

Exercise 4

$$① f(x) = \frac{x}{x+1}$$

$$f(x_0) = \frac{x_0}{x_0+1}$$

$$f'(x) = \frac{\frac{d}{dx}(x) \cdot (x+1) - x \cdot \frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{1 \cdot (x+1) - x \cdot (1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$f''(x) = \frac{1}{(x+1)^2} = (x+1)^{-2}$$
$$= -2(x+1)^{-3}$$

$$= -\frac{2}{(x+1)^3}$$

$$\therefore f(x_0) = \frac{x_0}{x_0+1} + \frac{1}{(x_0+1)^2} (x-x_0) - \frac{2}{2! \frac{1}{(x+1)^3}} (x-x_0)^2 + \mathcal{O}(x-x_0)^3$$

$$\therefore f(x_0) = \frac{x_0}{x_0+1} + \frac{1}{(x_0+1)^2} (x-x_0) - \frac{2}{(x_0+1)^3} (x-x_0)^2 + \mathcal{O}(x-x_0)^3$$

$$② f(x) = \frac{2x^2}{x^4} = 2x^{-2}$$

$$f(x_0) = 2(x_0)^{-2}$$

$$f'(x_0) = -4(x_0)^{-3}$$

$$f''(x_0) = 12(x_0)^{-4}$$

$$\therefore f(x_0) = 2(x_0)^{-2} - 4(x_0)^{-3} (x-x_0) + 12(x_0)^{-4} (x-x_0)^2 + \mathcal{O}(x-x_0)^3$$

$$③ \quad f(x) = \ln x^2 \\ = 2 \ln x$$

$$f(x_0) = 2 \ln(x_0)$$

$$f'(x_0) = \frac{2}{x_0}$$

$$f''(x_0) = -2x_0^{-2}$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + O((x-x_0)^3) \\ = 2 \ln(x_0) + \frac{2}{x_0}(x-x_0) - \frac{1}{x_0^2}(x-x_0)^2 + O((x-x_0)^3)$$

$$f^{(3)}(x) = \frac{4}{x^3} \Rightarrow f^{(3)}(x_0) = \frac{4}{x_0^3} \Rightarrow \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 = \frac{2}{3x_0^3}(x-x_0)^3$$

$$f^{(4)}(x) = -\frac{12}{x^4} \Rightarrow f^{(4)}(x_0) = -\frac{12}{x_0^4} \Rightarrow -\frac{1}{x_0^4}(x-x_0)^4$$

$$f^{(5)}(x) = \frac{20}{x^5} \Rightarrow f^{(5)}(x_0) = \frac{20}{x_0^5} \Rightarrow \frac{4}{x_0^5}(x-x_0)^5$$

$$f^{(6)}(x) = -\frac{120}{x^6} \Rightarrow f^{(6)}(x_0) = -\frac{120}{x_0^6} \Rightarrow -\frac{20}{x_0^6}(x-x_0)^6$$

$$f^{(7)}(x) = \frac{280}{x^7} \Rightarrow f^{(7)}(x_0) = \frac{280}{x_0^7} \Rightarrow \frac{40}{x_0^7}(x-x_0)^7$$

$$f^{(8)}(x) = -\frac{560}{x^8} \Rightarrow f^{(8)}(x_0) = -\frac{560}{x_0^8} \Rightarrow -\frac{70}{x_0^8}(x-x_0)^8$$

$$f^{(9)}(x) = \frac{1680}{x^9} \Rightarrow f^{(9)}(x_0) = \frac{1680}{x_0^9} \Rightarrow \frac{140}{x_0^9}(x-x_0)^9$$