PHYS 326: Lecture # 12

Cliff Sun

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Rigid Body Mechanics

Definition 0.1. *A rigid body is a collection of n particles with constant distance between them. That is, the shape does not change.*

Definition 0.2. *The center of mass is defined as*

$$\vec{R} = \frac{1}{M}\sum_{a=1}^{N} m_a r_a \tag{1}$$

This is the mass weighted with respect to its vector. The continuous limit is

$$\vec{R} = \frac{1}{M}\int dmr \tag{2}$$

Definition 0.3. *The momentum is*

$$\sum_a m_a \dot{r}_a \iff M\dot{\vec{R}} \tag{3}$$

The external force is

$$F_{ext} = M\ddot{\vec{R}} \tag{4}$$

Definition 0.4. *The total angular momentum is*

$$L_O = \sum_a \vec{r}_a \times m_a \dot{\vec{r}}_a \tag{5}$$

Total external torque is

$$R_O = \sum_a \vec{r}_a \times F_{a,\, ext} \tag{6}$$

Theorem 0.5.

$$\frac{dL_O}{dt} = R_O \tag{7}$$

Proof. Calculate dL_O/dt.

$$\frac{dL_O}{dt} = \frac{d}{dt}\left[\sum_a \vec{r}_a \times m_a \dot{\vec{r}}_a\right] \tag{8}$$

$$= \sum_a \dot{\vec{r}}_a \times m_a \dot{\vec{r}}_a + \sum_a \vec{r}_a \times m_a \ddot{\vec{r}}_a \tag{9}$$

$$= \sum_a \vec{r}_a \times \left(F_a^{ext} + F_a^{int}\right) \tag{10}$$

$$= \sum_a \vec{r}_a \times F_a^{ext} \tag{11}$$

$\square$

Theorem 0.6.

$$L_O = \vec{R} \times (M\dot{\vec{R}}) + L^{CM} \tag{12}$$

Where L^{CM} is the angular momentum around the center of mass.

Proof. Define $\rho = r_a - R$ where R is the location of the center of mass, then calculate torque and note that R has no a subscript. □

Theorem 0.7.

$$\frac{dL^{CM}}{dt} = \vec{\Gamma}^{CM} \tag{13}$$

The change in the angular momentum around the center of mass is the torque around the center of mass.

Theorem 0.8.

$$\frac{d}{dt}L_p = \Gamma_p^{actual} - \vec{r}_{p \to COM} \times M\vec{a}_p \tag{14}$$

Definition 0.9. *The Levi-Civita symbol is*

$$\epsilon_{ijk} = -1^p \tag{15}$$

Where p is the parity of the permutation of ijk.