PHYS 325: Lecture 8

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Lecture Span

- Conservative Forces

Conservative Forces

Definition 0.1. *A force $\vec{F}(r)$ is conservative if it can be written as a gradient of a potential $U(r)$, that is*

$$\vec{F}(r) = -\vec{\nabla}U(r) \tag{1}$$

Consequence 1

Curl of conservative force vanishes, but note, not every force has

$$\nabla \times F = 0 \tag{2}$$

In general, every force can be expressed as the following:

$$\vec{F} = \vec{F}_{\text{cons.}} + \vec{F}_{\text{diss}} \tag{3}$$

Thus

$$\nabla \times \vec{F} = \nabla \times \vec{F}_{\text{cons.}} + \nabla \times \vec{F}_{\text{diss}} \neq 0 \tag{4}$$

Consequence 2

Let's consider the work along a path, from $p_1 \to p_2$. To compute the work along this path:

$$W = \int_{p_1}^{p_2} \vec{F} \cdot d\vec{r} \tag{5}$$

If $\vec{F}$ is conservative, then we get the following relationship

$$W = -\int_{p_1}^{p_2} \nabla U \cdot d\vec{r} \iff U(p_1) - U(p_2) \tag{6}$$

If path is closed, that is $(p_1 \to p_1)$, then

$$W = -(U(p_1) - U(p_1)) = 0 \tag{7}$$

Conservative forces do NO work along a closed path.

Energy Conservation

If force is conservative, that is $\vec{F}(r) = -\nabla U$, then E = KE + PE is constant. We call this constant of the motion. But how do we go about this? An idea would be to take the time derivative of E, and see what happens.

We are trying to prove:

$$D_t(E) = D_t(T + U) = 0 \tag{8}$$

Thus,

$$D_t T = \frac{1}{2}m|v|^2 = mv \cdot \frac{dv}{dt} \iff m\vec{v} \cdot \vec{a} \iff \vec{v} \cdot \vec{F} \tag{9}$$

Note, $\vec{v} \cdot \vec{F}$ can be non-zero.

$$D_t(U(t, \vec{r}(t))) = \partial_t U + (\partial_x U \partial_t x + \partial_y U \partial_t y + \partial_z U \partial_t z) \tag{10}$$

$$D_t(U(t, \vec{r}(t))) = \partial_t U + \partial_t \vec{r} \cdot \nabla U \iff \partial_t U + -\vec{v} \cdot \vec{F} \tag{11}$$

$$D_t(E) = \vec{v} \cdot \vec{F} + -\vec{v} \cdot \vec{F} + \partial_t U \iff \partial_t U \tag{12}$$

If U has no explicit time dependence, then $D_t E = 0$, thus

$$E = c \text{ for some fixed } c \in \mathbb{R} \tag{13}$$

How to find the potential given the force

1. Indefinitely integrate $\vec{F} = -\nabla U$
2. Definitely integrate $\vec{F} = -\nabla U \iff \int_{p_1}^{p_2} dU \iff -\int_{p_1}^{p_2} \vec{F} \cdot d\vec{r}$

Example

$$\text{Force } = (2xy + 1)\hat{\imath} + (x^2 + 2)\hat{\jmath} \tag{14}$$

Trick: choose a convenient path Γ

$$-U = \int_\Gamma \vec{F} \cdot d\vec{r} = \int_\Gamma (F_x dx + F_y dy + F_z dz) \tag{15}$$

We decompose our arbitrary path into its counterparts:

$$\int_0^x F_{x'}(x', 0, 0)dx' + \int_0^y F_{y'}(x, y', 0)dy' + \int_0^z F_z(x, y, z')dz' \tag{16}$$

$$= \int_0^x dx' + \int_0^y (x^2 + 2)dy' + \int_0^z 0dz' \tag{17}$$

$$= x + (x^2 y + 2y) \tag{18}$$

$$U = -x - x^2 y - 2y \tag{19}$$

Central Forces

Definition 0.2. *A Central Force that is directed away from a center point. Could be centripetal, etc. We depict this as*

$$\vec{F} = f(r, \theta, \phi)\vec{e_r} \tag{20}$$

Conservation of Angular Momentum

Definition 0.3. *Angular Momentum, or $\vec{L}$, is defined with respect to a refernece point. (this means that this is a pseudovector!) Then we define*

$$\vec{L} = \vec{r} \times m\vec{v} \tag{21}$$

Thus the magnitude of the angular momentum is

$$|\vec{L}| = m|\vec{r}||\vec{v}| \sin \alpha \tag{22}$$

This vector is orthogonal to the $\vec{r} - \vec{v}$ plane.

Notes

1. If reference point is at the center of a central force field, then

$$\vec{L} = c \; c \in \mathbb{Z} \tag{23}$$

   Or would be constant with respect to motion.