MTH 416: Lecture 15

Cliff Sun

October 17, 2024

Lecture Span

- 2×2 determinants
- $n \times n$ determinants

Recall

1. $\det(A) \neq 0 \iff$ A is invertible
2. $\det(AB) = \det(A)\det(B)$
3. L_A scales volumes by factor of $|\det(A)|$
4. det is not linear, but <u>"multilinear"</u>

Let $R = [0, t] \times [0, t]$, such that $t > 0$. Then performing $T(R)$ yields

$$T(R) = \text{ parallelogram scaled by t} \tag{1}$$

Thus

$$\text{area}(T(R)) = t^2 |\det(A)| \tag{2}$$

So in general, let $S =$ some general area in $\mathbb{R}^2$. Then we claim that

$$\text{area}(T(S)) = \text{area(S)} \cdot |\det(A)|$$

We can divide up the region S into a bunch of little parallelograms, then applying T to the paralleograms scales each parallelogram by a factor $\det(A)$. Thus,

$$\text{area}(T(S)) = \text{area(S)} \cdot |\det(A)|$$

Theorem 0.1. *(multilinearity for 2×2 det) Let*

$$\det : M_{2\times 2}(\mathbb{R}) \to \mathbb{R} \tag{3}$$

Becomes a linear transformation if we view it as a function of only one row, and treat the other rows as constants. That is, if u, v, w are row vectors in $\mathbb{R}^2$ and $k \in \mathbb{R}$. Then

$$\det \begin{pmatrix} ku + v \\ w \end{pmatrix} = k \det \begin{pmatrix} u \\ w \end{pmatrix} + \det \begin{pmatrix} v \\ w \end{pmatrix} \tag{4}$$

Similarly

$$\det \begin{pmatrix} w \\ ku + v \end{pmatrix} = k \det \begin{pmatrix} w \\ u \end{pmatrix} + \det \begin{pmatrix} w \\ v \end{pmatrix} \tag{5}$$

Proof. Let $u = (a_1, a_2)$, $v = (b_1, b_2)$, and $w = (c_1, c_2)$. For the first equation, we expand it out to be

$$\begin{pmatrix} ka_1 + b_1 & ka_2 + b_2 \\ c_1 & c_2 \end{pmatrix} \tag{6}$$

Calculating the determinant yields

$$(ka_1 + b_1)c_2 - (ka_2 + b_2)c_1 \tag{7}$$

$$k(a_1c_2 - a_2c_1) + (b_1c_2 - b_2c_1) \tag{8}$$

$$= k \det \begin{pmatrix} u \\ w \end{pmatrix} + \begin{pmatrix} v \\ w \end{pmatrix} \tag{9}$$

The second calculation is similar. □

n x n determinants

Definition 0.2. *The determinant of an $n \times n$ matrix ($n > 1$) is*

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det(\tilde{A}_{1j}) \tag{10}$$

Where $\tilde{A}$ is the matrix given by deleting row 1 and column j from A. This is called <u>*cofactor expansion*</u> *on the first row.*

Example, n=2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{11}$$

Then

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det(\tilde{A}_{1j}) \tag{12}$$

$$= (-1)^{1+1} A_{11} \det(A_{22}) + (-1)^{1+2} A_{12} \det(A_{21}) \tag{13}$$

$$A_{11}A_{22} - A_{12}A_{21} \tag{14}$$

$$ad - bc \tag{15}$$

Example, n=3

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 5 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix} \tag{16}$$

$$\det(A) = 2 \det \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} - 1 \det \begin{pmatrix} 5 & 2 \\ 1 & 1 \end{pmatrix} + 3 \det \begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix} \tag{17}$$

Theorem 0.3. *The $n \times n$ determinant is multilinear in the nrows. That is suppose we have matirces $A, B, C \in M_{2\times 2}(\mathbb{R})$ which are identical except in row r, where*

$$a_r = kb_r + c_r \text{ for some } k \in \mathbb{R} \tag{18}$$

Then

$$\det(A) = k \det(B) + \det(C) \tag{19}$$

Proof. Induct on n.

Base Case: n = 1

This just says that the 1×1 determinant is linear, which we already know to be true.

Inductive Step

Assume that this theorem is true for all $n \times n$ matrices, then we must prove it for $(n+1) \times (n+1)$ matrices A, B, C as well. Let A, B, C be as above, then there are 2 cases. When $r = 1$ and $r \neq 1$.

r=1

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det(\tilde{A}_{1j}) \tag{20}$$

$$= \sum_{j=1}^{n+1} (-1)^{j+1} (kB_{1j} + C_{1j}) \det(\tilde{A}_{1j}) \tag{21}$$

But note that

$$\det(\tilde{A}_{1j}) = \det(\tilde{B}_{1j}) = \det(\tilde{C}_{1j}) \tag{22}$$

$$= k \sum_{j=1}^{n+1} -1^{j+1} B_{1j} \det(\tilde{B}_{1j}) + \sum_{j=1}^{n+1} -1^{j+1} C_{1j} \det(\tilde{C}_{1j}) \tag{23}$$

$$= k \det(B) + \det(C) \tag{24}$$

r¿1

$$\det(A) = \sum_{j=1}^{n+1} (-1)^{j+1} A_{1j} \det(\tilde{A}_{1j}) \tag{25}$$

The matrices $\tilde{A}_{1j}$, $\tilde{B}_{1j}$, $\tilde{C}_{1j}$ are identical except for the $r-1$ row. So

$$\det(A) = \sum_{j=1}^{n+1} (-1)^{j+1} A_{1j} \det(\tilde{A}_{1j}) \tag{26}$$

$$= \sum (-1)^{j+1} A_{1j} (k \det(\tilde{B}_{1j}) + \det(\tilde{C}_{ij})) \tag{27}$$

$$= \sum (-1)^{j+1} B_{1j} \det(\tilde{B}_{1j}) + \sum (-1)^{j+1} C_{1j} \det(\tilde{C}_{1j}) \tag{28}$$

This proves the theorem. □

Theorem 0.4. *Let $A \in M_{2\times 2}(\mathbb{R})$ and let $1 \leq r \leq n$. Then*

$$\det(A) = \sum_{j=1}^{n} (-1)^{r+j} A_{rj} \det(\tilde{A}_{rj}) \tag{29}$$

Lemma 0.5. *If A is a $n \times n$ matrix, with row $r = e_j$. Then*

$$\det(A) = (-1)^{r+j} \det(\tilde{A}_{rj}) \tag{30}$$