Quantum Phase Estimation

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IBDA4221 – Selected Topic in Computer Technology

Quantum Computing

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Capaian Pembelajaran

- Quantum Fourier Transform
- Quantum Phase Estimation
- Beberapa aplikasi QPE

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Quantum Fourier Transform

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Fourier Transform

- Fourier transform adalah suatu metode untuk mengubah basis awal menuju basis komplementer (misalnya dari waktu→frekuensi, posisi→momentum)
- Banyak digunakan untuk signal processing (misalnya noise canceling)
- Sangat berguna untuk algoritma kuantum (misalnya Shor dan QPE), karena setiap qubit memiliki phase dan frekuensi
- Persamaan:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x}\, dx, \quad \forall\, \xi \in \mathbb{R}.$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{i2\pi\xi x}\, d\xi, \quad \forall\, x \in \mathbb{R},$$

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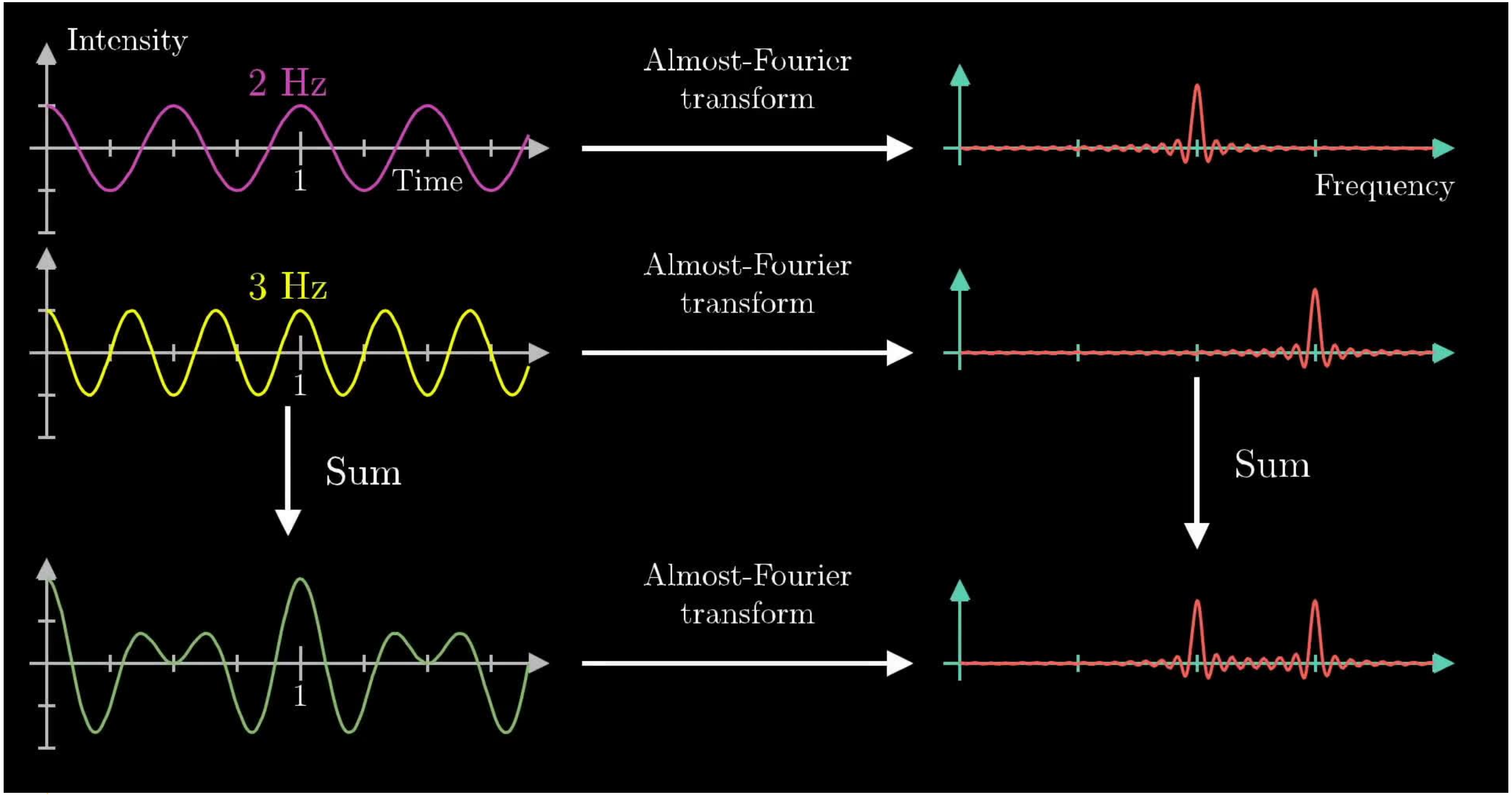
Fourier Transform



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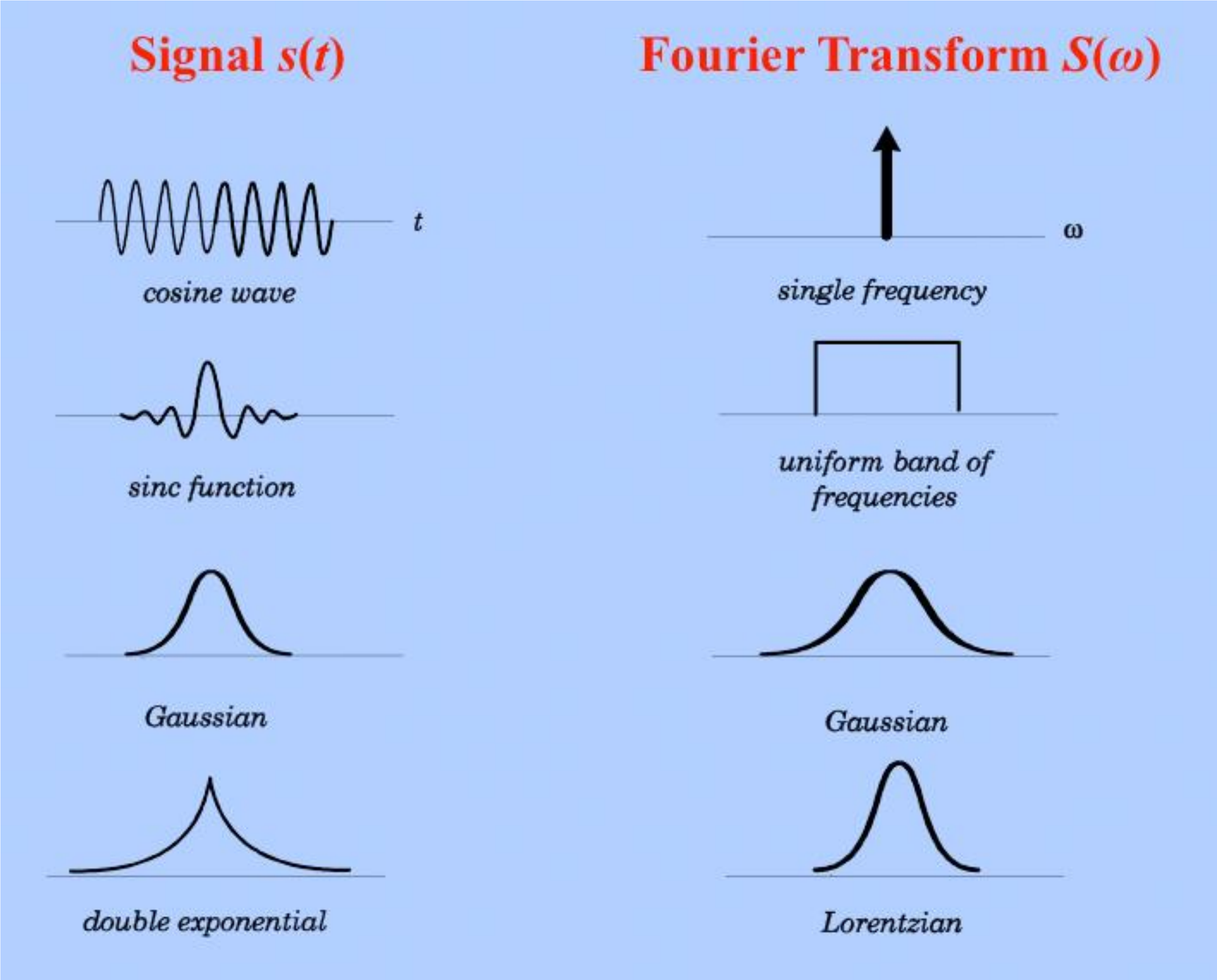
Fourier Transform

- Time space → Frequency space



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Quantum Fourier Transform

- Transformasi Fourier Diskrit memetakan vector $(x_0, \ldots, x_{N-1})$ → vector $(y_0, \ldots, y_{N-1})$:

$$y_k = \frac{1}{\sqrt{N}} \sum_{j}^{N-1} x_j w_N^{jk}\,, dimana\ w_N^{jk} = e^{2\pi i \frac{jk}{N}}$$

- Transformasi Fourier Kuantum memetakan state $|X\rangle = \sum_{j=0}^{N-1} x_j |j\rangle \rightarrow |Y\rangle = \sum_{k=0}^{N-1} y_k |k\rangle$

$$|j\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} w_N^{jk} |k\rangle$$

$$U_{QFT} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} w_N^{jk} |k\rangle\langle j|$$

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Contoh 1-qubit QFT

- Untuk qubit apapun $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:

$$y_0 = \frac{1}{\sqrt{2}}\left(\alpha e^{2\pi i\frac{0\times 0}{2}} + \beta e^{2\pi i\frac{1\times 0}{2}}\right) = \frac{1}{\sqrt{2}}(\alpha + \beta)$$

$$y_1 = \frac{1}{\sqrt{2}}\left(\alpha e^{2\pi i\frac{0\times 1}{2}} + \beta e^{2\pi i\frac{1\times 1}{2}}\right) = \frac{1}{\sqrt{2}}(\alpha - \beta)$$

$$QFT|\psi\rangle = \frac{1}{\sqrt{2}}(\alpha + \beta)|0\rangle + \frac{1}{\sqrt{2}}(\alpha - \beta)|1\rangle$$

- Operasi ini identic dengan operasi Hadamard terhadap qubit tersebut:

$$H|\psi\rangle = \alpha H|0\rangle + \beta H|1\rangle = \frac{1}{\sqrt{2}}(\alpha + \beta)|0\rangle + \frac{1}{\sqrt{2}}(\alpha - \beta)|1\rangle \equiv \tilde{\alpha}|0\rangle + \tilde{\beta}|1\rangle$$

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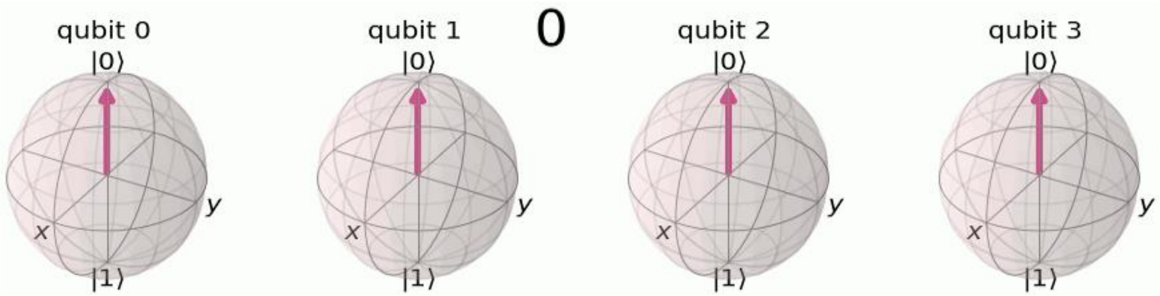
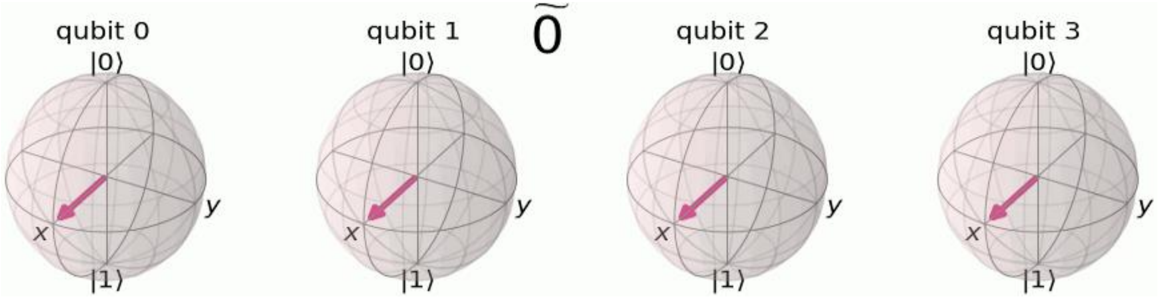
Transformasi Basis

- QFT transformasi computational (Z) basis → fourier basis: $QFT|x\rangle = |\tilde{x}\rangle$
- H-gate merupakan single-qubit QFT: Z-basis ($|0\rangle, |1\rangle$)→X-basis ($|+\rangle, |-\rangle$)
- Dalam computational basis, kita menyimpan angka dalam binary $|0\rangle, |1\rangle$



- Dalam fourier basis, kita menyimpan angka menurut rotasi terhadap z-axis
- $|\tilde{0}\rangle$ →Ketika semua qubit $|+\rangle$, $|\tilde{x}\rangle$ →qubit berfase $\frac{x}{2^n}, \frac{2x}{2^n}, \ldots, \frac{nx}{2^n}$



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QFT

- Untuk $N = 2^n$ atau n-qubit:

$$QFT_N|x\rangle = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} w_N^{xy}|y\rangle = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{\frac{2\pi ixy}{2^n}}|y\rangle$$

- Dalam notasi binary yang sudah dipecah-pecah:

$$= \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{2\pi ix\left(\sum_{k=1}^{n}\frac{y_k}{2^k}\right)}|y_1 \dots y_n\rangle = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}\prod_{k=1}^{n} e^{\frac{2\pi ixy_k}{2^k}}|y_1 \dots y_n\rangle$$

- Dalam notasi tensor product dapat disederhanakan menjadi:

$$\frac{1}{\sqrt{N}}\otimes_{k=1}^{n}\left(|0\rangle + e^{\frac{2\pi ix}{2^k}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{N}}\left(|0\rangle + e^{\frac{2\pi ix}{2^1}}|1\rangle\right)\otimes\left(|0\rangle + e^{\frac{2\pi ix}{2^2}}|1\rangle\right)\otimes\cdots\otimes\left(|0\rangle + e^{\frac{2\pi ix}{2^n}}|1\rangle\right)$$

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Uji Pemahaman

- Berapakah:
  - $QFT|0\rangle$
  - $QFT|1\rangle$
  - $QFT|00\rangle$
  - $QFT|10\rangle$
  - $QFT|000\rangle$
  - $QFT|100\rangle$

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Komponen QFT

- QFT hanya menggunakan H-gate (hadamard) dan CR-gate (controlled rotation)
- Efek H-gate terhadap qubit state $|x_k\rangle$:

$$H|x_k\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{\frac{2\pi i}{2}x_k}|1\rangle\right)$$

- Efek CR-gate terhadap $|x_l x_j\rangle$:

$$CR_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\frac{2\pi i}{2^k}} \end{pmatrix}$$

$$CR_k|0x_j\rangle = |0x_j\rangle$$

$$CR_k|1x_j\rangle = e^{\frac{2\pi i}{2^k}x_j}|1x_j\rangle$$

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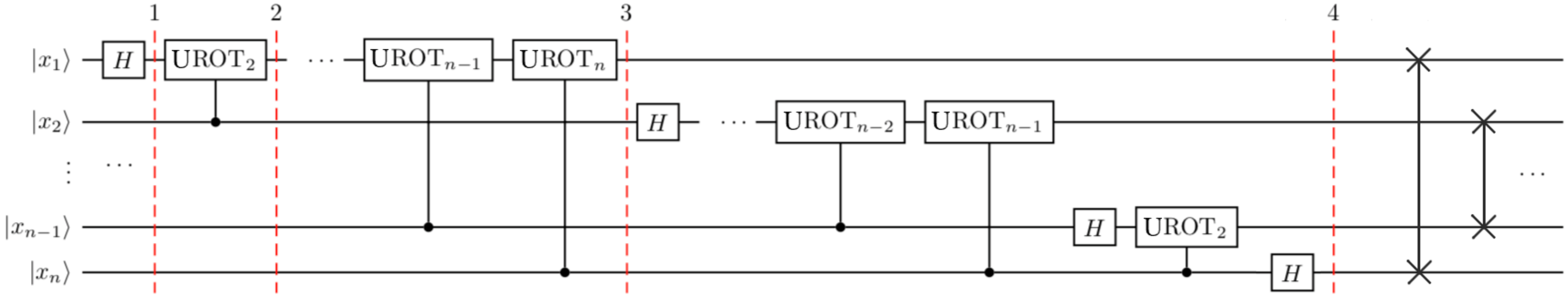
Sirkuit QFT



- State setelah H-gate:

$$|\psi_1\rangle = H|x_1x_2 \dots x_n\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\frac{2\pi i}{2^1}x_1}|1\rangle\right] \otimes |x_2x_3 \dots x_n\rangle$$

- State setelah CR-gate pertama:

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^2}x_2+\frac{2\pi i}{2^1}x_1\right)}|1\rangle\right] \otimes |x_2x_3 \dots x_n\rangle$$

- State setelah CR-gate ke-n:

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^n}x_n+\frac{2\pi i}{2^{n-1}}x_{n-1}+\dots+\frac{2\pi i}{2^2}x_2+\frac{2\pi i}{2^1}x_1\right)}|1\rangle\right] \otimes |x_2x_3 \dots x_n\rangle$$

- Dengan menggunakan definisi $x = 2^{n-1}x_1 + 2^{n-2}x_2 + \dots + 2^1x_{n-1} + 2^0x_n$:

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^n}x\right)}|1\rangle\right] \otimes |x_2x_3 \dots x_n\rangle$$

- State setelah H-gate terakhir:

$$|\psi_4\rangle = \frac{1}{\sqrt{N}}\left(|0\rangle + e^{\frac{2\pi i x}{2^1}}|1\rangle\right) \otimes \left(|0\rangle + e^{\frac{2\pi i x}{2^2}}|1\rangle\right) \otimes \dots \otimes \left(|0\rangle + e^{\frac{2\pi i x}{2^n}}|1\rangle\right)$$

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Contoh: 3-qubit QFT

- State setelah H-gate:

$$|\psi_1\rangle = |x_3\rangle \otimes |x_2\rangle \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\frac{2\pi i}{2^1}x_1}|1\rangle\right]$$

- State setelah CR-gate pertama:

$$|\psi_2\rangle = |x_3\rangle \otimes |x_2\rangle \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^2}x_2+\frac{2\pi i}{2^1}x_1\right)}|1\rangle\right]$$

- State setelah CR-gate ke-n:

$$|\psi_3\rangle = |\psi_3\rangle \otimes |x_2\rangle \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^3}x_3+\frac{2\pi i}{2^2}x_2+\frac{2\pi i}{2^1}x_1\right)}|1\rangle\right]$$

- State setelah H-gate terakhir:

$$|\psi_4\rangle = \frac{1}{\sqrt{N}}\left(|0\rangle + e^{\frac{2\pi i}{2^3}x_3}|1\rangle\right) \otimes \left(|0\rangle + e^{\frac{2\pi i}{2^2}x_3+\frac{2\pi i}{2^1}x_2}|1\rangle\right) \otimes \left(|0\rangle + e^{\frac{2\pi i}{2^3}x_3+\frac{2\pi i}{2^2}x_2+\frac{2\pi i}{2^1}x_1}|1\rangle\right)$$

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Uji Pemahaman

- Berapakah:
  - $QFT|101\rangle$
  - $QFT|110\rangle$

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Aktifitas

- 3-qubit QFT

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Quantum Phase Estimation

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QPE

- QPE merupakan subroutines paling penting dalam banyak algoritma kuantum
- QPE mengukur fase θ dari persamaan eigen unitary berikut:
$$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$$
- Dengan menggunakan CU-gates, qubit lain akan berputar karena phase kickback. Maka kita dapat menyimpan data dalam bentuk fase, lalu inverse QFT dapat mengubah basis ini menjadi computational basis
- Kombinasi beberapa CU-gates dan inverse QFT dapat digunakan untuk encode bilangan apapun

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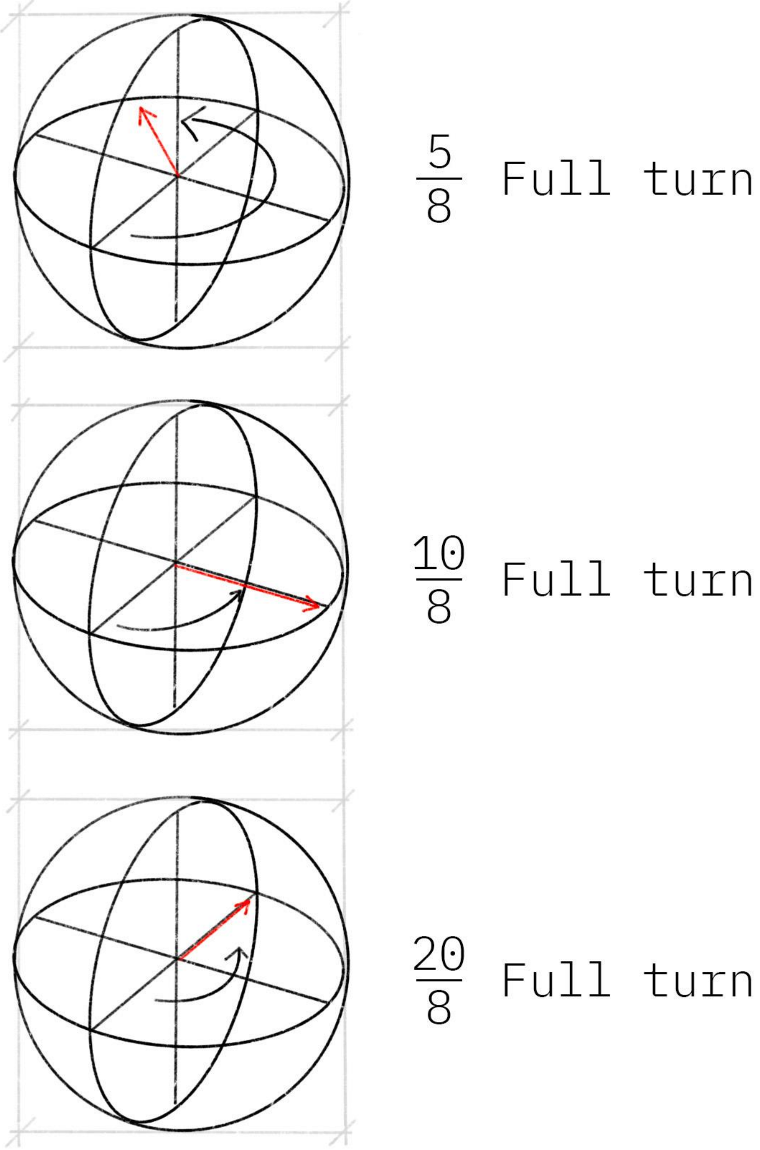
Intuisi

5 in the
fourier basis
(on 3 qubits)

- Kickback QPE menyimpan fase U dalam register t
- Kickback membuat fase qubit control berputar $e^{2i\pi\theta}$
- Fase ini disimpan dalam basis fourier
- Inverse QFT merubah basis fourier ke basis komputasi



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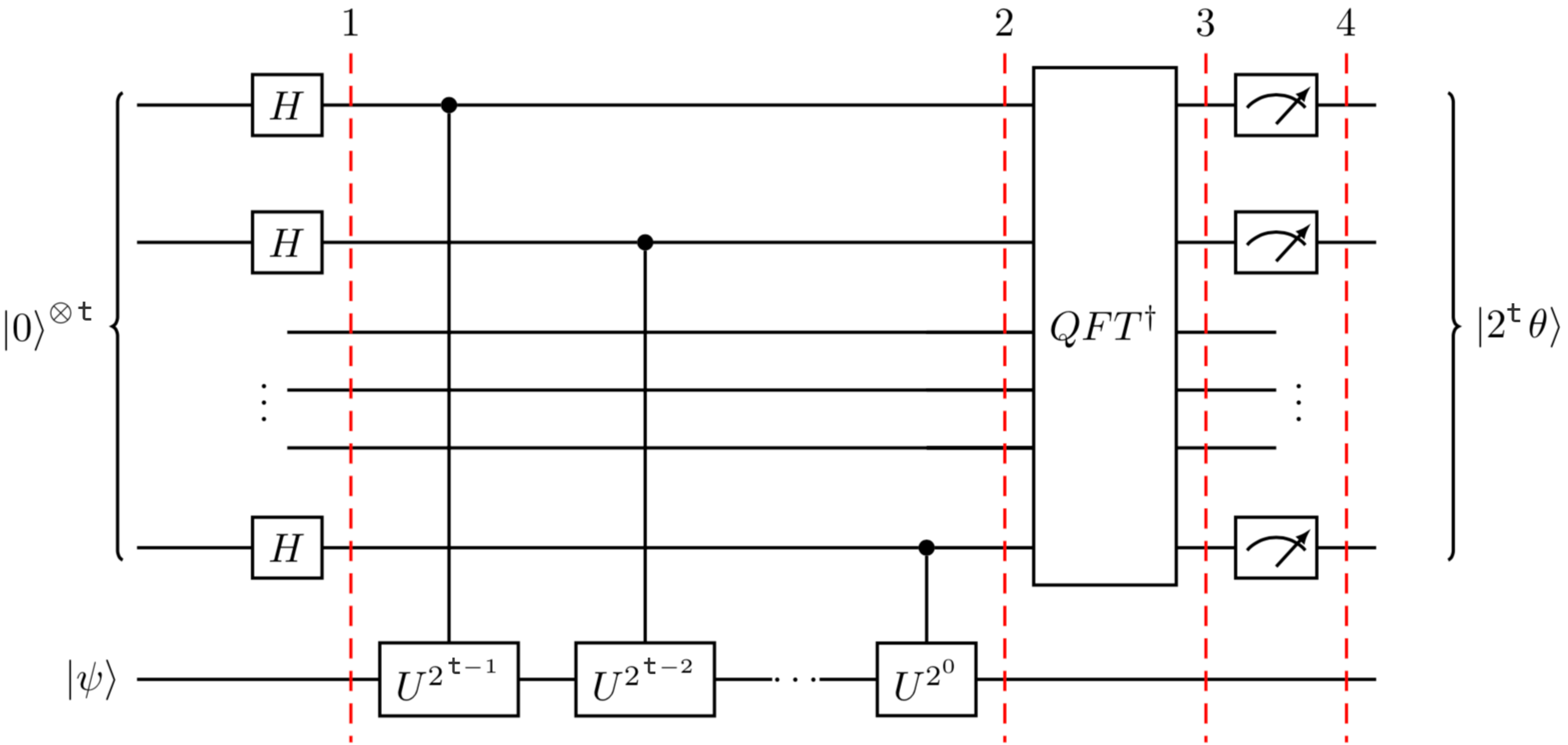
Sirkuit



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Langkah

- Setup:

$$|\psi_0\rangle = |0\rangle^{\otimes n}|\psi\rangle$$

- Superposition:

$$|\psi_1\rangle = \frac{1}{2^{\frac{n}{2}}}(|0\rangle + |1\rangle)^{\otimes n}|\psi\rangle$$

- Controlled Unitary Operations:

$$U^{2^j}|\psi\rangle = U^{2^j-1}U|\psi\rangle = U^{2^j-1}e^{2\pi i\theta}|\psi\rangle = \cdots = e^{2\pi i 2^j\theta}|\psi\rangle$$

$$CU(|+\rangle \otimes |\psi\rangle) = \big(|0\rangle + e^{2\pi i\theta}|1\rangle\big) \otimes |\psi\rangle \rightarrow CU^{2^j}(|+\rangle \otimes |\psi\rangle) = \left(|0\rangle + e^{2\pi i\theta 2^j}|1\rangle\right) \otimes |\psi\rangle$$

$$|\psi_2\rangle = \frac{1}{2^{\frac{n}{2}}}\big(|0\rangle + e^{2\pi i\theta 2^{n-1}}|1\rangle\big) \otimes \cdots \otimes \big(|0\rangle + e^{2\pi i\theta 2^0}|1\rangle\big) \otimes |\psi\rangle = \frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{2\pi i\theta k}|k\rangle \otimes |\psi\rangle$$

- Inverse Fourier Transform:

$$|\psi_3\rangle = QFT_n^{\dagger}\left(\frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{2\pi i\theta k}|k\rangle\right) \otimes |\psi\rangle = \frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{-\frac{2\pi i k}{2^n}(x-2^n\theta)}|k\rangle \otimes |\psi\rangle$$

- Measurement:

$$|\psi_4\rangle = |2^n\theta\rangle \otimes |\psi\rangle$$

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Uji Pemahaman

- Apakah algoritma DJ adalah sejenis QPE?

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Aktivitas

- QPE untuk menemukan fase θ dari T-gate

$$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$$

$$T|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{\frac{i\pi}{4}}|1\rangle$$

$$\theta = \frac{1}{8}$$

- Presisi tambahan

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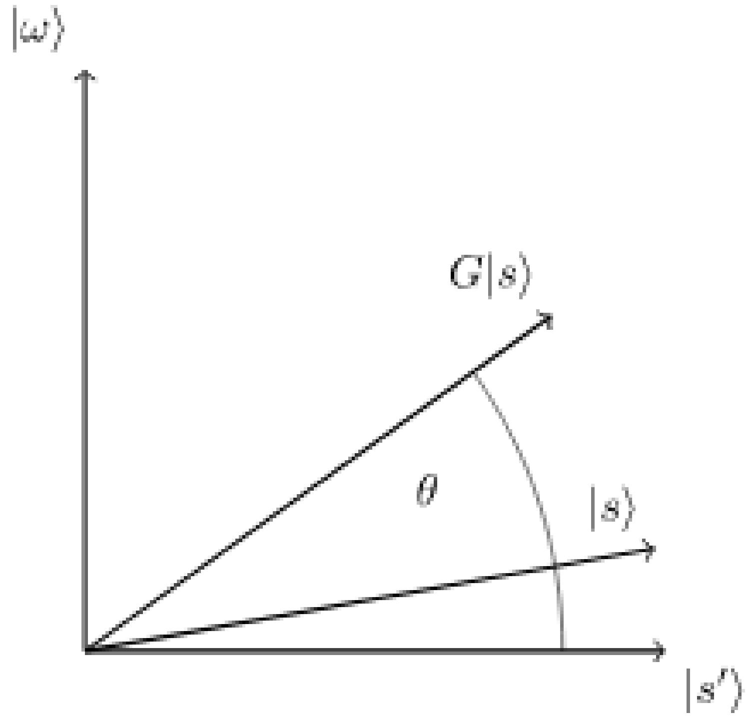
Beberapa Aplikasi QPE

Quantum Counting

- Algoritma Grover berusaha menemukan solusi dari oracle, sedangakan quantum counting memberitahu berapa banyak solusi yang ada
- Presentasi jumlah solusi mempengaruhi perbedaan antara $|s\rangle$ dan $|s'\rangle$
- Grover iterator dapat ditulis sebagai matriks:

$$G = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- Eigenvalues dari Grover iterator adalah $e^{\pm i\theta}$
- Eigenvectors dari Grover iterator adalah: $\begin{pmatrix} -i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}$



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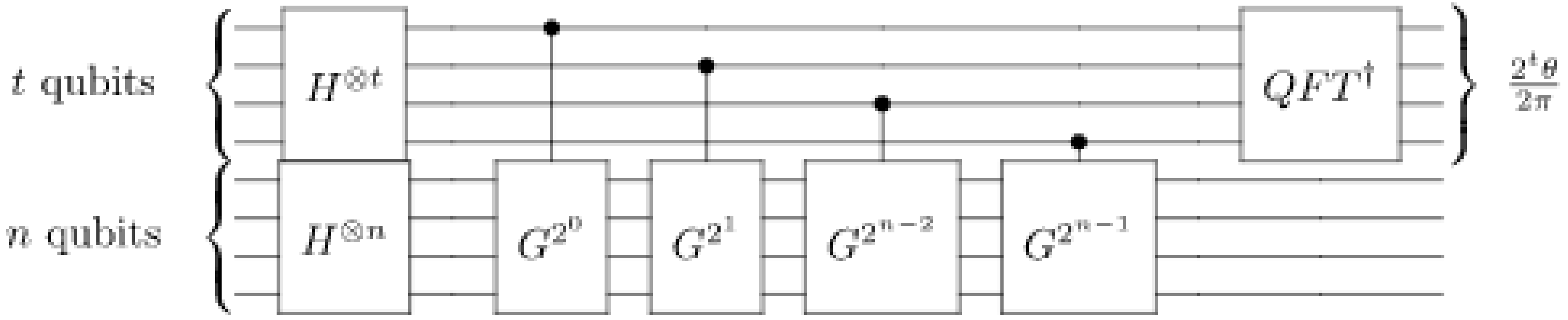
Sirkuit

- Dengan menggunakan QPE, maka fase θ dapat diketahui
- Melalui θ, maka jumlah solusi M dapat diperoleh



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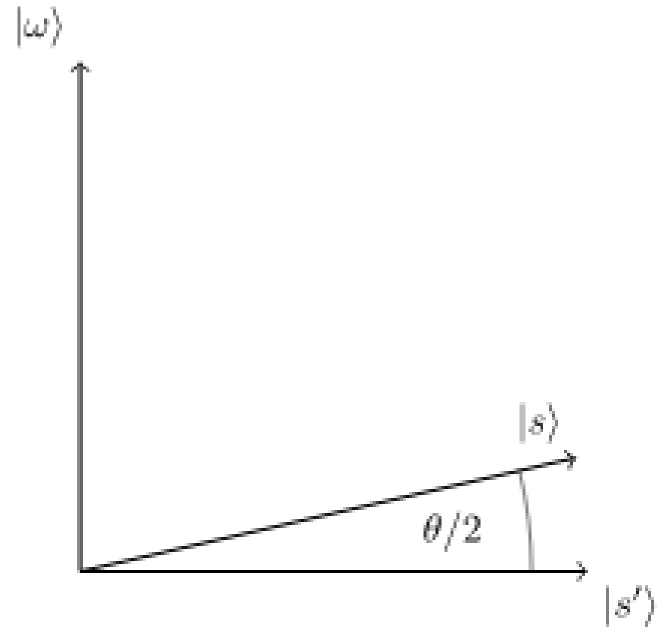
Jumlah Solusi



- Dengan menggunakan relasi antara $|s\rangle$ dan $|s'\rangle$:

$$|s\rangle = \sin\frac{\theta}{2}|w\rangle + \cos\frac{\theta}{2}|s'\rangle$$

$$|s\rangle = \sqrt{\frac{M}{N}}|w\rangle + \sqrt{\frac{N-M}{M}}|s'\rangle$$

- Inner product antara $|s\rangle$ dan $|s'\rangle$:

$$\langle s'|s\rangle = \sqrt{\frac{N-M}{M}} = \cos\frac{\theta}{2}$$

$$N - M = N\cos^2\frac{\theta}{2} \rightarrow M = N\sin^2\frac{\theta}{2}$$

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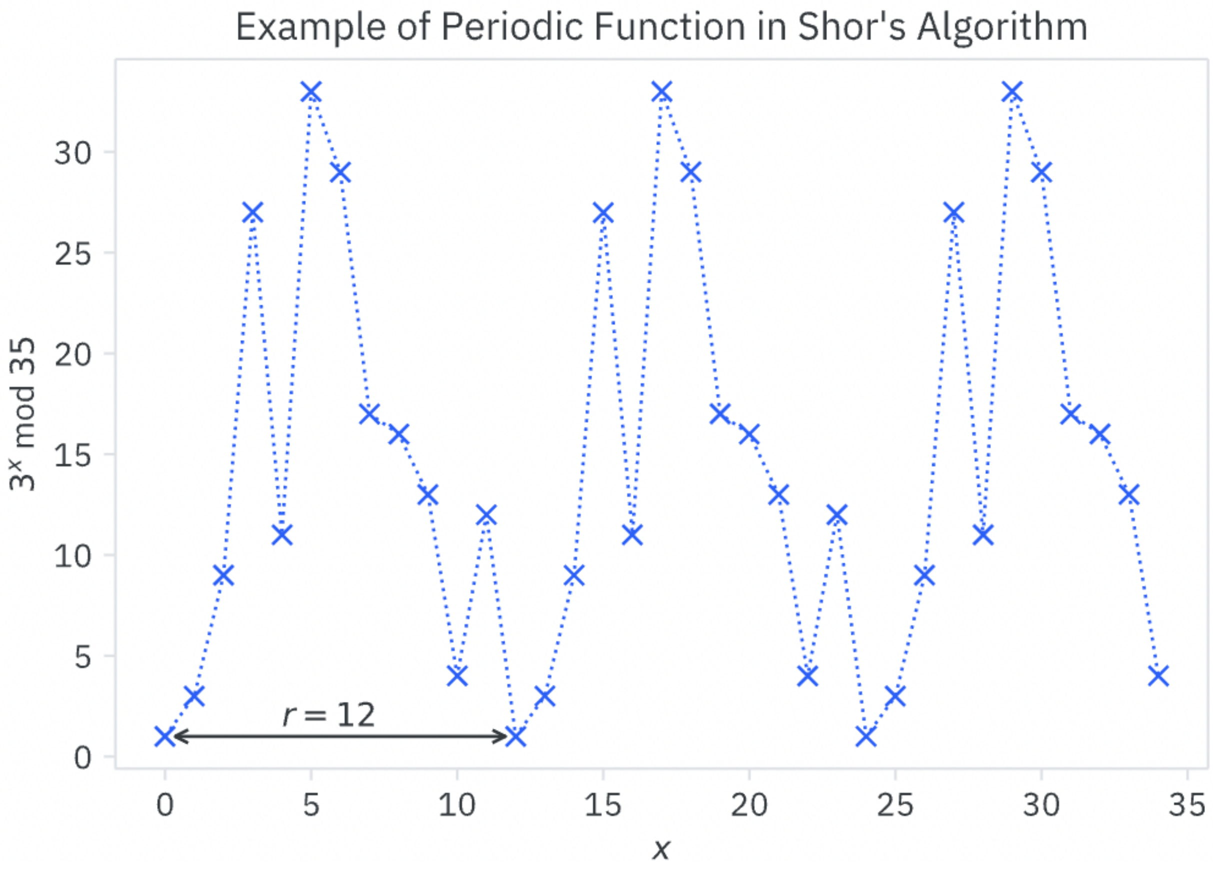
Algoritma Shor

- Algoritma shor dapat digunakan untuk faktorisasi bilangan prima (RSA)
- Algoritma Shor memanfaatkan pola berulang dari bilangan dengan menggunakan QPE



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Tuhan Memberkati

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