MTH 416: Lecture 11

Cliff Sun

October 3, 2024

Lecture Span

- Matrix multiplication & composition

Matrix Multiplication & composition

Recall

$$T : V \to W \tag{1}$$

a linear transformation where $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ for now. Given an ordered basis

$$\beta = \{e_1, \dots, e_n\} \text{ for } \mathbb{R}^n \tag{2}$$

$$\gamma = \{e_1, \dots, e_m\} \text{ for } \mathbb{R}^m \tag{3}$$

Which means we can write T as a matrix $[T]_\beta^\gamma$. Then if $V = W$, then $\beta = \gamma$ then $[T]_\beta \iff [T]_\beta^\beta$. We note that the columns of $[T]_\beta^\gamma$ is $T(e_1), \dots, T(e_n)$. In general

$$[T]_\beta^\gamma = ([T(v_1)]_\gamma, \dots, [T(v_n)]_\gamma) \tag{4}$$

Then we have that $T(v) = [T]_\beta^\gamma v$ for any column vector $v \in \mathbb{R}^n$. Then if $v =< a_1, \dots, a_n$, then

$$T(v) = a_1 T(e_1) + \cdots + a_n T(e_n) \tag{5}$$

It follows that

- $R(T) = \{T(v) : v \in \mathbb{R}^n\}$ = span(columns of A) = columnspace of A or $(C(A))$
- $N(T) = \{v \in \mathbb{R}^n : T(v) = 0\} = N(A)$ = kernel of a Linear Transformation

This gives a fresh perspective on the rank nullity theorem. That is if we row-reduce A, then

- $\dim R(T) = \#$ of columns that contain pivots
- $\dim N(T) = \#$ of non-pivot columns

Composition of Linear Transformations

Suppose V, W, X are vector spaces and we have that

$$V \to^T W \to^U X \tag{6}$$

Theorem 0.1. *$U \cdot T$ $V \to X$ is a linear transformation.*

Proof. Recall from HW 4,

$$\text{Some function } T \text{ is linear} \iff T(cv_1 + v_2) = cT(v_1) + T(v_2) \text{ for all } v_1, v_2, c$$

Calculate

$$U \cdot T(cv_1 + v_2) \iff U(T(cv_1 + v_2)) \tag{7}$$

$$\iff U(cT(v_1) + T(v_2)) \tag{8}$$

$$\iff cU(T(v_1)) + U(T(v_2)) \tag{9}$$

$$\iff c[U \cdot T(v_1)] + U \cdot T(v_2) \tag{10}$$

Thus $U \cdot T$ is linear. $\square$

Now suppose we're given ordered basis α ofr V, β for W, and γ for X. Then we have the matrices

$$[T]_{\alpha}^{\beta},\ [U]_{\beta}^{\gamma}, \&\ [U \circ T]_{\alpha}^{\gamma} \tag{11}$$

Note this is matrix multiplication. Also a transformation from $\mathbb{R}^n \to \mathbb{R}^n$ means that $\beta = \beta$

Definition 0.2. *Suppose $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$. The product $AB \in M_{m\times p}(\mathbb{R})$ is defined by*

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} \tag{12}$$

Note, this only makes sense for $(m \times n) \cdot (n' \times p)$ where $n = n'$.

Note: if $B = (b_1, \ldots, b_p)$, then

$$AB = (Ab_1, \ldots, Ab_p) \tag{13}$$

Theorem 0.3. *Given $V, W, X, T, U, \alpha, \beta, \gamma$ as before, then*

$$[U \cdot T]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma} \cdot [T]\alpha^{\beta} \tag{14}$$

Proof. We'll calculate the RHS, recall

1. $$[T]_{\gamma}^{\beta} = ([T(v_1)]_{\gamma}, \ldots, [T(v_n)]_{\gamma}) \tag{15}$$

2. For any $w \in W$, we have that
$$[U(w)]_{\gamma} = [U]_{\beta}^{\gamma} \cdot [w]_{\beta} \tag{16}$$

Thus,

$$[U]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta} = \Big([U]_{\beta}^{\gamma}[v]_{\alpha}, \ldots\Big) \tag{17}$$
$$\iff [U \cdot T]_{\alpha}^{\gamma} \tag{18}$$

□

In general,

1. Matrix multiplication is not generally communutative. That is
$$AB \neq BA \tag{19}$$
Say if $A = (2 \times 3)$ and $B = (3 \times 4)$, then $AB = (2 \times 4)$, but BA doesn't exist.

2. Even if A, B are borth $n \times n$, then AB is usually not equal to BA.

Theorem 0.4. *Matrix multiplication is*

1. Associative, meaning
$$A(BC) \iff (AB)C \tag{20}$$

2. Distributive, meaning
$$A(B + C) \iff AB + AC \tag{21}$$
and
$$(A + B)C \iff AC + BC \tag{22}$$

Note: (1) corresponds to the fact that for linear transformations

$$V \to^{S} W \to^{T}\to^{U} Y \tag{23}$$

Then

$$U \circ (T \circ S) = (U \circ T) \circ S \tag{24}$$

For any $a, b \in W$, we have that the zero matrix

$$0_{a\times b} = \big(0 \ldots 0\big) \tag{25}$$

Similarly, the identity matrix is

$$I_{a\times b} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots \end{pmatrix} \tag{26}$$

Facts: Whenever the equations (matrix multiplication) makes sense, we have:

$$A0 = 0 \tag{27}$$

$$0B = 0 \tag{28}$$

$$AI = A \tag{29}$$

$$IB = B \tag{30}$$

Let $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, and $A \in M_{m\times n}(\mathbb{R})$, then

Definition 0.5. *The left-multiplication operation by A is the function*

$$L_A : \mathbb{R}^n \to \mathbb{R}^m \tag{31}$$

$$L_A(v) = Av \tag{32}$$

Theorem 0.6. *1. L_A is linear*

2. $[L_A]_\beta^\gamma = A$ where β, γ are the same ordered basis for $\mathbb{R}^n$ and $\mathbb{R}^m$.

3. Given matrices $A = M_{m\times n}(\mathbb{R})$ and $B = M_{m\times p}(\mathbb{R})$, then we have that

$$L_{AB} = L_A \circ L_B \tag{33}$$