Homework #2

Caroline Hughes

March 6, 2018

** Please visit https://github.com/clhughes/math228b-spring2018 for gifs. **

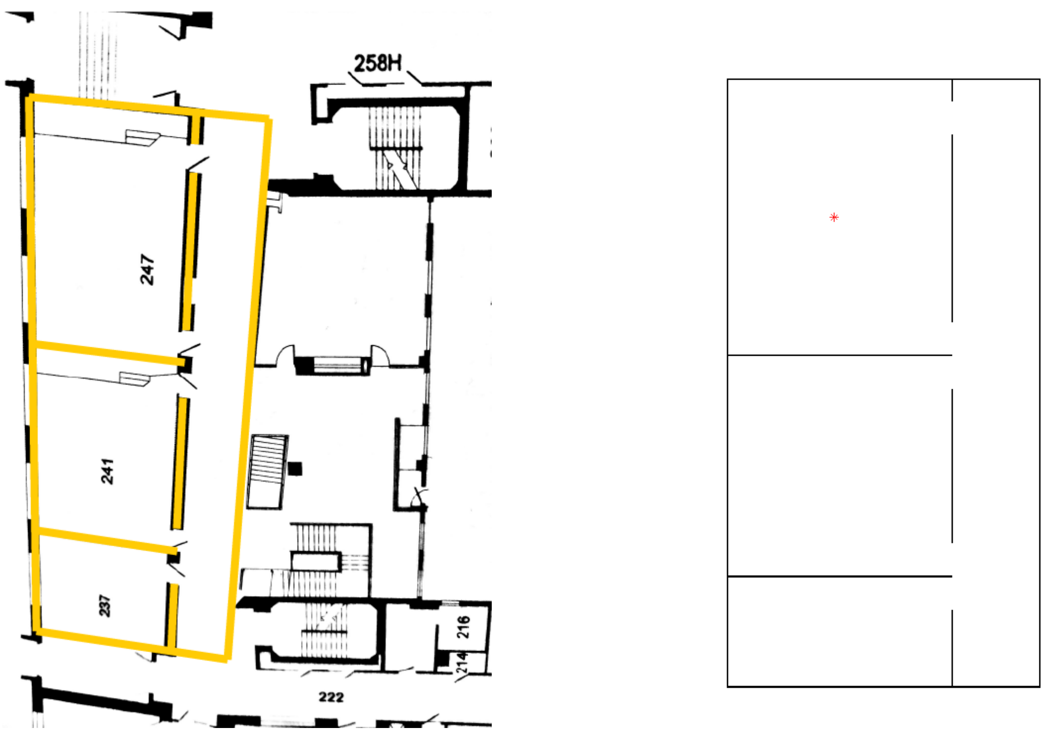
A bag of chips opened in Cory Hall room 247 will eventually tempt students in room 241 next door, though perhaps a pot of freshly-brewed coffee might be particularly enticing to graduate students. Regardless, the scent will propagate from 247 to 241 via diffusion and can be modeled using Newton's method in time and central differences in space:

$$u_t = \alpha \nabla^2 u$$
$$D_+ u = \alpha D_+ D_- u$$

To solve this problem, I will assume that the problem is two-dimensional, since diffusion in the z-direction will not move the scent from 247 to 241. I based my model on the Cory Hall floor plan (shout out to the folks in the office for the picture!) in Fig. 1(a), which I modeled as Fig. 1(b).

Figure 1: Cory Hall floor plan

(a) Scanned floor plan (b) Modeled floor plan



Numerically, this can be approached several ways. I first wanted to adapt my alternating direction implicit (ADI) method from the second homework from three to two dimensions. ADI involves solving in two iterations for each time step

$$\left(1 - \frac{\alpha k}{2}\delta_x^2\right) u_{i,j}^{n+1/2} = \left(1 + \frac{\alpha k}{2}\delta_y^2\right) u_{i,j}^n \tag{1a}$$

$$\left(1 - \frac{\alpha k}{2}\delta_y^2\right) u_{i,j}^{n+1} = \left(1 + \frac{\alpha k}{2}\delta_x^2\right) u_{i,j}^{n+1/2} \tag{1b}$$

where $u_{i,j}^n$ is ordered first in y and then in x and vectors in the intermediate time steps are ordered the other way around.

I encountered a strange issue upon implementation in two dimensions. After each half time step I had to reorder my vector manually. I really tried my best to resolve this issue and still don't understand what was going on since this did work for the pool problem, but could not figure it out. For this reason, I chose to use an explicit scheme instead:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{k} = \frac{\alpha}{h^2}\left[\left(u_{i-1,j}^n - 2u_{i,j}^n + u_{i+1,j}^n\right) + \left(u_{i,j-1}^n - 2u_{i,j}^n + u_{i,j+1}^n\right)\right]$$
$$u_{i,j}^{n+1} - u_{i,j}^n = \lambda u_{i,j-1}^n + \lambda\left(u_{i-1,j}^n - 2u_{i,j}^n + u_{i+1,j}^n\right)\lambda u_{i,j+1}^n$$

where $n = 0, \ldots, N$, $i = 0, \ldots, H_x$, and $j = 0, \ldots, H_y$ and I ensured that $h_x = h_y = h$. In matrix form, this equation can be represented as

$$\boldsymbol{u}^{n+1} = (I + \lambda A)\boldsymbol{u}^n$$

$$\text{where } A = \begin{pmatrix} T & I & & & \\ I & T & I & & \\ & \ddots & \ddots & \ddots & \\ & & I & T & I \\ & & & I & T \end{pmatrix} \text{ and } T = \begin{pmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{pmatrix}$$

Here, $\boldsymbol{u}^n = (\boldsymbol{u}_0^n, \ldots, \boldsymbol{u}_{H_y}^n)^T$, where $\boldsymbol{u}_j^n = (\boldsymbol{u}_{0,j}^n, \ldots, \boldsymbol{u}_{H_x,j}^n)^T$.

I built the model of the two classrooms in stages. I first modeled 247 with its doors closed to implement an insulating boundary condition (BC) on each wall. I then opened the doors to the classrooms to allow the scent to propagate out of 247 and into 241. Finally, I added a fan in the hallway to see how an additional term in the heat equation affected the

1 Insulating Boundary Conditions

I first modeled 247 with all doors and windows closed to implement the insulating BCs. I took two different approaches to modeling the very outer BCs. I first placed the grid points *on* the wall and imposed insulating BCs using central differences:

$$\left.\frac{du}{dn}\right|_{\gamma=0} = \frac{u_1 - u_{-1}}{2h} = 0 \quad \Longrightarrow \quad u_{-1} = u_1$$
$$\left.\frac{du}{dn}\right|_{\gamma=H} = \frac{u_{H+1} - u_{H-1}}{2h} = 0 \quad \Longrightarrow \quad u_{H+1} = u_{H-1}$$

Since these BCs are the same in both the x and y direction, I will consider both simultaneously using the indexing $\gamma = 0 \dots H$ where $\gamma = 0$ and $\gamma = H$ are on the exterior wall, as shown in Fig. 2(a). Now, the second-order partial differential equation (PDE) at $\gamma = 0$ and $\gamma = H$ can be written:

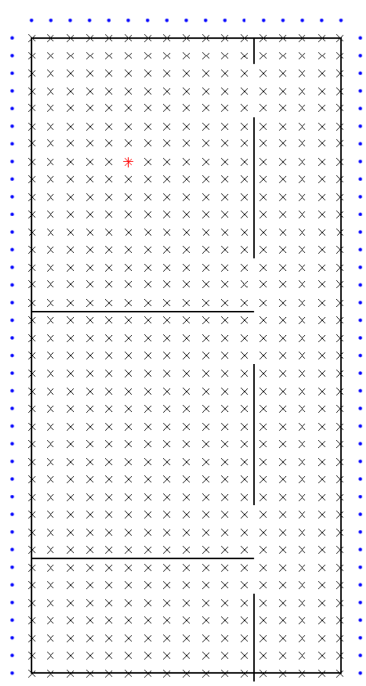
$$\left.\frac{\partial^2 u}{\partial n^2}\right|_{\gamma=0} = \frac{u_{-1} - 2u_0 + u_1}{h^2} = \frac{-2u_0 + 2u_1}{h^2}$$
$$\left.\frac{\partial^2 u}{\partial n^2}\right|_{\gamma=H} = \frac{u_{H-1} - 2u_H + u_{H+1}}{h^2} = \frac{2u_{H-1} - 2u_H}{h^2}$$

I quickly realized that this became more complicated when adding the interior walls – if I was solving for a grid point *on* the wall, how could I ensure insulation on either side of the wall? And what does non-zero scent inside the wall *mean*? To avoid tackling these issues, I restructured my mesh to straddle the wall as shown in Fig. 2(b).

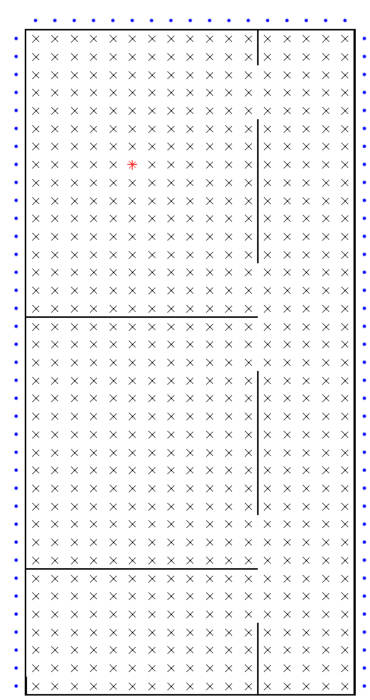
I still use a spatial step size of h in both x and y and index $\gamma = 0 \dots H$, but now there is a space of $h/2$ between u_0 and u_H and the closest walls. Again using central differences to apply the insulating BCs, I find:

$$\left.\frac{du}{dn}\right|_{\gamma=-1/2} = \frac{u_0 - u_{-1}}{h} = 0 \quad \Longrightarrow \quad u_{-1} = u_0$$
$$\left.\frac{du}{dn}\right|_{\gamma=H+1/2} = \frac{u_{H+1} - u_H}{h} = 0 \quad \Longrightarrow \quad u_{H+1} = u_H$$

(a) Geometry with exterior mesh points *on* the wall



(b) Geometry with exterior mesh points *off* the wall



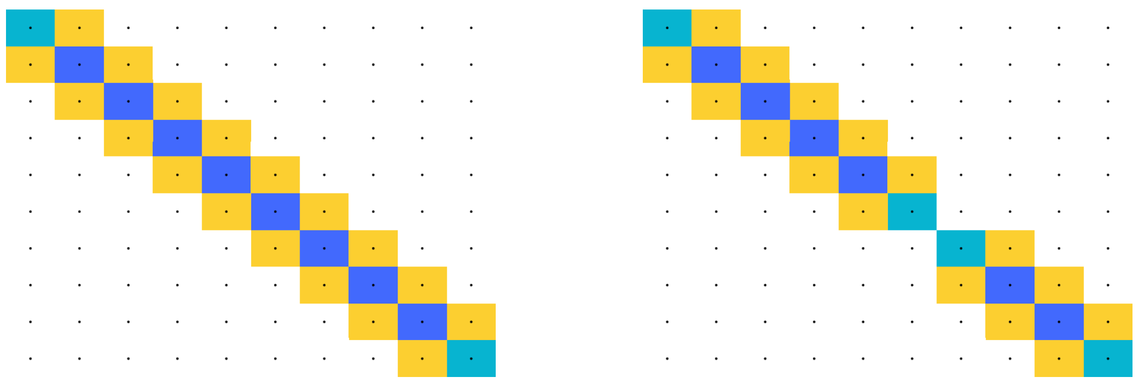
So the second-order PDE at $\gamma = 0$ and $\gamma = H$ can be written:

$$\left.\frac{\partial^2 u}{\partial n^2}\right|_{\gamma=0} = \frac{u_{-1} - 2u_0 + u_{+1}}{h^2} = \frac{-u_0 + u_1}{h^2} \tag{2a}$$

$$\left.\frac{\partial^2 u}{\partial n^2}\right|_{\gamma=0} = \frac{u_{H-1} - 2u_H + u_{H+1}}{h^2} = \frac{-u_H + u_{H+1}}{h^2} \tag{2b}$$

When adding the interior walls, the same boundary conditions apply, using Equation 2b before a wall and Equation 2a after a wall. Fig. 3(a) graphically represents the T matrix for unobstructed diffusion through a doorway and Fig. 3(b) shows this matrix for diffusion hindered by a wall.

(a) T matrix at a doorway (b) T matrix at a wall



2 A Fan in the Hallway

If a fan happened to be blowing at either end of the hallway, this could either speed up or slow down the diffusion into 241. I represented this by adding a directional derivative term in y, so that the diffusion equation is now given by

$$u_t = \alpha \nabla^2 u \pm \beta u_y \tag{3}$$

where α is the diffusion coefficient and $\beta \in \mathbb{R}$ is the air speed, with direction indicated by the sign attached to this parameter. Positive air flow moves down the hallway from room 247 to 241 and negative airflow moves up the hallway from 241 to 247.

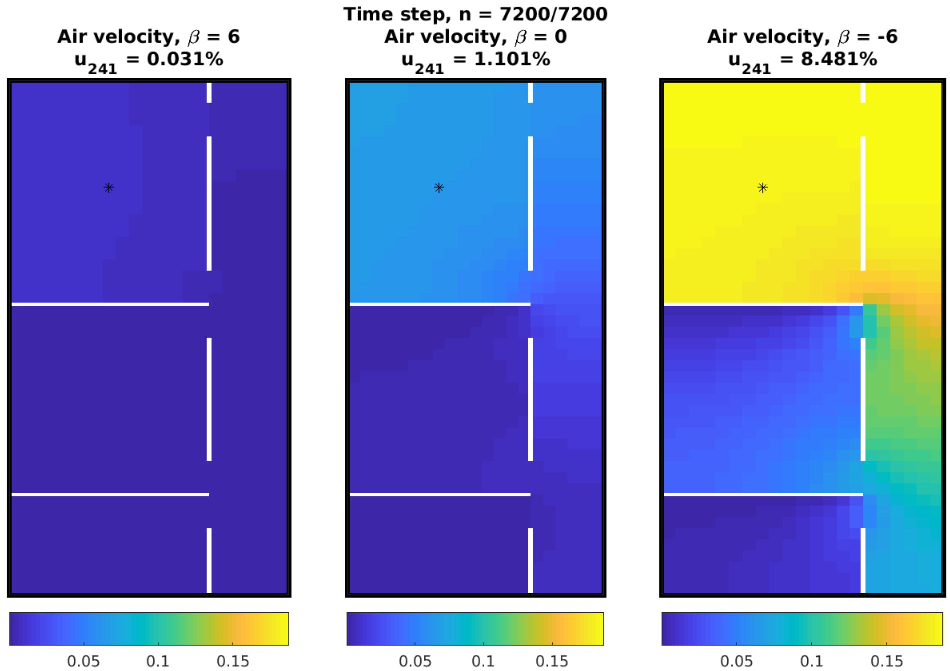
I discretized this using central difference for the second derivative in space, but either forward or backward differences for the first derivative in space. If the airflow moved downward ($\boldsymbol{\beta} = -|\beta|$) toward 241, I used backward differences; if it moved upward toward 247 ($\boldsymbol{\beta} = +|\beta|$), I used forward differences. I made these choices so that the information flow in the scheme used reflected the airflow in the physical scenario.

$$\frac{\partial u}{\partial y} = \alpha \left(\frac{u^n_{i,j-1} - 2u^n_{i,j} + u^n_{i,j+1}}{h^2} \right) - |\beta| \left(\frac{u^n_{i,j} - u^n_{i,j-1}}{h} \right) \tag{4a}$$

$$\frac{\partial u}{\partial y} = \alpha \left(\frac{u^n_{i,j-1} - 2u^n_{i,j} + u^n_{i,j+1}}{h^2} \right) + |\beta| \left(\frac{u^n_{i,j+1} - u^n_{i,j}}{h} \right) \tag{4b}$$

I implemented this method for a range of values of β over a time interval $t = 0 \ldots 3$ so that I could see both when the scent concentration in room 241 reached 1 % and when it reached its maximum before

Figure 4: Scent concentration at $t = 3$ for $beta = 6, 0, -6$



dissipating back into the hallway. I visualized this in gifs, available for select values of β at https://github.com/clhughes/math228b-spring2018. Fig. 4 shows the final concentration at $t = 3$ for $beta = 6, 0, -6$.

Fig. 5 shows the concentration in room 241 as a function of time for each of these β values.

Figure 5: Potato chip scent concentration in room 241 vs. time for varying air velocities

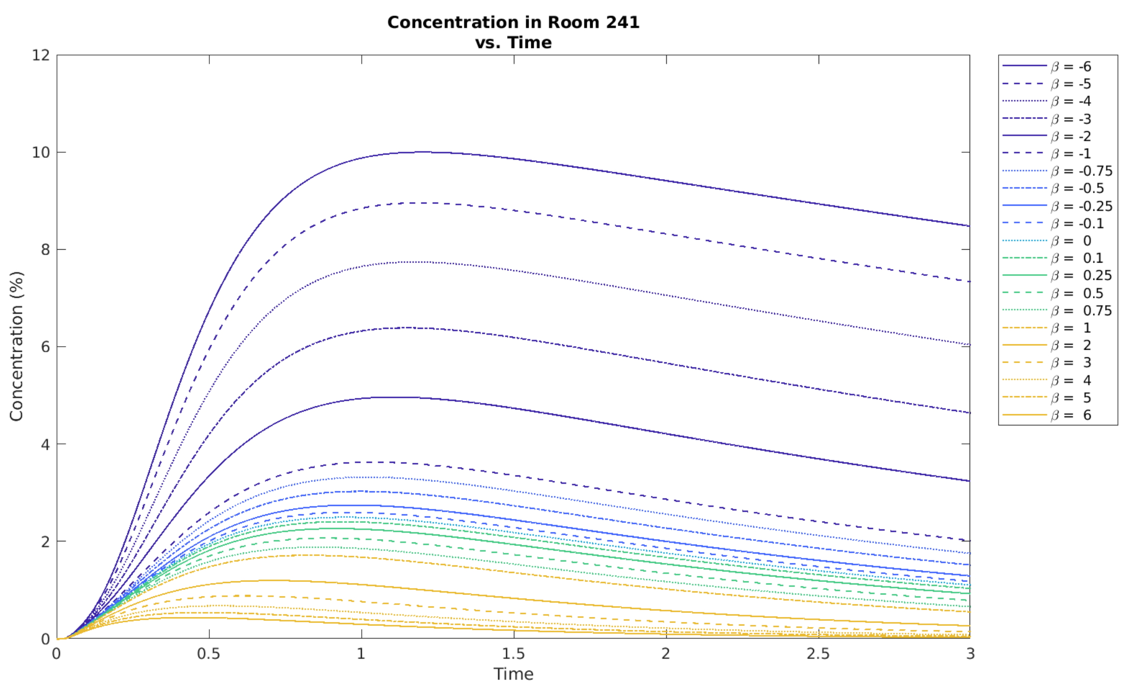
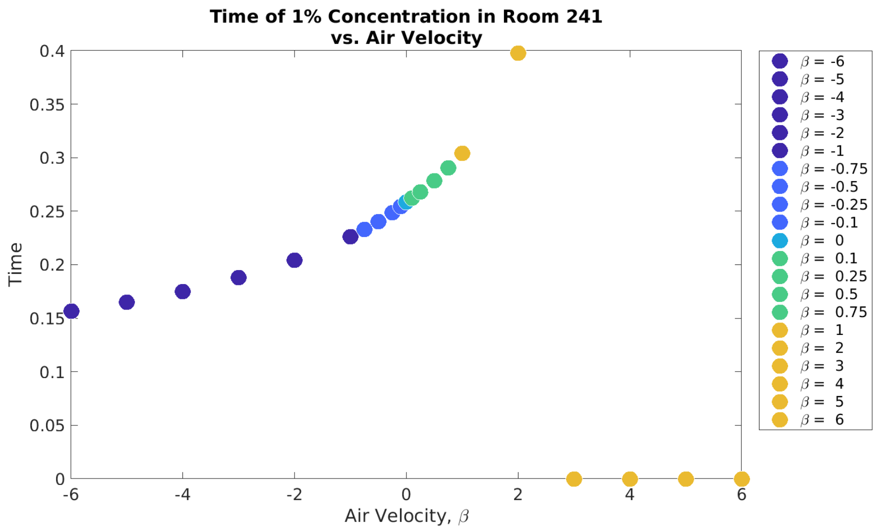


Fig. 6(a) shows the time at which the concentration in room 241 reached 1 % as a function of air velocity and Fig. 6(b) shows the time at which the concentration in room 241 is maximized as a function of air velocity.

Figure 6

(a) Time at which the concentration in room 241 reached 1 % vs. air velocity



(b) Time at which the concentration in room 241 is maximized vs. air velocity