MTh 416: Lecture 22

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Lecture Span

- Sum decomposition
- Inner product spaces

Definition 0.1. *If A is diagonalizable, then $\mathbb{R}^n$ is a direct sum of the eigenspaces of A. That is every $v \in \mathbb{R}^n$ can be uniquely expressed as*

$$w_1 + \cdots + w_k \tag{1}$$

where $w_i \in E_{\lambda_i}$.

Inner product spaces

Inner products will abstract away from dot product on $\mathbb{R}^n$. Note, up until now, nearly everything mentioned works over any field

$$F = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \ldots \tag{2}$$

Exception: Fundamental theorem of Algebra, which implies that matrices have enough eigenvalues. For inner products, we really only care about $\mathbb{R}$ and $\mathbb{C}$. Then the following abstracts from the dot product, then

Definition 0.2. *Let V be a vector space over $\mathbb{R}$. Then an inner product on V is a function which given $x, y \in V$, produces a scalar $c \in \mathbb{R}$. Denoted as*

$$< x, y >= c \in \mathbb{R} \tag{3}$$

It satisfies the following properties:

1. $\langle x + z, y \rangle =< x, y > + < z, y >$
2. $< cx, y >= c < x, y >$
3. $< x, y >=< y, x >$
4. *If* $x \neq 0$, *then* $< x, x >> 0$

Example

Let

$$V = C[0, 1] = \{\text{continuous functions from 0 to 1 to } \mathbb{R}\} \tag{4}$$

We define

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx \tag{5}$$

Non-examples

Let $V = \mathbb{R}^2$, then we define $\langle x, y \rangle$ as

1. $a_1^2 b_1^2 + a_2^2 b_2^2$ (not linear)
2. $a_1 b_1 + a_1 b_2 + a_2 b_2$ (not symmetric)
3. $-a_1 b_1 - a_2 b_2$

Definition 0.3. *Let V be a vector space over $\mathbb{C}$, an inner product on V is a function that takes two vector inputs and produces a complex number scalar, that is*

$$\langle x, y \rangle = c \in \mathbb{C} \tag{6}$$

We impose the following properties:

1. *As same as before*
2. *As same as before*
3. *$\langle x, y \rangle * = \langle x, y \rangle$ where this star is the complex conjugate of such.*
4. *As same as before (note $\langle x, x \rangle = \langle x, x \rangle * \iff \langle x, x \rangle \in \mathbb{R}$)*

Non Example

Let $V \in \mathbb{C}$, then if we define

$$\langle x, y \rangle = a_1 b_1 + \dots \tag{7}$$

Then if $x = y = (i, 0)$, then this inner product evaluates to -1 which doesn't make sense. Thus we want to define

$$\langle x, y \rangle = a_1 \overline{b_1} + a_2 \overline{b_2} + \dots \tag{8}$$

So this works out now.

Example

Let $V = M_{n \times n}(\mathbb{C})$, and if $A, B \in V$, then

$$\langle A, B \rangle = \sum \sum a_{ij} \overline{b_{ij}} \tag{9}$$

This is called the Forbenius inner product, it has the following reinterpretation

Definition 0.4. *If $A \in M_{n \times n}(\mathbb{C})$, then conjugate transpose, or the adjoint matrix, is the denoted as*

$$\overline{A} = \overline{A^T} \tag{10}$$

Definition 0.5. *The trace of a matrix $A \in F$ is the sum of its diagonal entries:*

$$Tr(A) = \sum a_{ii} \tag{11}$$

Fact

The Forbenius inner product is equivalent to

$$\langle A, B \rangle = tr(\overline{B} A) \tag{12}$$

Definition 0.6. *An inner product space is a pair $(V, \langle , \rangle)$ where V is a vector space over $\mathbb{R}$ or $\mathbb{C}$ and $\langle , \rangle$ is an inner product of V.*

Fact

For any inner product space, the following holds for all $x, y, z \in V$ and $c \in F(= \mathbb{R} \vee \mathbb{C})$.

1. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
2. $\langle x, cy \rangle = \overline{c} \langle x, y \rangle$
3. $\langle x, 0 \rangle = 0$
4. $\langle x, x \rangle = 0 \iff x = 0$

Definition 0.7. *If V is an inner product space and $x \in V$, then the length/norm of x is*

$$||x|| = \sqrt{\langle x, x \rangle} \tag{13}$$

Theorem 0.8. *Let V be an inner product space for all vectors $x, y \in V$ and $c \in F$,*

1. $||cx|| = |c| \cdot ||x||$
2. $||x|| = 0 \iff x = 0$
3. *Cauchy-Schwarz inequality:* $|\langle x, y \rangle| \leq ||x|| \cdot ||y||$
4. *Triangle inequality:* $|x + y| \leq |x| + |y|$