NE 255: Numerical Simulation in Radiation Transport
Homework #6

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1. Derive sampling algorithms for the three following situations. For reference, the continuous and discrete probability distribution function (PDF) and continuous distribution function (CDF)

	Continuous	Discrete
PDF	$p\{a \le x \le b\} = \int_a^b p(x)dx$ $p(x) \ge 0$ $\int_{-\infty}^{\infty} p(x)dx = 1$	$p(x = x_k) = p_k = p(x_k)$ $k = 1, \dots, N$ $p_k \ge 0$ $\sum_{k=1}^{N} p_k = 1$
CDF	$P\{x' \le x\} = P(x) = \int_{-\infty}^{x} p(x')dx'$ $P(-\infty) = 0, P(\infty) = 1$ $0 \le P(x) \le 1$ $\frac{dP}{dx} \ge 0$	$P\{x' \le x\} = P_k = P(x_k) = \sum_{j=1}^{k} P_j$ $k = 1, \dots, N$ $P_0 = 0, P_N = 1$ $0 \le P_k \le 1$ $P_k \ge P_{k-1}$

(a) The neutron direction in 3D if the neutron source is isotropic.

Referencing Professor Vujic's NE 250 notes . . .

$$\frac{d\hat{\mathbf{\Omega}}}{4\pi} = -\frac{d(\cos\theta)d\phi}{4\pi} = -\frac{d\mu d\phi}{4\pi} \tag{1}$$

Since μ and ϕ are independent, the PDF and CDF are separable such that $p(\hat{\mathbf{\Omega}}) = p_1(\mu)p_2(\phi)$ and $P(\hat{\mathbf{\Omega}}) = P_1(\mu)P_2(\phi)$. The PDFs and corresponding CDFs are given below:

$$p_1(\mu) = \frac{1}{2}, \quad \mu \in [-1, 1]$$
$$P_1(\mu) = \int_{-1}^{\mu} p_1(\mu')d\mu' = \frac{1}{2}(\mu + 1) = \xi_1$$

$$p_2(\phi)\frac{1}{2\pi}, \quad \phi \in [0, 2\pi]$$
$$P_2(\phi) = \int_0^{\phi} p_2(\phi')d\phi' = \frac{\phi}{2\pi} = \xi_2$$

Solving for ξ_1 and ξ_2 in terms of μ and ϕ:

$$\phi = 2\pi\xi_2$$
$$\mu = 2\xi_1 - 1$$

It is possible to invert these values to solve for a unit velocity with Cartesian coordinates (μ, η, ζ). The remaining coordinates, η and ϕ are given in terms of μ and ϕ below:

$$\eta = \sqrt{1 - \mu^2}\cos(\phi)$$
$$\zeta = \sqrt{1 - \mu^2}\sin(\phi)$$

The following algorithm can be used to implement this:

function SAMPLE
  $(\xi_i, \xi_2) \leftarrow$ random numbers $\in [0, 1]$
  $\phi_i \leftarrow 2\pi\xi_2$ ▷ Find angle ϕ_i
  $\mu_i \leftarrow 2\xi_1 - 1$ ▷ Find Cartesian coordinates (μ_i, η_i, ζ_i)
  $\eta_i \leftarrow \sqrt{1 - \mu_i^2}\cos\phi_i$
  $\zeta_i \leftarrow \sqrt{1 - \mu_i^2}\sin\phi_i$
  return (μ, η, ζ)
end function

(b) The distance to the next collision in the direction of neutron motion if the neutron is in the center of the spherical volume that consists of three concentric layers with radii R_1, R_2, and R_3, each made of different materials with total cross sections Σ_{t1}, Σ_{t2}, and Σ_{t3}, respectively.

Referencing Professor Slaybaugh's NE 255 notes . . .

$$p_c(s) = \Sigma_t(s)(s)e^{-\Sigma_t(s)s}$$
$$P_c(s) = \int_0^s \Sigma_t(s)e^{-\Sigma_t(s)s'}ds' = 1 - e^{-\Sigma_t(s)s}$$

Substitute $n = \Sigma_t(s)s, dn = \Sigma_t(s)ds$ to transform variables and measure in units of "number of mean free path," n_c until the next collision c.

$$p_c(n_c) = e^{-n_c}$$
$$P_c(n_c) = \int_0^{n_c} e^{-n_c'}dn_c' = 1 - e^{-n_c} = \xi$$

Invert to find n and the distance between collisions s_c:

$$n == -\ln(1-\xi)$$
$$s = \frac{n}{\Sigma_t}$$

For this problem, there are concentric layers with radii R_1, R_2, and R_3, each made of different materials with total cross sections Σ_{t1}, Σ_{t2}, and Σ_{t3}, respectively. The numbers of mean free paths n_i from the inner boundary to the outer boundary of each layer $i = 1, 2, 3$ can be found as follows:

$$n_{bi} = R_i \Sigma_{ti}$$

Require: R_1, R_2, R_3 and $\Sigma_{t1}, \Sigma_{t2}, \Sigma_{t3}$ are given

```
n_b1 ← R_1 Σ_t1
n_b2 ← R_2 Σ_t2
n_b3 ← R_3 Σ_t3
function SAMPLE(n_b1, n_b2, n_b3)
    if n_c < n_b1 then a collision occurs
        Sample for a new n_c
        n_b ← s_b Σ_t                          ▷ s_b is in the direction of the boundary
    elsea boundary crossing occurs
        n_c ← n_c − n_b
    end if
end function
```

(c) The type of collision if it is assumed that the neutron can have both elastic and inelastic scattering and can be absorbed in fission or (n, γ) capture interactions. Assume monoenergetic neutron transport.

From Professor Slaybaugh's NE 255 notes ...

$$\Sigma_{tj} = \sum_{x=1}^{R} \Sigma_{xj} = \sum_{x=1}^{4} \Sigma_{xj}$$

Where j refers to which nuclide in the material and x refers to the type of scattering that occurs. A discrete PDF can be used to determine which reaction occurs:

2. A Monte Carlo rejection method can be used to estimate $\pi \sim 3.141\,59$

```
for i = 1 → N do
    (x_i, y_i) ← random numbers ∈ [0, 1]
    x_i ← (b − a)x_i + a
    y_i ← g(x_i)y_i
    if y_i < f(x_i) then
        accept (x_i, y_i)
        count ← count + 1
    else
```

reject (x_i, y_i)

end if

end for

return $\pi \leftarrow 4(count/N)$

(a) Eq. (4) can be used to approximate π.

$$\pi = 4\int_a^b p(x)dx = 4\int_0^1 \sqrt{1-x^2}dx \tag{2}$$

(a) $N = 10$ (b) $N = 100$ (c) $N = 1000$ (d) $N = 10\,000$

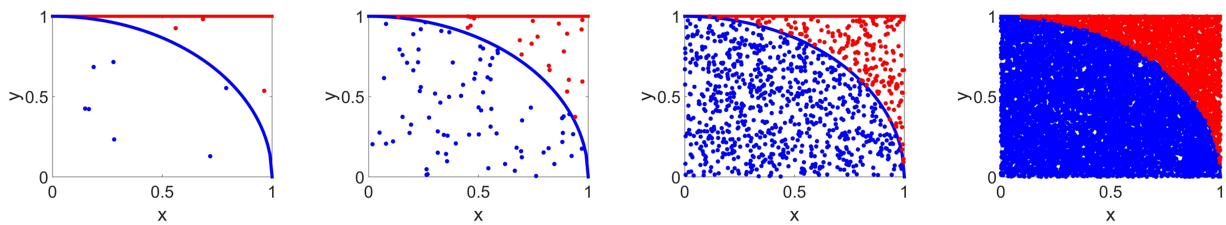

Figure 1: $\pi/4$ estimated using Eq. (4)

(b) Eq. (5) can be used to approximate π.

$$\pi = 4\int_a^b q(x)dx = 4\int_0^1 \frac{1}{1+x^2} \tag{3}$$

(a) $N = 10$ (b) $N = 100$ (c) $N = 1000$ (d) $N = 10\,000$

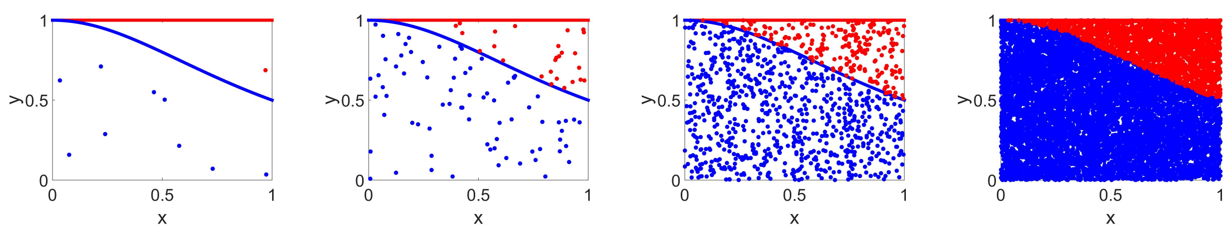

Figure 2: $\pi/4$ estimated using Eq. (5)

Table 1: Relative Error

Samples	Error	
	$p(x) = \sqrt{1-x^2}$	$q(x) = (1+x^2)^{-1}$
10	0.1087	0.1459
100	0.0069	0.0576
1000	0.0250	0.0056
10 000	0.0027	0.0014

(c)

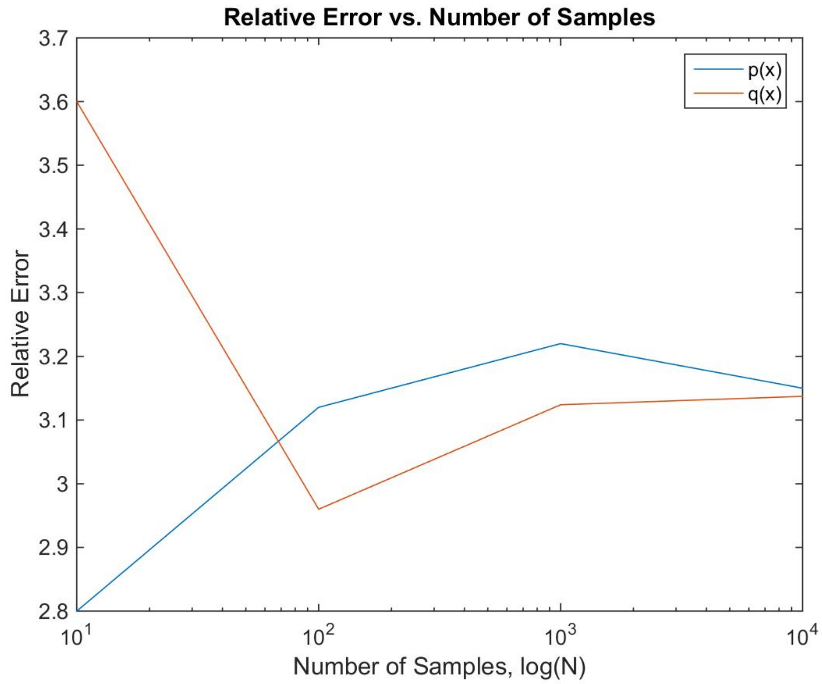

Figure 3: Relative error vs. N

(d) The relative error decreases as a function of the number of samples, N