# MTH 553: Lecture # 3

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January 26, 2026

## Last time

Studied quasilinear equations

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{1}$$

Such that

$$\begin{aligned} dx/dt &= a(x, y, u) \\ dy/dt &= b(x, y, u) \\ dz/dt &= c(x, y, u) \end{aligned}$$

If we define $u(x, y) = z(t)$, then

$$c = dz/dt = u_x a(x, y, u) + u_y b(x, y, u) \tag{2}$$

## Lecture Span

1. semi-linear example ($u$ does not appear in the coefficients)

$$xu_x + u_y = u^2 \tag{3}$$

$$u(x, 0) = k(x) \text{ initial condition is on the } x \text{ axis} \tag{4}$$

We first parameterize $\Gamma$,

$$\Gamma : \begin{cases} x = s \\ y = 0 \\ z = k(s) \end{cases}$$

We also have to check that

$$< f', g' > \nparallel < a, b > \in \Gamma \tag{5}$$

Here, $f'$ is the derivative of $x$ and $g'$ is the derivative of $y$. Therefore,

$$< 1, 0 > \nparallel < x, 1 >=< s, 1 > \tag{6}$$

This is true. Next, we solve the characteristic system:

$$\begin{aligned} dx/dt &= x \\ dy/dt &= 1 \\ dz/dt &= z^2 \end{aligned}$$

We can first solve

$$y = t + c_2(s)$$

Note, $c_2(s)$ is the constant of coefficient, but can depend on where you start on the initial curve. Then we solve

$$x = c_1(s)e^t$$

Then we can solve

$$z = \frac{1}{c_3(s) - t}$$

We then apply the initial condition that $c_2(s) = 0$, $c_1(s) = s$, and $c_3(s) = k(s)$. Therefore, we get

$$\begin{aligned} x &= se^t \\ y &= t \\ z &= \frac{1}{1/k(s) - t} = \frac{k(s)}{1 - tk(s)} \end{aligned}$$

Here,

$$u(x, y) = z = \frac{k(s)}{1 - tk(s)} \tag{7}$$

We want to solve for $s, t$ in terms of $x, y$. Here,

$$y = t \tag{8}$$

$$s = xe^{-y} \tag{9}$$

Therefore,

$$u(x, y) = \frac{k(xe^{-y})}{1 - yk(xe^{-y})} \tag{10}$$

We need this denominator to be $\neq 0$, therefore, the solution exists when

$$yk(xe^{-y}) \neq 1 \tag{11}$$

This curve does contain the initial condition. Moreover, if $k$ is bounded ($|k(x)| \leq A \quad \forall x$), then the domain contains strip around axis (what does this mean??).

# General practical difficulties

1. Can ODEs (of characteristic equations) be solved explicitly?
2. Can $s, t$ be found in terms of $x, y$?
3. On what domain in the x-y plane is the solution defined?

# Example: Inviscid burger's equation

$$uu_x + u_y = 0 \tag{12}$$

$$u(x, 0) = k(x) \tag{13}$$

Interpretation: stream of particles with constant velocity where $u(x, y)$ is the velocity of particle at location $x$, time $y$. In other words, for all $x, y$, then given $\tau$

$$u(x + \tau u(x, y), y + \tau) = u(x, y) \tag{14}$$

Where $\tau$ has units of time.

Differentiate w.r.t $\tau$ and let $\tau = 0$, then we get $uu_x + u_y = 0$. Solve

$$
\begin{aligned}
dx/dt &= z \\
dy/dt &= 1 \\
dz/dt &= 0
\end{aligned}
$$

Note that

$$dy/dx = \frac{1}{z} \tag{15}$$

In the homework, if $k'(x) \geq 0$, then the particles speeds go up. In particular, then $k(x)$ increasing and $1/k(x)$ is decreasing. The slope of the characteristic decreases from left to right. We have

$$u(x_0) = k(x_0) \tag{16}$$

Then the slope of the characteristic is $1/k(x_0)$ (FIGURE THIS OUT). These projected characteristics give a global solution. But what if $k'(x) \not\geq 0$, say $k(x_1) > k(x_2)$ for some $x_1 < x_2$. Then, some of the characteristic curves will intersect. What do you do? Should $u$ at this intersection point be equal to $k(x_1)$ or $k(x_2)$? These particles are colliding, and we see shock formation.