8.1-8.2

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Theorem 0.1. $\mathbb{Q}$ *is countably infinite.*

First, we prove that $\mathbb{Q}^+$ is countably infinite.

Proof. We can list all the positive rational numbers in a 2-d grid, that is

$$\begin{array}{ccc} 1/1 & 2/1 & \dots \\ 1/2 & 2/2 & \dots \\ 1/3 & 2/3 & \dots \end{array}$$

We can cross out all the repeat values, then this grid reduces to the following list:

n	1	2	3
f(n)	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{1}{3}$

This list would go on forever, and encapsulate all the possible rational numbers. It follows that this function $f(n)$ is a bijection from $\mathbb{N} \Rightarrow \mathbb{Q}^+$. Thus

$$|\mathbb{Q}^+| = |\mathbb{N}| = \aleph_0 \tag{1}$$

Thus, this concludes this proof. □

Similarly, we can also use this proof when constructing a function that maps from $\mathbb{Z} \to \mathbb{Q}$, so that $|\mathbb{Z}| = |\mathbb{Q}|$. But it follows that $|\mathbb{Z}| = \aleph_0$. Thus it follows that the rational numbers is in fact countably infinite.

Theorem 0.2. *In regards to countable sets, we have the following theorems:*

1. *If $A \subseteq B$, then $|A| \leq |B|$.*
2. *The union of countably many countable sets is countable.*
3. *The cartesian product of finitely many countable sets is always countable.*

Applications:
Let $\mathbb{Z}[x]$ denote the set of polynomials in x with integer coefficients. Then this set is countable.

Proof. For each $n \in \mathbb{N}$, let

$$A_n = \{f \in \mathbb{Z}[x] : deg(f) \leq n\} \tag{2}$$

Then we can prove that each A_n is countable using a bijection:

$$g : \mathbb{Z}^{n+1} \to A_n \tag{3}$$

(Think of it like a dot product) Then since $\mathbb{Z}^{n+1}$ is countable, and since $\mathbb{Z}[x]$ is the union of all A_n, it follows that $\mathbb{Z}[x]$ is countable. □

Lemma 0.3. *There are only countably many algebraic numbers.*

Recall: a number if algebraic if it's a root of a non-zero polynomial in $\mathbb{Z}[x]$.

Proof. For each nonzero $f \in \mathbb{Z}[x]$, let

$$B_f = \{\text{Zeros of F}\} \tag{4}$$

Each B_f is finite, thus the union of all B_f is countable. Thus, we conclude the proof. □

Theorem 0.4. $[0, 1]$ *is uncountable.*

Proof. We must do 2 things

1. Prove that $\exists f : \mathbb{N} \to [0, 1]$ is injective.
2. Prove that $\nexists g : \mathbb{N} \twoheadrightarrow [0, 1]$ is surjective.

For (1), define a function $f = \frac{1}{n}$

For (2), assume that such a function does exist for the sake of contradiction. □