HOMEWORK #8 SOLUTION

SECTION 4.1

1/ $f(t) = \begin{cases} -1 & t<1 \\ 1 & t \geq 1 \end{cases}$

$$\{f(t)\} = \int_0^1 -e^{-st}\,dt + \int_1^\infty e^{-st}\,dt = \frac{1}{s}e^{-st}\Big|_0^1 - \frac{1}{s}e^{-st}\Big|_1^\infty$$

$$= \frac{1}{s}e^{-s} - \frac{1}{s} - \left(0 - \frac{1}{s}e^{-s}\right) = \frac{2}{s}e^{-s} - \frac{1}{s}, \quad s>0.$$

3/ $f(t) = \begin{cases} t & 0 \leq t < 1 \\ 1 & t \geq 1 \end{cases}$

$$\{f(t)\} = \int_0^1 te^{-st}\,dt + \int_1^\infty e^{-st}\,dt = \left(-\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\right)\Big|_0^1 - \frac{1}{s}e^{-st}\Big|_1^\infty$$

$$= \left(-\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s}\right) - \left(0 - \frac{1}{s^2}\right) - \frac{1}{s}\left(0 - e^{-s}\right) = \frac{1}{s^2}\left(1 - e^{-s}\right), \quad s>0$$

6/ $f(t) = \begin{cases} \sin t & 0 \leq t < \pi/2 \\ 0 & t \geq \pi/2 \end{cases}$

$$\{f(t)\} = \int_0^{\pi/2} \sin t\, e^{-st}\,dt = -\frac{e^{-st}(s \cdot \sin t + \cos t)}{s^2+1}\Bigg|_0^{\pi/2}$$

$$= -\frac{e^{-\frac{\pi s}{2}}\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cancelto{0}{\cos\pi/2}\right)}{s^2+1} + \frac{e^0(\cos 0)}{s^2+1}.$$

$$= \frac{-\frac{\pi}{2}e^{-\frac{\pi s}{2}} + 1}{s^2+1} = \frac{1 - \frac{\pi}{2}e^{-\frac{\pi s}{2}}}{s^2+1}$$

7/ $f(t) = \begin{cases} 0 & 0 < t < 1 \\ t & t > 1 \end{cases}$

$$\{f(t)\} = \int_1^\infty t e^{-st}\,dt = \left(-\frac{1}{s} t e^{-st} - \frac{1}{s^2} e^{-st}\right)\Big|_1^\infty = \frac{1}{s} e^{-s} + \frac{1}{s^2} e^{-s}, \quad s > 0$$

8/ $f(t) = \begin{cases} 0 & 0 < t < 1 \\ 2t-2 & t > 1 \end{cases}$

$$\{f(t)\} = 2 \int_1^\infty (t-1) e^{-st}\,dt = 2\left(-\frac{1}{s}(t-1)e^{-st} - \frac{1}{s^2} e^{-st}\right)\Big|_1^\infty$$

$$= \frac{2}{s^2} e^{-s}, \quad s > 0.$$

10/ $f(t) = \begin{cases} 0 & 0 < t < a \\ c & a < t < b \\ 0 & t > b \end{cases}$

$$\{f(t)\} = \int_a^b c e^{-st}\,dt = -\frac{c}{s} e^{-st}\Big|_a^b = \frac{c}{s}\left(e^{-sa} - e^{-sb}\right), \quad s > 0$$

20 $f(t) = t^5$

$$\mathcal{L}\{f(t)\} = \frac{5!}{s^6}$$

22 $f(t) = 7t + 3$

$$\{7t+3\} = \frac{7}{s^2} + \frac{3}{s}$$

33 $f(t) = \sinh kt$

$$\{\sinh kt\} = \frac{k}{s^2 - k^2}$$

37 $f(t) = \sin 2t \cos 2t$

$$\{\sin 2t \cos 2t\} = \{\tfrac{1}{2}\sin 4t\} = \frac{2}{s^2+16}$$

40 $f(t) = 10\cos(t - \pi/6)$

$$\cos(t-\pi/6) = \cos t \cos\frac{\pi}{6} + \sin t \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\cos t + \frac{1}{2}\sin t$$

$$\Rightarrow \{10\cos(t-\pi/6)\} = 10\left(\frac{\sqrt{3}}{2}\{\cos t\} + \frac{1}{2}\{\sin t\}\right)$$

$$= 5\sqrt{3}\,\frac{s}{s^2+1} + 5\,\frac{1}{s^2+1}$$

41

(a) $\Gamma(a+1) = \int_0^\infty t^a e^{-t}\,dt$ (Integration by parts, $a>0$)

$$= -t^a e^{-t}\Big|_0^\infty + a\int_0^\infty t^{a-1} e^{-t}\,dt = a\Gamma(a)$$

(b)

$\{t^a\} = \int_0^\infty e^{-st} t^a\,dt$ Let $u = st \Rightarrow du = s\,dt$

$$\Rightarrow \{t^a\} = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^a \frac{1}{s}\,du = \frac{1}{s^{a+1}}\Gamma(a+1), \quad a > -1$$

SECTION 4.2

2/ $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{6}t^3$

4/ $\mathcal{L}^{-1}\left\{\left(\frac{2}{s}-\frac{1}{s^3}\right)^2\right\} = \mathcal{L}^{-1}\left\{4\frac{1}{s^2} - \frac{4}{6}\frac{3!}{s^4} + \frac{1}{120}\frac{5!}{s^6}\right\} = 4t - \frac{2}{3}t^3 + \frac{t^5}{120}$

6/ $\mathcal{L}^{-1}\left\{\frac{(s+2)^2}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + 4\frac{1}{s^2} + 2\frac{2}{s^3}\right\} = 1 + 4t + 2t^2$

8/ $\mathcal{L}^{-1}\left\{\frac{4}{s} + \frac{6}{s^5} - \frac{1}{s+8}\right\} = \mathcal{L}^{-1}\left\{4\frac{1}{s} + \frac{1}{4}\frac{4!}{s^5} - \frac{1}{s+8}\right\} = 4 + \frac{1}{4}t^4 - e^{-8t}$

10/ $\mathcal{L}^{-1}\left\{\frac{1}{5s-2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{1}{(s-2/5)}\right\} = \frac{1}{5}e^{2t/5}$

12/ $\mathcal{L}^{-1}\left\{\frac{10s}{s^2+16}\right\} = 10\cos 4t$

14/ $\mathcal{L}^{-1}\left\{\frac{1}{4s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\left(\frac{1/2}{s^2+1/4}\right)\right\} = \frac{1}{2}\sin\frac{1}{2}t$

16/ $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{s^2+2}\right\} = \cos\sqrt{2}t + \frac{\sqrt{2}}{2}\sin\sqrt{2}t$

18/ $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{4}\cdot\frac{1}{s} + \frac{5}{4}\frac{1}{s-4}\right\} = -\frac{1}{4} + \frac{5}{4}e^{4t}$

20/ $\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{9}\frac{1}{s-4} - \frac{1}{9}\frac{1}{s+5}\right\} = \frac{1}{9}e^{4t} - \frac{1}{9}e^{-5t}$

22 $\mathcal{L}^{-1}\left\{\dfrac{s-3}{(s-\sqrt{3})(s+\sqrt{3})}\right\} = \mathcal{L}^{-1}\left\{\dfrac{s-3}{s^2-3}\right\} =$

$= \mathcal{L}^{-1}\left\{\dfrac{s}{s^2-3} - \sqrt{3}\,\dfrac{\sqrt{3}}{s^2-3}\right\} = \cosh\sqrt{3}t - \sqrt{3}\sinh\sqrt{3}t$

24 $\mathcal{L}^{-1}\left\{\dfrac{s^2+1}{s(s-1)(s+1)(s-2)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{2}\,\dfrac{1}{s} - \dfrac{1}{s-1} - \dfrac{1}{3}\,\dfrac{1}{s+1} + \dfrac{5}{6}\,\dfrac{1}{s-2}\right\}$

$= \dfrac{1}{2} - e^{t} - \dfrac{1}{3}e^{-t} + \dfrac{5}{6}e^{2t}$

26 $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+2)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{4}\,\dfrac{s}{s^2+4} + \dfrac{1}{4}\,\dfrac{2}{s^2+4} - \dfrac{1}{4}\,\dfrac{1}{s+2}\right\}$

$= \dfrac{1}{4}\cos 2t + \dfrac{1}{4}\sin 2t - \dfrac{1}{4}e^{-2t}$

28 $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4-9}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{6\sqrt{3}}\,\dfrac{\sqrt{3}}{s^2-3} - \dfrac{1}{6\sqrt{3}}\,\dfrac{\sqrt{3}}{s^2+3}\right\} =$

$= \dfrac{1}{6\sqrt{3}}\sinh\sqrt{3}t - \dfrac{1}{6\sqrt{3}}\sin\sqrt{3}t$

30 $\mathcal{L}^{-1}\left\{\dfrac{6s+3}{s^4+5s^2+4}\right\} = \mathcal{L}^{-1}\left\{2\,\dfrac{s}{s^2+1} + \dfrac{1}{s^2+1} - 2\,\dfrac{s}{s^2+4} - \dfrac{1}{2}\,\dfrac{2}{s^2+4}\right\}$

$= 2\cos t + \sin t - 2\cos 2t - \dfrac{1}{2}\sin 2t$

SECTION 4-3

2 $\mathcal{L}\{te^{-6t}\} = \frac{1}{(s+6)^2}$

5 $\mathcal{L}\{t(e^{t}+e^{2t})^2\} = \mathcal{L}\{te^{2t}+2te^{3t}+te^{4t}\} = \frac{1}{(s-2)^2}+\frac{2}{(s-3)^2}+\frac{1}{(s-4)^2}$

7 $\mathcal{L}\{e^{t}\sin 3t\} = \frac{3}{(s-1)^2+9}$

13 $\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+1}\right\} = e^{3t}\sin t$

14 $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{(s+1)^2+2^2}\right\} = \frac{1}{2}e^{-t}\sin 2t$

16 $\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\} = \mathcal{L}^{-1}\left\{2\frac{s+3}{(s+3)^2+5^2}-\frac{1}{5}\frac{5}{(s+3)^2+5^2}\right\} =$

$= 2e^{-3t}\cos 5t - \frac{1}{5}e^{-3t}\sin 5t$

19 $\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2(s+1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{s}-\frac{1}{s^2}-\frac{5}{s+1}-\frac{4}{(s+1)^2}-\frac{3}{2}\frac{2}{(s+1)^3}\right\}$

$= 5-t-5e^{-t}-4te^{-t}-\frac{3}{2}t^2e^{-t}$