PHYS 326: Lecture 3

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2 pendula connected by spring

Assume 2 pendula connected by a spring of constant k. Each spring has a length L and mass m. θ_i describes the i-th pendulum. We describe its equation of motion using torque. On the first mass, we have

$$\tau_g = mgL\sin\theta_1 \tag{1}$$

$$\tau_k = (k\Delta x)L\sin\left(\frac{\pi}{2} - \theta_i\right) \tag{2}$$

Using small angle approximation, we have

$$\tau_g = mgL\theta_1 \tag{3}$$

$$\tau_k = kL^2(\theta_1 - \theta_2) \tag{4}$$

We use

$$I\ddot{\theta} = \sum \tau \tag{5}$$

$$\iff mL^2\ddot{\theta_1} - mgL\theta_1 - kL^2(\theta_1 - \theta_2) = 0 \tag{6}$$

Matrix form:

$$\begin{pmatrix} mL^2 & 0 \\ 0 & mL^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta_1} \\ \ddot{\theta_2} \end{pmatrix} + \begin{pmatrix} mgL + kL^2 & -kL^2 \\ -kL^2 & mgL + kL^2 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7}$$

Let $m = g = L = 1$, then define

$$\epsilon = \frac{kL}{mg} \iff k = \epsilon\frac{mg}{L} \tag{8}$$

Then

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9}$$

$$K = \begin{pmatrix} 1+\epsilon & -\epsilon \\ -\epsilon & 1+\epsilon \end{pmatrix} \tag{10}$$

Then we have that when $\omega^2 = 1$, then $a_1 = a_2$. And when $\omega^2 = 1 + 2\epsilon$, then $a_1 = -a_2$. These motions originate from the initial placement of the pendulum.

Weak Coupling

We assume weak coupling, that is

$$\epsilon = \frac{kL}{mg} << 1 \tag{11}$$

Then $\omega_1 = 1$ and $\omega_2 = \sqrt{1+2\epsilon} \approx 1 + \epsilon$. With initial conditions, $\theta_1 = 1$, $\dot{\theta_1} = 0$, $\theta_2 = 0$, $\dot{\theta_2} = 0$, then we obtain

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (1+2\epsilon)\left(B_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \tag{13}$$

THen $A_1 = A_2 = \frac{1}{2}$ and $B_1 = B_2 = 0$. Then

$$\theta_1 = \frac{1}{2}\left[\cos(t) + \cos\left(1+\epsilon\right)t\right] \tag{14}$$

$$\theta_2 = \frac{1}{2}\left[\cos(t) - \cos\left(1+\epsilon\right)t\right] \tag{15}$$

We convert them:

$$\theta_1(t) = \cos(\left(1 + \frac{\epsilon}{2}\right)t)\cos(\frac{\epsilon}{2}t) \tag{16}$$

$$\theta_2(t) = \sin(\left(1 + \frac{\epsilon}{2}\right)t)\sin(\frac{\epsilon}{2}t) \tag{17}$$

3 degrees of freedom

Assume 3 masses all connected together by strings of length L. Assume that the strings don't stretch and that there is constant tension. Assume that the masses are connected to a pole with springs of constant k_i. Then for the first mass, we have

$$m\ddot{x}_1 = T\sin\theta_2 - T\sin\theta_1 \tag{18}$$

We have

$$\sin\theta_2 \approx \frac{x_2 - x_1}{L} \tag{19}$$

$$\sin\theta_1 \approx \frac{x_1}{L} \tag{20}$$