# MTH 447: Lecture 4

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$\mathbb{Q}$ has holes. Completeness axiom for $\mathbb{R}$.

**Definition 0.1.** *Let $S$ be a non-empty set of numbers. IF $\exists M \in S$ s.t. for any $x \in S$, if*

$$x \leq M \tag{1}$$

*Then we say that $M$ is the maximum of $S$. Similarly, $\exists m \in S$ s.t.*

$$m \leq x \tag{2}$$

*Then $m$ is the minimum of $S$. Note that $m, M$ must be in $S$.*

**Definition 0.2.** *Let $S$ be a set of numbers. Then if $\exists M \in \mathbb{R}$ s.t.*

$$x \leq M \tag{3}$$

*For all $x \in S$, we say that $M$ is an upper bound for $S$. If $S$ has an upper bound, then we say that $S$ is bounded above.*

**Definition 0.3.** *$S$ is bounded if bounded above and bounded below.*

We want to define a minimum of out of all upper bounds.

**Definition 0.4.** *Let $S$ be bounded above. If there is a least upper bound, then we call this the supremum of $S$.*

Given $S$ bounded above:

$$U = \{ \text{ all upper bounds of } S \ \} \tag{4}$$

Then

$$\sup S = \min U \tag{5}$$

**Definition 0.5.** *Let $S$ be bounded below, then if there is a greatest lower bound, then we call that the infimum of $S$. Denoted as $\inf S$.*

**Definition 0.6.** *BIG AXIOM (Completeness axiom of real numbers): If $S$ is a set of real numbers, bounded above. Then $\sup S$ exists and is a real number.*

Let's say that $S$ is not bounded above, then we say that $\sup S = +\infty$.