Homework #4

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** Please visit https://github.com/clhughes/math228b-spring2018 for gifs. **

Table of Contents

1 One-Dimensional Wave Equation

Equation 1 describes the second-order wave equation, a hyperbolic partial differential equation (PDE), which here has a time domain $t \geq 0$ and a spatial domain of $x \in [0, 1]$. Here, we can apply a Dirichlet boundary condition (BC) at $x = 0$ and a Neumann BC at $x = 1$, and an initial condition (IC) on both u and u_t at $t = 0$.

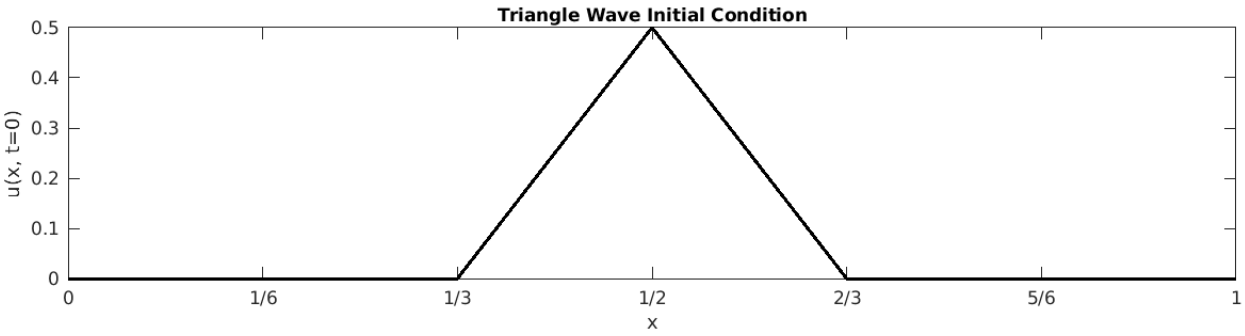
$$\begin{cases} u_{tt} = \alpha^2 u_{xx}, & t \in [0,1], \quad x \in [0,1] \\ u(0,t) = 0, & t \geq 0 \\ u_x(1,t) = 0, & t \geq 0 \\ u(x,0) = f(x), & x \in [0,1] \\ u_t(x,0) = 0, & x \in [0,1] \end{cases} \tag{1}$$

where $f(x)$ describes the triangle wave plotted in Fig. 1.

$$f(x) = \begin{cases} 3\left(x - \frac{1}{3}\right) \\ 3\left(\frac{2}{3} - x\right) \\ 0 \end{cases}$$

The solution can be approximated numerically by discretizing $u(x,t)$ as $u_j^n = u(hj, nk)$ where $j = 0, 1, \ldots, H$ and $n = 0, 1, \ldots, N$.

Figure 1



I analyzed the solutions by examining the potential, kinetic, and total energies of the waves as functions of time. Additionally, I examined the height of the wave at the center point. I chose this point to see how the wave diffused as a function of time, if at all.

The solution wave has both potential and kinetic energy, given by U and T calculated using Equations 2a and 2b [1], where the total energy of the system can be found by adding these two values.

$$T = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2 dx \tag{2a}$$

$$U = \frac{\alpha^2}{2}\int_{-\infty}^{\infty} u_x^2 dx \tag{2b}$$

I approximated u_t and u_x numerically using central differences and integrated using the trapezoidal rule, given in Equation 3

$$\int_{-a}^{b} f(x, t = nk)dx \approx \frac{\Delta x}{2}\left(f_0^n + 2\sum_{j=1}^{H-1} f_j^n + f_H^n\right) \tag{3}$$

1.1 Method of Characteristics

The exact solution to the wave equation can be given by Equation 4a, where the functions f and g come from the notation used in Equation 1. In the case that the wave begins with zero initial velocity, Equation 4a becomes Equation 4b, which is a special case known as d'Alembert's solution [2].

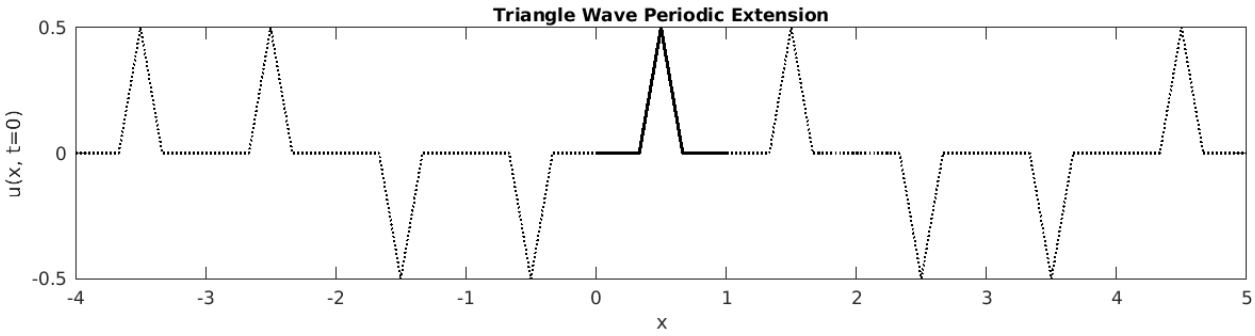
$$u(x,t) = \frac{1}{2}\left(f\left(x - t/\alpha\right) + f\left(x + t/\alpha\right) + \int_{x-t/\alpha}^{x+t/\alpha} g(\xi)d\xi\right) \tag{4a}$$

$$u(x,t) = \frac{1}{2}\Big(f(x - t/\alpha) + f(x + t/\alpha)\Big) \tag{4b}$$

This was the method I implemented in Matlab to solve the wave equation using the method of characteristics.

This became extremely complicated when implementing the BCs, which here impose a Dirichlet BC at $x = 0$ and a Neumann BC at $x = 1$. Implementing this required extending the initial triangle wave as shown in Fig. 2. I accomplished this by following Herman's notes and example on how to do this very carefully [3].

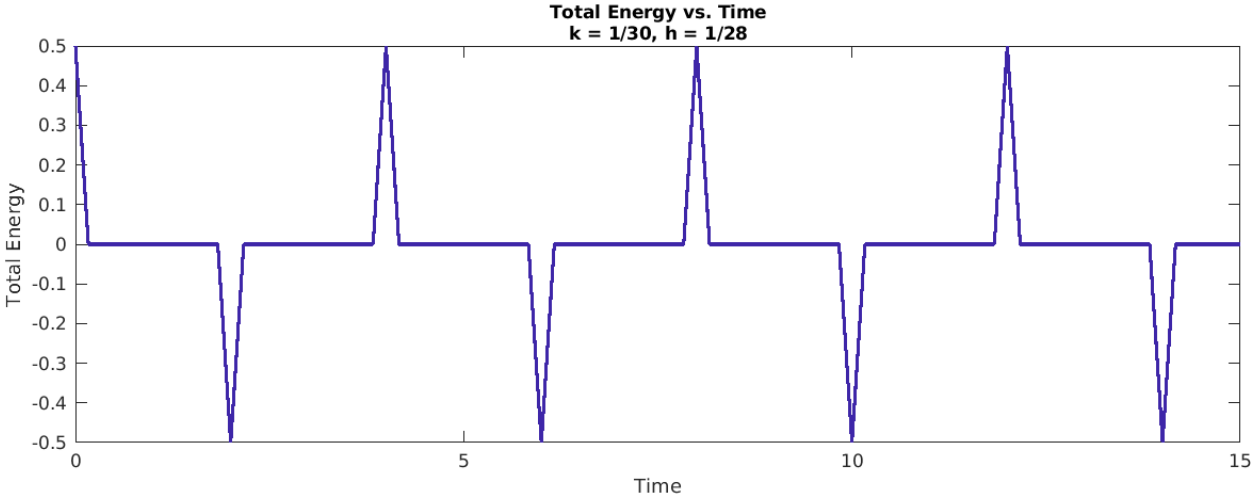
Figure 2



Results

The method of characteristics produces an *exact* solution to the PDE. This solution does not diffuse over time, resulting in $u(x = 0.5, t)$ shown in Fig. 3.

Figure 3

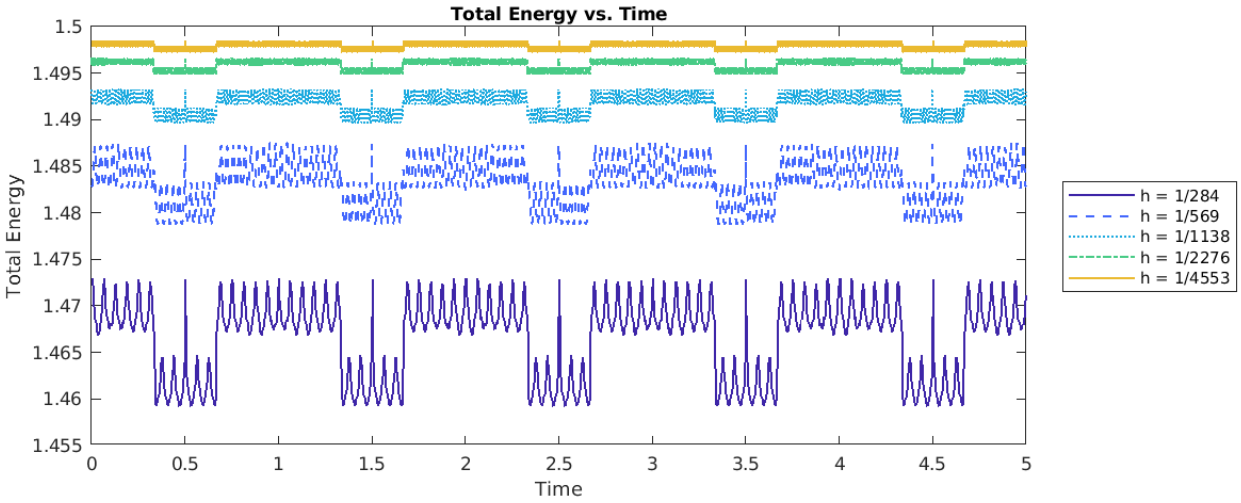


The total energy of the solution remains more-or-less constant, as shown for five different step sizes in Fig. 4. The oscillations in energy around where the wave interacts with the boundaries can be explained by the selection of step size since these oscillations become less extreme for smaller step sizes.

1.2 Leapfrog

This leapfrog method was born from my first attempt at this homework problem: I was not sure where to begin and just wanted to write a method that *worked*, so following our strategies developed in lecture, central differences in both time and space made sense. Upon further research into how to solve this problem, I discovered that this method is actually called the "leapfrog" method for the second-order wave equation

Figure 4



and renamed this section accordingly [4].

$$\frac{u_j^{n-1} - 2u_j^n + u_j^{n+1}}{k^2} = \alpha^2 \left(\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2} \right), \qquad s \equiv \frac{\alpha^2 k^2}{h^2}$$

$$u_j^{n+1} = \left(s u_{j-1}^n - 2(1-s) u_j^n + s u_{j+1}^n \right) - u_j^{n-1} \tag{5a}$$

Boundary Conditions Special consideration has to be taken at the boundaries to implement the Dirichlet BC at $x = 0$ and Neumann BC at $x = 1$. I do not need to solve for u_0^n, since the BCs dictate that $u_0^n = 0$. At $j = 1$, I can apply the BC $u(0,t) = 0$, such that Equation 5a becomes:

$$u_1^{n+1} = (-2(1-s)u_1^n + s u_2^n) - u_1^{n+1} \tag{5b}$$

At $j = H$, I can use a central difference scheme to apply the BC $u_x(1,t) = 0$:

$$\frac{u_{H+1}^n - u_{H-1}^n}{2h} = u_x(1,t) = 0 \quad \Longrightarrow \quad u_{H+1}^n = u_{H-1}^n$$

Now Equation 5a can be rewritten:

$$u_H^{n+1} = \left(2s u_{H-1}^n - 2(1-s) u_H^n \right) - u_H^{n-1} \tag{5c}$$

Combining Equations 5a to 5c, I can solve for $\boldsymbol{u}^{n+1}$ using Equation 6a for $n > 1$.

$$\boldsymbol{u}^{n+1} = A\boldsymbol{u}^n - \boldsymbol{u}^{n-1} \tag{6a}$$

$$A = \begin{pmatrix} 2(1-s) & s & & & \\ s & 2(1-s) & s & & \\ & \ddots & \ddots & \ddots & \\ & & s & 2(1-s) & s \\ & & & 2s & 2(1-s) \end{pmatrix}$$

Initial Condition A problem arises when solving for $\boldsymbol{u}^1$, since Equation 5a requires the solution at $\boldsymbol{u}^{-1}$. I can address this by using a central difference scheme to apply the IC $u_t(x,0) = 0$.

$$\frac{\boldsymbol{u}^1 - \boldsymbol{u}^{-1}}{2k} = g(x) = 0 \quad \Longrightarrow \quad \boldsymbol{u}^{-1} = \boldsymbol{u}^1$$
$$\boldsymbol{u}^1 = \frac{1}{2} A \boldsymbol{u}^1 \tag{6b}$$

Interface Condition So far, I have assumed uniform α in my approach to solving Equation 1. However, it is possible to have a discontinuous coefficient, for example,

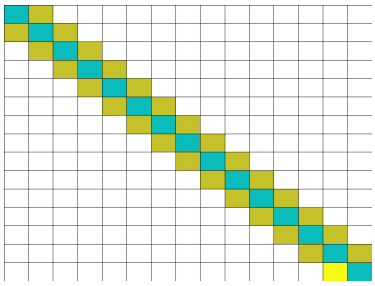
$$\alpha^2 = \begin{cases} 1, & x \in \left[0, \frac{1}{2}\right] \\ 2, & x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & r & 2(1-r) & \frac{r+s}{2} & & & \\ & & r & 2\left(1 - \frac{r+s}{2}\right) & s & & \\ & & & \frac{r+s}{2} & 2(1-s) & s & \\ & & & & \ddots & \ddots & \end{pmatrix}$$

Fig. 5 shows the A matrices for both smooth and discontinuous α graphically. I selected the step size h based on the value of α in each given region to maintain $s \approx r < 1$. Fig. 6 shows how adjusting step size based on α in each region effects the spatial discretization of the problem.

Figure 5: A matrices for smooth and discontinuous α

(a) A matrix for continuous α



(b) A matrix for discontinuous α

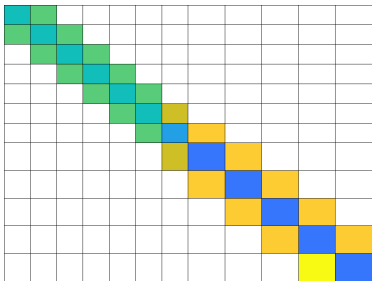
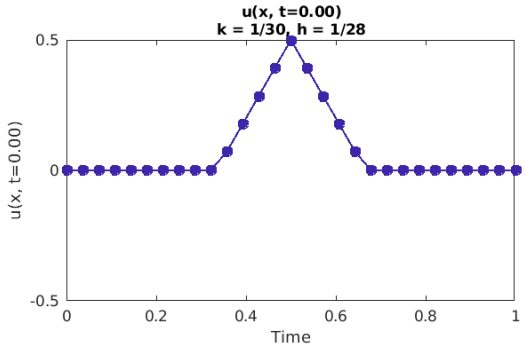
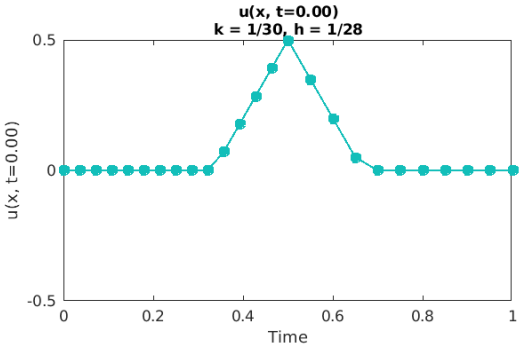


Figure 6: $u(x, t = 0)$ for smooth and discontinuous α

(a) $u(x, t = 0)$ for smooth α



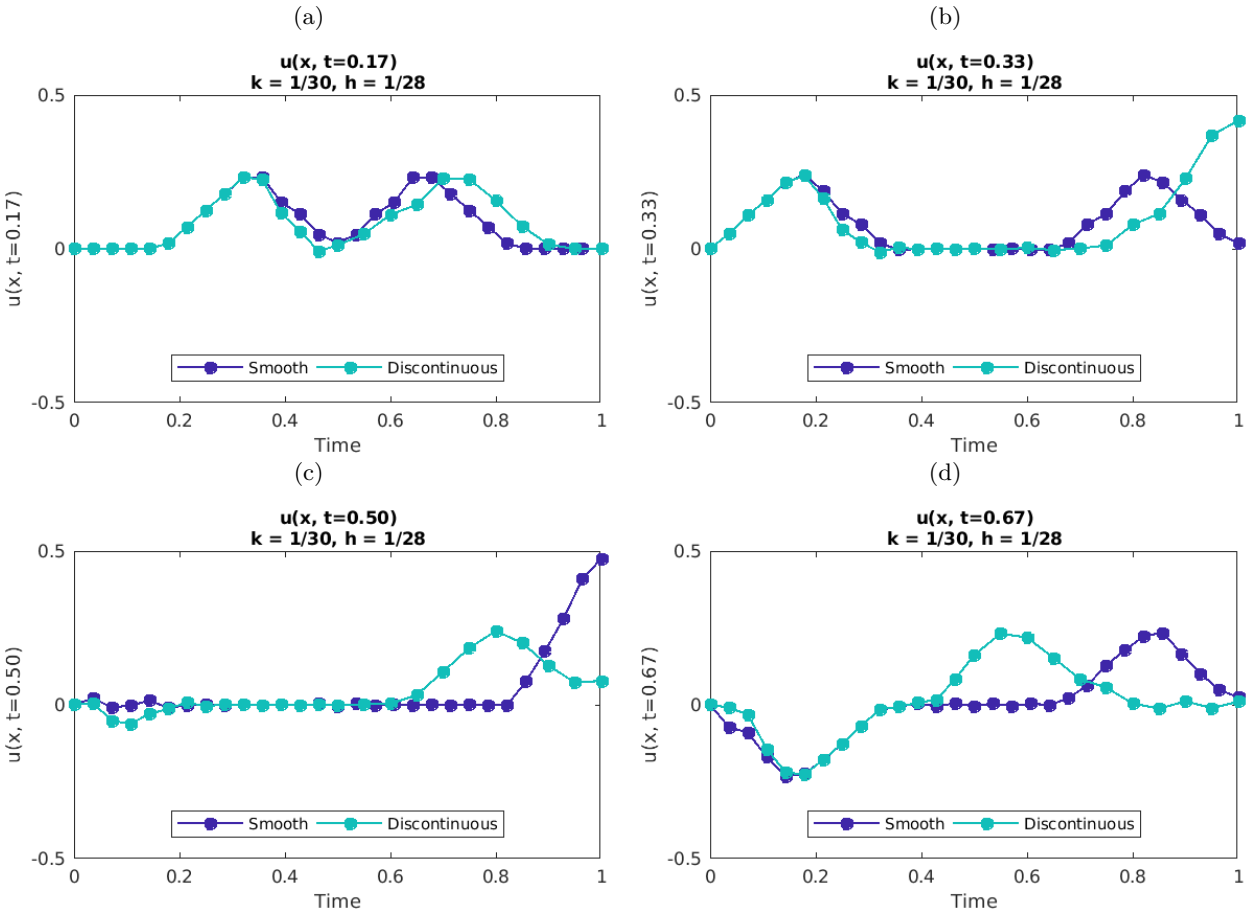
(b) $u(x, t = 0)$ for discontinuous α



Results

The difference in the solution using smooth and discontinuous α can be seen clearly in plots of the two functions at select time steps in Fig. 7.

Figure 7



When α is discontinuous, the pulse shifts such that it is no longer centralized at $x = 0.5$. This is clearly shown in Fig. 8, which shows $u(x = 0.5, t)$ in the case of both smooth and discontinuous α.

Fig. 9 compares the total energy as a function of time for both smooth and discontinuous α. This value is roughly consistent, but drops off when the waves interact with the boundaries. This is because at the boundary at $x = 1$, I used first-order central differences to apply the Neumann BC. This is a first order method, which hurts the overall order of the leapfrog method which is itself a second-order method in both time and space.

Figure 8

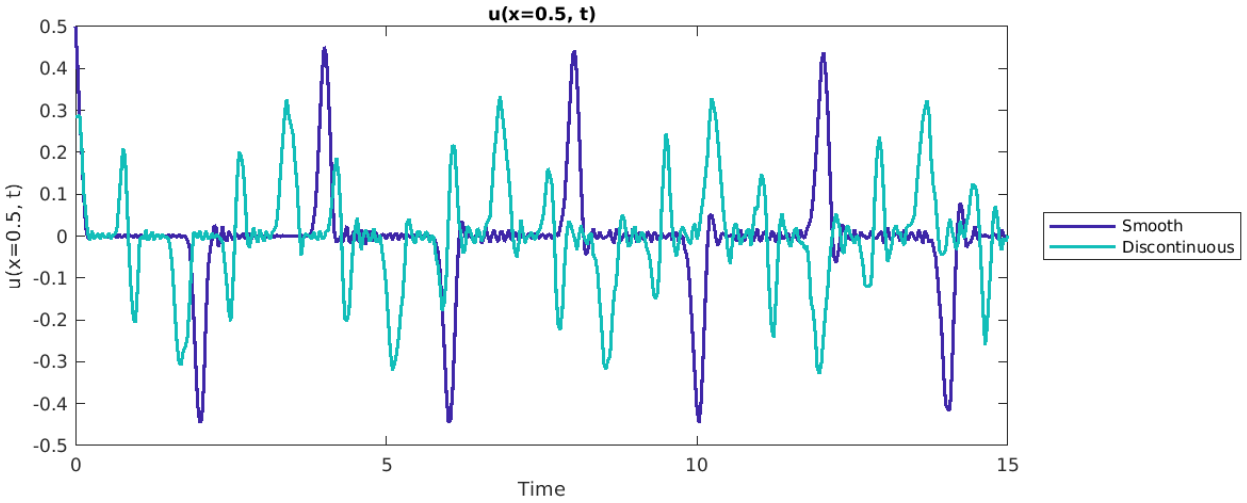
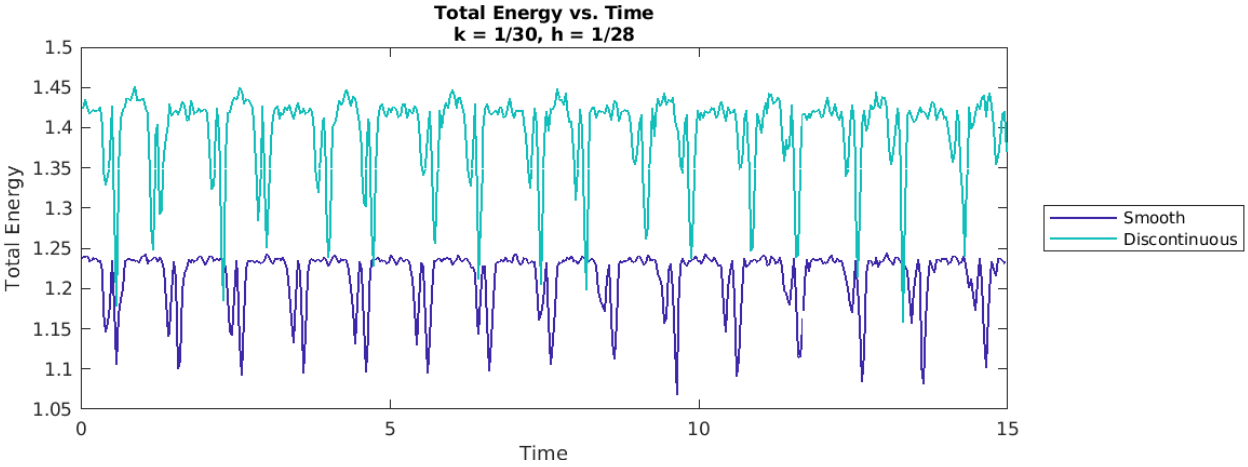


Figure 9



1.3 Lax-Wendroff

The Lax-Wendroff method for solving first-order PDEs can be applied to second-order PDEs by reformulating the problem as a system of coupled first-order PDEs [5]. The wave equation in Equation 1 can be expressed:

$$v_{tt} - \alpha^2 v_{xx} = 0$$
$$\left(\partial_t^2 - \alpha^2 \partial_x^2\right) v = \left(\partial_t - \alpha \partial_x\right)\left(\partial_t + \alpha \partial_x\right) v = 0$$

This now contains the first order wave equations $u_t = \pm \alpha u_x$, which have the solutions $u(x,t) = h(x \pm \alpha t)$. These equations describe characteristic curves along which the solution is always constant.

By defining $u_1 = u_t$ and $u_2 = u_x$, the PDE $u_{tt} = \alpha^2 u_{xx}$ can be split into a system of equations so that

the wave equation can be rewritten as Equation 7a with the initial condition given by Equation 7b.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \overbrace{\begin{pmatrix} 0 & \alpha^2 \\ 1 & 0 \end{pmatrix}}^{\equiv A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x \tag{7a}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} g(x) \\ f'(x) \end{pmatrix} \tag{7b}$$

The A matrix can be decomposed $A = T\Lambda T^{-1}$ where Λ contains A's eigenvalues and T contains the associated eigenvectors:

$$T = \begin{pmatrix} -\alpha & \alpha \\ 1 & 1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix}, \qquad T^{-1} = \frac{1}{2\alpha}\begin{pmatrix} -1 & \alpha \\ 1 & \alpha \end{pmatrix}$$

Now, Equation 1 can be written

$$\begin{aligned} \boldsymbol{u}_t &= T\Lambda T^{-1}\boldsymbol{u}_x \\ T^{-1}\boldsymbol{u}_t &= \Lambda T^{-1}\boldsymbol{u}_x \\ \left(T^{-1}\boldsymbol{u}\right)_t &= \Lambda\left(T^{-1}\boldsymbol{u}\right)_x \end{aligned}$$

This can be simplified further by defining $w \equiv T^{-1}u$ where w_1 is the right-travelling wave associated with the $-\alpha$ eigenvalue and w_2 is the corresponding left-travelling wave associated with the $+\alpha$ eigenvalue.

First Order Lax-Wendroff Method In general, the Lax-Wendroff method can be implemented to solve the first-order wave equation.

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + k\alpha D_0\boldsymbol{u}^n + \frac{k^2}{2}\alpha^2 D_+ D_-\boldsymbol{u}^n \tag{8}$$

Expanding the operators D_0, D_+, and D_-, the general equation for each element u_j^{n+1} is given by:

$$\begin{aligned} u_j^{n+1} &= u_j^n + k\alpha\left(\frac{u_{j+1}^n - u_{j-1}^n}{2h}\right) + \frac{k^2}{2}\alpha^2\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}\right) \\ &= \frac{\lambda}{2}\alpha\left(\lambda\alpha - 1\right)u_{j-1}^n + \left(1 - \lambda^2\alpha^2\right)u_j^n + \frac{\lambda}{2}\alpha\left(\lambda\alpha + 1\right)u_{j+1}^n \end{aligned}$$

where $\lambda \equiv k/h$.

This formulation of the problem can be applied to the transformed system of coupled first-order PDEs to solve for w. After each iteration, $v = Tw$ can be used to find the solution v to the coupled first-order PDEs, one of which then must be integrated to find the solution u to the second-order PDE.

Boundary Conditions Equation 8 applies to the center of the grid, and does not apply to the boundaries at $x = 0$ and $x = 1$. I formulated this problem such that w_1 solves the PDE in the case of $-\alpha$ and w_2 solves it for $+\alpha$. This means that w_1 travels right and w_2 travels left. At the $x = 0$ BC, w_2 can be calculated directly whereas w_1 must be calculated relative to w_2. At the $x = 1$ BC, the opposite is true: w_1 can be calculated directly and w_2 must be calculated relative to w_1.

When calculated directly using the characteristic lines,

$$(w_2)_0^{n+1} = (w_2)_1^n$$
$$(w_1)_H^{n+1} = (w_1)_{H-1}^n$$

$(w_1)_1$ and $(w_2)_H$ must be calculated relative to one another by applying the BCs:

$$u = Tw$$
$$\begin{pmatrix} v_t \\ v_x \end{pmatrix} = \begin{pmatrix} -\alpha & \alpha \\ 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$v_t = 0$ at $x = 0$ and $v_x = 0$ at $x = 1$. Solving these simultaneously using the above equations,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\alpha & \alpha \\ 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \implies \begin{array}{ll} x = 0, & w_1 = +w_2 \\ x = 1, & w_2 = -w_1 \end{array}$$

Integration The final solution can be found by integrating either u_t over time or u_x over space. Here, I chose to use the Forward Euler method, given in Equation 9, because this is easiest.

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + k(\boldsymbol{u}_t)^n \tag{9}$$

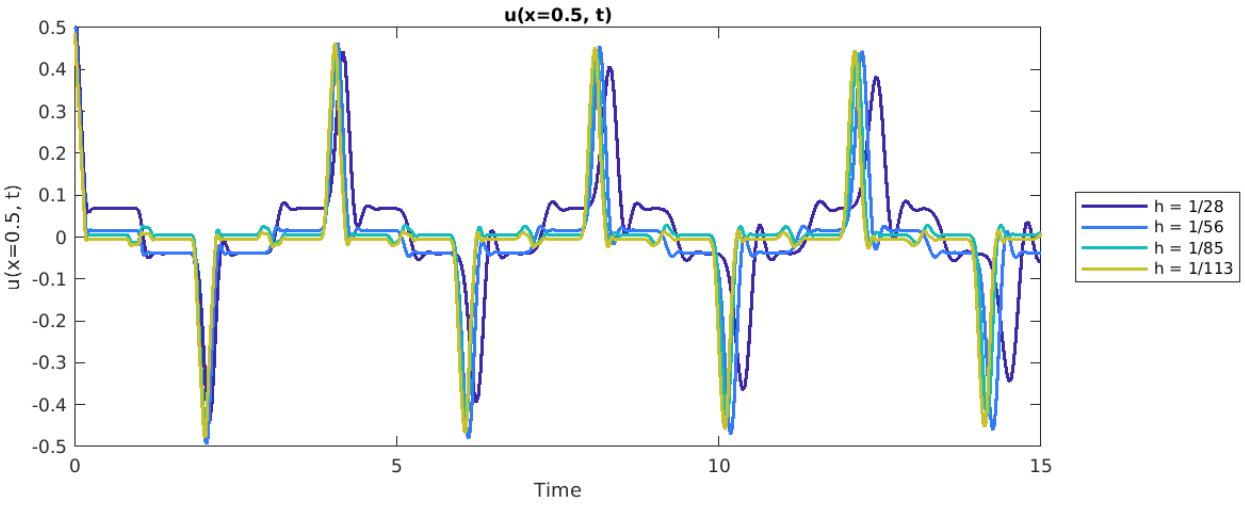
If I was looking for nicer results, I would need to consider that Lax Wendroff is second-order in both space and time, while Forward Euler is first-order in either space or time, depending on the end to which it is implemented. Thus, I *should* select a second-order integration scheme for the best possible results.

Results

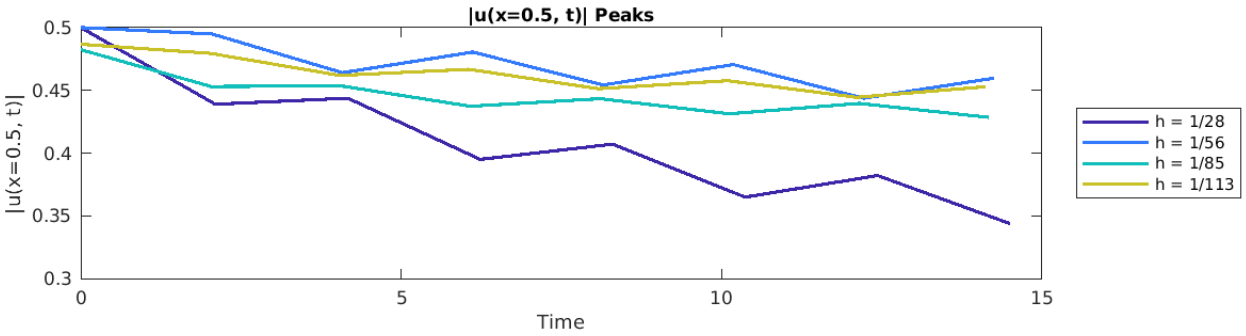
Of the three schemes I examined here, the solution found using this one experienced the most diffusion. I can see this here by examining the solution at the center point as a function of time for four different step sizes. Fig. 10(a) shows the solution at the center point and Fig. 10(b) shows the magnitude of each peak to show how the solution diffuses. It is also interesting to note that the location of the peak drifts later in time for larger step sizes.

Figure 10

(a)



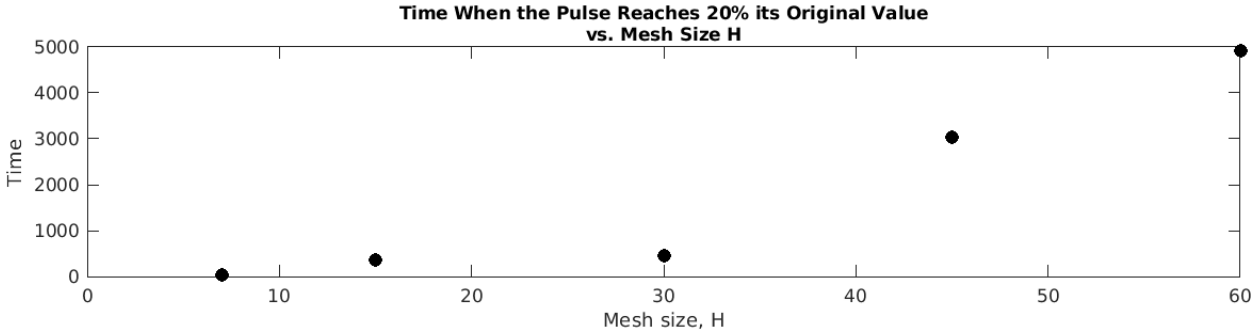
(b)



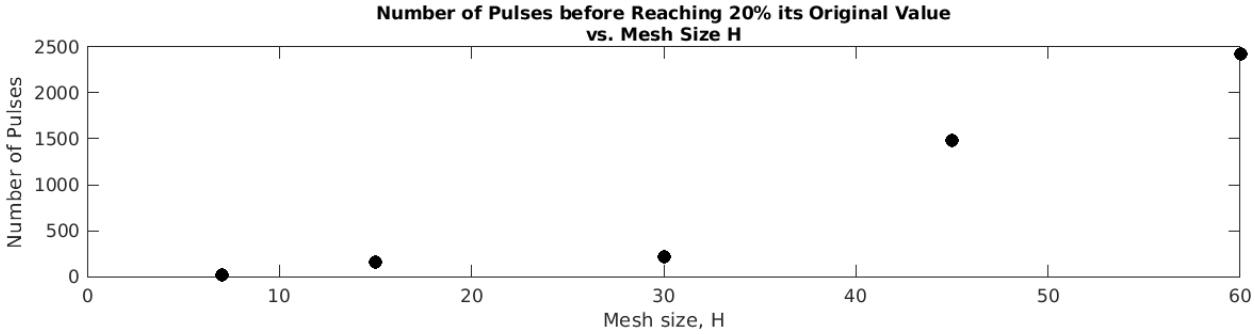
As expected, this diffusion becomes worse for larger step sizes. This diffusion becomes worse and worse until eventually, the pulse is gone. Fig. 11 shows the time and number of pulses it takes for the solution to reach 20% its original value for five different mesh sizes.

Figure 11: Pulse Diffusion

(a)



(b)



1.4 Method Comparison

Here I will directly compare the three schemes. Fig. 12 shows $u(x = 0.5, t)$ for each of the three schemes. Here it is possible to see that the method of characteristics solves the PDE exactly, leapfrog finds a solution close to the exact solution but with wiggles in between the peaks, and Lax Wendroff incurs diffusion and numerical error evident in the decreasing peak magnitude and oscillations between peaks. Fig. 13 shows the energy as a function of time for each method. The energy found using method of characteristics and the leapfrog schemes oscillates around the boundary conditions for each pulse but overall remains roughly constant, as expected. The Lax Wendroff scheme, however, diffuses the pulse, thus decreasing the total energy over time.

Figure 12

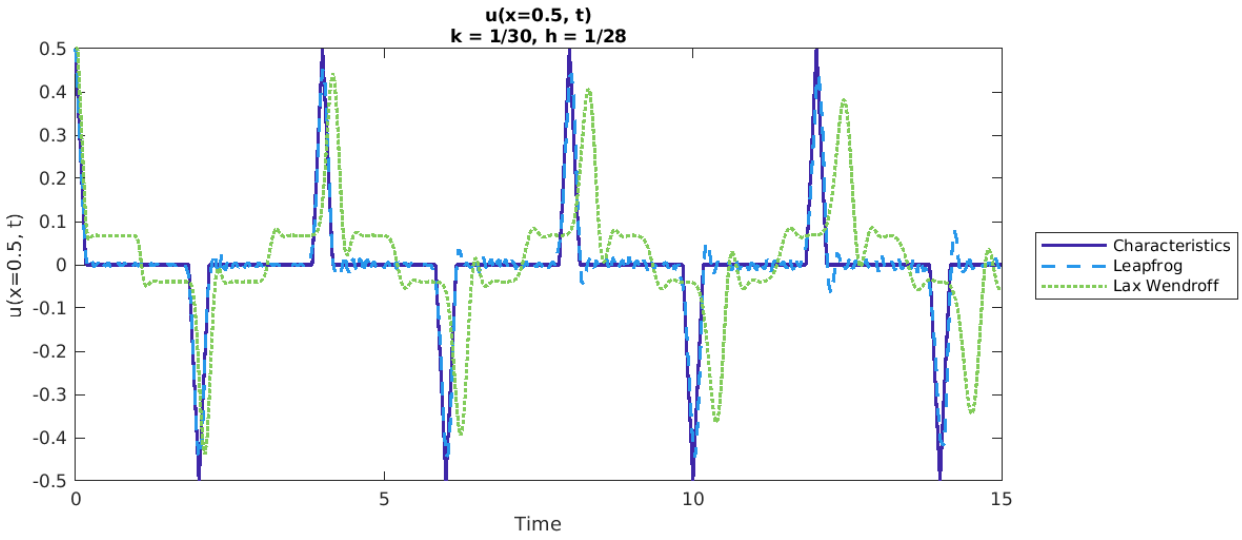
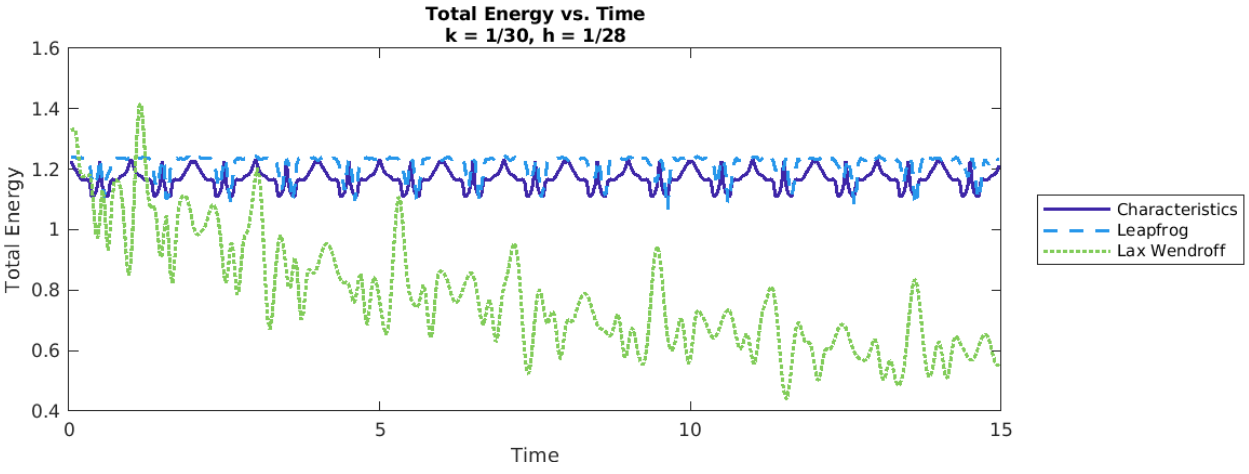


Figure 13



Method of Characteristics This method is great for *simple* PDEs in one dimension since it gives the exact solution to the problem, does not diffuse, and is fast. Imposing the BCs was extremely difficult even for this case where the left end of the string was fixed and the right side was free to oscillate.

Leapfrog This method also reached a well-behaved solution that incurs wiggles, but the pulse itself does not diffuse badly. The major downside to this method is that it is slow and requires working with large matrices. Although, of the three methods examined here, this one is the most straightforward to implement.

Lax Wendroff This scheme was difficult for me personally to understand, but now that I understand it, actually implementing it was not difficult. This method has several major downsides. It requires integrating to find the solution to the second-order PDE, thus incurring additional numerical error. In this case, I used Forward Euler to integrate, losing an order of accuracy in the process. This would have to be examined in future iterations of this method.

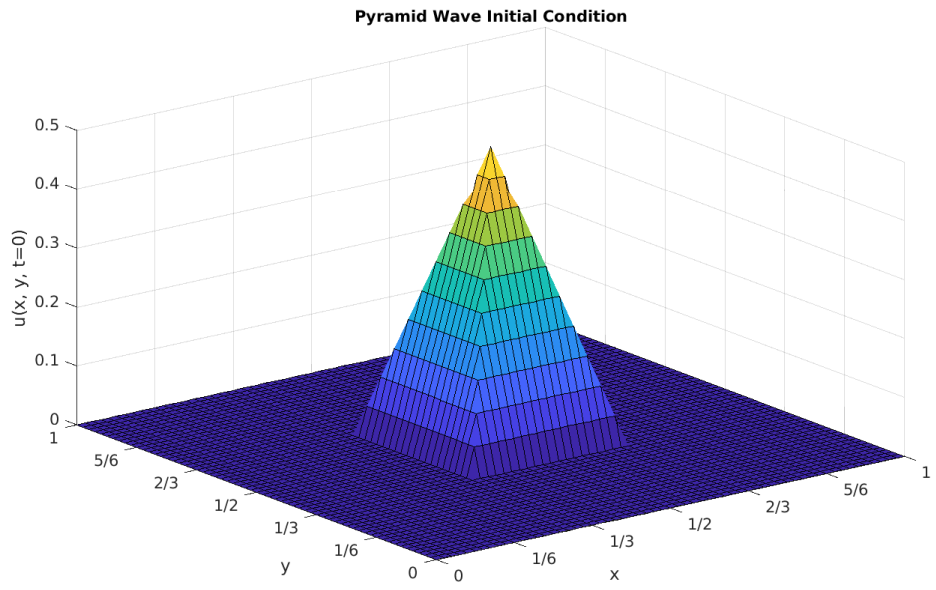
2 Two-Dimensional Wave Equation

In two dimensions, the wave equation is expressed as Equation 10

$$u_{tt} = \alpha^2 \nabla^2 u = \alpha^2 \left(u_{xx} + u_{yy} \right) \tag{10}$$

with a pyramid wave initial condition shown in Fig. 14 and a Dirichlet BC $u = 0$.

Figure 14



2.1 Leapfrog

I applied expanded my approach to the leapfrog method in Section 1.2 to apply to solve in both x and y, using central differences to approximate each second derivative.

$$\left(\frac{u_{i,j}^{n-1} - 2u_{i,j}^{n} + u_{i,j}^{n+1}}{k^2}\right) = \alpha^2\left(\frac{u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i+1,j}^{n}}{h^2}\right) + \alpha^2\left(\frac{u_{i,j-1}^{n} - 2u_{i,j}^{n} + u_{i,j+1}^{n}}{h^2}\right)$$

$$u_{i,j}^{n+1} = \left[su_{i,j-1}^{n} + \left(+su_{i-1,j}^{n} + 2(1-2s)u_{i,j}^{n} + su_{i+1,j}^{n}\right) + su_{i,j-1}^{n}\right] - u_{i,j}^{n-1} \tag{11}$$

2.2 Results

I analyzed the solutions by examining the potential, kinetic, and total energies of the waves as functions of time. Additionally, I examined the height of the wave at the center point. I chose this point to see how the wave diffused as a function of time, if at all.

Energy The solution wave has both potential and kinetic energy, given by U and T calculated using Equations 2a and 12 [1], where the total energy of the system can be found by adding these two values.

$$T = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2 dx \tag{2a}$$

$$U = \frac{\alpha^2}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(u_x^2 + u_y^2\right)dxdy \tag{12}$$

I approximated u_t, u_x, and u_y numerically using central differences and integrated using the trapezoidal rule in two dimensions [6]. Fig. 15(a) shows the energy as a function of time for the leapfrog method and Fig. 15(b) shows the center solution as a function of time for the leapfrog method.

Figure 15

(a) Energy vs. time for the leapfrog method

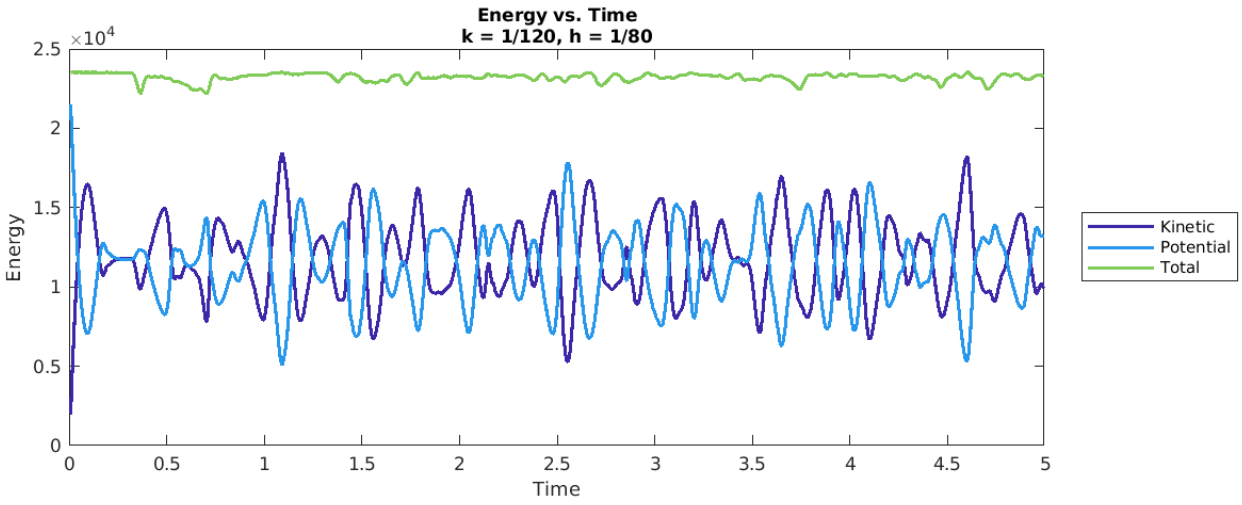
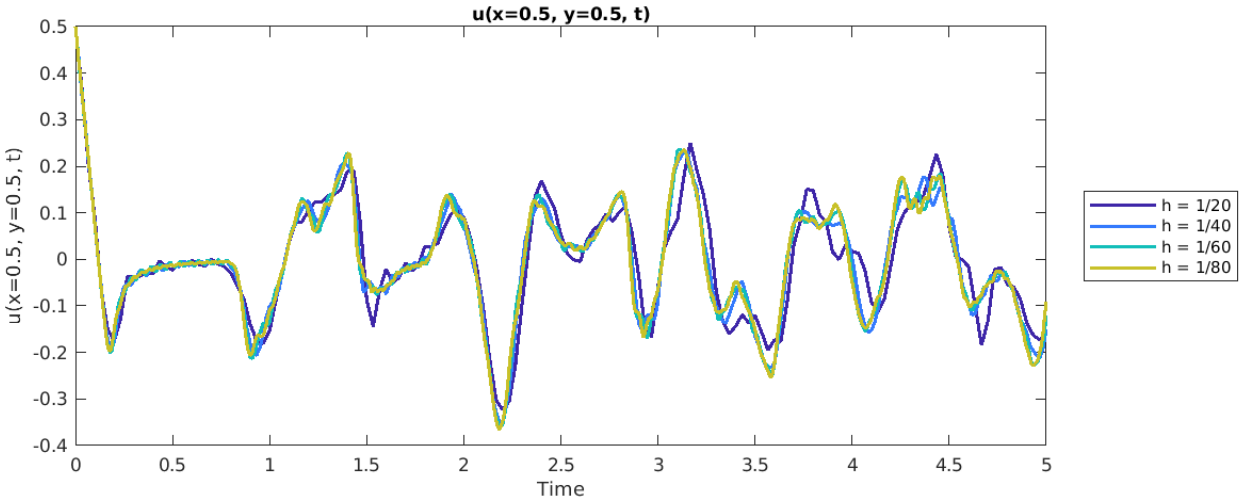


Fig. 15(a) shows that the total energy remains consistent throughout the problem. This supports my theory that the oscillations in total energy when the wave interacts with the $x = 1$ boundary result from

(b) Center solution vs. time for the leapfrog method



using a first-order method (central differences) to apply the Neumann BC, as described in Section 1.2. Since this two-dimensional problem has only Dirichlet BCs, this is not an issue.

Fig. 15(b) shows $u(x = 0.5, y = 0.5, t)$ for four different step sizes. For every case, it seems that the wave does not "pulse" like it did in the one-dimensional problem, but rather smooths out quickly, becoming unrecognizable after $t \approx 2$. It would be interesting to compare this scheme with another scheme such as Lax Wendroff or the method of characteristics in two dimensions.

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