CS201A: Endsem Examination

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Question 1. (10 marks) Consider the following arguement:

> Let $\mathbb{U}$ be the set of all sets. Define a partial ordering on $\mathbb{U}$ by inclusion: $A \leq B$ iff $A \subseteq B$ for $A, B \in \mathbb{U}$. Consider a chain C of $\mathbb{U}$ under this partial ordering: $C : A_1 \leq A_2 \leq A_3 \leq \cdots$. Define $B = \bigcup_{i \geq 1} A_i$. Clearly, $B \in \mathbb{U}$ and it is an upper bound of the chain C. Hence, Zorn's Lemma implies that $\mathbb{U}$ has a maximal element, say M.

The argument is clearly wrong since M is not a maximal element: $M \subset \{M, \{M\}\} \in \mathbb{U}$. Identify which step in the argument is wrong and why.

Solution 1.

Let there be a set $\mathbb{A}$ and R is partial order defined on $\mathbb{A}$. According to Zorn's Lemma, if every chain of $(\mathbb{A}, R)$ has an upper bound, then $(\mathbb{A}, R)$ has a maximal element.

Claim: The set of all sets, $\mathbb{U}$, does not exist.

Proof. Consider the power set of $\mathbb{U}$ to be P. Let the cardinality of $\mathbb{U}$ be $\mathcal{N}_0$.

- We know the cardinality of power set > cardinality of the set itself. Let cardinality of P be $\mathcal{N}_1$, therefore we have $\mathcal{N}_1 > \mathcal{N}_0$.
- Since every element of the power set P is also a set, it will be an element of $\mathbb{U}$ which is the set of all sets. That is, $Y \in \mathbb{U} \,\forall\, Y \in P \Rightarrow P \subseteq \mathbb{U}$.
- But any subset of $\mathbb{U}$ will have its cardinality $\leq \mathcal{N}_0$. Since $P \subseteq \mathbb{U}$, therefore $\mathcal{N}_1 \leq \mathcal{N}_0$.

We have obtained two inequalities:

$$\mathcal{N}_1 > \mathcal{N}_0$$
$$\mathcal{N}_1 \leq \mathcal{N}_0$$

Which is clearly a contradiction, hence such $\mathbb{U}$ cannot exist. □

Therefore, the first step in the argument that is "Let $\mathbb{U}$ be the set of all sets" is incorrect, as such a $\mathbb{U}$ cannot exist.

□ □ □

Question 2. (20 marks) Let $(G, \cdot)$ be a finite group with the property that there exists only one element, $a_2 \in G$ such that $a_2 \neq e$ and $a_2^2 = a_2 \cdot a_2 = e$. Define a bipartite graph $H = (G,\ G,\ E)$ as follows.

Edge $(a, b) \in E$ if $a \neq e$ and $b = a^k$ for some $1 < k \leq s$, or $a = e$ and $b = a_2$.

Here, s is the smallest number greater than zero such that $a^s = e$. Prove that the graph H has a perfect matching.

<u>Solution 2.</u> We intend to find a bijective function $f : G \to G$, if it exists, and convert it into a perfect matching in H. This is possible for the following reasons-

- Any perfect matching in $H = (G,\ G,\ E)$ can easily be converted to a bijective function f by assigning $f(a) = b$ if $(a,\ b)$ is an edge in the perfect matching.
  f will be bijective because each $a \in \boldsymbol{G}$ has exactly one edge in the perfect matching of $H = (\boldsymbol{G},\ G,\ E)$, i.e., a has a unique image in f. Further, each $b \in \boldsymbol{G}$ also has exactly one edge in the perfect matching of $H = (G,\ \boldsymbol{G},\ E)$, i.e., b has a unique pre-image in f.

- Any bijective f can be converted to a perfect matching by drawing an edge $(a,\ f(a))$, given the constraint that $(a,\ f(a)) \in E$.

Lemma 2.1: For every element $a \in G$, the set $S_a = \{a^r \mid r \in \mathbb{N}\}$ is finite.

Proof. We have $a \in G$, and by closure property over multiplication we obtain

$$a^r \in G \,\forall\, r \in \mathbb{N}$$

$\therefore$ Every element of S_a is an element of G. $\Rightarrow S_a \subseteq G$
Since G is finite, S_a has to be finite. $\square$

Lemma 2.2: For each $a \in G$, $\exists\, 1 < x \in \mathbb{N}$ such that $a^x = e$.

Proof. Consider the set $S_a = \{a^r \mid r \in \mathbb{N}\}$. By *Lemma 2.1*, we know S_a is finite. Let the cardinality of S_a be $\mathcal{N}$.
Consider any $\mathcal{N} + 1$ distinct natural numbers. For each number n we obtain a^n. Clearly these will be $\mathcal{N} + 1$ in number too. Taking these as pigeons, and elements of S_a as holes, by pigeon-hole principle, there are 2 distinct numbers, say i, j such that $a^i = a^j$. (Assume $i > j$ without loss of generality)

Since G is a group, inverse of every element in G exists. Let the inverse of a^j be denoted by a^{-j}.

$$\therefore\ a^j \cdot a^{-j} = e$$

We have $a^i \ = \ a^j$

$$\begin{aligned}
&\Rightarrow a^i \cdot a^{-j} \ = \ a^j \cdot a^{-j} && \text{(Multiply by } a^{-j} \text{ both sides)}\\
&\Rightarrow a^i \cdot a^{-j} \ = \ e\\
&\Rightarrow (a^{i-j} \cdot a^j) \cdot a^{-j} \ = \ e\\
&\Rightarrow a^{i-j} \cdot (a^j \cdot a^{-j}) \ = \ e\\
&\Rightarrow a^{i-j} \cdot e \ = \ e\\
&\Rightarrow a^{i-j} \ = \ e
\end{aligned}$$

$0 < (i-j) \in \mathbb{N}$. We choose $x = i - j$, and we are done, as $a^x \ = \ e$. □

We choose the minimum of these numbers x for a given a and denote it by $\mathcal{O}(a)$. It is easy to see that $\mathcal{O}(a)$ is equal to s mentioned in the question.

Observation 2.1 No element, other than a_2 and e, is its own inverse. Follows from the uniqueness of e and a_2, such that $a_2^2 = e$.

Corollary 2.1 Each element a is distinct from its inverse a^{-1}, subject to the constraint that $a \notin \{a_2, e\}$. Or, in other words,

$$a \neq a^{-1} \ \forall \ a \in G \backslash \{e, a_2\}$$

Corollary 2.2 From *Lemma 2.2* and *Obs 2.1*, we obtain that $\mathcal{O}(a) > 2 \ \forall \ a \in G\backslash\{e, a_2\}$, as only $\mathcal{O}(e) = 1$, and only $\mathcal{O}(a_2) = 2$, but $\mathcal{O}(a)$ exists and is finite $\forall \ a \in G$.

Lemma 2.3: $(g, \ g^{-1}) \in E \ \forall \ g \in G\backslash\{e, a_2\}$

Proof. By *Lemma 2.2* and *Cor 2.2*, for each $g \in G\backslash\{e, a_2\} \subseteq G$, we have $2 < \mathcal{O}(g) \in \mathbb{N}$ such that $e = g^{\mathcal{O}(g)}$.
On multiplying both sides with g^{-1}, we get,

$$g^{\mathcal{O}(g)-1} = g^{-1}$$

$\therefore$ Choose $b \leftarrow g^{-1}, a \leftarrow g, k \leftarrow (\mathcal{O}(g) - 1)$. From *Cor 2.2*, it easy to see that,

$$1 < \mathcal{O}(g) - 1 = k < \mathcal{O}(g) = s$$

As all the conditions are satisfied, we get

$$(g, g^{-1}) = (a, b) \in E \ \forall \ g \in G\backslash\{e, a_2\}$$

□

Corollary 2.3 $(a^{-1}, a) \in E \ \forall \ a \in G \backslash \{e, a_2\}$. Follows from *Lemma 2.3* on choosing $g \leftarrow a^{-1}$.

We define the intended bijective mapping f as follows:

$$f(a) = \begin{cases} a_2 & \text{If } a = e \qquad \because (e, a_2) \in E \\ e & \text{If } a = a_2 \qquad \because e \ = \ a_2^2 \\ a^{-1} & \text{Otherwise} \qquad \text{Follows from } \textit{Lemma 2.3} \end{cases}$$

Onto: Let a be any element in the co-domain, G, of f. If $a = e$, then its pre-image is easily seen to be a_2, while if $a = a_2$, then its pre-image is seen to be e. Otherwise, its pre-image is simply a^{-1}.
$\because$ Each element in the co-domain, G, has a pre-image, thus f is *onto*.

One-One: Let $a \in G \backslash \{e, a_2\}$. Then f maps a to its inverse a^{-1} which cannot be in $\{e, a_2\}$ as e, and a_2 are their own inverses, and we know that inverse of any element in a group is unique.
If $a = e$, then f maps it to $a_2 (\neq e)$ which is not in $G \backslash \{e, a_2\}$, else, if $a = a_2$, then f maps it to $e (\neq a_2)$ which is again, not in $G \backslash \{e, a_2\}$.
Hence, f is *one-one*.

So, there is at least one bijective mapping f from G to G which *"respects"* the edge set E. This bijective mapping can be converted to a perfect matching as shown *here*.

Alternate

Using *Lemma 2.3* and uniqueness of inverses, and using the fact that $\{(e, a_2), (a_2, e)\} \subseteq E$, we can take any $U \subseteq G$, then we are guaranteed that $|N(U)| \geq |U|$, $N(U)$ being the neighbor set of U, because each element in U will have an edge with at least its (unique) inverse, except for e and a_2, for which, the candidate edges have been swapped.

Hence, we can invoke the theorem that

> A bipartite graph $G = (V_1, V_2, E)$ has a perfect matching if and only if $|V_1| = |V_2|$, and for every $U \subseteq V_1$, $|N(U)| \geq |U|$.

By taking, $V_1 \leftarrow G$, $V_2 \leftarrow G$, so $|V_1| = |V_2|$ follows trivially, and $|N(U)| \geq |U|$ follows for reasons mentioned above.

Hence, $H = (G, G, E)$ has a perfect matching.

$\square \square \square$

Question 3. (5+5+5+10+5 marks) Let R be a ring and $a \in R$.

Define $(a) = \{b \cdot a \mid b \in R\}$.

- Prove that (a) is an ideal of R.

Let polynomial $C(x,\, y) = (x^2 + y^2 - 1) \cdot x$. The curve $C(x, y) = 0$ is a unit circle plus y-axis on the plane. $(C) = \{Q(x,\, y) \cdot C(x,\, y) \mid Q(x,\, y) \in \mathbb{R}[x,\, y]\}$ is an ideal of the ring $\mathbb{R}[x,\, y]$, the ring of polynomials in two variables with coefficients in $\mathbb{R}$.

Define $R = \mathbb{R}[x,\, y]/(C)$. For any point $P \in \mathbb{R} \times \mathbb{R}$ on the plane, define $R_P = \{\frac{f}{g} \mid f,\, g \in R$ and $g(P) \neq 0\}$ and $I_P = \{\frac{f}{g} \mid f,\, g \in R$ and $g(P) \neq 0$ and $f(P) = 0\}$. Prove that

- R_P is a ring.
- I_P is a maximal ideal of R_P.
- For point $P = (1, 0)$, $I_P = (y)$.
- For point $P = (0, 1)$, $(x) \subseteq I_P$.

It can be shown that $I_P \neq (x)$. Therefore, ring R_P contains information about whether curve C is *degenerate* at point P.

<u>Solution 3.</u>

- $(a) \coloneqq \{b \cdot a \mid b \in R\}$ is an ideal of R.
  Let $(R, \cdot, +)$ be the ring with $+$ as the addition operation and $\cdot$ as the multiplication operation.
  - Let $a_0 \in (a)$. Then $a_0 = b \cdot a$ for some $b \in R$. As $\cdot$ has closure in R, $a_0 = b \cdot a \in R$. Hence, $a_0 \in R \;\forall\; a_0 \in (a) \Rightarrow (a) \subseteq R$.
  - Let $a_1,\ a_2 \in (a)$. $\therefore a_1 = b_1 \cdot a$ and $a_2 = b_2 \cdot a$ for some $b_1,\ b_2 \in R$. $a_1 + a_2 \in R$-

$$\begin{aligned} a_1 + a_2 &= b_1 \cdot a + b_2 \cdot a \\ &= (b_1 + b_2) \cdot a && \{\text{From Distributivity of } \cdot \text{ over } + \text{ in } R \\ &= (b) \cdot a && \{\text{From Closure property of } + \text{ in } R \\ &= b \cdot a \in (a) \end{aligned}$$

  - Let $c \in R$ and $a_0 \in (a)$. $\therefore a_0 = d \cdot a$ for some $d \in R$. $c \cdot a_0 \in (a)$-

$$\begin{aligned} c \cdot a_0 &= c \cdot (d \cdot a) \\ &= (c \cdot d) \cdot a && \{\text{From Associativity of } \cdot \text{ in } R \\ &= (b) \cdot a && \{\text{From Closure of } \cdot \text{ in } R \\ &= b \cdot a \in (a) \end{aligned}$$

  Hence, (a) is an ideal of R.

Throughout the rest of the discussion, we restrict any point to be only from the set

$$S = \{P \mid C(P) = 0\}$$

We also use $[f]$ to denote the equivalence class of $f \in \mathbb{R}[x,\ y]$ under R, $+$ to denote the addition operation on R and $\cdot$ to denote the multiplication operation on R.

Lemma 3.1 (Evaluating Equivalence Classes at Points)**.** For a fixed $[f] \in \mathbb{R}[x,\, y]/(C)$, and any polynomial $f \in [f]$, the value of $f(P)$ remains constant.

Proof. Any $f \in [f]$ can be represented as

$$f = f_0 + C \cdot h$$

where f_0 is the residue polynomial modulo C and h is any polynomial in $\mathbb{R}[x,\, y]$.
For any point $P \in S$,

$$\begin{aligned} f(P) &= f_0(P) + C(P) \cdot h(P) \\ &= f_0(P) + Q(P) \cdot 0 \\ &= f_0(P) \end{aligned}$$

Hence, we can define $[f](P) = f_0(P)$ □

Note This allows us to refer to any equivalence class $[f]$ by its *"residue"* polynomial f_0 (residue with respect to C).

- $R_p := \{\frac{f}{g} \mid f,\, g \in R \text{ and } g(P) \neq 0\}$ is a ring, $P \in S$

  Addition on R_P:

Closure: Let $r_1 = \frac{[f_1]}{[g_1]},\ r_2 = \frac{[f_2]}{[g_2]} \in R_P$. Then,

$$r_1 + r_2 = \frac{[f_1]}{[g_1]} + \frac{[f_2]}{[g_2]} = \frac{[f_1] \cdot [g_2] + [f_2] \cdot [g_1]}{[g_1] \cdot [g_2]} = \frac{[f_1 \cdot g_2] + [f_2 \cdot g_1]}{[g_1 \cdot g_2]} = \frac{[f_1 \cdot g_2 + f_2 \cdot g_1]}{[g_1 \cdot g_2]}$$

Now, $[g_1](P) \neq 0$, and $[g_2](P) \neq 0 \Rightarrow [g_1 \cdot g_2](P) \neq 0$, and of course, $([f_1] \cdot [g_2] + [f_2] \cdot [g_1]) \in R$ and $[g_1 \cdot g_2] \in R$ by closure of addition and multiplication on R. Thus $r_1 + r_2 \in R_P$.

Associative: Let $r_1 = \frac{[f_1]}{[g_1]},\ r_2 = \frac{[f_2]}{[g_2]},\ r_3 = \frac{[f_3]}{[g_3]} \in R_P$. Then,

$$\begin{aligned} (r_1 + r_2) + r_3 &= \left(\frac{[f_1]}{[g_1]} + \frac{[f_2]}{[g_2]}\right) + \frac{[f_3]}{[g_3]} = \frac{[f_1] \cdot [g_2] + [f_2] \cdot [g_1]}{[g_1] \cdot [g_2]} + \frac{[f_3]}{[g_3]} \\ &= \frac{[f_1] \cdot [g_2] \cdot [g_3] + [f_2] \cdot [g_1] \cdot [g_3] + [f_3] \cdot [g_1] \cdot [g_2]}{[g_1] \cdot [g_2] \cdot [g_3]} \\ r_1 + (r_2 + r_3) &= \frac{[f_1]}{[g_1]} + \left(\frac{[f_2]}{[g_2]} + \frac{[f_3]}{[g_3]}\right) = \frac{[f_1]}{[g_1]} + \frac{[f_2] \cdot [g_3] + [f_3] \cdot [g_2]}{[g_2] \cdot [g_3]} \\ &= \frac{[f_1] \cdot [g_2] \cdot [g_3] + [f_2] \cdot [g_3] \cdot [g_1] + [f_3] \cdot [g_2] \cdot [g_1]}{[g_1] \cdot [g_2] \cdot [g_3]} \end{aligned}$$

Thus, $(r_1 + r_2) + r_3 = r_1 + (r_2 + r_3)$ follows from commutativity of multiplication in R.

Commutative: Let $r_1 = \frac{[f_1]}{[g_1]},\ r_2 = \frac{[f_2]}{[g_2]} \in R_P$. Then,

$$r_1 + r_2 = \frac{[f_1]}{[g_1]} + \frac{[f_2]}{[g_2]} = \frac{[f_1]\cdot[g_2] + [f_2]\cdot[g_1]}{[g_1]\cdot[g_2]} = \frac{[f_1\cdot g_2] + [f_2\cdot g_1]}{[g_1\cdot g_2]} = \frac{[f_1\cdot g_2 + f_2\cdot g_1]}{[g_1\cdot g_2]}$$
$$r_2 + r_1 = \frac{[f_2]}{[g_2]} + \frac{[f_1]}{[g_1]} = \frac{[f_2]\cdot[g_1] + [f_1]\cdot[g_2]}{[g_2]\cdot[g_1]} = \frac{[f_2\cdot g_1] + [f_1\cdot g_2]}{[g_2\cdot g_1]} = \frac{[f_2\cdot g_1 + f_1\cdot g_2]}{[g_2\cdot g_1]}$$

Thus, $r_1 + r_2 = r_2 + r_1$ follows from commutativity of addition and multiplication in R.

Identity: $\frac{[0]}{[1]}$ serves as the identity element of addition over R_P, as, for any $\frac{[f]}{[g]} \in R_P$, we have

$$\frac{[f]}{[g]} + \frac{[0]}{[1]} = \frac{[f]\cdot[1] + [g]\cdot[0]}{[g]\cdot[1]} = \frac{[f\cdot 1] + [g\cdot 0]}{[g\cdot 1]} = \frac{[f] + [0]}{[g]} = \frac{[f]}{[g]}$$

Most of the simplifications follow from $[0]$ being the identity of addition and $[1]$ being the identity of multiplication, except for $[g]\cdot[0] = [0]$, which follows from,

$$[g](P)\cdot[0](P) = g_0(P)\cdot 0(P) = g_0(P)\cdot 0 = 0 \in [0]$$

where $g_0 \in [g]$ and 0 is the zero polynomial.

Inverse: Let $a = \frac{[f]}{[g]} \in R_P$, then, it is easy to see that $b = \frac{[-f]}{[g]} \in R_P$, is the inverse of a. More formally, we evaluate,

$$a + b = \frac{[f]}{[g]} + \frac{[-f]}{[g]} = \frac{f_0}{g_0} + \frac{-f_0}{g_0} = \frac{f_0 - f_0}{g_0} = \frac{0}{g_0} = 0 = \frac{[0]}{[1]}$$

Hence, every element has a unique inverse associated to it in R_P.

Multiplication on R_P

Closure: Let $r_1 = \frac{[f_1]}{[g_1]},\ r_2 = \frac{[f_2]}{[g_2]} \in R_P$. Then,

$$r_1\cdot r_2 = \frac{[f_1]}{[g_1]}\cdot\frac{[f_2]}{[g_2]} = \frac{[f_1]\cdot[f_2]}{[g_1]\cdot[g_2]} = \frac{[f_1\cdot f_2]}{[g_1\cdot g_2]}$$

Again, $[f_1\cdot f_2],\ [g_1\cdot g_2] \in R$ by closure of multiplication over R, and $[g_1](P) \neq 0$ and $[g_2](P) \neq 0 \Rightarrow [g_1\cdot g_2](P) \neq 0$. Thus, $r_1\cdot r_2 \in R_P$.

Associative: Let $r_1 = \frac{[f_1]}{[g_1]},\ r_2 = \frac{[f_2]}{[g_2]},\ r_3 = \frac{[f_3]}{[g_3]} \in R_P$. Then,

$$(r_1\cdot r_2)\cdot r_3 = (\frac{[f_1]}{[g_1]}\cdot\frac{[f_2]}{[g_2]})\cdot\frac{[f_3]}{[g_3]} = (\frac{[f_1]\cdot[f_2]}{[g_1]\cdot[g_2]})\cdot\frac{[f_3]}{[g_3]} = \frac{[f_1]\cdot[f_2]\cdot[f_3]}{[g_1]\cdot[g_2]\cdot[g_3]}$$
$$r_1\cdot(r_2\cdot r_3) = \frac{[f_1]}{[g_1]}\cdot(\frac{[f_2]}{[g_2]}\cdot\frac{[f_3]}{[g_3]}) = \frac{[f_1]}{[g_1]}\cdot(\frac{[f_2]\cdot[f_3]}{[g_2]\cdot[g_3]}) = \frac{[f_1]\cdot[f_2]\cdot[f_3]}{[g_1]\cdot[g_2]\cdot[g_3]}$$

Thus, $(r_1\cdot r_2)\cdot r_3 = r_1\cdot(r_2\cdot r_3)$ is a direct consequence of associativity of multiplication over R.

Identity: $\frac{[1]}{[1]}$ serves as the identity of multiplication over R_P, as for any $\frac{[f]}{[g]} \in R_P$, we have

$$\frac{[f]}{[g]} \cdot \frac{[1]}{[1]} = \frac{[f] \cdot [1]}{[g] \cdot [1]} = \frac{[f \cdot 1]}{[g \cdot 1]} = \frac{[f]}{[g]}$$

Follows from [1] being the identity of in R.

Distributive: Let $r_1 = \frac{[f_1]}{[g_1]}$, $r_2 = \frac{[f_2]}{[g_2]}$, $r_3 = \frac{[f_3]}{[g_3]} \in R_P$. Then,

$$\begin{aligned} r_1 \cdot (r_2 + r_3) &= \frac{[f_1]}{[g_1]} \cdot \big(\frac{[f_2]}{[g_2]} + \frac{[f_3]}{[g_3]}\big) = \frac{[f_1]}{[g_1]} \cdot \big(\frac{[f_2] \cdot [g_3] + [f_3] \cdot [g_2]}{[g_2] \cdot [g_3]}\big) \\ &= \frac{[f_1] \cdot ([f_2] \cdot [g_3] + [f_3] \cdot [g_2])}{[g_1] \cdot [g_2] \cdot [g_3]} \\ &= \frac{[f_1] \cdot [f_2] \cdot [g_3] + [f_1] \cdot [f_3] \cdot [g_2]}{[g_1] \cdot [g_2] \cdot [g_3]} \\ r_1 \cdot r_2 + r_1 \cdot r_3 &= \frac{[f_1]}{[g_1]} \cdot \frac{[f_2]}{[g_2]} + \frac{[f_1]}{[g_1]} \cdot \frac{[f_3]}{[g_3]} = \frac{[f_1] \cdot [f_2]}{[g_1] \cdot [g_2]} + \frac{[f_1] \cdot [f_3]}{[g_1] \cdot [g_3]} \\ &= \frac{[f_1] \cdot [f_2] \cdot [g_3]}{[g_1] \cdot [g_2] \cdot [g_3]} + \frac{[f_1] \cdot [f_3] \cdot [g_2]}{[g_1] \cdot [g_2] \cdot [g_3]} \\ &= \frac{[f_1] \cdot [f_2] \cdot [g_3] + [f_1] \cdot [f_3] \cdot [g_2]}{[g_1] \cdot [g_2] \cdot [g_3]} \end{aligned}$$

Thus, $r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$.

- $I_P = \{\frac{f}{g} \mid f, g \in R \text{ and } g(P) \neq 0 \text{ and } f(P) = 0\}$ is a maximal ideal of R_P.
  We first show that I_P is an ideal of R_P.

  - Let $k = \frac{[f_1]}{[g_1]}$, $h = \frac{[f_2]}{[g_2]} \in I_P$, then,

    $$k + h = \frac{[f_1]}{[g_1]} + \frac{[f_2]}{[g_2]} = \frac{[f_1] \cdot [g_2] + [f_2] \cdot [g_1]}{[g_1] \cdot [g_2]} = \frac{[f_1 \cdot g_2 + f_2 \cdot g_1]}{[g_1 \cdot g_2]} = \frac{[f]}{[g]} \in I_P$$

    Clearly, $[g](P) \neq 0$. Further $[f](P) = 0$, because,

    $$\begin{aligned} [f](P) = [f_1 \cdot g_2 + f_2 \cdot g_1](P) &= [f_1](P) \ \cdot \ [g_2](P) \ + \ [f_2](P) \ \cdot \ [g_1](P) \\ &= 0 \cdot [g_2](P) + 0 \cdot [g_1](P) = 0 + 0 = 0 \end{aligned}$$

    Hence, for any $k, h \in I_P$, we have $k + h \in I_P$
  - Let $a = \frac{[f_a]}{[g_a]} \in R_P$, and $h = \frac{[f_h]}{[g_h]} \in I_P$, then,

    $$a \cdot h = \frac{[f_a]}{[g_a]} \cdot \frac{[f_h]}{[g_h]} = \frac{[f_a] \cdot [f_h]}{[g_a] \cdot [g_h]} = \frac{[f_a \cdot f_h]}{[g_a \cdot g_h]} = \frac{[f]}{[g]} \in I_P$$

    Clearly, $[g](P) \neq 0$. Further $[f](P) = 0$, because,

    $$[f](P) = [f_a \cdot f_h](P) = [f_a](P) \cdot [f_h](P) = [f_a](P) \cdot 0 = 0$$

    Hence, for any $a \in R_P$ and $h \in I_P$, we obtain $a \cdot h \in I_P$.

Thus, I_P is an ideal of R_P. Further, to show that I_P is a maximal ideal of R_P, we assume towards contradiction that $\exists\ J$, an ideal of R_P, such that $I_P \subset J \subset R_P$.
We intend to show that $\frac{[1]}{[1]} \in J$.
As $I_P \subset J$, and $[f](P) = 0\ \forall\ \frac{[f]}{[g]} \in I_P$, there must be $\frac{[f_c]}{[g_c]} \in J$, such that $[f_c](P) \neq 0$.
Also, $[g_c](P) \neq 0 \because \frac{[f_c]}{[g_c]} \in J \subset R_P$.
Clearly, $\frac{[g_c]}{[f_c]} \in R_P\ \because [f_c](P) \neq 0$. We choose, $a_c = \frac{[g_c]}{[f_c]} \in R_P$, and $h_c = \frac{[f_c]}{[g_c]} \in J$, and observe their product,

$$a_c \cdot h_c = \frac{[g_c]}{[f_c]} \cdot \frac{[f_c]}{[g_c]} = \frac{g_{c_0}(P)}{f_{c_0}(P)} \cdot \frac{f_{c_0}(P)}{g_{c_0}(P)} = 1 = \frac{1}{1}$$

$\because J$ is an ideal, $\therefore a_c \cdot h_c = \frac{[1]}{[1]} \in J$. But, the existence of identity of multiplication in $J \Rightarrow J = R_P$. A contradiction!
Thus, I_P is indeed, a maximal ideal of R_P.

- For $P = (1, 0)$, $I_p = (y)$.

  Lemma 3.2. $(y) \subseteq I_p$

  Proof. For any $a \in (y)$, let $a = \frac{[y]\cdot[f]}{[g]}$, where $\frac{[f]}{[g]} \in R_p$.
  Now, looking at the numerator, $[y \cdot f](P) = [y](P) \cdot [f](P) = 0$, as $[y](P)$ is trivially 0.
  The denominator, $[g](P) \neq 0$, by definition of R_p.

  $$\therefore a \in I_p$$

  $$\Rightarrow (y) \subseteq I_p$$

  □

  Lemma 3.3. $[x - 1] \in (y)$, where $[x - 1] \in R$.

  Proof. We need $[x - 1] = [y] \cdot r$ for some $r \in R$. Let $r = \frac{[f]}{[g]}$

  $$\begin{aligned}
  [x-1]\cdot[g] &= [y]\cdot[f] \\
  [(x-1)\cdot g] &= [y\cdot f] \\
  [(x-1)\cdot g] - [y\cdot f] &= [(x-1)\cdot g - y\cdot f] = 0 \\
  \therefore (x-1)\cdot g - y\cdot f &= a \in (C) \\
  \Rightarrow (x-1)\cdot g - y\cdot f &= a = C\cdot h \qquad\qquad \text{where h} \in \mathbb{R}[x,\ y] \\
  (x-1)g - yf &= h((x-1)x(x+1) + xy^2) \\
  (x-1)(g + hx(x+1)) &= y(f + xy)
  \end{aligned}$$

  We, rather arbitrarily, set $h = 1$. Now, inspecting the factors, we set:

  $$\begin{aligned}
  f + xy &= x - 1 \\
  f &= x - xy - 1
  \end{aligned}$$

$$g + x(x+1) = y$$
$$g = y - x^2 - x$$

Additionally, the only constraint we have is that $[g](P) \neq 0$. Indeed, $[g](P) = -2$ in our case.

$$\therefore [x-1] = [y] \cdot \frac{[x - xy - 1]}{[y - x^2 - x]} \Rightarrow [x-1] \qquad \in (y)$$

Hence proved. $\square$

Lemma 3.4. For any $f \in \mathbb{R}[x, y]$, if $f(1,0) = 0$, then f can be expressed as a linear combination of $(x-1)$ and y.

Proof. We can separate out the terms in f into the two sets - one which only contains powers of x, and the other terms. Note that the constant term is also to be considered in the first set. A y can be factored out from the terms in the second set. We rewrite f as follows

$$f(x,y) = q(x) + yh(x,y)$$

Now,

$$f(1,0) = q(1) + 0 \cdot h(1,0)$$
$$q(1) = 0$$

By remainder theorem, $(x-1)$ is a factor of $q(x)$. We write $q(x) = (x-1)q'(x)$.

$$\therefore f(x,y) = (x-1)q'(x) + yh(x,y)$$

Hence proved. $\square$

Finally, for any $r \in I_p$, let $r = \frac{[f]}{[g]}$

$$r = \frac{[(x-1)q' + yh]}{[g]} \qquad \text{(From lemma 3.4)}$$

$$r = \frac{[x-1][q']}{[g]} + \frac{[y][h]}{[g]}$$

Since, $[x-1] \in (y)$ (lemma 3.3) and $\frac{[q']}{[g]} \in R_p$, $\frac{[x-1][q']}{[g]} \in (y)$ as (y) is an ideal

Let $y_1 = \frac{[x-1][q']}{[g]} \in (y)$ and $y_2 = \frac{[y][h]}{[g]} \in (y)$

$$r = y_1 + y_2$$

$$r \in (y) \qquad \text{(Since } (y) \text{ is an ideal)}$$

This implies $I_p \subseteq (y)$, but from lemma 3.2, we get

$$I_p = (y)$$

Hence proved.

- For point $P = (0,1)$, $(x) \subseteq I_P$.
  It is easy to see that $(x)(P) = 0$ simply because $x(P) = 0$. More formally, let $a = [x] \cdot \frac{[f]}{[g]} \in (x)$ where $\frac{[f]}{[g]} \in R_P$. We re-write $[x]$ as $\frac{[x]}{[1]}$, since, $[1]$ is the identity of multiplication. Then, we have,

  $$a = \frac{[x]}{[1]} \cdot \frac{[f]}{[g]} = \frac{[x] \cdot [f]}{[1] \cdot [g]} = \frac{[x \cdot f]}{[g]}$$

  We claim that $a \in I_P$ because, $[g](P) \neq 0$, as $\frac{[f]}{[g]} \in R_P$, and

  $$[x \cdot f](P) = [x](P) \cdot [f](P) = 0 \cdot [f](P) = 0$$

  Hence, $a \in I_P \; \forall \; a \in (x) \Rightarrow (x) \subseteq I_P$.

$\square\square\square$