MTH 447: Lecture #18

Cliff Sun

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Functions

Definition 0.1. *A function is defined as*

$$\mathbb{R} \to \mathbb{R} \tag{1}$$

Definition 0.2. *Let $F : A \to B$. Let $x_0 \in A$. We say that $f(x)$ is continuous at x_0 if for any sequence (x_n), $x_n \in A$ and $\forall n$ and*

$$\lim_{n\to\infty} x_n = x_0 \implies \lim_{n\to\infty} f(x_n) = f(x_0) \tag{2}$$

This definition can be modified to use the $\epsilon - \delta$ definition.

Definition 0.3. *If F is continuous $\forall x_0 \in S \subseteq A$, then F is continuous on S.*

Theorem 0.4. *$f : [a, b] \to \mathbb{R}$, f is cont. Then*

1. *f is bounded*
2. *f attains its maximum & minimum. That is $\exists x_{max}, x_{min} \in [a, b]$ s.t. $\forall x \in [a, b]$*

$$f_{x_{min}} \leq f(x) \leq f(x_{max}) \tag{3}$$

Proof. (1). Let f be continuous, $f : [a, b] \to \mathbb{R}$. And assume $\{f(x)\}$ is unbounded above. This means for any n, $\exists x$ such that $f(x) > n$. Then $\lim f(x_n) = \infty$. But (x_n) is bounded, in particular, between $[a, b]$. Then x_{n_k} is a convergent subsequence. Then $x_{n_k} \to x_0$. Then $f(x_{n_k}) \to \infty$. This is a contradiction. This concludes the proof. □

Proof. (2). Let $M = \sup(f(x))$. This exists and is finite. Then for all n, there exists y_n such that

$$M - \frac{1}{n} \leq f(y_n) \leq M \tag{4}$$

Note, $y_{n_k} \to y_0$. □