MTH 553: Lecture # 14

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Last time

From last time, for the wave equation in three dimensions, we claim that the Kirchhoff formula solves

$$u(x,t) = \partial_t \left(t \int_{S^2} g(x+ct\xi) \frac{d\xi}{4\pi} \right) + t \int_{S^2} h(x+ct\xi) \frac{d\xi}{4\pi} \tag{1}$$

Remarks:

1. It involves the derivatives of g as well as g. (unlike d'Alembert)
2. Domain of dependence for (x,t) is the sphere with center x and radius ct. This means that as time progresses, more and more parts of S^2 influence x. This lives in the x space.
3. Range of influence for $\tilde{x}$ is a surface of a cone

$$|x - \tilde{x}| = ct$$

, which is a union of spheres with radius ct. This lives in the x-t plane.

4. Waves have finite propagation speed c
5. Signals are sharp in 3-dim since the range of influence is a 2 dimension surface at each time.

HWE in 2 dimensions

$u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solves

$$\begin{aligned} u_{tt} &= c^2 \nabla^2 u \\ u(x,0) &= g(x_1,x_2) \\ u_t(x,0) &= h(x_1,x_2) \end{aligned}$$

iff

$$u(x,t) = \frac{1}{4\pi} \frac{\partial}{\partial t} \left(2t \int_{B^2} g(x+cty) \frac{dy}{\sqrt{1-|y|^2}} \right) + \frac{1}{4\pi} 2t \int_{B^2} h(x+cty) \frac{dy}{\sqrt{1-|y|^2}} \tag{2}$$

Where B^2 is the unit disk in $\mathbb{R}^2$.

Proof. Use Kirchhoff's formula. u solves HWE in three dimensions, just not depending on x_3. Then take

$$U(x_1,x_2,x_3,t) = u(x_1,x_2,t) \tag{3}$$

$$G(x_1,x_2,x_3) = g(x_1,x_2) \tag{4}$$

$$H(x_1,x_2,x_3) = h(x_1,x_2) \tag{5}$$

Where U satisfies the wave equation in three dimensions. Now, apply Kirchhoff. In the upper hemisphere of S^2, consider a point on the disk B^2. Now, projecting that onto S^2 corresponds to a unit vector $\xi = (y_1, y_2, \sqrt{1-|y|^2})$. Then $d\xi$ is the area of the graph of $z = \sqrt{1-|y|^2}$. Then

$$d\xi = \sqrt{1 + \left(\frac{\partial z}{\partial y_1}\right)^2 + \left(\frac{\partial z}{\partial y_2}\right)^2} \tag{6}$$

$$= \frac{dy}{\sqrt{1-|y|^2}} \tag{7}$$

Here, this is the 3d projected area element of our 2d problem. Also, then

$$x + ct\xi = (x_1 + cty_1, x_2 + cty_2, x_3 + ct\sqrt{1-|y|^2}) \tag{8}$$

Finally, the sphere has two hemispheres so we collect a factor of 2 (upper and lower are identical by symmetry). Therefore, we obtain equation (2). □

Remarks

1. This is called Hadamard's method of descent for solving 2-d WE using Kirchhoff's formula
2. Domain of dependence for (x,t) is a disk $B^2(x, ct)$ (disk centered at x with radius ct). What points of x influenced this point?
3. Range of influence, solid cone going positively. Namely

   $$|x - \tilde{x}| \leq ct \tag{9}$$

   This is a union of disks with radius of ct. Note, this is because in the domain of dependence in 2-d is a circle, but in 3-d is the boundary of a ball (also called sphere)
4. Propagation speed c
5. No sharp signals (meaning signals live every in B^2), but they do decay in time.

HWE in $\mathbb{R}^n$, $n \geq 4$

If n is even, WE behaves like 3-d with sharp signals. If n is odd, then WE behaves like 2-d with non-sharp signals. This is called Huygen's principle.

NHWE in $\mathbb{R}^n$

APply Duhamel to handle the forcing term. Use Kirchhoff, Hadamard as appropriate depending on dimensionality to compute $Z(x, t-s; s)$.

Next time - Conservation of Energy

Example: n=1 $u_{tt} = c^2 u_{xx}$. Assume $u(x,t)$ has compact support for each t. Then multiply the WE by u_t and integrate with respect to x.

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} u_t(u_{tt} - c^2 u_{xx}) dx \\ &= \int_{-\infty}^{\infty} \left(u_t u_{tt} + c^2 u_x u_{xt}\right) dx \text{ by parts} \\ &= \frac{d}{dt}\frac{1}{2}\int_{-\infty}^{\infty} \left(u_t^2 + c^2 u_x^2\right) dx \end{aligned}$$

This is the conservation of energy.