PHYS 435: Lecture 2

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Potential Theory

We restate Maxwell's Equations:

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0} \tag{1}$$

$$\vec{\nabla} \times \vec{E}(\vec{r}) = -\frac{\partial}{\partial t}\vec{B} \tag{2}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\vec{E} \tag{4}$$

Assume static conditions, then all $\frac{\partial}{\partial t} f = 0$ for all functions f. Then if

$$\vec{\nabla} \times \vec{E} = 0 \tag{5}$$

Then we can find $\vec{E}$ by defining a potential function. The issue is that potential functions are not uniquely defined. Then

$$\int_{P_i} d\vec{l} \cdot \vec{E} \tag{6}$$

is path independent for all paths P_i. Then let P_1 go from a point O to a point R from P_2 goes from R to O. Then

$$\int_{P_1} + \int_{P_2} = \oint_{\partial S} d\vec{l} \cdot \vec{E} = \int_S \vec{\nabla} \times \vec{E} = 0 \tag{7}$$

Thus

$$\int_{P_1} = -\int_{P_2} \tag{8}$$

Without loss of generality. Then we define a vector potential function relative to a point O to be

$$V(\vec{r}) = -\int_O^{\vec{r}} d\vec{l'} \cdot \vec{E} \tag{9}$$

Note this is a potential function. We define the potential energy function of a charged particle in the presence of an E and B field as

$$u(\vec{a}) - u(\vec{b}) = -\int_a^b d\vec{l} \cdot \vec{F} \tag{10}$$

Where F is the lorentz force. We note that the magnetic field does no work on the particle.

Proof. Let F_B be the force exerted by the B Field. That is

$$F_B = q\vec{v} \times \vec{B} \tag{11}$$

Then the work done is

$$W = q\int_a^b d\vec{l} \cdot \vec{v} \times \vec{B} \tag{12}$$

We note that

$$\frac{d\vec{l}}{dt} = \vec{v} \tag{13}$$

Then the work turns into

$$q\int_a^b d\vec{l} \cdot \frac{d\vec{l}}{dt} \times \vec{B} \tag{14}$$

We multiply this integral by $\frac{dt}{dt}$ to obtain

$$q\int_a^b dt\frac{d\vec{l}}{dt} \cdot \frac{d\vec{l}}{dt} \times \vec{B} \tag{15}$$

But

$$\frac{d\vec{l}}{dt} \perp \frac{d\vec{l}}{dt} \times \vec{B} \tag{16}$$

Thus this integral evaluates to 0. $\square$

We first define

$$V(\vec{r}) = -\int_O^{\vec{r}} d\vec{l'} \cdot \vec{E} \tag{17}$$

Let

$$\vec{r} \to \vec{r} + dx\hat{x} \tag{18}$$

Then

$$V(\vec{r} + dx\hat{x}) \approx V(\vec{r}) + dx\hat{x} \cdot \vec{E} \tag{19}$$

Because this is how work is defined. Then

$$\frac{V(\vec{r} + \hat{x}dx) - V(\vec{r})}{dx} = E_x(\vec{r}) \tag{20}$$