Squeeze Theorem and Continuity of Algebraic Operations

Cliff Sun

April 24, 2024

Squeeze Theorem

Claim:

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0 \tag{1}$$

Proof. We use the squeeze theorem, with $a_n = \frac{-1}{n}$, $b_n = \frac{1}{n}$ and $x_n = \frac{\cos n}{n}$. Since we have that

$$a_n \leq x_n \leq b_n \tag{2}$$

For all n, and that

$$\lim a_n = 0 \wedge \lim b_n = 0 \tag{3}$$

It follows that

$$\lim x_n = 0 \tag{4}$$

□

Theorem 0.1. *Suppose*

$$a_n \leq x_n \leq b_n \tag{5}$$

for all n. Then if

$$\lim a_n = a \tag{6}$$

$$\lim b_n = b \tag{7}$$

and

$$a \neq b \tag{8}$$

Then if $\lim x_n = x$*, then we have that*

$$a \leq x \leq b \tag{9}$$

Theorem 0.2. *Suppose that* (x_n) *and* (y_n) *are sequences that converge to* x, y *respectively. Then*

1. $\lim(x_n + y_n) = x + y$
2. $\lim(x_n - y_n) = x - y$
3. $\lim(x_n \cdot y_n) = x \cdot y$
4. *If* y *and all* y_n *are not zero, then* $\lim(\frac{x_n}{y_n}) = \frac{x}{y}$

This tells us that addition, multiplication, subtraction, and division are all continuous functions. That is if x_n *is close to* x *and* y_n *is close to* y*. Then* $x_n + y_n$ *is close to* $x + y$*.*

To begin, we prove statement (1).

Proof. Suppose $\lim x_n = x$ and $\lim y_n = y$, we claim that $\lim(x_n + y_n) = x + y$. Let $\epsilon > 0$, plugging in $\frac{\epsilon}{2}$ for both x_n and y_n, then we get M_1 and M_2. Choosing $M' = \max(M_1, M_2)$, we have that

$$|x_n - x| < \frac{\epsilon}{2} \tag{10}$$

and

$$|y_n - y| < \frac{\epsilon}{2} \tag{11}$$

We rewrite this to be the following:

$$-\frac{\epsilon}{2} + x < x_n < \frac{\epsilon}{2} + x \tag{12}$$

and

$$-\frac{\epsilon}{2} + y < y_n < \frac{\epsilon}{2} + y \tag{13}$$

Adding the equations together yields

$$-\epsilon < x_n + y_n - (x + y) < \epsilon \tag{14}$$

This concludes the proof. □

lim sup and lim inf

Recall that if (x_n) converges, then (x_n) is bounded. But the converse is clearly not true. Then what is the long-term behavior of a bounded divergent sequence?

Then we define lim inf to be the lower limit of the interval in which the sequence oscillates long term and lim sup similarly.

Definition 0.3. *Let (x_n) be a bounded sequence such that*

1. $a_n = \sup\{x_k : k \geq n\}$
2. $b_n = \inf\{x_k : k \geq n\}$
3. $\limsup(x_n) = \lim a_n$
4. $\liminf(x_n) = \lim b_n$