Exam 2 Review

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Topics covered on the exam:

1. Separation of variables
2. Fourier Series
3. Harmonic Functions

For (1), we first assume that

$$u(x, y) = f(x)g(y) \tag{1}$$

For (2), we've discussed the full, sine, and cosine Fourier Series. For Full Fourier Series with period 2l, we have that

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos(\frac{n\pi x}{l}) + B_n \sin(\frac{n\pi x}{l})) \tag{2}$$

Where

$$A_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\cos(\frac{n\pi x}{l})dx \tag{3}$$

and

$$B_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\sin(\frac{n\pi x}{l})dx \tag{4}$$

For the Sine Fourier Series defined on the interval $[0, l]$ is the following:

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}) \tag{5}$$

with

$$A_n = \frac{2}{l}\int_{0}^{l} \phi(x)\sin(\frac{n\pi x}{l})dx \tag{6}$$

And for Cosine defined on the interval $[0, l]$, we have the following

$$\phi(x) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cos(\frac{n\pi x}{l}) \tag{7}$$

and

$$B_n = \frac{2}{l}\int_{0}^{l} \phi(x)\cos(\frac{n\pi x}{l})dx \tag{8}$$

$$B_0 = \frac{2}{l}\int_{0}^{l} \phi(x)dx \tag{9}$$

For full fourier series to work, the function must have a period of 2l. You can always redefine l to fit the criteria.]

Convergence

Let $\Psi(x)$ be a function of period $2l$ defined by equation (2), then we have the following theorems for Convergence.

Theorem 0.1. *If $\Psi(x)$ is differentiable at $x = x_0$, then the Fourier Series convergences to $\Psi(x)$ at $x = x_0$. (Pointwise Convergence)*

Theorem 0.2. *If $\Psi(x)$ is continuously differentiable, then the Fourier Series converges to $\Psi(x)$ on $\mathbb{R}$. (Uniform Convergence)*

Theorem 0.3. *The Fourier Series converges to $\Psi(x)$ on L^2 on $[-l, l]$. We also have Parseval's equality:*

$$\int_{-l}^{l} |\Psi(x)|^2 dx = l[\frac{A_0^2}{2} + \sum_{n=1}^{\infty}(A_n^2 + B_n^2)] \tag{10}$$

(L^2 convergence)

Here are Parseval's equalities for sine and cosine:
Sine:

$$\int_{0}^{l} |\Psi(x)|^2 dx = \frac{l}{2}[\sum_{n=1}^{\infty} A_n^2] \tag{11}$$

Cosine:

$$\int_{0}^{l} |\Psi(x)|^2 dx = \frac{l}{2}[\frac{A_0^2}{2} + \sum_{n=1}^{\infty} B_n^2] \tag{12}$$

Harmonic Functions

Harmonic Functions in a Rectangle D. Assume that the edges are the following: top = $g_2(x)$, bottom = $g_1(x)$, left = $h_1(y)$, right = $h_2(y)$. Then the unique function is as follows:

$$u(x, y) = \sum \alpha_n e^{\frac{n\pi}{a}y} sin(\frac{n\pi x}{l}) + \beta_n e^{-\frac{n\pi}{a}y} sin(\frac{n\pi x}{l}) + \text{ the sine series for the other boundary.} \tag{13}$$

An easy form of this formula will be on the test. The sin's variable depends on the location of the zeros. That is if the zeros go up and down, then sin's variable is y, and vice versa. The length is the length of the rectangle in the direction of propagation.

Maximum Principle

$$max_{\bar{D}} u = max_{\partial D} u \tag{14}$$

Invariance of Laplace Equation

Laplace equation is invariant under rigid motion. But not covered on test.