MTH 447: Lecture # 15

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Series

Given a sequence (a_n), then

$$s_n = \sum_{k=1}^{n} a_k \tag{1}$$

Then

$$\lim_{n\to\infty} s_n = ? \tag{2}$$

If s_n is increasing

$$s_{n+1} - s_n = a_{n+1} \geq 0 \tag{3}$$

If this sum is bounded, then $s_n \to S$. If unbounded, then $\to \infty$.

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \tag{4}$$

Definition 0.1. *We say that the sum $\sum_{n=1}^{\infty} a_n$ satisfies the Cauchy criterion if $\forall \epsilon > 0$, there exists N s.t. if $n, m > N$, then*

$$\left| \sum_{k=m+1}^{n} a_k \right| < \epsilon \tag{5}$$

Theorem 0.2. *$\sum a_n$ convergs $\iff$ it satisfies the Cauchy Criterion.*

Proof.

$$S_n = \sum_{k=1}^{n} a_n \tag{6}$$

Then

$$S_n - S_m = \left| \sum_{k=m+1}^{n} a_k \right| < \epsilon \text{ (because this sum converges)} \tag{7}$$

This is saying that a sequence of partials sums is cauchy. □

Corollary 0.3. *If $\sum a_n$ converges, then $a_n \to 0$*

Proof. This follows from the Cauchy Criterion. Choose $n = m + 1$, then

$$|a_{m+1}| < \epsilon \tag{8}$$

□

What about the converse? This is false.

Theorem 0.4. *If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, and if $|b_n| \leq a_n$, then $\sum b_n$ converges.*

Theorem 0.5. *If $a_n \leq b_n$ and $\sum a_n = \infty$, then $\sum b_n = \infty$.*

Proof.

$$\left|\sum_{k=m+1}^{n} b_k\right| \leq \sum_{k=m+1}^{n} |b_n| \leq \sum_{k=m+1}^{n} a_k \tag{9}$$

If a_n satifies the Cauchy Criterion, then so does b_n. Thus, b_n converges. □

Definition 0.6. *We say that a series $\sum a_k$ is absolutely convergent if*

$$\sum_{k=1}^{\infty} |a_n| \text{ is convergent} \tag{10}$$

Corollary 0.7. *If $\sum a_k$ is absolutely convergent, then this sum converges.*