

# PHYS 486: Catch-up Edition

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## Angular Momentum

Angular momentum is defined as

$$L = \hat{r} \times \hat{p} = -i\hbar (\hat{r} \times \vec{\nabla}) \quad (1)$$

This produces

$$L_x = y p_z - z p_y, \dots \quad (2)$$

As well,

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k \quad (3)$$

## Ladder Operators

Let  $L_{\pm} = L_x \pm iL_y$ , then if  $\psi$  is an eigenfunction of  $L^2$  and  $L_z$ , then  $L_{\pm}\psi$  is also an eigenfunction.

In particular, suppose

$$L^2\psi = \lambda\psi \quad L_z\psi = \mu\psi \quad (4)$$

Then,

$$L^2(L_{\pm}\psi) = \lambda(L_{\pm}\psi) \quad (5)$$

$$L_z L_{\pm}\psi = (\mu \pm \hbar)(L_{\pm}\psi) \quad (6)$$

Note that because  $L_z$  and  $L_{\pm}$  don't commute, then  $L_{\pm}$  directly affects the measurement of  $L_z$ .

We can also compute what  $\lambda$  is by taking inspiration from the quantum simple harmonic oscillator. In particular,

$$L_{\pm}L_{\mp} = L^2 - L_z^2 \pm \hbar L_z \quad (7)$$

One can rearrange this equation in terms of  $L^2$ , and given the boundary conditions,

$$L_z\psi_t = \hbar l\psi_t \implies \mu_{\max} = \hbar l \quad (8)$$

$$L_z\psi_b = \hbar \bar{l}\psi_b \implies \mu_{\min} = \hbar \bar{l} \quad (9)$$

Then we obtain that

$$L^2\psi_t = \hbar^2 l(l+1)\psi_t \wedge L^2\psi_b = \hbar^2 \bar{l}(\bar{l}-1) \quad (10)$$

This means that  $\bar{l} = -l$  and the boundaries of  $\mu$  range from

$$\mu = -\hbar l, -\hbar(l+1), \dots, \hbar l \quad (11)$$

Then in total,

$$L^2\psi_l^m = \hbar^2 l(l+1)\psi_l^m \wedge L_z\psi_l^m = \hbar m\psi_l^m \quad (12)$$

The reason why this wave function is indexed by both  $m$  and  $l$  is because the  $l$  terms represents the upper bound on the  $L_z$  eigenvalue and do not change. However,  $m$  does change and therefore needs to be indexed by a different number. In total, there are  $2l + 1$  values of  $m$ . This means that  $l$  can either be  $n$  or  $n + 1/2$  for an integer  $n$ .

## The Spherical Equation Solution

To solve the spherical equation, we use the notion that

$$L^2 Y = \hbar^2 l(l+1)Y \quad (13)$$

solving for  $L^2$ , we obtain

$$L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right] \quad (14)$$

Letting

$$Y = \Omega(\theta)\Lambda(\phi) \quad (15)$$

We obtain

$$\frac{1}{\Lambda} \frac{d^2 \Lambda}{d\Lambda^2} = -m^2 \implies \Lambda(\phi) = e^{im\phi} \quad (16)$$

But noting that  $\Lambda(\phi) = \Lambda(\phi + 2\pi)$ , we get that  $m$  must be an integer. However,  $l$  is an integer or integer  $+1/2$ . In general, the total angular momentum allows for integer or integer  $+1/2$ . But the orbital angular momentum is only integer.

The theta equation is a bit more complicated, but

$$\Omega(\theta) = AP_l^m(\cos \theta) \quad (17)$$

Where  $P_l^m$  is the associated Legendre function. Therefore,

$$Y_l^m = Ae^{im\phi} P_l^m(\cos \theta) \quad (18)$$

Where

$$A = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} \quad (19)$$

Moreover,

$$\langle Y_l^m | Y_{l'}^{m'} \rangle = \delta_{ll'} \delta_{mm'} \quad (20)$$

## The Radial Equation solution

The radial equation is

$$\left\{ \frac{1}{R} \frac{d}{dR} \left( r^2 \frac{dR}{dr} \right) - \frac{2mr^2}{\hbar^2} [V(r) - E] \right\} = l(l+1) \quad (21)$$

Letting  $R = u/r$ , then we can simplify this formula to be

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[ V + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u = E_n \quad (22)$$

Where  $E_n$  is some energy indexed by  $n$ . For the case of the infinite well, where

$$V(r) = \begin{cases} 0 & r \leq a \\ \infty & r > a \end{cases} \quad (23)$$

Then inside the well, we obtain

$$\frac{d^2 u}{dr^2} = \left[ \frac{l(l+1)}{r^2} - k^2 \right] u \quad (24)$$

With the boundary condition of  $u(a) = 0$ . Then

$$u(r) = A \sin(kr) + B \cos(kr) \implies B = 0 \quad (25)$$

In general,

$$u(r) = Arj_l(kr) + Bru_l(kr) \implies B = 0 \quad (26)$$

Where  $j_l$  are the spherical bessel functions and  $u_l$  are the spherical Neumann functions. Therefore,

$$Y_{nlm} = A_{nl} j_l(\beta_{NL} \frac{r}{a}) Y_l^m(\theta, \phi) \quad (27)$$

Where  $k = \beta_{NL}/a$ .

## Hydrogen Atom

For the case of the Hydrogen Atom,

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} \quad (28)$$

Then the radial equation becomes

$$-\frac{\hbar^2}{2m} \frac{d^2u}{dr^2} + \left[ -\frac{e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2}{2m_e} \frac{l(l+1)}{r^2} \right] u = E_n \quad (29)$$

Defining  $\rho = kr$  and  $\rho_0 = \frac{m_e e^2}{2\pi\epsilon_0 \hbar^2 k}$  where  $k = \frac{\sqrt{-2m_e E}}{\hbar}$ , then

$$\frac{d^2u}{d\rho^2} = \left[ 1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] u \quad (30)$$

We can guess an ansatz by studying the asymptotes of this equation. In particular, if  $\rho \rightarrow \infty$ , then

$$u(\rho) \sim Ae^{-\rho} \quad (31)$$

Similarly, for  $\rho \rightarrow 0$ , then

$$u(\rho) \sim C\rho^{l+1} \quad (32)$$

Therefore, we guess that

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho) \quad (33)$$

Plugging this into the radial equation where  $v(\rho) = \sum_i c_i \rho^i$ , we obtain

$$c_{j+1} = \left[ \frac{2(j+l+1) - \rho_0}{(j+1)(j+2l+2)} \right] c_j \quad (34)$$

For large  $j$ , we have that

$$c_j \underset{j!}{\approx} \frac{2j}{j!} c_0 \implies c_0 e^{2\rho} \quad (35)$$

But this makes the wavefunction diverge for large  $\rho$ , therefore the series must end. This means that  $\exists N$  such that  $c_{N-1} \neq 0$  but  $c_N = 0$ . In this case,

$$2(N+l) - \rho_0 = 0 \implies \rho_0 = 2n \quad (36)$$

Then in particular, the series must terminate at  $n-l-1$  from the recursion formula. Note, that  $\rho = kr$  with

$$\rho_0 = \frac{m_e e^2}{2\pi\epsilon_0 \hbar^2 k} \quad (37)$$

But since  $\rho_0 = 2n$ , we have that

$$k = \frac{m_e e^2}{4\pi\epsilon_0 \hbar^2 n} \quad (38)$$

Defining

$$\frac{1}{a} = \frac{m_e e^2}{4\pi\epsilon_0 \hbar^2} \quad (39)$$

We have that

$$\rho = \frac{r}{an} \quad (40)$$

Now, suppose  $n = 1$ ,  $l = 0$ , and  $m = 0$ , then

$$R(r) = \frac{c_0}{a} e^{-r/a} \quad (41)$$

and

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \quad (42)$$

Therefore,

$$\psi_{100} = \frac{c_0}{\sqrt{4\pi a}} e^{-r/a} \quad (43)$$

It can be computed that  $c_0 = \frac{2}{\sqrt{\pi a}}$ .

## Spin

We call a particle to be a Boson if its spin is  $s = 1$ , and a particle is a fermion of its spin is  $s = 1/2$ .

We define

$$\vec{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} \quad (44)$$

And

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k \quad (45)$$

Let the wave function be  $|s, m\rangle$ , then

$$S^2 |s, m\rangle = \hbar^2 s(s+1) |s, m\rangle \quad (46)$$

$$S_z |s, m\rangle = \hbar m |s, m\rangle \quad (47)$$

$$S_{\pm} |s, m\rangle = \hbar \sqrt{s(s+1) - m(m \pm 1)} |s, m \pm 1\rangle \quad (48)$$

The eigenstates of  $S_z$  are

$$|0\rangle = \chi_+ = \left| \frac{1}{2}, \frac{1}{2} \right\rangle = |\uparrow\rangle \quad (49)$$

$$|1\rangle = \chi_- = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = |\downarrow\rangle \quad (50)$$

In general,  $\chi = \alpha\chi_+ + \beta\chi_-$ . We can also find that given

$$S^2 \chi_{\pm} = \frac{3}{4} \hbar^2 \chi_{\pm} \implies S^2 = \frac{3}{4} \hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (51)$$

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (52)$$

As well,

$$S_+ \chi_- = \hbar \chi_+ \wedge S_+ \chi_+ = 0 \quad (53)$$

And vice versa. we can also compute that

$$S_x = \frac{\hbar}{2} \sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (54)$$

$$S_y = \frac{\hbar}{2} \sigma_y = \frac{\hbar}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad (55)$$

And

$$S = \frac{\hbar}{2} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \quad (56)$$

In general, we can use spin to solve for the time evolution of  $S_x$  in an external magnetic field. Suppose  $B = B_0 \hat{z}$ , then

$$H = -\mu \cdot B \implies -\gamma B_0 \hat{z} \cdot S \implies \frac{\gamma B_0 \hbar}{2} \sigma_z \quad (57)$$

This is the energy operator of the entire system. Computing  $\langle \chi | H | \chi \rangle$  will get the average energy of the spin state  $|\chi\rangle$ . We let  $E_+ = -\frac{\gamma B_0 \hbar}{2}$  and  $E_- = \frac{\gamma B_0 \hbar}{2}$ . At time  $t = 0$ , the arbitrary spin can be described as a bloch sphere,

$$|\chi\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle \quad (58)$$

And according to the shrodinger wave equation, we have that

$$|\chi\rangle(t) = \cos \frac{\theta}{2} |0\rangle e^{-iE_+ t/\hbar} + \sin \frac{\theta}{2} e^{i\phi} |1\rangle e^{-iE_- t/\hbar} \quad (59)$$

Where  $|0\rangle = \chi_+$  and  $|1\rangle = \chi_-$ . Then we can compute

$$\langle S_x \rangle = \langle \chi(t) | S_x | \chi(t) \rangle = \frac{\hbar}{2} \sin \theta \cos(\gamma B_0 t) \quad (60)$$

**Note:** all these calculations were done for a spin-1/2 particle, or a fermion.

## Stern-Gerlach Experiment

Assume that

$$\chi = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) \quad (61)$$

Generally,

$$|\pm\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \pm |\downarrow\rangle) \quad (62)$$

We define a composite state

$$|s_1, s_2, m_1, m_2\rangle = |s_1, m_1\rangle \otimes |s_2, m_2\rangle \quad (63)$$

then we define  $S_{x,y,z}^j$  and  $(S^2)^j$  where the operator only acts on the  $j$ -th composite state. An example:

$$(S^2)^1 |s_1, s_2, m_1, m_2\rangle = s_1(s_1 + 1)\hbar^2 |s_1, s_2, m_1, m_2\rangle \quad (64)$$

Defining the total angular momentum  $S = S^1 + S^2$ , we try to measure  $S_z$

$$S_z |s_1, s_2, m_1, m_2\rangle = \hbar(m_1 + m_2) |s_1, s_2, m_1, m_2\rangle \quad (65)$$

Where  $m = m_1 + m_2$ . We will observe some degeneracy if  $m_1$  and  $m_2$  are anti-symmetric, or are  $\pm 1/2$  and  $\mp 1/2$ .