Excercise 4

① $f(x) = \frac{x}{x+1}$

$f(x_0) = \frac{x_0}{x_0+1}$

$$f'(x) = \frac{\frac{d}{dx}(x)\cdot(x+1) - x\cdot\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{1\cdot(x+1) - x\cdot(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$f''(x) = \frac{1}{(x+1)^2} = (x+1)^{-2}$$
$$= -2(x+1)^{-3}$$

$$= -\frac{2}{(x+1)^3}$$

$$\therefore f(x_0) = \frac{x_0}{x_0+1} + \frac{1}{(x_0+1)^2}\cdot(x-x_0) - \frac{\frac{2}{(x+1)^3}}{2!}(x-x_0)^2 + O(x-x_0)^3$$

$$\therefore f(x_0) = \frac{x_0}{x_0+1} + \frac{1}{(x_0+1)^2}(x-x_0) - \frac{4}{(x+1)^3}(x-x_0)^2 + O(x-x_0)^3$$

② $f(x) = \frac{2x^2}{x^4} = 2x^{-2}$

$f(x_0) = 2(x_0)^{-2}$

$f'(x_0) = -4(x_0)^{-3}$

$f''(x_0) = 12(x_0)^{-4}$

$$\therefore f(x_0) = 2(x_0)^{-2} - 4(x_0)^{-3}(x-x_0) + 24(x_0)^{-4}(x-x_0)^2 + O(x-x_0)^3$$

③ $f(x) = \ln x^2$
$= 2\ln x$

$f(x_0) = 2\ln(x_0)$

$f'(x_0) = \frac{2}{x_0}$

$f''(x_0) = -2x_0^{-2}$

$f(x_0) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + O(x-x_0)^3$
$= 2\ln|x_0| + \frac{2}{x_0}(x-x_0) - \frac{4}{x_0^2}(x-x_0)^2 + O(x-x_0)^3$