6.1 & 6.2: Cartesian Product and Power sets

Cliff Sun

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Previously, we showed that induction implies the well ordering of the natural numbers.

Remark: Strong induction is equivalent to this case as well. That is, the validity of Strong induction implies the well-ordering of the natural numbers.

Cartesian Products

Theorem 0.1. *Let A and B be sets. Then their Cartesian product is*

$$A \times B = \{(a, b), a \in A, b \in B\} \tag{1}$$

You can also take the product of multiple sets A, B, C where the definition would be:

$$A \times B \times C = \{(a, b, c), a \in \mathbb{A}, b \in \mathbb{B}, c \in \mathbb{C}\} \tag{2}$$

Remark: $A \times B$ is not equal to $B \times A$ since the group $(a, b) \neq (b, a)$

That means that

$$(A \times B) \times C \neq A \times (B \times C) \neq A \times B \times C \tag{3}$$

However, there are bijections between them such that

$$f : A \times B \to B \times A \tag{4}$$

or

$$(b, a) \to (a, b) \tag{5}$$

Generally speaking the cardinality of $A \times B$ is the same as multiplying the cardinality of A and B.

Proof. Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$, then $A \times B$ can be written as:
$\{(a_1, b_1), (a_1, b_2), \dots, (a_1, b_n) \dots (a_m, b_1)\}$.
Since there are a total of $m \times n$ number of elements, it follows that the cardinality of $A \times B$ is just $|A||B|$. □

Theorem 0.2. *If $A \subseteq C$ and and $B \subseteq D$, then it follows that $A \times B \subseteq C \times D$*

Proof. Suppose that $A \subseteq C$ and $B \subseteq D$. We claim that $A \times B \subseteq C \times D$. To prove this, consider an element of $A \times B$. Then this element would be an ordered pair. This has the form of (a, b) for a in A and b in B. Since $A \subseteq C$, then a is an element in C. Similarly, we can apply the same argument for b. Therefore, (a, b) is an ordered pair where the first entry is an element in C and the second entry is in D. Thus, by definition, (a, b) is an element in $C \times D$. This concludes the proof. □

What is the compliment of $A \times B$ in the context of $C \times D$?

Suppose that $(x, y) \in C \times D$, and $(x, y) \not\in A \times B$. But that implies that $x \in C - A$ and $y \in D - B$. Then that implies that

$$(x, y) \in (C - A) \times D \cup (C) \times (D - B) \tag{6}$$

Power Sets

Theorem 0.3. *Let A be a set. Then the power set of A is*

$$P(A) = \{B : B \subseteq A\} \tag{7}$$

Such that

$$B \subseteq P(A) \iff B \subseteq A \tag{8}$$

This can include vectors consisting of multiple elements.

Theorem 0.4. *If —A— is n, then the cardinality of $P(A) = 2^n$*

Proof. We will use induction on n. As a base case, $|A| = 0$, thus the power set of A has cardinality 1. For the inductive step, suppose the theorem is true for all sets of cardinality n, and let $|A| = n + 1$, then B = set of all elements in A until a_n and C = just the last element are subsets of A. $\square$