PHYS 326: Lecture 6

Cliff Sun

February 6, 2025

Symmetries

Definition 0.1. *A symmetry is defined as a change of coordinates that leaves the lagrangian unchanged.* $\boldsymbol{S}$ *is a symmetric matrix if*

$$\boldsymbol{S}^T K \boldsymbol{S} = K \tag{1}$$

$$\boldsymbol{S}^T M \boldsymbol{S} = M \tag{2}$$

We define a new set of coordinates $\vec{q} = \boldsymbol{S}\vec{p}$*, then*

$$L(q) = L(p) \tag{3}$$

Then

$$PE = \frac{1}{2} q^T \mathbf{K} q \tag{4}$$

$$= \frac{1}{2} (Sp)^T \mathbf{K} Sp \tag{5}$$

$$= \frac{1}{2} p \mathbf{K} p \tag{6}$$

We have various types of symmetries, namely cyclic and mirror symmetries.

Theorem 0.2. *An eigenvector of the system is also an eigenvector of the symmetry matrix.*

Then if

$$\mathbf{K}\vec{u} = \omega^2 \mathbf{M}\vec{u} \tag{7}$$

Then

$$\mathbf{S}\vec{u} = \lambda\vec{u} \tag{8}$$

Proof. Given

$$\mathbf{K}\vec{u} = \omega^2 \mathbf{M}\vec{u} \tag{9}$$

Multiply by $\mathbf{S}^T$ and insert $\mathbf{S}\mathbf{S}^{-1}$:

$$\mathbf{S}^T\mathbf{K}\mathbf{S}\mathbf{S}^{-1}\vec{u} = \omega^2 \mathbf{S}^T\mathbf{M}\mathbf{S}\mathbf{S}^{-1}\vec{u} \tag{10}$$

$$\mathbf{K}\mathbf{S}^{-1}\vec{u} = \omega^2 \mathbf{M}\mathbf{S}^{-1}\vec{u} \tag{11}$$

Therefore,

$$S^{-1}u \tag{12}$$

obeys the E.O.M. with same eigenvalue as $\vec{u}$. In other words, the 2 vectors are same up to a constant. Therefore

$$\mathbf{S}u = \lambda u \tag{13}$$

$\square$

We look into reflection, mirror, or parity symmetries. When applied twice, they recover the original state. That is $S^2 = I$. Then

$$Sx = \lambda x \tag{14}$$

$$S^2 x = \lambda^2 x \tag{15}$$

Thus

$$\lambda = \pm 1 \tag{16}$$