

MTH 417: Lecture 15

Cliff Sun

September 29, 2025

$$S_3 = \{e, (12), (13), (23), (123), (132)\} \quad (1)$$

All elements in this set partition S_3 .

Definition 0.1. Let $H \leq G$ be a subgroup. Let $g \in G$. A subset of the form

$$gH = \{gh \mid h \in H\} \quad (2)$$

is called a left coset of H in G . The right coset is defined as

$$Hg = \{hg \mid h \in H\} \quad (3)$$

Note, this is also a function of g where g is a fixed element of G .

In general, the left and right cosets are different.

Proposition 0.2. Let $H \leq G$, and $a, b \in G$. The following statements are all equivalent (TFAE):

1. $a \in bH$
2. $b \in aH$
3. $b^{-1}a \in H$
4. $a^{-1}b \in H$
5. $aH = bH$

Proof. Prove that 1 \implies 2. Suppose $a \in bH$. This means that $a = bh \iff ah^{-1} = b \in aH$. \square

Proof. Prove that 2 \implies 3. Suppose $b = ah \iff h^{-1} = b^{-1}a \in H$. \square

Proof. Prove that 3 \implies 4. Suppose $b^{-1}a = h \iff h^{-1} = a^{-1}b \in H$. \square

Proof. Prove that 4 \implies 5. Suppose $a^{-1}b = h$. Then let $\tilde{h} \in aH$, then $a\tilde{h} = bh^{-1}\tilde{h} \in bH$. A similar exercise for $bH \subseteq aH$. \square

Proof. Prove that 5 \implies 1. Suppose $aH = bH$. In particular, $ae \in aH \iff ae \in bH$. \square

Proposition 0.3. $H \leq G$, $a, b \in G$. Then, these are consequences:

1. Either $aH = bH$ or $aH \cap bH = \emptyset$
2. The function $f : aH \rightarrow bH$ such that $f(x) = ba^{-1}x$. This is a bijection.
3. $G = \cup$ Left Cosets

Proof. We prove 3. Note that left coset $\subseteq G$. Then let $g \in G$, then in particular, $g \in$ some left coset because every left coset has a the element $ge = g$. \square

Proof. We prove 2. Assume that $aH \cup bH \neq \emptyset$. Then $\exists x \in aH \cup bH$. Then in particular, $x \in aH$ means that $aH = xH$. Similarly, $bH = xH$. The result is trivial. \square

Theorem 0.4. Lagrange's Theorem: Let G be a finite group. Let $H \leq G$. Then the order of H divides the order of G and the quotient is the number of left cosets of H in G .

Definition 0.5. For any $H \leq G$, the # of left cosets of H in G is the index of H in G . Denoted $[G : H]$

Then Lagrange tells us that if $|G| \leq \infty$, then $|G| = |H|[H : G]$

Proof. Suppose a_1, \dots, a_k be representatives of the left cosets of H in G . Then

$$G = \bigcup_i a_i H \quad (4)$$

$$\implies |G| = \sum_i |a_i H| \quad (5)$$

because they are disjoint.

$$\sum_i |a_i H| = \sum_i |H| \quad (6)$$

$$|G| = k|H| \quad (7)$$

This means that the order of H divides G and the integer is the number of left cosets. \square