7.4-7.5 Paritions and Modular Arithmetic

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Partitions

Lemma 0.1. *Let $\sim$ be an equivalence relation on a set X. Then for $x, y \in X$, we have that $x \sim y \iff [x] = [y]$. That is the equivalence class of x is equal to that of y.*

Proof. $\implies$, Suppose that $x \sim y$, we must prove that $[x] \subseteq [y]$ and that $[y] \subseteq [x]$. Suppose that $z \in [x]$, which implies that $z \sim x$. But by definition of the equivalence class, then $z \sim y$ by transitivity. Thus $z \in [y]$. Conversely, suppose that $z \in [y]$, that implies that $z \sim y$ which means that $z \sim x$ by symmetry of the equivalence class. Thus $z \in [x]$ by transitivity. This proves the $\implies$ direction.

$\impliedby$, next suppose that $[x] = [y]$, that is for every element in $[x]$, it also exists in $[y]$. Then because $\sim$ is reflexive, we have that $x \sim x$, then that implies that $x \in [x]$. But since $[x] = [y]$, it follows that $x \in [y]$ which by definition of the equivalence class, means that $x \sim y$. This concludes the full proof. □

Theorem 0.2. *Let X be a set, then:*

1. *If $\sim$ is an equivalence relation on X, then its equivalences classes paritition X.*
2. *If $\{A_n : n \in I\}$ forms a partition, then there exists some equivalence relation that relates the values in that partition.*

For 2, a more concrete definition is that

$$x \sim y \iff \exists n \in I : x \in A_n \wedge y \in A_n \tag{1}$$

This relation is a equivalence relation.

Proof. This is a proof of 1 in the theorem. Let $\sim$ be an equivalence relation, then we must prove that

1. Every $x \in X$ is in some equivalence class.
2. That given an 2 equivalence classes, they are either the same or disjoint.

For the 1st statement above, it follows that x is in its own equivalence class ($[x]$) by reflexivity.

Next, for the 2nd statement above, suppose that we are given two equivalence classes. We can prove this by stating that if they have a common element, then they must share the same elements. So suppose $[x]$ and $[y]$ share a common element z. Then we must show that $[x] = [y]$. But this statement that $z \in [x]$ and $z \in [y]$ states that $z \sim x$ and $z \sim y$. Then the lemma states that $[x] = [y] = [z]$. □

Proof. This is a proof of 2 in the theorem. We must show that the relation showed in the theorem is an equivalence relation. We begin first by proving reflexivity,

Reflexive: For any $x \in X$, it follows that $x \sim x$ since $x \in A_n$ and $x \in A_n$ by definition of partitions.

Symmetric: If x and y are in the same parition, then it follows that $y \sim x$ since y and x are in the same partition.

Transitivity: Suppose that $x \sim y$, and that $y \sim z$. This implies that for some $m, n \in I$, x and y share a partition and that z and y share a partition. We define A_n to be the partition that x and y share for some $n \in I$, then it follows that y and z share that same partition since y lives in A_n by definition of the relation. Thus $x \sim y \sim z$. This concludes the proof. □

Recall that $X/\sim\ = \{[x] : x \in X\}$. Let's make a mobius strip. Suppose X is a rectangle. That is $X = [0,6] \times [0,1]$. Let's glue the ends of this rectangle together. More specifically, for all $(0,y) to (6,1-y)$ The parition that does this is that

1. $A_{(x,y)} = x, y$ for all $x \in (0,6)$ and $y \in (0,1)$.
2. $A_{(0,y)} = (0,y), (6,1-y)$ for $y \in [0,1]$.