MTH 447: Lecture 5

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Definition 0.1. *A sequences with entries in S is a function*

$$f : \mathbb{N} \to \mathbb{S} \tag{1}$$

With

$$(x_1, x_2, x_3, \cdot, x_n) \tag{2}$$

Definition 0.2. *A sequence (x_n) converges to L or has a limit L if*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \tag{3}$$

Such that if

$$n > N, |x_n - L| < \epsilon \tag{4}$$

Theorem 0.3. *If $x_n \to L$ and $x_n \to M$, then*

$$L = M \tag{5}$$

That is limits are unique.

Proof. Let $\epsilon = \frac{|L-M|}{2}$. Then assume $\epsilon > 0$ and $L \neq M$. Then $x_n \to L$, $\exists N_1 \in \mathbb{N}$ such that $n > N_1$

$$|x_n - L| < \epsilon \tag{6}$$

Since $x_n \to M$, $\exists N_2$ such that $n > N_2 \implies |x_n - M| < \epsilon$. Let $N = \max(N_1, N_2)$, then

$$n > N \implies |x_n - L| < \epsilon \wedge |x_n - M| < \epsilon \tag{7}$$

Then

$$|L - M| \tag{8}$$

$$\iff |L - x_n + x_n - M| \tag{9}$$

$$\iff |(L - x_n) + (x_n - M)| \tag{10}$$

$$\leq |x_n - L| + |x_n - M| \tag{11}$$

$$< 2\epsilon \tag{12}$$

$$\iff |L - M| \tag{13}$$

This is a contradiction, this concludes the proof. ☐

Definition 0.4. *Let A be a finite set,*

$$A^n = \text{ all strings of length } n \text{ from } A \tag{14}$$

$$\tilde{A}^n = \cup_{k=1}^{n} A^k \tag{15}$$

$$A^\infty = \text{ all finite strings from } A \tag{16}$$

$$\iff \cup_{k=1}^{\infty} A_n \tag{17}$$

$$A^* = \text{ all infinite strings from } A \tag{18}$$

Definition 0.5. *We say a set S is countable if*

1. *it is finite*
2. *S is infinite and its cardinality is $\aleph_0$.*

Theorem 0.6.

1. *A^n and $\tilde{A}^n$ are finite sets*
2. *A^∞ is countable*
3. *A^* is not countable*