Math 442 Exam #1 Review

Cliff Sun

February 27, 2024

Topics:

1. What is a PDE?
2. What is a degree of a PDE?
3. Whether a PDE is linear or not
4. First order linear equation - Method of characteristics, change of variables
5. Homogenous and non-homogenous Wave equation
6. Heat Equation - maximum principle, diffusion on the half line

Notes

First order linear pde:

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = 0 \tag{1}$$

Wave equation:

$$u_{tt} - c^2 u_{xx} = f(x,t) \tag{2}$$

$$u(x,0) = \phi(x) \tag{3}$$

$$u_t(x,0) = \Psi(x) \tag{4}$$

The general solution is:

$$u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(s)ds \tag{5}$$

The heat equation is:

$$u_t(x,t) - ku_{xx}(x,t) = f(x,t) \text{for} (x,t) \in \mathbb{R} \times (0,\infty) \tag{6}$$

$$u(x,0) = \psi(x) \tag{7}$$

The general solution is:

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\psi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds \tag{8}$$

Where

$$S(x,y) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \tag{9}$$

The error function is the following:

$$Erf(x) = \frac{2}{\pi} \int_0^x e^{-p^2} dp \tag{10}$$

Example of linearity within the heat equation:

$$u_t - ku_{xx} = \cos(x) \tag{11}$$

$$u(x, 0) = \sin(x) \tag{12}$$

We know that the solution to the homogenous heat equation with the same initial condition is just:

$$v(x, t) = e^{-kt} \sin(x) \tag{13}$$

Thus applying linearity, we have that:

$$u(x, t) - v(x, t) = w(x, t) \tag{14}$$

Applying this yields:

$$w_t - kw_{xx} = \cos(x) \tag{15}$$

Choosing $w_1(x, t)$ to be $\frac{1}{k} \cos(x)$, we see that now:

$$w_2 = w - w_1 \tag{16}$$

Then, we have that the new heat equation becomes homogenous with an intial condition of $-\frac{1}{k} \cos(x)$. Now applying linearity, we can rederive the full solution.
The maximum principle states that

$$\text{Max}_\gamma(u(x, t)) = \text{Max}_\mathbb{R}(u(x, t)) \tag{17}$$

Such that $\gamma \in \mathbb{R}$ and that γ is the boundary of $\mathbb{R}$.