MTH 416: Lecture 17

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October 24, 2024

Lecture Span

- Properties of determinants
- Elementary matrices

Theorem 0.1. *Let E be an $n \times n$ elementary matrix, given by applying some elementary row operation to the identity matrix. Then for any $A \in M_{n\times n}$, applying the same row operation to A produces EA.*

Corollary 0.2. *Every elementary matrix E is invertible, and the inverse are also elementary matrices.*

Proof. Recall that every row operation can be undone by another row operation. In particular, if E is an elementary matrix, then the matrix is the product of doing some elementary operation to the identity matrix. Therefore, we can build another elementary matrix E' that corresponds to the inverse row operation. We calculate that

$$E'(EA) = A = E(E'A) \tag{1}$$

Letting $A = I_n$, we have that

$$E'E = I_n = EE' \tag{2}$$

This concludes the proof. □

In general, an inverse matrix encodes the different types of row operations needed in order to bring A down to I_n.

Theorem 0.3. *An $n \times n$ matrix A is invertible $\iff$ it is a product of some number of elementary matrices.*

Proof. ($\Longleftarrow$) Suppose $A = E_1, \dots, E_n$ where E_n is some invertible matrix. Then we claim that A^{-1} is the reversed order of inverses for this matrix. We calculate

$$E_n^{-1} \dots E_1^{-1} E_1 \dots E_n = I_n \tag{3}$$

($\Longrightarrow$) Suppose A is invertible, then it row reduces to the identity matrix. Then we construct some A^{-1} composed of elementary matrices multiplied together corresponding to the row operations needed to reduce A down to I_n. Then

$$E_n \dots E_1 A = I_n \tag{4}$$

Then

$$A = E_1^{-1} \dots E_n^{-1} I_n \tag{5}$$

So A is a product of elementary matrices. □

Theorem 0.4. *For any $A, B \in M_{n\times n}$, then*

$$\det(AB) = \det(A)\det(B) \tag{6}$$

Proof. First assume A is an elementary matrix, then

$$\det(A) = c \cdot \det(I_n) \tag{7}$$

Where $c \neq 0$, that depends on the type of row operation being performed. We also have

$$\det(AB) = c \cdot \det(B) \tag{8}$$

Because this same row operation is being performed to B. Thus,

$$\det(AB) = c \cdot \det(B) = \det(A)\det(B) \tag{9}$$

In general, there are 2 cases:

1. A is an invertible matrix, then $A = E_1 \dots E_k$ for some elementary matrices E_i. Then

$$\det(AB) = \det(E_1 \dots E_k B) = \det(E_1) \cdot (E_2 \dots E_k B) \tag{10}$$

   We can perform this recursively,

$$= \det(A)\det(B) \tag{11}$$

2. If A isn't invertible, then $\det(A) = 0$, so we want to prove that $\det(AB) = 0$. In other words, AB is non invertible either. Since A isn't invertible, then L_A is not an isomorphism. Then L_A is neither one-to-one or onto. Then L_{AB} is $L_A \cdot L_B$. Then this isn't onto, because L_B is both one-to-one and onto, thus all $\mathbb{R}^n$ vectors are mapped to $\mathbb{R}^n$ vectors. But these same vectors aren't mapped onto $\mathbb{R}^n$ by L_A, thus this isn't one-to-one or onto due to this bottleneck. Therefore, AB is not invertible. Thus its determinant is 0.

□

Theorem 0.5. *For any $A \in M_{n\times n}$, then*

$$\det(A) = \det(A^T) \tag{12}$$

Why do we care?: Collectively, we've proved the following statements:

1. Cofactor expansion on the r-th row
2. $\det(A)$ and row-reduction
3. det is multilinear as a function of one row

The theorem implies that this all works with rows and columns switched.

Lemma 0.6. *If $A \in M_{m\times n}$, and $B \in M_{n\times p}$, then*

$$(AB)^T = B^T A^T \tag{13}$$

Note, in general, you can't multiply $A^T B^T$.

Proof. Note that $(AB)^T$ and $B^T A^T$ are both $p \times m$ matrices. We first see that

$$(AB)^T_{ki} = AB_{ik} \iff \sum_{j=1}^{n} A_{ij} B_{jk} \tag{14}$$

But

$$(B^T A^T)_{ki} = \sum_{j=1}^{n} (B^T)_{kj} A^T_{ji} \iff \sum_{j=1}^{n} B_{jk} A_{ij} \tag{15}$$

Which is equal to equation 14. Thus this concludes the proof. □

We now prove that $\det(A^T) = \det(A)$.

Proof. We first assume that A is an elementary matrix, then $\det(A) = \det(A^T)$ by HW 8 (bruh). In general, there are 2 cases:

1. A is invertible $\iff$ A is the product of elementary matrices. Then

$$\det(A^T) = \det(E_n^{T^{-1}} \dots E_1^{-1^T}) \iff \det(E_n)\dots\det(E_1) \tag{16}$$

2. Suppose A is non invertible, then $\det(A)$ is 0. Then we prove that A^T is also not invertible. We prove two cases of this

Proof. If A is non-invertible, then $rank(A) < n$. This means that dim(column space) $= rank(A)$. And taking the transpose of the pivots still comes up with white space, thus A^T is non-invertible. □

3. We can also prove this with contrapositive, that is suppose A^T is invertible, then A is also invertible.

□

Theorem 0.7. *For any $A \in M_{n\times n}$, the linear transformation L_A scales volumes by a factor of $|\det(A)|$. And L_A reverses orientation $\iff \det(A) < 0$.*

What if A is an elementary matrix?