Mathematical Foundations of IV Surface Modeling and Optimal Stopping + Next Steps

1. Black-Scholes Option Pricing Formula

The price of a European put option under the Black-Scholes model is:

$$P(S, K, T, r, \sigma) = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

2. Implied Volatility Inversion

Implied volatility σ_{impl} is defined as the unique volatility such that:

$$P_{\text{market}} = P_{\text{BS}}(S, K, T, r, \sigma_{\text{impl}})$$

This is typically solved numerically via root-finding methods like bisection or Newton-Raphson.

3. Log-Moneyness

We define log-moneyness as:

$$x = \log\left(\frac{K}{F}\right) = \log\left(\frac{K}{Se^{rT}}\right)$$

or in forward-adjusted cases, simply:

$$x = \log\left(\frac{K}{F}\right)$$

4. Longstaff-Schwartz Algorithm (LSMC)

Given simulated paths $\{S_t^m\}_{m=1}^M$, we proceed backward in time:

- At time t, consider only in-the-money paths.
- Regress future discounted payoffs Y_{t+1} onto basis functions $\phi_j(S_t)$:

$$\mathbb{E}[Y_{t+1} \mid S_t] \approx \sum_{j=0}^{d} \beta_j \phi_j(S_t)$$

- Exercise if:

$$\text{Payoff}(S_t) > \sum_{j=0}^{d} \hat{\beta}_j \phi_j(S_t)$$

5. SABR Model for Implied Volatility

The SABR approximation from Hagan et al. is:

$$\sigma_{\mathrm{BS}}(K) = \frac{\alpha}{(FK)^{(1-\beta)/2}} \cdot \frac{z}{x(z)} \cdot \left[1 + \left(\frac{(1-\beta)^2\alpha^2}{24(FK)^{1-\beta}} + \frac{\rho\beta\nu\alpha}{4(FK)^{(1-\beta)/2}} + \frac{\nu^2(2-3\rho^2)}{24}\right)T\right]$$

where:

$$z = \frac{\nu}{\alpha}(FK)^{(1-\beta)/2}\log(F/K), \quad x(z) = \log\left(\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right)$$

6. Bayesian Update of IV Surface

Combining a model fit $\sigma_{\mathrm{fit}}(x)$ and observed market IV $\sigma_{\mathrm{obs}}(x)$, the Bayesian posterior is:

$$\sigma_{\mathrm{Bayes}}(x) = \frac{\sigma_{\mathrm{fit}}(x)/\tau^2 + \sigma_{\mathrm{obs}}(x)/\sigma^2}{1/\tau^2 + 1/\sigma^2}$$

where τ^2 is prior variance and σ^2 is observation noise.

7. Neural Network Approximation of IV or Policy Surface

Let the input features be $x = (\log(K/F), T)$. The network learns:

$$\hat{y}(x) = f_\theta(x) \approx \begin{cases} \mathbb{P}(\text{exercise at } x), & \text{for optimal stopping} \\ \sigma_{\mathrm{IV}}(x), & \text{for volatility surface} \end{cases}$$

Trained using:

$$\mathcal{L}_{\mathrm{MSE}} = \frac{1}{N}\sum_{i=1}^{N}(\hat{y}(x_i) - y_i)^2 \quad \text{or} \quad \mathcal{L}_{\mathrm{BCE}} = -\frac{1}{N}\sum_{i=1}^{N}[y_i \log \hat{y}(x_i) + (1-y_i)\log(1-\hat{y}(x_i))]$$

Connecting Optimal Exercise Policy to Implied Volatility

The Longstaff-Schwartz Monte Carlo (LSMC) method approximates the **optimal stopping rule** for American options, identifying when the continuation value falls below immediate exercise value.

1. LSMC-Implied Pricing Surface

Given simulated paths $\{S_t^m\}_{m=1}^M$ and an optimal policy $\pi(S_t, t) \in \{0, 1\}$, define the discounted payoff:

$$P_{\mathrm{LSMC}}(K,T) = \mathbb{E}\left[e^{-r\tau}(K - S_\tau)^+\right] \quad \text{where } \tau = \inf\{t \mid \pi(S_t, t) = 1\}$$

This gives us **model-implied prices** $P_{\mathrm{LSMC}}(K,T)$ across a grid of strikes K and maturities T, effectively producing a **synthetic price surface**.

2. Extracting Implied Volatility Surface from LSMC

Using the synthetic price surface, we invert the Black-Scholes formula to recover **implied volatility**:

$$\sigma_{\mathrm{LSMC}}(K,T) \quad \text{such that} \quad P_{\mathrm{BS}}(S, K, T, r, \sigma_{\mathrm{LSMC}}) = P_{\mathrm{LSMC}}(K,T)$$

This is done numerically using a root-finding algorithm (e.g., Brent, bisection, or Newton-Raphson). The result is:

$$\sigma_{\mathrm{LSMC}}(K,T) = \text{IV surface implied by optimal stopping behavior}$$

3. Smoothing and Modeling the IV Surface

To remove simulation noise and prepare for further modeling or arbitrage analysis, we fit a surface $\hat{\sigma}(x, T)$ where $x = \log(K/F)$. This can be done via:

- Neural networks: $\hat{\sigma}(x, T) = f_\theta(x, T)$
- Parametric models: SABR, SVI, or polynomial regression
- Bayesian smoothing with a prior surface

4. Summary

$$\text{LSMC} \Rightarrow P_{\text{LSMC}}(K, T) \Rightarrow \sigma_{\text{LSMC}}(K, T) \Rightarrow \text{Smoothed/Model IV Surface}$$

This allows us to **build realistic IV surfaces** consistent with exercise behavior under optimal stopping, bridging simulation-based pricing with volatility modeling.

LSMC-Derived Optimal Stopping Region (Log-Moneyness Domain)

We simulate M paths of the underlying asset $S_t^{(m)}$ using a discretized process (e.g., GBM). For each path, we evaluate the payoff at each timestep and regress continuation value against basis functions.

1. Simulation of Underlying Paths:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t, \quad S_0 = S$$

2. Discrete Time Grid: $t_0 = 0 < t_1 < \cdots < t_N = T$

3. Discounted Cashflows at Each Step: Let payoff be $h(S_t) = (K - S_t)^+$. We estimate the continuation value via regression:

$$C_t(S_t) \approx \mathbb{E}[e^{-r\Delta t} V_{t+1} \mid S_t]$$

4. Optimal Stopping Rule:

$$\tau^{(m)} = \inf \left\{ t_n \,:\, h(S_{t_n}^{(m)}) \geq C_{t_n}(S_{t_n}^{(m)}) \right\}$$

5. Exercise Heatmap over Log-Moneyness:
For each stopping time $\tau^{(m)}$, define the point $(\log(S_{\tau^{(m)}}/K),\, T - \tau^{(m)})$. Aggregate over all paths into a 2D histogram.

$$x = \log\left(\frac{S_{\tau^{(m)}}}{K}\right), \quad y = T - \tau^{(m)}$$

This yields an empirical estimate of the **exercise frequency density** in the $(\log(S/K),\, T)$ space.

Next Steps

Creating a order book matching system to host adversarial trades between competing strategies with additional random noise to simulate retail traders. The strategies will populate the order book which will then be projected on the log-moneyness x IV x TTE space to analyze the apparent geometric topologies evolution throughout simulated trading to gain insight into adversarial dynamics in HFT.