PHYS 326: Lecture 2

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Coupled Oscillators (2 degrees of freedom)

Imagine 2 masses m_A and m_B, with 3 springs, one spring k_1 connecting m_A to the wall, k_2 connecting m_A and m_B, and k_3 connecting m_B to the wall. We have that x_1 and x_2 describe the motion for m_A and m_B respectively. How to we find the equations of motion? Then

$$KE = \frac{1}{2}m_1 x_1^2 + \frac{1}{2}m_2 x_2^2 \tag{1}$$

$$PE = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_3 x_2^2 + \frac{1}{2}k_2(x_1 - x_2)^2 \tag{2}$$

We find E.L equations for this problem:

$$m_1 \ddot{x_1} = -k_1 x_1 + k_2(x_2 - x_1) \tag{3}$$

$$m_2 \ddot{x_2} = -k_3 x_2 - k_2(x_2 - x_1) \tag{4}$$

We write this in matrix form:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x_1} \\ \ddot{x_2} \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{5}$$

We rewrite this equation as

$$M\ddot{\vec{x}} + K\vec{x} = 0 \tag{6}$$

Note that M and K are symmetric matrices, that is $M^t = M$ and $K^t = K$. Ansatz: normal mode solution. We guess

$$\vec{x}(t) = \vec{a}e^{i\omega t} \tag{7}$$

Then

$$(-M\omega^2 + K)\vec{a}e^{i\omega t} = 0 \tag{8}$$

We expand

$$(-M\omega^2 + K)\vec{a} = \begin{pmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_1 + k_3 - m_2\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \tag{9}$$

Then

$$(k_1 + k_2 - m_1\omega^2)a_1 - k_2 a_2 = 0 \tag{10}$$

$$-k_2 a_1 + (k_1 + k_3 - m_2\omega^2)a_2 = 0 \tag{11}$$

If we brute force solve this, we get $a_1 = a_2 = 0$. To find the non-trivial solution, we demand that

$$\det(K - M\omega^2) = 0 \tag{12}$$

So that both equations are just multiples of each other. We solve for ω, and then find $\vec{a}$ or eigenvectors. Then the general solutions must be a linear combination of the eigenfrequncies. We demand this determinant condition because then it implies that some of the vectors are linearly dependent, which we can then use to construct a non-trivial vector that finds this linear dependence.

Normal Coordinates

Define new coordinates for describing the motion:

$$\zeta_1 = \frac{1}{2}(x_1 + x_2) \tag{13}$$

$$\zeta_2 = \frac{1}{2}(x_1 - x_2) \tag{14}$$

$$\zeta = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\vec{x} \tag{15}$$

We can write E.O.M of equal springs and equal masses in ζ,

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) \tag{16}$$

$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) \tag{17}$$

We sum the equations

$$m(\ddot{x}_1 + \ddot{x}_2) = -k(x_1 + x_2) \tag{18}$$

$$\zeta_1 = -\frac{k}{m}\zeta_1 \tag{19}$$

We then subtract the equations and yield

$$\zeta_2 = -\frac{3k}{m}\zeta_2 \tag{20}$$