CS203B: Mathematics For Computer Science-III

Assignment 1

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Question 1

A company asks you to set up a mechanism to send single bits (0 or 1) between two points A and B. The company has n identical communication channels available. However, these channels are not perfect. Each of these channels flips the bits send through it (0 becomes 1 or 1 becomes 0) with probability p. The company gives you two options for using these channels to connect A and B: either the channels can be connected in series between A and B or they can be connected in parallel between A and B. In the series mode, the output of the channel at the receiving end is taken as the received bit, while in the parallel mode, the bit which appears in the majority of the channel outputs is taken as the received bit. Answer the following questions:

(a) If $p = \frac{1}{3}$ and $n = 5$, which mode of connecting the channels maximizes the probability that the correct bit is received at the receiving end?

Solution: We make the keen observation that to retain the original bit, we need an even number of flips in the series mode (because even number of flips preserve the polarity), while we need a (strict) minority of flips in the parallel mode (by definition). We denote a "flip" by an f, and an "unflip" by a u. Clearly, $P(f) = p = \frac{1}{3}$ and $P(u) = 1 - p = \frac{2}{3}$. Also, a string such as *"ufuff"* is used to mean that the *1*st and *3*rd channel do not flip the bit (u), the *2*nd, *4*th and *5*th channel flip the bit.

Parallel: For the *parallel* mode, since we require lesser number of "flips" (in comparison to "unflips"), the events relevant to receiving the correct bit (excusing permutations), are *uuuff*, *uuuuf* and *uuuuu*.

The probability for the *uuuuu* case is ,

$$P(uuuuu) = (1-p)^5 = \frac{2^5}{3^5}$$

The events *uuuff* and *uuuuf* actually act as umbrellas for different outcomes ($\binom{5}{2}$ and $\binom{5}{1}$ outcomes respectively) because of the permutations arising from the position(s) of *f('s)* and *u('s)*. Since the permutations of a particular event are probabilistically similar (due to independence of towers on "flipping"), we compute the probability of a single outcome and multiply it with the number of permutations possible in the corresponding event. Thus,

$$P(uuuff) = \binom{5}{2} \cdot (1-p)^3 \cdot p^2 = 10 \cdot \frac{2^3}{3^5}$$

$$P(uuuuf) = \binom{5}{1} \cdot (1-p)^4 \cdot p = 5 \cdot \frac{2^4}{3^5}$$

Combining these expressions, we have,

$$\begin{aligned} P(\text{Correct Bit})_{Parallel} &= P(uuuff) + P(uuuuf) + P(uuuuu) \\ &= \frac{10 \cdot 2^3}{3^5} + \frac{5 \cdot 2^4}{3^5} + \frac{2^5}{3^5} = \boxed{\frac{192}{243}} \end{aligned}$$

Series: For the *series* mode, since we require an even number of "flips", the events relevant to receiving the correct bit (excusing permutations), are *uffff*, *uuuff* and *uuuuu*. Again, the events *uffff* and *uuuff* act as umbrellas for different outcomes ($\binom{5}{4}$ and $\binom{5}{2}$ outcomes respectively) because of the permutations arising from the position(s) of *f('s)* and *u('s)*. Following the same argument as given above, we compute the probability of a single outcome and multiply it with the number of permutations possible in the corresponding event. Thus,

$$\begin{aligned}
P(uffff) &= \binom{5}{4} \cdot (1-p)^1 \cdot p^4 = 5 \cdot \frac{2^1}{3^5} \\
P(uuuff) &= \binom{5}{2} \cdot (1-p)^3 \cdot p^2 = 10 \cdot \frac{2^3}{3^5} \\
P(uuuuu) &= (1-p)^5 = \frac{2^5}{3^5}
\end{aligned}$$

Combining these, we get,

$$\begin{aligned}
P(\text{Correct bit})_{Series} &= P(uffff) + P(uuuff) + P(uuuuu) \\
&= \frac{5 \cdot 2^1}{3^5} + \frac{10 \cdot 2^3}{3^5} + \frac{2^5}{3^5} = \boxed{\frac{122}{243}}
\end{aligned}$$

It is clearly evident now, that the *Parallel* method maximizes the probability of receiving the correct bit while the *Series* method does not.

(b) Assume that $p = \frac{1}{3}$, $n = 3$ and the parallel mode is used for connecting A and B. A chooses the bit 0 with probability $\frac{2}{3}$ and the bit 1 with probability $\frac{1}{3}$ and sends this random bit to B. The majority of the bits received at B turns out to be equal to 1. Given this fact, what is the probability that the original bit sent by A was also equal to 1?

Solution: We denote the probability that A had chosen 0 by $p_0 = \frac{2}{3}$ and that A had chosen 1 by $p_1 = \frac{1}{3}$. Further, we use the notation $A = i$ to mean that i was chosen by A, and $B = j$ to mean that j was the resultant bit received by B.
Now, we split the calculation into two cases depending on what A had chosen, and then calculate the probability that the majority of the bits received at B is 1.

- $A = 1$ and $B = 1$
  In this case, since the majority is the same as the sent bit, there must have been a majority of "unflips" in the channels. The list of events leading to this are (excusing the permutations) *uuf* and *uuu*. We use the same argument and calculation as in part a to compute $P(uuf)$.

$$\begin{aligned}
P(uuf) &= \binom{3}{1} \cdot (1-p)^2 \cdot p = 3 \cdot \frac{2^2}{3^3} \\
P(uuu) &= (1-p)^3 = \frac{2^3}{3^3}
\end{aligned}$$

  These terms allow us to calculate $P(B = 1 \mid A = 1)$,

$$\begin{aligned}
P(B = 1 \mid A = 1) &= P(uuf) + P(uuu) \qquad \because \textit{uuf} \text{ and } \textit{uuu} \text{ are disjoint events} \\
&= \frac{3 \cdot 2^2}{3^3} + \frac{2^3}{3^3} \\
&= \boxed{\frac{20}{27}}
\end{aligned}$$

- $A = 0$ and $B = 1$
  In this case, since the majority is different than the sent bit, there must have been a majority

of "flips" in the channels. The list of events leading to this are (excusing the permutations) *uff* and *fff*. We use the same argument and calculation as in *part a* to compute $P(uff)$.

$$P(uff) = \binom{3}{1} \cdot (1-p) \cdot p^2 = 3 \cdot \frac{2}{3^3}$$
$$P(fff) = p^3 = \frac{1}{3^3}$$

These terms allow us to calculate $P(B = 0 \mid A = 1)$,

$$\begin{aligned} P(B = 0 \mid A = 1) &= P(uff) + P(fff) && \because uff \text{ and } fff \text{ are disjoint events} \\ &= \frac{3 \cdot 2}{3^3} + \frac{1}{3^3} \\ &= \boxed{\frac{7}{27}} \end{aligned}$$

Finally, we have all the values we need to compute $P(A = 1 \mid B = 1)$ except for one small component, $P(B = 1)$ which we shall be writing as

$$P(B = 1) = P(B = 1 \mid A = 1) \cdot P(A = 1) + P(B = 1 \mid A = 0) \cdot P(A = 0)$$

since $A = 1$ and $A = 0$ are mutually exclusive and collectively exhaustive events.
Now, we invoke Bayes' Theorem to compute $P(A = 1 \mid B = 1)$,

$$\begin{aligned} P(A = 1 \mid B = 1) &= \frac{P(B = 1 \mid A = 1) \cdot P(A = 1)}{P(B = 1)} \\ &= \frac{P(B = 1 \mid A = 1) \cdot p_1}{P(B = 1 \mid A = 1) \cdot p_1 + P(B = 1 \mid A = 0) \cdot p_0} \\ &= \frac{\frac{20}{27} \cdot \frac{1}{3}}{\frac{20}{27} \cdot \frac{1}{3} + \frac{7}{27} \cdot \frac{2}{3}} \\ &= \frac{20}{20 + 7 \cdot 2} = \frac{10}{17} \end{aligned}$$

Thus, the probability that the original bit sent by A was also 1 when the majority bit received by B is 1 is $\frac{10}{17}$.

Question 2

Suppose U is a continuous random variable with the probability density function ($c \in \mathbb{R}$)

$$g(u) = \begin{cases} c - |u|, & |u| < \frac{1}{2}. \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

<u>*Solution:*</u> $\because$ g is a probability density function, we invoke the necessary condition that,

$$\int_{-\infty}^{\infty} g(u)\ \mathrm{d}u = 1$$

Using the way $g(u)$ has been defined, we split the number-line (and consequently the limits of the integral) into some disjoint, exhaustive sets which would make evaluating the integral easier. These sets are $(-\infty, -\frac{1}{2}]$, $(-\frac{1}{2}, 0]$, $(0, \frac{1}{2})$, and $[\frac{1}{2}, \infty)$. The integral thus becomes,

$$\begin{aligned}
\int_{-\infty}^{\infty} g(u)\ \mathrm{d}u &= \int_{-\infty}^{-\frac{1}{2}} 0\ \mathrm{d}u + \int_{-\frac{1}{2}}^{0} (c - (-u))\ \mathrm{d}u + \int_{0}^{\frac{1}{2}} (c - (u))\ \mathrm{d}u + \int_{\frac{1}{2}}^{\infty} 0\ \mathrm{d}u \\
&= [0]\Big|_{-\infty}^{-\frac{1}{2}} + \left[cu + \frac{u^2}{2}\right]\Bigg|_{-\frac{1}{2}}^{0} + \left[cu - \frac{u^2}{2}\right]\Bigg|_{0}^{\frac{1}{2}} + [0]\Big|_{\frac{1}{2}}^{\infty} \\
&= [0] + \left[c \cdot \frac{1}{2} - \frac{1}{8}\right] + \left[c \cdot \frac{1}{2} - \frac{1}{8}\right] + [0] \\
&= c \cdot \left(\frac{1}{2} + \frac{1}{2}\right) - \frac{1}{8} - \frac{1}{8} = c - \frac{1}{4} \\
&\implies c - \frac{1}{4} = 1 \implies \boxed{c = \frac{5}{4}}
\end{aligned}$$

(b) The cumulative distribution function of a random variable X is the function $F_X(x) = P(X \leq x)$ for every $x \in \mathbb{R}$. Find the cumulative distribution function F_U of U.

<u>*Solution:*</u> We make use of the previous split of the number-line and calculate F_U piece-wise for each of the split sets and then combine the results.
We use the expression, $F_U(u) = P(U \leq u) = \int_{-\infty}^{u} g(x)\ \mathrm{d}x$ to evaluate F_U. Since the limits of the integral can be split as per our convenience, we split them so that we may use results from other sets. It is helpful to observe that

$$\begin{aligned}
F_U(u_1) = P(U \leq u_1) &= P((U \leq u_2) \cup (u_2 < U \leq u_1)) \\
&= P(U \leq u_2) + P(u_2 < U \leq u_1) \\
&= F_U(u_2) + \int_{u_2}^{u_1} g(u)\ \mathrm{d}x & \forall\, u_2 < u_1
\end{aligned}$$

which will help in using previous results. The proof is simple and follows from the fact that the sets $(-\infty, u_2]$ and $(u_2, u_1]$ are disjoint.
We also plug the value of c into the expression of $g(u)$ wherever applicable.

- $u \in \left(-\infty, -\frac{1}{2}\right]$

$$\begin{aligned}
F_U(u) &= \int_{-\infty}^{u} g(x)\ \mathrm{d}x \\
&= \int_{-\infty}^{u} 0\ \mathrm{d}x = \boxed{0} && \because\ g(u) = 0 \text{ for } u \leq -\frac{1}{2}
\end{aligned}$$

- $u \in \left(-\frac{1}{2}, 0\right]$

$$\begin{aligned}
F_U(u) &= F_U\left(-\frac{1}{2}\right) + \int_{-\frac{1}{2}}^{u} g(x)\ \mathrm{d}x \\
&= [0]\Big|_{u=-\frac{1}{2}} + \int_{-\frac{1}{2}}^{u} \left[\frac{5}{4} - (-x)\right]\ \mathrm{d}x \\
&= 0 + \left[\frac{5x}{4} + \frac{x^2}{2}\right]\Bigg|_{-\frac{1}{2}}^{u} \\
&= \boxed{\frac{u^2}{2} + \frac{5u}{4} + \frac{1}{2}}
\end{aligned}$$

- $u \in \left(0, \frac{1}{2}\right)$

$$\begin{aligned}
F_U(u) &= F_U(0) + \int_{0}^{u} g(x)\ \mathrm{d}x \\
&= \left[\frac{u^2}{2} + \frac{5u}{4} + \frac{1}{2}\right]\Bigg|_{u=0} + \int_{0}^{u} \left[\frac{5}{4} - (x)\right]\ \mathrm{d}x \\
&= \frac{1}{2} + \left[\frac{5x}{4} - \frac{x^2}{2}\right]\Bigg|_{0}^{u} \\
&= \boxed{\frac{1}{2} + \frac{5u}{4} - \frac{u^2}{2}}
\end{aligned}$$

- $u \in \left[\frac{1}{2}, \infty\right)$

$$\begin{aligned}
F_U(u) &= F_U\left(\frac{1}{2}\right) + \int_{\frac{1}{2}}^{u} g(x)\ \mathrm{d}x \\
&= \left[-\frac{u^2}{2} + \frac{5u}{4} + \frac{1}{2}\right]\Bigg|_{u=\frac{1}{2}} + \int_{\frac{1}{2}}^{u} 0\ \mathrm{d}x \\
&= 1 + [0]\Big|_{\frac{1}{2}}^{u} = \boxed{1}
\end{aligned}$$

Finally, combining all these results, we obtain F_U as,

$$\mathrm{F}_U(u) = \begin{cases} 0, & u \leq -\frac{1}{2} \\ \frac{u^2}{2} + \frac{5u}{4} + \frac{1}{2}, & -\frac{1}{2} < u \leq 0 \\ -\frac{u^2}{2} + \frac{5u}{4} + \frac{1}{2}, & 0 < u < \frac{1}{2} \\ 1, & \frac{1}{2} \leq u \end{cases}$$

(c) Evaluate the conditional probability $Pr(\frac{1}{8} < U < \frac{2}{5} \mid \frac{1}{10} < U < \frac{1}{5})$.

<u>*Solution:*</u> We re-write the term as $Pr\Big(U \in (\frac{1}{8}, \frac{2}{5}) \mid U \in (\frac{1}{10}, \frac{1}{5})\Big)$.
We note that $U \in (0, \frac{1}{2})$ in the term, so we fix $g(u) = c - (u) = \frac{5}{4} - u$.

$$\begin{aligned}
Pr\left(U \in \left(\frac{1}{8}, \frac{2}{5}\right) \,\middle|\, U \in \left(\frac{1}{10}, \frac{1}{5}\right)\right) &= \frac{Pr(U \in (\frac{1}{8}, \frac{2}{5}) \cap (\frac{1}{10}, \frac{1}{5}))}{Pr(U \in (\frac{1}{10}, \frac{1}{5}))} \\
&= \frac{Pr(U \in (\frac{1}{8}, \frac{1}{5}))}{Pr(U \in (\frac{1}{10}, \frac{1}{5}))} \\
&= \int_{\frac{1}{8}}^{\frac{1}{5}} g(u)\ \mathrm{d}u \Big/ \int_{\frac{1}{10}}^{\frac{1}{5}} g(u)\ \mathrm{d}u \\
&= \int_{\frac{1}{8}}^{\frac{1}{5}} \left[\frac{5}{4} - u\right] \mathrm{d}u \Big/ \int_{\frac{1}{10}}^{\frac{1}{5}} \left[\frac{5}{4} - u\right] \mathrm{d}u \\
&= \left[\frac{5u}{4} - \frac{u^2}{2}\right]\Bigg|_{\frac{1}{8}}^{\frac{1}{5}} \Big/ \left[\frac{5u}{4} - \frac{u^2}{2}\right]\Bigg|_{\frac{1}{10}}^{\frac{1}{5}} \\
&= \left[\frac{23}{100} - \frac{19}{128}\right] \Big/ \left[\frac{23}{100} - \frac{3}{25}\right] = \frac{261}{352}
\end{aligned}$$

Thus, we have,

$$Pr\left(\frac{1}{8} < U < \frac{2}{5} \,\middle|\, \frac{1}{10} < U < \frac{1}{5}\right) = \frac{261}{352}$$

Question 3

Alice has an unbiased 5-sided die and 5 different coins with her. The probabilities of obtaining a head on tosses of these coins are $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}$ and $\frac{5}{6}$ respectively. She likes to observe patterns in subsequent tosses of these coins. Alice performs the following experiment. She rolls the 5-sided die and if the i^{th} side turns up, she chooses the i^{th} coin and starts tossing this coin repeatedly.

(a) What is the expected number of tosses required for obtaining 6 consecutive heads given that the side 1 turned up during the roll of the die?

<u>Solution:</u> Denote the probability of getting a head on tossing the i^{th} coin by $p_i^H = \frac{i}{6}$ and getting a head by $p_i^T = \frac{6-i}{6}$, the probability of getting i on rolling the die by $p_i = \frac{1}{5}$, the event of getting i on rolling the die by $D = i$, and X_i be the number of tosses to get 6 consecutive heads on the i^{th} coin.
We solve parts *a* and *b* in a method similar to that used to solve the *3-door miner* problem, mentioned in the class notes, because the advent of a Tail essentially "resets" our progress. We define the events, where the

- First toss results in a Tail, followed by any side of the coin. $(T\ldots)$
  Probability of this case: p_1^T
  Expectation of getting *6* consecutive heads: $1 + E[X_1]$
  As 1 toss was "wasted" as a result of the reset.
- First toss was a Head, followed by a Tail, followed by any. $(HT\ldots)$
  Probability of this case: $(p_1^H) \cdot p_1^T$
  Expectation of getting *6* consecutive heads: $2 + E[X_1]$
  As 2 tosses were "wasted" as a result of the reset.
- First two tosses were Heads, followed by a Tail, followed by any. $(HHT\ldots)$
  Probability of this case: $(p_1^H)^2 \cdot p_1^T$
  Expectation of getting *6* consecutive heads: $3 + E[X_1]$
  As 3 tosses were "wasted" as a result of the reset.
- First three tosses were Heads, followed by a Tail, followed by any. $(HHHT\ldots)$
  Probability of this case: $(p_1^H)^3 \cdot p_1^T$
  Expectation of getting *6* consecutive heads: $4 + E[X_1]$
  As 4 tosses were "wasted" as a result of the reset.
- First four tosses were Heads, followed by a Tail, followed by any. $(HHHHT\ldots)$
  Probability of this case: $(p_1^H)^4 \cdot p_1^T$
  Expectation of getting *6* consecutive heads: $5 + E[X_1]$
  As 5 tosses were "wasted" as a result of the reset.
- First five tosses were Heads, followed by a Tail, followed by any. $(HHHHHT\ldots)$
  Probability of this case: $(p_1^H)^5 \cdot p_1^T$
  Expectation of getting *6* consecutive heads: $6 + E[X_1]$
  As 6 tosses were "wasted" as a result of the reset.

- First six tosses were Heads, followed by any. $(HHHHHH\dots)$
  Probability of this case: $(p_1^H)^6$
  Expectation of getting *6* consecutive heads: 6
  As the 6 tosses have resulted in 6 consecutive heads.

Let all these events be stored in the set S.
Observe that these include all the possible prefixes and are thus, collectively exhaustive events. Since they are disjoint as well, we may write,

$$
\begin{aligned}
E[X_1] &= \sum_{C \in S} E[X_1 \mid C] \cdot P(C) \\
&= (1 + E[X_1]) \cdot p_1^T + (2 + E[X_1]) \cdot (p_1^H) \cdot p_1^T + (3 + E[X_1]) \cdot (p_1^H)^2 \cdot p_1^T \\
&+ (4 + E[X_1]) \cdot (p_1^H)^3 \cdot p_1^T + (5 + E[X_1]) \cdot (p_1^H)^4 \cdot p_1^T \\
&+ (6 + E[X_1]) \cdot (p_1^H)^5 \cdot p_1^T + (6) \cdot (p_1^H)^6 \qquad \because p_1^T = \frac{5}{6} \,\&\, p_1^H = \frac{1}{6} \\
&= 5 \cdot \left(\frac{1}{6} + \frac{2}{6^2} + \frac{3}{6^3} + \frac{4}{6^4} + \frac{5}{6^5} + \frac{6}{6^6} \right) \\
&+ E[X_1] \cdot \left(\frac{5}{6} + \frac{5}{6^2} + \frac{5}{6^3} + \frac{5}{6^4} + \frac{5}{6^5} + \frac{5}{6^6} \right) + \frac{6}{6^6} \\
E[X_1] \cdot \frac{1}{6^6} &= \frac{9331}{6^5} \\
E[X_1] &= 9331 \cdot 6 = 55986
\end{aligned}
$$

(b) What is the expected number of tosses required for obtaining 6 consecutive heads while performing this random experiment?

Solution: It is important to observe that $E[X_1]$ is actually equal to $E[X \mid D = 1]$, and similarly, $E[X_i]$ equals $E[X \mid D = i]$. We compute $E[X_i]$ for $i \in 2, 3, 4, 5$ in a way similar to that in *part a*.
To avoid unnecessary arithmetic and cluttering, the individual values have been calculated and listed below for each i:

$$
\boxed{
\begin{aligned}
E[X_1] &= 55986 \\
E[X_2] &= 1092 \\
E[X_3] &= 126 \\
E[X_4] &= \frac{1995}{64} \\
E[X_5] &= \frac{186186}{15625}
\end{aligned}
}
$$

Using the disjoint and mutually exhaustive nature of $D = 1, D = 2, \dots, D = 5$, we write,

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{5} E[X|D = i] \cdot P(D = i) \\
&= \sum_{i=1}^{5} E[X_i] \cdot P(D = i) \\
&= \frac{1}{5} \cdot \sum_{i=1}^{5} E[X_i] \qquad \because P(D = i) = \frac{1}{5}
\end{aligned}
$$

$$= \frac{1}{5}\left(55986 + 1092 + 126 + \frac{1995}{64} + \frac{186186}{15625}\right)$$
$$= \boxed{\frac{57247087779}{5 \cdot 10^6} \approx 11449.42}$$

(c) What is the probability that in the first n tosses, she obtains n consecutive heads?

Solution: We split the situation into multiple cases, one each for the five faces of the die. We denote the event of getting n consecutive heads in the first n tosses by nH.
Suppose, i is the result of the die roll, then the probability of getting a head consecutively n times is $(p_i^H)^n$, since the tosses are independent of each other. More, precisely, we have, $P(nH \mid D = i) = (p_i^H)^n = (\frac{i}{6})^n$.
Now, using the disjoint and exhaustive nature of the rolls of the die, we write,

$$\begin{aligned}
P(nH) &= \sum_{i=1}^{5} P(nH \mid D = i) \cdot P(D = i) \\
&= \sum_{i=1}^{5} \left(p_i^H\right)^n \cdot (p_i) \\
&= \frac{1}{5} \cdot \sum_{i=1}^{5} \left(\frac{i}{6}\right)^n && \because p_i = \frac{1}{5} \text{ and } p_i^H = \frac{i}{6} \\
&= \boxed{\left[\frac{1^n + 2^n + 3^n + 4^n + 5^n}{5 \cdot 6^n}\right]}
\end{aligned}$$

(d) In the first n-tosses, she obtains n consecutive heads. Given this outcome, calculate the probability that the i^{th} side turned up during the roll of the die (the closed form expression for arbitrary i). How do these probabilities behave as $n \to \infty$?

Solution: Let $P(D = i \mid nH)$ denote the probability that the i^{th} side had turned up on the dice, given that the first n-tosses were all heads. Then we use Bayes' Theorem to calculate it,

$$\begin{aligned}
P(D = i \mid nH) &= \frac{P(nH \mid D = i) \cdot P(D = i)}{P(nH)} \\
&= \frac{\left(\left(\frac{i}{6}\right)^n \cdot \frac{1}{5}\right)}{\left(\frac{1^n + 2^n + 3^n + 4^n + 5^n}{5 \cdot 6^n}\right)} \\
&= \frac{i^n}{1^n + 2^n + 3^n + 4^n + 5^n} \\
&= \frac{(\frac{i}{5})^n}{(\frac{1}{5})^n + (\frac{2}{5})^n + (\frac{3}{5})^n + (\frac{4}{5})^n + (\frac{5}{5})^n} && \{\text{Dividing both Num. and Den. by } 5^n
\end{aligned}$$

Now,

$$\boxed{\lim_{n \to \infty} \left(\frac{i}{5}\right)^n = \begin{cases} 0, & 1 \le i \le 4 \\ 1, & i = 5 \end{cases}}$$

The numerator is the same as the limit calculated above while the denominator is

$$\lim_{n\to\infty}\left(\left(\frac{1}{5}\right)^n+\left(\frac{2}{5}\right)^n+\left(\frac{3}{5}\right)^n+\left(\frac{4}{5}\right)^n+\left(\frac{5}{5}\right)^n\right)=0+0+0+0+1=1$$

Thus, we have,

$$\boxed{\lim_{n\to\infty} P(D=i\,|\,nH)=\begin{cases}0, & i\in 1,2,3,4\\ 1, & i=5\end{cases}}$$

Question 4

In this programming exercise, let us explore the behavior of the averages,

$$\frac{X_1 + X_2 + X_3 + \cdots + X_n}{n}$$

of independent and identically distributed random variables X_i as $n \to \infty$.

(a) Implement the following program using any language and graph plotting library of your choice (You are encouraged to use Python with Matplotlib for this assignment). Consider the random variable X which takes the values $0, 1, 2, 3, \ldots m-1$ with the respective probabilities $p_0, p_1, p_2, p_3 \ldots p_{m-1}$ such that $p_0 + p_1 + p_2 + p_3 \cdots + p_{m-1} = 1$. Your program should take as inputs, the value m, the probabilities $p_0, p_1, p_2, p_3 \ldots p_{m-1}$ and n which is the number of samples to be generated. Generate n samples according to the distribution of X and calculate the average value of the samples generated. Repeat this sampling and averaging process for a fairly large number of iterations and store the average values obtained. Round each of the average values to the nearest integer and generate a plot of the frequency of the rounded averages thus obtained against the range of possible values. The above programs should be included in the submitted `.zip` file with the name `[Your roll number]_Q4` and the appropriate extension according to your choice of the programming language (for e.g. `20xxxx_Q4.py`). You should also include a `readme` document explaining how this program can be executed. *Note:* The averages have been rounded off to one decimal digit instead of rounding off to integers since it provided an appropriate trade-off between the range of averages and their frequencies.

(b) The following questions should be answered in your main answer script. Give a brief account of how you implemented random sampling according to the required distributions in your program. Also, use your program to answer the following questions:

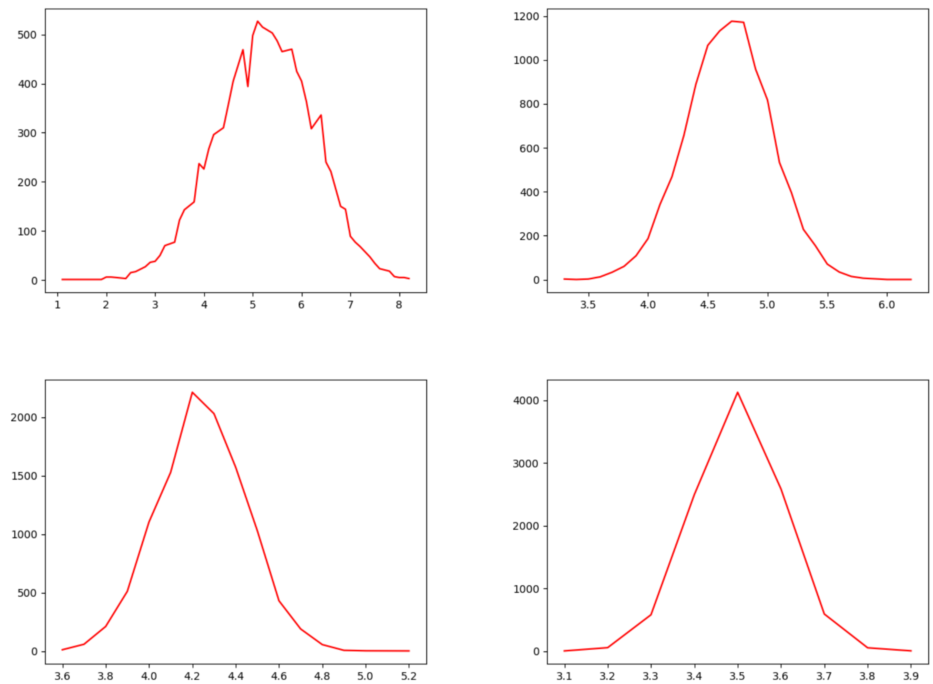
Implementation: For implementing the random sampling, with respect to the given probability distribution, `random.uniform` was used to generate a random number (say r) in the range $[0, 1]$. The given probability distribution was converted to a cumulative distribution, and then r was searched in this distribution, and accordingly, the random choice in $[0, m)$ was returned.
For example, let $p = (0.1, 0.2, 0.3, 0.2, 0.2) \to p = (0.1, 0.3, 0.6, 0.8, 1.0)$. Now, suppose $r = 0.55$, then since 0.6 is the number which just exceeds r and its index in the cumulative distribution is 2, the random number we shall return is 2. Similarly, the return value for $r = 1$ is $m - 1$ and $r = 0.1$ is 0.
This retains the original probability distribution because of the linear nature of cumulation.

i. How does the frequency plot of the averages behave as $n \to \infty$?

<u>*Answer:*</u> As the number of samples, n, is increased the graph becomes thinner. More formally, the variance decreases and the graph gets concentrated more around the "Expected" value of X_i's. Conversely, for smaller n, the graph is more "spread out", because smaller n allows outliers more easily.
The Law of Large Numbers can be seen in action here, both with the admittance of outliers for smaller n as well as the concentration of the graph near the expected value for larger n.
Some *plots* have been included at the end of the question to portray the variation with respect to n.

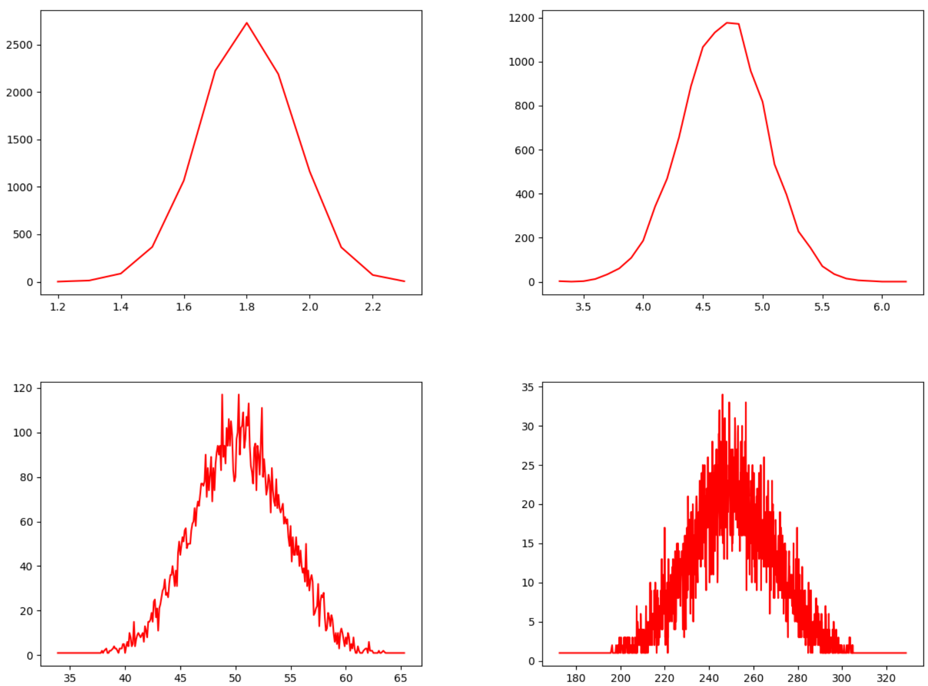
ii. Does the shape of the frequency plot change on varying m or the values of the probabilities? Can you interpret the shape of the plots for these distributions in terms of any of the concepts that were discussed in class?

Answer: On keeping the number of iterations fixed (currently fixed in the range $[10^4, 1.1 \cdot 10^4]$), and increasing m, the graph flattens and the variance increases. But this can be resolved by scaling the number of iterations with m accordingly.
Some *plots* have been included at the end of the question to portray the variation with respect to m.
The graph also varies with the probability distribution, in that there are more peaks for smaller n if the probability distribution is spread apart (for example, $(0.5, 0, 0, \ldots, 0.5)$). But again, it will become more uniform for larger n and more number of iterations.
The shape of the graph is akin to a bell curve, which arises from the Central Limit Theorem. Thus, the graph looks like a normal distribution regardless of the probability distribution for a sufficiently large number of iterations which is in agreement with the theorem.



(i) Graphs for $m = 10$; iterations $\approx 10^4$;
(clockwise from top-left) $n = (8, 50, 200, 800)$

Clearly, the graph is concentrating towards a particular value (expected value).



(ii) Graphs for $n = 50$; iterations $\approx 10^4$;
(clockwise from top-left) $m = (5, 10, 100, 500)$

Clearly, the graph's peak is becoming less prominent and the graph itself is spreading more with increasing m.

Appendix

- Although the sub-parts of a question have been solved separately, they have not always been solved independently. Some results/explanations from other parts have been used.
- Each question has been started from a new page.
- Minor results (if any) have been $\boxed{\text{boxed}}$. These are used in the computation of the final result.
- The final answers/results have been summarised at the end of each sub-part and they have been colored differently.
- Local links have been colored in *Navy Blue*, while external links have been colored in Cerulean.
- Comments/ Minor explanations have been colored in *Gray*.

Acknowledgements

- I had discussed the answers for the first *3* questions with *Soham Samaddar (200990)*.
- I had discussed parts *a* and *b* of Question 3 with *Akhil Agrawal (200076)*.