PHYS 326: Lecture 7

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Symmetries

Crystal Lattice

Infinite number of degrees of freedom. All masses m and spring constants k. Consider the spatial coordinates q_n where q_n is the middle one. Each mass represents a single atom, springs are the forces between neighboring atoms. Then

$$L = \frac{m}{2}\left(\cdots + \dot{q}_{n-2}^2 + \dots\right) - \frac{k}{2}\left(\cdots + (q_{n-2} - q_{n-1})^2 + (q_{n-1} - q)^2 + \dots\right) \tag{1}$$

Thus

$$m\ddot{q}_n + k(2q_n - q_{n+1} - q_{n-1}) = 0 \text{ for each } n \tag{2}$$

Seek solution, let

$$\bar{q}(t) = \bar{u}e^{i\omega t} \tag{3}$$

Solve

$$\left[\mathbf{K} - \omega^2\mathbf{M}\right]\bar{u} = 0 \tag{4}$$

Using parity symmetry. Move left or right and reproduce the exact same system. Let

$$\bar{u} = \begin{bmatrix} \dots \\ \dots \\ 1 \\ a \\ b \\ c \\ \dots \end{bmatrix} \tag{5}$$

Apply a symmetry matrix:

$$\mathbf{S}\bar{u} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ 1 \\ a \\ b \\ \dots \end{bmatrix} = \lambda\bar{u} \tag{6}$$

Thus

$$1 = \lambda a \iff a = \frac{1}{\lambda} \tag{7}$$

$$a = \lambda b \iff b = \frac{1}{\lambda^2} \tag{8}$$

Thus

$$\bar{u} = \begin{bmatrix} \dots \\ 1 \\ \frac{1}{\lambda} \\ \frac{1}{\lambda^2} \\ \dots \end{bmatrix} \tag{9}$$

For infinite system, $|\lambda| = 1$, otherwise $\bar{u}$ is not physical. If $\lambda = 1$, then all atoms are moving together, or $\omega = 0$. If $\lambda = 1$, then

$$u_{n+1} = u_n \tag{10}$$

Generally, $\lambda = e^{-i\phi}$. Thus

$$q_n = e^{i(\omega t - n\phi)} \tag{11}$$

We find the frequencies:

$$m\ddot{q}_n + k(2q_n - q_{n-1} - q_{n+1}) = 0 \tag{12}$$

Then

$$-m\omega^2 + k(-e^{i\phi} - e^{i\phi}) = 0 \tag{13}$$

$$\omega^2 = \frac{2k}{m}(1 - \cos\phi) \tag{14}$$