MTH 553: Lecture # 1

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Lecture Span

Todo for Friday:

- Visit Canvas and read materials
- Send prof emails
- Read notation (pg.7) and skim pg.1-6 (R mcOwen, PDEs, 2nd edition)

PDEs terminology - Examples

1. Laplace's equation - $\triangle u = 0$ where $\triangle = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2 = \nabla^2$

We say that u is harmonic.

1. Say $n = 1$, then $\partial_x^2 u = 0 \iff u(x) = ax + b$.
2. If $n = 2$, then $\partial_x^2 u + \partial_y^2 u = 0$ [$\Longleftarrow$ $u(x, y) = ax + by + c$]

We call $\triangle u = 0$ a 2nd order PDE, and also a linear function of u and its derivatives. But we do not require a PDE to be linear with respect to the x_i's. Example:

$$x_2^2 u_{x_1,x_1,x_1} - \sin(x_1) = 0 \tag{1}$$

This a 3rd order PDE.

Semi-linear Poisson Equation

$$\triangle u = u^5 \tag{2}$$

This equation is non-linear, but we call it semi-linear because its derivatives are linear, but not u. Another example is

$$tu_t + u_x = u^2 \tag{3}$$

Inviscid Burger's Equation

Inviscid means no viscosity. We consider

$$u_t + uu_x = 0 \tag{4}$$

Not semi-linear, but its highest order derivatives occur linearly with coefficients x, t.

Eikonal Equation

$$u_x^2 + u_y^2 = 1 \tag{5}$$

This is fully non-linear because its highest order derivatives is non-linear. In this class, we will focus one

linear → semi-linear → quasi-linear → fully non-linear

Notation (pg.7)

1. $\mathbb{Z}$ - integers
2. $\mathbb{N} = \{1, 2, 3, \ldots\}$
3. $(x, y) \in \mathbb{R}$
4. Ω is the domain
5. Closure of $\Omega = \Omega \cup \partial\Omega$ (locations of which Ω reach-ish)
6. $C(\Omega)$ = collection of continuous functions (does not have to be bounded)
7. $C^1(\Omega)$ = continuous functions whos first derivatives are continuous.
8. $C^k(\Omega)$, defined similarly
9. $C^\infty(\Omega)$
10. $C^1(\bar{\Omega}) = \{f \in C^1(\Omega) : f_{x_1,\ldots,x_n}$ can be extended continuously to $\bar{\Omega}\}$
11. The support of a function $\text{supp}(f) = \{x : f(x) \neq 0\}$, then take the closure. Note that the support of a function is away from the boundary of Ω
12. $C_0^1(\Omega) = \{f \in C^1(\Omega); \text{supp}(f) \subset \Omega\}$ (doesn't include $\partial\Omega$, i.e. functions that must equal 0 near $\partial\Omega$)

Examples

1. With mathematica:
$$u(x,t) = \frac{1}{t^{1/3}}\left(1 - \frac{x^2}{12t^{2/3}}\right) \tag{6}$$
This solves the porous medium equation, e.g. $u_t = (u^2)_{xx}$ (similar to the heat equation). When $t > 0$ and $|x| < \sqrt{12}t^{1/3}$
2. Sketch $U(x,t)$ as a function of x at $t = 1$ and $t = 8$
3. Show that the area under the graph is the same for each t (conserved fluid?)