PHYS 486: Lecture # 19

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Recap: Harmonic Oscillator

The Hamiltonain is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \tag{1}$$

Introduced a couple of new oscillators:

$$a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp i\hat{p} + m\omega\hat{x}\right) \tag{2}$$

This means that

$$\hat{H} = (a_+a_- + \frac{1}{2})\hbar\omega \tag{3}$$

Note that

$$\left|0\right\rangle, \left|1\right\rangle = a_+\left|0\right\rangle, \ldots \tag{4}$$

The energy difference is always $\hbar\omega$. Note that a_+ (creation operator) and a_- (annihilation operator) are always called ladder operators. Solving

$$\hat{H}\left|0\right\rangle = \underbrace{\frac{\hbar\omega}{2}}_{E_0}\left|0\right\rangle$$

This E_0 is interpreted as some sort of quantum noise. This is inevitable. Introduce

$$a_+a_- = \frac{H}{\hbar\omega} - \frac{1}{2} = \hat{n} \text{ (number operator)} \tag{5}$$

This is the number of energy levels there are. Note that

$$\left\langle n\middle|\hat{H}\middle|n\right\rangle = \frac{1}{2}\hbar\omega + n\bar{\omega} \tag{6}$$

Then

$$\langle a_+a_-\rangle = n \tag{7}$$

Since this value is Hermitian, this means that it is measurable. We compute

$$\langle n|a_+a_-|n\rangle \neq 0 \tag{8}$$

Since we are in an orthonormal eigenbasis, and this inner product is $\neq 0$, we can infer that

$$\left(\left|a_-n\right\rangle\right)^\dagger = \left(a_-\left|n\right\rangle\right)^\dagger = \left\langle n\right|(a_-)^\dagger = \left\langle n\right|a_+ \tag{9}$$

Therefore,

$$a_-^\dagger = a_+ \tag{10}$$

Normalization

We know that $\langle n|n\rangle = 1$. But how do we normalize this? We show that:

$$\begin{aligned}
a_+ |n\rangle &= c_{n+1} |n+1\rangle \\
\langle n|a_- a_+|n\rangle &= \langle n+1| c_{n+1}^* c_{n+1} |n+1\rangle \\
&= \langle n|\frac{\hat{H}}{\hbar\omega} + \frac{1}{2}|n\rangle \\
&= \langle n|\frac{n\hbar\omega}{\hbar\omega} + \frac{1}{2}|n\rangle \\
&= n+1
\end{aligned}$$

This means that $c_{n+1} = \sqrt{n+1}$. This is nothing but a mathematical necessity. We repeat with a_-:

$$\begin{aligned}
a_- |n\rangle &= c_{n-1} |n-1\rangle \\
c_{n-1} &= \sqrt{n-1}
\end{aligned}$$

Brief Summary:

1. $|0\rangle$ is the ground state
2. a_+: $a_+ |n\rangle = \sqrt{n+1}\,|n+1\rangle$
3. a_-: $a_- |n\rangle = \sqrt{n-1}\,|n-1\rangle$
4. $a_+^\dagger = a_-$
5. $\hat{n} = a_+ a_-$
6. $\frac{\hat{H}}{\hbar\omega} = a_+ a_- + \underbrace{\frac{1}{2}}_{ZPE}$ (ZPE = zero point energy)
7. $[a_-, a_+] = 1$

S.S in the position basis. The boundary conditions are

$$a_-\psi_0(x) = \frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{\partial}{\partial x} + m\omega x\right)\psi_0(x) = 0 \tag{11}$$

This means that

$$\frac{\partial}{\partial x}\psi_0(x) = -\frac{m\omega x}{\hbar}\psi_0(x) \tag{12}$$

$$\implies \psi_0(x) = A\exp\left(-\frac{m\omega x^2}{2\hbar}\right) \tag{13}$$

The phase space of the position and momentum is aligned. We first normalize.

$$\langle\psi_0(x)|\psi_0(x)\rangle \implies A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \tag{14}$$

We find $\psi_1(x)$:

$$a_+\psi_0 = \sqrt{0+1}\psi_1 \tag{15}$$

$$\psi_1(x) = \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{\partial}{\partial x} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \tag{16}$$

Following an iterative procedure, we obtain

$$\psi_n = \left(\prod_{k=1}^{n} \frac{a_+}{\sqrt{k}}\right) \psi_0 = \frac{1}{\sqrt{n!}} a_+^n \psi_0 \tag{17}$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(x) e^{-x^2/2} \tag{18}$$