MTH 447: Lecture 3

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Definition 0.1. *We say*

$$\frac{p}{q} \leq \frac{r}{s} \tag{1}$$

iff

$$ps \leq rq \tag{2}$$

In the rational numbers, we have

1. $a, b \in \mathbb{Q}$, $a \leq b$ or $b \leq a$
2. $a \leq b \wedge b \leq a \iff a = b$
3. $a \leq b \wedge b \leq c \implies a \leq c$
4. $a \leq b \implies a + c \leq b + c$
5. $a \leq b \wedge c \geq 0 \implies ac \leq bc$

Properties in $\mathbb{Q}$:

1. $a + c = b + c \implies a = b$
2. $a \cdot 0 = 0$
3. $(-a)(b) = -ab$
4. $(-a)(-b) = ab$
5. $ac = bc \wedge c \neq 0 \implies a = b$
6. $ab = 0 \implies a = 0 \vee b = 0$
7. $a \leq b \implies b \leq a$
8. $a \leq b \wedge c \leq 0 \implies ac \leq bc$
9. $a \geq 0, b \geq 0 \implies ab \geq 0$
10. $a^2 \geq 0$
11. $0 < 1$
12. $a > 0 \implies a^{-1} > 0$
13. $0 < a < b \implies 0 < b^{-1} < a^{-1}$

Proof. Show that $a \cdot 0 = 0$, then

$$a \cdot 0 \iff a \cdot (0 + 0) \iff a \cdot 0 + a \cdot 0 = a \cdot 0 \tag{3}$$

$$\iff a \cdot 0 = 0 \tag{4}$$

□

Definition 0.2.

$$|a| = \begin{cases} a & a > 0 \\ 0 & a = 0 \\ -a & a < 0 \end{cases} \tag{5}$$

Theorem 0.3. *1.* $|a| > 0$ *&* $|a| = 0 \iff a = 0$

2. $|ab| = |a| \cdot |b|$

3. $|a + b| \leq |a| + |b|$

Note:

$$-|a| \leq a \leq |a| \tag{6}$$

$$-|b| \leq b \leq |b| \tag{7}$$

Then

$$-(|a| + |b|) \leq a + b \leq |a| + |b| \tag{8}$$

Lemma 0.4. *If* $M > 0$ *and*

$$-M \leq x \leq M \tag{9}$$

Then

$$|x| \leq M \tag{10}$$

Thus we see that

$$|a + b| \leq |a| + |b| \tag{11}$$