PHYS 326: Lecture # 15

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Symmetrical Top, Subject to Torque

In the presence of torque, we work in a fixed lab frame. Write the Lagrangian. Define generalized coordinates: Euler Angles. We define Orientation/Rotation Matrices Q. Consider a coordinate system $\hat{i}, \hat{j}, \hat{k}$, then transform to new unit vectors: $\hat{i}', \hat{j}', \hat{k}'$. Then write

$$\hat{i}' = Q_{xx}\hat{i} + Q_{xy}\hat{j} + Q_{xz}\hat{k} \tag{1}$$

$\hat{j}'$ and $\hat{k}'$ are defined similarly. Note, the elements of Q are just angles, that is

$$\hat{i}' \cdot \hat{j} = \underbrace{|\hat{i}'||\hat{j}|}_{1}\cos\theta_{i'j} = Q_{xy} \tag{2}$$

Write

$$\hat{e}_i' = \sum_j Q_{ij}\hat{e}_j \tag{3}$$

Require that $\hat{e}_i'$ be orthonormal basis, then

$$\delta_{ik} = \hat{e}_i' \cdot \hat{e}_k' = \sum_j Q_{ij}\hat{e}_j \sum_m Q_{km}\hat{e}_m' \tag{4}$$

$$\iff \sum_{jm} Q_{ij}Q_{km}\delta_{jm} \tag{5}$$

$$\iff \sum_j Q_{ij}Q_{km} \iff \sum_j Q_{ij}Q_{jk}^T \implies I = QQ^T \tag{6}$$

Note, the eigenvalues of Q is $\lambda = 1$, and $\lambda = e^{\pm i\alpha}$. Where α is the angle of rotation. Consider infinitesimal rotations by small angle ωdt, then

$$Q = Q + \epsilon \tag{7}$$

Then consider

$$Q^TQ = I \iff (I + \epsilon^T)(I + \epsilon) \implies \epsilon^T + \epsilon = 0 \tag{8}$$

This means ϵ is anti-symmetric. Then consider

$$Q(t + dt) = (I + \omega dt)Q(t) \tag{9}$$

Then

$$\frac{dQ}{dt} = \frac{Q(t+dt) - Q(t)}{dt} = \omega Q \tag{10}$$

Then

$$\omega = \frac{dQ}{dt}Q^T \tag{11}$$