MTH 553: Lecture # 12

Cliff Sun

February 16, 2026

Lecture Span

- Continution of L

From last time:

1. $Lu = f$ classicaly, then $Lu = f$ weakly
2. $Lu = f$ weakly and u is smooth, then $Lu = f$ classically
3. Uniqueness theorem

Proof. Proof of (2). For all $v \in C_0^m(\Omega)$, then consider

$$\int_\Omega (Lu - f)v dx = 0 \tag{1}$$

Hence $Lu = f$ pointwise, and also smoothly. □

Proof. Proof of (3). $Lu = f_1$ and $Lu = f_2$, then $f_1 = f_2$. □

Proof. Proof of (4) If $Lu = f$ weakly in $\tilde{\Omega}$ for any $\tilde{\Omega} \in \Omega$. If $v \in C_0^m(\Omega) \implies v \in C_0^m(\tilde{\Omega})$. □

Definition 0.1. *We call f the <u>weak derivative</u> of u (for a function u on $\Omega = [a, b]$) if $u' = f$ weakly. I.e.*

$$\int_a^b (Lu - f)v dx = 0 \tag{2}$$

With $L = d/dx$ and $L' = -d/dx$. In other words,

$$-\int_a^b uv' dx = \int_a^b fv dx \tag{3}$$

Example

We claim that below is the weak derivative of $|x|$.

$$\frac{d}{dx}(|x|) = \begin{cases} 1 & x < 0 \\ -1 & x > 0 \end{cases} \tag{4}$$

Proof. $\forall v \in C_0^1(a, b)$. Then

$$\begin{aligned} -\int_{-\infty}^{\infty} uv' dx &= \int_{-\infty}^0 xv'(x) dx - \int_0^\infty xv'(x) dx \\ &= -\int_{-\infty}^0 v(x) dx + \int_0^\infty v(x) dx \text{ boundary terms during IBP vanish } x = 0 \text{ at } x = 0 \text{ and } v(0) \text{ at } x = \pm\infty \\ &= \int_{-\infty}^\infty fv dx \end{aligned}$$

□

Therefore, $|x|$ is not classically differentiable, but it is weakly differentiable.

Transmission Conditions

Suppose $Lu = f$ weakly. what can we say about potential jumps in u or its derivatives across a curve γ? I.e. we seek an analogue of the jump condition for shocks.

Example

Suppose $u_{xy} = 0$ weakly in $\mathbb{R}^2$. And $\gamma = \{\text{y - axis}\}$. And suppose $u_{xy} = 0$ classically in Ω_1 and Ω_2. Then, $\forall v \in C_0^2(\mathbb{R}^2)$:

$$0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv_{xy}dydx$$

Since $u_{xy} = 0$ weakly. Then we integrate by parts twice.

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv_{xy}dydx \\
&= \int_{0}^{\infty}\int_{-\infty}^{\infty} uv_{xy}dydx + \int_{-\infty}^{0}\int_{-\infty}^{\infty} uv_{xy}dydx \\
&= -\int_{0}^{\infty}\int_{-\infty}^{\infty} u_y v_x dydx - \int_{-\infty}^{0}\int_{-\infty}^{\infty} u_y v_x dydx \\
&= -\int_{-\infty}^{\infty}\int_{0}^{\infty} u_y v_x dxdy - \int_{-\infty}^{\infty}\int_{-\infty}^{0} u_y v_x dxdy \\
&= \int_{-\infty}^{\infty} u_y(0+,y)v(0,y)dy - \int_{-\infty}^{\infty} u_y(0-,y)v(0,y)dy \\
&= \int_{-\infty}^{\infty} \frac{d}{dy}\left[u(0+,y) - u(0-,y)\right]v(0,y)dy
\end{aligned}$$

If this is valid for all v, then this integrand must be 0. i.e.

$$u(0+,y) - u(0-,y) = \text{const.} \tag{5}$$

Along the y axis. I.e. the jump across the y axis must be the same for any value on the y axis!
This is an example of the transmission condition.

Transmission condition for the wave equation

If $u_{\mu\eta} = 0$ weakly, if u is a smooth solution except for constant jumps along vertical or horizontal lines (in the μ, η coordinate system.)

Then reverting to the x, y coordinate plane, u becomes a weak solution of

$$u_{tt} - c^2 u_{xx} = 0 \tag{6}$$

If it is a smooth function except for constant jumps along the characeristics $x \pm ct$. This means that a traveling square wave is a weak solution.

General method

Given Ω and a curve γ that slices through Ω. Then find the transmission condition by

1. Writing the definition of the weak solution
2. Split integrals into Ω_1 and Ω_2
3. Integrate by parts using divergence theorem (retain boundary terms)
4. Use $Lu = f$ classically on Ω_1 and Ω_2 and cancel out
5. Stare at what remains (bruh)