MTH 553: Lecture # 10

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Lecture Span

- Continue working on non-homogenous wave equation with arbitrary boundary conditions

Recall that we reduced the wave equation to

$$\begin{aligned} w_{tt} - c^2 w_{xx} &= 0 \\ w(x,0) &= g_2(x) \\ w_t(x,0) &= h_2(x) \\ aw(0,t) - bw_x(0,t) &= 0 \\ c^* w(L,t) + dw_x(L,t) &= 0 \end{aligned}$$

Assume $g_2 = 0$ which corresponds to zero initial displacement, then

$$w(x,t) = \sum_n (a_n \sin(\lambda_n x) + b_n \cos(\lambda_n x)) \cdot \sin(\lambda_n ct) \tag{1}$$

Note that given the w above, we can verify that:

$$w_{tt} = c^2 w_{xx} \tag{2}$$

The, we can indeed check that:

$$w(x,0) = 0 \tag{3}$$

So now, the question is what λ_n do we choose? We can determine λ_n from the boundary conditions. First, assume that $b = d = 0$, then define

$$\phi_n(x) = a_n \sin(\lambda_n x) + b_n \cos(\lambda_n x) \tag{4}$$

Then we have the dirichlet boundary conditions:

$$\begin{aligned} \phi_n(0) &= 0 \\ \phi_n(L) &= 0 \end{aligned}$$

The first boundary condition requires that $b_n = 0$, and the second one requires $\sin(\lambda_n L) = 0$, therefore,

$$\lambda_n = \frac{n\pi}{L} \tag{5}$$

Then, we have that

$$w(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) \tag{6}$$

If $a = c^* = 0$, then we get the neumann boundary conditions,

$$\partial_x \phi_n(0) = 0$$
$$\partial_x \phi_n(L) = 0$$

Then $a_n = 0$ and we obtain that $\sin(\lambda_n L) = 0$ and

$$\lambda_n = \frac{n\pi}{L} \tag{7}$$

Therefore,

$$w(x,t) = \sum_{n=0}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) \tag{8}$$

Now, we go back to the Dirichlet example, we can determine a_n using the initial conditions. Here, $g_2 \equiv 0$, $\lambda = n\pi/L$, and $b_n = 0$ with

$$w(x,t) = \sum_n a_n \sin(\lambda_n x)\sin(\lambda_n ct) \tag{9}$$

Then we impose IC,

$$h_2(x) = w_t(x,0) = \sum_n (\lambda_n c a_n)\sin(\lambda_n x) \tag{10}$$

To find a_n, we multiply by $\sin(\lambda_m x)$ and integrate (i.e. take L^2 inner product with the basis elements). Then we obtain,

$$\int_0^L h_2(x)\sin(\lambda_m x)\,dx = \sum_{n=0}^{\infty} (\lambda_n c a_n) \int_0^L \sin(\lambda_n x)\sin(\lambda_m x)\,dx \tag{11}$$

But

$$\int_0^L \sin(\lambda_n x)\sin(\lambda_m x)\,dx = \frac{L}{2}\delta_{nm} \tag{12}$$

Therefore, we obtain

$$\int_0^L h_2(x)\sin(\lambda_m x)\,dx = \lambda_m c a_m \frac{L}{2} \tag{13}$$

Therefore,

$$a_m = \frac{2}{m\pi c}\int_0^L h_2(x)\sin(\lambda_m x)\,dx \tag{14}$$

Step 3b - solve 2b by characteristics

Recall that

$$w(x,t) = F(x+ct) + G(x-ct) \tag{15}$$

On a characteristic parallelogram on the x,t plane with points A,B,C,D, here, B connects to A and C connects to D with the slope $x - ct = d$ where d is a constant. Similarly, B connects to C and A connects to D with a slope of $x + ct = e$ where e is a constant. Then consider the wave equation at point A and C, we calculate

$$\begin{aligned} w(A) + w(C) &= F(A) + G(A) + F(C) + F(D) \\ &= F(D) + G(B) + F(B) + G(D) \\ &= W(B) + W(D) \end{aligned}$$

Therefore,

$$w(A) + w(C) = w(B) + w(D) \tag{16}$$

Find w in triangle R_1 (which extends back to the initial condition) using D'Alembert on interval $[0, L]$. Then assume there is a point A that we don't know, then we can pick 3 other points that form a parallelogram on the x,t plane. Then, we obtain

$$w(A) = w(B) + w(D) - w(C) \tag{17}$$

Where B rests on the boundary, then $w(B) = 0$.