PHYS 325: Lecture 11

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Lecture Span

- 2 Body problem

Midterm

1. Force in 3 dimensions
2. Curvilinear coordinates
3. Central force
4. Check out the email she sent

2 Body problem

Kepler's Laws

1. Orbits around the sun are planar ellipses
2. The line between the sun and planet produces the same area in equal time. That is given 2 points in the orbit separated by a time dt, then we have that the area between these two points (should be like a triangle) is the same everywhere given the same dt.
3. Orbital period P around the sun is proportional to $\frac{3}{2}$ power of the semi-major axis. Ellipses have 2 axis, the semi-minor and semi-major. The semi-major is the longer radius in an ellipse and the semi-minor is the smaller radius in an ellipse.

Characterize Orbits

1. We first introduce **angular momentum per unit mass**

$$L = mr^2\dot{\phi} \implies l = \frac{L}{m} \iff r^2\dot{\phi} \tag{1}$$

   Thus

$$\dot{\phi} = \frac{l}{r^2} \tag{2}$$

   Note: The angle between $\vec{r}$ and $\vec{v}$ is NO LONGER 90 DEGREES!

2. **Energy per unit mass** conservation

$$\epsilon = \frac{E}{m} \iff \frac{1}{m}(T + U_{grav}) \tag{3}$$

$$\iff \frac{1}{2}|\vec{v}|^2 - \frac{GM}{r} \tag{4}$$

$$\iff \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{GM}{r} \tag{5}$$

$$\epsilon = \frac{1}{2}\dot{r}^2 + \frac{l^2}{2r^2} - \frac{GM}{r} \tag{6}$$

   This is a one-dimesional problem with an effective potential.

3. Perigee $[r_p]$ (minimal distance to focal point (M)) and Apogee $[r_a]$ (maximal distance to focal point (M))
   - Velocity vectors $\vec{v}_p$ and $\vec{v}_a$ are perpendicular to $\vec{r}_p$ and $\vec{r}_a$. Thus
   $$\vec{r_p} \cdot \vec{v_p} = 0 \tag{7}$$
   - Angular momentum $\vec{l} = \vec{r} \times \vec{v}$. At the Perigee and Apogee
   $$\vec{l} = \vec{r} \times \vec{v} = |r_p||v_p| = |r_a||v_a| \tag{8}$$
4. Energy per unit mass in r_p and r_a in Apogee and Perigee, we have that $\dot{r} = 0$, thus
$$\epsilon_{p,a} = \frac{l^2}{2r_{p,a}^2} - \frac{GM}{r_{p,a}} \tag{9}$$
$$\epsilon_{p,a} = \frac{1}{2}v_{p,a}^2 - \frac{GM}{r_{p,a}} \tag{10}$$
$$\epsilon_{p,a} = \frac{1}{2}v_{p,a}^2 - \frac{GM}{l}v_{p,a} \tag{11}$$
Thus
$$v_{p,a} = \frac{GM}{l} \pm \sqrt{(\frac{GM}{l})^2 + 2\epsilon} \tag{12}$$
Note, plus is for perigee and minus is for the apogee.
$$r_{p,a} = \frac{l}{v_{p,a}} = l\left[\frac{GM}{l} \pm \sqrt{(\frac{GM}{l})^2 + 2\epsilon}\right]^{-1} \tag{13}$$

Effective Potential and Orbits

$$\epsilon = \frac{1}{2}\dot{r}^2 + U_{eff} \tag{14}$$
$$U_{eff} = \frac{l^2}{2r^2} - \frac{GM}{r} \tag{15}$$

1. Circular Orbit with radius r_c
   - Minimum in effective potential
   $$U'_{eff}|_{r=r_c} = 0 \tag{16}$$
   In this scenario, we have that
   $$\frac{mv^2}{r} = -\frac{GMm}{r^2} \tag{17}$$
   $$v = \sqrt{\frac{GM}{r}} \tag{18}$$
   Thus,
   $$P = \frac{2\pi}{\omega} \implies \omega = \frac{v}{r} \implies 2\pi\sqrt{\frac{r^3}{GM}} \tag{19}$$
   We have that
   $$l = rv \implies v = \frac{GM}{l} \iff r = \frac{l^2}{GM} \tag{20}$$
   Energy
   $$\epsilon = \frac{1}{2}\dot{r}^2 - \frac{GM}{r} \tag{21}$$
   $$\iff -\frac{1}{2}v^2 = -\frac{1}{2}(\frac{GM}{l})^2 \tag{22}$$
2. Elliptical orbits $\epsilon_{central} < \epsilon_{elliptical} < 0$
   - Bound orbit with
   $$-\frac{1}{2}(\frac{GM}{l})^2 = \epsilon_c < \epsilon_e < 0 \tag{23}$$
3. Parabolic "orbit" $\epsilon = 0$

- The orbiting mass gets "flung" by the larger mass M (like the sun).
- Perigee:

$$V_p = \frac{2GM}{l} \tag{24}$$

$$r_p = \frac{l^2}{2GM} \tag{25}$$

- Apogee, $v_a \to 0$ and $r_a = \frac{l}{r_a} \to \infty$

4. Hyperbola $\epsilon > 0$

- Unbound orbit ("scattering hyperbola")

$$v_{p,a} = \frac{GM}{l} \pm \sqrt{(\frac{GM}{l})^2 + 2\epsilon} \tag{26}$$

For $v, r > 0$, hyperbola, $v < 0$ unphysical

Two body problem