MTH 447: Lecture 8

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Theorem 0.1. *If $x_n \to L$ and $y_n \to M$, then $x_n y_n \to LM$, then*

$$\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n) \tag{1}$$

Proof.

$$|x_n y_n - LM| \tag{2}$$
$$= |x_n y_n - x_n M + x_n M - LM| \tag{3}$$
$$= |x_n(y_n - M) + M(x_n - L)| \tag{4}$$
$$\leq |x_n(y_n - M)| + |M(x_n - L)| \tag{5}$$
$$= |x_n||(y_n - M)| + |M||x_n - L| \tag{6}$$

Note that $|x_n|$ is bounded, namely $|x_n| < Q$, then $\forall n$

$$< Q|y_n - M| + |M||x_n - L| \tag{7}$$

Then $\forall \epsilon > 0$, $\exists N_1$ such that

$$|x_n - L| < \frac{\epsilon}{2|M|} \tag{8}$$

Similarly for $y_n - L$. then

$$|x_n y_n - L| < \epsilon \tag{9}$$

□

Theorem 0.2. *Suppose $x_n \to L$, $y_n \to L$, and $\exists N$ where $\forall n \in N$,*

$$x_n \leq z_n \leq y_n \tag{10}$$

then $z_n \to L$

Proof. We want to prove

$$|z_n - L| < \epsilon \tag{11}$$

□

Theorem 0.3. *$x_n \to L$ and $y_n \to M$, $M \neq 0$, then*

$$\frac{x_n}{y_n} \to \frac{L}{M} \tag{12}$$

Lemma 0.4. *if $y_n \to M$, $M \neq 0$, then $\frac{1}{y_n} \to M$*

Proof.

$$|\frac{1}{y_n} - \frac{1}{M}| < \epsilon \tag{13}$$
$$\frac{|M - y_n|}{|y_n M|} \tag{14}$$
$$= |y_n - M|\frac{1}{|M|}\frac{1}{|y_n|} \tag{15}$$

□

Lemma 0.5. *If $y_n \to M$, $M \neq 0$, then $\exists N$ such that $\frac{1}{|y_n|}$ is bounded.*