MTH 416: Lecture 23

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Lecture Span

- Inner product spaces
- Orthonormal bases & Gram-Schmidt

Recall: Inner product axioms

1. $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$
2. $\langle cx, y\rangle = c\langle x, y\rangle$
3. $\overline{\langle x, y\rangle} = \langle y, x\rangle$
4. If $x \neq 0$, then $\langle x, x\rangle > 0$ and $||x|| = \sqrt{\langle x, x\rangle}$

Theorem 0.1. *Let V be an inner product space, for all $x, y \in V$, we have that*

1. *(Cauchy-Schwarz)* $|\langle x, y\rangle| \leq ||x||||y||$
2. *(Triangle Inequality)* $||x+y|| \leq ||x|| + ||y||$

Proof. We can prove 2 assuming 1. We first square both sides:

$$||x+y||^2 = \langle x+y, x+y\rangle \tag{1}$$

$$= \langle x, x+y\rangle + \langle y, x+y\rangle \tag{2}$$

$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \tag{3}$$

Note:

$$\langle x, y\rangle + \langle y, x\rangle = 2\mathbb{R}(\langle x, y\rangle) \tag{4}$$

$$= \langle x, x\rangle + 2\mathbb{R}(\langle x, y\rangle) + \langle y, y\rangle \tag{5}$$

We note that if $2\mathbb{R}(\langle x, y\rangle) = 2a$, then

$$2|a| \leq |a+bi| + |a-bi| \tag{6}$$

$$2|a| \leq 2\sqrt{a^2+b^2} \tag{7}$$

Then the whole equation simplifies down to

$$\leq \langle x, x\rangle + 2|\langle x, y\rangle| + \langle y, y\rangle \tag{8}$$

$$\leq ||x||^2 + 2||x||||y|| + ||y||^2 \tag{9}$$

$$= (||x|| + ||y||)^2 \tag{10}$$

□

We prove (1), then

Proof. It is clearly true if $y = 0$, thus we assume that $y \neq 0$. For any $c \in F$, then

$$0 \leq \langle x - cy, x - cy\rangle \tag{11}$$

We first expand out the first value:

$$= \langle x, x - cy\rangle - c\langle y, x - cy\rangle \tag{12}$$
$$= \overline{\langle x - cy, x\rangle} - c\overline{\langle x - cy, y\rangle} \tag{13}$$
$$= \langle x, x\rangle - \overline{c\langle y, x\rangle} - c\overline{\langle x, y\rangle} + c\overline{c\langle y, y\rangle} \tag{14}$$
$$= \langle x, x\rangle - \bar{c}\langle x, y\rangle - c\overline{\langle x, y\rangle} + c\bar{c}\langle y, y\rangle \tag{15}$$

Choose

$$c = \frac{\langle x, y\rangle}{\langle y, y\rangle} \tag{16}$$

Then we have

$$\bar{c}\langle x, y\rangle = \frac{\overline{\langle x, x\rangle}}{\langle y, y\rangle}\langle x, y\rangle = \frac{\langle x, y\rangle\langle y, x\rangle}{\langle y, y\rangle} \tag{17}$$

$$c\overline{\langle x, y\rangle} = \frac{\langle x, y\rangle\langle y, x\rangle}{\langle y, y\rangle} \tag{18}$$

$$c\bar{c}\langle y, y\rangle = \frac{\langle x, y\rangle\langle y, x\rangle}{\langle y, y\rangle} \tag{19}$$

So

$$0 \leq \langle x, x\rangle - \frac{\langle x, y\rangle\langle y, x\rangle}{\langle y, y\rangle} \tag{20}$$

Thus

$$\langle x, y\rangle\langle y, x\rangle \leq \langle x, x\rangle\langle y, y\rangle \tag{21}$$
$$= (|\langle x, y\rangle|)^2 \leq ||x||^2||y||^2 \tag{22}$$

Then, because all numbers are positive, we have that

$$|\langle x, y\rangle| \leq ||x||||y|| \tag{23}$$

This is the Cauchy-Schwarz inequality. □

Definition 0.2. *Let V be an inner product space with $x, y \in V$, then*

1. *x and y are orthogonal if $\langle x, y\rangle = 0$*
2. *A subset $S \subseteq V$ is orthogonal if all pairs of distinct $x, y \in S$ are orthogonal*
3. *x is a unit vector if $\langle x, x\rangle = 1 \iff ||x|| = 1 = \sqrt{\langle x, x\rangle}$*
4. *A set $S \subseteq V$ is orthonormal if it is orthogonal and consists entirely of unit vectors.*

Example 1

Let $V = \mathbb{C}^2$, then

1. $\{e_1, e_2\}$ are still orthonormal.
2. $v = \begin{pmatrix} 2+i \\ i \end{pmatrix}$ and $w = \begin{pmatrix} -1 \\ 1+2i \end{pmatrix}$ are orthogonal. Then

$$\langle x, w\rangle = (2+i)(-1) + i(\overline{1+2i}) = 0 \tag{24}$$

Example 2

Let $V = P_1(\mathbb{R})$ with

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx \tag{25}$$

Claim vectors that

$$f(x) = \frac{1}{\sqrt{2}} \tag{26}$$

and

$$g(x) = \sqrt{\frac{3}{2}}x \tag{27}$$

are orthonormal basis. We check:

$$\langle f, g\rangle \int_{-1}^{1} \frac{1}{\sqrt{2}}\sqrt{\frac{3}{2}}xdx \tag{28}$$

This is an odd function, thus this evaluates to 0.

$$\langle f, f\rangle = \int_{-1}^{1} \frac{1}{2}dx \iff 2 \cdot \frac{1}{2} = 1 \tag{29}$$

$$\langle g, g\rangle = \int_{-1}^{1} \frac{3}{2}x^2dx \iff x^3(\frac{1}{2}) = \frac{1}{2} \cdot 2 = 1 \tag{30}$$

Goal:

Let's try to orthonormal bases for a given inner product space. Then

Theorem 0.3. *Every finite dimensional inner product space has an orthonormal basis.*

Theorem 0.4. *Given an orthogonal set*

$$S = \{v_1, \ldots, v_k\} \text{ with } 0 \notin S \tag{31}$$

with

$$y = a_1v_1 + \cdots + a_kv_k \tag{32}$$

$$a_i = \frac{\langle y, v_i\rangle}{\langle v_i, v_i\rangle} \tag{33}$$

Proof. Given

$$y = a_1v_1 + \cdots + a_kv_k \tag{34}$$

Calculate

$$\langle y, v_i\rangle = a_1\langle v_1, v_i\rangle + \ldots \tag{35}$$

Because this is an orthongal basis, we have that

$$\langle v_j, v_i\rangle = 0 \iff j \neq i \tag{36}$$

Then the only one standing is

$$\langle y, v_i\rangle = a_i\langle v_i, v_i\rangle \tag{37}$$

In other words

$$a_i = \frac{\langle y, v_i\rangle}{\langle v_i, v_i\rangle} \tag{38}$$

□

Corollary 0.5. *If $S \subseteq V$ is an orthogonal set in V, and $0 \notin S$, then S is linearly independent.*

Proof. If we have any linear dependency, that is

$$a_1v_1 + \cdots + a_kv_k = 0 = y \tag{39}$$

Then

$$a_i = \frac{\langle y, v_i\rangle}{\langle v_i, v_i\rangle} \tag{40}$$

This means that

$$a_i = 0 \tag{41}$$

□

Fourier Series

Let $V = C[-\pi, \pi]$ where C represents continuous functions in the interval $[-\pi, \pi]$. We define

$$\langle f, g\rangle = \frac{1}{\pi}\int_{\pi}^{-\pi} f(x)g(x)dx \tag{42}$$

then

Fact

The following set is orthonormal in V:

$$S = \{\frac{1}{\sqrt{2}}\} \cup \{\cos(nx)\ n \in \mathbb{N}\} \cup \{\sin(nx)\ n \in \mathbb{N}\} \tag{43}$$

We can solve for the coefficients using

$$a_i = \frac{\langle y(x), v_i\rangle}{\langle v_i, v_i\rangle} \tag{44}$$

S is not a basis of V, but most functions in practice can be expressed as Fourier Series.