MTH 416: Lecture 26

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Lecture Span

- Adjoint, Least squares
- Jordan Canonical Form

Lemma 0.1. *Let $F = \mathbb{R}, \mathbb{C}$, and $A \in M_{m\times n}(\mathbb{F})$. For any column vectors $x \in F^n$ and $y \in F^m$. Then we have that*

$$\langle Ax, y\rangle_m = \langle x, A^*y\rangle_n \tag{1}$$

Note:

$$A^* \equiv \overline{A^T} \tag{2}$$

Definition 0.2. *Suppose $T : V \to W$, where V, W are inner product spaces. An adjoint of T is a linear transformation $T^* : W \to V$ such that for all $v \in V$ and $w \in W$, we have*

$$\langle T(v), w\rangle_W = \langle v, T^*(w)\rangle_V \tag{3}$$

Theorem 0.3. *If T is a linear transformation between 2 finite dimensional vector spaces, then it has a unique adjoint.*

Proof Idea: Fix $w \in W$, then we want to define $T^*(w)$. The left hand side of equation 3 is a linear transformation

$$f : V \to F \tag{4}$$

$$f(v) = \langle T(v), w\rangle_W \tag{5}$$

Fact: Every linear transformation $f : V \to F$ can be expressed as $\langle v, x\rangle_V$ for some unique $x \in V$. Define $T^*(w) = x$. This then proves the theorem.

Fact: If we are given an orthonormal basis β for V and γ for W, then

$$[T^*]_\gamma^\beta = \overline{\left([T]_\beta^\gamma\right)}^T \tag{6}$$

Lemma 0.4. *Suppose $A \in M_{m\times n}(F)$ has rank n. Then the $n \times n$ matrix A^*A is invertible.*

Proof. Let A be given, apply rank nullity to $L_A : F^n \to F^m$, then

$$\dim(F^n) = rank(A) + null(A) \tag{7}$$

$$\iff n = n + 0 \tag{8}$$

So

$$N(A) = \{0\} \tag{9}$$

We now prove that $N(A^*A) = \{0\} \iff A^*A$ is invertible. We claim that

$$A^*Ax = 0 \implies x = 0 \tag{10}$$

Proof. Suppose $x \in F^n$ and $A^*Ax = 0$, then

$$0 = \langle A^*Ax, x\rangle_n = \langle Ax, (A^*)^*x\rangle_m = \langle Ax, Ax\rangle_m = Ax = 0 \tag{11}$$

$\square$

So $Ax = 0$, thus it follows that $x = 0$. So $N(A^*A) = \{0\}$ which means that A^*A is invertible. $\square$

This has applications in least squares, recall that we're given n data points $(x_i, y_i) \in \mathbb{R}^2$. We want the 'best fit' line $y = mx + b$. We convert this into vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \end{pmatrix} \wedge y = \begin{pmatrix} y_1 \\ y_2 \\ \dots \end{pmatrix} \wedge u = \begin{pmatrix} 1 \\ 1 \\ \dots \end{pmatrix} \tag{12}$$

Then the least square parameters are given by

$$proj_W(y) = mx + bu \tag{13}$$

Where $W = span(\{x, u\})$. We convert this into matrices:

$$A = \begin{pmatrix} x_1 & 1 \\ \dots & \dots \end{pmatrix} \in M_{n\times 2}(\mathbb{R}) \tag{14}$$

We define

$$v = \begin{pmatrix} m \\ b \end{pmatrix} \in \mathbb{R}^2 \tag{15}$$

So

$$Av = mx + bu \tag{16}$$

We must find $v_0 = \begin{pmatrix} m_0 \\ b_0 \end{pmatrix} \in \mathbb{R}^2$ such that

$$Av_0 = proj_W(y) \tag{17}$$

To minimize the cost, we need that

$$Av_0 - y \in W^\perp \tag{18}$$

$$\iff \langle Av_0, Av_0 - y \rangle = 0 \tag{19}$$

Then

$$\iff \langle v_0, A^*(Av_0 - y) \rangle \tag{20}$$

This can only happen is

$$A^*(Av_0 - y) = 0 \tag{21}$$

We multiply it out

$$A^*Av_0 = A^*y \tag{22}$$

Because A^*A is invertible, we have that

$$v_0 = (A^*A)^{-1}A^*y \tag{23}$$

But we need to know whether

$$rank(A) = 2 \tag{24}$$

This is always true unless all (x_i, y_i) have the same x_i value.

Jordan Canonical Form

Recall: if $A \in M_{n\times n}(F)$, then there are 2 things that prevent A from being diagonalizable.

1. Not enough eigenvalues
2. Not enough eigenvectors (for some λ_i, geo multi (dimension of eigen space) < alg multi (degree of polynomial associated with λ_i))

If $F = \mathbb{C}$, then 1. cannot happen by the fundamental theorem of Algebra.

Question: If A is a matrix with problem (2), how close can we get to a diagonalizing it?

Is there soem form of "almost diagonal" matrix such that every A is similar to some matrixx of the given form. Yes, this is called the Jordan canonical form.

Definition 0.5. *1. A Jordan block is a $k \times k$ (for $k \geq 1$) matrix of the form*

$$\begin{pmatrix} \lambda_i & 1 & 0 & 0\dots \\ 0 & \lambda_i & 1 & 0\dots \\ \dots \end{pmatrix} \tag{25}$$

All the λ_i's are the same.

2. A matrix B is in Jordan Canonical Form if it has the form

$$\begin{pmatrix} B_1 & 0\dots & \\ 0 & B_2 & \dots \\ \dots \end{pmatrix} \tag{26}$$

Where B_i is a Jordan block. Note: since we allow 1×1 Jordan blocks, all diagonal matrices are in JCF.

Claim: A Jordan block of size $k \geq 1$ is not diagonalizable.

Proof. Let A be a $k \times k$ Jordan block. Then

$$\det(A - tI) = (\lambda - t)^k \tag{27}$$

So the only eigenvalue is λ with alg multi $= k$. We note that

$$\text{geo multi} = 1$$

□

More generally, a JCF matrix B is diagonalizable iff all blocks are 1×1. Which iff it is already diagonal....

Theorem 0.6. *Let $A \in M_{n\times n}(F)$. A is similar to a matrix in JCF if and only if char poly(A) splits over F.*