# Data Structure: Graph

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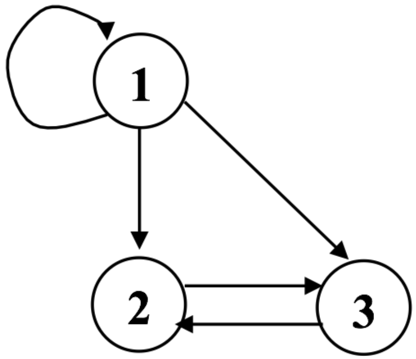
# graphs

- a graph G = (V, E)
  - V: a set of vertices (or nodes)
  - E: a set of edges (or arcs)

    each edge is represented as (v, w) where v, w $\in$ V
- directed graph (Digraph): a graph with directed edges
- undirected graph: a graph with undirected edges

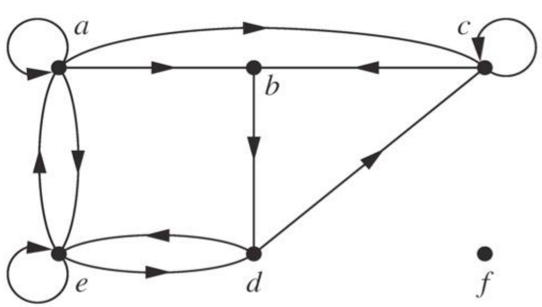
**V = {1, 2, 3}**
**E = {(1,1), (1,2), (2,3), (3,2), (1,3)}**



# directed graphs

- Let G = (V, E) be an directed graph
  - edge (u, v)
    - u is adjacent to v
    - u is initial vertex of (u, v)
    - v is terminal vertex of (u, v)
- degree of edges
  - in-degree of a vertex v
    - the number of edges with v as their terminal vertex
  - out-degree of a vertex v
    - the number of edges with v as their initial vertex

$$\sum_{v \in V} indeg(v) = \sum_{v \in V} outdeg(v) = |E|$$



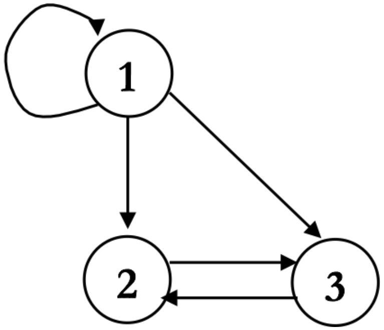

indeg(a) = 2 outdeg(a) = 4
indeg(b) = 2 outdeg(b) = 1
indeg(f) = 0 outdeg(f) = 0

# connectivity of graphs

- For a digraph $G = (V, E)$, $n = |V|$, $e = |E|$, $e \leq n^2$
- path is a sequence of vertices $x_1, x_2, \ldots, x_{n-1}$
- the length of path is the number of edges in the path
- cycle begins and ends at the same vertex
- a path or cycle is simple if it does not contain the same edge more than once, the first and the last could be the same
- DAG (Directed Acyclic Graph): a digraph with no cycles.

# graph representation: adjacency matrices

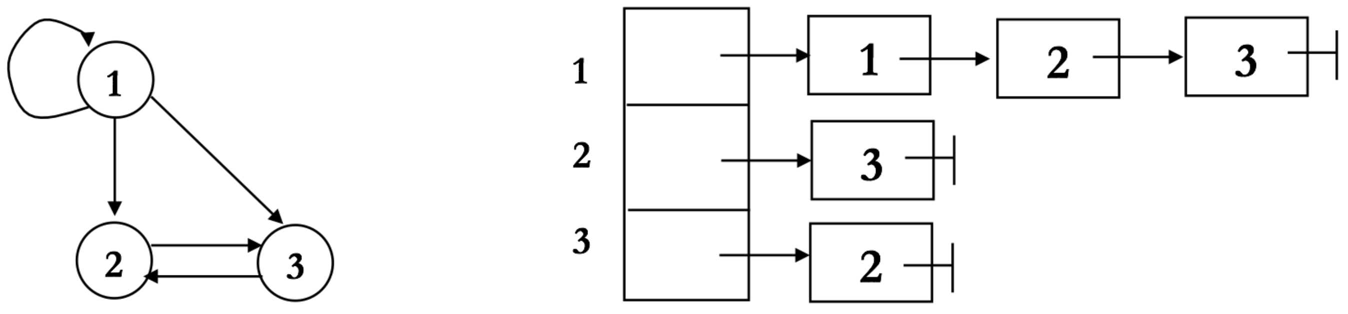
- *A*[*u*][*v*] = *w* if an edge exists between vertices *u* and *v*
  *A*[*u*][*v*] = 0 otherwise.
- *w* = 1 or an arbitrary weight associated with edges
- use a table of size |*V*| to store a mapping from vertex names to array indices
- simple but Θ (|*V*|$^2$) space is needed
- appropriate for dense graphs with |*E*| approaching to (|*V*|$^2$).




| | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 1 |
| 3 | 0 | 1 | 0 |

# graph representation: adjacency lists

- for each vertex, keep a list of adjacent vertices.
- space requirement: $O( |E| + |V| )$
- appropriate for sparse graphs.



# partial orderings

- a relation R on a set S is partial order if it is reflexive, antisymmetric, and transitive
- (S, R): a set S with a partial ordering R is a partially ordered set (poset)

$\geq$ is a partial ordering on the set of integers

reflexive: $a \geq a$ for every integer a

antisymmetric: $a \geq b$ and $b \geq a$ then $a = b$

transitive: $a \geq b$ and $b \geq c$ imply $a \geq c$

# total orderings

- if (S, $\preccurlyeq$) is a poset and every two elements of S are comparable, S is a totally ordered, linearly ordered set, or chain

poset ($Z$, $\leq$) is totally ordered because $a \leq b$ or $b \leq a$

poset ($Z^+$, $|$) is not totally ordered because $5 \nmid 7$ and $7 \nmid 5$

# topological sorting

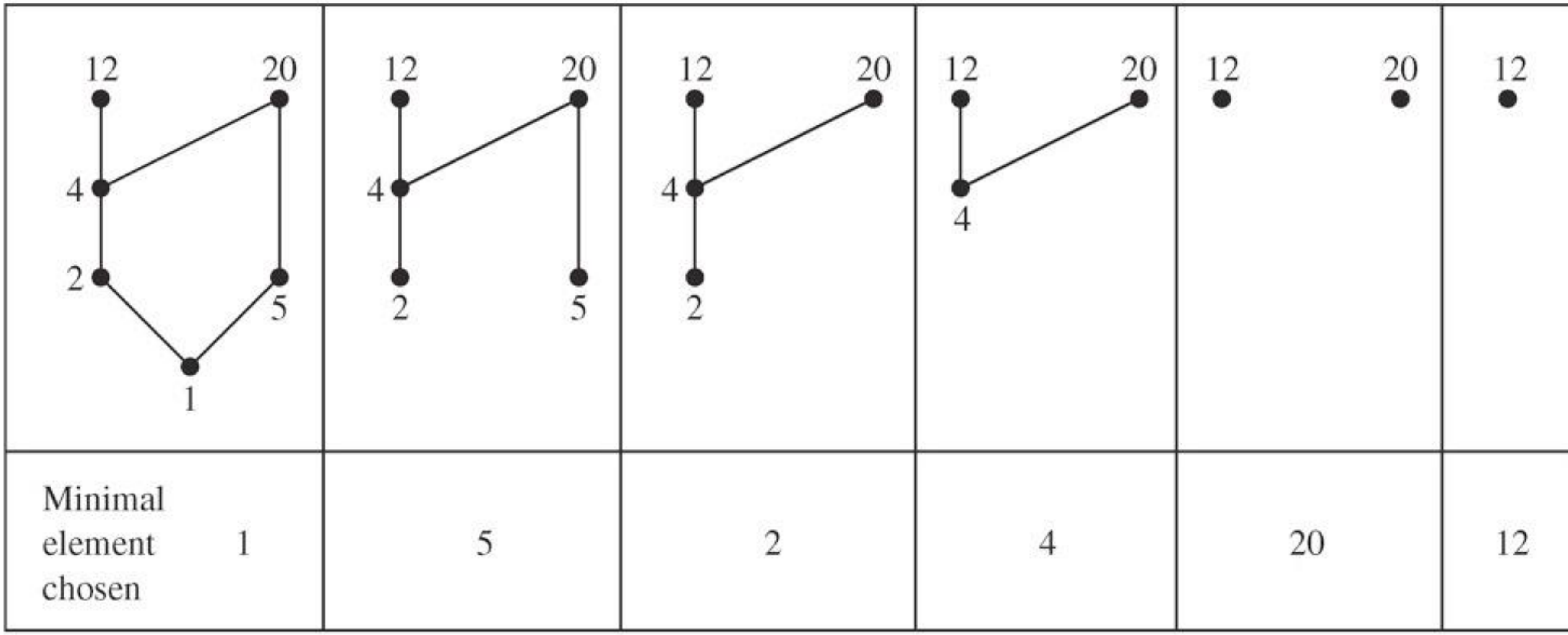
- topological sorting is constructing a compatible total ordering from a partial ordering

procedure topological sorting ((S, $\preccurlyeq$))

k := 1
while S $\neq \emptyset$
    $a_k$ := a minimal elements of S
    S := S - $\{a_k\}$
    k := k + 1
return $a_1, a_2, \ldots a_n$ $\{a_1, a_2, \ldots, a_n$ is a compatible total ordering of S$\}$

# topological sorting

Find a compatible total ordering for the poset({1, 2, 4, 5, 12, 20}, |)

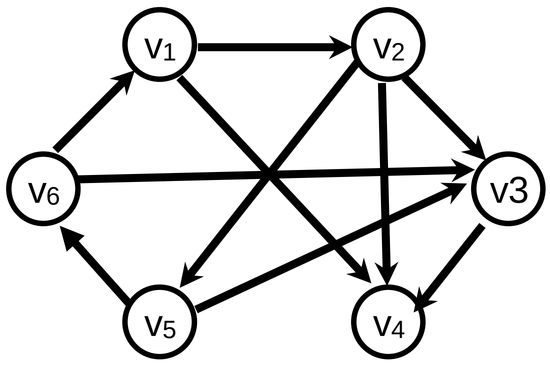



$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12$

# topological sort

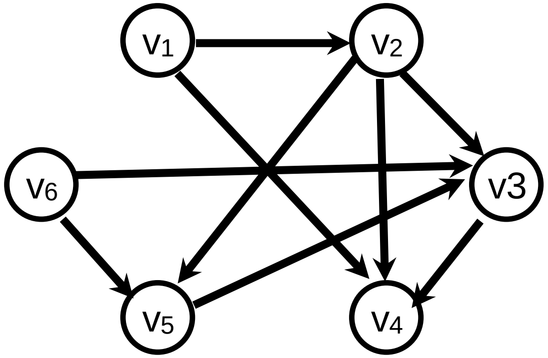
- ordering of vertices in a DAG, such that if there exists a path from $v_i$ to $v_j$, then $v_j$ appears after $v_i$
- Example: a topological ordering of courses
  - any course sequence that does not violate the prerequisite requirement

# topological sort



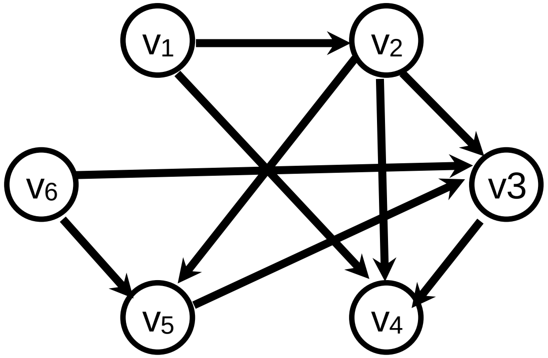

Can you perform topological sort?

# topological sort

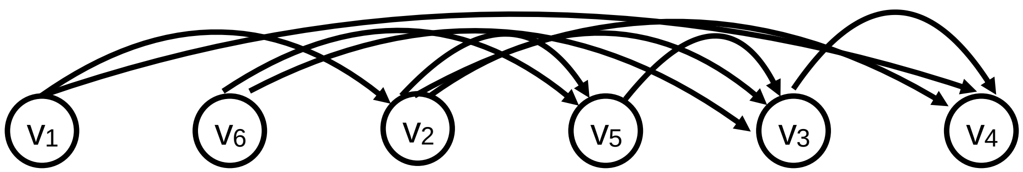



Can you perform topological sort?

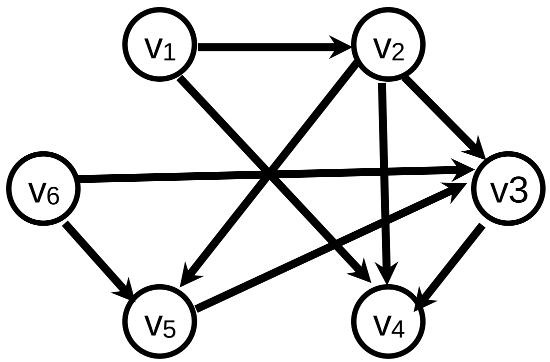
# topological sort



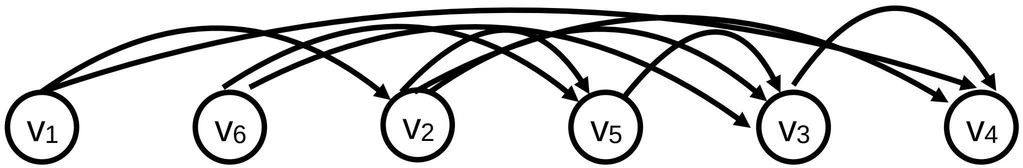

Can you perform topological sort?

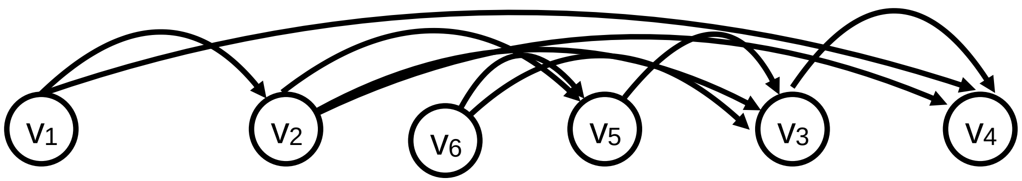


# topological sort




Can you perform topological sort?



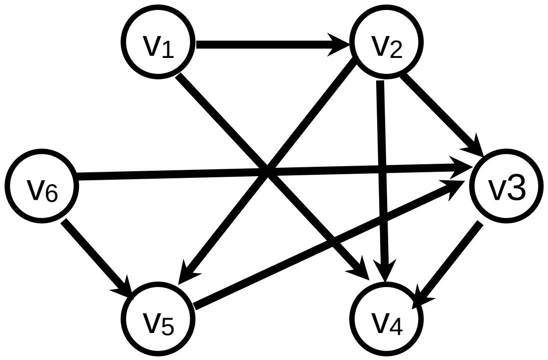



# topological sort

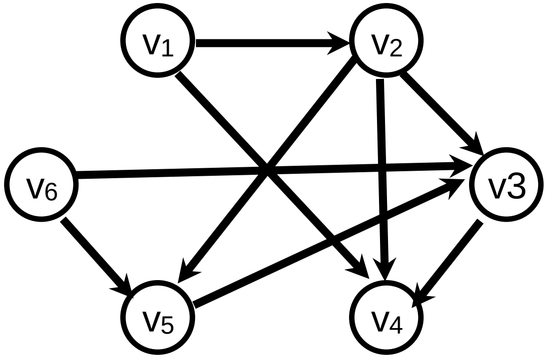
- algorithm:
  - for each vertex *v* whose in-degree is zero,
    - print *v*
    - remove *v* and its outgoing edges (which leads to decrementing the in-degree value of *v*'s adjacent vertices)

- use either stack or queue to keep track of vertices with in-degree = 0
- use adjacency list representation or matrix

# topological sort

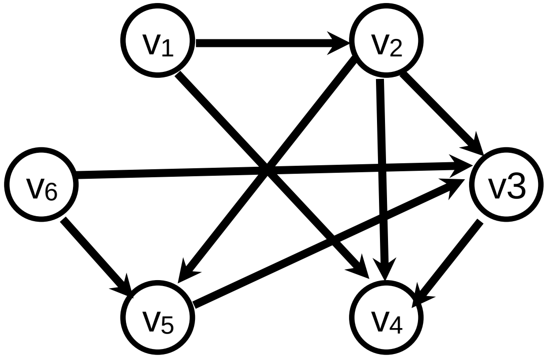


# topological sort




| | v1 | v2 | v3 | v4 | v5 | v6 |
|---|---|---|---|---|---|---|
| v1 | 0 | 1 | 0 | 1 | 0 | 0 |
| v2 | 0 | 0 | 1 | 1 | 1 | 0 |
| v3 | 0 | 0 | 0 | 1 | 0 | 0 |
| v4 | 0 | 0 | 0 | 0 | 0 | 0 |
| v5 | 0 | 0 | 1 | 0 | 0 | 0 |
| v6 | 0 | 0 | 1 | 0 | 1 | 0 |

# topological sort

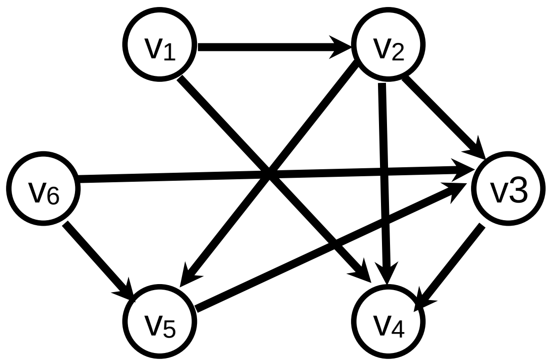



# of in-dgree

| | | | | | | |
|---|---|---|---|---|---|---|
| v1 | 0 | | | | | |
| v2 | 1 | | | | | |
| v3 | 3 | | | | | |
| v4 | 3 | | | | | |
| v5 | 2 | | | | | |
| v6 | 0 | | | | | |
| queue | | | | | | |
| dequeue | | | | | | |

| | v1 | v2 | v3 | v4 | v5 | v6 |
|---|---|---|---|---|---|---|
| v1 | 0 | 1 | 0 | 1 | 0 | 0 |
| v2 | 0 | 0 | 1 | 1 | 1 | 0 |
| v3 | 0 | 0 | 0 | 1 | 0 | 0 |
| v4 | 0 | 0 | 0 | 0 | 0 | 0 |
| v5 | 0 | 0 | 1 | 0 | 0 | 0 |
| v6 | 0 | 0 | 1 | 0 | 1 | 0 |

# topological sort

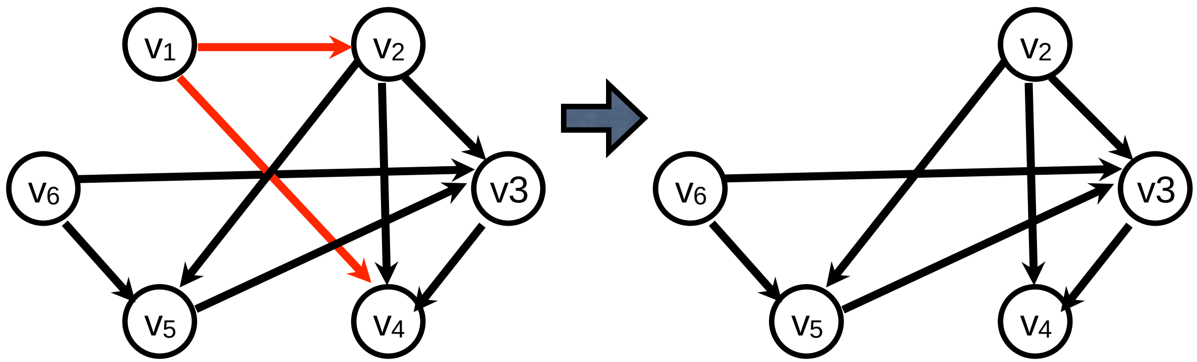



# of in-dgree

| | | | | | | |
|---|---|---|---|---|---|---|
| v1 | 0 | | | | | |
| v2 | 1 | | | | | |
| v3 | 3 | | | | | |
| v4 | 3 | | | | | |
| v5 | 2 | | | | | |
| v6 | 0 | | | | | |
| queue | **v1, v6** | | | | | |
| dequeue | | | | | | |

| | v1 | v2 | v3 | v4 | v5 | v6 |
|---|---|---|---|---|---|---|
| v1 | 0 | 1 | 0 | 1 | 0 | 0 |
| v2 | 0 | 0 | 1 | 1 | 1 | 0 |
| v3 | 0 | 0 | 0 | 1 | 0 | 0 |
| v4 | 0 | 0 | 0 | 0 | 0 | 0 |
| v5 | 0 | 0 | 1 | 0 | 0 | 0 |
| v6 | 0 | 0 | 1 | 0 | 1 | 0 |

# topological sort



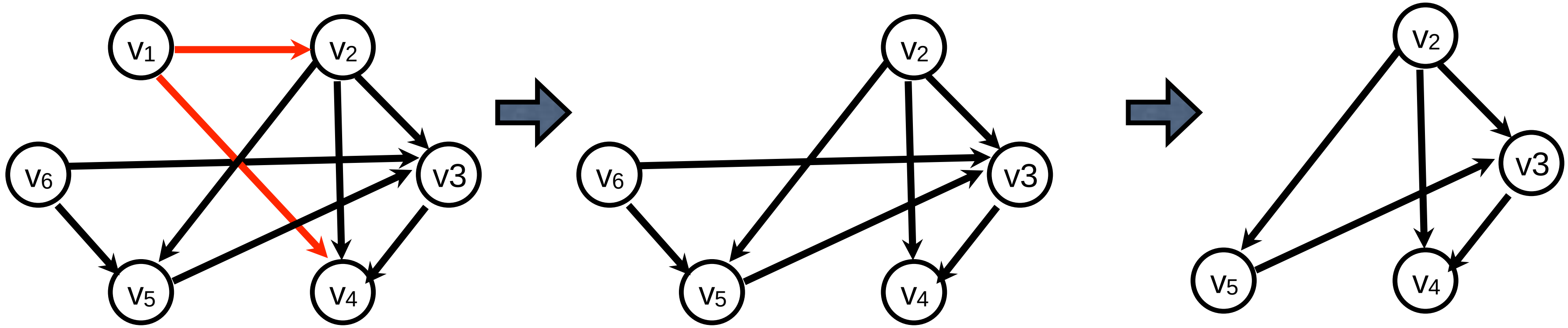

# of in-dgree

| | | | | | | |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | | | | |
| v2 | 1 | 0 | | | | |
| v3 | 3 | 3 | | | | |
| v4 | 3 | 2 | | | | |
| v5 | 2 | 2 | | | | |
| v6 | 0 | 0 | | | | |
| queue | v1, v6 | | | | | |
| dequeue | **v1** | | | | | |

queue v6

| | v1 | v2 | v3 | v4 | v5 | v6 |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | 0 | 0 | 0 | 0 |
| v2 | 0 | 0 | 1 | 1 | 1 | 0 |
| v3 | 0 | 0 | 0 | 1 | 0 | 0 |
| v4 | 0 | 0 | 0 | 0 | 0 | 0 |
| v5 | 0 | 0 | 1 | 0 | 0 | 0 |
| v6 | 0 | 0 | 1 | 0 | 1 | 0 |

# topological sort



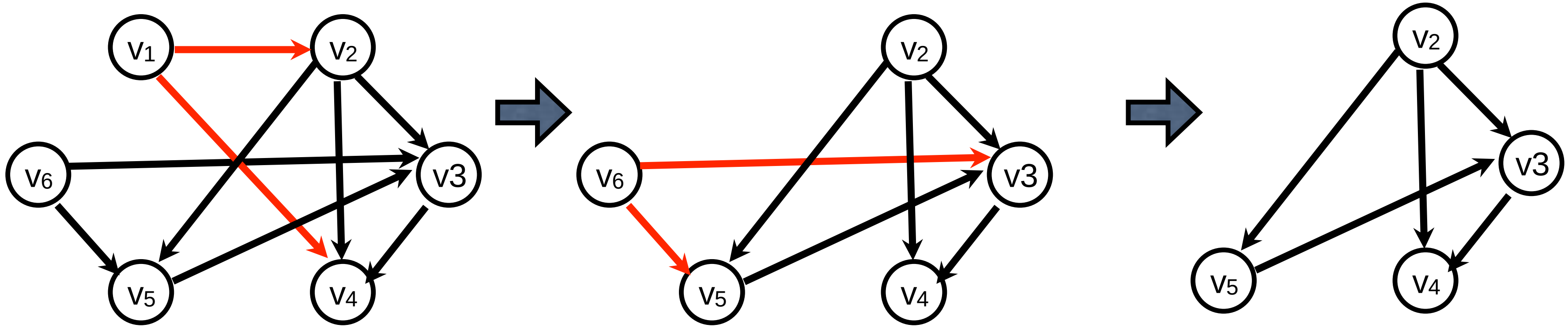

# of in-dgree

| | | | | | | |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | | | | |
| v2 | 1 | 0 | | | | |
| v3 | 3 | 3 | | | | |
| v4 | 3 | 2 | | | | |
| v5 | 2 | 2 | | | | |
| v6 | 0 | 0 | | | | |
| queue | v1, v6 | v6, **v2** | | | | |
| dequeue | **v1** | | | | | |

queue v6

| | v1 | v2 | v3 | v4 | v5 | v6 |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | 0 | 0 | 0 | 0 |
| v2 | 0 | 0 | 1 | 1 | 1 | 0 |
| v3 | 0 | 0 | 0 | 1 | 0 | 0 |
| v4 | 0 | 0 | 0 | 0 | 0 | 0 |
| v5 | 0 | 0 | 1 | 0 | 0 | 0 |
| v6 | 0 | 0 | 1 | 0 | 1 | 0 |

# topological sort



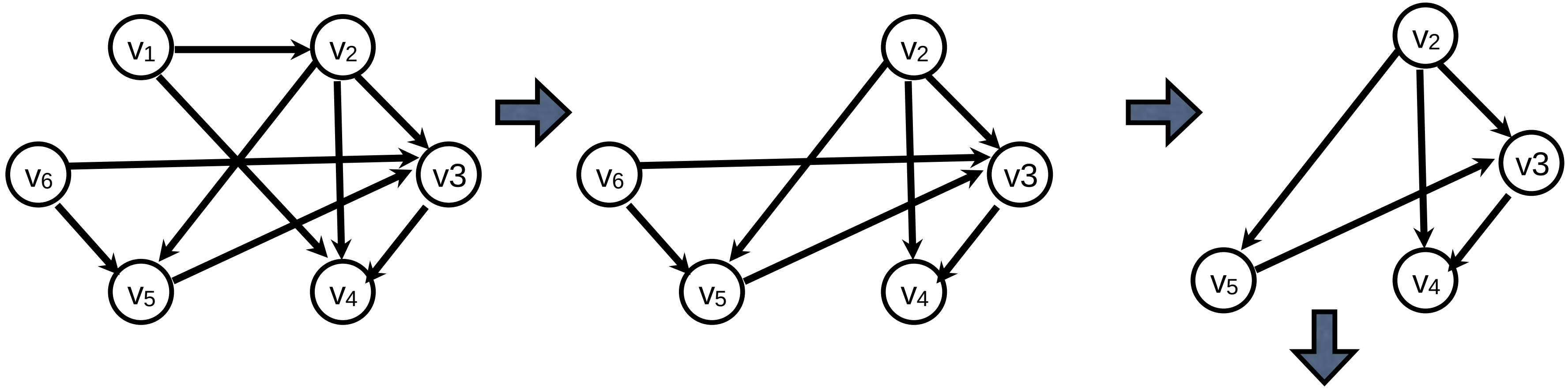

# of in-dgree

| | | | | | | |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | | | | |
| v2 | 1 | 0 | | | | |
| v3 | 3 | 3 | | | | |
| v4 | 3 | 2 | | | | |
| v5 | 2 | 2 | | | | |
| v6 | 0 | 0 | | | | |
| queue | v1, v6 | v6, **v2** | | | | |
| dequeue | **v1** | v6 | | | | |

queue v6 v2

| | v1 | v2 | v3 | v4 | v5 | v6 |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | 0 | 0 | 0 | 0 |
| v2 | 0 | 0 | 1 | 1 | 1 | 0 |
| v3 | 0 | 0 | 0 | 1 | 0 | 0 |
| v4 | 0 | 0 | 0 | 0 | 0 | 0 |
| v5 | 0 | 0 | 1 | 0 | 0 | 0 |
| v6 | 0 | 0 | 1 | 0 | 1 | 0 |

# topological sort



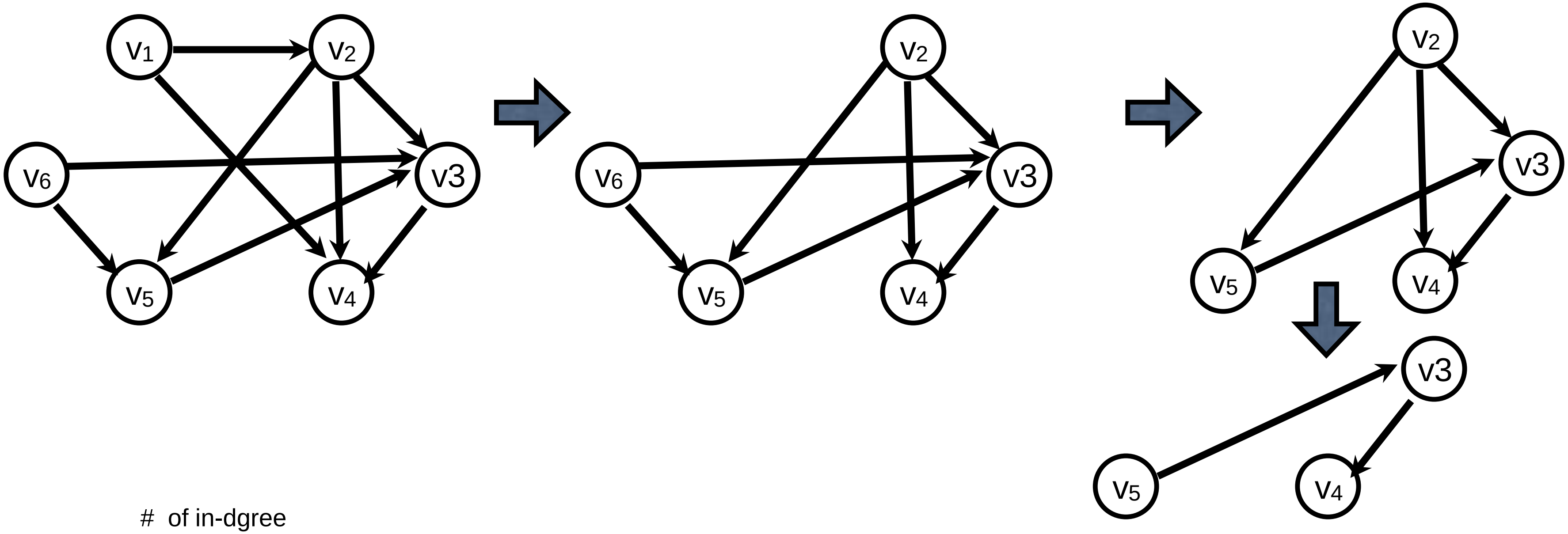

\# of in-dgree

| | | | | | | |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | 0 | | | |
| v2 | 1 | 0 | 0 | | | |
| v3 | 3 | 3 | 2 | | | |
| v4 | 3 | 2 | 2 | | | |
| v5 | 2 | 2 | 1 | | | |
| v6 | 0 | 0 | 0 | | | |
| queue | v1, v6 | v6, v2 | v2 | | | |
| dequeue | v1 | v6 | v2 | | | |

queue v6 v2 none

| | v1 | v2 | v3 | v4 | v5 | v6 |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | 0 | 0 | 0 | 0 |
| v2 | 0 | 0 | 1 | 1 | 1 | 0 |
| v3 | 0 | 0 | 0 | 1 | 0 | 0 |
| v4 | 0 | 0 | 0 | 0 | 0 | 0 |
| v5 | 0 | 0 | 1 | 0 | 0 | 0 |
| v6 | 0 | 0 | 0 | 0 | 0 | 0 |

# topological sort



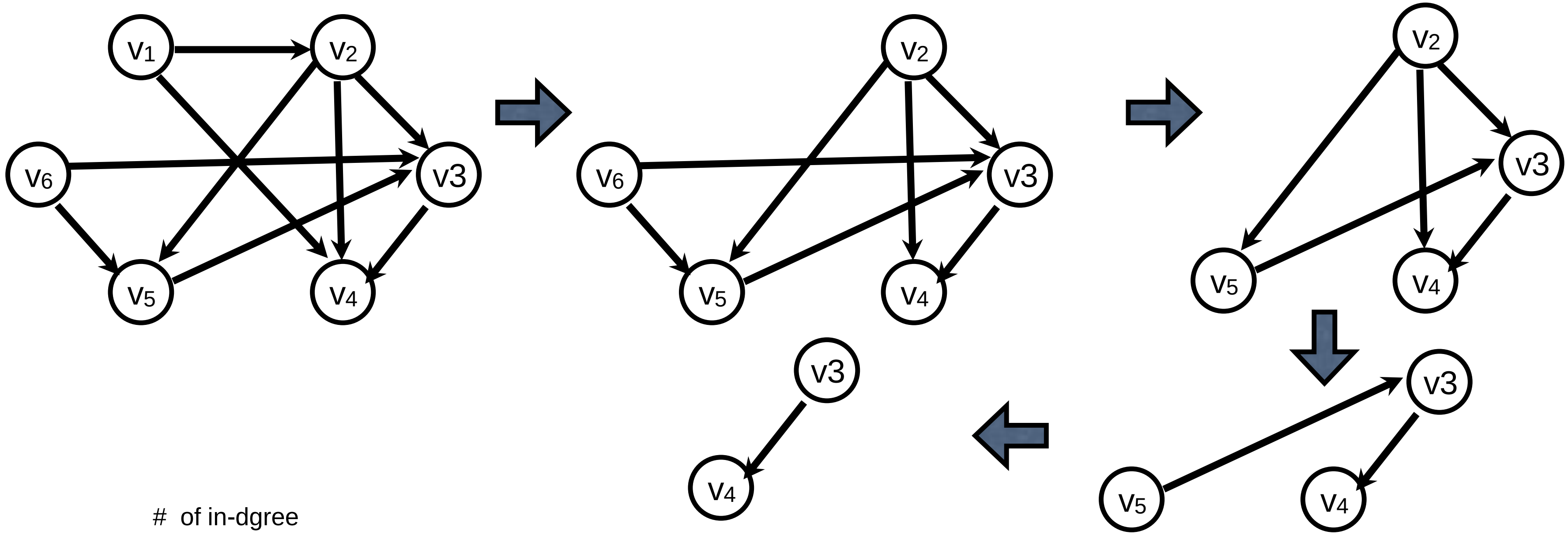

\# of in-dgree

| | | | | | | |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | 0 | 0 | | |
| v2 | 1 | 0 | 0 | 0 | | |
| v3 | 3 | 3 | 2 | 1 | | |
| v4 | 3 | 2 | 2 | 1 | | |
| v5 | 2 | 2 | 1 | 0 | | |
| v6 | 0 | 0 | 0 | 0 | | |
| queue | v1, v6 | v6, v2 | v2 | v5 | | |
| dequeue | v1 | v6 | v2 | v5 | | |

queue v6 v2 none none

| | v1 | v2 | v3 | v4 | v5 | v6 |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | 0 | 0 | 0 | 0 |
| v2 | 0 | 0 | 0 | 0 | 0 | 0 |
| v3 | 0 | 0 | 0 | 1 | 0 | 0 |
| v4 | 0 | 0 | 0 | 0 | 0 | 0 |
| v5 | 0 | 0 | 1 | 0 | 0 | 0 |
| v6 | 0 | 0 | 0 | 0 | 0 | 0 |

# topological sort



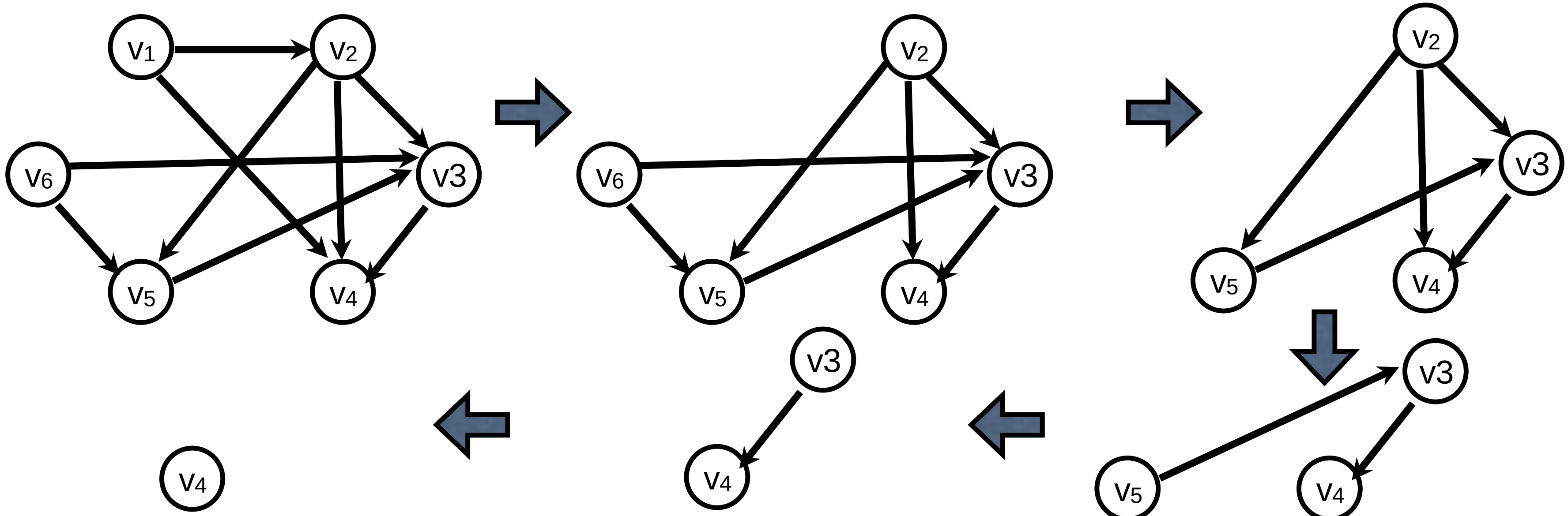

# of in-dgree

| | | | | | | |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | 0 | 0 | 0 | |
| v2 | 1 | 0 | 0 | 0 | 0 | |
| v3 | 3 | 3 | 2 | 1 | 0 | |
| v4 | 3 | 2 | 2 | 1 | 1 | |
| v5 | 2 | 2 | 1 | 0 | 0 | |
| v6 | 0 | 0 | 0 | 0 | 0 | |
| queue | v1, v6 | v6, v2 | v2 | v5 | v3 | |
| dequeue | v1 | v6 | v2 | v5 | v3 | |

queue v6 v2 none none none

| | v1 | v2 | v3 | v4 | v5 | v6 |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | 0 | 0 | 0 | 0 |
| v2 | 0 | 0 | 0 | 0 | 0 | 0 |
| v3 | 0 | 0 | 0 | 1 | 0 | 0 |
| v4 | 0 | 0 | 0 | 0 | 0 | 0 |
| v5 | 0 | 0 | 0 | 0 | 0 | 0 |
| v6 | 0 | 0 | 0 | 0 | 0 | 0 |

# topological sort




# of in-dgree

| | | | | | | |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | 0 | 0 | 0 | 0 |
| v2 | 1 | 0 | 0 | 0 | 0 | 0 |
| v3 | 3 | 3 | 2 | 1 | 0 | 0 |
| v4 | 3 | 2 | 2 | 1 | 1 | 0 |
| v5 | 2 | 2 | 1 | 0 | 0 | 0 |
| v6 | 0 | 0 | 0 | 0 | 0 | 0 |
| queue | v1, v6 | v6, v2 | v2 | v5 | v3 | v4 |
| dequeue | v1 | v6 | v2 | v5 | v3 | v4 |

queue v6 v2 none none none none

| | v1 | v2 | v3 | v4 | v5 | v6 |
|---|---|---|---|---|---|---|
| v1 | 0 | 0 | 0 | 0 | 0 | 0 |
| v2 | 0 | 0 | 0 | 0 | 0 | 0 |
| v3 | 0 | 0 | 0 | 0 | 0 | 0 |
| v4 | 0 | 0 | 0 | 0 | 0 | 0 |
| v5 | 0 | 0 | 0 | 0 | 0 | 0 |
| v6 | 0 | 0 | 0 | 0 | 0 | 0 |

# topological sort

```
void Topsort (Graph G)
{
      Queue Q;
      Vertex V, W;
      int *Indegree;

      Q = CreateQueue();
      checkIndegree(Indegree);

      for each vertex V
          if( Indegree[V] == 0 )
                                    Enqueue(V, Q);

      while( !IsEmtpy(Q) )
      {
          V = Dequeue(Q);
          for each W adjacent to V
               if ( --Indegree[W] == 0 )
                                                Enqueue(W, Q);
      }

}
```

# topological sort

- n = |V|, e = |E|
- the number of iterations of the while loop is at most n
- the number of iterations of the for loop is proportional to outdeg(v) and each iteration takes constant time
- since outdeg(v) may be zero and we need to spend some time updating loop variables, etc. in this case, the time is Θ(outdeg(v) + 1)
- the running time is

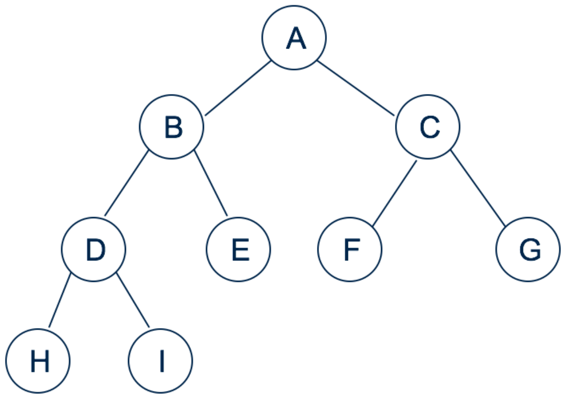
$$T(n) = n + \sum_{v \in V} (outdeg(v) + 1)$$

$$= n + \sum_{v \in V} outdeg(v) + \sum_{v \in V} 1 = n + e + n \in \Theta(n + e)$$

# tree traversal

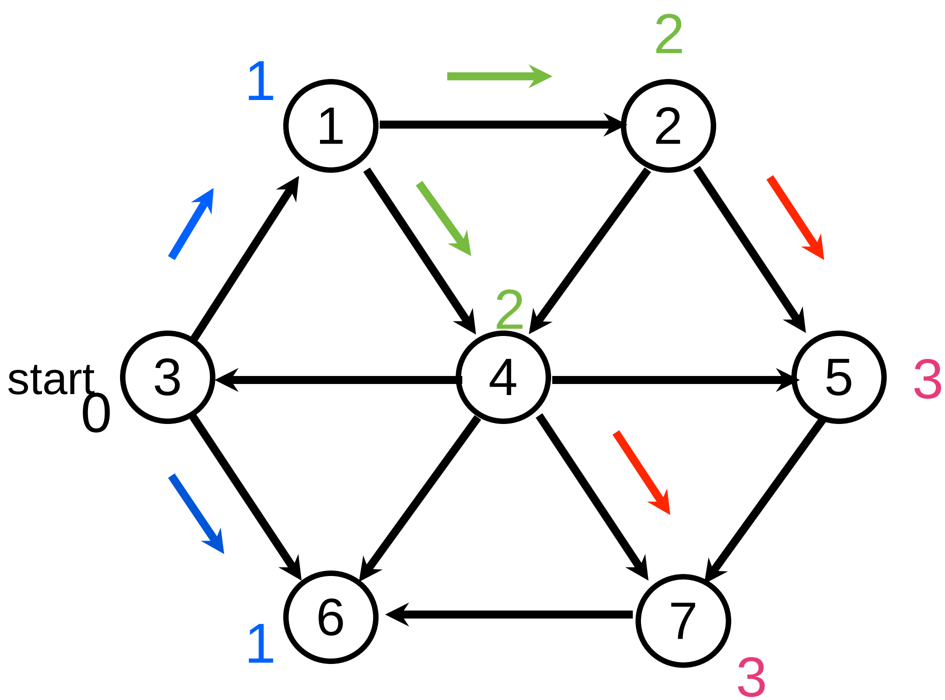
- level-order traversal

```
void levelOrder (Tree ptr)   {
        int front = rear = 0;
        Tree queue[MAX];
        if (! ptr)    return;
        addq(ptr);
        for (; ; )  {
                ptr = deleteq();
                if (ptr)  {
                        printf("%d", ptr->data);
                        if (ptr -> leftChild)
                                addq(ptr -> leftChild);
                        if (ptr -> rightChild)
                                addq(ptr -> rightChild);
                }
                else break;
        }
}
```



# breadth-first search

- vertices closest to the start are evaluated first, and the most distant vertices are evaluated last
- use Queue to keep track of vertices to evaluate



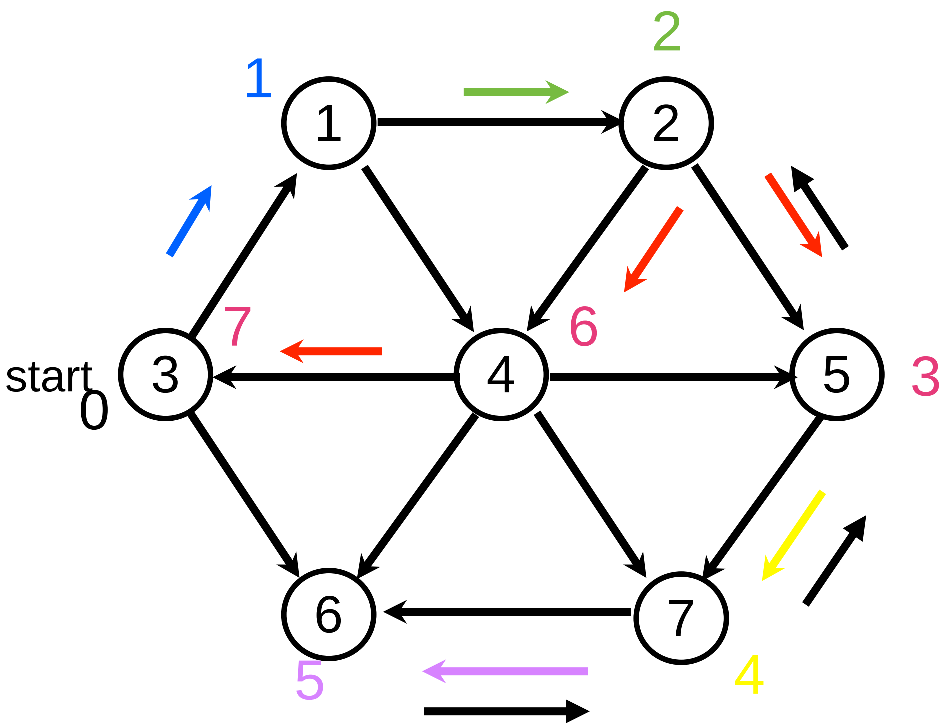

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 6 | | | | | | |
| 1 | 6 | 2 | 4 | | | | | |
| 6 | 2 | 4 | | | | | | |
| 2 | 4 | 5 | | | | | | |
| 4 | 5 | 7 | | | | | | |
| 5 | 7 | | | | | | | |
| 7 | | | | | | | | |

# breadth-first search

```
BFS(Table T)
{
    Q = CreateQueue(NumVertex);
    MakeEmpty(Q);
    Enqueue(s, Q);
    while( !IsEmtpy(Q) ) do
    {
          v = Dequeue(Q);
          for each w adjacent to v.
                if( d[w] == infinity ) {
                    /* d[w] : depth, d[w] == infinity means “not visited yet” */
                              d[w] = d[v] + 1;
                              pred[w] = v;
                              Enqueue(w, Q);
                      }
           }
           DisposeQueue(Q);
}
```

# depth-first search

- travel as deep as possible from neighbour to neighbour before backtracking
- use Stack to keep track of vertices to evaluate



# depth-first search: recursive implementation

```
void DFS (G, u){

        while u has an unvisited neighbour in G
                v = an unvisited neighbour of u
                mark v visited
                DFS(G, v)
}
```

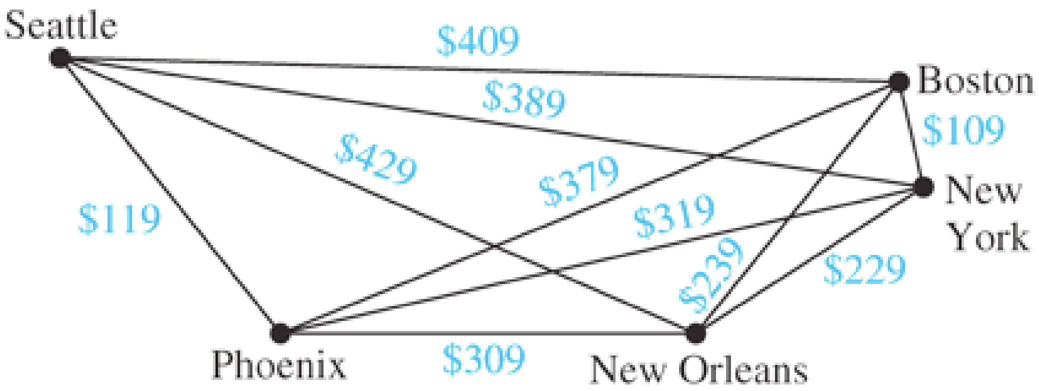
# depth-first search: iterative implementation using stack

```
void DFS (G, u){

        S = stack initialized
        S.push (u)
        while S is not empty
                v = S.pop()
                if v not visited
                        mark  v as visited
                        for  w is a neighbour of v
                                S. push(w)
```

# shorted path algorithms

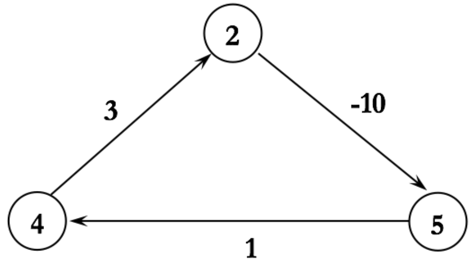
- a weighted graph: a graph G = (V, E), where a cost $c_{ij}$ is associated with each edge
  - weighted path length: $\sum_{i=1}^{n-1} c_{i,i+1}$ for the path $v_1, v_2, \ldots v_n$
  - unweighted path length: the number of edges on the path, i.e. $c_{i,i+1} = 1$
- Single source shortest-path problem
  - given as input a weighted graph G and a distinguished vertex s as source
  - find the shortest weighted path from s to every other v vertex in G



# shorted path algorithms

- In many practical applications, we consider finding the shortest path from one vertex s to another t
  - currently no algorithm can find the path from one source to one vertices (ie. s to t) any faster than finding the path from one source to all vertices

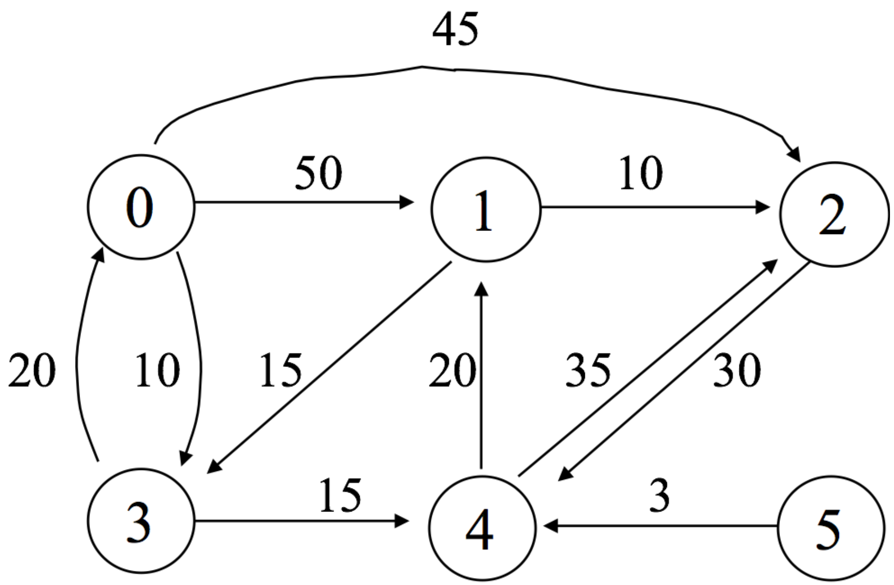
- When negative-cost cycles are present in the graph, the shortest path may be undefined




5 → 4: 1
5 → 4 → 2 → 5 → 4: -5

# shorted path algorithms

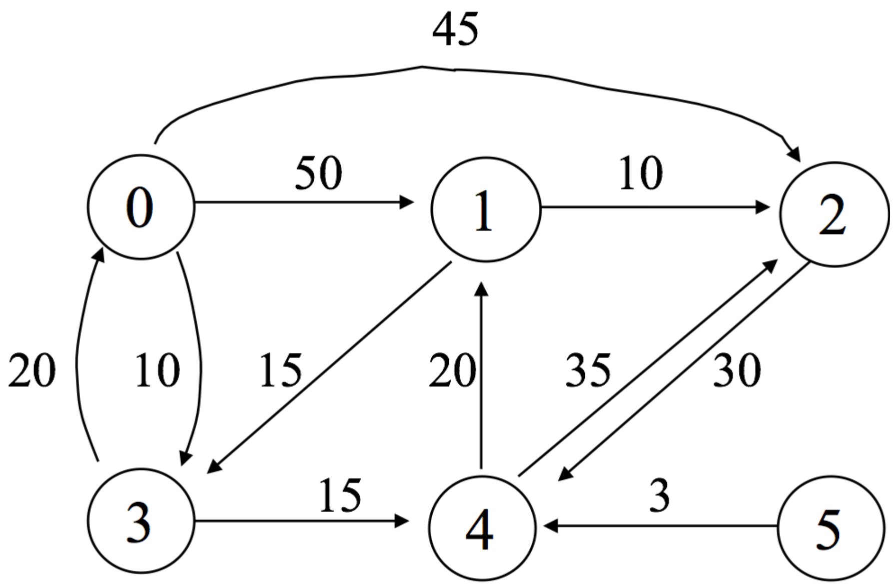
shortest path from 0 to 1?




(a)

# shorted path algorithms

shortest path from 0 to 1?



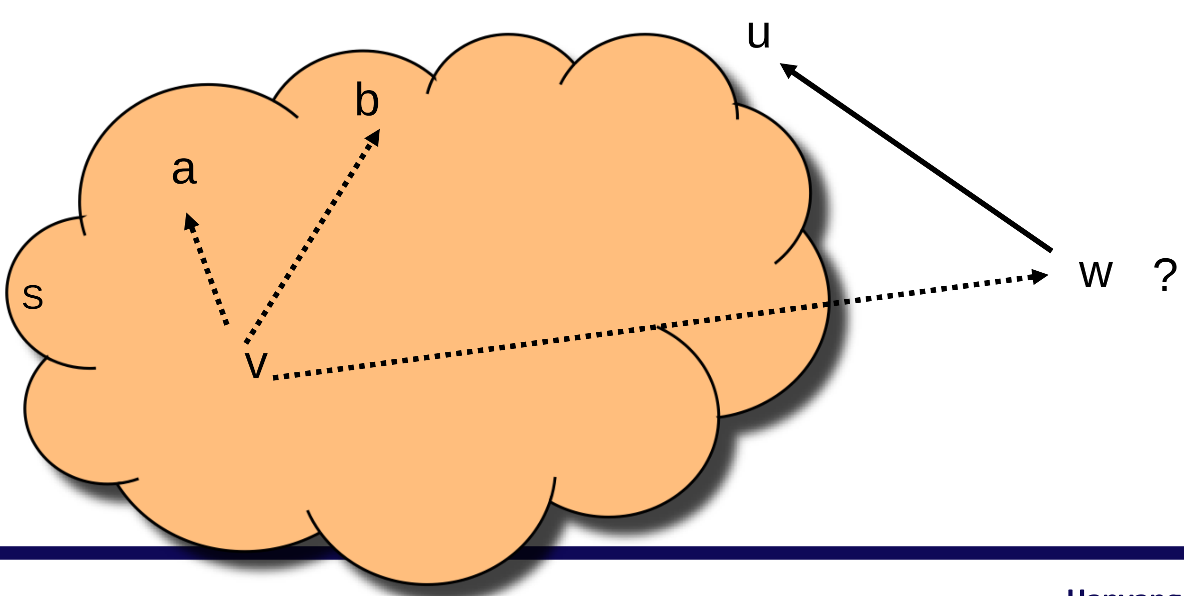

(a)

| path | length |
|---|---|
| 1) 0,3 | 10 |
| 2) 0,3,4 | 25 |
| 3) 0,3,4,1 | 45 |
| 4) 0,2 | 45 |

(b)

# shorted path algorithms

- S is a set of vertices that have the shortest path from v to those vertices
- We generate the paths in non-descending order of length
- When the next shortest path is to vertex u, it possible to have a shortest path v to u through w?
- if vertex u is chosen, it has the minimum distance among all the vertices not in S



# weighted single-source shorted path : Dijkstra's algorithm

- length of a path: sum of edge weights along the path
- finding minimum length of the path from u to v: δ(s, v)
- given a directed graph with non-negative edge weights G = (V, E), and a special source vertex s ∈ V, determine the distance from the source vertex to every vertex in G
    - d[v]: shortest path from the source to v
    - pred[v]: previous vertex of v in the path
  - each node is one of the status, permanent or temporary
    - the status of a node is permanent if its distance value is equal to the shortest distance from node s
    - otherwise, the status of a node is temporary

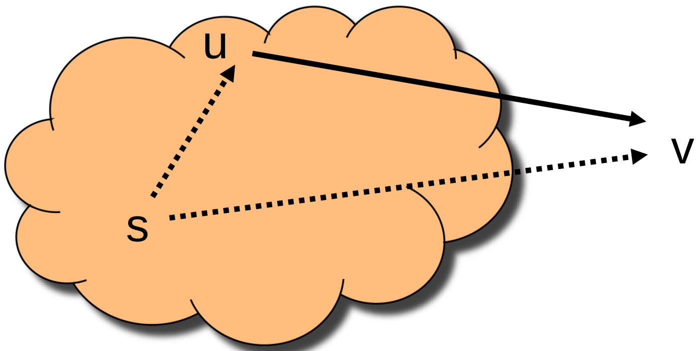
# weighted single-source shorted path : Dijkstra's algorithm

- how does the algorithm work
  - start by assigning some initial values for the distance d[v] from a node s to every other node v in the graph
  - at each step, update the distance to every node and determine a node j that has the smallest distance value $d_j$ among all nodes in the temporary sets
  - if all nodes are labeled as permanent, stop

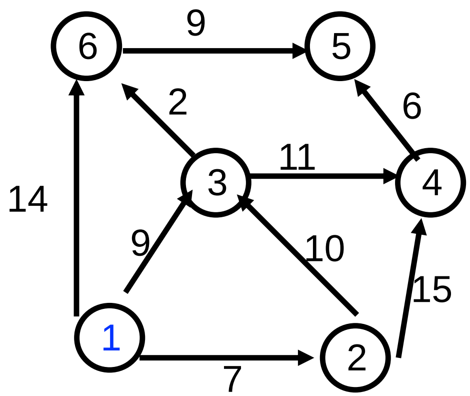
# weighted single-source shorted path : Dijkstra's algorithm

- relaxation (update) process
  - d[v]: shortest path from the source to v
  - pred[v]: previous vertex of v in the path
  - initially d[s] = 0,  d[v] = ∞    v: all other nodes except the starting node
  - d[v] is updated until d[v] is converged to minimum distance δ(s, v)
  - implemented with a priority queue: every operation (insert, delete_min, decrease_key) can be done in Θ (log n) time

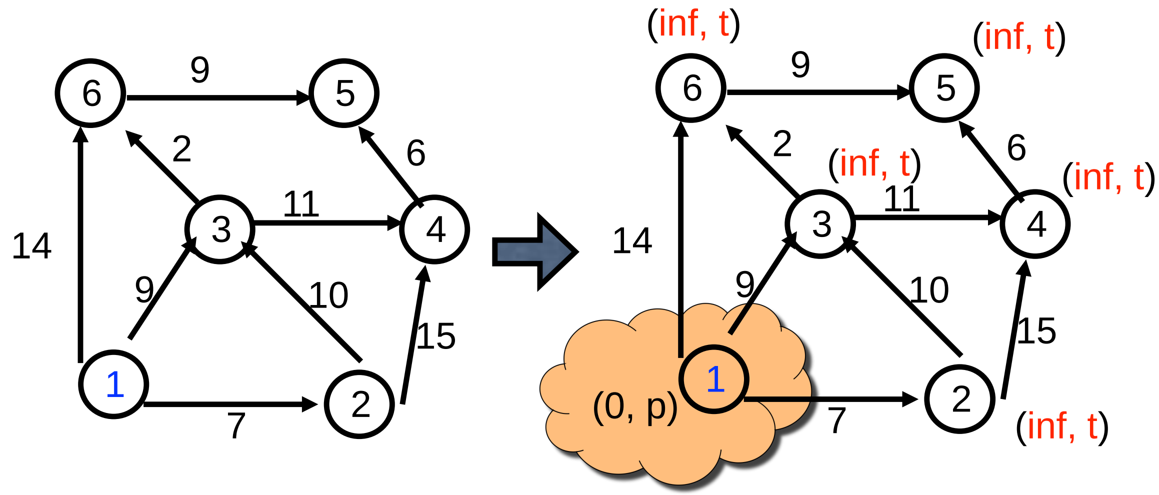
```
if (d[u] + w[u, v] < d[v])
{
   d[v] = d[u] + w[u, v];
   pred[v] = u;
}
```



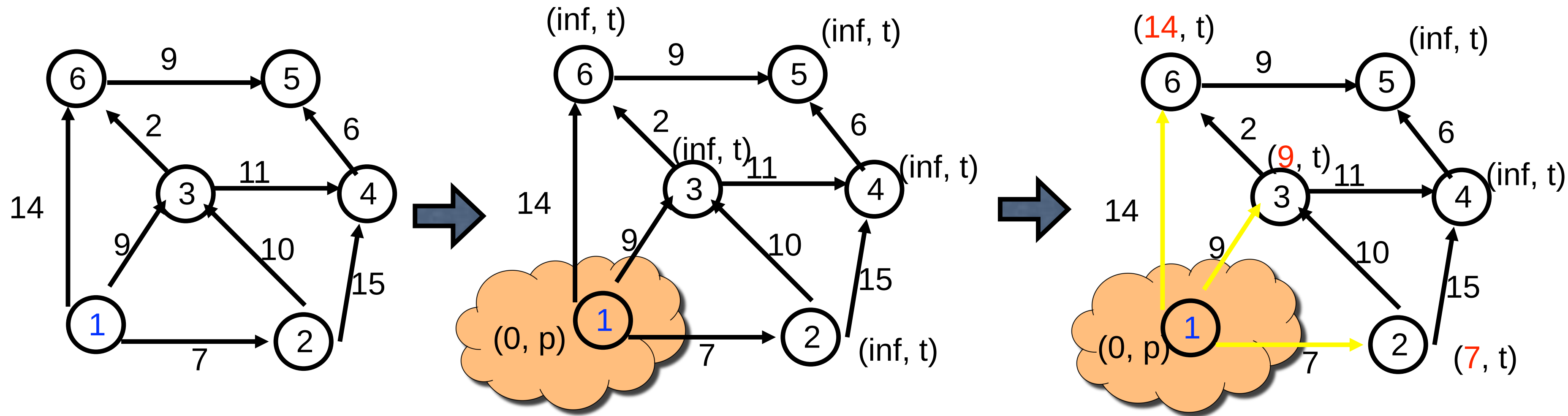
# Dijkstra's algorithm



# Dijkstra's algorithm

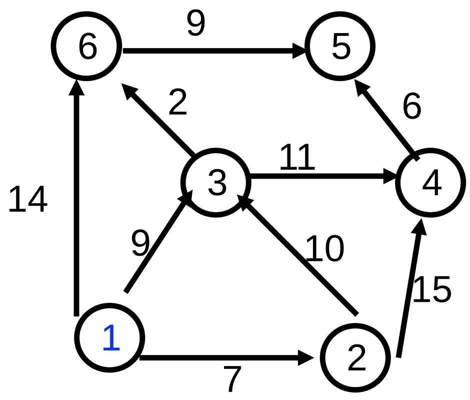


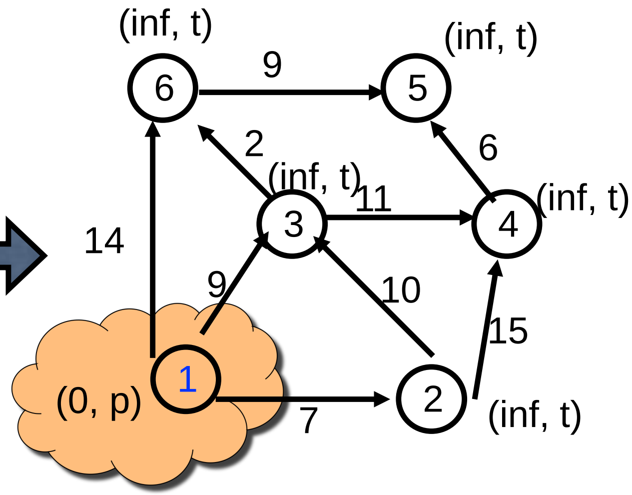
# Dijkstra's algorithm

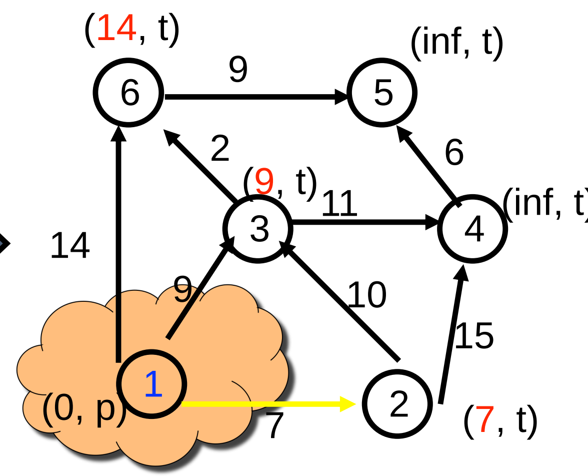



d2 = min{inf, 0+7) = 7
d3 = min{inf, 0+9) = 9
d6 = min{inf, 0+14) = 14

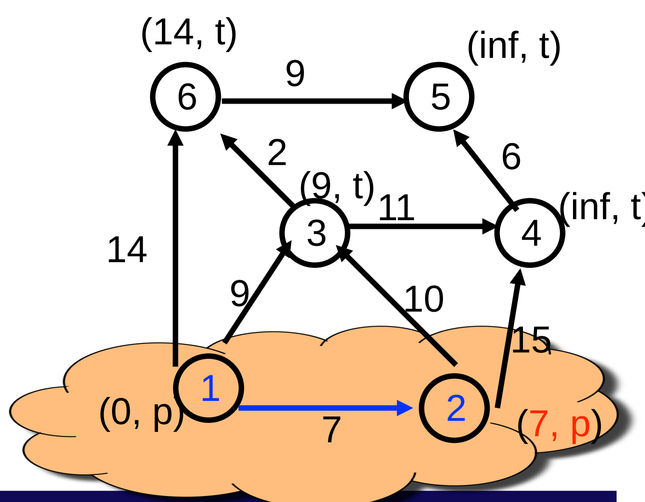
# Dijkstra's algorithm



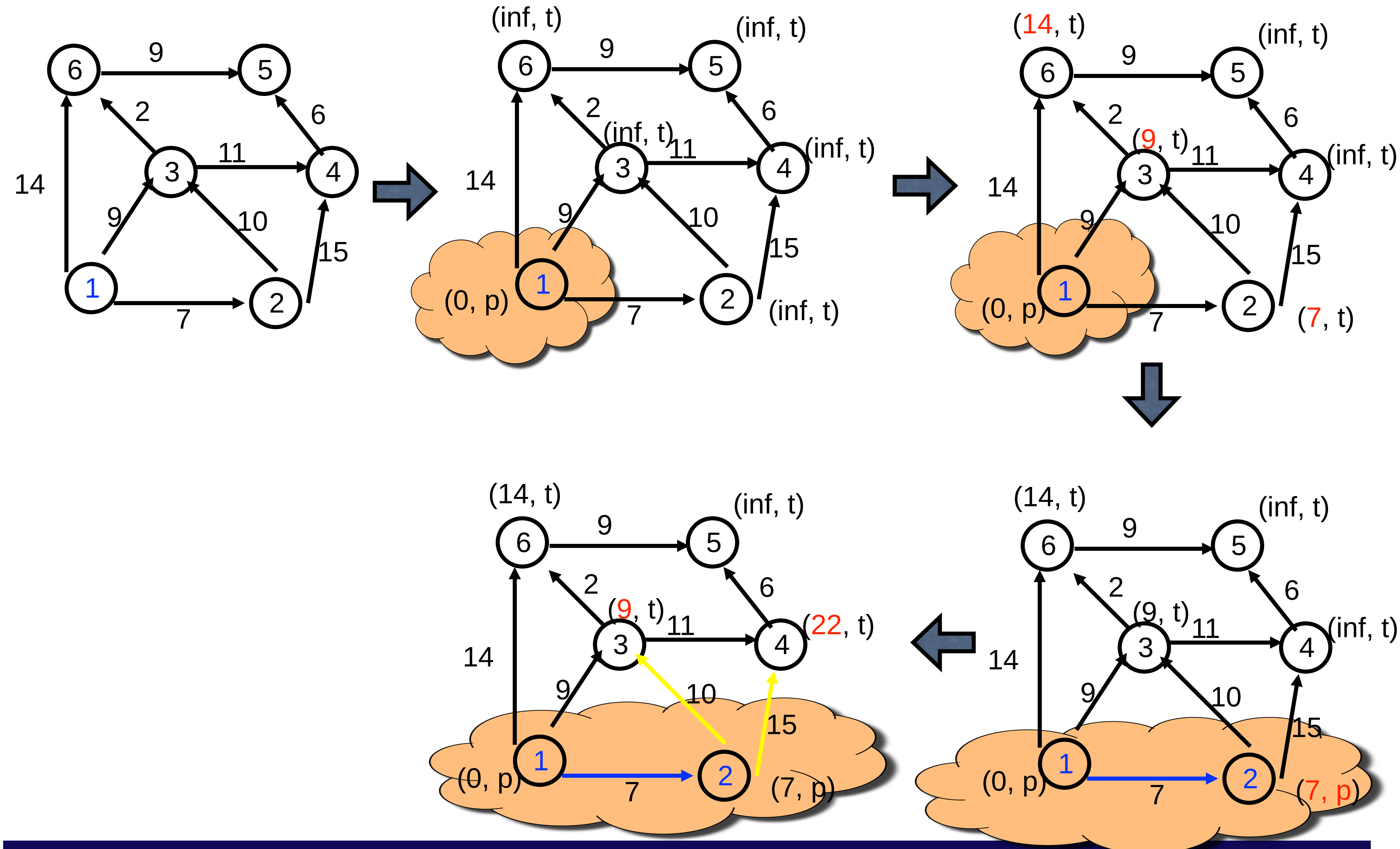




among all temporary nodes,
node 2 is the smallest



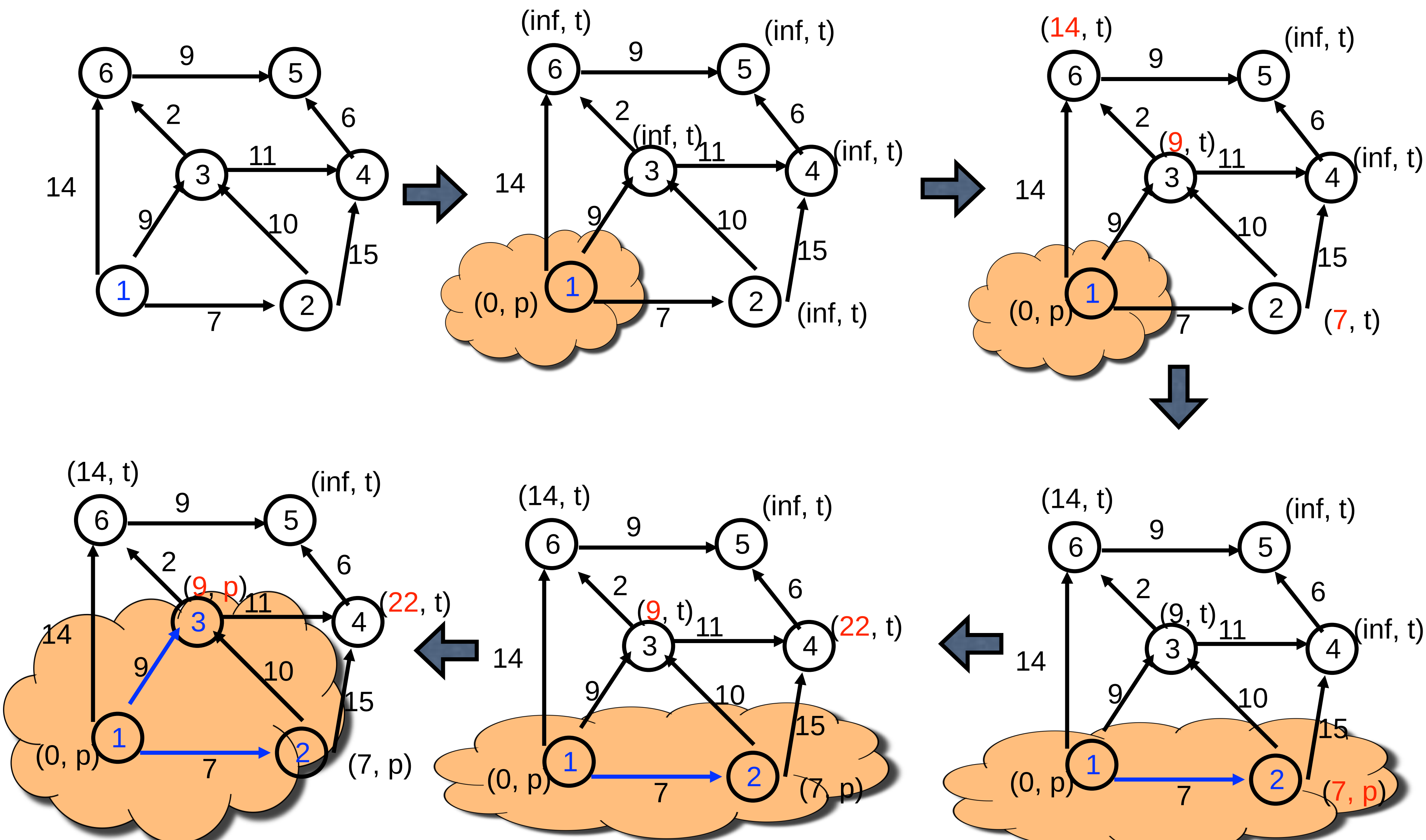
# Dijkstra's algorithm




d3 = min{9, 7+10) = 9

d4 = min{inf, 7+15) = 22

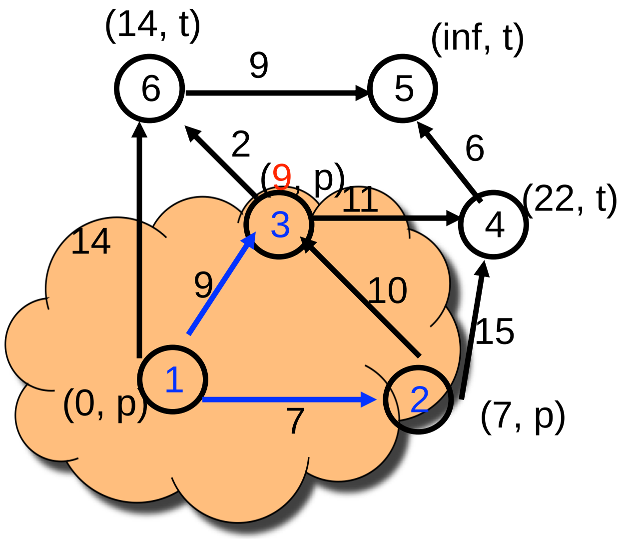
# Dijkstra's algorithm



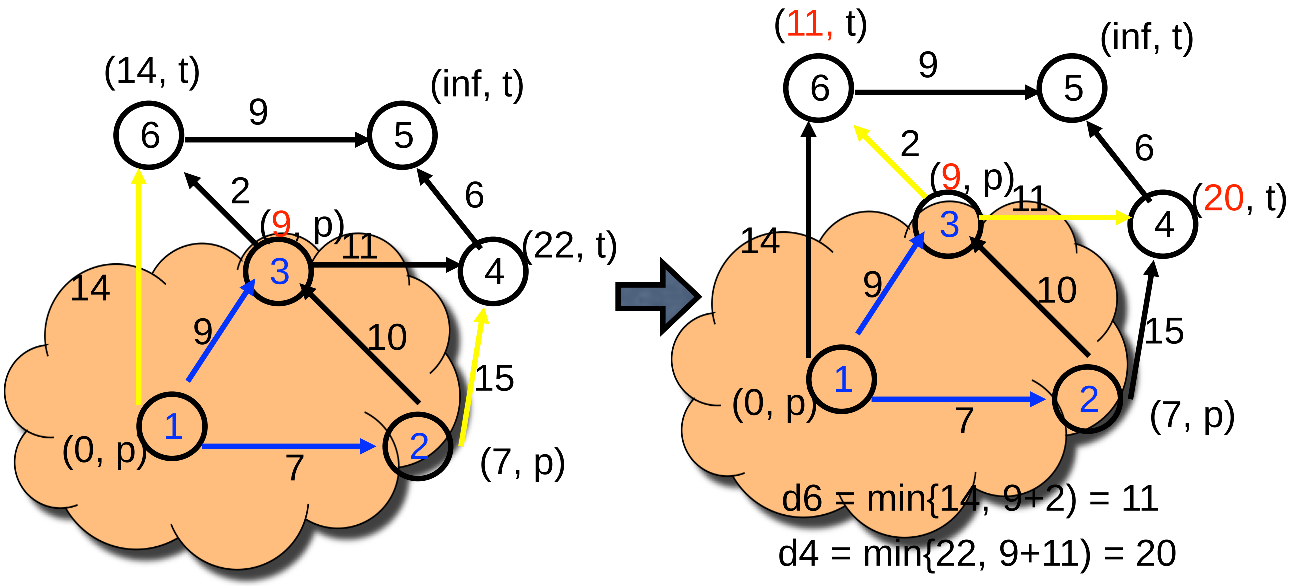
d3 = min{9, 7+10) = 9

d4 = min{inf, 7+15) = 22

# Dijkstra's algorithm



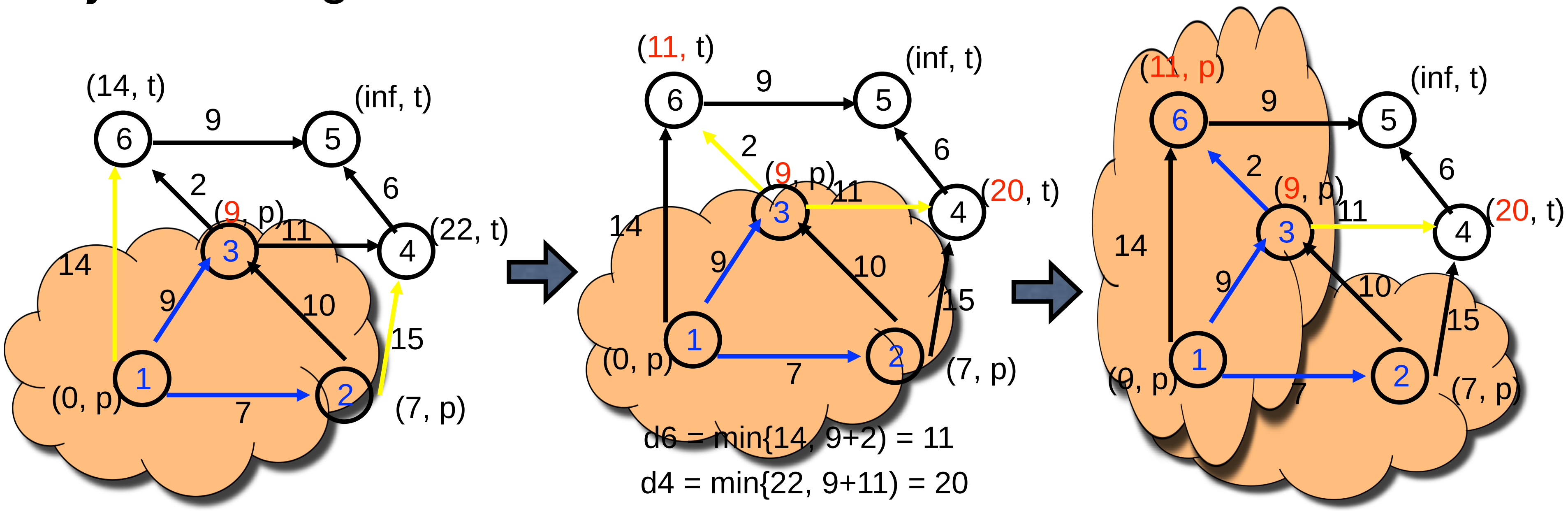
# Dijkstra’s algorithm



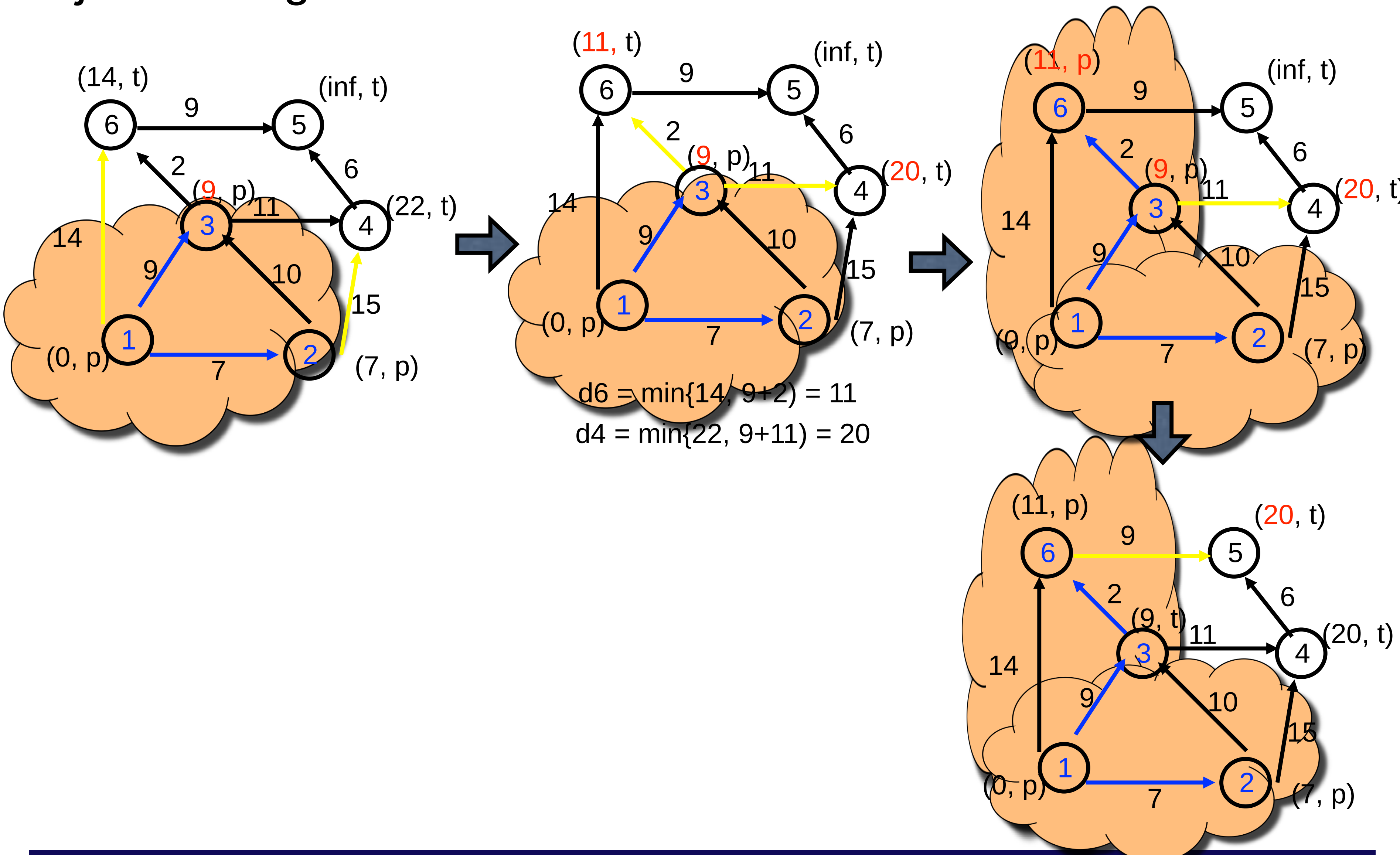

d6 = min{14, 9+2) = 11

d4 = min{22, 9+11) = 20

# Dijkstra's algorithm

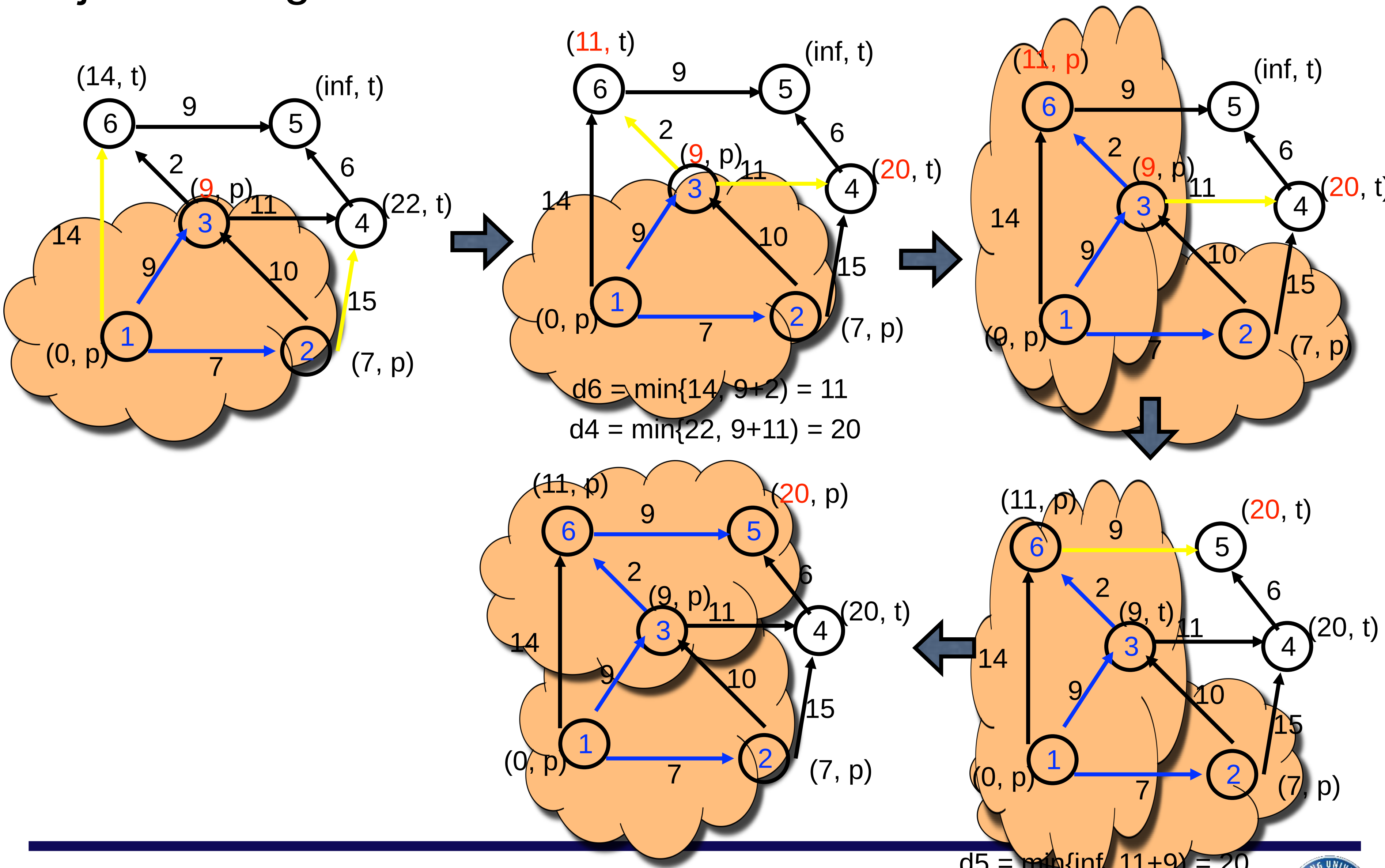


# Dijkstra's algorithm

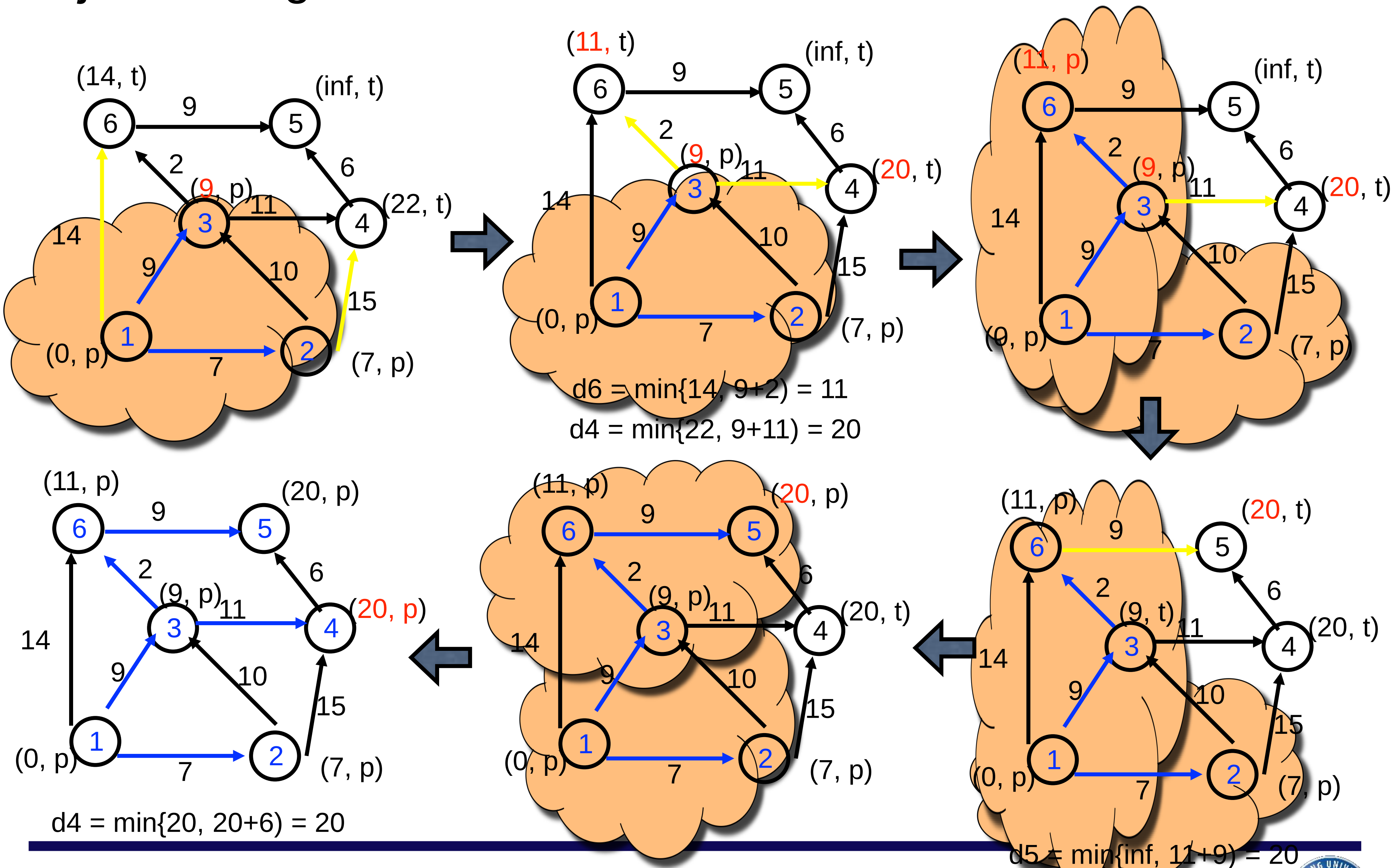



d5 = min{inf, 11+9) = 20

# Dijkstra's algorithm



# Dijkstra’s algorithm



# Dijkstra's algorithm

```
Dijkstra(G=(V, E, w), s)
{
                SP = {};
                for each v in V do  {
                            d[v] = +infinity;       pred[v] = nil;
                }
            d[s] = 0;
            for each v in Adj[s] do {
                            d[v] = w[s, v];         pred[v] = s;
                }

            Add each vertex to priority queue Q;

            While (Q is not empty) do  {       /* for each node*/
                            u = Delete_Min(Q);
                            SP = SP + {u};
                  for each v in Adj[u] do   {   /* for each outdeg(u)*/
                            if (d[u] + w(u, v) < d[v]) then  {
                                            d[v] = d[u] + w(u, v);
                                            pred[v] = u;
                                            Decrease_Priority(Q, v);
                                       }
                }
            }
}
```

Θ(n)

Θ(log n)

Θ(log n)

$$\sum_{u\in V}(\log n + 1 + outdeg(u)\times\log n) =$$

$$n\log n + n + \left(\sum_{u\in V} outdeg(u)\right)\log n \in \Theta((n+e)\log n)$$

# Dijkstra’s algorithm