MTH 416: Lecture 16

Cliff Sun

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Lecture Span

- $n \times n$ determinant
- Properties & calculating with row operations

Recall

Theorem 0.1. *For any $1 \leq r \leq n$, then*

$$\det(A) = \sum_{j=1}^{n} (-1)^{r+j} A_{rj} \det(\tilde{A_{rj}}) \tag{1}$$

then

Lemma 0.2. *If A has some row that is e_j, then*

$$\det(A) = (-1)^{r+j} \det(\tilde{A}_{rj}) \tag{2}$$

Proof. Using the Lemma, let $A \in M_{2\times 2}(\mathbb{R})$, then define new matrices B_j such that B_j is identical to A except that row r is replaced by e_j. Then

$$\det(A) = \sum A_{rj} \det(B_j) \tag{3}$$

$$\det(A) = \sum (-1)^{r+j} A_{rj} \det(\tilde{A_{rj}}) \tag{4}$$

□

To prove the lemma, we induct on n.

Proof. Base cases: $n = 1$ and $n = 2$. For $n = 1$, we define the determinant of the 0×0 matrix to be 1.

Induction

Assume true for some n, then let A be some form $(n+1) \times (n+1)$. Then we use cofactor expansion on the first row:

$$\det(A) = \sum (-1)^{1+j} A_{1j} \det(\tilde{A_{1j}}) \tag{5}$$

Each $\tilde{(}A_{1j})$ has two forms:

1. If we cross out column $j \neq k$ (or not where the standard basis appears), then we can use the IH to calculate the determinant.
2. If we do cross out column $j = k$, then we can use $\tilde{A_{1j}}$ has a row of all $0's$, then we can use the following theorem:

   Theorem 0.3. *If A has a row of all $0's$, then the determinant is 0.*

   Proof. If row r of A is all 0's, then row $r = 0\cdot$(row r), so multilinearity states that

   $$\det(A) = 0 \cdot \det(A) = 0 \tag{6}$$

   □

□

Triangular matrices

Calculate the determinant of this matrix:

$$\begin{pmatrix} 2 & 1 & 4 & 5 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 3 \end{pmatrix} \tag{7}$$

Expand this out using the last row:

$$\det(A) = -0 + 0 - 0 + 3\det\begin{pmatrix} 2 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \tag{8}$$

$$\det(A) = 3 \cdot 4 \cdot \det\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \tag{9}$$

$$= 3 \cdot 4 \cdot 1 \cdot 2 \tag{10}$$

These are the diagonal entries of the matrix.

Theorem 0.4. *If A is upper or lower triangular, then*

$$\det(A) = \textit{ product of the diagonal entries} \tag{11}$$

Theorem 0.5. *If A has two identical rows, then the determinant of $A = 0$.*

Proof. Induct on n. The base case is $n = 2$ is trivial.

Induction

Suppose the theorem is true for some n, then let A be $(n+1) \times (n+1)$ with 2 identical rows. Then we calculate the determinant of A using cofactor expansion using any other rows other than the two rows.

$$\det(A) = \sum_{j=1}^{n+1} (-1)^{r+j} A_{rj} \det(\tilde{A_{rj}}) \tag{12}$$

But $\tilde{A_{rj}}$ still has two identical rows, which means that we can apply the base case. Thus, by the IH, this determinant is 0. $\square$

Theorem 0.6. ***Main Theorem*** *Suppose A and B are $M_{2\times 2}(\mathbb{R})$, and B is the result of applying one elementary row operation to A. Then,*

1. *Switching 2 rows, then* $\det B = -\det A$
2. *Scaling a row by factor $c \neq 0$, then* $\det B = c \det A$
3. *Adding cR_i to R_j, then* $\det(B) = \det(A)$

Corollary 0.7. $\det(A) \neq 0 \iff$ *A is invertible.*

Proof. If A is invertible, then it row-reduces to I_n. Since $\det(I_n) = 1$, then it follows that $\det(A) \neq 0$. If A is not invertible, then it reduces to some RREF matrix with # of pivots less than n. This has a row of all zeros, so $\det(B) = 0$, and the $\det(A) = 0$. $\square$

Theorem 0.8. *If A has 2 identical rows, then* $\det(A) = 0$.

Proof. Use multilinearity and above theorem.

Property 2

This is just multilinearity.

Property 3

Let

$$A = \begin{pmatrix} \dots \\ R_i \\ \dots \\ R_j \\ \dots \end{pmatrix} \tag{13}$$

Then

$$B = \begin{pmatrix} \dots \\ R_i \\ \dots \\ R_j + cR_i \\ \dots \end{pmatrix} \tag{14}$$

Taking this determinant, we notice that

$$\det(B) = \det \begin{pmatrix} \dots \\ R_i \\ \dots \\ R_j \\ \dots \end{pmatrix} + c \det \begin{pmatrix} \dots \\ R_i \\ \dots \\ R_i \\ \dots \end{pmatrix} \tag{15}$$

The first matrix is A, and the second matrix has 2 identical rows, thus its determinant is 0. Therefore,

$$\det(B) = \det(A) \tag{16}$$

Property 1

Let

$$A = \begin{pmatrix} \dots \\ R_i \\ \dots \\ R_j \\ \dots \end{pmatrix} \tag{17}$$

Then we switch two rows

$$B = \begin{pmatrix} \dots \\ R_j \\ \dots \\ R_i \\ \dots \end{pmatrix} \tag{18}$$

Trick: consider

$$C = \begin{pmatrix} \dots \\ R_i + R_j \\ \dots \\ R_i + R_j \\ \dots \end{pmatrix} \tag{19}$$

Thus, $\det(C) = 0$. But we can also calculate the determinant of C using multilinearity.

$$\det(C) = \det(\begin{pmatrix} \dots \\ R_i \\ \dots \\ R_i + R_j \\ \dots \end{pmatrix} + \begin{pmatrix} \dots \\ R_j \\ \dots \\ R_i + R_j \\ \dots \end{pmatrix}) \tag{20}$$

$$\det(C) = \det(\begin{pmatrix} \dots \\ R_i \\ \dots \\ R_j \\ \dots \end{pmatrix}) + \det \begin{pmatrix} \dots \\ R_i \\ \dots \\ R_i \\ \dots \end{pmatrix} + \det \begin{pmatrix} \dots \\ R_j \\ \dots \\ R_j \\ \dots \end{pmatrix} + \det \begin{pmatrix} \dots \\ R_j \\ \dots \\ R_i \\ \dots \end{pmatrix} \tag{21}$$

Thus

$$\det \begin{pmatrix} \dots \\ R_j \\ \dots \\ R_i \\ \dots \end{pmatrix} = -\det(\begin{pmatrix} \dots \\ R_i \\ \dots \\ R_j \\ \dots \end{pmatrix}) \tag{22}$$

□