PHYS 325: Lecture 27

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Lagrange Multipliers

Extremize a functional S_1 subject to a global constraint S_2. Concrete example:

Minimize Area under curve given a fixed length of the curve

Given

$$C = S_1 - \lambda S_2 \tag{1}$$

Where λ is the Lagrange multiplier, we have that

$$\delta C = \delta S_1 - \lambda \delta S_2 \tag{2}$$

We have that

$$\delta S_1 = 0 \tag{3}$$

Because we want to extremize S_1, similarly, we have that

$$\delta S_2 = 0 \tag{4}$$

Because S_2 is constant. Thus

$$\delta S_1 - \lambda \delta S_2 = 0 \tag{5}$$

Hanging Chain

Given a chain of mass density $\rho = \frac{dm}{dl}$, we have that

$$m = \int \rho dl \iff \int \rho \sqrt{x'^2 + 1} dy \tag{6}$$

Thus

$$PE = -mgh \iff -\int \rho g y \sqrt{x'^2 + 1} dy = S_1 \tag{7}$$

This is our first functional. Our second functional fixes the length of the chain to be a length L, that is

$$L = \int \sqrt{x'^2 + 1} dy = S_2 \tag{8}$$

We choose $\tilde{\lambda} = g\rho\lambda$, we get

$$C = PE - \tilde{\lambda} L = -g\rho \int dy (y - \lambda) \sqrt{x'^2 + 1} \tag{9}$$

Local Constraints

Consider functionals of 2 or more functions, that is

$$F(x(t), y(t)) = \int f(x, x', y, y'; z)dz \tag{10}$$

with conditions

$$g(x, y; z) = 0 \tag{11}$$

Similarly,

$$C = f - \lambda g \tag{12}$$

Extremize

$$C = \int (f - \lambda g)dz \tag{13}$$

Lagrangian Mechanics

$$L = T - U \tag{14}$$