Convergence of Sequences of Functions

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Recap, we first have pointwise convergence which operates off the epsilon delta defintion. Next is uniform convergence which operates off the sup norm notation. Now, we define something called the L^2 convergence.

Definition 0.1. *Let $I \in \mathbb{R}$ be an interval, and let $f(x)$ and $g(x)$ be functions. Then we define L^2 distance to be the following:*

$$|f-g|_2 = [\int_I |f(x)-g(x)|^2 dx]^{1/2} \tag{1}$$

Definition 0.2. *Let $I \in \mathbb{R}$ be an interval. Let $f(x)$ be a function on I and $f_m(x)$ be a sequence of functions on I. Then we say that $f_m(x)$ converges to $f(x)$ on I in L^2 if*

$$\lim_{m\to\infty} |f_m - f|_2 = 0 \tag{2}$$

Where this subscript 2 represents the L^2 distance between the functions.

But f_m can converge to multiple functions because the integral would get rid of discontinuities. So we define a notation $\sim$ such that

$$f \sim g \iff f(x) = g(x) \text{ almost everywhere} \tag{3}$$

But L^2 convergence states that two different continuous functions cannot converge to each other. But a discontinous and a continuous function can converge to each other.

One thing to note, uniform convergence implies pointwise convergence.

We apply this to the Full Fourier Series:

Theorem 0.3. *Fix $l > 0$. Let $\Psi(x)$ be a function on $\mathbb{R}$ with a period of $2l$. That is $\Psi(x+2l) = \Psi(x)$ for all x in $\mathbb{R}$. Then the full fourier series is the following:*

$$\Psi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n \cos(\frac{n\pi x}{l}) + B_n \sin(\frac{n\pi x}{l})] \tag{4}$$

Where

$$A_n = \frac{1}{l}\int_{-l}^{l} \Psi(x)\cos(\frac{n\pi x}{l})dx \tag{5}$$

and

$$B_n = \frac{1}{l}\int_{-l}^{l} \Psi(x)\sin(\frac{n\pi x}{l})dx \tag{6}$$

Then we can define convergence in the context of the Fourier Series as the following:

Definition 0.4.

1. *If $\Psi(x)$ is differentiable at x_0, then the Fourier Series converges pointwisely.*
2. *If $\Psi(x)$ is continuously differentiable, the the Fourier Series converges uniformly on $\mathbb{R}$*
3. *If $\int_{-l}^{l} |\Psi(x)|^2 dx < \infty$, then the full fourier series converges L^2 on $(-l, l)$*