CS656

Algorithmic Game Theory

Algorithmic: Study the models using algorithms, reductions, complexity, etc.

Game Theory: A model to study multiple (rational) agents with conflicting objective functions

Decision Problem

- A · Actions : Alternatives available to agents
- X · Outcomes : Possible consequences of agents' actions
- $\succeq$ · Preferences : Ranking of the set of possible outcomes
- Rationality can be assumed
- Example of a DP : Prisoner's Dilemma, Battle of Sexes

	S	C
Silent	-2,-2	-5,-1
Confess	-1,-5	-4,-4

	Opera	Football
O	2,1	0,0
F	0,0	1,2

Stronger Reasoning:
- PD's outcome – (C, C) because always better to confess.

Weaker Reasoning:
- BoS's outcome – (O, O) or (F, F) because if they are on (O, O), neither of them want to change

- Eg: Traveler's Dilemma
  - Bet b/w 2 - 100
  - If (m, m) ⟶ Both get m
  - If (m, m') ⟶ (m < m') ⟶ Get (m+2, m-1)
  - Eq[m] is (2, 2) by iterative elimination
- Eg: Guess 2/3 of the average
  - n players pick $x_i \in [0, 100]$
  - $y = \sum_i x_i / n$
  - j wins if x_j closest to y
  - Eq[m] is (0, 0)

Games & Logic

- The model checking problem for a logic L

  Given a structure A and a formula $\alpha \in L$, does $A \vDash \alpha$?

- Reduction to a game:

  Reduce the model checking problem to the strategy problem in the game $G(A, \alpha)$ consisting of two players V (verifier) and F (Falsifier)

  $\therefore A \vDash \alpha$ iff V has a winning strategy in $G(A, \alpha)$

Mechanism Design

Given an outcome, what game should be played so that it is the most optimal?

Preference Relation

- Weak Preference Relation

  $x \succeq y$: x is at least as good as y

- Strict Preference Relation

  $x \succeq y$ but $y \not\succeq x$: x is strictly better than y

- Indifference Relation

  $x \approx y$: $x \succeq y$ & $y \succeq x$

1) Completeness : $\forall\, x, y \in X$, either $x \succeq y$ or $y \succeq x$
2) Reflexivity : $\forall\, x \in X$, $x \succeq x$
3) Transitivity : $\forall\, x, y, z \in X$, if $x \succeq y$ & $y \succeq z$ then $x \succeq z$

Rational Preference Ordering

A preference relation which is complete, reflexive and transitive.

UTILITY THEORY

Utility Function

A utility function (or payoff function) $u: X \to \mathbb{R}$ represents the preference relation $\succeq$ if for any pair $x, y \in X$,

$u(x) \geq u(y)$ iff $x \succeq y$

Lottery

- Lotteries over X is $\mathcal{L} = \{[p_1(x_1), p_2(x_2), \ldots, p_k(x_k)] \mid p_i \geq 0$ and $p_1 + p_2 + p_3 + \ldots + p_k = 1\}$
- Easy to see that deterministic outcomes are also lotteries.
- So, the definition of a utility function can be updated to include lotteries as well:

A utility function u represents the preference relation $\succeq$ if for any pair of lotteries L, L', we have $u(L) \geq u(L')$ iff $L \succeq L'$

★ von Neumann-Morgenstern utility function (linear utility function)

A utility function u is said to be linear if for every lottery $\mathcal{L} = \{[p_1(x_1), \ldots, p_k(x_k)]\}$, the following holds:

$$u(L) = p_1 u(x_1) + p_2 u(x_2) + \ldots + p_k u(x_k)$$

- Which preference relation(s) can be represented by a linear utility function?
  Need more constraints than just rational preference relation: von-Neumann axioms.

Compound Lottery

$$\hat{L} = [q_1(L_1),\ q_2(L_2),\ \ldots,\ q_m(L_m)]$$

Ex: $\hat{L} = [3/4(L_1),\ 1/4(L_2)]$, $L_1 = [1/3(x_1),\ 2/3(x_2)]$

$L_2 = [1/2(x_1),\ 1/2(x_2)]$

- A utility function u represents the preference relation $\succeq$ if for any pair of lotteries $\hat{L}, \hat{L}'$, we have $u(\hat{L}) \geq u(\hat{L}')$ iff $\hat{L} \succeq \hat{L}'$

AXIOMS OF UTILITY THEORY

A1 Continuity

For every triplet of outcomes $x \succeq y \succeq z$, there exists a $\theta \in [0, 1]$ such that $y \approx [\theta(x),\ (1-\theta)(z)]$

A2 Monotonicity

Let $\alpha, \beta \in [0, 1]$ and suppose that $x \succ y$ then

$[\alpha(x),\ (1-\alpha)(y)] \succeq [\beta(x),\ (1-\beta)(y)]$ iff $\alpha \geq \beta$

Lemma: If a preference relation satisfies the above two axioms, and if $x \succsim y \succsim z$ and $x \succ z$, then the value of θ defined in the axiom of Continuity is unique.

Corollary: If a preference relation $\succsim$ satisfies the above two axioms and if $x_k \succ x_1$, then for each $j = 1, \dots, k$ there exists a unique θ_j such that

$$x_j \approx [\theta_j(x_k), (1-\theta_j)(x_1)]$$

A3 Simplification

Let $\hat{\mathcal{L}} = [q_1(\mathcal{L}_1), q_2(\mathcal{L}_2), \dots, q_m(\mathcal{L}_m)]$ and for each $j : 1 \leq j \leq m$, let $\mathcal{L}_j = [p_1^j(x_1), \dots, p_k^j(x_k)]$.

For each $i = 1, \dots, k$, let $r_i = \sum_{j=1}^{j=m} q_j p_i^j$.

Consider the lottery $\mathcal{L} = [r_1(x_1), \dots, r_k(x_k)]$, then $\hat{\mathcal{L}} = \mathcal{L}$

A4 Independence

Let $\hat{\mathcal{L}} = [q_1(\mathcal{L}_1), q_2(\mathcal{L}_2), \dots, q_m(\mathcal{L}_m)]$ and $\mathcal{M}$ be a simple lottery. If $\mathcal{L}_j \approx \mathcal{M}$, then $\hat{\mathcal{L}} \approx [q_1(\mathcal{L}_1), \dots, q_j(\mathcal{M}), q_{j+1}(\mathcal{L}_{j+1}), \dots, q_m(\mathcal{L}_m)]$

A Characterization Theorem

If the preference relation $\succsim$ over $\hat{\mathcal{L}}$ is complete, transitive and satisfies the four von-Neumann Morgenstern axioms then $\succsim$ can be represented by a linear utility function

Proof. Assume that the most desired outcome $x_k \succ x_1$. By Lemma 1, for each outcome x_j, we have $x_j \approx [\theta_j(x_k), (1-\theta_j)(x_1)]$

Utility Function.

Suppose $\hat{L} = [q_1(L_1), \ldots, q_m(L_m)]$ and for each $j : 1 \leq j \leq m$, let

$$L_j = [p_1^j(x_1), p_2^j(x_2), \ldots, p_k^j(x_k)]$$

Then for each $i = 1, \ldots, k$, $r_i := q_1 p_i^1 + \ldots + q_m p_i^m$

can think of u as $u(x_1)=0$, $u(x_k)=1$

$$u(\hat{L}) := r_1\theta_1 + r_2\theta_2 + \ldots + r_k\theta_k$$

For every simple lottery

$L = [p_1(x_1), p_2(x_2), \ldots, p_k(x_k)]$, $u(L) = \sum_{j=1}^{j=k} p_j\theta_j$

Outcome x_j is same as the lottery $L = [1(x_j)]$, which is the same as $\hat{L} = [1(L)]$. So, outcome of $\hat{L}$ is x_j with probability 1. So, we have

$$r_i = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \longrightarrow u(x_j) = \theta_j$$

To show: The function u is linear.

$\Rightarrow$ Need to show that for each simple lottery $L = [p_1(x_1), p_2(x_2), \ldots, p_k(x_k)]$,

$$u(L) = \sum_{j=1}^{j=k} p_j u(x_j)$$

We have,

$$u(L) = \sum_{j=1}^{j=k} p_j\theta_j, \quad u(x_j) = \theta_j$$

$$= \sum_{j=1}^{j=k} p_j u(x_j)$$

To show: The function u is a utility function.

$\Rightarrow$ Need to show that for any pair of compound lotteries $\hat{L}$ & $\hat{L}'$,

$\hat{\mathcal{L}} \succeq \hat{\mathcal{L}}'$ iff $u(\hat{\mathcal{L}}) \geq u(\hat{\mathcal{L}}')$

Claim 1. $\hat{\mathcal{L}} \approx [u(\hat{\mathcal{L}})(x_k),\ (1-u(\hat{\mathcal{L}}))(x_1)]$ for every $\hat{\mathcal{L}}$.

Assuming the claim, the result follows from monotonicity of $\succeq$. $\square$

Proof.

Let $\hat{\mathcal{L}} = [q_1(\mathcal{L}_1), \ldots, q_m(\mathcal{L}_m)]$ and for each $j : 1 \leq j \leq m$, let

$$\mathcal{L}_j = [p_1^j(x_1), p_2^j(x_2), \ldots, p_k^j(x_k)]$$

Let $r_i = \sum_{j=1}^{j=m} q_j p_i^j \Rightarrow \hat{\mathcal{L}} \approx [r_1(x_1), \ldots, r_k(x_k)]$

and let $M_i = [\theta_i(x_k), (1-\theta_i)(x_1)]$ for every $1 \leq i \leq k$.

By definition, we have $x_i \approx M_i$. Thus, k applications of the independence axiom yields:

$$\hat{\mathcal{L}} \approx [r_1(M_1), \ldots, r_k(M_k)]$$

Let r^* be the total probability of x_k in the lottery on RHS. Then,

$$r^* = \sum_{i=1}^{i=k} r_i \theta_i = u(\hat{\mathcal{L}})$$

By simplification axiom,

$$\hat{\mathcal{L}} \approx [r^*(x_k), (1-r^*)(x_1)]$$
$$= [u(\hat{\mathcal{L}})(x_k), (1-u(\hat{\mathcal{L}}))(x_1)]$$

$\square$

Fundamental Assumption

Preferences can always be reduced to Utility Functions $\rightarrow$ Need only study Utility Functions.

(This assumption is the reason why a lot of GT books directly use numbers in examples)

STRATEGIC GAMES

Normal Form Games

- $N = \{1, 2, 3, \dots, n\}$
- S_i : Finite set of strategies for player i, $i \in N$
- $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$: The game
- $s_i \in S_i$: A strategy for player i
- $s \in S = S_1 \times S_2 \times \dots \times S_n$: A strategy profile

Notation. For $i \in N$ and $s \in S$,

$\boxed{s_{-i}} := (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ Every strategy excluding i's strategy s_i

So, $s = (s_{-i}, s_i)$

Utility Function $u_i : S_1 \times S_2 \times \dots \times S_n \rightarrow \mathbb{R}$

Ex:

Column Player

Row Player / Player 1

	X	Y	Z
A	(3, 2)	(1, 1)	(2, 0)
B	(1, 0)	(3, 1)	(1, 2)

Prisoner's Dilemma

	D	C
D	(4, 4)	(0, 5)
C	(5, 0)	(1, 1)

D: Don't Confess
C: Confess

$S_1 = \{A, B\}, \quad S_2 = \{X, Y, Z\}$

Strict Dominance

s_i is strictly dominated if there exists $t_i \in S_i$ s.t. for every strategy profile s_{-i} :

$$u_i(s_i, s_{-i}) < u_i(t_i, s_{-i})$$

Ex: In Prisoner's Dilemma, D_1 is strictly dominated by C_1.

* Strongly dominated strategies can be

eliminated under the assumptions:

- All players are rational
- Rational players do not choose strictly dominated strategies.

Ex:

	X	Y	Z
A	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

An elimination sequence is Z (Y), B (A), X (Y)

$\Rightarrow$ Outcome : (A, Y)

* Requires another assumption for iterative elimination:

Common Knowledge of Rationality:

- Everyone knows everyone else is rational.
- Everyone knows everyone knows everyone is rational.
- ...

Restriction of a Game:

G' is a **restriction** of the game G if $G' = (N, (R_i)_{i \in N}, (u_i)_{i \in N})$, $R_i \subseteq S_i$

- Given two restrictions G_1 & G_2 of G, s.t. $G_1 \neq G_2$, $G_2 \subseteq G_1$, define the relation:

$\boxed{\rightarrow_s}$:= $G_1 \rightarrow_s G_2$ if $\forall\ i \in N$ $\forall\ s_i \in G_1 \setminus G_2$, $\exists\ s_i' \in G_1$ s.t. s_i is strictly dominated by s_i' in G_1.

- Note that $\rightarrow_s$ is not transitive (example can be found above)

- Let us instead define $\rightarrow_s^*$ as the reflexive transitive closure of $\rightarrow_s$
  $\{ G \rightarrow_s^* G'$ iff $G = G'$ or $\exists$ a sequence of games $(G_1, G_2, \dots, G_k)$ s.t.
  $G \rightarrow_s G_1 \rightarrow_s G_2 \dots \rightarrow_s G_k \rightarrow_s G' \}$

IESDS

– Iterative Elimination of Strongly Dominated Strategies

- Let G' be obtained from G if $G \rightarrow_s^* G'$

* If for no restriction G'' of G, it holds that $G' \rightarrow_s G''$, then G' is the **outcome** of IESDS

Ex:

	X	Y	Z
A	3,2	2,0	3,1
B	1,1	1,2	3,0
C	2,2	1,1	2,1

$Z, C, B, Y \rightarrow (A, X)$

$C, Z, B, Y \rightarrow \underbrace{(A, X)}$

Outcome is the same

Theorem: IESDS is order-independent

Ex:

	X	Y
A	1,2	2,3
B	2,2	2,0

A is weakly-dominated by B $\rightarrow (B, X)$

Weak Dominance

$s_i \in S_i$ is **weakly dominated** if $\exists$ a $t_i \in S_i$ s.t.

- $\forall s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) \leq u_i(t_i, s_{-i})$

- $\exists \; t_{-i} \in S_{-i}, \quad u_i(s_i, s_{-i}) < u_i(t_i, t_{-i})$

IEWDS

Ex:

	X	Y	Z
A	1,2	2,3	0,3
B	2,2	2,1	3,2
C	2,1	0,0	1,0

AZCY : (B, X)
CXYA : (B, Z)
AYZ : (B, X), (C, X)

Sealed Bid Auctions

- 1 seller, one object for sale
- n bidders, each bidder has a value $v_i \geq 0$ for the object.
- Process:
  - Bidders choose a bid (b_i) and submit it in a sealed envelope
  - Auctioneer opens the envelopes and selects the highest bidder as the winner.

* First price auction: Winner pays what he bids.

N-bidders $\quad (v_i)_{i \in N}$ — valuation
$\quad (b_i)_{i \in N}$ — bid

$$u_i(b) = \begin{cases} v_i - b_i & \text{if } i = \text{argsmax } b \\ 0 & \text{otherwise} \end{cases}$$

- For a tuple $b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ denote the least l s.t. $b_l = \max_{k \in N} b_k$ by argsmax b

* Second Price Auction

$$u_i(b) = \begin{cases} v_i - \max\limits_{j \neq i} b_j & \text{if } i = \operatorname{argmax} b \\ 0 & \text{otherwise} \end{cases}$$

- When the highest 2 bids co-incide, the utilities are the same as in first price auction.
- Winner's curse is possible
  - When $v_i < b_i$ and some other player bids between (v_i, b_i)

Theorem. In second price auction, $b_i = v_i$ $\forall\ i \in N$ weakly dominates all other strategies.

Proof.

Consider $i \in N$ whose valuation is v_i.
Divide $S_i = [0, \infty)$ into 3 sets:

Strategies: $b_i \in [0, v_i)$
$b_i = v_i$
$b_i \in (v_i, \infty)$

Let $B_{-i} = \max\limits_{i \neq j} b_j$ and

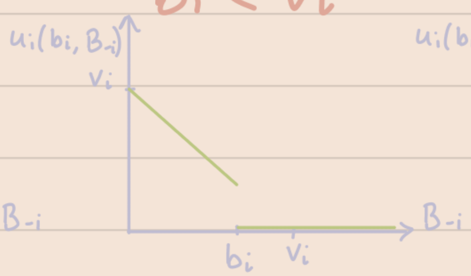
$$u_i(b) = \begin{cases} v_i - B_{-i} & b_i > B_{-i} \\ 0 & b_i < B_{-i} \end{cases}$$

The dependency that u_i has on b_{-i} is only through B_{-i} — denote $u_i(b_i, B_{-i})$

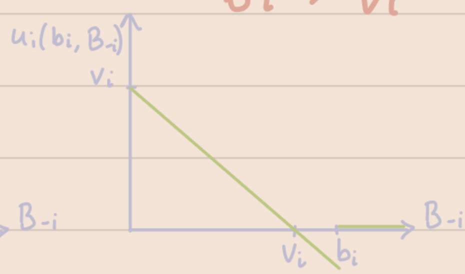
- If

$b_i = v_i$

$b_i < v_i$



$b_i > v_i$



$\therefore$ $b_i = v_i$ weakly dominates all other strategies

The proof is complete. $\square$

- In the general case, VCG mechanism can be used.

Nash Equilibrium

$s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is a **Nash Equilibrium** if for all $i \in N$, for all $s_i \in S_i$,

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*)$$

Ex:

	x	y	z
A	0,6	6,0	4,3
B	6,0	0,6	4,3
C	3,3	3,3	5,5

→ (C, Z) is an NE

* $s_i' \in S_i$ is a **profitable deviation** of player i from $s \in S$, if

$$u_i(s_i', s_{-i}) > u_i(s)$$

- s^* is an NE if no player has a profitable deviation from s.

- $s_i \in S_i$ is a **best response** to $s_{-i} \in S_{-i}$ if

$$u_i(s_i, s_{-i}) = \max_{t_i \in S_i} u_i(t_i, s_{-i})$$

- s^* is an NE if s_i^* is a best response to s_{-i}^* for all $i \in N$

Ex:

	D	C
D	4,4	0,5
C	5,0	1,1

	C	F
C	2,1	0,0
F	0,0	1,2

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

NO NE

Security

Ex:

	X	Y
A	2,1	2,-20
B	3,0	-10,1
C	-100,2	3,3

- NE : (C, Y)
- If R player does not trust C, and wants to be safe instead. will play A
- Similarly, C player would prefer X for safety ∵ outcome is non-negative.
- Maxmin value : (2, 0), strategy: (A, X)

* If player i chooses s_i, then the worst utility that he can get is

$$\min_{t_{-i} \in S_{-i}} u_i(s_i, t_{-i})$$

- Maxmin value or Security value for player i :

$$\underline{v_i} = \max_{s_i \in S_i} \min_{t_{-i} \in S_{-i}} u_i(s_i, t_{-i})$$

Maxmin Strategy

s_i^* that guarantees the maxmin value. So,

$$u_i(s_i^*, t_{-i}) \geq \underline{v_i} \quad \forall \; t_{-i} \in S_{-i}$$

Ex:

	X	Y
A	3,1	0,4
B	2,3	1,1

(X, B) ⟶ (2, 3)

or (Y, B) ⟶ (1, 1)

Theorem. A strategy of player i that weakly dominates all other strategies is a maxmin strategy for that player (it is a best response to any strategy profile of others)

Corollary. In G, if every player has a strategy that weakly dominates all other strategies, the profile of such weak dominant strategies is a maxmin profile and a Nash equilibrium.

Theorem. If every $i \in N$ has a strategy s_i^* that strictly dominates all of i's other strategies, then $s^* = (s_1^*, \dots, s_n^*)$ is a unique Nash equilibrium and a unique maxmin profile.

Theorem. Every Nash equilibrium s^* satisfies the condition

$$u_i(s_i^*) \geq \underline{v}_i \quad \forall\ i \in N$$

Proof.

- $\forall\ s_i \in S_i,\quad u_i(s_i, s_{-i}^*) \geq \min\limits_{s_{-i}} u_i(s_i, s_{-i})$

- $u_i(s^*) = \max\limits_{s_i \in S_i} u_i(s_i, s_{-i}^*) \geq \max\limits_{s_i} \min\limits_{s_{-i}} u_i(s_i, s_{-i}) = \underline{v}_i$

$\square$

Theorem. Let $\hat{s}_j \in S_j$ be a weakly dominated strategy of player j. Let $\hat{G}$ be the game obtained from G by elimination of $\hat{s}_j$. Then the maxmin value of j in $\hat{G}$ is equal to maxmin of j in G.

Proof.

$$\underline{v}_j = \max_{s_j} \min_{s_{-j}} u_j(s_j, s_{-j}) \text{ in } G$$

$$\underline{\hat{v}}_j = \max_{s_j,\, s_j \neq \hat{s}_j} \min_{s_{-j}} u_j(s_j, s_{-j})$$

maxmin of j in $\hat{G}$

- Let t_j weakly dominate $\hat{s}_j$ in G. Then,

$$u_j(\hat{s}_j, s_{-j}) \leq u_j(t_j, s_{-j}) \quad \forall s_{-j} \in S_{-j}$$

and,

$$\min_{s_{-j}} u_i(\hat{s}_j, s_{-j}) \leq \min_{s_{-j}} u_i(t_j, s_{-j}) \leq \max_{s_j} \min_{s_{-j}} u_j(s_j, s_{-j})$$

$$\underline{v}_j = \max_{s_j} \min_{s_{-j}} u_j(s_j, s_{-j})$$

$$= \max \left\{ \max_{s_j \neq \hat{s}_j} \min_{s_{-j}} u_j(s_j, s_{-j}),\ \min_{s_{-j}} u_j(\hat{s}_j, s_{-j}) \right\}$$

$$= \max_{s_j \neq \hat{s}_j} \min_{s_{-j}} u_j(s_j, s_{-j}) = \underline{\hat{v}}_j$$

$\square$

Theorem. Let $\hat{G}$ be a restriction of G ($\hat{G}$ is obtained from G by elimination of strategies, $\hat{S}_i \subseteq S_i \; \forall i \in N$). If s^* is a Nash equilibrium in G and $s_i^* \in \hat{S}_i \;\; \forall \; i \in N$ then s^* is an NE in $\hat{G}$.

Proof. $\because$ s^* is an NE in G, we have $\forall i \in N$, $u_i(s_i, s_{-i}^*) \leq u_i(s^*) \;\; \forall \; s_i \in S_i$
$\because$ $\hat{S}_i \subseteq S_i \;\; \forall \; i \in N$, we have

$$u_i(s_i, s_{-i}^*) \leq u_i(s^*) \quad \forall \; s_i \in \hat{S}_i$$

$\because$ s^* is a profile in $\hat{G}$, it means that s^* is an NE in $\hat{G}$.

- Elimination of strategies can introduce new NE's.

Theorem. If weakly dominated strategies are eliminated then no new NE are introduced.

Theorem. Let $\hat{s}_j \in S_j$ be a weakly dominated strategy of player j in G. Let $\hat{G}$ be the game obtained by eliminating $\hat{s}_j$. Then every NE in $\hat{G}$ is also a NE in G.

That is: $NE(\hat{G}) \subseteq NE(G)$

Proof.

$$\hat{S}_i = \begin{cases} S_i & \text{if} \quad i \neq j \\ S_j \setminus \{\hat{s}_j\} & \text{if} \quad i = j \end{cases}$$

Suppose $s^* \in NE(\hat{G})$. Then by

definition of NE

$u_i(s_i, s^*_{-i}) \le u_i(s^*) \quad \forall\, i \neq j, \ \forall s_i \in \hat{S}_i \ (= S_i)$

$u_j(s_j, s^*_{-j}) \le u_j(s^*) \quad \forall\, s_j \in \hat{S}_j$

To show: $s^* \in NE(G)$

- For $i \neq j$, $\hat{S}_i = S_i$. So, i has no profitable deviation
- j does not have a profitable deviation to any strategy in $\hat{S}_j = S_j \setminus \{\hat{s}_j\}$

To show: Player j does not have a profitable deviation to $\hat{s}_j$

$\because$ $\hat{s}_j$ is weakly dominated, $\exists\ t_j \in S_j$ that weakly dominates $\hat{s}_j$

$\therefore$ $t_j \neq \hat{s}_j$, $t_j \in \hat{S}_j$

$u_j(\hat{s}_j, s_{-j}) \le u_j(t_j, s_{-j}) \quad \forall\, s_{-j} \in S_{-j}$

$u_j(\hat{s}_j, s_{-j}) \le u_j(t_j, s^*_{-j})$

$\le u_j(s^*_j, s^*_j)$

$\therefore$ $s^* \in NE(G)$ $\square$

Corollary. Let $\hat{G}$ be derived from G by IEWDS. Then, for every $s^* \in NE(\hat{G})$, we have $s^* \in NE(G)$

- If G can be solved by IEWDS, with outcome s^* then $s^* \in NE(G)$

- Elimination of weakly dominated strategies can result in the loss of NE's.
- Is it the case that $NE(G) \subseteq NE(\hat{G})$? No.

$$\begin{array}{|c|c|} \hline 0,0 & 2,1 \\ \hline 3,2 & 1,2 \\ \hline \end{array} \Rightarrow \begin{array}{|c|} \hline 2,1 \\ \hline 1,2 \\ \hline \end{array}$$

Theorem. Let $\hat{s_j} \in S_j$ be a strictly dominated strategy of $j \in N$. Let $\hat{G}$ be derived from G by eliminating $\hat{s_j}$. Then $NE(G) = NE(\hat{G})$

Corollary. A strictly dominated strategy cannot be an NE of a game.

Theorem. Let $\hat{G} = (N, (\hat{S}_i)_{i \in N}, (u_i)_{i \in N})$ be a restriction of G. If $s^* \in NE(G)$ and $s_i^* \in \hat{S}_i \ \forall \ i \in N$, then $s^* \in NE(\hat{G})$.

Theorem. Let G be a game and let $\hat{s_j} \in S_j$ be a strictly dominated strategy of $j \in N$. Let $\hat{G}$ be derived from G by eliminating $\hat{s_j}$. Then, $NE(G) = NE(\hat{G})$.

Proof.

Observation. $NE(\hat{G}) \subseteq NE(G)$

To show. $NE(G) \subseteq NE(\hat{G})$

Let $s^* \in NE(G)$

Suffices to show: $s^* \in \hat{G}$

$\because$ $\hat{G}$ is derived from G by eliminating $\hat{s_j}$.

Suffices to show: $s_j^* \neq \hat{s_j}$

By assumption, $\hat{s_j}$ is strictly dominated in G. So $\exists$ $t_j \in S_j$ s.t.

$$u_j(\hat{s_j}, s_{-j}) < u_j(t_j, s_{-j}) \quad \forall \ s_{-j} \in S_{-j}$$

$\because$ $s^* \in NE(G)$, we have,

$$u_j(\hat{s_j}, s_{-j}^*) < u_j(t_j, s_{-j}^*) \leq u_j(s_j^*, s_{-j}^*) \Rightarrow \hat{s_j} \neq s_j^* \quad \square$$

Corollary. If G can be solved by IESDS — with outcome s^*, then s^* is a unique NE in G.

Zero-Sum Games

	x	y	z
A	3, -3	-5, 5	-2, 2
B	1, -1	4, -4	1, -1
C	6, -6	-3, 3	-5, 5

NE as well as Maxmin

☆ Also, ∵ sum is zero, can just use $u = u_1$, $u_2 = -u$

☆ $\underline{v}_1 = \max_{s_1} \min_{s_2} \underset{u_1 = -u}{u}(s_1, s_2)$,

$\underline{v}_2 = \max_{s_2} \min_{s_1} \underset{u_2 = -u}{u_2}(s_1, s_2) = -\min_{s_2} \max_{s_1} u(s_1, s_2)$

maxmin value

$$\boxed{\underline{v} := \max_{s_1} \min_{s_2} u(s_1, s_2)}$$

minmax value

$$\boxed{\bar{v} := \min_{s_2} \max_{s_1} u(s_1, s_2)}$$

* Two player zero-sum game:

$G = (N = \{1, 2\}, (S_i)_{i \in N}, u)$

$u : S \to \mathbb{R}$

Player 1's maxmin value $\underline{v}_1$

Player 2's maxmin value $\underline{v}_2$

As defined above

* $\underline{v}$: maxmin value of G

$\bar{v}$: minmax value of G

If $\underline{v} = \bar{v}$, then the two player zero-sum game has a value $v = \underline{v} = \bar{v}$

The maxmin and minmax strategies

are called the optimal strategies.

Ex:

	x	y	z	$\min_{s_2} u(s_1, s_2)$
A	3	-5	-2	-5
B	1	4	1	1
C	6	-3	-5	-5 / $1 = \max_{s_1} \min_{s_2} u = \underline{v}$
$\max_{s_1} u$	6	4	1 / $1 = \min_{s_2} \max_{s_1} u = \bar{v} = \underline{v} \Rightarrow v = 1$	

	x	y	$\max_{s_2}$
A	1	-1	1
B	-1	1	1 / 1
$\min_{s_1} u$	-1	-1	/ -1 $\quad \underline{v} \neq \bar{v}$

Obs. 1. It could be that $\underline{v} \neq \bar{v}$

Obs. 2. It is always the case that $\underline{v} \leq \bar{v}$

Theorem. If a two player zero-sum game has a value v and if s_1^* and s_2^* are optimal strategies then $s^* = (s_1^*, s_2^*)$ is an NE with payoff $(v, -v)$.

Proof.
$$u(s_1^*, s_2) \geq v \quad \forall \; s_2 \in S_2$$
$$u(s_1, s_2^*) \leq v \quad \forall \; s_1 \in S_1$$
$$s_2 \leftarrow s_2^*, \; s_1 \leftarrow s_1^* : \quad u(s_1^*, s_2^*) \leq v \leq u(s_1^*, s_2^*)$$
$$\Rightarrow \quad u(s_1^*, s_2^*) = v$$

Also,
$$u(s_1, s_2^*) \leq u(s_1^*, s_2^*) \quad \forall \; s_1 \in S_1$$
$$\& \; -u(s_1^*, s_2) \leq -u(s_1^*, s_2^*) \quad \forall \; s_2 \in S_2$$
$\therefore \; (s_1^*, s_2^*)$ is an NE with payoff $(v, -v)$ $\square$

Theorem. If (s_1^*, s_2^*) is an NE of a 2P-ZSG then the game has a value $v = u(s_1^*, s_2^*)$ and the strategies s_1^* and s_2^* are optimal strategies.

Proof. $\because \; (s_1^*, s_2^*)$ is an NE, we have,

(1) $u(s_1, s_2^*) \leq u(s_1^*, s_2^*) \quad \forall\ s_1 \in S_1$

(2) $u(s_1^*, s_2) \geq u(s_1^*, s_2^*) \quad \forall\ s_2 \in S_2$

Let $v = u(s_1^*, s_2^*)$

From (1), $u(s_1, s_2^*) \leq v \quad \forall\ s_1 \in S_1$

$\Rightarrow \max\limits_{s_1} u(s_1, s_2^*) \leq v$

$\Rightarrow \min\limits_{s_2} \max\limits_{s_1} u(s_1, s_2) \leq v$

$\Rightarrow \bar{v} \leq v$

Similarly, from (2), $\underline{v} \geq v$

Now, $\underline{v} \leq \bar{v}$ & $\underline{v} \geq v \geq \bar{v}$

Thus, v is the value of the game, and s_1^*, s_2^* are optimal strategies. $\square$

Consequence. In 2P-ZSG's, the concept of stability (NE) and security (Maxmin) coincide.

↪ Expected ∵ Nash used von-Neumann's notions of Maxmin in 2P-ZSG & generalized them to the notion of NE in more general games

Corollary. In a 2P-ZSG, if (s_1^*, s_2^*) & (t_1^*, t_2^*) are two Nash Equilibria, then

1. $u(s_1^*, s_2^*) = u(t_1^*, t_2^*)$
2. (s_1^*, t_2^*) and (t_1^*, s_2^*) are also Nash Equilibria.

Saddle Point

A pair of strategies (s_1^*, s_2^*) is a saddle point of a function $u : S_1 \times S_2 \to \mathbb{R}$ if

- $u(s_1^*, s_2^*) \geq u(s_1, s_2^*) \quad \forall\ s_1 \in S_1$
- $u(s_1^*, s_2^*) \leq u(s_1^*, s_2) \quad \forall\ s_2 \in S_2$

- In other words,

  $u(s_1^*, s_2^*)$: lowest value on the row s_1^*

  : highest value on the column s_2^*

Theorem. In a 2P-ZSG, (s_1^*, s_2^*) is a saddle point of u iff (s_1^*, s_2^*) are optimal strategies and $u(s_1^*, s_2^*)$ is the value of the game.

MIXED STRATEGIES

$G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$

$\hookrightarrow u_i : S \to \mathbb{R}$

$\hookrightarrow$ finite set of (pure) strategies

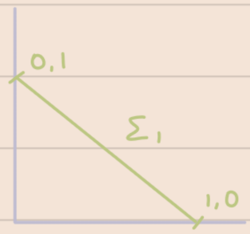
Mixed Strategy for player i

$$\Sigma_i = \{\sigma_i : S_i \to [0,1] \mid \sum_{s_i \in S_i} \sigma_i(s_i) = 1\}$$

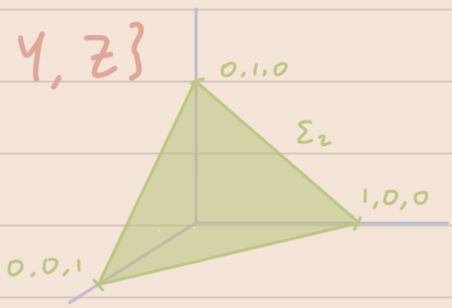
$\sigma_i(s_i)$

$\hookrightarrow$ probability of playing the pure strategy s_i

Ex: $S_1 = \{H, T\}$



$S_2 = \{X, Y, Z\}$



Mixed extension of a Game

- Mixed extension of G is $\Gamma = (N, (\Sigma_i)_{i\in N}, (u_i)_{i\in N})$

  Σ_i : Set of mixed strategies of player i

$$\Sigma = \Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n$$
$$u_i : \Sigma \to \mathbb{R}$$
$$u_i(\sigma) = \sum_{s \in S} u_i(s) \prod_{j \in N} \sigma_j(s_j)$$

If all, but one, variables are kept fixed, then the function is linear in the last variable

Theorem. u_i is a multi-linear function in n-variables. For every $i \in N$, $\forall\, \sigma_i, \sigma_i' \in \Sigma_i$ and for every $\lambda \in [0,1]$

$$u_i(\lambda \sigma_i + (1-\lambda)\sigma_i', \sigma_{-i}) = \lambda u_i(\sigma_i, \sigma_{-i}) + (1-\lambda) u_i(\sigma_i', \sigma_{-i}) \quad \forall\, \sigma_{-i} \in \Sigma_{-i}$$

Consequence 1. $u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i)\, u_i(s_i, \sigma_{-i})$

Consequence 2. u_i is a continuous function in the extension to mixed strategies (for all finite strategic games)

Nash Equilibrium

A mixed strategy profile σ^* is an NE if $\forall\, i \in N$, $\forall\, \sigma_i \in \Sigma_i$,

$$u_i(\sigma^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

Ex:

σ_1 \ σ_2	$\frac{1}{2}$ H	$\frac{1}{2}$ T
$\frac{1}{2}$ H	1, -1	-1, 1
$\frac{1}{2}$ T	-1, 1	1, -1

$$u_1(\sigma_1, \sigma_2) = 0$$
$$u_2(\sigma_1, \sigma_2) = 0$$

Theorem. Let $G = (N, (S_i), (u_i))$ and let Γ be the mixed extension of G.
$\sigma^* \in NE(G)$ iff $\forall\, i \in N$, $\forall\, s_i \in S_i$,

$$u_i(\sigma^*) \geq u_i(s_i, \sigma_{-i}^*)$$

Proof. ($\Rightarrow$) follows from the definition
($\Leftarrow$) Suppose σ^* satisfies

$$u_i(\sigma^*) \geq u_i(s_i, \sigma_{-i}^*) \quad \forall\, i \in N, \ \forall\, s_i \in S_i$$

Then, for each mixed strategy $\sigma_i \in \Sigma_i$,

$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}^*) &= \sum_{s_i \in S_i} \sigma_i(s_i)\, u_i(s_i, \sigma_{-i}^*) \\ &\leq \sum_{s_i \in S_i} \sigma_i(s_i)\, u_i(\sigma_i^*, \sigma_{-i}^*) \\ &= u_i(\sigma^*) \sum_{s_i} \sigma_i(s_i) = u_i(\sigma^*) \end{aligned}$$

$\square$

Theorem (Nash).
Every finite game has a Nash Equilibrium in mixed strategies.

Corollary [Minmax Theorem].
Every finite two player zero-sum game has a value in mixed stratgies.

$$v = \min_{\sigma_2} \max_{\sigma_1} u(\sigma_1, \sigma_2) = \max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2)$$

Theorem [Indifference Principle]
Let σ^* be an NE and s_i, $\hat{s}_i$ be two pure strategies of player i. If $\sigma_i^*(s_i) > 0$ and $\sigma_i^*(\hat{s}_i) > 0$, then

$$u_i(s_i, \sigma_{-i}^*) = u_i(\hat{s}_i, \sigma_{-i}^*)$$

Proof Idea.
Suppose there is a strict inequality.
Define σ_i:

$$\sigma_i(t_i) = \begin{cases} \sigma_i^*(t_i), & t_i \notin \{s_i, \hat{s}_i\} \\ 0, & t_i = \hat{s}_i \\ \sigma_i^*(s_i) + \sigma_i^*(\hat{s}_i), & t_i = s_i \end{cases}$$

Argue $u_i(\sigma_i, \sigma_{-i}^*) > u_i(\sigma^*)$

* A strategy $s_i \in S_i$ is **strictly dominated** by $\sigma_i \in \Sigma_i$ if $\forall\ s_{-i} \in S_{-i}$,

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$$

Ex:

	L	R
T	3,1	0,0
M	0,1	3,2
B	1,1	1,2

- B is not dominated by any other pure strategy, but is strictly dominated by the mixed strategy $[\frac{1}{2}(T), \frac{1}{2}(M)]$

Theorem. Let $G = (N, S_i, u_i)$. If $s_i \in S_i$ is strictly dominated by a mixed strategy $\sigma_i \in \Sigma_i$, then in every (mixed) NE in G, s_i is chosen with probability zero.

Proof. Let s_i be strictly dominated by σ_i. Let $\hat{\sigma} \in \Sigma$ where $\hat{\sigma}_i(s_i) > 0$.

To show: $\hat{\sigma} \notin NE(G)$

Suffices: $\hat{\sigma}_i$ is not a best response to $\hat{\sigma}_{-i}$

Define σ_i':

$$\sigma_i'(t_i) = \begin{cases} \hat{\sigma}_i(s_i)\,\sigma_i(s_i), & t_i = s_i \\ \hat{\sigma}_i(t_i) + \hat{\sigma}_i(s_i)\,\sigma_i(t_i) & t_i \neq s_i \end{cases}$$

Claim: $\sum_{s_i} \sigma_i(s_i) = 1$

$$\hookrightarrow \sum_{t_i \neq s_i} \hat{\sigma}_i(t_i) + \underbrace{\sum_{t_i \neq s_i} \hat{\sigma}_i(s_i)\,\sigma_i(t_i) + \hat{\sigma}_i(s_i)\,\sigma_i(s_i)}_{\sum_{t_i} \sigma_i(t_i) = 1 \Rightarrow \text{Simplified to } \hat{\sigma}_i(s_i)}$$

Claim: σ_i' is a better response to $\hat{\sigma}_{-i}$ $\left\{ \text{as compared to } \hat{\sigma}_i \right.$

That is, $u_i(\hat{\sigma}_i, \hat{\sigma}_{-i}) < u_i(\sigma_i', \hat{\sigma}_{-i})$

Use $u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i} \sigma_i(s_i)\, u_i(s_i, \sigma_{-i})$

$\square$

Nash's Theorem

- Every finite game has an NE in mixed strategies.
- **Brouwer's Fixed Point Theorem**

  Let X be a convex, compact set in d-dimensional Euclidean space. Let $f: X \rightarrow X$ be a continuous function. Then, there is an $x \in X$ s.t. $f(x) = x$
- In the 1-D case, it reduces to IVT
- Proving the theorem:

  Obs. $\Sigma = \Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n$ convex and compact subset of the Euclidean space $\mathbb{R}^{m_1 + m_2 + \dots + m_n}$

  Proof Sketch. Define $F: \Sigma \rightarrow \Sigma$ and show that
  - F is continuous
  - Every fixed point of F is an NE

  Nash's theorem then follows from the Brouwer's Fixed point theorem

  Proof. To define $F: \Sigma \rightarrow \Sigma$, $\left\{ \forall\ i \in N, \text{ let } S_i = \{s_i^1, s_i^2, \dots, s_i^{m_i}\} \right.$

  $$F(\sigma) = (F_i(\sigma))_{i \in N}$$

  $$g_i^j : \Sigma \rightarrow [0, \infty) \quad \forall\ i \in N,\ \forall\ j \in \{1, \dots, m_i\}$$

  Picks only the profitable deviations

  $$g_i^j = \max\{0,\ u_i(s_i^j, \sigma_{-i}) - u_i(\sigma)\}$$

  Obs. σ is an NE iff $g_i^j(\sigma) = 0$ $\forall\ i \in N,\ \forall\ j \in \{1, 2, \dots, m_i\}$

  Obs. $\forall\ i \in N,\ \forall\ j \in \{1, 2, \dots, m_i\}$, g_i^j is a continuous function.

  Gives more weightage to the strategies that offer profitable deviations using g_i

  $$F_i^j(\sigma) = \frac{\sigma_i(s_i^j) + g_i^j(\sigma)}{1 + \sum_{k=1}^{k=m_i} g_i^k(\sigma)}$$

Claim. For all σ, $F(\sigma) \in \Sigma$,

1. $F_i^j(\sigma) \geq 0 \quad \forall\ i \in N, \forall j \in \{1, \dots, m_i\}$
   - $g_i^j(\sigma)$ is non-negative. So, denominator of F_i^j is atleast 1
   - Numerator is non-negative
   - So $F_i^j(\sigma) \geq 0$
2. $\sum_{j=1}^{j=m_i} F_i^j(\sigma) = 1 \quad \forall\ i \in N$

Can be shown by expanding

Claim. F is a continuous function.

- Both numerator & denominator are continuous.
- Denominator is atleast 1.
- F is the ratio between two continuous function whose denominator is always positive.

Lemma. Let σ be a fixed point of F. Then $g_i^j(\sigma) = \sigma_i(s_i^j) \sum_{k=1}^{m_i} g_i^k(\sigma)$ $\forall\ i \in N, \forall\ j \in \{1, \dots, m_i\}$

Proof. $F(\sigma) = \sigma \Rightarrow F_i^j(\sigma) = \sigma_i(s_i^j)$

$$\Rightarrow \frac{\sigma_i(s_i^j) + g_i^j(\sigma)}{1 + \sum_{k=1}^{k=m_i} g_i^k(\sigma)} = \sigma_i(s_i^j) \quad \forall\ i \in N, \forall j \in \{1, \dots, m_i\}$$

$$\Rightarrow g_i^j(\sigma) = \sigma_i(s_i^j) \sum_{k=1}^{m_i} g_i^k(\sigma)$$

Theorem. Let σ be a fixed point of F. Then σ is a Nash Equilibrium

Proof. Assume σ is not an NE.

Then, $\exists\ i \in N$, $\ell \in \{1, \dots, m_i\}$ s.t. $g_i^\ell(\sigma) > 0 \Rightarrow \sum_{k=1}^{m_i} g_i^k(\sigma) > 0$

$\Rightarrow \sigma_i(s_i^j) > 0$ iff $g_i^j(\sigma) > 0 \ \forall j$

$\because g_i^{\ell}(\sigma) > 0 \Rightarrow \sigma_i(s_i^{\ell}) > 0$

$$u_i(\sigma) = \sum_{j=1}^{m_i} \sigma_i(s_i^j)\, u_i(s_i^j, \sigma_{-i})$$

$$u_i(\sigma) \sum_{j=1}^{m_i} \sigma_i(s_i^j) = \sum_{j=1}^{m_i} \sigma_i(s_i^j)\, u_i(s_i^j, \sigma_{-i})$$

$$\Rightarrow 0 = \sum_j \sigma_i(s_i^j)\left[u_i(s_i^j, \sigma_{-i}) - u_i(\sigma)\right]$$

$$= \sum_{\{j : \sigma_i(s_i^j) > 0\}} \sigma_i(s_i^j)\, \underbrace{\left[u_i(s_i^j, \sigma_{-i}) - u_i(\sigma)\right]}_{= g_i^j(\sigma) \text{ as } \sigma_i(s_i^j) > 0}$$

$\Rightarrow 0 = \sum_{\sigma_i(s_i^j) > 0} \sigma_i(s_i^j)\, g_i^j(\sigma) > 0 \Rightarrow\Leftarrow$

$\therefore \sigma$ must be an NE

Can anyways ignore such terms $\because$ they're equal to zero

$\sigma_i(s_i^j) = 0$
$\Rightarrow \sigma_i(s_i^j)(u_i(s_i^j, \sigma_{-i}) - u_i(\sigma)) \le 0$

$\sigma_i(s_i^j) > 0 \Leftrightarrow g_i^j(\sigma) > 0$
$\Leftrightarrow u_i(s_i^j, \sigma_{-i}) > u_i(\sigma)$

Order Independence

- Let D be a function that assigns to every G', a restriction of G, a subset of strategy profiles in G': $D_{G'} \subseteq \bigcup_{i \in N} S_i'$
- $s_i \in S_i$ is **D-dominated** in G' if $s_i \in D_{G'}$. $G' \rightarrow_D G''$ if $\forall\, i \in N$, $s_i \in G' \setminus G''$, s_i is D-dominated in G'.
- D is **order-independent** if for all initial games G, all iterations of $\rightarrow_D$ gives the same final outcome.
- D is **hereditary** if for all initial games G, all restrictions G' and G'' s.t. $G' \rightarrow_D G''$ and $s_i \in G''$, if s_i is D-dominated in G' then s_i is D-dominated in G''
- **Reduction System** $(A, \rightarrow)$
  - where A is a set and $\rightarrow \subseteq A \times A$
  - $b \in A$ is a **normal form** of $a \in A$ if $a \rightarrow^* b$ and there is no $c \in A$ s.t. $b \rightarrow c$
- $(A, \rightarrow)$ has the **unique normal form property** if every element in A has a unique normal form.

- $\rightarrow$ is ==weakly confluent== if for all $a, b, c \in A$, if $a \rightarrow b$ and $a \rightarrow c$ then $\exists\, d \in A$ s.t. $b \rightarrow^* d$ and $c \rightarrow^* d$
- Lemma [Newman]. Suppose $(A, \rightarrow)$ satisfies the following conditions:
  1. There exists no infinite sequence
  2. $\rightarrow$ is weakly confluent

  then $\rightarrow$ satisfies the unique normal form property.
- Lemma. The relation of being strictly dominated is hereditary.

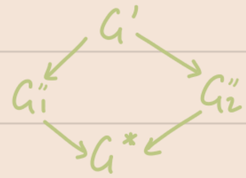
  Proof. Suppose $s_i \in G''$ is strictly dominated in G' and $G' \rightarrow_{SD} G''$. Then $\exists\, s_i'$ that strictly dominates s_i in G' and s_i' is not strictly dominated in G'. So, s_i' is not eliminated by $\rightarrow_{SD}$. That is, $s_i' \in G_i''$. Now, $G'' \subseteq G'$, so s_i' strictly dominates s_i in G''.
- Theorem. Every hereditary dominance relation is order independent.

  Proof. Suppose $G' \rightarrow_D G''$ for some hereditary dominance relation D and restrictions G' and G''. Let G^* be the restriction of G' obtained by removing all strategies that are D-dominated in G'. By definition, $G^* \subseteq G''$.

  If they are equal, then $G'' \rightarrow_D G'' = G^*$ trivially.

  Suppose $G'' \neq G^*$. Choose any $s_i \in G'' \setminus G^*$. s_i is D-dominated in G'. $\because D$ is hereditary, s_i is D-dominated in G'' as well $\therefore G'' \to_D G^*$

$\therefore \to_D$ is weakly confluent.

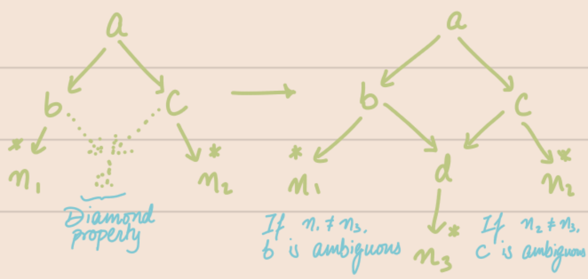


From Newman's Lemma,
$\to_D$ satisfies the unique normal form property
$\therefore D$ is order-independent.

- Proof of Newman's Lemma:

  Every element $a \in A$ has a normal form ($\because \nexists$ infinite sequences).
  Suffices to prove uniqueness.
  $a \in A$ is ambiguous if it has atleast 2 different normal forms.
  We show: for every ambiguous $a \in A$, $\exists\ b \in A$ s.t. $a \to b$ and b is ambiguous. } $\Rightarrow$ If $\exists$ ambiguous element, then $\exists$ infinite sequence $\Rightarrow \Leftarrow$ for Newman's Lemma

  Suppose a is ambiguous, then a has 2 distinct normal forms $n_1 \neq n_2$

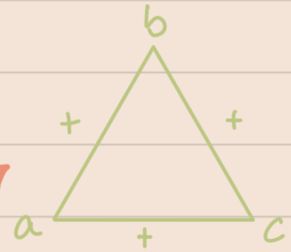


A Graph Problem

Hopfield network problem
More generally, the class is known as "Local Search Problems".

- An undirected graph $H = (V, E)$
- Integer valued weights on edges w_e — +/-
- State of a node $u \in V$ is $+1/-1$
- Configuration $s: V \to \{+1, -1\}$
- Each edge imposes a requirement on its endpoints: $e = (u, v)$ and
  if $w_e < 0$, then $s(u) = s(v)$
  $w_e > 0$, $s(u) \neq s(v)$

Q. Is there always a configuration satisfying all the edge requirements? No:

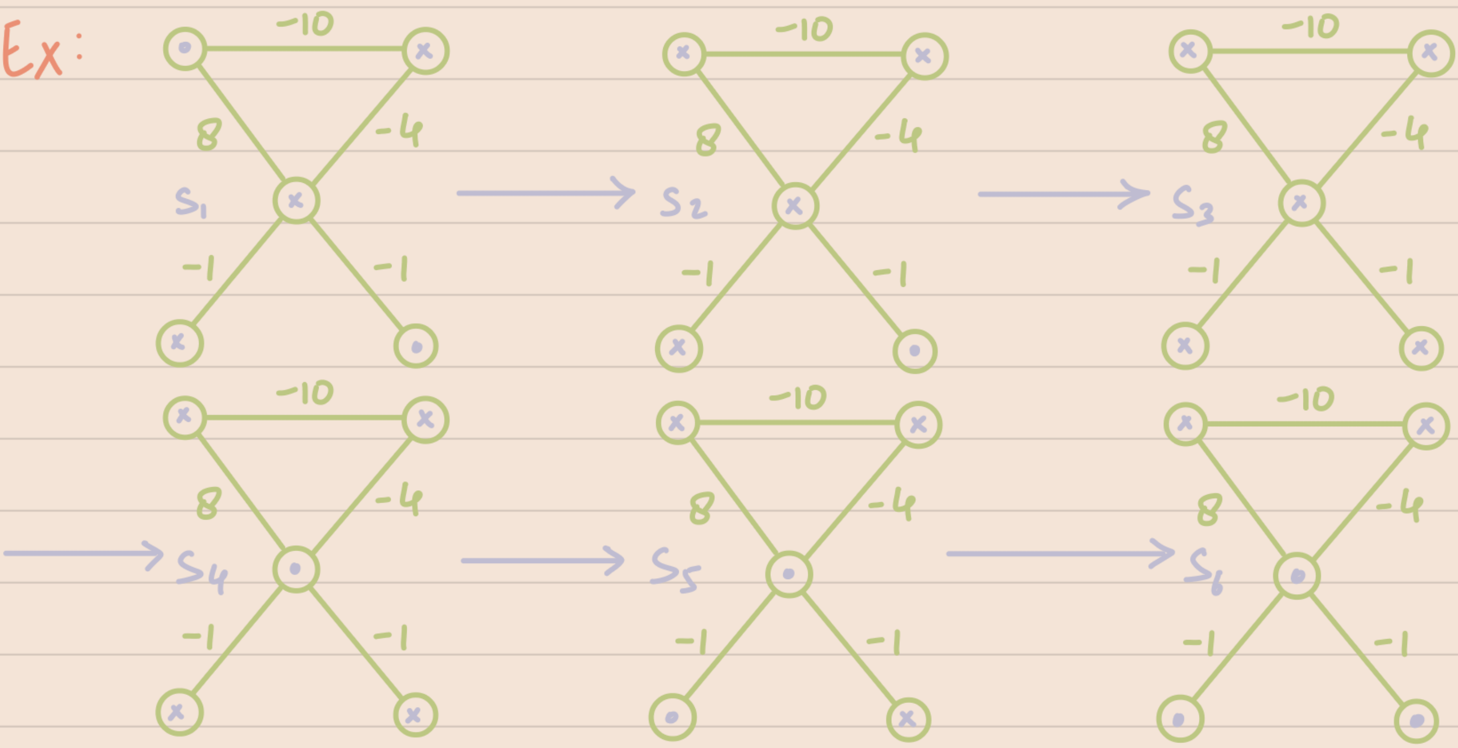


- Given a configuration s, we say that an edge $e = (u, v)$ is
  - good : $w_e < 0$, $s(u) = s(v)$ / $w_e > 0$, $s(u) \neq s(v)$ $\Big\}$ $w_e\, s(u) s(v) < 0$
  - bad : otherwise
- Node u is satisfied if the total absolute weight of all good edges incident on u is at least as large as the total absolute weight of all the bad edges incident on u:

$$\sum_{v:\, e=(u,v) \in E} w_e\, s_u\, s_v \leq 0$$

- A configuration is stable if all nodes are satisfied.

Ex:



Algorithm - State Flip
1. Start with an arbitrary configuration
2. While the configuration is not stable:
   2.1. Choose an unsatisfied node u
   2.2. Flip the state of u
   End-While

· Observation: If State-Flip terminates then there is a stable configuration.

Progress measure $\Phi(s)$

· No. of unsatisfied nodes: $s_3 \to s_4$ ✗
· Every node changes state at most once: ✗
$s_2 \to s_3 \to^* s_5 \to s_6$
· $\Phi(s) = \sum_{e:\text{good}} |w_e|$ (or $\sum_e w_e s_u s_v$)
· $\Phi(s) \geq 0$, $\Phi(s) \leq W = \sum_e |w_e|$

· Consider an arbitrary configuration s which is not stable. Let u be the unsatisfied node in s.

$s \to s'$ [u flips its state]

$\phi(s') = \phi(s) - g_u + b_u$

g_u: Abs. weight of all good edges incident on u
b_u: — bad —

Note: $b_u > g_u$ ⟶ u is not satisfied
$b_u \geq g_u + 1$ ⟶ integer weights of all edges

To show: $\underbrace{\phi(s') > \phi(s)}_{\substack{\phi \text{ is bounded and } \uparrow \text{ w/ each flip} \\ \Rightarrow \text{Algo always terminates}}}$ {Follows from the note}

· Theorem. Every local maxima of Φ is a stable configuration.

COMPUTATION OF NASH EQUILIBRIA

Algorithm [Best / Better Response Dynamics]

- Input : A game G

1. Choose an arbitrary strategy profile s.
2. Repeat until s is a Nash Equilibrium.
   2.1. Choose a player i, s.t. s_i is not a best response to s_{-i}
   2.2. Choose a better response s_i' for player i
   2.3. Set s_i to s_i'
3. Return s.

- A path is a sequence $s^1 s^2 \ldots$ such that for all $k > 1$, $\exists\ s_i' \neq s_i^k$ and $s^{k+1} = (s_i', s_{-i}^k)$
- An improvement path is a maximal path where for all $k > 1$, $u_i(s^{k+1}) > u_i(s^k)$ } This relation should hold atleast for the player i s.t. $s_i^k \neq s_i^{k+1}$
- G has the $\underbrace{\text{FIP}}$ if every improvement path is finite. Finite Improvement Property
- Observation : If G has the FIP then G has a Nash Equilibrium.

Potential Game

$G = (N, (S_i), (u_i))$

- A function $\phi : S \rightarrow \mathbb{R}$ is an (exact) potential function for G if (ϕ is independent of players.)

$$u_i(s_i', s_{-i}) - u_i(s) = \phi(s_i', s_{-i}) - \phi(s)$$

The equation only holds for the player changing their strategy

$\forall\ i \in N,\ \forall\ s_{-i} \in S_{-i},\ \forall\ s_i' \in S_i$

- A game that has a potential function is a **potential game**.
- Lemma. Every finite exact potential game has the FIP.

Consider an arbitrary improvement path $p = s^1, s^2, \dots$ we have

$$\phi(s^1) < \phi(s^2) < \dots$$

$\because$ S is finite, p has to be finite

Q. Does every game with the FIP have an exact potential function? No.

	X	Y
A	2,2	0,3
B	3,0	1,2

	X	Y
A	0	1
B	1	2

Ordinal Potential

$\phi : S \to \mathbb{R}$ is an **ordinal potential** function for G if it satisfies the property:

Δu_i should have the same sign as $\Delta\phi \equiv sig(\Delta u_i) = sig(\Delta\phi)$

$$u_i(s_i', s_{-i}) - u_i(s_i, s_{-i}) > 0 \quad \text{iff}$$
$$\phi(s_i', s_{-i}) - \phi(s_i, s_{-i}) > 0$$
$$\forall\ i \in N,\ \forall\ s_{-i} \in S_{-i},\ \forall\ s_i' \in S_i$$

- Lemma. Every finite ordinal potential game has the FIP! Same as before

Q. Does the existence of FIP imply the existence of an ordinal potential? No

	X	Y
A	1,0	2,0
B	2,0	0,1

	X	Y
A	0	3
B	1	2

$$\phi(A, X) < \phi(B, X) < \phi(B, Y) < \phi(A, Y) = \phi(A, X)$$

Generalised Ordinal Potential

$\phi : S \to \mathbb{R}$ is a generalised ordinal potential for G if it satisfies the property:

Δu_i and $\Delta\phi$ have the same sign if $\Delta u_i \neq 0 \equiv sig(\Delta u_i) \neq 0 \Rightarrow sig(\Delta u_i) = sig(\Delta\phi)$

$u_i(s_i', s_{-i}) - u_i(s_i, s_{-i}) > 0$ then
$\phi(s_i', s_{-i}) - \phi(s_i, s_{-i}) > 0$
$\forall\ i \in N,\ \forall\ s_{-i} \in S_{-i},\ \forall\ s_i' \in S_i$

· Lemma. Every finite generalised ordinal potential game has the FIP. Same as before

Theorem. G has the FIP iff G is a generalised ordinal potential game.

Proof. ($\Leftarrow$) Covered by the previous lemma.
($\Rightarrow$) Define $< \subseteq S \times S$: i.e., a Relation

$s > t$ iff $s \neq t$ and there exists a finite improvement path starting at t and ending at s.

Obs: $>$ is transitive ($\because$ G has the FIP)

$Z \subseteq S$ is represented if $\exists\ F : Z \to \mathbb{R}$ s.t. $\forall\ x, y \in Z$,
$x > y$ implies $F(x) > F(y)$

Let Z be the maximal represented subset of S.

Claim: $Z = S$

Suppose $\exists\ s \notin Z$

If $s > z\ \forall\ z \in Z$, $F(s) = \max\limits_{z \in Z} F(z) + 1$

If $s < z\ \forall\ z \in Z$, $F(s) = \min\limits_{z \in Z} F(z) - 1$

Otherwise, $F(s) = (\max\{F(z) \mid z \in Z, s > z\} + \min\{F(z) \mid z \in Z, s < z\})/2$ $\square$

* Our algorithms should be polynomial time; but polynomial in what?

Let $G = (N, (S_i), (u_i))$, with $|N| = n$, $|S_i| \sim m$ $\forall i \in N$, then $|u_i| \sim m^n$ $\forall i \in N$

The most naïve algorithm (i.e. bruteforce) can find all NE's in $O(\underbrace{n \cdot m \cdot m^n}_{|N| \cdot |S_i| \cdot |S|})$ time under this representation.

$\because |S| \sim |u_i|$
$\Rightarrow$ Polynomial

- So, need to find a more compact representation of $(u_i)_{i \in N}$ and correspondingly a meaningful class under the representation, as well as algorithms for such games.

- TLDR: Algorithms' runtimes should be polynomials in n and m.

CONGESTION MODELS

$M = (N, F, (X_i)_{i \in N}, (C_f)_{f \in F})$

- N - Set of players
- F - Set of facilities
- $X_i \subseteq 2^F$ - a subset of facilities
- $C_F : \mathbb{N} \rightarrow \mathbb{R}$ - cost function for each facility

A player can choose to use some of the facilities available

Cost of a facility depends only on the number of people using it.

- For a profile $\bar{x} = (x_1, x_2, \dots x_n) \in X$, the cost incurred for the usage of facility $f \in F$: $c_f(n_f(x))$
  where $n_f(x) = |\{ i \in N \mid f \in x_i \}|$
- Congestion Game: Given $M = (N, F, (X_i), (c_f))$, $G_M = (N, (S_i)_{i \in N}, (C_i)_{i \in N})$, $C_i : S \to \mathbb{R}$, $S_i = X_i$, $C_i(s) = \sum_{f \in s_i} c_f(n_f(s))$

- Can provide the congestion model (M) from which the game (G) can be inferred, a much more compact representation.

Theorem. Congestion games are exact potential games.

Proof. Let $G = (N, (S_i)_{i \in N}, (C_i)_{i \in N})$ → Congestion Game
and $\phi : S \to \mathbb{R}$

$$\phi(s) = \sum_{f \in F} \sum_{k=1}^{n_f(s)} c_f(k)$$

To prove: $\phi(s_i', s_{-i}) - \phi(s_i, s_{-i})$
$= C_i(s_i', s_{-i}) - C_i(s_i, s_{-i})$
$\forall i \in N, \ \forall s_i, s_i' \in S_i, \ \forall s_{-i} \in S_{-i}$

$$\phi(s_i', s_{-i}) = \underbrace{\sum_{f \in F} \sum_{k=1}^{n_f(s)} c_f(k)}_{\phi(s_i, s_{-i})} + \underbrace{\sum_{f \in s_i' \setminus s_i} c_f(n_f(s)+1) - \sum_{f \in s_i \setminus s_i'} c_f(n_f(s))}_{= (C_i(s_i', s_{-i}) - C_i(s_i, s_{-i}))}$$

$$C_i(s_i', s_{-i}) = C_i(s_i, s_{-i}) + \sum_{f \in s_i' \setminus s_i} c_f(n_f(s)+1) - \sum_{f \in s_i \setminus s_i'} c_f(n_f(s))$$

$\square$

Characterization of Exact Potential Games

Definition. A strategic game G is a:

- **Coordination game** if $\exists$ a function $u : S \to \mathbb{R}$ s.t. $u_i = u \;\; \forall\, i \in N$
- **Dummy game** if $\forall\, i \in N$, $\forall\, s_{-i} \in S_{-i}$ and $\forall\, s_i, s_i' \in S_i$, $u_i(s_i, s_{-i}) = u_i(s_i', s_{-i})$

Theorem. Let G be a finite strategic game. G is an exact potential game iff $\exists$ functions $(c_i)_{i \in N}$ and $(d_i)_{i \in N}$ s.t.

- $u_i(s) = c_i(s) + d_i(s) \quad \forall\, i \in N$
- $(N, (S_i)_{i \in N}, (c_i)_{i \in N})$ is a coordination game
- $(N, (S_i)_{i \in N}, (d_i)_{i \in N})$ is a dummy game

Proof.

$(\Leftarrow)$ $\phi(s) = c(s)$. Consider an arbitrary $i \in N$ and $s \in S$. For every $s_i' \in S_i$,

$$
\begin{aligned}
u_i(s_i', s_{-i}) - u_i(s_i, s_{-i}) &= (c_i(s_i', s_{-i}) - c_i(s_i, s_{-i})) \\
&\quad + (d_i(s_i', s_{-i}) - d_i(s_i, s_{-i})) \\
&= \phi(s_i', s_{-i}) - \phi(s_i, s_{-i}) + 0
\end{aligned}
$$

$(\Rightarrow)$ Suppose ϕ is an exact potential function for G.

$$u_i(s) = +\phi(s) + (u_i(s) - \phi(s))$$

Claim 1. $(N, (S_i)_{i \in N}, (\phi)_{i \in N})$ is a co-ordination game.

Follows from ϕ being used $\forall\, i \in N$

Claim 2. $(N, (S_i)_{i \in N}, (u_i - \phi)_{i \in N})$ is a dummy game.

Take $i \in N$, $s_{-i} \in S_{-i}$. $\because G$ is an exact potential game.

$\forall \; s_i, s_i' \in S_i$,

$$u_i(s_i', s_{-i}) - u_i(s_i, s_{-i}) = \phi(s_i', s_{-i}) - \phi(s_i, s_{-i})$$

$$\Leftrightarrow u_i(s_i', s_{-i}) - \phi(s_i', s_{-i}) = u_i(s_i, s_{-i}) - \phi(s_i, s_{-i})$$

$\therefore (N, (S_i), (u_i - \phi))$ is a dummy game. $\square$

* Let $G_1 = (N, (S_i), (u_i))$, $G_2 = (N, (T_i), (v_i))$. G_1 and G_2 are isomorphic if $\forall \; i \in N$, $\exists$ a bijection $\varphi_i : S_i \to T_i$ s.t. $\forall \; s = (s_1, s_2, \ldots, s_n) \in S$,

$$u_i(s_1, s_2, \ldots, s_n) = v_i(\varphi(s_1), \varphi(s_2), \ldots, \varphi(s_n))$$

Theorem. Every exact potential game is isomorphic to a congestion game.

Lemma 1. Every coordination game is isomorphic to a congestion game.

Lemma 2. Every dummy game is isomorphic to a congestion game.

Proof (Th.). Decompose the exact potential game to a coordination game and a dummy game.

Apply L1 and L2 to obtain two congestion games G_1 & G_2

(with disjoint facilities). Then take the union of these games. The union represents the isomorphism of the exact potential game to a congestion game. $\square$

Proof (L1). Let $G = (N, (S_i), (u))$ be a coordination game. For every $s \in S$, introduce a new facility $F(s)$. Congestion model $M = (N, F, (T_i)_{i \in N}, (c_f)_{f \in F})$.

$F = \{ F(s) \mid s \in S \}$

$T_i = \{ \alpha_i(s_i) \mid s_i \in S_i \}$

$\alpha_i(s_i) = \{ F(s_i, s_{-i}) \mid s_{-i} \in S_{-i} \}$

$$c_{F(s)}(k) = \begin{cases} u(s), & k = n \\ 0, & \text{otherwise} \end{cases}$$

$F(s)$ has cost $u(s)$ if every player uses this facility.

Let G' be the congestion game obtained from M.

Define bijection $\varphi_i : s_i \mapsto \alpha_i(s_i)$

Obs. For every $s \in S$, $\bigcap_{i \in N} \alpha_i(s_i) = F(s)$

Thus, for every $(s_1, s_2, \dots, s_n) \in S$, there is exactly one facility used by all the n players in G'.

Thus $\forall\ i \in N$,

$$\underbrace{u(s)}_{\text{in } G} = c_{F(s)}(n) = \underbrace{c_i(\alpha_1(s_1), \alpha_2(s_2), \dots, \alpha_n(s_n))}_{\text{in } G'}$$

$\therefore$ G & G' are isomorphic. $\square$

REPRESENTATION OF STRATEGIC GAMES

Suppose there are n players and k strategies for each player. Then $n \cdot k^n$ parameters are required to explicitly represent the utility function.

Congestion games have a compact representation in terms of the congestion model.

- A pure NE always exists in congestion games.

Q. Can we efficiently compute a pure NE in congestion games?

(We shall show) Computing a pure NE in congestion games is as hard as solving any other local search problem (LSP).

* Canonical problem: The maximum cut problem

Input — Undirected graph $G = (V, E)$, with non-negative weights $w_e \geq 0$ for each $e \in E$.

Feasible solutions — Cuts $(X, \bar{X})$ where $(X, \bar{X})$ is a partition of V.

Objective — Maximise the total weight of the cut edges.

Cut edges — Edges with one end

point in X and other in $\bar{X}$.

- Obs: Max-Cut problem is NP-hard. (MIS - Maximum Independent Set can be reduced to Max-Cut)
  Then, assuming $P \neq NP$, there is no polynomial time algorithm to solve max-cut.

Local Search for Max-Cut

1. Initialise with an arbitrary cut $(X, \bar{X})$
2. WHILE there is an improving local move DO update the cut with an arbitrary such move.

Local move: Shift a single vertex v from X to $\bar{X}$ or from $\bar{X}$ to X

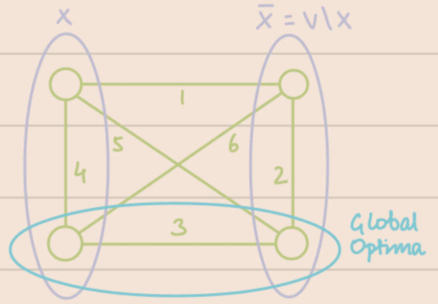
- If v is moved from X to $\bar{X}$, the change in the objective function:

$$\underbrace{\sum_{\substack{x \in X \\ (x,v) \in E}} w_{xv}}_{\text{Newly Cut}} - \underbrace{\sum_{\substack{x \in \bar{X} \\ (x,v) \in E}} w_{xv}}_{\text{Newly Uncut}}$$

If this difference is positive, then this is an improving local move.

- Local Search procedure stops at a local optimum.

Q. Is every local optima also a global optima?
The cut $(X, \bar{X})$ is a



local optimum with obj. 15, while global optimum has obj. 17.

$\therefore$ A local optimum need not be a global optimum.

$\Rightarrow$ There are (possibly) more local optima than global optima.

Computing a local optima might be easier.

- Special Case: All edge weights are 1.
    - Computing a global optima is NP-hard.
    - Computing a local optima is in P
        - Obj. function takes values in the set $\{0, 1, \dots, |E|\}$
        - Each iteration $\uparrow$es obj. by atleast 1 $\Rightarrow$ max. $|E|$ iterations

Local Search Problem (LSP)

- A maximisation/minimisation problem.
- Specified by three algorithms:

Find a feasible solution in PTIME

1. Polynomial time algorithm that takes as input an instance and outputs a feasible solution

Obj. value

Evaluate a feasible solution in PTIME

2. Polynomial time algorithm that takes as input an instance and a feasible solution and computes the value of the obj. function.

Report local optimality or suggest a better feasible solution.

3. Polynomial time algorithm that

takes as input an instance and a feasible solution and outputs "locally optimal" or generates another feasible solution with a "better" objective function value.

- Computing a PNE in congestion games is as hard as any other LSP.

Formal Definition

- A set of instances $\mathcal{I}$
- For every $I \in \mathcal{I}$
  - A set of feasible solutions $F(I)$
  - An objective function $c: F(I) \to \mathbb{Z}$
  - For every $y \in F(I)$, a neighborhood of y, $N(y, I) \subseteq F(I)$

PLS

An LSP is in PLS if the following can be done in polynomial time:

1. Compute the initial feasible solution $y \in F(I)$
2. For each feasible solution $y \in F(I)$, compute the obj. function $c(y)$
3. For each $y \in F(I)$, determine if y is locally optimal. If not, find a "better" solution in the neighborhood of y.

PLS - reduction

Let $\Pi_1 = (\mathcal{I}_1, F_1, C_1, N_1)$;
$\Pi_2 = (\mathcal{I}_2, F_2, C_2, N_2)$

Π_1 is **PLS-reducible** to Π_2 if $\exists$ two polynomial time computable functions f and g s.t.

- f maps every instance $I \in \mathcal{I}_1$ to an instance $f(I) \in \mathcal{I}_2$
- g maps every tuple (S_2, I) with $S_2 \in F_2(f(I))$ to a solution $S_1 \in F_1(I)$
- $\forall\ I \in \mathcal{I}_1$, if S_2 is a local optimum of $f(I)$, then $g(S_2, I)$ is a local optimum of I.

$$\begin{tikzcd}
\Pi_1 \arrow[d, "\text{PLS reducible}"] & I \in \mathcal{I}_1 \arrow[d, "f"] & S_1 = g(S_2, I) \in F_1(I) \\
\Pi_2 & f(I) \in \mathcal{I}_2 & S_2 \in F_2(f(I)) \arrow[u, "g\ (\text{Local Optima preserving})"']
\end{tikzcd}$$

- An LSP Π is **PLS-complete** if
  - Π is in PLS
  - Every problem in PLS is PLS-reducible to Π

Max Cut : PLS
SAT : NP
Canonical problems

Theorem. Max-Cut is PLS-complete.

Theorem. Computing a local maximum in a max-cut instance with non-negative weights using local search can require an exponential (in $|V|$) number of iterations irrespective of how an improvement is chosen in each iteration.

Theorem. Computing a PNE in congestion games is PLS-complete.

Obs. Computing a PNE in congestion games is in PLS.

- Need to reduce max cut to the problem of computing a PNE in congestion games.

  Input: $G = (V, E)$, non-negative edge weights $\{w_e\}_{e \in E}$

  To Do: Define the associated congestion game:
  - Players are the vertices
  - For each $e \in E$, there are 2 facilities r_e and $\bar{r}_e$
  - Player $v \in V$ has 2 possible strategies:

    $$S_v^1 = \{r_e\}_{e \in \delta(v)}$$
    $$S_v^2 = \{\bar{r}_e\}_{e \in \delta(v)}$$

    $\delta(v)$: Set of edges incident on v

    Each player has 2 possible strategies consisting of $|\delta(v)|$ facilities.
  - r_e or $\bar{r}_e$ for $e = (u, v)$ can only be used by players u & v

    Cost:

    $$C_{r_e}(1) = 0 = C_{\bar{r}_e}(1)$$
    $$C_{r_e}(2) = w_e = C_{\bar{r}_e}(2)$$

- Transformation:
  Cuts in $G \longleftrightarrow$ Strategy profiles

Given $(X, \bar{X})$, define s as:
$\forall\ v \in X$, choose the strategy containing the resource r_e
$\forall\ v \in \bar{X}$, ... $\bar{r}_e$

- Obs: Above translation is a bijection
  Fix a cut $(X, \bar{X})$;

For each edge e that is cut in $(X, \bar{X})$, each facility r_e & $\bar{r}_e$ are used by exactly 1 player;
For an edge e not cut by $(X, \bar{X})$, either r_e or $\bar{r}_e$ is used by two players.

- Recall: $\phi(s) = \sum_{f \in F} \sum_{k=1}^{n_f(s)} c_f(k)$
  $\therefore$ Contribution to ϕ is either w_e or 0

Thus, value of potential function for the corresponding outcome is
$W - w(X, \bar{X}) = \sum_{e \in E} w_e - w(X, \bar{X})$

Total weight of edges not cut by $(X, \bar{X})$

$\therefore$ Cuts of larger weights correspond to outcomes with smaller potential value.
Local minima of $\phi(s)$ has a 1-1 correspondence w/ PNE of the congestion game.

Symmetric Congestion Game

Every agent has the same set of strategies.

Theorem. The problem of computing a PNE in symmetric congestion games is PLS-complete.

Proof. Reduce from congestion games:

Given a congestion game with $|N|$ players, facilities F and strategies (S_i) $(H_1 = (N, F, S_i, C))$, we construct a symmetric game $H_2 = (N, F', S, C')$.

- $N = N$
- $F' = F \cup \{r_i\}_{i \in N}$
- $C'(f) = C(f) \quad \forall f \in F$

  $C'(r_i) = \begin{cases} 0 & \text{if } 1 \text{ player uses } r_i \\ \infty & \text{if } >1 \text{ players use } r_i \end{cases}$
- $S = \{ s_i \cup \{r_i\} \mid i \in N, s_i \in S_i \}$, the common strategy set
- Agent choosing a strategy with r_i — corresponds to adopting the identity of player i in H_1.
- At NE, no two players will choose strategies with the same r_i's. There are n r_i's, so each player can choose a different identity.

Network Congestion Game

Directed graph $G = (V, E)$: two nodes $a_i, b_i \in V$ for each $i \in N$

- Delay / cost function on edges.

- Strategies are subsets of E — all paths from a_i to b_i
- A network congestion game is symmetric if all players have the same start and end points $\underbrace{a, b}$ Common for all $i \in N$
- Some results:

	Network CG	General CG
Symmetric	PTIME	PLS-complete
Not Symmetric	PLS-complete	PLS-complete

Compact Representation

Towards compact representation of subclasses of strategic games.

- A strategic game can be viewed as a pair $(N, (M_i)_{i\in N})$ where each M_i is a payoff matrix.
- Ex: 2 player bimatrix representation
  $G = (N, M_1, M_2)$
  Player 2's payoff matrix
  Player 1's payoff matrix

Graphical Games

- Each player is a vertex in an undirected graph $H = (N, E)$
- $N(i) = \{ j \in N \mid (i, j) \in E \} \subseteq \{1, 2, \dots, n\} = N$
  $\underbrace{N(i)} \longrightarrow$ Neighbourhood of player i
- For a strategy profile $s \in S$, let s^i denote the restriction of s to $N(i)$
- Graphical Game — $(H, \underbrace{(M_i)_{i\in N}}_{\text{Set of } n \text{ local games}})$
- Payoff of $i \in N$ in $s \in S$ is $M_i(s^i)$
  — depends only on the strategies of

of $j \in N(i)$

- Strategic form game representation is exponential in the number of players.
  In graphical games, the representation is exponential in the size of the largest local neighbourhood — call it d. If $d \ll n$, graphical games provide a compact representation.

Polymatrix Games

- Player interaction is captured by a neighbourhood graph.
- n-player games $u_i : S \to \mathbb{R}$
- $\forall\ i, j \in N$, $\exists$ a partial utility function $u_{ij} : S_i \times S_j \to \mathbb{R}$
- For $i \in N$ and $s \in S$, $u_i(s) = \sum_{j \neq i} u_{ij}(s)$
- View it as a graph where each edge is labelled with a bi-matrix game: $G = (N, (S_i)_{i \in N}, (u_{ij})_{i,j \in N})$

  $\to 2k^2 \cdot \binom{n}{2}$ — size of a matrix · # of matrices

Q. ($\exists$ PNE) Given a strategic form game G, does G have a pure Nash equilibrium? Is $NE(G) = \phi$?

- Observation: $\exists \text{PNE} \in \text{NP}$
  Guess a profile s, check if $s \in NE(G)$, which can be checked in PTIME

* If $\forall\ i, j \in N$ and $\forall\ s \in S$, if $u_{ij}(s_i, s_j) \in \{0, 1\}$, call it 0/1 polymatrix games.

- Define a subclass of polymatrix games: Colouring games.
- Result: $\exists$ PNE in coloring games is NP-hard.
  Corollary: $\exists$ PNE in 0/1 polymatrix games is NP-hard.

Colouring Game

- Directed graph $H = (N, E, w)$ where $w : E \to \mathbb{N}$.

  M is a finite set of colours.

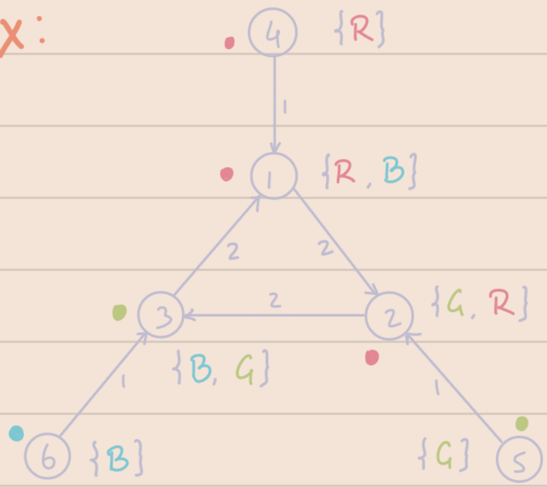
  $C : N \to 2^M$

  $S_i = \{ m \in M : m \in C(i) \} = C(i)$

  $N(i) = \{ j : \exists (j, i) \in E \}$ i.e. $\underbrace{j \to i}$ All incoming edges

  $u_i(s) = \sum_{\substack{j \in N(i) \\ s_i = s_j}} w_{(j,i)}$ $\left(= |\{ j \mid j \in N(i), s_j = s_i \}| \right)$ for unweighted graphs

Ex:



Set of colours $M = \{R, G, B\}$

Consider the profile s labelled in the figure above:

$u_1(s) = 1, \quad u_2(s) = 2,$

$u_3(s) = u_4(s) = u_5(s) = u_6(s) = 0$

Q. Does a PNE always exist in colouring games?

No, consider the above game.

Improvement Cycle:

$(R, R, \underline{G}) \longrightarrow (\underline{R}, R, B) \longrightarrow (B, \underline{R}, B)$

$(R, \underline{G}, G) \longleftarrow (\underline{B}, G, G) \longleftarrow (B, G, \underline{B})$

$\therefore$ This game does not have a PNE.

- Colouring games $\subseteq$ Polymatrix games
  Unweighted colouring games $\subseteq$ 0/1 - polymatrix games

Theorem 1. $\exists$ PNE for colouring games is in NP.

Theorem 2. $\exists$ PNE for unweighted colouring games is NP-hard.

Corollary. $\exists$ PNE for unweighted colouring games is NP-complete.

- Colouring games form a subclass of polymatrix games.
- Unweighted colouring games form a subclass of 0/1 polymatrix games.

- An NP-complete problem: 3SAT

Definition. Propositional formula over n variables (propositions) $x_1, \dots, x_n$ in CNF where each clause contains three literals.

$$\phi = \bigwedge_{i=1}^{i=k} (a_i \vee b_i \vee c_i)$$

a_i, b_i, c_i are literals over $\{x_1, \dots, x_n\}$ — either x_j or $\neg x_j$

3SAT: Given such a 3-CNF ϕ, is ϕ satisfiable?

Theorem. 3SAT is NP-complete

- To prove Theorem 2, we give a reduction from 3SAT.

  To Do: Given a 3-CNF, construct a colouring game G_ϕ s.t. the following holds:

  ϕ is satisfiable

  $\exists$ a valuation v s.t. $v \models \phi$ iff $\exists$ a PNE in G_ϕ

- Variables / Propositions $\{x_1, x_2, \dots, x_n\}$

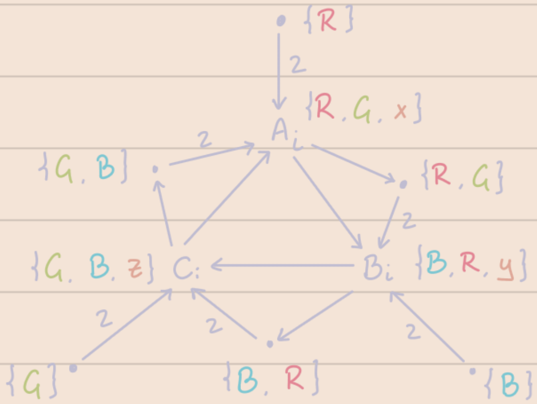
  CNF : Conjunction of clauses

  Clause : Disjunction of literals

  Literal : x_i or $\neg x_i$

- A Gadget : $D_i(x, y, z)$

  Gadget D_i with parameter x, y and z where
  $x, y, z \in \{\top, \bot\}$



- Observation. For all parameter values x, y, z, $D_i(x, y, z)$ does not have a PNE.

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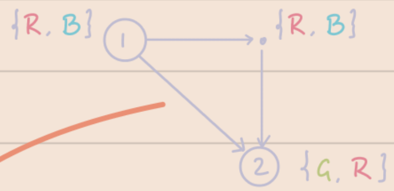
Translation to unweighted edges



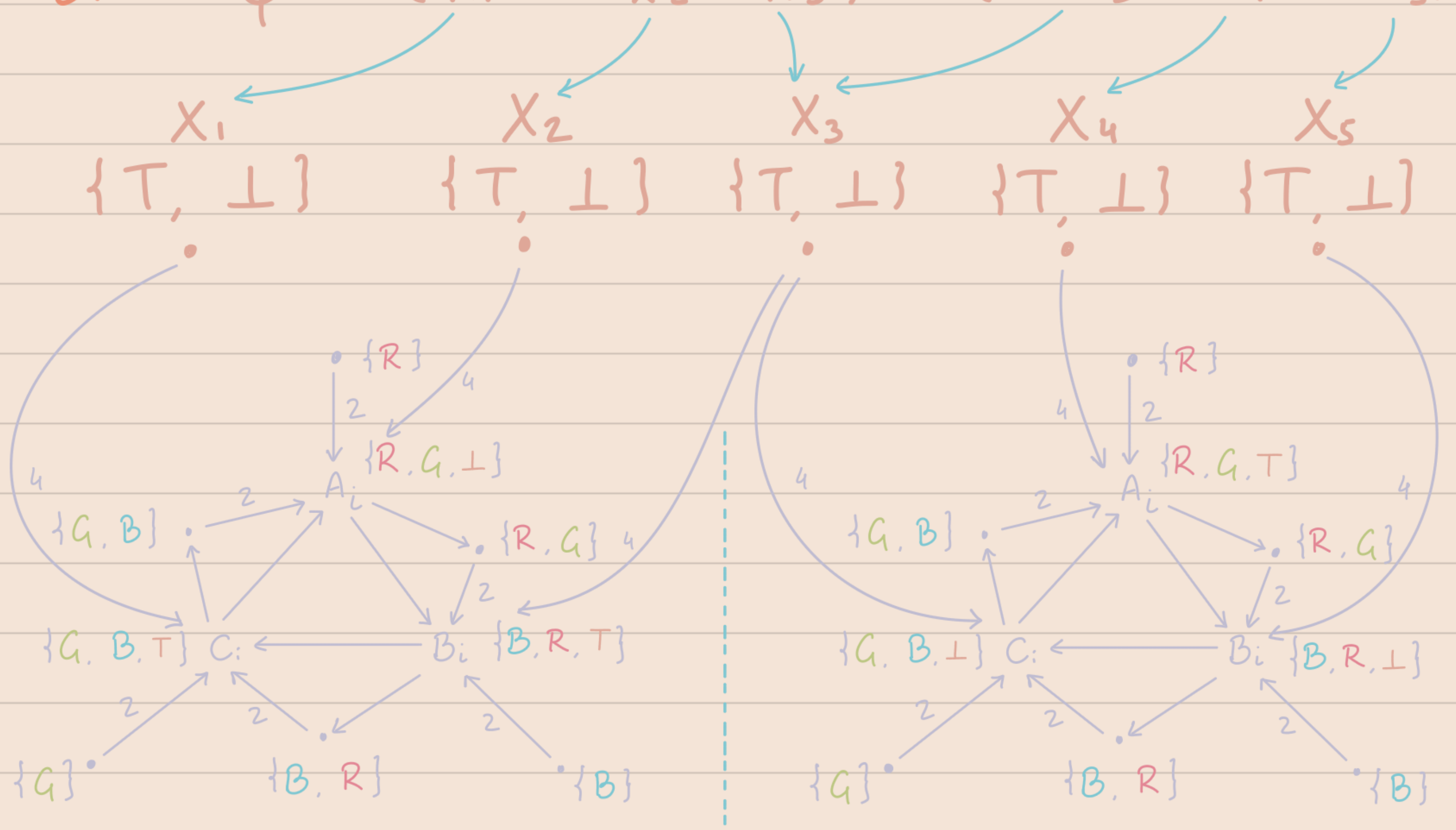
Convention: Weights of value 1 are omitted

Using the translation, weighted graphs

can be transformed to unweighted ones.

- The associated colouring game:
  - For each variable x_i — associate a node X_i in G_ϕ with colour set $\{\top, \bot\}$
  - For a literal l, let
    $$pos(l) = \begin{cases} \top & \text{if } l \text{ is positive} \\ \bot & \text{if } l \text{ is negative} \end{cases}$$
  - For each clause $(a_i \vee b_i \vee c_i)$ in ϕ, add to G_ϕ the gadget
    $$D_i(pos(a_i), pos(b_i), pos(c_i)).$$
  - For each literal a_i, b_i, c_i — which is x_j or $\neg x_j$ for some j, add edge from X_j to A_i or B_i or C_i with weight 4.

Ex: $\phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee x_4 \vee \neg x_5)$



In PNE, X_j can be anything, but atleast one out of $\{A_i, B_i, C_i\}$ have the same truth value as their corresponding X_j in each gadget D_i. Once in any D_i a node picks the color $\{T, \perp\}$ (and it is the same as its X_j, then it does not have a prof. dev, but it consequently fixes others' colors)

Note how the gadget does not have a PNE w/o $\{\perp, T\}$ colors

3SAT instance is in SAT iff the corresponding game has a PNE.

Proof:

($\Leftarrow$) Suppose G_ϕ has an NE, i.e., $s \in NE(G_\phi)$

Consider $v : \{x_1 \dots x_n\} \to \{\perp, T\}$ where $v(x_i) = s_{x_i}$

To show: $v \models \phi_{\beta_i}$ for all clauses β_i

That is, v satisfies one of the literals a_i, b_i, c_i $\forall$ clauses β_i

By observation on the gadget; in s, at least one of the nodes A_i, B_i or C_i select the same colour as its neighbour X_j.

Suppose (wlog) it is node A_i. Only colour A_i and X_j have in common is $pos(a_i)$.

So, $s_{x_j} = pos(a_i)$. By definition of v, this is $v(x_j)$. By construction, x_j is the variable of the literal a_i.

Thus, $v(x_j) = pos(a_i)$ implies that v satisfies a_i.

($\Rightarrow$) Suppose ϕ is satisfiable.

Let $v : \{x_1 \dots x_n\} \to \{\perp, T\}$ s.t. $v \models \phi$

To Do: Define a strategy profile s_v; Show that $s_v \in NE(G_\phi)$

For all j, assign to node X_j the colour $v(x_j)$.

Claim: This partial assignment can be

extended to a strategy that is an NE

For $i \in \{1 \dots k\}$, consider the gadget $D_i(pos(a_i), pos(b_i), pos(c_i))$. v makes $a_i \vee b_i \vee c_i$ true. Suppose $v \models a_i$. The unique best response for node A_i is to select the colour $pos(a_i)$.

By definition, $a_i = x_j$ or $a_i = \neg x_j$ for some j. Since $v \models a_i$, we have $v(x_j) = pos(a_i)$. So, when A_i chooses $pos(a_i)$, the colour assigned to x_j, its utility is 4. This partial assgn. can further be extended in the gadget towards an NE

$\square$

Corollary: $\exists$ PNE in 0/1 polymatrix game is NP-complete. We reduced 3SAT to a coloring game w/ bounded weights (≤ 4). $\therefore$ This coloring game can further be reduced in PTIME to a 0/1 polymatrix game.

* Colouring games with restricted neighbourhood structures.

DAG's

- Computation: What is the complexity of computing a PNE (if it exists) in colouring games over DAG's?
- Existence: Does colouring games over DAG's always have a PNE?
- Let π be the ordering of N resulting from a topological sort of H (the underlying graph).

  $\pi = i_0 i_1 \dots i_n$ where $i_j \in N$

  Process the players based on the ordering π and assign their best response.

Obs: Colouring games over DAG's are weakly acyclic.

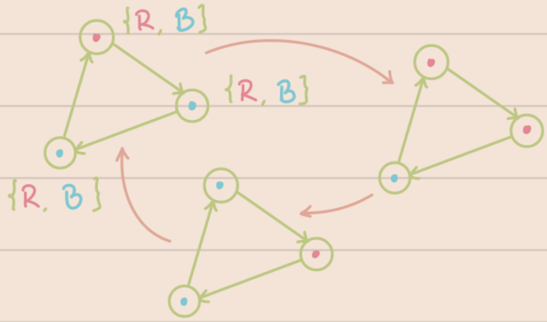
Obs: A PNE always exists and it can be computed in polynomial time.

Q: Are colouring games on DAG's potential games?
Yes.

Simple Cycles

- Computation: What is the complexity of computing a PNE (if it exists)?
- Existence: Does colouring games on simple cycles always have a PNE?

Q: Are colouring games on simple cycles potential games?



- Proposition: Colouring games on simple cycles need not have the FIP.

Q: Are colouring games on simple cycles weakly acyclic?
Yes.

$$S \xrightarrow{\text{Phase 1}} S_1 \xrightarrow{\text{Phase 2}} S_2$$

Phase 1: Iterate over nodes sequentially and change colour to BRF if possible

Phase 2: Repeat it but termination is guaranteed ∵ all but 1 node use BRF strategy

$\forall s, \exists \rho \Rightarrow WAG + \exists PNE$
$\exists s, \exists \rho \Rightarrow \exists PNE$
finite

- Games with FIP:
  $\forall s \in S$, $\forall$ improvement paths ρ starting at s, ρ is finite.
- FIP $\Rightarrow$ PNE(G) $\neq \phi$

- Weakly acyclic games [Peyton-Young]
  A game G is weakly acyclic if $\forall s \in S$, $\exists$ an improvement path ρ starting at s s.t., ρ is finite.
  If G is weakly acyclic, then PNE(G) $\neq \phi$

EFFICIENCY OF EQUILIBRIUM

- Prisoner's Dilemma

	D	C
D	(4,4)	(0,6)
C	(6,0)	(1,1)

Nash equilibrium is strictly Pareto inefficient

There is another outcome where all players achieve a better utility (this is "better" for all the players)

- Define objective functions:
  - Utilitarian := Sum of players' cost/utility
    Social Welfare
  - Egalitarian := Max player cost/

Min player utility

Inefficiency

Ratio between the objective function value of an equilibrium outcome of the game and that of an "optimal" outcome

subject to the obj. function chosen

- Define the objective functions
- Define "equilibrium"
- If there are multiple equilibria, which one should be considered?

- We choose the utilitarian objective function in the following discussions.

Price of Anarchy (PoA)

Ratio between the worst objective function value of an equilibrium of the game and that of an optimal outcome.

- Cost function Cost : $S \rightarrow \mathbb{R}$ (minimise)

$$PoA := \frac{\max_{s \in NE(G)} Cost(s)}{\min_{s \in S} Cost(s)}$$

- Utility function SW (maximise)

$$PoA := \frac{\max_{s \in S} SW(s)}{\min_{s \in NE(G)} SW(s)}$$

Price of Stability (PoS)

- $$PoS := \frac{\max_{s \in S} SW(s)}{\max_{s \in NE(G)} SW(s)}$$ — utility (maximise)

- $\text{PoS} := \dfrac{\min_{s \in NE(G)} \text{Cost}(s)}{\min_{s \in S} \text{Cost}(s)}$ — cost (minimise)

- $1 \leq \text{PoS} \leq \text{PoA}$

- Fair Cost Sharing game
  A special case of congestion games
  - Each facility $f \in F$ has a cost $d_f \in \mathbb{R}$ and cost function
    $$c_f(n_f(s)) = \frac{d_f}{n_f(s)}$$
  - Cost function $c_f : \mathbb{N} \to \mathbb{R}$ for each $f \in F$ is obtained by dividing d_f equally between the players using f.
    Thus, $$c_i(s) = \sum_{f \in s_i} \frac{d_f}{n_f(s)}$$

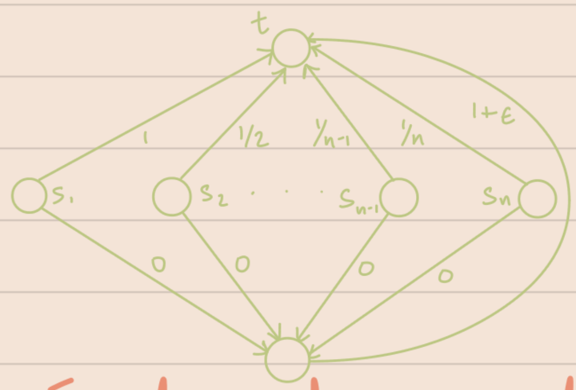
- Connection Games
  - A directed graph $G = (V, E)$
  - Non-negative edge costs $c : E \to \mathbb{R}_+$
  - Set of players N; $\forall\ i \in N$ a pair $(s_i, t_i) \in V \times V$
  - Goal: Choose a directed path from s_i to t_i of min. cost
    $$c_i(s) = \sum_{e \in s_i} c_e / n_e(s)$$
    $$\text{Cost}(s) = \sum_{i \in N} c_i(s) = \sum_{e \in s} c_e$$

Ex: 

n-players; 2 facilities: $S_i = \{\{1\},\{2\}\}$

- If all n players choose facility 2, the cost for every player is 1. Social cost is n.
- All players choosing F2 is an NE.
- Social optimum: All players choose F1
- $\therefore \forall n \in \mathbb{N}$, PoA of the class of n player fair cost sharing games is at least n.

We have provided an NE above, $\therefore$ $n \leq \max_{s \in NE} Cost(s) \Rightarrow n \leq PoA$

Ex: 

- n players
- $2n+1$ facilities
- Each $i \in N$ has to go from s_i to t

$S_i = \{\{i\}, \{n+i, 2n+1\}\}$

- Each player choosing the "top edge" (strategy $\{i\}$) is an NE
- Claim: This is the unique NE with $Cost(s) = \sum_{i=1}^{i=n} 1/i = H_n$
- Optimum: i chooses $\{n+i, 2n+1\}$; $Cost(s^*) = 1+\epsilon$
- $\therefore \forall n \in \mathbb{N}$, the PoS of the class of n player FCSG is atleast $H(n)$

- Potential for Connection Games

$\phi_e(s) = c_e \cdot H_{n_f(s)}$: $H_k = \sum_{j=1}^{j=k} 1/j$

$\phi(s) = \sum_e \phi_e(s)$

- Lemma: $\text{Cost}(s) \leq \phi(s) \leq H_n \cdot \text{Cost}(s)$
- Potential function method
  - Bound of PoS
- Theorem. Suppose we have a potential game with potential function ϕ, and assume that for any $s \in S$, we have
$$\frac{\text{Cost}(s)}{A} \leq \phi(s) \leq B \cdot \text{Cost}(s)$$
for some $A, B > 0$. Then, the PoS is atmost $A \cdot B$.

Proof. Suppose s minimizes ϕ. Then, s is an NE.

To show: $\text{Cost}(s)$ is not much larger than $\text{Cost}(s^*)$ where s^* is of minimal cost.

By assumption, $\text{Cost}(s)/A \leq \phi(s)$
By definition, $\phi(s) \leq \phi(s^*)$
By assumption, $\phi(s^*) \leq \text{Cost}(s^*) \cdot B$
Combining them, $\frac{\text{Cost}(s)}{A} \leq \text{Cost}(s^*) \cdot B$

$\min_{s' \in S} \text{Cost}(s') = \text{Cost}(s^*)$

$\min_{s' \in NE} \text{Cost}(s') \leq \text{Cost}(s)$

$$\Rightarrow \text{PoS} := \frac{\min_{s' \in NE(G)} \text{Cost}(s')}{\min_{s' \in S} \text{Cost}(s')} \leq \frac{\text{Cost}(s)}{\text{Cost}(s^*)} \leq A \cdot B$$

- The same cannot be used to get a bound on PoA because of the way PoA is defined.