

MTH 553: Lecture # 4

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Last time

- Inviscid Burger's Equation

$$uu_x + u_y = 0 \quad (1)$$

$$u(x, 0) = k(x) \quad (2)$$

Lecture Span

1. Method of general solutions

Method of general solutions

Objective: Find two expressions of x, y that are constant along each characteristic. Consider an arbitrary function of these two expressions. Find Γ to determine the solution function.

Example

$$uu_x + yu_y = x \quad (3)$$

$$u(x, 1) = 2x \quad (4)$$

Here, the characteristic system is

$$dx/dt = z \quad (5)$$

$$dy/dt = y \quad (6)$$

$$dz/dt = x \quad (7)$$

Here, we can find

$$\frac{dz}{dx} = \frac{x}{z} \iff z dz = x dx \iff \frac{1}{2} z^2 = \frac{1}{2} x^2 + C_1(s) \quad (8)$$

Here, we can then define a constant function (dependent on where on Γ you start on)

$$\phi(x, y, z) = z^2 - x^2 \quad (9)$$

Lets try adding (5) + (7), then we obtain

$$\frac{d(x+z)}{dt} = (x+z) \quad (10)$$

Then consider

$$\frac{d(x+z)}{dy} = \frac{x+z}{y} \iff \frac{dy}{y} = \frac{d(x+z)}{x+z} \quad (11)$$

$$\iff x + z = C_2(s)y \quad (12)$$

We can then define another constant function (dependent on starting point on Γ) as

$$\psi(x, y, z) = \frac{x + z}{y} \quad (13)$$

Then a general solution is

$$F(u^2 - x^2, \frac{x + u}{y}) = 0 \quad (14)$$

Where F depicts a relationship between these values that originates on Γ .

We can determine F from Γ , that is $u(x, 1) = 2x$. Let $u = 2x$, then we obtain

$$F(3x^2, 3x) = 0 \quad (15)$$

Satisfied by $F(a, b) = a - 1/3b^2$, therefore, we obtain

$$u - x^2 - \frac{1}{3} \left(\frac{x + u}{y} \right)^2 = 0 \quad (16)$$

Solving for u , we obtain

$$u = \frac{x \pm 3xy^2}{3y^2 - 1} \quad (17)$$

To satisfy the initial condition, we choose $+$. Therefore, the final solution is

$$u = \frac{x + 3xy^2}{3y^2 - 1} \quad (18)$$

The solution exists for $y > 1/\sqrt{3}$ and $\forall x$. **NOTE:** There is no uniqueness theory (yet) for non-linear equations.

Weak Solutions

Jump condition

This determines where "the shock goes". Given some smooth function $G(z)$, then define

$$G(u)_x + u_y = 0 \quad (19)$$

Similarly,

$$G'(u)u_x + u_y = 0 \quad (20)$$

For the conservation law, consider

$$G(z) = \frac{1}{2}z^2 \implies G'(z) = z \quad (21)$$

Then this is the burger's equation. We call u a weak solution of (19) if it satisfies the x-integrated version of (19). That is, u satisfies:

$$G(u(b, y)) - G(u(a, y)) + \frac{d}{dy} \int_a^b u(x, y) dx = 0 \quad (22)$$

for all $a < b$ and all y . But what's so cool about this?

You can have discontinuities w.r.t x

If u is a smooth solution that satisfies (19), then it satisfies (22), by using the fundamental theorem of Calculus. But the converse is false, because u can have discontinuities and still be a weak solution.

Example

Specifically, suppose $x = \xi(y)$ as our shock curve, which defines where this discontinuity occur. Let $u = u_l(y)$ be to the left of the shock curve and $u = u_r(y)$ be similarly defined for the right. Now, suppose $u(x, y)$ jumps across a C^1 -smooth curve $x = \xi(y)$. But everywhere else, u is a smooth solution. Then suppose u solves the conservation law to the left and right of $x = \xi(y)$. Then, we plug this equation into (22).

$$G(u(b, y)) - G(u(a, y)) + \frac{d}{dy} \left[\int_a^{\xi(y)} u_l(x, y) dx + \int_{\xi(y)}^b u_r(x, y) dx \right] \quad (23)$$

$\forall a, b$ with $a < \xi(y) < b$ for all y . Need not check intervals of $[a, b]$ that don't contain $\xi(y)$.

$$\iff 0 = G(u(b, y)) - G(u(a, y)) + \int_a^{\xi(y)} u_y(x, y) dx + \int_{\xi(y)}^b u_y(x, y) dx + \xi'(y)u_l - \xi'(y)u_r \quad (24)$$

By fundamental theorem of Calculus and chain rule. Where

$$u_l = \lim_{x \rightarrow \xi^-} u$$

and u_r defined similarly. Moreover, $u_y = -G(u)_x$

$$\iff 0 = -G(u_l) + G(u_r) + \xi'(y)(u_l - u_r) \quad (25)$$

By evaluating the integrals. Then we obtain

$$\xi'(y) = \frac{G(u_l) - G(u_r)}{u_l - u_r} \quad (26)$$