PHYS 486: Lecture # 22

Cliff Sun

October 23, 2025

Hydrogen Atom

Connect $\hat{L}$ to the angular equation. Recall that the angular equation is

$$\frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right\}=l(l+1) \tag{1}$$

Note that

$$\vec{L}=-i\hbar\left(r\times\nabla\right) \tag{2}$$

Where ∇ is the gradient operator in spherical coordinates. Although, we cannot solve everything in spherical coordinates since we care about L_z. With $r=r\hat{r}+\dots$, we have that

$$L=-i\hbar\left(r(\hat{r}\times\hat{r})\partial_r+(\hat{r}\times\hat{\theta})\partial_\theta+(\hat{r}\times\hat{\phi})\frac{1}{\sin\theta}\partial_\phi\right) \tag{3}$$

Note that

$$\hat{r}\times\hat{\theta}=\hat{\phi},\ \hat{r}\times\hat{\phi}=-\hat{\theta} \tag{4}$$

In cartesian coordinates, we have

$$\hat{\theta}=\cos\theta\cos\phi\hat{i}+\cos\theta\sin\phi\hat{j}+\sin\theta\hat{k} \tag{5}$$

$$\hat{\phi}=-\sin\phi\hat{i}+\cos\phi\hat{j} \tag{6}$$

Then

$$L_x=-i\hbar\left(-\sin\phi\partial_\theta-\cos\phi\cot\theta\partial_\phi\right) \tag{7}$$

$$L_y=-i\hbar(\cos\phi\partial_\theta-\sin\phi\cot\theta\partial_\phi) \tag{8}$$

$$L_z=-i\hbar\partial_\phi \tag{9}$$

Defining the ladder operators, we have that

$$L_\pm=L_x\pm iL_y=\pm\hbar\exp(\pm i\phi)\left(\partial_\theta\pm i\cot\theta\partial_\phi\right) \tag{10}$$

$$L^2=L_+L_-+L_z^2-\hbar L_z \tag{11}$$

$$\implies -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\partial_\theta\right)+\frac{1}{\sin^2\theta}\partial_\phi^2\right] \tag{12}$$

Thus, our angular equation can be computed as

$$\hbar^2 l(l+1)Y=L^2Y \tag{13}$$

This is the eigenvalue equation for Y. To solve for $Y(\theta,\phi)$. Let $Y=\Omega(\theta)\Lambda(\phi)$, then we get

$$\frac{1}{\Omega}\left[\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Omega}{d\theta}+l(l+1)\sin^2\theta\right)\right]+\underbrace{\frac{1}{\Lambda}\frac{d^2}{d\phi^2}\Lambda}_{=-m^2}=0 \tag{14}$$

Then we solve

$$d_\phi^2\Lambda=-m^2\Lambda \implies \Lambda(\phi)=\exp(im\phi) \tag{15}$$

These are the eigenfunctions of L_z. We have to demand that $\Lambda(\phi)=\Lambda(\phi+2\pi)$, then we get that m must be an integer. In principle, A.M allows l = integer or integer + 1/2. However, orbital A.M. can only be an integer.

We next solve the θ equation. Note that this solutions are the Legendre Functions, thus,

$$\Omega\left(\theta\right)=Ap_l^m(\cos\theta) \tag{16}$$

Such that

$$P_l^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d}{dx}^m P_l(x) \tag{17}$$

Where P_l is the l-th legendre polynomial:

$$P_l=\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l(x^2-1)^l \tag{18}$$

With normalization, etc., we get that

$$Y_l^m\left(\theta,\phi\right)=\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}\exp(im\phi)P_l^m(\cos\theta) \tag{19}$$

These are called "spherical harmonics". As well,

$$\langle Y_l^m|Y_{l'}^{m'}\rangle=\delta_{ll'}\delta_{mm'} \tag{20}$$

Summary

$V=V(r)$ (central potential) leads to

$$L^2Y(\theta,\phi)=\hbar^2l(l+1)Y(\theta,\phi) \tag{21}$$

$$L_zY=\hbar mY \tag{22}$$