MTH 553: Lecture # 6

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February 2, 2026

Lecture Span

- Traffic Flow

Traffic Flow

Let $\rho(x)$ be the density of cars on the highway per unit length and $q(r,t)$ is the rightward flux of the cars per unit time. The conservation of cars between a, b is

$$q(b,t) - q(a,t) = -\frac{d}{dt}\int_a^b \rho(x,t)dx \tag{1}$$

Assume that the traffic speed is a function of ρ. Therefore, the flux is density times speed and therefore, a function of density.

That is, $q(x,t) = G(\rho(x,t))$. Then the conservation law states that ρ is a weak solution of

$$G(\rho)_x + \rho_t = 0 \tag{2}$$

The above is a conservation law.

Example

Assumption: Traffic speed is c (constant), a reasonable assumption for low density. Then

$$G(\rho) = c\rho \tag{3}$$

We then get the <u>transport equation</u>.

$$c\rho_x + \rho_t = 0 \tag{4}$$

The solutions are $x = ct + x_0$. So the initial density just gets translated along with the traffic.

Example

Under heavier traffic conditions, assume traffic speed is

$$c\left(1 - \frac{\rho}{\rho_{\max}}\right)$$

i.e. the speed decreases linearly with respect to density. Flux is density times speed. Therefore,

$$q = c\rho\left(1 - \frac{\rho}{\rho_{\max}}\right) \tag{5}$$

This is a concave function.

Example

Traffic flow after a red light turns green. Therefore,

$$\rho(x,0) = \begin{cases} \rho_{\max} & x < 0 \\ 0 & x > 0 \end{cases} \tag{6}$$

Characteristic: $x = G'(\rho)t + x_0$. Solution:

$$G'(\rho) = c\left(1 - \frac{2\rho}{\rho_{\max}}\right) \tag{7}$$

Then $G'(\rho_{\max}) = -c$ and $G'(0) = c$. In the fan, we try some function $\rho = v(x/t)$, note that x/t is constant out of the origin. Substituting into the conservation equation yields

$$c\left(1 - \frac{2\rho}{\rho_{\max}}\right)v'(x/t)/t + v'(x/t)(-x/t^2) = 0 \tag{8}$$

Therefore,

$$G'(\rho) = c\left(1 - \frac{\rho}{\rho_{\max}}\right) = \frac{x}{t} \tag{9}$$

We then get

$$\rho = \frac{\rho_{\max}}{2}\left(1 - \frac{x}{ct}\right) \tag{10}$$

Interpretation

The first car in the line travels at speed c. But any car at $x_0 = -ct_0$ must wait until time t_0 in order to move. What path does it follow?

Note: the characteristics are NOT the trajectory of the vehicles, but are rather the lines of constant density.

Definition 0.1. *G is uniformly convex if $G'' \geq a > 0$ for some constant a. Similar definition for concave $G'' \leq b < 0$ for b is a constant.*

If G is uniformly convex, then G' is strictly increasing. Then $u_l > u_r$. Vice versa for G uniformly concave.