Generic Homework

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Recall:

1. $\mathbb{Q}$ is countable.
2. $\bar{\mathbb{Q}} = \{\text{Algebraic numbers}\}$ is countable.
3. $[0, 1]$ is uncountable. This implies that $\mathbb{R}$ is also uncountable.

Lemma 0.1. *$\mathbb{R}/\mathbb{Q}$ and $\mathbb{R}/\bar{\mathbb{Q}}$ are uncountable.*

Proof. Suppose for contradiction that $\mathbb{R}/\mathbb{Q}$ is countable. Then $\mathbb{R}$ is the union of both of these countable sets. Thus $\mathbb{R}$ is also countable. This is a contradiction, thus $\mathbb{R}/\mathbb{Q}$ is uncountable. □

Theorem 0.2. *Cantor's theorem. For any set A, $|A| < |P(A)|$.*

Proof. We first must show that there exists an injective function from A to its powerset. Secondly, there cannot exist a bijective function from A to its powerset. We first choose

$$f(a) = \{a\} \tag{1}$$

This is injective if $\{a\} = \{b\}$, then $a = b$. Secondly, suppose for the sake of contradiction that g is a bijection from A to its powerset. In particular, g is surjective. However, we state that there exists some set X such that X is not an element of $g(a_i)$ for $a_i \in A$. We choose X to be:

$$X = \{a \in A : a \notin g(a)\} \tag{2}$$

Then we claim that X cannot be in $g(a_i)$.

Proof. By definition, b is in X iff

$$b \in X \iff b \notin g(b) \tag{3}$$

But that means that $X \neq g(b)$ since one contains b and the other does not. □

□

Theorem 0.3. *The following all have the same cardinalities, which we call $c = 2^{\aleph_0}$ (the cardinality of the continuum).*

1. $[0, 1]$ *or really any closed interval* $[a, b]$ *or the open interval* (a, b) *for* $a < b$.
2. $\mathbb{R}$
3. $P(\mathbb{N})$ *or the powerset of* S *for* S *countably infinite.*

Theorem 0.4.

$$|A| \leq |B| \leq |A| \implies |A| = |B| \tag{4}$$