MTH 553: Lecture # 5

Cliff Sun

January 30, 2026

Lecture Span

- Shock waves

Recap, the jump condition is

$$\xi'(y) = \frac{G(u_l) - G(u_r)}{u_l - u_r} \tag{1}$$

Burger's Example

Consider:

$$G(u) = \frac{1}{2}u^2$$
$$\left(\frac{1}{2}u^2\right)_x + u_y = 0$$
$$uu_x + u_y = 0$$

Consider $u_r = 0$ and $u_l = 1/2$. There is an intersection between u_l and u_r. Clearly, there will not be a smooth solution, so we will look for a weak solution with a shock $x = \xi(y)$ with $\xi(0) = 0$ (why?). This is because the shocks start infinitesimally close to $x = 0$.

$$\xi'(y) = \frac{G(2) - G(0)}{2} = 1 \tag{2}$$

Therefore,

$$\xi(y) = y \tag{3}$$

NOTE: we are finding $\xi(y)$. This is a weak solution with a shock. The jump condition singles out a meaningful solution when characteristics cross. For Burger's, we have that

$$\frac{G(u_l) - G(u_r)}{u_l - u_r} = \frac{1}{2}(u_l + u_r) = \text{ average of left and right particle velocities} \tag{4}$$

Numerically, this shock curve represents a physical path of where the particles collide, and the velocity of the particles along this $\xi(y)$ can be found.

Burger's Example (again)

Now, $u_l = -2$. So now, the characteristic paths leave some sort of gap on the x-y plane. We call this gap region W. Here, characteristics give us no information in W.

Solution 1: Try a new shock

Let $u_l = -2$ and $u_r = 0$, then

$$x'(y) = \xi'(y) = -1 \tag{5}$$

With $\xi(0) = 0$, then we get $x = \xi(y) = -y$. This shock is non-physical since particles are being created at the shock.

Solution 2

Look for a solution in W of the form

$$u(x, y) = v\left(\frac{x}{y}\right) \tag{6}$$

Note, v is a function of x/y because x/y is constant along each characteristic for this specific PDE. Substituting into Burger's equation, we obtain

$$\begin{aligned} 0 &= uu_x + u_y && (7)\\ &= v(x/y)v'(x/y)/y + v'(x/y)(-x/y^2) && (8)\\ &= v(x/y)/y + (-x/y^2) && (9)\\ v(x/y) &= x/y && (10) \end{aligned}$$

So we try this solution:

$$u(x, y) = \begin{cases} 0 & x > 0 \\ x/y & (x, y) \in W \\ -2 & y < -x/2 \end{cases} \tag{11}$$

We call this type of solution a "fan" or rarefaction wave.

Entropy Condition

Assume u is a weak solution of

$$G(u)_x + u_y = 0 \tag{12}$$

This is the conservation law. Also, u satisfies the jump condition

$$\xi'(y) = \frac{G(u_l) - G(u_r)}{u_l - u_r} \tag{13}$$

along the shock curve $x = \xi(y)$.

Definition 0.1. *We say that u satisfies the entropy condition if $G'(u_l) > \xi'(y) > G'(u_r)$ everywhere on the shock.*

Meaning

Characteristics are straight lines of the form $x = G'(u)y +$constant. So entropy condition means that characteristics can meet at a shock, but cannot form from one.

The entropy condition rules out non-physical shocks (like solution 1)