PHYS 326: Lecture 1

Cliff Sun

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Review: Simple Harmonic Oscillator

For a particle moving with potential $V(x)$, then

$$L = T - U = \frac{1}{2}mv^2 - V(x) \tag{1}$$

Then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{2}$$

$$\implies m\ddot{x} = -V'(x) = F(x) \tag{3}$$

To find the equilibrium points, we have

$$\ddot{x} = 0 \implies V'(x) = 0 \tag{4}$$

For a stable equilibrium, you need

$$\frac{\partial F}{\partial x} \leq 0 \text{ retrieving force} \tag{5}$$

Double Well potential

Picture a graph with 2 stable equilibrium. We look at small deviations at the stable equilibrium. Define

$$y(t) = x(t) - x_{eq} \tag{6}$$

$$L = \frac{1}{2}m\dot{x}^2 - V(x) \tag{7}$$

$$L(y) = \frac{1}{2}m\dot{y}^2 - V(x_{eq} + y) \tag{8}$$

Then

$$V(x_{eq} + y) \approx V(x_{eq}) + \frac{\partial V}{\partial x}y + \frac{1}{2}\frac{\partial^2 V}{\partial^2 x}y^2 \tag{9}$$

The 2nd term drops out, so then

$$L(y) = \frac{1}{2}m\dot{x}^2 - V(x_{eq}) - \frac{1}{2}ky^2 \tag{10}$$

We find Euler Lagrange for y:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) - \frac{\partial L}{\partial y} = 0 \tag{11}$$

$$\implies m\ddot{y} + ky = 0 \tag{12}$$

Let our ansatz be

$$y(t) = ae^{i\omega t} \tag{13}$$

then

$$\left[k - m\omega^2\right] ae^{i\omega t} = 0 \tag{14}$$

General non-trival solution:

$$y(t) = a_+e^{i\omega t} + a_-e^{-i\omega t} \tag{15}$$

Forced Motion of S.H.O.

Harmonic forcing

$$m\ddot{y} + ky = F(t) \tag{16}$$

If $F(t) = f_0 e_{i\omega t}$,

$$(-m\omega^2 + k)y_0 e^{i\omega_d t} = f_0 e_{i\omega t} \tag{17}$$

Then

$$y_0 = \frac{f_0}{m}\frac{1}{\omega_0^2 - \omega_d^2} \tag{18}$$

Impulsive Forcing

$$f(t) = I_0\delta(t) \tag{19}$$

Assume $y = \dot{y} = 0$ at $t = 0$, then the change in momentum is

$$\Delta P = \int f(t)dt = I_0 \tag{20}$$

Initial conditions:

$$y(t = 0^+) = 0 \tag{21}$$

$$y(t = 0^+) = \frac{\Delta P}{m} = \frac{I_0}{m} \tag{22}$$

We use these initial conditions to solve the SHO to get:

$$y(t) = \frac{I_0}{m\omega_0}\sin(\omega_0 t)\Theta(t) \tag{23}$$

Where $\Theta(t)$ is the heavyside function.

$$y(t) \iff I_0 G(t) \tag{24}$$

Where $G(t)$ is the impulse response to the system.

Arbitrary Forcing

$$m\ddot{y} + ky = f(t) \tag{25}$$

Solution is

$$y(t) = \int_{-\infty}^{t} f(\tau)G(t-\tau)d\tau \tag{26}$$

SHO Generalization

1. Non-linearity
2. Damping
3. Next: Multiple Coupled Oscillators