\rangle(t) \tag{1}$$

In this picture we can introduce quantum circuit diagram notation, whereby:

- States progress in time along horizontal parallel lines
- Time goes from left to right
- Gates denoted X,Y,Z are the single qubit pauli operators $\sigma_x, \sigma_y, \sigma_z$
- Gates can act on one or multiple qubits, whereby an X gate on qubit 1 in a 3-qubit system should be interpreted as:
  $(X \otimes \mathbb{I} \otimes \mathbb{I})(|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle)$

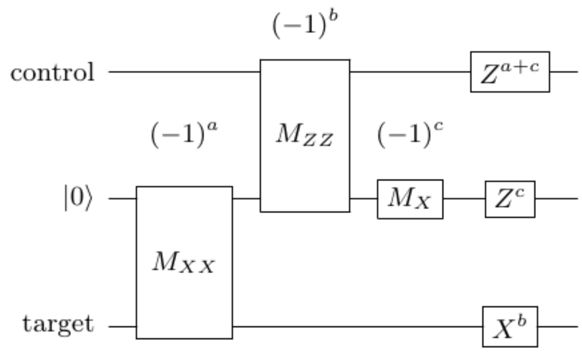

Figure 1: A Quantum Circuit to implement a measurement based Controlled-$X_{|\psi\rangle_{control} \to |\psi\rangle_{target}}$ Gate, where $|0\rangle$ is the $+1$ eigenstate in σ_z-basis.

As can be seen explicitly calculated in the familiar Schroedinger picture in Appendix 3.1, the circuit from figure 1 implements a CNOT-gate from the control qubit to the target qubit.

We will now analyze this circuit in the Heisenberg picture, finding that it results in an equal output.

1.1.2 Heisenberg Picture

In this picture, we focus on the time evolution of operators instead of states:

$$A = A(t) \tag{2}$$

By considering specifically the operators to which the input state space is part of those operators eigenstatespace, we can compute the output of any circuit:

$$Circuit(|\phi\rangle) = Circuit(A)|\phi\rangle \tag{3}$$

if $|\phi\rangle$ is an eigenstate of A.
This is true because for any unitary operation on a multi-qubit System applying A to a circuit implementing operator B:

$$|\psi\rangle \to^B B|\psi\rangle \tag{4}$$

$$\Rightarrow^A A|\psi\rangle \to^B AB|\psi\rangle = (ABA^\dagger)A|\psi\rangle \tag{5}$$

If $A|\psi\rangle = |\psi\rangle$, this simplifies to:

$$|\psi\rangle \rightarrow (ABA^\dagger)|\psi\rangle \tag{6}$$

Therefore the circuit operator $B \rightarrow (ABA^\dagger)$.

We call an operator/gate A, to which the input state is an eigenvector, a "Stabilizer" of that input state.
The "Stabilizer group" is a generating subset of the set of such operators.
One notable group of operators is the pauli group. It has the feature that it is "complete", in the sense that any transformation of a single qubit states into another can be achieved by sequential application of these transformations.
Also, for each $A \in P_G, n \in \mathbb{N} : A^{2n} = \mathbb{I}$.
We therefore use the Pauli Group as generating set for our stabilizers, so e.g. the input state in figure 1 is only stabilized by $\mathbb{I} \otimes Z \otimes \mathbb{I}$ (and trivially $\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I}$).
In our case, for the first Measurement in circuit 1 , A is $\mathbb{I} \otimes Z \otimes \mathbb{I}$ and B is $\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \pm \mathbb{I} \otimes X \otimes X$, so depending on wether the measurement result is ± 1 after the first Measurement, to obtain the evolved Operator we need to compute:

$$(\mathbb{I} \otimes Z \otimes \mathbb{I})(\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \pm \mathbb{I} \otimes X \otimes X)(\mathbb{I} \otimes Z \otimes \mathbb{I})^\dagger \tag{7}$$

This notation quickly becomes quite convoluted, so in the following we shall denote the Stabilizer as S, and the measurements as $M_1^\pm, M_2^\pm, M_3^\pm$, where
$M_1^\pm = \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \pm \mathbb{I} \otimes X \otimes X$,
$M_2^\pm = \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \pm Z \otimes Z \otimes \mathbb{I}$ and
$M_3^\pm = \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \pm \mathbb{I} \otimes X \otimes \mathbb{I}$.
After the second measurement the system operator evolves to:

$$M_2^{\dagger\pm} M_2^\pm S M_1^\pm \tag{8}$$

After the third it evolves to:

$$M_3^{\dagger\pm} M_3^\pm M_2^{\dagger\pm} M_2^\pm S M_1^\pm \tag{9}$$

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2 Conclusion

3 Appendix

3.1 Calculation 1

In the quantum circuit depicted in figure 1 the input state can be written as $|\psi_{control}\rangle \otimes |0\rangle \otimes |\psi_{target}\rangle$ and the measurement in the first timestep can be expressed as $\mathbb{I} \otimes X \otimes X$.

The initial state $|\phi_{t=0}\rangle = |\psi_{control}\rangle \otimes |\psi_{ancilla}\rangle \otimes |\psi_{target}\rangle$
where
$|\psi_{control}\rangle = \alpha|0\rangle + \beta|1\rangle$
$|\psi_{ancilla}\rangle = |0\rangle$
$|\psi_{target}\rangle = \gamma|0\rangle + \delta|1\rangle$
therefore:

$$|\phi_{t=0}\rangle = \alpha\left(\gamma|000\rangle + \delta|001\rangle\right) + \beta\left(\gamma|100\rangle + \delta|101\rangle\right) \tag{10}$$

If the first measurement result is +1, the state becomes:

$$\begin{aligned}|\phi^{+}_{t=1}\rangle &= \frac{1}{2}\left(\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes X \otimes X\right)|\phi_{t=0}\rangle \\ &= \alpha\left(\gamma\left(|000\rangle + |011\rangle\right) + \delta\left(|001\rangle + |010\rangle\right)\right) \\ &+ \beta\left(\gamma\left(|100\rangle + |111\rangle\right) + \delta\left(|101\rangle + |110\rangle\right)\right)\end{aligned}$$

if the result is -1, it becomes:

$$\begin{aligned}|\phi^{-}_{t=1}\rangle &= \frac{1}{2}\left(\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes X \otimes X\right)|\phi_{t=0}\rangle \\ &= \alpha\left(\gamma\left(|000\rangle - |011\rangle\right) + \delta\left(|001\rangle - |010\rangle\right)\right) \\ &+ \beta\left(\gamma\left(|100\rangle - |111\rangle\right) + \delta\left(|101\rangle - |110\rangle\right)\right)\end{aligned}$$

In the case of the +1 Measurement $\rightarrow$ a=0:

$$\begin{aligned}|\phi^{++}_{t=2}\rangle &= \frac{1}{2}\left(\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} + Z \otimes Z \otimes \mathbb{I}\right)|\phi^{+}_{t=1}\rangle \\ &= (|000\rangle\langle 000| + |001\rangle\langle 001| + |110\rangle\langle 110| + |111\rangle\langle 111|)|\phi^{+}_{t=1}\rangle \\ &= \alpha\left(\gamma|000\rangle + \delta|001\rangle\right) + \beta\left(\gamma|111\rangle + \delta|110\rangle\right)\end{aligned}$$

$$\begin{aligned}|\phi^{+-}_{t=2}\rangle &= \frac{1}{2}\left(\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} - Z \otimes Z \otimes \mathbb{I}\right)|\phi^{+}_{t=1}\rangle \\ &= (|010\rangle\langle 010| + |011\rangle\langle 011| + |100\rangle\langle 100| + |101\rangle\langle 101|)|\phi^{+}_{t=1}\rangle \\ &= \alpha\left(\gamma|011\rangle + \delta|010\rangle\right) + \beta\left(\gamma|100\rangle + \delta|101\rangle\right)\end{aligned}$$

In the case of the -1 Measurement $\rightarrow$ a=1:

$$\begin{aligned}|\phi_{t=2}^{-+}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} + Z\otimes Z\otimes\mathbb{I}\right)|\phi_{t=1}^{-}\rangle\\ &= \alpha\left(\gamma|000\rangle + \delta|001\rangle\right) - \beta\left(\gamma|111\rangle + \delta|110\rangle\right)\end{aligned}$$

$$\begin{aligned}|\phi_{t=2}^{--}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} - Z\otimes Z\otimes\mathbb{I}\right)|\phi_{t=1}^{-}\rangle\\ &= -\alpha\left(\gamma|011\rangle + \delta|010\rangle\right) + \beta\left(\gamma|100\rangle + \delta|101\rangle\right)\end{aligned}$$

Now the applied measurement is $\mathbb{I}\otimes X\otimes\mathbb{I}$, which means:

$$\begin{aligned}|\phi_{t=3}^{+++}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} + \mathbb{I}\otimes X\otimes\mathbb{I}\right)|\phi_{t=2}^{++}\rangle\\ &= \frac{1}{2}((|010\rangle + |000\rangle)\langle 000| + (|011\rangle + |001\rangle)\langle 001|\\ &+ (|000\rangle + |010\rangle)\langle 010| + (|001\rangle + |011\rangle)\langle 011|\\ &+ (|110\rangle + |100\rangle)\langle 100| + (|111\rangle + |101\rangle)\langle 101|\\ &+ (|100\rangle + |110\rangle)\langle 110| + (|101\rangle + |111\rangle)\langle 111|)|\phi_{t=2}^{++}\rangle\\ &= \frac{1}{2}(\alpha\left(\gamma(|000\rangle + |010\rangle) + \delta(|011\rangle + |001\rangle)\right)\\ &+ \beta\left(\gamma(|101\rangle + |111\rangle) + \delta(|100\rangle + |110\rangle)\right))\end{aligned}$$

In this case, a,b and c would each be zero, therefore no further gate would be applied.
As intended, this state is equivalent to $CNOT_{|\psi_{Control}\rangle\rightarrow|\psi_{Target}\rangle}|\phi_{t=0}\rangle$.
If the last measurement result is -1:

$$\begin{aligned}|\phi_{t=3}^{++-}\rangle &= \frac{1}{2}\left(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} - \mathbb{I}\otimes X\otimes\mathbb{I}\right)|\phi_{t=2}^{++}\rangle\\ &= \frac{1}{2}((|010\rangle + |000\rangle)\langle 000| + (|001\rangle - |011\rangle)\langle 001|\\ &+ (|010\rangle - |000\rangle)\langle 010| + (|011\rangle - |001\rangle)\langle 011|\\ &+ (|100\rangle - |110\rangle)\langle 100| + (|101\rangle - |111\rangle)\langle 101|\\ &+ (|110\rangle - |100\rangle)\langle 110| + (|111\rangle - |101\rangle)\langle 111|)|\phi_{t=2}^{++}\rangle\\ &= \frac{1}{2}(\alpha\left(TODOTODOTODOTODOTODO\gamma(|000\rangle + |010\rangle) + \delta(|011\rangle + |001\rangle)\right)\\ &+ \beta\left(\gamma(|101\rangle + |111\rangle) + \delta(|100\rangle + |110\rangle)\right))\end{aligned}$$

Notably, each measurement sequence has a differing resulting ancilla state, however we do not care since ancillas are meant to be discarded.
For now, the other 7 final computation steps are left as an exercise to the reader, however I probably will still finish that.