# MTH 416: Lecture 12

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## Lecture Span

- Invertibility
- Isomorphisms

## Recall

$$T : V \to W \equiv A = [T]_\beta^\gamma \tag{1}$$

But "left multiplication" does the following: given $A \in M_{m\times n}(\mathbb{R})$,

$$L_A : \mathbb{R}^n \to \mathbb{R}^m \tag{2}$$

$$L_A(v) = Av \tag{3}$$

Claim: given $\beta, \gamma$ are the standard ordered basis for $\mathbb{R}^n, \mathbb{R}^m$

$$[L_A]_\beta^\gamma = [A] \tag{4}$$

### Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \tag{5}$$

Then

$$L_A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff \begin{pmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{pmatrix} \tag{6}$$

Then

$$L_A(e_1) = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{7}$$

$$L_A(e_2) = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \tag{8}$$

$$L_A(e_3) = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \tag{9}$$

Thus

$$[L_A]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \iff [A] \tag{10}$$

# Inverse Functions

Suppose $f : S \to T$ is any function, then

**Theorem 0.1.** *$f$ has an inverse $\iff$ $f$ is bijective (one-to-one & onto), that is*

$$g \circ f = I_S \land f \circ g = I_T \tag{11}$$

*Then*

$$g(f(s)) = s \land f(g(t)) = t \tag{12}$$

*for all $s \in S$ and $t \in T$. Note, if $f$ has an inverse, then it is unique and we call it $f^{-1} = g$*

Goal: understand invertibility of linear transformations. Let $T : V \to W$ be a linear transformation, then

1. When does $T$ have an inverse?
2. What can we say about $T^{-1}$ when it exists?

## Example:

Let $V = W = \mathbb{R}^2$, and

$$T : \mathbb{R}^2 \to \mathbb{R}^2 : T(x, y) = (-y, x) \tag{13}$$

We define

$$U(x, y) = (y, -x) \tag{14}$$

then

$$T(U(x, y)) = T(y, -x) \to (-(-x), y) \iff (x, y) \tag{15}$$

similarly,

$$U(T(x, y)) = U(-y, x) \to (x, -(-y)) \iff (x, y) \tag{16}$$

Thus $U = T^{-1}$

**Theorem 0.2.** *If $T$ is invertible, then $T^{-1}$ is also a linear transformation.*

*Proof.* Suppose $T$ is an invertible linear transformation, then let $w_1, w_2 \in W$ and $c \in \mathbb{R}$. We claim that

$$T^{-1}(cw_1 + w_2) = cT^{-1}(w_1) + T^{-1}(w_2) \tag{17}$$

Let $v_1 = T^{-1}(w_1)$ and $v_2 = T^{-1}(w_2)$, then $T(v_1) = w_1$ and $T(v_2) = w_2$.

$$T(cv_1 + v_2) \iff cT(v_1) + T(v_2) \tag{18}$$

$$\iff cw_1 + w_2 \tag{19}$$

Therefore,

$$T^{-1}(cw_1 + w_2) = cv_1 + v_2 \tag{20}$$

But this is nothing but

$$\iff cT^{-1}(w_1) + T^{-1}(w_2) \tag{21}$$

□

**Theorem 0.3.** *Suppose $T : V \to W$ is an invertible linear transformation, and $\beta$ is a basis for $V$. Then*

$$\gamma = \{T(v) : v \in \beta\} \text{ is a basis for } W \tag{22}$$

**Corollary 0.4.** *If $V$ is $n$-dimensional, then $W$ is too.*

*Proof.* Proof of theorem in finite dimensional case. Suppose $T : V \to W$ is an invertible linear transformation. Let

$$\beta = \{v_1, \ldots, v_n\} \tag{23}$$

be a basis for $V$. Then we claim that

$$\gamma = \{T(v_1), \ldots, T(v_n)\} \tag{24}$$

is a basis for $W$.

**Spanning Proof**

Let $w \in W$, then we claim that $w$ is a linear combination of $\gamma$. Then let $v = T^{-1}(w)$, then

$$v = a_1 v_1 + \cdots + a_n v_n \tag{25}$$

for some constants $a_i$. Applying $T$ to both sides, we yield

$$T(v) = T(a_1 v_1 + \cdots + a_n v_n) \tag{26}$$

Then

$$w = a_1 T(v_1) + \cdots + a_n T(v_n) \tag{27}$$

So $w \in span(T(v_1), \ldots, T(v_n))$

**Linear Independent**

Suppose

$$a_1 T(v_1) + \cdots + a_n T(v_n) = 0 \tag{28}$$

for some $a_i \in \mathbb{R}$. We claim that all of these $a_i$'s are equal to 0. We take $T^{-1}$ of both sides:

$$T^{-1}(a_1 T(v_1) + \cdots + a_n T(v_n)) \iff a_1 v_1 + \cdots + a_n v_n = 0 \tag{29}$$

But because $\beta$ is linearly independent, we have that all $a_i$'s are 0. Thus, $\gamma$ is linearly independent. Therefore, $\gamma$ is a basis for $W$. □

**Definition 0.5.** *1. We call vector spaces $V$ and $W$ <u>isomorphic</u> if there exists an invertible linear transformation from one to another.*

*2. Any invertible linear transformation is called <u>isomorphism</u>*
*Notation: $V \cong W \iff V$ and $W$ are isomorphic.*

If $V \cong W$, then an isomorphism $T : V \to W$ allows us to translate any linear algebra fact about $V$ to one about $W$ and so one. In other words, any properties that we discover of $V$ can be applied to $W$ due to symmetry. "$V$ and $W$ are like the same vector space written in different ways".

## Example

$$P_2(\mathbb{R}) \cong \mathbb{R}^3 \tag{30}$$

We define an isomorphism $T$ to be

$$T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{31}$$

## Example

$$M_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4 \tag{32}$$

Then

$$U \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \tag{33}$$

is also an isomorphism.

**Note**

The relation $V \cong W$ is an equivalence relation, this implies

1. $V \cong V$
2. $V \cong W \implies W \cong V$
3. $V \cong W \wedge W \cong X \implies V \cong X$

**Theorem 0.6.** *Suppose $V$ is finite dimensional and $W$ is any vector space. Then*

$$V \cong W \iff \dim(W) = \dim(V)$$

*Proof.* $\implies$ is the corollary from earlier. Suppose $\dim(W) = \dim(V) = n$. We claim that

$$V \cong \mathbb{R}^n \cong W \tag{34}$$

Which implies $V \cong W$. For that, we choose an isomorphism

$$T(v) = [v]_\beta \tag{35}$$

So $V \cong \mathbb{R}^n$. Similarly for $W$. Thus, this proves the theorem. $\square$

Conclusion: all n-dimensional vector spaces are isomorphic to each other, in fact, they are only isomorphic to each other. That is, all n-dimensional vector spaces "look alike", that is they look like $\mathbb{R}^n$ if you choose coordinates.

# Note about matrix invertibility

**Definition 0.7.** *An $n \times n$ matrix $A$ is invertible if there exists an $n \times n$ matrix $B$ such that*

$$AB = BA = I \tag{36}$$

**Lemma 0.8.** *If $A^{-1}$ exists, then it is unique.*

*Proof.* Suppose $A$ has two inverses $B, C$. Then

$$AB = AC = BA = CA = I \tag{37}$$

Then

$$CAB = C(AB) \iff CI = (CA)B \iff IB \iff C = B \tag{38}$$

$\square$

## Connecting it to linear transformations

Let $T$ be a linear transformation from $V \to W$, where $\dim(V) = \dim(W) = n$. And let $\beta, \gamma$ be basis for $V, W$ respectively.

**Claim:** If $A$ is $[T]_\beta^\gamma$, then $A^{-1} = [T^{-1}]_\beta^\gamma$.

*Proof.* $AA^{-1} = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta = [T \circ T^{-1}]_\gamma^\gamma = I$ $\square$