MTH 416: Introduction & Vector Spaces

Cliff Sun

August 27, 2024

Lecture Span

- Introduction To Abstract Linear Algebra
- Vector Spaces

Textbook: Linear Algebra by Friedberg, Insel, & Spence

Free Online Textbook: A First Course in Linear Algebra by Robert Beezer

Introduction to Linear Algebra

Main Objects of Study

- Vector Spaces
- Linear Transformation
- Systems of Linear Equations
- Matrices & Determinants
- Eigenvalues & Eigenvectors
- Inner Product Spaces

Given a vector L such that

$$L = \{tv : t \in \mathbb{R}\} \tag{1}$$

Where

$$L = (t, t, t) \tag{2}$$

And given a vector

$$w = (1, 0, 0) \tag{3}$$

We have that

$$L + w = (2, 1, 1) \textbf{ such that } t = 1 \iff P \tag{4}$$

thus

$$P = \{t + u, t, t\} \quad t, u \in \mathbb{R} \tag{5}$$

Remark: P is a plane.

Definition 0.1. *A vector space V is any set which behaves algebraically like $\mathbb{R}^n$. In particular,*

1. *Given any $v, w \in V$, we can add them such that $v + w \in V$ for some vector space V*
2. *Given any $v \in V$ and $c \in \mathbb{R}$, we have that $cV \in \mathbb{V}$*
3. *These operations satisfy all usual rules of arithmetic*

Remarks

1. No cross product: this is not present in a general vector space
2. No dot product: this is a unique operation that goes beyond basic arithmetic & geometric intuition.
3. We don't know what vectors are

Why is this so abstract?

Short answer: Abstraction allows us to generalize vectors into a more applicable theory, (Physics - Quantum Mechanics, Computer Science - ML, etc.)

- "Vectors" can be functions, equivalence classes, etc.

As long as we can prove that these mathematical objects are vector spaces, then we can apply Abstract Linear Algebra onto it.

Definition 0.2. *Linear Transformations is a function that transforms a vector space into another vector space. Preserves vector addition & scalar multiplication geometrically.*

Example of L.T:

$$T(x, y) = (2x, 3y) \tag{6}$$

Recall: In Differential Calculus, we approximate functions of 1 variable with affine linear functions (Form: $ax + b$, namely add a constant) using derivatives. In multivariable Calculus, we approximate functions of multiple variables with linear transformations (Linear functions of $x, y, z \dots$) + constant vector.

Definition 0.3. *A vector space over $\mathbb{R}^n$ is a set equipped with 2 operations:*

1. *Addition: given 2 elements $v, w \in \mathbb{R}^n$, then $v + w$ produces a unique vector*
2. *Scalar multiplication: given a vector v and a scalar $c \in \mathbb{R}$, $cv \in \mathbb{R}^n$ produces some unique vector.*

They all satisfy the following 8 properties:

1. $\forall x, y \in V, \quad x + y \iff y + x$
2. $\forall x, y, z \in V, \quad (x + y) + z \iff x + (y + z)$
3. $\exists\, 0 \in V$ *such that for all vectors* $x \in V$, *we have that* $0 + x = x$
4. $\forall x \in V, \exists y \in V$ *such that* $x + y = 0$. *In other words,* $y = -x$
5. *For each* $x \in V$, $1 \cdot x = x$
6. *For each* $a, b \in \mathbb{R}$ *and* $x \in V$, $(ab)x = a(bx)$
7. *For each* $a \in \mathbb{R}$ *and* $x, y \in V$, $a(x + y) = ax + ay$
8. *For each* $a, b \in \mathbb{R}$ *and* $x \in V$, *we have that* $(a + b)x = ax + bx$

Terminology

Elements of V are called "vectors", and elements of $\mathbb{R}$ are called "scalars".

Conjecture 0.4. *For any $x \in V$,*

$$0 \cdot \vec{x} = \vec{0} \tag{7}$$

Example: $\mathbb{R}^n$ for any $n \geq 0$

1. As a set, $\mathbb{R}^n = \{(x_1, x_2, x_3, \dots)\} : x_i \in \mathbb{R}$
2. Given $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$, we define $x + y = (x_1 + y_1, \dots)$
3. Given $x \in \mathbb{R}^n$, and $c \in \mathbb{R}$, we define $cx = (cx_1, cx_2, \dots)$

Proof. Property 3: the 0 vector is $(0, 0, 0, \dots)$. Thus we have that $v + 0 = v$ for all $v \in V$ □

Proof. Property 4: Given $x \in \mathbb{R}^n$, we choose $y = -x \iff y = (-x_1, -x_2, \dots)$ which means that $x + y = 0 \iff (0, 0, 0, 0, \dots)$ □

Proof. Propety 6: Given $a, b \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we first calculate the left side.

$$(ab)x \iff ((ab)x_1, (ab)x_2, \dots) \tag{8}$$

We then manipulate equation (8) using the associate rule of scalar multiplication:

$$((ab)x_1, (ab)x_2, \dots) \iff (a(bx_1), a(bx_2), \dots) \iff a(bx) \tag{9}$$

□

Example 2: $m \times nmatrices$

1. As a set, $M_{m \times n}(\mathbb{R})$ represents a matrix of values with m rows and n columns.
2. Addition & scalar multiplication are defined entry wise.

Example $m \times n$ matrix:

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \tag{10}$$

Notation

Given a matrix $A \in M_{m \times n}(\mathbb{R})$, we let A_{ij} = the entry in i-th row, and j-th column where $0 \leq i \leq m$ and $0 \leq j \leq n$.

Claim: $M_{m \times n}(\mathbb{R})$ is a vector space on $\mathbb{R}$.

Proof. $0_{m \times n} =$

$$\begin{pmatrix} 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix}$$

We calculate that $A + 0_{m,n} = A$ for all $A \in M$ □

Example 3: Real-valued functions on a set

1. Fix a set S, and define $F(S, \mathbb{R}) = \{\text{Functions } f : S \to \mathbb{R}\}$
2. Given $f, g \in F(S, \mathbb{R})$, we define $(f + g)(s) = f(s) + g(s)$
3. Given $f \in F(S, \mathbb{R})$, and $c \in \mathbb{R}$, $(cf)(s) = c\dot{f}(s)$ for all $s \in S$

Claim: $F(S, \mathbb{R})$ is a vector space over $\mathbb{R}$, given any set S

Proof. Axiom 3: Let f_0 be the function $f : S \to 0$ □

Claim: For all $g \in F(S, \mathbb{R})$, we have that

Proof.

$$g + f_0 = g \tag{11}$$

For any $s \in S$

$$(g + f_0)(s) = g(s) + f_0(s) \iff g(s) + 0 \iff g(s) \tag{12}$$

□