PHYS 326: Lecture #16

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Euler Angles

Euler Angles describe any type of orientation for any object. Specify 3 principle axis of the body, called $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$. Use lab frame $(\hat{x}, \hat{y}, \hat{z})$. Assume that both coordinate systems are initially aligned.

1. Then rotate away from the lab frame by angle ϕ around $\hat{z}$.
2. Rotation by angle θ around $\hat{e}_2$. Then $(\hat{e}_1, \hat{e}_2, \hat{e}_3) \to (\hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$
3. Rotation by angle Ψ around $\hat{e}'_3$. Then $(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3) \to (\hat{e}''_1, \hat{e}''_2, \hat{e}''_3)$

Interpretation

1. ϕ = precession around lab vertical
2. θ = nutation away from lab vertical
3. Ψ = spin around body axis

Write Q matrix corresponding to this "Euler Procedure".

$$Q = Q_3 Q_2 Q_1 \tag{1}$$

$$= \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Express $\vec{\omega}$ vector (instantaneous vector) in terms of Euler Angles.

$$\vec{\omega} = \dot{\phi}\hat{e}_3 + \dot{\theta}\hat{e}'_2 + \dot{\Psi}\hat{e}''_3 \tag{2}$$

Will convert this vector into the double prime coordinate system. Note

$$\hat{e}''_2 = Q_3 \hat{e}'_2 \iff \hat{e}'_2 = Q_3^T \hat{e}''_2 \tag{3}$$

and

$$\hat{e}'_3 = Q_2 \underbrace{Q_1}_{I} \hat{e}_3 \iff \hat{e}_3 = Q_2^T \hat{e}'_3 = Q_3^T Q_2^T \hat{e}''_3 \tag{4}$$

Then

$$\vec{\omega} = (\dot{\theta}\sin\Psi - \dot{\phi}\sin\theta\cos\Psi)\hat{e}''_1 + (\dot{\phi}\sin\theta\sin\Psi + \dot{\theta}\cos\Psi)\hat{e}''_2 + (\dot{\Psi} + \dot{\phi}\cos\theta)\hat{e}''_3 \tag{5}$$