MTH 553: Lecture # 8

Cliff Sun

February 6, 2026

Lecture Span

- Wave equation

One dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0 \tag{1}$$

Here, u is the displacement of the string and c is the speed of the wave. **But why should we expect this equation?**

Plausibility

We consider a concave parabola with a $u_{xx} < 0$. Here, we expect the tension in the string to pull the string down. Here, this tension is a force, which means an acceleration, or u_{tt}. Therefore, if $u_{xx} < 0$, then $u_{tt} < 0$ since it would be pulling it down. Therefore, we naturally arrive at the wave equation, where the concavity of the wave is proportional to its acceleration.

$$c^2 u_{xx} = u_{tt} \tag{2}$$

This wave equation is called the homogenous wave equation, or HWE.

Solution on $\mathbb{R}$

Change variables, let $\mu = x + ct$ and $\eta = x - ct$. That is

$$\begin{aligned} x &= \frac{\mu + \eta}{2} \\ t &= \frac{\mu - \eta}{2c} \end{aligned}$$

Then, the wave equation in these new coordinates becomes

$$u_{\mu\eta} = 0 \tag{3}$$

We check: by the chain rule, we have that

$$\begin{aligned} u_\mu &= u_x x_\mu + u_y y_\mu \\ &= \frac{1}{2} u_x + \frac{1}{2c} u_t \end{aligned}$$

Similar for u_η. Therefore,

$$u_{\mu\eta} = \frac{1}{4c^2}\left(c^2 u_{xx} - u_{tt}\right) \tag{4}$$

Hence,

$$u = f(\mu) + g(\eta) \iff f(x+ct) + g(x-ct) \tag{5}$$

These are traveling waves. That is

$$u = \text{left-moving wave of shape f} + \text{right-moving wave of shape g}$$

IVP for HWE

This is the famous d'Alembert solution. Consider a wave equation with initial conditions

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(x,0) &= g(x) \\ u_t(x,0) &= h(x) \end{aligned}$$

Using equation 5, we have that

$$g(x) = u(x,0) = F(x) + G(x) \tag{6}$$

Then

$$h(x) = u_t(x,0) = c\left(F'(x) - G'(x)\right) \tag{7}$$

Integrating both sides yields

$$\frac{1}{c}\int_0^x h(\xi)d\xi + C = F(x) - G(x) \tag{8}$$

If we add and subtract these equations, we obtain

$$F(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_0^x h(\xi)d\xi + \frac{C}{2} \tag{9}$$

$$G(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_0^x h(\xi)d\xi - \frac{C}{2} \tag{10}$$

Then this means that

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)d\xi \tag{11}$$

So this solution is like the left and right moving "responses" to the initial condition.

Proposition 0.1. *Well-posedness*

1. *Existence, does a solutione exist? If $g \in C^k(\mathbb{R})$, $h \in C^{k-1}(\mathbb{R})$, for some $k \geq 2$, then use equation (5) gives a C^k solution of (11).*
2. *Uniqueness, d'Alembert (11) is unique C^k smooth solution.*
3. *Continuous dependence. u depends continuously on the data, with respect to the max-norm. This means that if $||g_1 - g_2||_\infty \leq \epsilon$, and $||h_1 - h_2||_\infty \leq \epsilon$, then $||u_1(;t) - u_2(;t)||_\infty \leq (1+t)\epsilon$. Here $||g||_\infty$ is $\max_{x\in\mathbb{R}} |g(x)|$.*

Proof. 1. Check if u is C^k smooth (has k derivatives) by using fundamental theorem of Calculus.

2. If u and v solve (4), then consider $w = u - v$

$$\begin{aligned} w_{tt} - c^2 w_{xx} &= 0 \\ w(x,0) &= 0 \\ w_t(x,0) &= 0 \end{aligned}$$

Which implies that w is the zero function which means that $u = v$.

3. homework 3, bruh.

□

Definition 0.2. *Consider a triangle and as cone with slope $x = \pm t$. Domain of dependence, for u is $[x - ct, x + ct]$ for the triangle. And for the cone, the range of influence (points that the wavefunction can later influence) has speed of $\pm c$ at vertex ξ.*

IVP for NHWE on $\mathbb{R}$

Now, we now consider the non-homoegenous wave equation:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x,t) \\ u(x,0) &= g(x) \\ u_t(x,0) &= h(x) \end{aligned}$$

Decompose $u = w + z$ where w is the homogenous solution and z is the non-homoegenous solution. Note

$$\begin{aligned} z_{tt} - c^2 z_{xx} &= f(x,t) \\ z(x,0) &= 0 \\ z_t(x,0) &= 0 \end{aligned}$$