PHYS 326: Lecture 9

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Non-linear Dynamics and Chaos

Linear systems are solvable and intuitive, but are only an approximation. Non-linear systems can be simple, but show complex and unpredictable behavior.

The Logistic Map

Define logistic map

$$x_{j+1} = \alpha x_j(1 - x_j) \tag{1}$$

The points of stability is

$$x = 0, x = 1 - \frac{1}{\alpha} \tag{2}$$

We search for equilibrium points, stay close to $1 - \frac{1}{\alpha} + \epsilon$, then

$$x_{j+1} = \alpha x_j(1 - x_j) \tag{3}$$

$$\iff \alpha(1 - \frac{1}{\alpha} + \epsilon)(1 - 1 + \frac{1}{\alpha} + \epsilon) \tag{4}$$

$$\iff 1 - \frac{1}{\alpha} + \epsilon(2 - \alpha) + O(\epsilon^2) \tag{5}$$

$$\iff 1 - \frac{1}{\alpha} + \epsilon' \tag{6}$$

Note

$$\epsilon' < \epsilon \implies \epsilon(2 - \alpha) < \epsilon \iff 1 < \alpha < 3 \tag{7}$$

Definition 0.1. *Chaos is defined as*

1. *Attractor = fractal*
2. *Attract is exponentially sensitive to initial conditions.*

Suppose two populations x_n and x'_n, then for sufficiently large n, then

$$|x_n - x'_n| \sim \epsilon e^{\lambda n} \tag{8}$$

Where λ is the Lyapunov exponent.