Archimedean Property and Sup/Inf

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Recall, suppose that $A \subseteq \mathbb{R}$ (A cannot be $\emptyset$), then

$$\sup(A) = \text{ least upper bound of A} \tag{1}$$

$$\inf(A) = \text{ greatest upper bound of A} \tag{2}$$

Claim, $\inf(\mathbb{R}^+) = 0$

Proof. We must prove that

1. $\forall x \in \mathbb{R}^+, 0 \leq x$
2. $\forall b \in \mathbb{R}$ such that b is a lower bound of $\mathbb{R}$, $b \leq 0$

To prove (2), suppose that $b \in \mathbb{R}^+$, we must prove that b is not a lower bound of $\mathbb{R}^+$. To prove this, note that $\frac{b}{2}$ is less than b, thus b cannot be a lower bound of $\mathbb{R}^+$. This is because $b > 0$. □

Claim: Let $A \subseteq \mathbb{R}$, then

1. If A has a smallest element x, then $x = \inf(A)$
2. If A has a largest element y, then $y = \sup(A)$

Proof. To prove (1), suppose that x is the smallest element of A. Then x is a lower bound of A since $x \leq a \forall a \in A$. Moreover, if b is a lower bound of A, then $b \leq a \forall a \in A$. Then in particular, $b \leq x$. Thus x is the greatest lower bound of A, that is $x = \inf(A)$. □

Theorem 0.1. $\mathbb{N}$ *is not bounded above.*

Proof. Suppose for the sake of contradiction that $\mathbb{N}$ is bounded above. In particular, by the least upper bound property, $\mathbb{N}$ must have a least upper bound. Then, we define n is a least upper bound of $\mathbb{N}$. Then we define $x > n - 1$ such that $x \in \mathbb{N}$, but it follows that $x + 1 \in \mathbb{N}$ by definition of the natural numbers. Thus, it follows that $\mathbb{N}$ doesn't have a upper bound. □

Corollary 0.2. *Let* $A = \{\frac{1}{n} : n \in \mathbb{N}\}$*, then* $\inf(A) = 0$.

Proof. We first must prove that 0 is a lower bound, and that 0 is also the greatest lower bound. To prove the first claim, we have that trivially that $0 \leq \frac{1}{n} \forall n \in \mathbb{N}$. Secondly, suppose that 0 is not the greatest lower bound for the sake of contradiction. Then there exists some $b \in \mathbb{N}$ such that $b \leq \frac{1}{n}$. Flipping both sides yields $\frac{1}{b} \geq n$, but this implies that the natural numbers is bounded. This is a contradiction. □

Theorem 0.3. *The Archimedean property states that for any* $x \in \mathbb{R}^+$ *and* $y \in \mathbb{R}$*, there exists some* $n \in \mathbb{N}$ *such that*

$$nx > y \tag{3}$$

Theorem 0.4. $\mathbb{Q}$ *is dense in* $\mathbb{R}$ *for all real numbers* $x < y$*, there exists some rational number* r *such that*

$$x < r < y \tag{4}$$

Proof. To prove the Archimedean property, given $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$, we want to find $n \in \mathbb{N}$ such that $nx > y$. Equivalently, $n > \frac{y}{x}$. We can find such a number since the natural numbers is not bounded. $\square$

Proof. To prove the theorem that follows from the Archimedean Property, we first assume that $0 \leq x < y$. Since $x < y$, we have that $y - x > 0$, in particular, $y - x > \frac{1}{n}$. Fix n, we claim that for some $m \in \mathbb{N}$, $x < \frac{m}{n} < y$. Let

$$A = \{k \in \mathbb{N} : \frac{k}{n} > x\} \tag{5}$$

Claim that A is nonempty, if it were empty, then that means that $\frac{k}{n} \leq x$ for all $k \in \mathbb{N}$, but that means that the natural numbers is bounded, thus this set must be nonempty. Since $\mathbb{N}$ is well-ordered, it follows that A contains a least element. We must now prove that $x < \frac{m}{n} < y$. By definition, $\frac{m}{n} > x$. But since $m - 1 \notin x$, thus $\frac{m-1}{n} < x$, or $\frac{m}{n} \leq x + \frac{1}{n} < y$. Thus, this proves the inequality. $\square$

Definition 0.5. *If $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then let*

$$xA = \{xa : a \in A\} \tag{6}$$

$$x + A = \{x + a : a \in A\} \tag{7}$$

Then

Theorem 0.6. *Suppose that $A \subseteq \mathbb{R}$ and that x is a real number. Assuming that $\sup(A)$ exists, then*

1. $\sup(x + A) = \sup(A) + x$
2. $\inf(x + A) = \inf(A) + x$
3. $\sup(xA) = x\sup(A)$ *if $x > 0$*
4. *etc.*

Proof. To prove (1), for any b in the real numbers, we have that b is a upper bound of A means that

$$b \geq a \forall a \in A \tag{8}$$

$$b + x \geq a + x \tag{9}$$

Thus $b + x$ is a upper bound of A. $\square$