PHYS 326: Lecture 10

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Non-linear Dynamics

Consider dynamics using

$$\frac{d\bar{x}}{dt} = \bar{F}(x) \tag{1}$$

Where $\bar{x}$ is the set of coordinates in n-dimensional "phase space" (or "state space"). Sufficient to determine the future state.

Example: Non-linear pendulum

Study

$$\ddot{\theta} + \omega_r^2 \sin\theta = 0 \tag{2}$$

No approximation. Denote $x = \theta$ and $y = \dot{\theta}$. Then $\bar{x} = \{x, y\} \iff \{\theta, \dot{\theta}\}$. Then

$$\frac{dx}{dt} = y \tag{3}$$

$$\frac{dy}{dt} = -\omega_0^2 \sin x \tag{4}$$

Let $\omega_0 = 1$, then

$$\frac{dx}{dt} = y \tag{5}$$

$$\frac{dy}{dt} = \sin x \tag{6}$$

To find fixed points, $y = 0$ and $x = n\pi$.

Van Der Pol Oscillator

$$\frac{d^2y}{dt^2} + E(y^2 - 1)\frac{dy}{dt} + y = 0 \tag{7}$$

Map to phase plane, define $v = \frac{dy}{dt}$, then

$$\dot{y} = v \tag{8}$$

$$\dot{v} = -y - E(y^2 - 1)v \tag{9}$$

Then $\bar{x} = \{y, v\}$. At small y, then the ODE turns into

$$\frac{d^2y}{dt^2} - E\frac{dy}{dt} + y = 0 \tag{10}$$

Which means the voltage grows. At large y, then the ODE turns into

$$\frac{d^2y}{dt^2} + Ey^2\frac{dy}{dt} + y = 0 \tag{11}$$

Which means the voltage shrinks. This ODE exhibits a cool behavior that all initial conditions converge to the same limit cycle.

Numerical Solutions to Differential Equations

Simple Harmonic Oscillator. Note,

$$\frac{dx}{dt} = v \tag{12}$$

$$\frac{dv}{dt} = -\frac{k}{m}x \tag{13}$$

Use taylor expansion to approximate.

Runge-Kutta Method

Let

$$\dot{x} = f(x) \tag{14}$$

Then

$$x_{n+1} = x_n + \frac{1}{6}\left(k_1 + 2k_2 + 3k_3 + k_4\right) \tag{15}$$

Where

$$k_1 = f(x_n)\Delta t, k_2 = f(x_n + \frac{1}{2}k_1)\Delta t, \ldots \tag{16}$$