

Math 442 Exam #1 Review

Cliff Sun

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Topics:

1. What is a PDE?
2. What is a degree of a PDE?
3. Whether a PDE is linear or not
4. First order linear equation - Method of characteristics, change of variables
5. Homogenous and non-homogenous Wave equation
6. Heat Equation - maximum principle, conservation of energy, diffusion on the half line

Notes

First order linear pde:

$$a(x, y)u_x(x, y) + b(x, y)u_y(x, y) = 0 \quad (1)$$

Wave equation:

$$u_{tt} - c^2 u_{xx} = f(x, t) \quad (2)$$

$$u(x, 0) = \phi(x) \quad (3)$$

$$u_t(x, 0) = \Psi(x) \quad (4)$$

The general solution is:

$$u(x, t) = \frac{1}{2}(\phi(x - ct) + \phi(x + ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \Psi(s) ds \quad (5)$$

The heat equation is:

$$u_t(x, t) - k u_{xx}(x, t) = f(x, t) \text{ for } (x, t) \in \mathbb{R} \times (0, \infty) \quad (6)$$

$$u(x, 0) = \psi(x) \quad (7)$$

The general solution is:

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t) \psi(y) dy + \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) f(y, s) dy ds \quad (8)$$

Where

$$S(x, y) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \quad (9)$$

The error function is the following:

$$\operatorname{Erf}(x) = \frac{2}{\pi} \int_0^x e^{-p^2} dp \quad (10)$$

Example of linearity within the heat equation:

$$u_t - ku_{xx} = \cos(x) \quad (11)$$

$$u(x, 0) = \sin(x) \quad (12)$$

We know that the solution to the homogenous heat equation with the same initial condition is just:

$$v(x, t) = e^{-kt} \sin(x) \quad (13)$$

Thus applying linearity, we have that:

$$u(x, t) - v(x, t) = w(x, t) \quad (14)$$

Applying this yields:

$$w_t - kw_{xx} = \cos(x) \quad (15)$$

Choosing $w_1(x, t)$ to be $\frac{1}{k} \cos(x)$, we see that now:

$$w_2 = w - w_1 \quad (16)$$

Then, we have that the new heat equation becomes homogenous with an initial condition of $-\frac{1}{k} \cos(x)$. Now applying linearity, we can rederive the full solution.

The maximum principle states that

$$\operatorname{Max}_\gamma(u(x, t)) = \operatorname{Max}_\mathbb{R}(u(x, t)) \quad (17)$$

Such that $\gamma \in \mathbb{R}$ and that γ is the boundary of \mathbb{R} .