

The method of characteristics applied to quasi-linear PDEs

18.303 Linear Partial Differential Equations

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1 Motivation

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Ref: Haberman §12.6

Most of the methods discussed in this course: separation of variables, Fourier Series, Green's functions (later) can only be applied to linear PDEs. However, the method of characteristics can be applied to a form of nonlinear PDE.

1.1 Traffic flow

Consider the idealized flow of traffic along a one-lane highway. Let $\rho(x, t)$ be the traffic density at (x, t) . The total number of cars in $x_1 \leq x \leq x_2$ at time t is

$$N(t) = \int_{x_1}^{x_2} \rho(x, t) dx \quad (1)$$

Assume the number of cars is conserved, i.e. no exits. Then the rate of change of the number of cars in $x_1 \leq x \leq x_2$ is given by

$$\begin{aligned} \frac{dN}{dt} &= \text{rate in at } x_1 - \text{rate out at } x_2 \\ &= \rho(x_1, t) V(x_1, t) - \rho(x_2, t) V(x_2, t) \\ &= - \int_{x_1}^{x_2} \frac{\partial}{\partial x} (\rho V) dx \end{aligned} \quad (2)$$

where $V(x, t)$ is the velocity of the cars at (x, t) . Combining (1) and (2) gives

$$\int_{x_1}^{x_2} \left(\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x} (\rho V) \right) dx = 0$$

and since x_1, x_2 are arbitrary, the integrand must be zero at all x ,

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x}(\rho V) = 0 \quad (3)$$

We assume, for simplicity, that velocity V depends on density ρ , via

$$V(\rho) = c \left(1 - \frac{\rho}{\rho_{\max}} \right)$$

where $c = \max$ velocity, $\rho = \rho_{\max}$ indicates a traffic jam ($V = 0$ since everyone is stopped), $\rho = 0$ indicates open road and cars travel at c , the speed limit (yeah right). The PDE (3) becomes

$$\frac{\partial \rho}{\partial t} + c \left(1 - \frac{2\rho}{\rho_{\max}} \right) \frac{\partial \rho}{\partial x} = 0 \quad (4)$$

We introduce the following normalized variables

$$u = \frac{\rho}{\rho_{\max}}, \quad \tilde{t} = ct$$

into the PDE (4) to obtain (dropping tildes),

$$u_t + (1 - 2u)u_x = 0 \quad (5)$$

The PDE (5) is called quasi-linear because it is linear in the derivatives of u . It is NOT linear in $u(x, t)$, though, and this will lead to interesting outcomes.

2 General first-order quasi-linear PDEs

The general form of quasi-linear PDEs is

$$A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial t} = C \quad (6)$$

where A, B, C are functions of u, x, t . The initial condition $u(x, 0)$ is specified at $t = 0$,

$$u(x, 0) = f(x) \quad (7)$$

To solve the PDE (6), we note that

$$(A, B, C) \cdot (u_x, u_t, -1) = 0$$

so that $(u_x, u_t, -1)$ is the normal to the solution surface $z = u(x, t)$. The vector (A, B, C) is the tangent to this solution surface.

The IC $u(x, 0) = f(x)$ is a curve in the $u - x$ plane. For any point on the initial curve, the vector (A, B, C) generates a curve on the solution surface, called a characteristic curve of the PDE. Once we find all the characteristic curves, we have a complete description of the solution $u(x, t)$.

2.1 Method of characteristics

We represent the characteristic curves parametrically,

$$x = x(r), \quad t = t(r), \quad u = u(r).$$

The tangent vector is

$$(A, B, C) = \left(\frac{dx}{dr}, \frac{dt}{dr}, \frac{du}{dr} \right)$$

The characteristic curves are found by solving

$$\frac{dx}{dr} = A, \quad \frac{dt}{dr} = B, \quad \frac{du}{dr} = C \quad (8)$$

where (A, B, C) depend on (x, t, u) in general. We take $r = 0$ to be the initial value of r at $t = 0$, so that $t(0) = 0$. We let $s = x(0)$ so that the initial value of u is $u(x(0), t(0)) = u(s, 0) = f(s)$. The initial conditions for the characteristic curves are thus

$$x(0) = s, \quad t(0) = 0, \quad u(0) = f(s). \quad (9)$$

3 Solution to traffic flow problem

The traffic flow PDE (5) is

$$u_t + (1 - 2u)u_x = 0 \quad (10)$$

and has form (6) with $(A, B, C) = (1 - 2u, 1, 0)$. The characteristic curves satisfy (8) and (9)

$$\begin{aligned} \frac{dx}{dr} &= 1 - 2u, & x(0) &= s, \\ \frac{dt}{dr} &= 1, & t(0) &= 0, \\ \frac{du}{dr} &= 0, & u(0) &= f(s). \end{aligned}$$

Integrating gives the parametric equations

$$t = r + c_1, \quad u = c_2, \quad x = (1 - 2u)r + c_3 = (1 - 2c_2)r + c_3$$

Imposing the ICs gives $c_1 = 0$, $c_2 = f(s)$, $c_3 = s$, so that

$$t = r, \quad u = f(s), \quad x = (1 - 2f(s))r + s = (1 - 2f(s))t + s \quad (11)$$

For a given time $t = t_0$, plotting the solution amounts to choosing an appropriate range of values for s , and then plotting $u(t_0, s)$ versus $x(t_0, s)$ in the xu -plane.

4 Example : Light traffic heading into heavier traffic

Consider light traffic heading into heavy traffic, and model the initial density as

$$u(x, 0) = f(x) = \begin{cases} \alpha, & x \leq 0 \\ \left(\frac{3}{4} - \alpha\right)x + \alpha, & 0 \leq x \leq 1 \\ \frac{3}{4}, & x \geq 1 \end{cases} \quad (12)$$

where $0 \leq \alpha \leq 3/4$. The lightness of traffic is parametrized by α . We consider the case of light traffic $\alpha = 1/6$ and moderate traffic $\alpha = 1/3$.

From (11), the characteristics are

$$x = \begin{cases} (1 - 2\alpha)t + s, & s \leq 0 \\ (1 - 2\alpha - 2(3/4 - \alpha)s)t + s, & 0 \leq s \leq 1 \\ -t/2 + s, & s \geq 1 \end{cases}$$

For $\alpha = 1/6$, we have

$$x = \begin{cases} \frac{2}{3}t + s, & s \leq 0 \\ \left(\frac{2}{3} - \frac{7}{6}s\right)t + s, & 0 \leq s \leq 1 \\ -\frac{1}{2}t + s, & s \geq 1 \end{cases}$$

For $\alpha = 1/3$, we have

$$x = \begin{cases} \frac{1}{3}t + s, & s \leq 0 \\ \left(\frac{1}{3} - \frac{5}{6}s\right)t + s, & 0 \leq s \leq 1 \\ -\frac{1}{2}t + s, & s \geq 1 \end{cases}$$

Again, for fixed times $t = t_0$, plotting the solution amounts to choosing an appropriate range of values for s , in this case $-2 \leq s \leq 2$ would suffice, and then plotting the resulting points $u(t_0, s)$ versus $x(t_0, s)$ in the xu -plane.

5 Shock formation and propagation

[Oct 25, 2004]

Notice that the characteristics overlap for sufficiently large time. The point at which characteristics first touch is called a shock. The shocks occur when and where the transformation $(r, s) \rightarrow (x, t)$ becomes multi-valued (i.e. non-invertible). Once a shock forms, parts of the solution we derived in the previous section become invalid.

Our goal in this section is to extend the solution beyond the time the first shock forms.

The transformation $(r, s) \rightarrow (x, t)$ is non-invertible if the determinant of the Jacobian matrix is zero,

$$\frac{\partial(x, t)}{\partial(r, s)} = \det \begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = \det \begin{pmatrix} 1 - 2f(s) & -2f'(s)r + 1 \\ 1 & 0 \end{pmatrix} = 2f'(s)r - 1 = 0. \quad (13)$$

Solving for r and noting that $t = r$ gives the time when the determinant becomes zero,

$$t = r = \frac{1}{2f'(s)}. \quad (14)$$

Since times in this problem are positive $t > 0$, then shocks occur if $f'(s) > 0$ for some s . The first such time where shocks occur is

$$t_{shock} = \frac{1}{2 \max \{f'(s)\}}. \quad (15)$$

In the example above, the time when a shock first occurs is given by substituting (12) into (15),

$$t_{shock} = \frac{1}{2 \max \{f'(s)\}} = \frac{1}{2 \left(\frac{3}{4} - \alpha\right)}.$$

Thus, lighter traffic (smaller α) leads to shocks sooner! The position of the shock at t_{shock} is given by

$$x_{shock} = (1 - 2\alpha)t_{shock} = \frac{\frac{1}{2} - \alpha}{\frac{3}{4} - \alpha}.$$

It is known from work on shock waves that for PDEs of the form

$$v_t + w_x = 0$$

the shock wave propagates at velocity

$$V_{shock} = \frac{dx_{shock}}{dt} = \frac{[w]}{[v]}$$

where square brackets $[]$ denote the jump in the quantity at the x -location of the shock, i.e.

$$[w] = w(x_{shock}^+, t) - w(x_{shock}^-, t)$$

where $w(x_{shock}^+, t)$ is the right limit (i.e. the limit from above) of $w(x, t)$

$$w(x_{shock}^+, t) = \lim_{x \rightarrow x_{shock}^+} w(x, t)$$

and $w(x_{shock}^-, t)$ is the left limit (i.e. the limit from below) of $w(x, t)$

$$w(x_{shock}^-, t) = \lim_{x \rightarrow x_{shock}^-} w(x, t).$$

We can rewrite the traffic flow PDE (5) as

$$u_t + (u - u^2)_x = 0$$

and hence the velocity of the shock is

$$V_{shock} = \frac{[u - u^2]}{[u]} \quad (16)$$

In the example above, the shock occurs where the characteristics with $s = 0$ and $s = 1$ meet. The value to the left of the shock is the value of u on the characteristic with $s = 0$, $u(x_{shock1}^-, t) = \alpha$; and to the right of the shock, the value of $s = 1$, $u(x_{shock1}^+, t) = 3/4$. Thus the speed of the first shock is, from (16),

$$V_{shock} = \frac{\left(\frac{3}{4} - \left(\frac{3}{4}\right)^2\right) - (\alpha - \alpha^2)}{\frac{3}{4} - \alpha} = \frac{1}{4} - \alpha.$$

Thus, for lighter traffic $\alpha = 1/6$, the shock propagates downstream at velocity $V_{shock} = 1/12$. For heavier traffic $\alpha = 1/3$, the shock propagates upstream at $V_{shock} = -1/12$. Note that if $\alpha = 1/4$, the shock would not move!

In the simple case considered here, the characteristics with $0 \leq s \leq 1$ meet at (t_{shock}, x_{shock}) and end there. The characteristics with $s > 1$ and $s < 0$ continue until they hit the shock, and then end. Thus, after time t_{shock} , $u = \alpha$ to the left of the shock and $u = 3/4$ to the right. The plots can thus be extended beyond the shock time t_{shock} , using the velocity V_{shock} of the shock.

It is important to note that shocks form because of the nonlinear term $u u_x$ in the quasi-linear PDE (10),

$$u_t + (1 - 2u) u_x = 0.$$

In contrast, the linear PDE

$$u_t + c_0 u_x = 0, \quad c_0 \text{ constant} \quad (17)$$

has the solution $u = f(x - c_0 t)$ for some function $f(x)$ (the initial wave form). The characteristics are $x - c_0 t = \text{const}$, which are parallel lines (never meet). Thus shocks do not form in the solutions of the linear PDE (17).

6 Example : multiple shocks in traffic density

In this example, multiple shocks will form, propagate and coalesce in the traffic flow density. Consider the initial traffic density

$$u(x, 0) = f(x) = \begin{cases} 1/3 & x \leq -2 \\ x/3 + 1 & -2 \leq x \leq -1 \\ 2/3 & -1 \leq x \leq 1 \\ x/12 + 7/12 & 1 \leq x \leq 2 \\ 3/4 & x \geq 2 \end{cases}$$

From (11), the characteristics are

$$x = (1 - 2f(s))t + s = \begin{cases} \frac{1}{3}t + s & s \leq -2 \\ -(1 + \frac{2}{3}s)t + s & -2 \leq s \leq -1 \\ -\frac{1}{3}t + s & -1 \leq s \leq 1 \\ -\frac{1}{6}(s+1)t + s & 1 \leq s \leq 2 \\ -\frac{1}{2}t + s & s \geq 2 \end{cases}$$

From (14), the shocks occur when

$$t = \frac{1}{2f'(s)}$$

Note that $f'(s) > 0$ on two intervals, $-2 \leq s \leq -1$, where $f'(s) = 1/3$ and

$$t_{shock1} = \frac{3}{2}$$

and on $1 \leq s \leq 2$, where $f'(s) = 1/12$ and

$$t_{shock2} = 6$$

The first shock occurs at time $t = t_{shock1}$, where the characteristics starting from $s = -2$ and $s = -1$ meet,

$$x_{shock1} = \frac{1}{3}t_{shock1} - 2 = -\frac{3}{2}$$

The value of u to the left of the shock is that on the characteristic with $s = -2$, $u(x_{shock1}^-, t) = 1/3$; and to the right of the shock, that $s = -1$, $u(x_{shock1}^+, t) = 2/3$. Thus the speed of the first shock is, from (16),

$$V_{shock1} = \frac{[u - u^2]}{[u]} = \frac{\left(\frac{2}{3} - \left(\frac{2}{3}\right)^2\right) - \left(\frac{1}{3} - \left(\frac{1}{3}\right)^2\right)}{\frac{1}{3} - \frac{2}{3}} = 0$$

Hence the shock does not propagate, and stays at $x = x_{shock1} = -3/2$. We can now update the xt -diagram with the characteristics until the second shock occurs at $t = t_{shock2} = 6$.

The second shock occurs at time $t = t_{shock2} = 6$, where the characteristics starting from $s = 1$ and $s = 2$ meet,

$$x_{shock2} = -\frac{1}{2}t_{shock2} + 2 = -1$$

The value of u to the left of the shock is that on the characteristic with $s = 1$, $u(x_{shock2}^-, t) = 2/3$; and to the right of the shock, that of $s = 2$, $u(x_{shock1}^+, t) = 3/4$. Thus the speed of the second shock is, from (16),

$$V_{shock2} = \frac{[u - u^2]}{[u]} = \frac{\left(\frac{3}{4} - \left(\frac{3}{4}\right)^2\right) - \left(\frac{2}{3} - \left(\frac{2}{3}\right)^2\right)}{\frac{3}{4} - \frac{2}{3}} = -\frac{5}{12}$$

A third shock will occur when the second shock hits the first (stationary) at $x = x_{shock1}$ at time

$$t = t_3 = t_{shock2} + \frac{x_{shock1} - x_{shock2}}{V_{shock2}} = 6 + \frac{-3/2 - (-1)}{-5/12} = \frac{36}{5}$$

We can now update the xt -diagram with the characteristics until shocks collide at $t = t_3 = 36/5$.

When the shocks collide, the characteristics meet. Since the characteristics are fixed, the shocks can't pass through each other. Instead, they coalesce into one new shock. Define x_{shock3} as the position at each time of the new shock. The value of u to the left of the new shock is that to the left of the first shock,

$$u(x_{shock3}^-, t) = u(x_{shock1}^-, t) = 1/3.$$

The value of u to the right of the new shock is that to the right of the second shock,

$$u(x_{shock3}^+, t) = u(x_{shock2}^+, t) = 3/4.$$

Thus the speed of the second shock is, from (16),

$$V_{shock3} = \frac{[u - u^2]}{[u]} = \frac{\left(\frac{3}{4} - \left(\frac{3}{4}\right)^2\right) - \left(\frac{1}{3} - \left(\frac{1}{3}\right)^2\right)}{\frac{3}{4} - \frac{1}{3}} = -\frac{1}{12}$$

We can now complete the xt -diagram for all times $t \geq t_3$.

Plots of u vs. x can also be drawn at various times (e.g. $t = 0, t_{shock1}, t_{shock2}, t_3$).