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Department of Mathematics,
University College Cork

SUPERVISOR

Prof. Bernard Hanzon

STUDENT

Colm Clery
118307903

Estimation of Non-Stationary, Linear Time Series

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Background

Definition: ARMA(p,q) process

An Autoregressive-Moving Average (ARMA) process of autoregressive order p and moving average order q is defined as:

$$y_t = \emptyset_1 y_{t-1} + \emptyset_2 y_{t-2} + \cdots + \emptyset_p y_{t-p} + \mathrm{u_t} + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \cdots + \theta_q u_{t-q}$$

$$u_t \sim N(0, \sigma^2)\ i.i.d, \qquad t \in Z, t \leq T$$

This can be equivalently expressed in terms of $\emptyset(L), \theta(L)$ the characteristic polynomial of the AR and MA component respectively.

$$\emptyset(L) y_t = \theta(L)\, u_t$$

$$\emptyset(L) = 1 - \emptyset_1 L - \emptyset_2 L^2 - \cdots - \emptyset_p L^p$$

$$\theta(L) = 1 + \theta_1 L + \theta_2 \mathrm{L}^2 + \cdots + \theta_q L^q$$

Definition: Stationary process

The time series is said to be stationary if its first and second moments are time invariant and the autocovariance function depends only on the difference in time h and not t. Formally, a stochastic process y_t is stationary if

$$E[y_t] = \mu \text{ and } E[(y_t - \mu)(y_{t-h} - \mu)] = \gamma_h = \gamma_{-h} \ \forall \ t, h \in Z$$

A non-stationary process violates one or both of these conditions.

An stable ARMA process has all roots of the AR characteristic polynomial $\emptyset(L)$ **outside** the complex unit circle.

An invertible ARMA process has at all roots of the MA characteristic polynomial $\theta(L)$ **outside** the complex unit circle.

Spectral Factorisation of Moving Average processes

For each non-invertible MA(q) process $\theta(L)u_t$, there exists a unique invertible MA(q) process $\theta^s(L)\, u_t^s$ such that the autocovariance functions of each process are equal and the processes are equivalent under this condition.

In the MA(1) case, it can be shown that for the non-invertible process $\theta(L)u_t$ the equivalent invertible process is given by

$$\mathrm{y_t} = \frac{1}{\theta} u_{t-1}^s + u_t^s \qquad u_t^s \sim N(0, \theta^2\sigma^2\,)\; i.i.d$$

Equivalence of ARMA(p,q) and AR(∞)

AR(∞) of an ARMA(1,1)

Consider an ARMA(1,1) process and isolate u_t

$$y_t = \phi y_{t-1} + \theta u_{t-1} + u_t \qquad (1)$$

$$\Leftrightarrow \; u_t = y_t - \phi y_{t-1} - \theta u_{t-1}$$

Applying the backshift operator to both sides we get

$$u_{t-1} = y_{t-1} - \phi y_{t-2} - \theta u_{t-2}$$

So, by substituting for u_{t-1} into equation (1) and similarly for u_{t-2}, u_{t-3}... we derive

$$y_t = \phi y_{t-1} + \theta\left(y_{t-1} - \phi y_{t-2} - \theta\left(y_{t-2} - \phi y_{t-3} - \theta(\ldots)\right)\right) + \mathrm{u_t}$$

$$= u_t + \sum_{i=1}^{\infty}(\phi + \theta)\theta^{i-1} y_{t-i}$$

Which is an AR(∞).

Conditions of AR(∞)

1. Are limited to finite data sets, and so we must truncate the infinite series at some $p^* \in N$ to form an AR(p*) process which can be fit to the time series.
2. We are required to condition the likelihood function on the initial p* observations.

AR(p*) approximation of an ARMA(1,1)

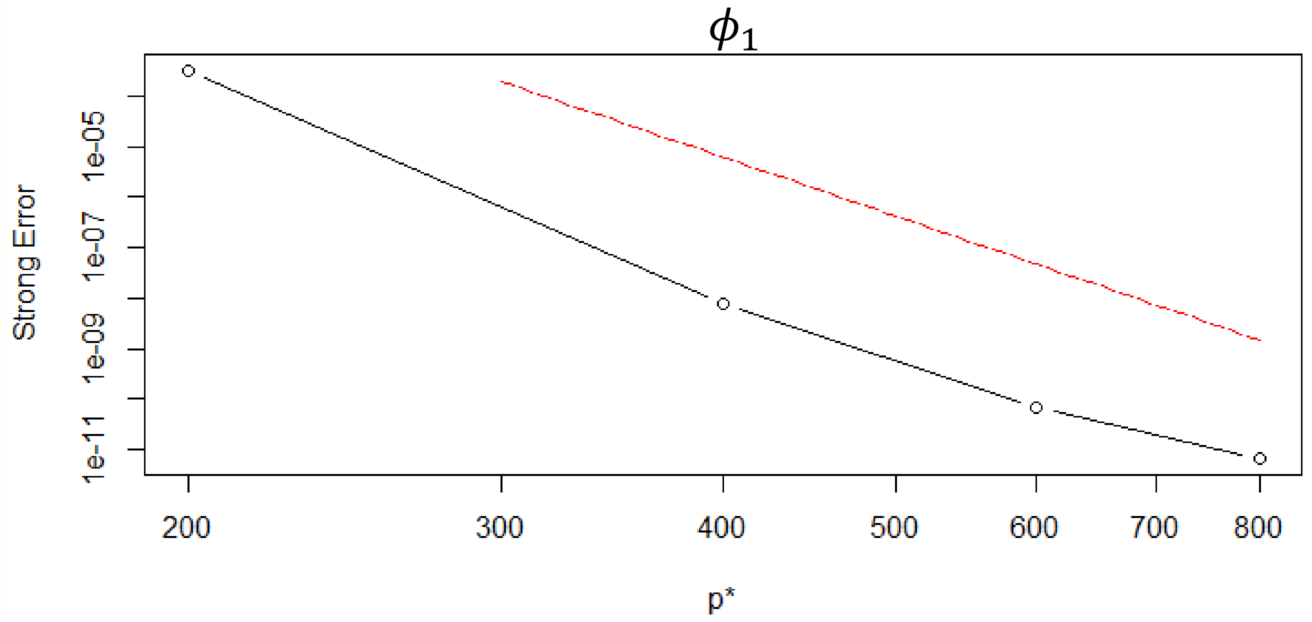
Given a set of observations ${\{y_t\}^T}_{t=1}$ and an ARMA(1, 1) process with gaussian error process $u_t \sim N(0, \sigma^2)\ i.i.d$, the resulting parameter estimates $\hat{\phi}, \hat{\theta}, \widehat{\sigma^2}$ are those which maximise the conditional maximum likelihood of the AR(p*) process.

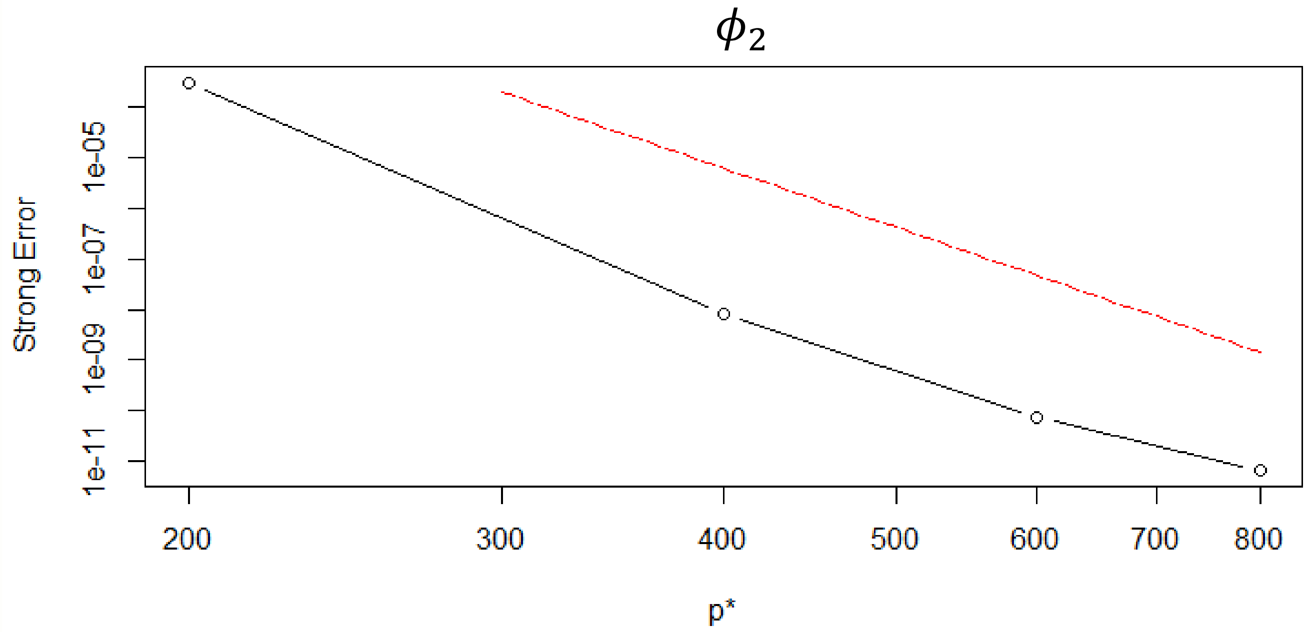
$$y_t = u_t + \sum_{i=1}^{p^*} (\phi + \theta)\theta^{i-1} y_{t-i}$$

Convergence of AR(p*) coefficient estimates

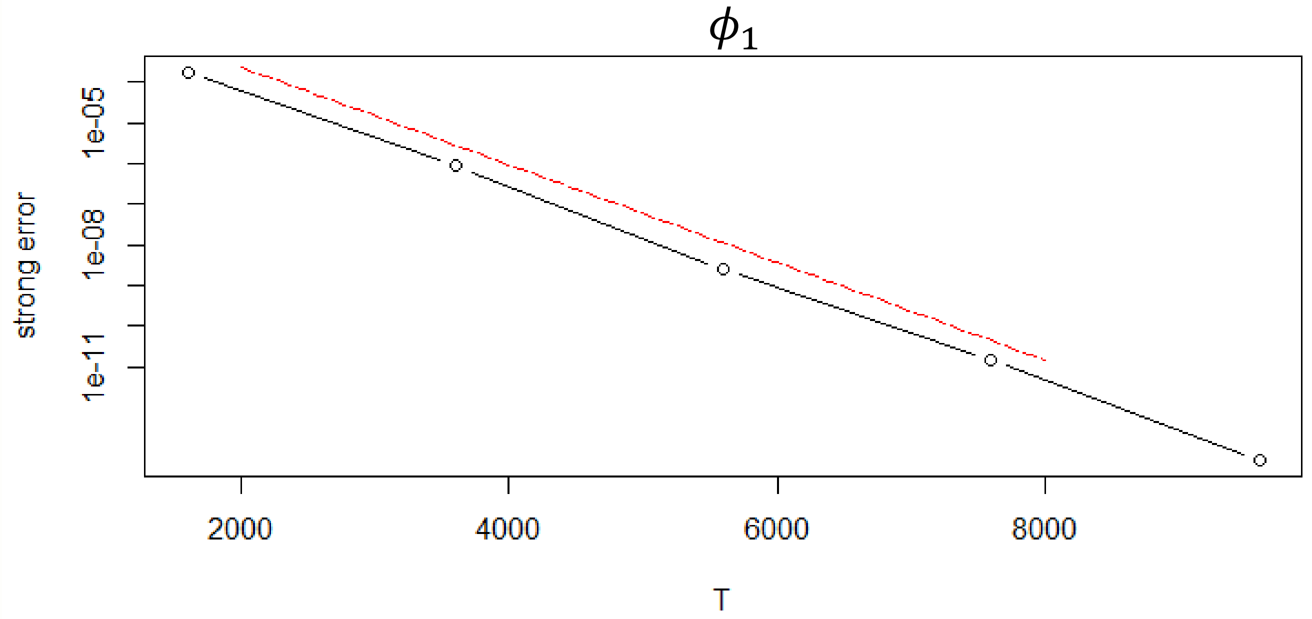
Convergence of autoregressive coefficient estimates

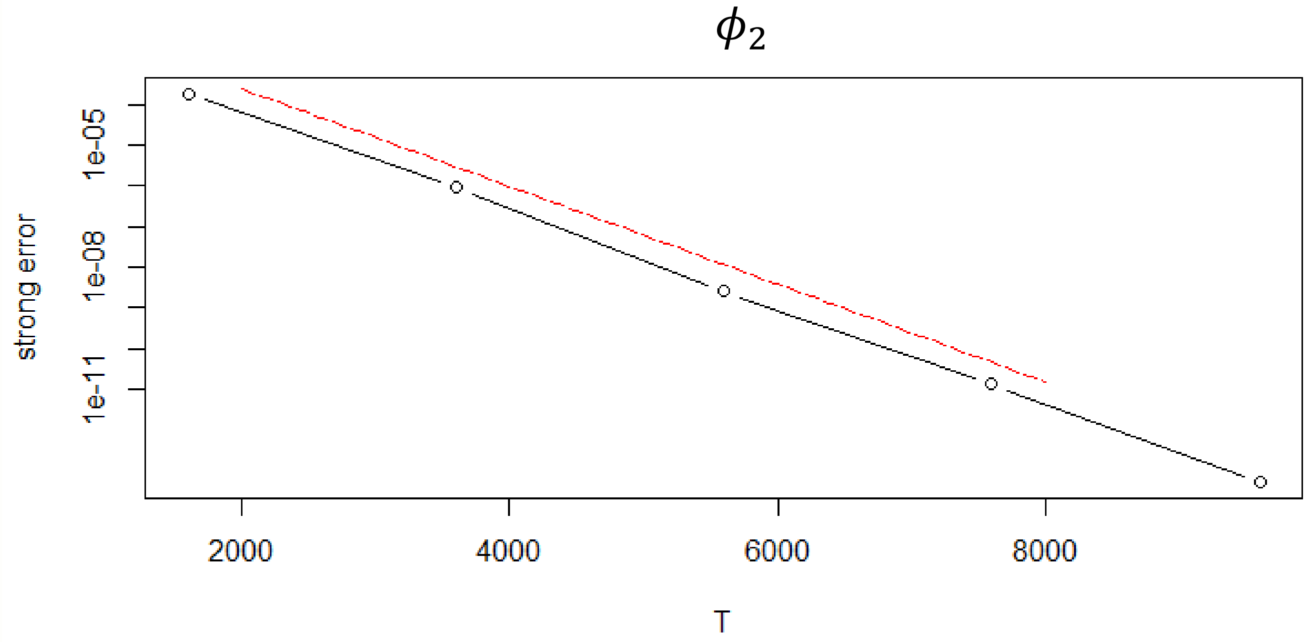
Strong error with respect to p* - purely explosive ARMA(2,1)





Strong error with respect to T - - purely explosive ARMA(2,1)





Convergence of autoregressive coefficient estimates

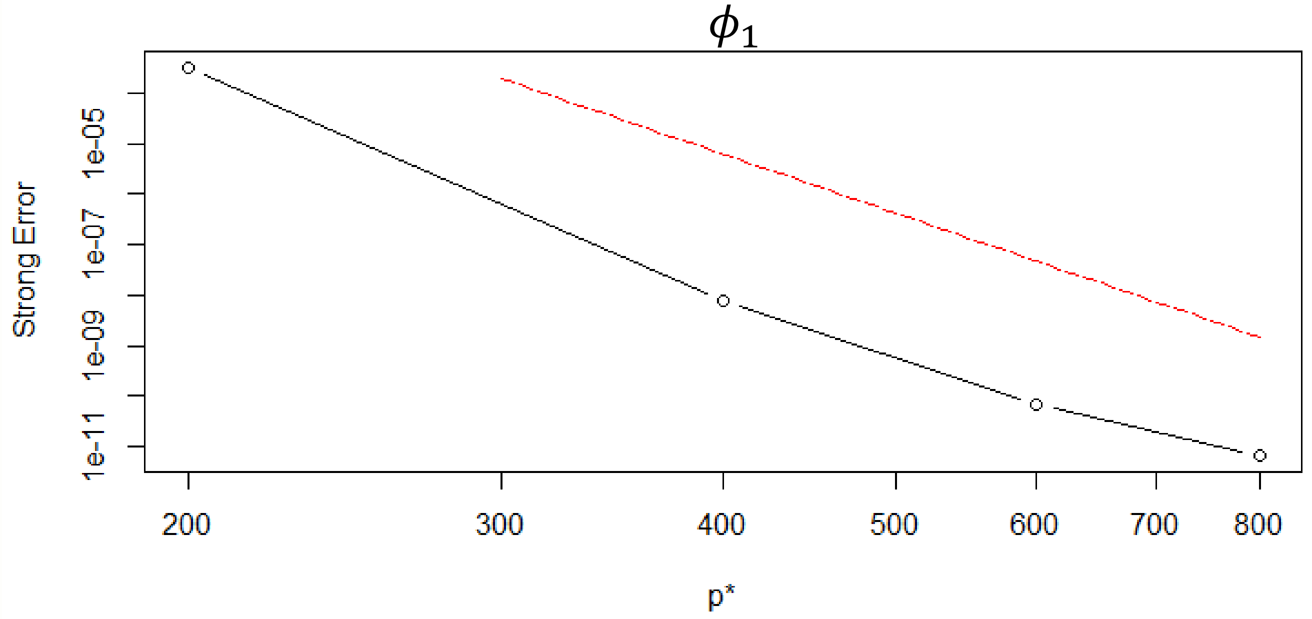
Hypothesis 1.

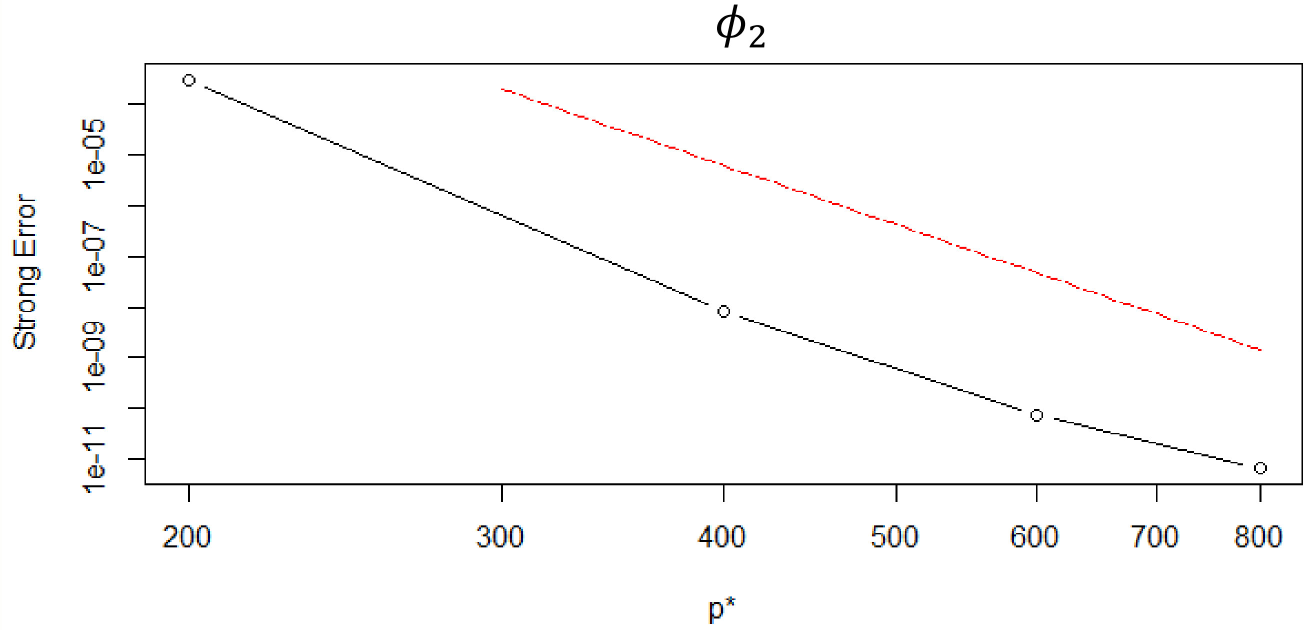
The autoregressive coefficient estimates $\hat{\phi}(L)$ of an non-stationary ARMA(2,q) process with purely explosive AR component obtained under the AR(p*) method converge strongly as $\mathrm{p}^*, T \rightarrow \infty$, such that

$$\lim_{p^*, T\rightarrow\infty} \hat{\phi}(L) \rightarrow \phi(L) \text{ a.s}$$

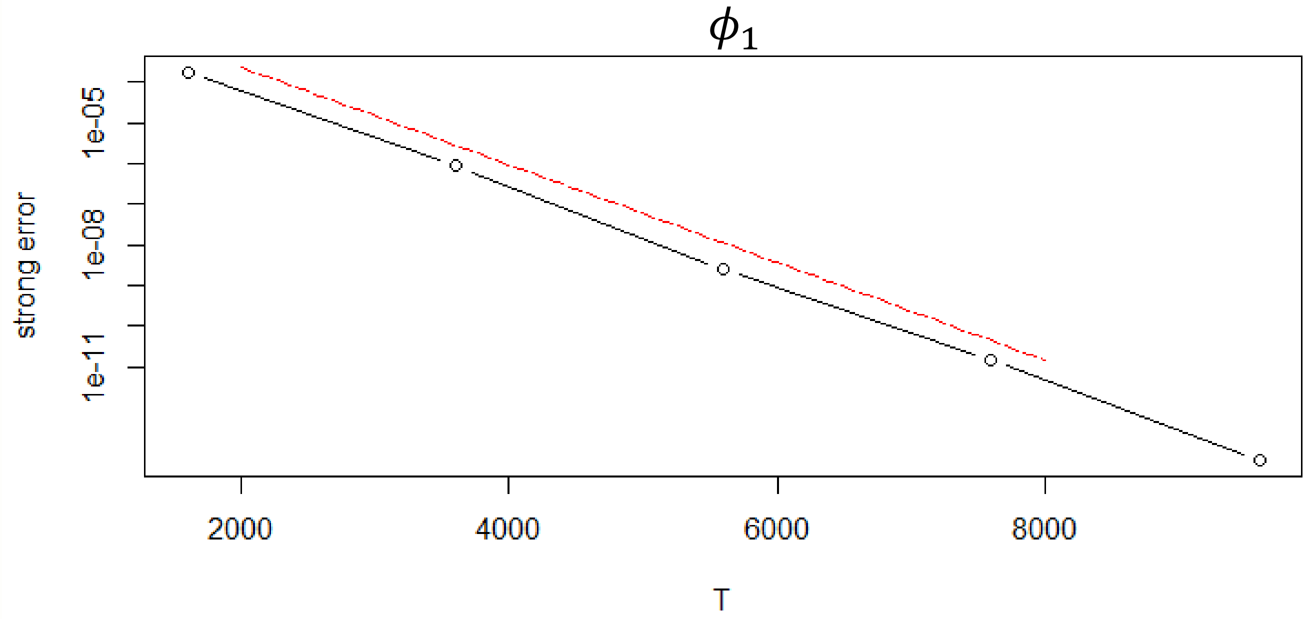
Convergence of autoregressive coefficient estimates

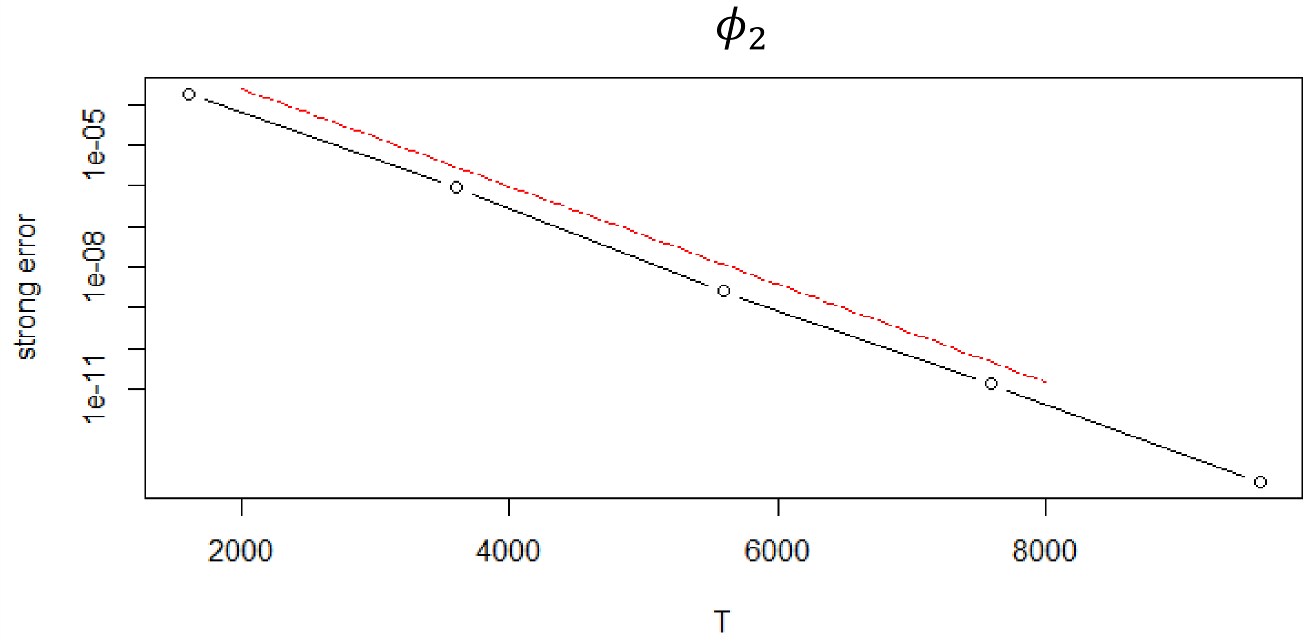
Strong error with respect to p* - ARMA(2,1)





Strong error with respect to T - ARMA(2,1)





Convergence of autoregressive coefficient estimates

Hypothesis 2.
For T sufficiently large, the strong order of convergence of the AR(p*) autoregressive coefficient estimates of an ARMA(2, q) with purely explosive AR component under the AR(p*) method with respect to p*

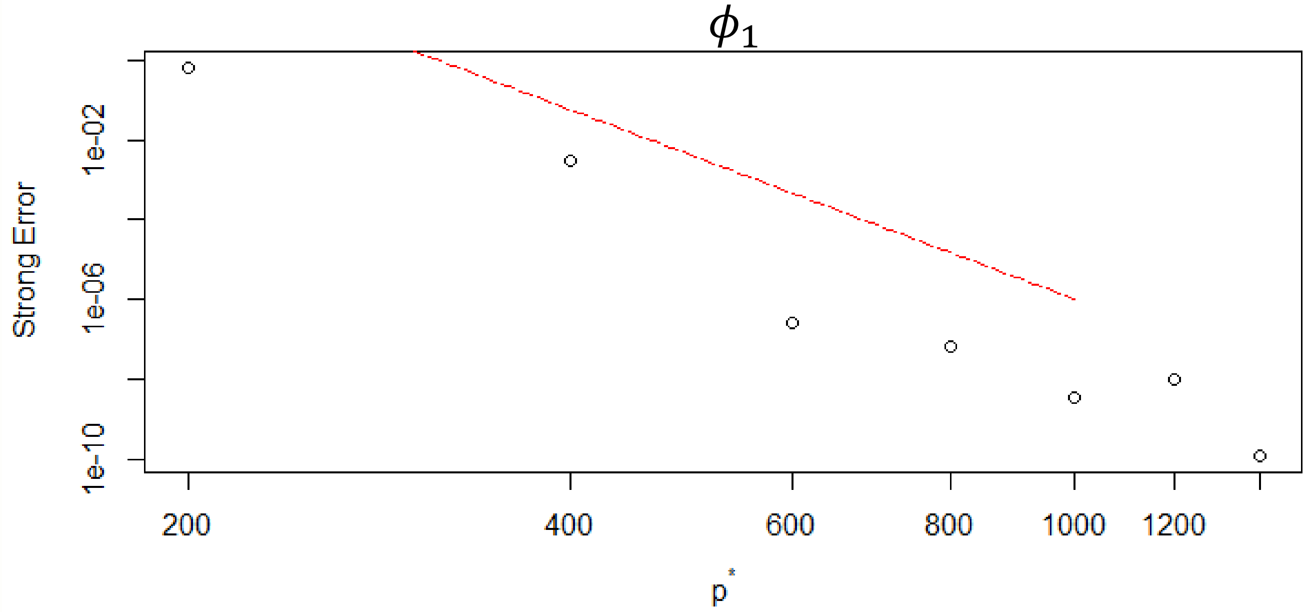
$$O(p^{-12})$$

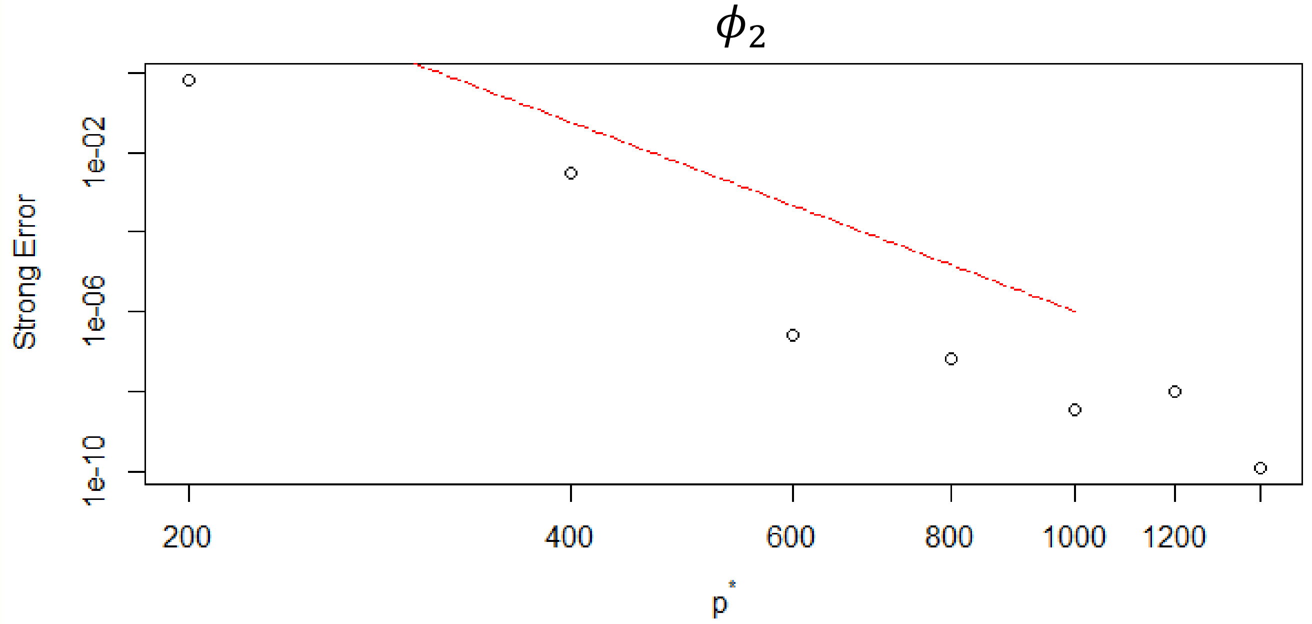
Hypothesis 3.
For p* sufficiently large, the strong order of convergence of the AR(p*) autoregressive coefficient estimates of an ARMA(2, q) process with purely explosive AR component under the AR(p*) method with respect to T is

$$O(\exp(-12T))$$

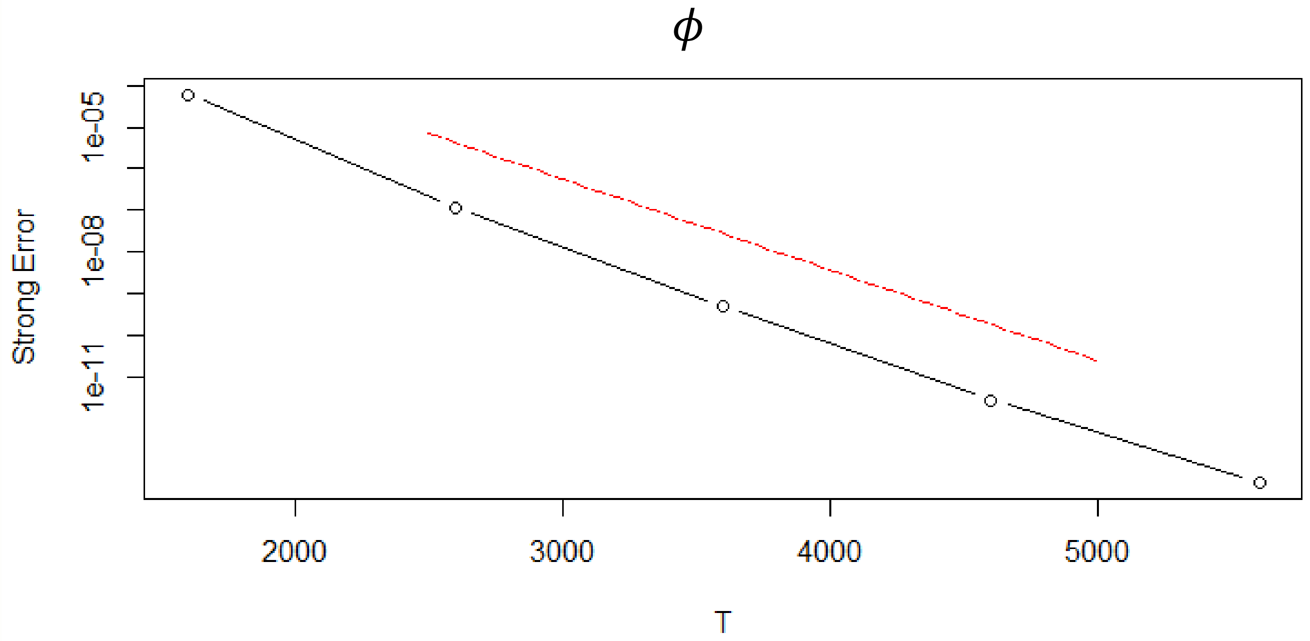
Convergence of autoregressive coefficient estimates

Strong error with respect to p* - ARMA(2,1) mixed root AR component



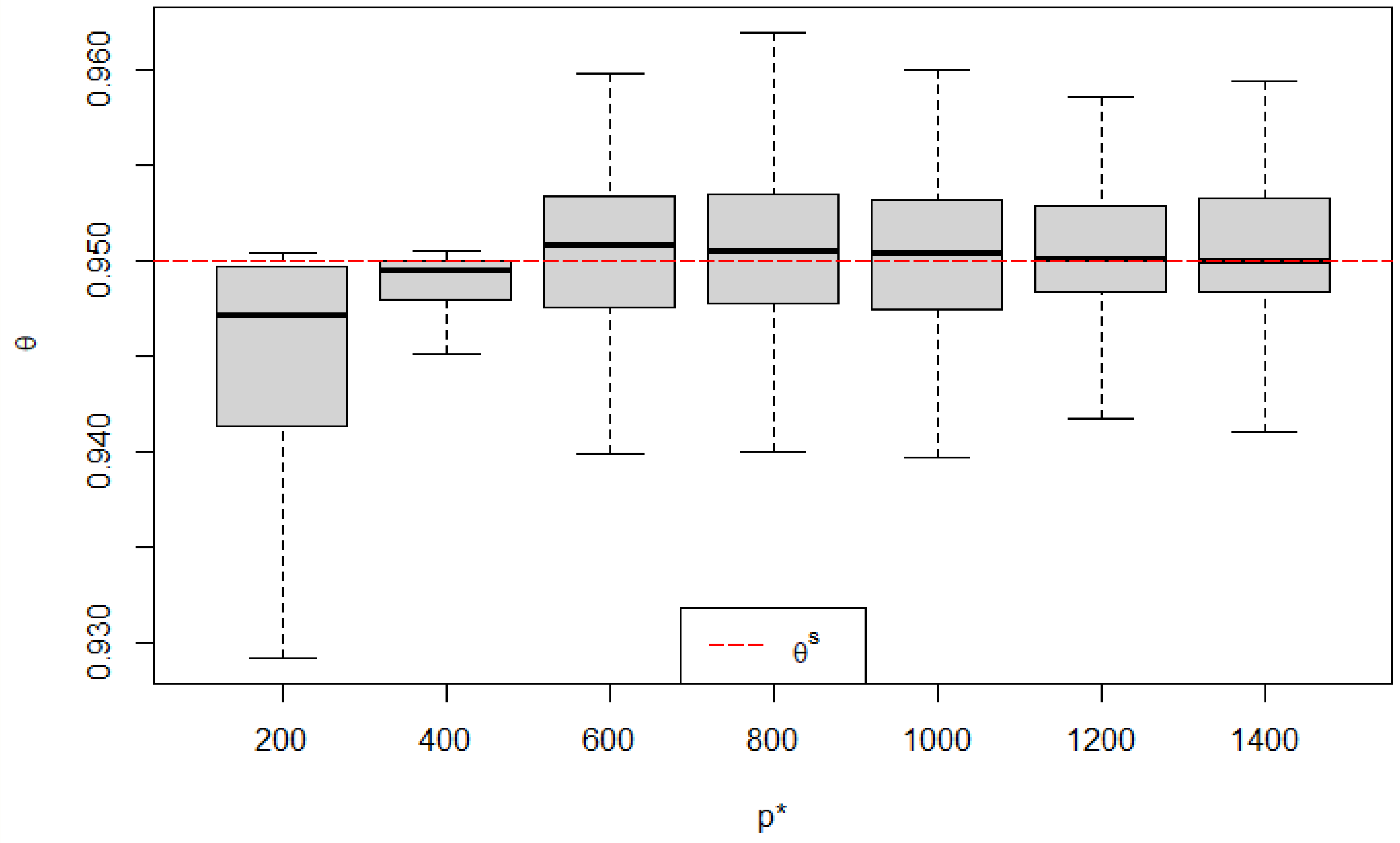


Strong error with respect to T - ARMA(1,1)



Convergence of Moving Average coefficient estimates

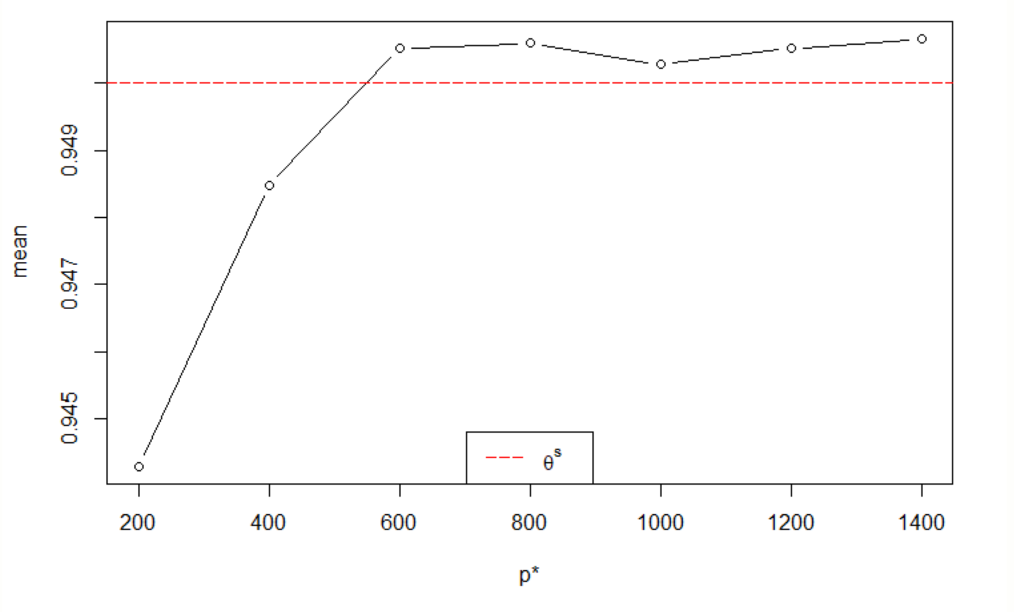
Distribution of MA coefficient estimates – ARMA(2,1)



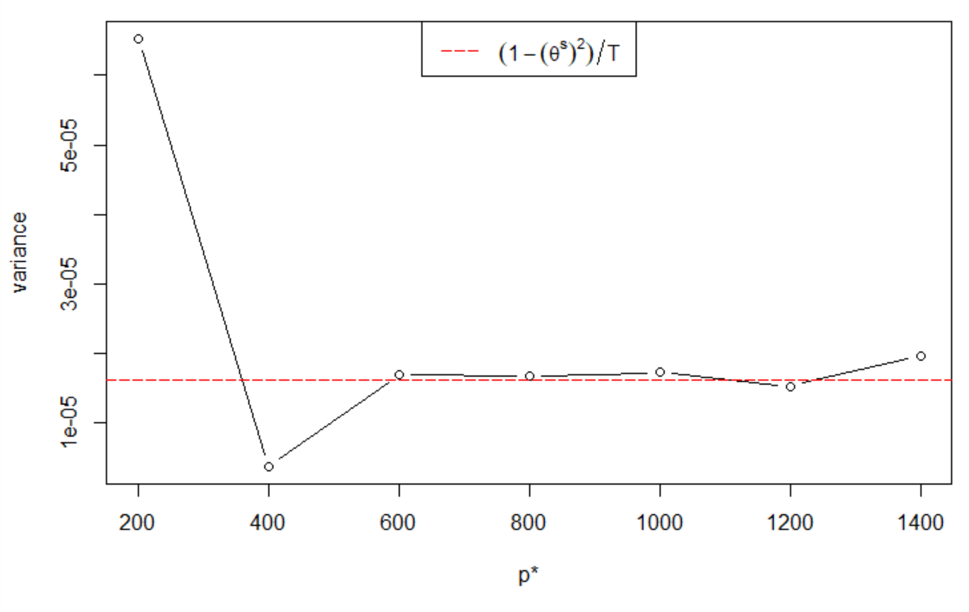
Convergence of Moving Average coefficient estimates

Moments of MA coefficient estimate Distributions – ARMA(2,1)

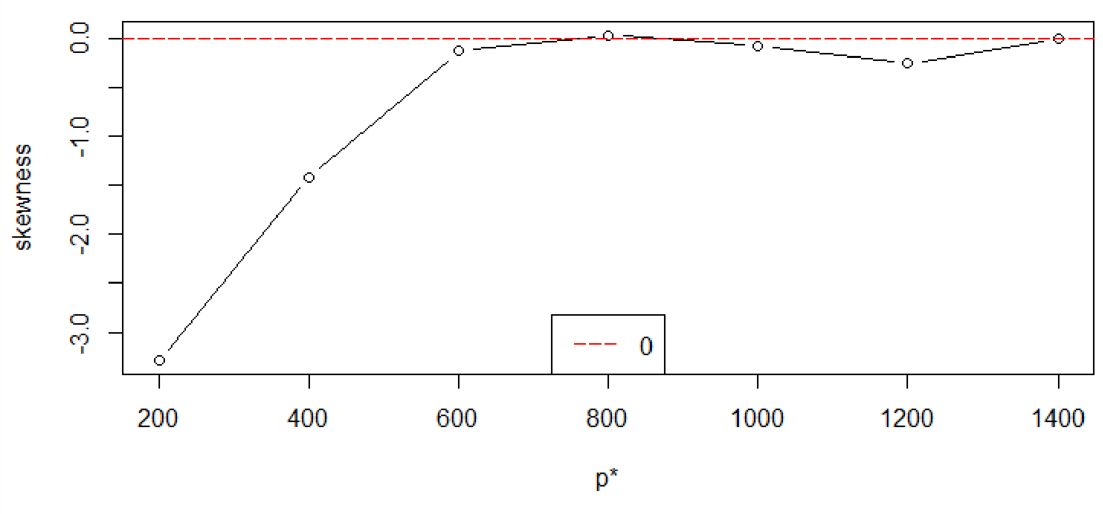
Mean



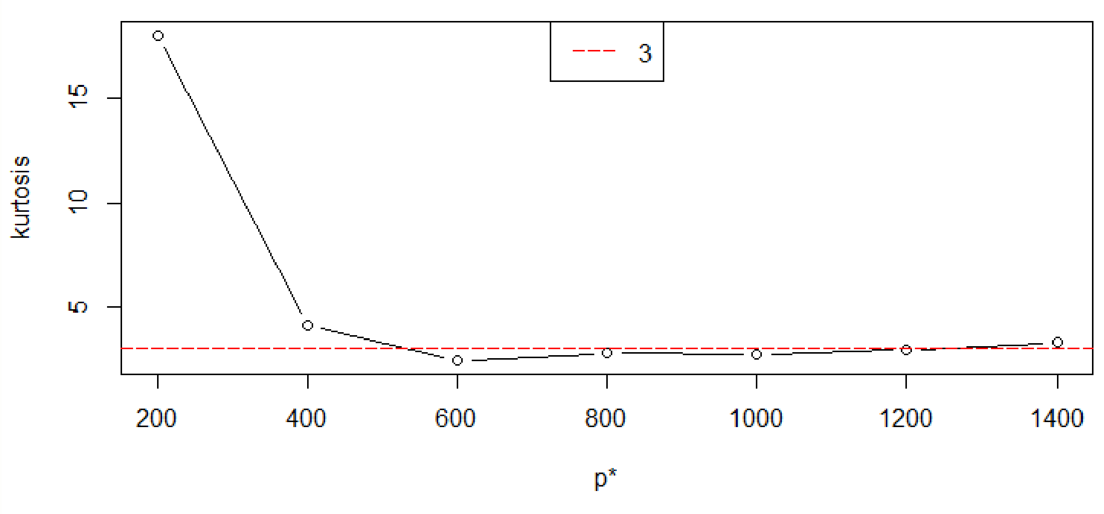
Variance



Skewness



Kurtosis



Convergence of Moving Average coefficient estimates

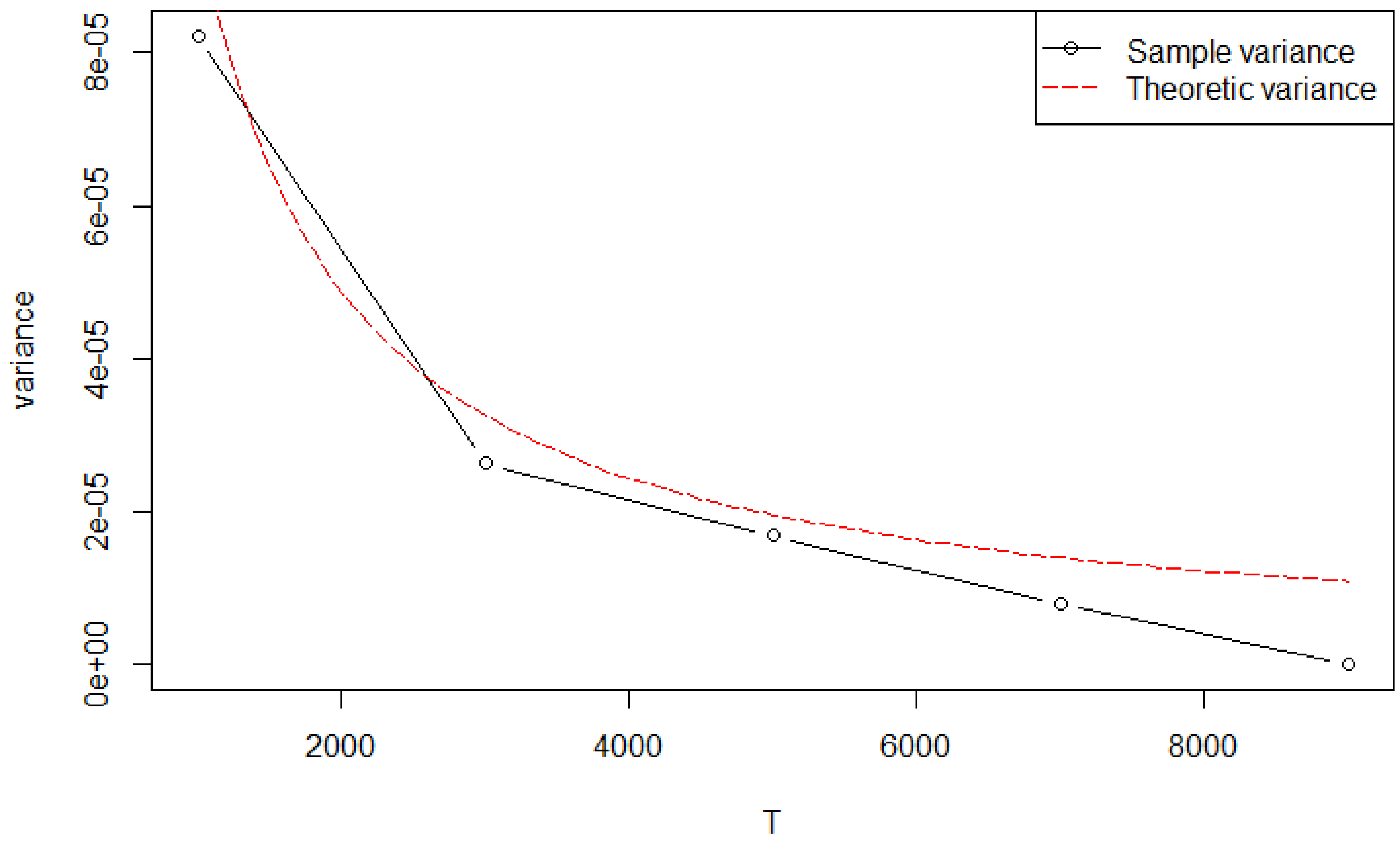
Hypothesis 4.

The moving average coefficient estimates $\hat{\theta}$ of an non-stationary ARMA(p,1) process obtained under the AR(p*) method converge in distribution as p* $\rightarrow \infty$ such that

$$\lim_{p^* \rightarrow \infty} \hat{\theta} \xrightarrow{d} N\left(\theta^s, \frac{1 - \theta^{s2}}{T - p^*}\right)$$

Convergence of Moving Average coefficient estimates

Variance of MA coefficient estimates – ARMA(2,1)



Convergence of Moving Average coefficient estimates

Hypothesis 5.
The moving average coefficient estimates $\hat{\theta}$ of a non-stationary ARMA(p,1) process obtained under the AR(p*) method converge in probability as T, p* $\rightarrow \infty$ such that

$$\lim_{p^*,T\to\infty} \hat{\theta} \xrightarrow{p} \theta^s$$

Using parameter estimates to create stationary time series

Purely Explosive ARMA process

Given a non-stationary process $\emptyset(L)y_t = \theta(L)\, u_t$ and set of observations ${\{y_t\}^T}_{t=1}$, define $z_t = \emptyset(L)y_t$ such that $z_t = \theta(L)u_t$ is a stationary process.

The filtering procedure is as follows:

Step 1:

Apply the AR(p*) method to the series ${\{y_t\}^T}_{t=1}$, and store estimates $\{\widehat{\phi_1}, \dots \widehat{\phi_p}\}$

Step 2:

Generate series $\{\widehat{z_t}\}_{t=1}^{T-p}$ where $\widehat{z_t} = y_{t+p+1} - \widehat{\phi_1}y_{t+p} - \dots \widehat{\phi_p}y_t$

Step 3:

Estimate the series $\{\widehat{z_t}\}_{t=1}^{T-p}$ as a stationary MA(q) process to obtain the moving average coefficient estimates $\{\widehat{\theta_1}, \dots \widehat{\theta_q}\}$.

The final estimated model is:

$$y_t = \widehat{\phi_1}y_{t-1} - \dots \widehat{\phi_p}y_{t-p} + u_t + \widehat{\theta_1}\, u_{t-1} + \widehat{\theta_q}\, u_{t-q}$$

.

Mixed Root ARMA process

Given a non-stationary process $\phi(L)y_t = \theta(L)\,u_t$ with mixed root AR component and set of observations $\{y_t\}^T{}_{t=1}$, define $\phi^s(L), \phi^u(L)$ such that

$$\phi^s(L) = \prod_{i=1}^{p^s}\left(\frac{1}{r_i^s}L - 1\right) \;;\; r_i^s > 1 \text{ for each i}$$

$$\phi^u(L) = \prod_{j=1}^{p^u}\left(\frac{1}{r_j^u}L - 1\right) \;;\; r_i^j \leq 1 \text{ for each j}$$

$$p^u + p^s = p$$

$$\phi(L) = \phi^s(L)\phi^u(L)$$

From here, define $z_t = \phi^u(L)y_t$ such that $\phi^s(L)z_t = \theta(L)u_t$ is a stationary process.

Mixed Root ARMA process

The filtering procedure for mixed root AR component is as follows:

Step 1:

Apply the AR(p*) method to the series $\{y_t\}^T{}_{t=1}$, and store estimates $\{\widehat{\phi_1}, \ldots \widehat{\phi_p}\}$

Step 2:

Calculate roots $\widehat{r_i^u}, \widehat{r_j^s}$ of the characteristic polynomial of coefficient estimates.

Step 3:

Generate series $\{\widehat{z_t}\}_{t=1}^{T-p^u}$ where $\widehat{z_t} = \widehat{\phi^u(L)} y_t$

Step 4:

Estimate the series $\{\widehat{z_t}\}_{t=1}^{T-p^u}$ as a stationary ARMA(p^s, q) process to obtain the remaining coefficient estimates $\{\widehat{\phi^s(L)}, \widehat{\theta_1}, \ldots \widehat{\theta_q}\}$.

The final estimated model is hence

$$\widehat{\phi^s(L)}\widehat{\phi^u(L)} y_t = + u_t + \widehat{\theta_1}\, u_{t-1} + \widehat{\theta_q}\, u_{t-q}$$

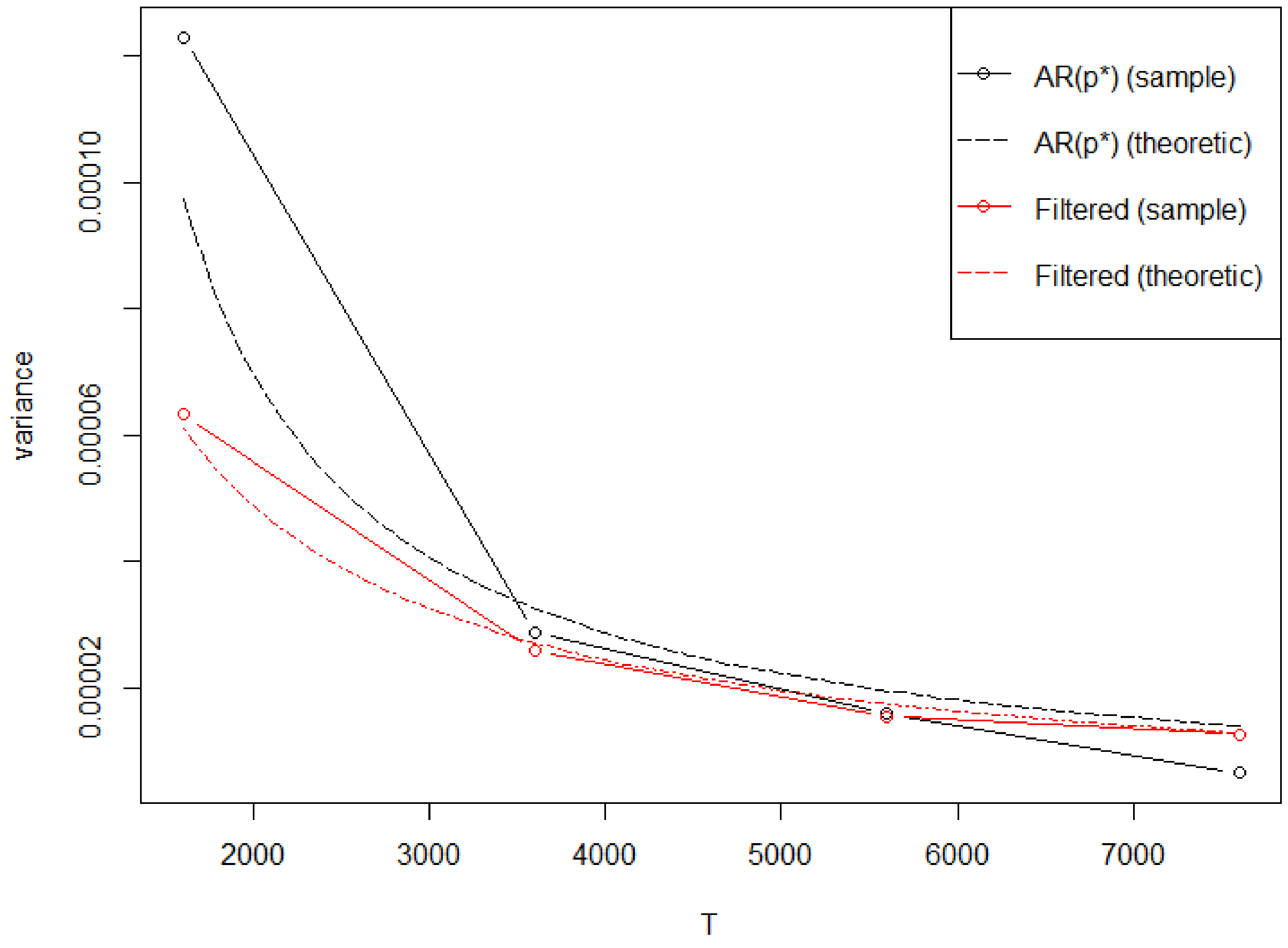
On AR(p*) filtering methods

- Common stationary time series methods can be applied to stationary series generated from filtering methods.

- Initial tests suggests that “Filtering method for purely explosive processes” results is lowest variance of coefficient estimates in purely explosive and mixed root cases!

On AR(p*) filtering methods

- Filtering increases fitting points available to stationary series than AR(p*), reducing standard errors particularly when T is small.



Variance of Moving Average coefficient estimates of ARMA(2,1) process with purely explosive AR component (p* = 600)

Computational Errors and inaccuracies

Non-stationary data and computational accuracy

Observations from non-stationary processes are explosive in nature. Such processes return observations with large absolute values and which differ largely from initial observations. Numerical overflow and round off error may occur as a result.

Optimization errors

The conditional negative log likelihood of a non-stationary ARMA process is known to have local minima apart from the global minimum. This remains the case in the AR(p*) estimation approximation method.

BFGS (Broyden, Fletcher, Goldfarb and Shanno) method was utilised throughout this project. Unusual values were investigated via multidimensional grid search.

Conclusions & Further Research

Theoretical Findings

This project is motivated by the work on Hanzon & Scherrer (2019) and serves to forming conjectures that may build upon their work. The results of this project are being considered with conjectures and further research being considered.

Applications to forecasting

The AR(p^*) and resulting filtering methods may prove useful in forecasting time series. Optimization is known to converge to local minima. In forecasting, more rigorous optimization methods may be necessary.

Generalization to higher order ARMA processes

Investigate whether higher order processes and SARMA models display the same behavior under the AR($p*$) method and if not, in what ways do they differ.
Further progression of this would be to vector ARMA processes and linear scalar ARMA processes with non-Gaussian errors.

Thursday, 19th May 2022

Department of Mathematics,
University College Cork

SUPERVISOR

Prof. Bernard
Hanzon

STUDENT

Colm Clery
118307903

Thank you for listening!

Questions are welcome!

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