PHYS 326: Lecture 4

Cliff Sun

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Multiple D.O.F: The general case

Consider a system with a generlized coordinates $q_\alpha(t)$ for $\alpha = 1, 2, \ldots, N$. Then

$$L = L(q_\alpha, \dot{q}_\alpha) \tag{1}$$

We want to find the equations of motion for small deviations from equilibrium. At equilibrium, $q_\alpha = c$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \tag{2}$$

At equilibrium, we have that nothing changes in time, therefore:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) \implies \frac{\partial L}{\partial q_\alpha} = 0 \tag{3}$$

We study small deviations from equilibrium:

$$q_\alpha = \bar{q_\alpha} + x_\alpha(t) \tag{4}$$

For x_α small and $\bar{q_\alpha}$ is the equilibrium point. We plug this into the lagrangian, and we only retain up to the 2nd order:

$$L(x_1, x_2, \ldots, \dot{x_1}, \dot{x_2}, \ldots) = \frac{1}{2}\sum_{\alpha,\beta=1}^{N} M_{\alpha\beta}\dot{x}_\alpha \dot{x}_\beta + \frac{1}{2}\sum_{\alpha\beta=1}^{N} \Gamma_{\alpha\beta}\dot{x}_\alpha x_\beta - \frac{1}{2}\sum_{\alpha,\beta=1}^{N} K_{\alpha\beta}x_\alpha x_\beta \tag{5}$$

In classical physics, Γ is usually 0. Then we plug into E.L. equations:

$$= \sum_\beta \left(\frac{M_{\gamma\beta+M_{\beta\gamma}}}{2}\ddot{x}\right) + \sum_\beta \left(\frac{\Gamma_{\gamma\beta} + \Gamma_{\beta\gamma}}{2}\dot{x}\right) + \sum_\beta \left(\frac{K_{\gamma\beta} + K_{\beta\gamma}}{2}x\right) \tag{6}$$

$$-\frac{\partial L}{\partial x_\gamma} = -\frac{\partial}{\partial x_\gamma}[L] \tag{7}$$

Note that $\frac{\partial x_\mu}{\partial x_\nu} = \delta_{\mu\nu}$. See lecture notes for derivation. But the key thing is that

$$M_{\alpha\beta} = \frac{\partial}{\partial x_\alpha \partial x_\beta} T \tag{8}$$

Since α and β are dummy variables, we have that M must be symmetric. In a similar fashion, K is also symmetric.

Cart + Spring + Pendulum