# MTH 416: Lecture 3

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## Lecture Span

- Linear combinations
- Systems of linear equations

## Recall

**Theorem 0.1.** *If $W$ is a subset of a vector space $V$, then*

$$0 \in W \wedge W \textit{ is closed under } + \textit{ and } \cdot \iff W \textit{ is a Vector Space with the same operations as } V \tag{1}$$

Last time, we prove ($\implies$), this time we want to prove ($\impliedby$). That is, if either left or right is true, then $W$ is a subspace of $V$.

*Proof.* Suppose $W$ is a vector space with the same operations as $V$, that is $W$ satisfies all 8 axioms. Then we must prove that

1. $0 \in W$
2. $W$ is closed under $+$ and $\cdot$

We first prove (2), as it is the easiest to start with.

*Proof.* If it weren't closed under these operations, then $+$ and $\cdot$ doesn't make sense as operations under $W$. □

We then prove (1),

*Proof.* $W$ is a vector space, so it contains a $0_W$ vector. Such that

$$w + 0_W = w \text{ for all } w \in W \tag{2}$$

Since $V$ is a vector space, it also contains $0_V$, such that

$$v + 0_V = v \text{ for all } v \in V \tag{3}$$

We claim that $0_W = 0_V$.

*Proof.*

$$w + 0_W = w = w + 0_V \text{ because } w \in V \tag{4}$$

Using the cancellatin theorem we have that

$$0_W = 0_V \tag{5}$$

□

□

□

## Linear combinations

**Definition 0.2.** *Let $u_1, \ldots, u_n$ be vectors in a vector space $V$, then*

1. *A linear combination of the vectors $u_i$ is any vector that can be written as the following form:*

$$u_i = a_1u_1 + \cdots + a_nu_n \tag{6}$$

2. *A set of all linear combinations $u_i$ is called the span of $u_i$.*

**Theorem 0.3.** *If $u_1, \cdots, u_n$ are vectors in a vector space $V$, then $span(u_1, \cdots, u_n)$ is always going to be a subspace of $V$.*

*Proof.* We must show that this span has the 0 vector and is closed under addition & scalar multiplication.

1. $0 = 0u_1 + \cdots + 0u_n$
2. Assume we are given $v = a_1u_1 + \cdots a_nu_n$ and $w = b_1u_1 + \cdots + b_nu_n$, we have that prove that $v + w$ is also a linear combination. In other words, $v + w \in span(u_i)$. We first expand this out:

$$v + w \iff (a_1u_1 + \cdots a_nu_n) + (b_1u_1 + \cdots b_nu_n) \iff (a_1 + b_1)u_1 \cdots (a_n + b_n)u_n \tag{7}$$

   This an element of $span(u_i)$.

3. Suppose $v = a_1u_1 + \cdots + a_nu_n$ and $c \in \mathbb{R}$. Then,

$$cv = c(a_1u_1 + \cdots + a_nu_n) \iff ca_1u_1 + \cdots + ca_nu_n \in span(u_i) \tag{8}$$

□

Moreover, $span(u_i)$ is the smallest subspace of $V$ containing all $u_1, \cdots u_n$. That is, there cannot be a smaller subspace that also contains all $u_1, \cdots, u_n$.

1. $u_1, \cdots, u_n \in span(u_i)$ (Choose $a_i = 1$, all other 0)
2. If $W$ is any subspace of V also containing all $u_i$, then $span(u_i) \in W$

## Systems of Linear Equations/Linear System

**Definition 0.4.** *A system of $m$ linear equations with $n$ unknowns is a system of the form:*

$$a_{11}x_+ \cdots + a_{1n}x_n = b_1 \tag{9}$$

$$a_{21}x_+ \cdots + a_{2n}x_n = b_2 \tag{10}$$

$$a_{m1}x_+ \cdots + a_{mn}x_n = b_m \tag{11}$$

*Where $a_{ij}$ and $b_i$ are scalars.*

### Goals

1. Determine whether if there is a solution $(x_1, \cdots, x_n)$
2. Find all solutions

### Notation

We can associate to a linear system the augmented matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{2n} & b_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix}$$

Where the $m \times n$ matrix are all the coefficients and the right hand $b$ are the constraints. We denote this as

$$LS(A|b) \tag{12}$$