

MTH 416: Lecture 4

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Lecture Span

- Solving systems of linear equations
- Reduced row echelon form
- Types of solution sets

Systems of linear equations (SLEs)

We saw that SLEs corresponded to augmented matrices. That is

$$x_1 - x_2 = 4 \tag{1}$$

$$x_1 + x_2 = 6 \tag{2}$$

$$-x_1 + 2x_2 = -3 \tag{3}$$

corresponded to

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 1 & 1 & 6 \\ -1 & 2 & -3 \end{array} \right]$$

Theorem 0.1. *Any system of linear equations over \mathbb{R} has either zero, one, or infinite solutions.*

Elementary row operations

Definition 0.2. *The following 3 types of operations on an augmented matrix, called Elementary Row Operations. None of these operations change the solution set of the linear equations.*

1. Switch the order of 2 rows
2. Scale any row by a non-zero constant
3. Add some constant multiple of 1 row to any different row

Proof. 1. Switching two rows just trivially reorders the equations.

2. Scaling a row with a non-zero constant is nothing but scaling that equation by a non-zero constant.
3. This adds $c \cdot$ some row to another row for some $c \in \mathbb{R}$,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \tag{4}$$

$$a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \tag{5}$$

Adding row 1 and $c \cdot$ row 2 yields row 2':

$$(a_{11} + c \cdot a_{21})x_1 + \cdots = b_1 + b_2 \tag{6}$$

If some vector was a solution to row 1 & row 2, then it would also be a solution to row 1 & row 2'. That is scaling and adding true equations yields true equations. Conversely, if (x_1, \dots, x_n) was a solution to row 1 and row 2', then it is a solution to row 2 = row 2' - row 1.

□

Definition 0.3. Two matrices M, N of the same size are row equivalent iff there are some finite sequence of elementary row operations that can transform $M \rightarrow N$ WLOG.

Corollary 0.4. If M, N are row equivalent, then the corresponding linear systems have the same solution set.

Question: is

$$M \sim N \iff M \text{ \& } N \text{ are row-equivalent} \quad (7)$$

an equivalence relation? YES!

Example: Find all solutions to

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & -1 \\ 0 & 2 & 4 & -8 \end{array} \right]$$

We define R_1, R_2, R_3 to be the rows of this matrix, we first perform $R_2 - 2R_1$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 2 & 4 & -8 \end{array} \right]$$

$R_3 = R_3/2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 1 & 2 & -4 \end{array} \right]$$

Switch R_2, R_3

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & -3 \end{array} \right]$$

Subtract $R_2 - 2R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array} \right]$$

Subtract $R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array} \right]$$

This corresponds to

$$x_1 = -1, x_2 = 2, x_3 = -3 \quad (8)$$

Definition 0.5. A matrix is in Reduced Row Echelon Form (RREF) if it satisfies the following 4 conditions:

1. Any rows of all 0's are at the bottom
2. The leftmost non-zero number in every row is a 1 (a leading 1 or pivot)
3. Every leading 1 is the only non-zero number in its column
4. If there are leading 1's at positions (i, j) and (s, t) with $i < s$, then $j < t$.

So why do we care about matrices in RREF?

1. If $(A|b)$ is in RREF, then we will be able to solve its linear system easily.
2. Every $m \times n$ matrix is row-equivalent to some $m \times n$ matrix in RREF.

In RREF, the pivot variables only appear once each in the entire SLEs. We can rearrange the equations in terms of the pivot variables. The variables that make up the pivot variables are thus the degrees of freedoms. Example

$$x_1 = 2 - 2x_2 - x_4 \quad (9)$$

$$x_3 = 3 - 3x_4 \quad (10)$$

$$x_5 = 4 \quad (11)$$

For any $x_2, x_4 \in \mathbb{R}$. Suppose we have an augmented matrix $(A|b)$, which is in reduced row echelon form.

Case 1

If there is a pivot in the last column (b), then the system is inconsistent (or has no solution).

Proof. We would have some row that looks as follows:

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array} \right]$$

This is the only case of which the system is inconsistent. □

Case 2

If there is no pivot in b , but there are all pivots in A , then we claim that there is only 1 solution.

Proof. Then the matrix would look like the following:

$$\left[\begin{array}{ccc|c} 1 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & b_2 \\ \cdots & \cdots & 1 & b_n \end{array} \right]$$

This assigns each variable to exactly one constant, thus there can only be 1 solution. □

Case 3

Suppose that there are no pivots in b , and there are $r < n$ pivots in A . Such that $r = \#$ pivot columns and $n = \#$ non-zero columns. Then there are infinitely many solutions and they're all specified by $n - r$ parameters.