MTH 447: Lecture 6

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Previously, we defined a sequence (x_n) and the epsilon-delta definition of a limit.

Example

Proof. Show

$$\frac{1}{n^p} \to 0 \tag{1}$$

Then

$$|\frac{1}{n^p} - 0| < \epsilon \tag{2}$$

$$\implies \frac{1}{n^p} < \epsilon \tag{3}$$

$$n > \frac{1}{\epsilon^{\frac{1}{p}}} \tag{4}$$

Then let

$$N = \lceil \epsilon^{-\frac{1}{p}} \rceil \tag{5}$$

□

Example

Proof. Prove limit of

$$x_n = \frac{3n+5}{7n-1} \tag{6}$$

Guess $L = \frac{3}{7}$. Note that

$$x_n = \frac{3 + \frac{5}{n}}{7 - \frac{1}{n}} \tag{7}$$

We can use the previous theorem to note that

$$\lim \frac{5}{n} = \lim \frac{1}{n} = 0 \tag{8}$$

But we use the limit definition, namely

$$\left| \frac{3n+5}{7n-1} - \frac{3}{7} \right| < \epsilon \tag{9}$$

Then

$$\left| \frac{21n + 35 - 21n + 3}{49n - 7} \right| \tag{10}$$

$$\frac{38}{49n-7} < \epsilon \tag{11}$$

$$\frac{1}{49}\left(\frac{38}{\epsilon} + 7\right) < n \tag{12}$$

Then

$$N = \lceil \frac{1}{49}\left(\frac{38}{\epsilon} + 7\right) \rceil \tag{13}$$

□

Example

Proof. Show limit of

$$x_n = \frac{\sin(3n)}{n} \tag{14}$$

Claim $L = 0$, then

$$|\frac{\sin(3n)}{n} - 0| < \epsilon \tag{15}$$

Then

$$\frac{|sin(3n)|}{n} < \epsilon \tag{16}$$

Then

$$\frac{|sin(3n)|}{n} \leq \frac{1}{n} \tag{17}$$

Then

$$\frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon} \tag{18}$$

IF $n > \frac{1}{\epsilon}$, then

$$\frac{1}{n} < \epsilon \tag{19}$$

Then

$$\frac{|\sin(3n)|}{n} < \epsilon \tag{20}$$

$$= |x_n - 0| < \epsilon \tag{21}$$

□

Example

Proof.

$$x_n = \frac{2n+4}{3n^4 - 2n + 7} \tag{22}$$

$$\left|\frac{2n+4}{3n^4 - 2n + 7}\right| < \epsilon \tag{23}$$

We replace

$$2n + 4 \leq 3n \tag{24}$$

$$\iff 4 \leq n \tag{25}$$

and

$$3n^4 - 2n + 7 \geq 2n^4 \tag{26}$$

$$n^4 \geq 2n - 4 \tag{27}$$

$$n^4 \geq 2n \tag{28}$$

$$n^3 \geq 2 \tag{29}$$

$$n \geq 2 \tag{30}$$

Then if $n > \max(2, 4)$, then

$$\frac{2n+4}{3n^4 - 2n + 7} \leq \frac{3n}{2n^4} \leq \frac{3}{2}n^{-3} \tag{31}$$

Then

$$\frac{3}{2}n^{-3} < \epsilon \tag{32}$$

Then

$$n > \left(\frac{3}{2\epsilon}\right)^{\frac{1}{3}} \tag{33}$$

Then

$$N = \max(\lceil \left(\frac{3}{2\epsilon}\right)^{\frac{1}{3}} \rceil, 4) \tag{34}$$

□