Lim sup and Lim Inf; Cauchy Sequence

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Definition 0.1. *Let (x_n) be a valid sequence:*

1. $a_n = \sup\{x_n, x_{n+1}, \cdots\}$
2. $b_n = \inf\{x_n, x_{n+1}, \cdots\}$
3. $\limsup x_n = \lim a_n$
4. $\liminf x_n = \lim b_n$

An example would be $x_n = \frac{1}{n}$. Its $a_n = x_n$ because it is a monotone decreasing sequence and $b_n = 0$. So we have that:

$$\limsup x_n = 0 \tag{1}$$

$$\liminf x_n = 0 \tag{2}$$

General facts about $\limsup$ and $\liminf$

Suppose that (x_n) is a bounded sequence.

1. (a_n) is decreasing, (b_n) is increasing, and both converge.
2. (x_n) converges to some number x, if and only if the $\limsup = \liminf = x$
3. There exists subsequences (x_{n_i}) and (x_{m_i}) that converge to the $\limsup$ and $\liminf$ respectively.
4. $\limsup x_n$ and $\liminf x_n$ are the largest and smallest limits of subsequences.

Proof. This a proof of statement (1). To prove that (a_n) is decreasing, recall that

$$a_n = \sup\{x_n, x_{n+1}, \cdots\} \tag{3}$$

In particular, a_n is an upper bound. Then consider a_{n+1}, thus we have that all possible candidates for a_{n+1} are also candidates for a_n. Thus, we have that $a_n \geq a_{n+1}$. Thus (a_n) is decreasing. Same argument for b_n. By monotone convergence theorem, if (a_n) is bounded below, and if (b_n) is bounded above, then both are convergent. But (x_n) is bounded, thus both converge. □

Proof. This is a proof of (2)($\Longleftarrow$). Suppose that $\limsup x_n = \liminf x_n = x$. By definition that means that $\lim a_n = \lim b_n = x$. Since we have that

$$b_n \leq x_n \leq a_n \tag{4}$$

By the squeeze theorem, we have that $\lim x_n = x$. □

Proof. Proof that (3) $\implies$ (2)($\implies$). Suppose that (x_n) has subsequences converging to $\limsup x_n$ and $\liminf x_n$ and $\lim x_n = x$. But all the subsequences of (x_n) must also converge to x, thus in particular, $\limsup x_n = \liminf x_n = x$. □

Proof. This is a proof of (3). We claim that there exists a subsequence that converges to $\limsup x_n = x$. To prove this, we find a sequence of indicies (n_i) such that

1. (n_i) is increasing
2. $(a_{n_i}) - \frac{1}{i} < x_{n_i} \leq a_{n_i}$

Using this, we can build a sequence recursively. We claim that $\lim x_{n_i} = \limsup x_n$.

Proof. We first note that $\lim a_{n_i} = \lim a_n = \limsup x_n$. Saying that the $\frac{1}{i}$ goes to 0, we have that

$$\lim a_{n_i} = \limsup x_n = x \tag{5}$$

□

□

Corollary 0.2. *This is a corallary of* (3). *Every bounded sequence has a convergent subsequence.*