APMA1930V Final Project

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1 Introduction

Options are financial derivatives that give buyers the right to buy or sell assets at a certain price, called the strike price, at a later time in the future. There are various types of options that differ in when the right to buy or sell can be exercised, how the payoff is determined, or in other ways. Here, we focus call options, specifically European call options, which only allow the buyer to execute the trade at a specified time, and Asian call options, whose payoff is determined by the average price of the asset over a determined period of time. We want to determine how to price these options, and one particular method is to calculate the payoff of acquiring such an option and pricing accordingly.

While some styles of call options have analytic solutions to the expected payoff, others do not, or are quite complicated to compute, thus making simulation a helpful technique for option pricing. If we let

- T be the expiration date of the option
- K be strike price
- $B(t)$ be a Brownian motion
- μ be the drift parameter
- σ be the volatility

then we can model the price of an asset across a time period of length T using a geometric Brownian motion

$$S(t) = e^{\mu + \sigma B(t)}.$$

This can then be discretized over time points $0 = t_0 < \cdots < t_k = T$ by letting

$$L_i = e^{\mu(t_i - t_{i-1}) + \sigma\sqrt{t_i - t_{i-1}} Z_i}$$

$$S(t_j) = \prod_{i=1}^{j} L_j$$

where $Z_i \sim N(0, 1)$.

Here, we look specifically at how to simulate the payoff of European and Asian call options, where in the case of the European call option we want to estimate

$$\mathbb{E}[(S(T) - K)^{+}],$$

while in the case of the Asian call option, we want to estimate

$$\mathbb{E}\left[\left(\frac{1}{k}\sum_{i=1}^{k} S(t_i) - K\right)^{+}\right].$$

2 European Call Option

We can determine the payoff of a European call option by determining

$$\mathbb{E}[(S(T) - K)^{+}].$$

Here we explore three different methods of estimating this expected value: standard Monte Carlo, antithetic sampling, and importance sampling. In each of the simulations, the following parameters were used:

- $T = 1$
- $K = 1.5$
- $\mu = 0.01$
- $n = 10^6$

With these choices of parameters, $e^{\mu T} < K$.

2.1 Naive Monte Carlo

With straightforward Monte Carlo, we can leverage the fact that as $n \to \infty$, $\frac{1}{n}\sum_{i=1}^{n} H(X_i) \to E[H(X)]$, where X_i are identically and independently distributed from the same distribution as X, by the law of large numbers. Then by drawing independent samples X_i from the distribution of X, we can approximate $E[H(X)]$ with $\frac{1}{n}\sum_{i=1}^{n} H(X_i)$. In this case, $H(X) = (S(T) - K)^{+}$.

To simulate $S(T)$, n samples were first randomly drawn from the $N(0, 1)$ distribution, as $B(T) = B(T) - B(0) \sim N(0, T)$ so $\sqrt{T}X \sim N(0, T)$. Then $(S(T))_i = e^{\mu T + \sigma\sqrt{T}X_i}$, where X_i is our i-th randomly drawn sample. From there, the final estimator for $E[S(T) - K)^{+}]$ is

$$\hat{l}_0 = \frac{1}{n}\sum_{i=1}^{n}((S(T))_i - K)^{+}.$$

This procedure was carried out for various levels of sigma, ranging from 0.01 to 2. The following results were obtained:

σ	est	var	95% CI	relative error
2.00	6.5872	0.0035	(6.4713, 6.7030)	0.0090
1.00	0.6933	3.8120×10^{-6}	(0.6895, 0.6972)	0.0027
0.50	0.1197	1.2439×10^{-7}	(0.1190, 0.1204)	0.0029
0.25	0.0101	3.3693×10^{-9}	(0.0100, 0.0102)	0.0058
0.20	0.0029	6.6925×10^{-10}	(0.0029, 0.0030)	0.0089
0.15	3.0044×10^{-4}	4.1942×10^{-11}	(0.0003, 0.0003)	0.0216
0.10	1.6572×10^{-6}	1.1030×10^{-13}	(0.0000, 0.0000)	0.2004
0.05	0	0	(0, 0)	NaN
0.02	0	0	(0, 0)	NaN
0.01	0	0	(0, 0)	NaN

Table 1: The estimate, sample variance, 95% confidence intervals, and relative error of the expected payoff of a European call option for various σ values using straight-forward Monte Carlo simulation. NaN means the relative error could not be calculated due to the estimate being 0.

When the relative error is *"NaN"*, this means that our estimate is exactly 0, so the relative error cannot be calculated. We can see that for small values of σ, our straightforward Monte Carlo simulation breaks down and is unable to produce a non-zero estimate. As σ gets smaller, $\sigma\sqrt{T}$ gets smaller, so we need larger X to have $e^{\mu T + \sigma\sqrt{T}X} > K$ and thus a non-zero estimate, since $e^{\mu T} < K$. But the probability of large X, especially in the tail of the normal distribution has a low density, thus making the event that $e^{\mu T + \sigma\sqrt{T}X} > K$ rare. This means as σ gets smaller, more samples would need to be generated to get non-zero samples and thus a more accurate estimate.

2.2 Antithetic Sampling

In order to improve variance, another method used was antithetic sampling. With straightforward Monte Carlo, we get that the variance of the estimator is

$$Var\left[\frac{1}{n}\sum_{i=1}^{n} H(X_i)\right] = \frac{1}{n}\sum_{i=1}^{n} Var[H(X)]$$

.

However, if we choose $Y_i \sim Y$ such that each Y has the same distribution as X, then we can see that

$$\begin{aligned} E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{H(X)+H(Y)}{2}\right] &= \frac{1}{2n}\sum_{i=1}^{n}\left(E\left[H(X_i)\right] + E\left[H(Y_i)\right]\right) \\ &= \frac{1}{2n}\sum_{i=1}^{n} 2E[H(X)] = E[H(X)]. \end{aligned}$$

Furthermore,

$$\begin{aligned}
&Var\left[\frac{1}{n}\sum_{i=1}^{n}\frac{H(X)+H(Y)}{2}\right] \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} Var\left[H(X_i)+H(Y_i)\right] \\
&= \frac{1}{4n}\sum_{i=1}^{n}(Var[H(X)]+Var[H(Y)]+2Cov(H(X),H(Y))) \\
&= \frac{1}{4n}\sum_{i=1}^{n}(2Var[H(X)]+2Cov(H(X),H(Y))) \\
&= \frac{1}{2n}\sum_{i=1}^{n}(Var[H(X)]+Cov(H(X),H(Y))) \\
&= \frac{1}{2n}\sum_{i=1}^{n}Var[H(X)]+\frac{1}{2n}\sum_{i=1}^{n}Cov(H(X),H(Y))
\end{aligned}$$

so if X and Y are negatively correlated, $Cov(H(X),H(Y)) \leq 0$ and we get

$$Var\left[\frac{1}{n}\sum_{i=1}^{n}\frac{H(X)+H(Y)}{2}\right] \leq \frac{1}{2n}\sum_{i=1}^{n}Var[H(X)],$$

which reduces the variance of our estimator by at least two-fold as compared to our naive Monte Carlo estimator.

For our case here, since $X \sim N(0,1)$ and $-X \sim N(0,1)$, we can just choose $Y = -X$. Then letting $S_X(T) = e^{\mu T+\sigma\sqrt{T}X}$ and $S_Y(T) = e^{\mu T+\sigma\sqrt{T}Y} = e^{\mu T-\sigma\sqrt{T}X}$, $S_X(T)$ and $S_Y(T)$ are negatively correlated since increasing $S_X(T)$ means increasing X_i, which means decreasing $S_Y(T)$. Then since K is fixed here, as $(S_X(T)-K)^+$ increases, $(S_Y(T)-K)^+$ must decrease. This suggests negative correlation between $H(X)$ and $H(-X)$ and thus a negative covariance.

Then similar to the naive Monte Carlo method, we can first randomly generate n samples from $N(0,1)$ and calculate $(S_X(T))_i = e^{\mu T+\sigma\sqrt{T}X_i}$. However, this time, we also calculate $(S_Y(T))_i = e^{\mu T-\sigma\sqrt{T}X_i}$. Then our estimator for $\mathbb{E}[(S(T)-K)^+]$ becomes

$$\hat{l}_1 = \frac{1}{n}\sum_{i=1}^{n}\frac{((S_X(T))_i-K)^+ + ((S_Y(T))_i-K)^+}{2}.$$

Our results show that the relative error is smaller, while the estimates are roughly the same, so we do indeed get a reduction in the variance with antithetic sampling.

	Naive MC		Antithetic Sampling	
σ	**estimate**	**var**	**estimate**	**var**
2.00	6.5872	0.0035	6.5363	0.0014
1.00	0.6933	3.8120×10^{-6}	0.6903	1.6480×10^{-6}
0.50	0.1197	1.2439×10^{-7}	0.1195	5.5000×10^{-8}
0.25	0.0101	3.3693×10^{-9}	0.0100	1.6205×10^{-9}
0.20	0.0029	6.6925×10^{-10}	0.0029	3.2670×10^{-10}
0.15	3.0044×10^{-4}	4.1942×10^{-11}	3.0114×10^{-4}	2.0666×10^{-11}
0.10	1.6572×10^{-6}	1.1030×10^{-13}	1.4884×10^{-6}	4.1733×10^{-14}
0.05	0	0	0	0
0.02	0	0	0	0
0.01	0	0	0	0

Table 2: The estimate and sample variance for the expected payoff of a European call option for various σ values using straight-forward Monte Carlo simulation vs antithetic sampling.

	Naive MC		Antithetic Sampling	
σ	**95% CI**	**rel err**	**95% CI**	**rel err**
2.00	(6.4713, 6.7030)	0.0090	(6.4633, 6.6094)	0.0057
1.00	(0.6895, 0.6972)	0.0027	(0.6877, 0.6928)	0.0019
0.50	(0.1190, 0.1204)	0.0029	(0.1190, 0.1200)	0.0020
0.25	(0.0100, 0.0102)	0.0058	(0.0100, 0.0101)	0.0040
0.20	(0.0029, 0.0030)	0.0089	(0.0028, 0.0029)	0.0063
0.15	(0.0003, 0.0003)	0.0216	(0.0003, 0.0003)	0.0151
0.10	(0.0000, 0.0000)	0.2004	(0.0000, 0.0000)	0.1373
0.05	(0, 0)	NaN	(0, 0)	NaN
0.02	(0, 0)	NaN	(0, 0)	NaN
0.01	(0, 0)	NaN	(0, 0)	NaN

Table 3: The 95% confidence intervals and relative errors of the estimate of the expected payoff of a European call option for various σ values using straight-forward Monte Carlo simulation vs antithetic sampling.

We can see that while the relative error is reduced antithetic sampling also breaks down for small σ, around the same levels as with the naive Monte Carlo method. Since $X, Y = -X \sim N(0,1)$, to get a non-zero sample, we would need $S_X(T) > K$ or $S_Y(T) > K \Rightarrow |X| > \frac{\ln(K)-\mu T}{\sigma\sqrt{T}}$, which has low density for small σ. Antithetic sampling here would also require generating many samples for small values of σ.

2.3 Importance Sampling

Importance sampling is another method used when traditional Monte Carlo fails. With traditional Monte Carlo, we are trying to estimate $E_f[H(X)]$, where H is some function of the random variable X and f refers to the pdf of the distribution of X. Taking Ω to be the sample space of X, we can note that

$$E_f[H(X)] = \int_\Omega H(x)f(x)dx = \int_\Omega \frac{H(x)f(x)}{g(x)} g(x)dx = E_g\left[\frac{H(X)f(X)}{g(X)}\right],$$

where g is the pdf of some alternative distribution.

Then in the cases where the event of interest is rare, we can instead choose to sample from an alternative distribution g that makes this rare event more common. In this case here, we are interested in the event $(S_T - K)^+ > 0$ or $S_T > K$, since only then will we will get a non-zero estimate.

With exponential families, a natural choice is often the exponential tilt of that family. In the case of $N(0,1)$, the exponential tilt would be $g^*(x) \sim N(\theta, 1)$, where, based on the Gibbs conditioning principle, θ is chosen such that $g^*(x)$ resembles $f(x|S_T > K)$, or the conditional distribution given the rare event occurs:

$$\begin{aligned} f(x|S_T > K) &\propto f(S_T > K|x)f(x) \\ &= f(x)\mathbb{1}_{S_T > K} \\ &= f(x)\mathbb{1}_{e^{\mu T + \sigma\sqrt{T}x} > K} \\ &= f(x)\mathbb{1}_{x > \frac{\ln(K)-\mu T}{\sigma\sqrt{T}}} \end{aligned}$$

Since $e^{\mu T} < K$, we can see $f(x|S_T > K) > 0$ only for $x > \frac{\ln(K)-\mu T}{\sigma\sqrt{T}} > 0$. Then since $f(x)$ monotonically decreases on $(0, \infty)$, $f(x|S_T > K)$ is greatest when $x = \frac{\ln(K)-\mu T}{\sigma\sqrt{T}}$. Then a good choice of θ would be $\frac{\ln(K)-\mu T}{\sigma\sqrt{T}}$, giving us $g^* \sim N\left(\frac{\ln(K)-\mu T}{\sigma\sqrt{T}}, 1\right)$.

Thus, our estimator becomes

$$\hat{l}_2 = \frac{1}{n}\sum_{i=1}^{n} \frac{((S(T))_i - K)^+ f(x_i)}{g^*(x_i)}.$$

We can see from the results, that indeed, importance sampling fares better in that we are able to get a non-zero estimate for smaller σ.

	Naive MC		Importance Sampling	
σ	**estimate**	**var**	**estimate**	**var**
2.00	6.5872	0.0035	6.5334	0.0013
1.00	0.6933	3.8120×10^{-6}	0.6895	1.3293×10^{-6}
0.50	0.1197	1.2439×10^{-7}	0.1195	2.0838×10^{-8}
0.25	0.0101	3.3693×10^{-9}	0.0101	1.2536×10^{-10}
0.20	0.0029	6.6925×10^{-10}	0.0029	1.1047×10^{-11}
0.15	3.0044×10^{-4}	4.1942×10^{-11}	3.0618×10^{-4}	1.4224×10^{-13}
0.10	1.6572×10^{-6}	1.1030×10^{-13}	1.3411×10^{-6}	3.7678×10^{-18}
0.05	0	0	1.2007×10^{-17}	6.1634×10^{-40}
0.02	0	0	3.8145×10^{-90}	1.6801×10^{-184}
0.01	0	0	0	0

Table 4: The estimate and sample variance for the expected payoff of a European call option for various σ values using straight-forward Monte Carlo simulation vs importance sampling.

	Naive MC		Importance Sampling	
σ	**95% CI**	**rel err**	**95% CI**	**rel err**
2.00	(6.5076, 6.7114)	0.0090	(6.4641, 6.6027)	0.0054
1.00	(0.6884, 0.6960)	0.0028	(0.6873, 0.6918)	0.0017
0.50	(0.1185, 0.1199)	0.0029	(0.1192, 0.1197)	0.0012
0.25	(0.0100, 0.0103)	0.0058	(0.0100, 0.0101)	0.0011
0.20	(0.0029, 0.0030)	0.0089	(0.0029, 0.0029)	0.0011
0.15	(0.0003, 0.0003)	0.0216	(0.0003, 0.0003)	0.0012
0.10	(0.0000, 0.0000)	0.2004	(0.0000,0.0000)	0.0014
0.05	(0, 0)	NaN	(0.0000,0.0000)	0.0021
0.02	(0, 0)	NaN	(0.0000,0.0000)	0.0034
0.01	(0, 0)	NaN	(0, 0)	NaN

Table 5: The 95% confidence intervals and relative errors of the estimate of the expected payoff of a European call option for various σ values using straight-forward Monte Carlo simulation vs importance sampling.

Since we are making larger values of X more likely, we can get non-zero samples, and thus non-zero estimates, for smaller values of σ, than we are able to with antithetic sampling and naive Monte Carlo. Still, though, importance sampling fails for small enough σ, in this case here, for $\sigma = 0.01$.

2.4 Summary

We can see that overall, importance sampling performed best of the three methods, and was able to provide a non-zero estimate for smaller σ values than naive Monte Carlo or Monte Carlo with antithetic sampling. Antithetic sampling, however, was able to reduce the relative error compared to naive Monte Carlo.

3 Asian Call Option

For an Asian call option, we want to determine

$$\mathbb{E}\left[\left(\frac{1}{k}\sum_{i=1}^{k} S(t_i) - K\right)^+\right].$$

Again, we explore naive Monte Carlo, antithetic sampling, and importance sampling as three possible methods of estimating the expected payoff. We will use the same parameters as we did for estimating the European call option payoff with the additional parameters:

- $T = 1$
- $K = 1.5$
- $\mu = 0.01$
- $n = 10^6$
- $k = 5$
- $\Delta \coloneqq t_i - t_{i-1} = \frac{T}{k}$ for $i = 1, 2, ..., k$

With these parameters, $\sum_{i=1}^{k} e^{\mu t_i} < kK$.

3.1 Naive Monte Carlo

For determining the payoff for an Asian call option, we can take

$$H(\mathbf{X}) = \left(\frac{1}{k}\sum_{j=1}^{n} S(t_j) - K\right)^+,$$

where $\mathbf{X} = (X_1, X_2, ..., X_k)$. Then applying a similar method as with the European call option, we can simulate samples of $S(t_j)_i$, by generating $\mathbf{X}_j \sim N(0, I_k)$, where I_k is the $k \times k$ identity matrix, and letting $S(t_j) = S(t_{j-1}) \cdot L(t_j)$. Our estimator here is

$$\hat{l}_0 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k} S(t_j) - K\right)^+.$$

With the parameters specified above, we get the following estimates and relative errors for the various levels of σ:

σ	est	var	95% CI	rel err
2.00	2.8574	0.0027	(2.8288, 2.8861)	0.0051
1.00	0.3278	3.7452×10^{-6}	(0.3260, 0.3295)	0.0027
0.50	0.0397	1.2464×10^{-7}	(0.0394, 0.0400)	0.0038
0.25	9.2302×10^{-4}	3.3778×10^{-9}	(0.0009, 0.0009)	0.0141
0.20	9.4675×10^{-5}	6.6957×10^{-10}	(0.0001, 0.0001)	0.0345
0.15	1.5995×10^{-6}	4.1184×10^{-11}	(0.0000,0.0000)	0.1864
0.10	0	0	(0, 0)	NaN
0.05	0	0	(0, 0)	NaN
0.02	0	0	(0, 0)	NaN
0.01	0	0	(0, 0)	NaN

Table 6: The estimate, sample variance, 95% confidence intervals and relative errors of the expected payoff of an Asian call option for various σ values using straight-forward Monte Carlo simulation.

Similar to the European call option, we can see that naive Monte Carlo simulation fails for small values of σ, specifically here for $\sigma \leq 0.10$.

3.2 Antithetic Sampling

We can also apply antithetic sampling here as well. For the same reasons as with the European call option, we can see that $L_X(t_j)$ and $L_Y(t_j)$ are negatively correlated for each $j = 1, ..., k$. Then $S_X(t_j)$ and $S_Y(t_j)$ are also negatively correlated, which means that $\frac{1}{k}\sum_{j=1}^{k} S_X(t_j)$ and $\frac{1}{k}\sum_{j=1}^{k} S_Y(t_j)$ are negatively correlated. Finally, we can see then that $\left(\frac{1}{k}\sum_{j=1}^{k} S_X(t_j) - K\right)^+$ and $\left(\frac{1}{k}\sum_{j=1}^{k} S_Y(t_j) - K\right)^+$ must also be negatively correlated.

Then our final estimator is

$$\hat{l}_1 = \frac{1}{n}\sum_{i=1}^{n} \frac{\left(\frac{1}{k}\sum_{i=j}^{k}(S_X(T))_i - K\right)^+ + \left(\frac{1}{k}\sum_{i=j}^{k}(S_Y(T))_i - K\right)^+}{2}.$$

	Naive MC		Antithetic Sampling	
σ	**estimate**	**var**	**estimate**	**var**
2.00	2.8574	0.0027	2.8597	0.0012
1.00	0.3278	3.7452×10^{-6}	0.3274	1.6855×10^{-6}
0.50	0.0397	1.2464×10^{-7}	0.0396	5.5065×10^{-8}
0.25	9.2302×10^{-4}	3.3778×10^{-9}	9.2211×10^{-4}	1.6079×10^{-9}
0.20	9.4675×10^{-5}	6.6957×10^{-10}	1.0645×10^{-4}	3.3517×10^{-10}
0.15	1.5995×10^{-6}	4.1184×10^{-11}	1.2668×10^{-6}	2.1102×10^{-11}
0.10	0	0	0	6.5910×10^{-14}
0.05	0	0	0	0
0.02	0	0	0	0
0.01	0	0	0	0

Table 7: The estimate and sample variance for the expected payoff of an Asian call option for various σ values using straight-forward Monte Carlo simulation vs antithetic sampling.

	Naive MC		Antithetic Sampling	
σ	**95% CI**	**rel err**	**95% CI**	**rel err**
2.00	(2.8288, 2.8861)	0.0051	(2.8402, 2.8792)	0.0035
1.00	(0.3260, 0.3295)	0.0027	(0.3262, 0.3285)	0.0018
0.50	(0.0394, 0.0400)	0.0038	(0.0394, 0.0398)	0.0026
0.25	(0.8975, 0.0009)	0.0141	(0.0009, 0.0009)	0.0099
0.20	(0.0001, 0.0001)	0.0345	(0.0001, 0.0001)	0.0239
0.15	(0.0000,0.0000)	0.1864	(0.0000, 0.0000)	0.1582
0.10	(0, 0)	NaN	(0, 0)	NaN
0.05	(0, 0)	NaN	(0, 0)	NaN
0.02	(0, 0)	NaN	(0, 0)	NaN
0.01	(0, 0)	NaN	(0, 0)	NaN

Table 8: The 95% confidence intervals and relative errors of the estimate of the expected payoff of an Asian call option for various σ values using straight-forward Monte Carlo simulation vs antithetic sampling.

We can see that while antithetic sampling does reduce the relative error, and thus variance, it still fails for small values of σ.

3.3 Importance Sampling

Just as with the European call option, we can use importance sampling to improve estimates for smaller σ. In this case, the event of interest that becomes rare as σ gets small is $\sum_{i=1}^{n} S(t_i) > kK$. We can note, however, that $S(t_i) > K$ for all $i = 1, 2, ..., k$ implies that $\sum_{i=1}^{n} S(t_i) > kK$, so we can choose $g^*(\mathbf{x})$ to make the event that $S(t_i) > K$ for all $i = 1, 2, ..., k$ more likely.

Then we get the conditional distribution

$$
\begin{aligned}
f(\mathbf{x}|S(t_i) > K) &\propto f(S(t_i) > K|\mathbf{x})f(\mathbf{x}) \\
&= \left(\prod_{i=1}^{k} f(S(t_i) > K|S(t_{i-1}), ..., S(t_1) > k, \mathbf{x})\right) f(\mathbf{x}) \\
&= f(S(t_1) > K|\mathbf{x}) \left(\prod_{i=2}^{k} f(S(t_i) > K|S(t_{i-1}) > k, \mathbf{x})\right) f(\mathbf{x}) \\
&= f(S(t_1) > K|\mathbf{x}) \left(\prod_{i=2}^{k} f(L(t_i) > K|\mathbf{x})\right) f(\mathbf{x}) \\
&= (f(x_1)\mathbb{1}_{L(t_1)>K}) \prod_{i=1}^{k} f(x_i)\mathbb{1}_{L(t_i)>1}
\end{aligned}
$$

$L(t_1) > K$ means that $x_1 > \frac{\ln(K)-\mu\Delta}{\sigma\sqrt{\Delta}}$. $L(t_i) > 1$ means that $x_i > \frac{-\mu\Delta}{\sigma\sqrt{\Delta}}$. But

$$
\begin{aligned}
\sum_{i=1}^{k} e^{\mu t_i} &= \sum_{i=1}^{k} e^{\mu\Delta} \\
&= ke^{\mu\Delta} < kK \\
&\Rightarrow e^{\mu\Delta} < K
\end{aligned}
$$

so $\frac{\ln(K)-\mu\Delta}{\sigma\sqrt{\Delta}} > 0$, while $\frac{-\mu\Delta}{\sigma\sqrt{\Delta}} < 0$, so $f(x_1)\mathbb{1}_{L(t_1)>K}$ will be greatest at $\frac{\ln(K)-\mu\Delta}{\sqrt{\Delta}\sigma}$, while $f(x_i)\mathbb{1}_{L(t_i)>1}$ will be greatest at 0. Then, we can see that a good choice for $g^*(\mathbf{x})$ might be $g^*(x_1) \sim N\left(\frac{\ln(K)-\mu\Delta}{\sigma\sqrt{\Delta}}, 1\right)$ and $g^*(x_i) \sim N(0,1)$, for $i = 2, ..., k$.

Indeed we can see that under this new distribution, we are able to achieve non-zero estimates for smaller σ values than we were able to with naive Monte Carlo or antithetic sampling.

	Naive MC		Importance Sampling	
σ	**estimate**	**var**	**estimate**	**var**
2.00	2.8574	0.0027	2.8739	0.0014
1.00	0.3278	3.7452×10^{-6}	0.3278	1.3451×10^{-6}
0.50	0.0397	1.2464×10^{-7}	0.0396	2.0840×10^{-8}
0.25	9.2302×10^{-4}	3.3778×10^{-9}	9.0380×10^{-4}	1.2532×10^{-10}
0.20	9.4675×10^{-5}	6.6957×10^{-10}	8.70385×10^{-5}	1.1032×10^{-11}
0.15	1.5995×10^{-6}	4.1184×10^{-11}	1.4846×10^{-6}	1.4221×10^{-13}
0.10	0	0	1.9036×10^{-12}	3.7526×10^{-18}
0.05	0	0	1.7317×10^{-55}	6.1541×10^{-40}
0.02	0	0	7.0686×10^{-238}	1.6816×10^{-184}
0.01	0	0	0	0

Table 9: The estimate and sample variance for the expected payoff of an Asian call option for various σ values using straight-forward Monte Carlo simulation vs importance sampling.

	Naive MC		Importance Sampling	
σ	**95% CI**	**rel err**	**95% CI**	**rel err**
2.00	(2.8288, 2.8861)	0.0051	(2.8522, 2.8956)	0.0039
1.00	(0.3260, 0.3295)	0.0027	(0.3265, 0.3291)	0.0020
0.50	(0.0394, 0.0400)	0.0038	(0.0393, 0.0399)	0.0041
0.25	(0.8975, 0.0009)	0.0141	(0.0008, 0.0010)	0.0362
0.20	(0.0001, 0.0001)	0.0345	(0.0001, 0.0001)	0.0692
0.15	(0.0000, 0.0000)	0.1864	(0.0000, 0.0000)	0.2896
0.10	(0, 0)	NaN	(0.0000, 0.0000)	0.5871
0.05	(0, 0)	NaN	(0.0000, 0.0000)	0.4894
0.02	(0, 0)	NaN	(0.0000, 0.0000)	0
0.01	(0, 0)	NaN	(0, 0)	NaN

Table 10: The 95% confidence intervals and relative errors of the estimate of the expected payoff of an Asian call option for various σ values using straight-forward Monte Carlo simulation vs importance sampling.

Here, though the relative errors may be larger using importance sampling for the larger values of σ than the naive Monte Carlo method, we are able to get non-zero estimates for the expected payoff for smaller values of σ than with naive Monte Carlo. Specifically here, we can see that importance sampling fails at $\sigma \leq 0.01$, whereas naive Monte Carlo failed for $\sigma \leq 0.10$.

3.4 Summary

Similar to the European call option, we can see that of the three methods of simulation, the importance sampling performed best in that it was able to get non-zero estimates for smaller σ values than the naive Monte Carlo and antithetic sampling methods. With antithetic sampling, though, we were able to see a reduction in the relative error compared to the naive Monte Carlo estimator, just as we did with the European call option. Also similar to the European call option, all three methods eventually broke down for small enough σ.