MTH 416: Lecture 14

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Lecture Span

- Change of Coordinates
- 2×2 determinants

Change of coordinates

To write a vector in a different coordinate system, we multiply it by the change in coordinates matrix Q^{-1}. Where the columns represent the new basis in terms of the old one. To write a lin transformation $T : V \to V$ in a new coordinate system, we use

$$[T]'_\beta = Q^{-1}[T]_\beta Q \tag{1}$$

Definition 0.1. *Two matrices $A, B \in M_{n\times n}(\mathbb{R})$ are similar if $B = Q^{-1}AQ$ for some invertible matrix $Q \in M_{n\times n}(\mathbb{R})$. They describe the same linear transformation, but in two different coordinate systems.*

Example

Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Then we claim that A and B are similar. Namely choose

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

Note, $Q = Q^{-1}$, then we have that

$$Q^{-1}AQ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

Then the result is

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \iff B \tag{4}$$

Fact: if A and B are similar, then they have the same rank.

Reason: Rank is an intrinsic property of linear transformations, so it doesn't depend on the linear transformation.

Application of change of coordinates

Let

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \tag{5}$$

Let

$$W : \{< x, y, z >\in \mathbb{R}^3 : x + y + z = 0\} \tag{6}$$

Basis of W:

$$\{v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\} \tag{7}$$

Define

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \tag{8}$$

to be an orthogonal projection onto W, that is

$$T(v) = \text{the closest vector to } v \text{ in } W$$

Then what is $[T]_\beta$ with respect to $\beta = \{e_1, e_2, e_3\}$. We first write $[T]_{\beta'}$ as

$$\beta' = \{v_1, v_2, v_3\} \tag{9}$$

Where $v_3 = \{1, 1, 1\}$. So we calculate

$$T(v_1) = v_1 \tag{10}$$

$$T(v_2) = v_2 \tag{11}$$

$$T(v_3) = 0 \tag{12}$$

So

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

Then

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q \tag{14}$$

then

$$Q[T]_{\beta'}Q^{-1} = [T]_\beta \tag{15}$$

Where

$$Q = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \tag{16}$$

Turns out,

$$Q^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \tag{17}$$

Thus,

$$[T]_\beta = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{18}$$

Determinants

We will construct a function

$$\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R} \tag{19}$$

satisfying the following properties:

1. $\det(A) \neq 0 \iff A$ is invertible
2. $\det(AB) = \det(A)\det(B)$
3. $\det(A)$ tells how $L_A : \mathbb{R}^n \to \mathbb{R}^n$ scales volumes.
4. $\det(A)$ is <u>not</u> a linear transformation (except for $n = 1$), but is <u>multi-linear</u> (will elaborate on this later on)

Start with n = 1

$$\det : M_{1\times 1}(\mathbb{R}) \to \mathbb{R} \tag{20}$$

$$\det(a) = a \tag{21}$$

Let's check properties:

1. The 1×1 matrix is invertible $\iff a \neq 0$. Then the inverse is $\frac{1}{a}$.
2. If $A = (a)$ and $B = (b)$, then $\det(AB) = ab$
3. If $A = (a)$, then L_A is the function

$$L_A = \mathbb{R} \to \mathbb{R} \tag{22}$$

$$L_A(x) = ax \tag{23}$$

Note, L_A scales "volumes" by $|a|$. If the determinant is < 0, then L_A reverses orientation of the line.

n = 2

$$\det : M_{2\times 2} \to \mathbb{R} \tag{24}$$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \tag{25}$$

1. Claim: For any $A \in M_{2\times 2}$, $\det(A) \neq 0 \iff A$ is invertible.

Proof. ($\Longleftarrow$) Suppose A is invertible, then

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = 1 \tag{26}$$

Thus the determinant of A is non-zero.
($\Longrightarrow$) Suppose A is non-zero, then we claim that A is invertible. Claim

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{27}$$

Multiplying this out yields the identity matrix. $\square$

2. Claim: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ scales the volume by $|\det(A)| = |ad - bc|$. Namely

$$ad - bc \text{ is positive} \iff v, w \text{ are positively oriented.}$$

In other words, v to w is a counterclockwise rotation less than 180 degrees.

Proof. We will first prove the claim in 2 special cases.

(a) v is pointing along the positive x axis. Then $v = \begin{pmatrix} a \\ 0 \end{pmatrix}$, then $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ Then area is base times height, which is

$$a|d| \iff |ad| = |ad - bc| = \det(A) \tag{28}$$

(b) A is a rotation matrix. Then rotation matrices don't affect area or orientation. What is $A(\theta)$?

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{29}$$

Then $\det(A) = \cos^2 + \sin^2 = 1$

(c) Now let v, w be any basis and $A = \begin{pmatrix} v & w \end{pmatrix}$ Let $B = v', w'$, then $A = CB$ for some rotation matirx C. Then

$$\det(L_A) = \det(L_C L_B) \tag{30}$$

$$\det(L_B) \iff \det(A) \iff \det(B) \tag{31}$$

$\square$