MTH 553: Lecture # 11

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Lecture Span

- Reflection Method

Reflection Method

Similar to the parallelogram method. Let g, h be defined for $[0, L]$

Dirichlet Boundary Conditions

Extend g, h by odd reflection over $[-L, L]$ with period $2L$ and extend periodically over $\mathbb{R}$. Solve with d'Alembert's formula. Here, imagine a picture where $h \equiv 0$ and g_2 is a rectangular pulse. Now, imagine performing an odd reflection on g so now there is an inverted rectangular pulse over the y axis. Now, imagine that there is a period of $2L$.

You can imagine that as t progresses, the rectangular pulses cancel out at $x = 0$. So now, to the observer, it looks like the square has been reflected upside down.

Neumann Boundary Conditions

Now, let's do an even extension over $[-L, L]$ and extend periodically. Then, the square is reflected the same way up, but reflected over the y axis. (like a parity transformation)

Note: for physical intuition of the BCs, look at Farlow handout on Canvas (week 4).

Adjoints and weak solutions

Theorem 0.1. *Divergence theorem:*

$$\int_\Omega \nabla \cdot \vec{F} dx = \int_{\partial\Omega} \vec{F} \cdot \vec{v} dS \tag{1}$$

Where $\vec{v}$ is a normal vector to $\partial\Omega$. This is a version of the fundamental theorem of Calculus but in three dimensions.

Next, we apply the div theorem to derive a generalized integration by parts formula with $u, r \in C^1(\overline{\Omega})$ and $\vec{F} = \langle 0, \ldots, uv, \ldots, 0 \rangle$. So

$$\nabla \cdot \vec{F} = \partial_{x_n}(uv) = (\partial_{x_n} u)v + u(\partial_{x_n} v) \tag{2}$$

Next, we have that

$$F \cdot v = uvv_k \tag{3}$$

Therefore, we can derive an integration by parts formula

$$\int_\Omega \frac{\partial u}{\partial x_k} v dx = -\int_\Omega u \frac{\partial v}{\partial x_k} dx + \int_{\partial\Omega} uvv_k dS \tag{4}$$

If $v = 0$ on $\partial\Omega$, then we have that

$$\int_\Omega \frac{\partial u}{\partial x_k} v dx = -\int_\Omega u \frac{\partial v}{\partial x_k} dx \tag{5}$$

Repeating, then we get

$$\int_\Omega (D^\alpha u) v dx = (-1)^{|\alpha|} \int_\Omega u (D^\alpha v) dx \tag{6}$$

For all $u \in C^{|\alpha|}(\Omega)$ and for all $v \in C_0^{|\alpha|}(\Omega)$ where v is zero for some neighborhood around the boundary. Here, $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\alpha_i \in \mathbb{Z}^+ \geq 0$.

Where α_i determines the number of times that you take the partial derivative with respect to x_i. In other words,

$$D^\alpha \equiv \Pi^n \left(\partial_{x_i}\right)^{\alpha_i} \tag{7}$$

And

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \tag{8}$$

is the order of the multi-index α. As an example,

$$u_{x_1,x_2} = D^{(1,1)} u \tag{9}$$

Where $\alpha = (1,1)$. Then

$$\sum_{\alpha:|\alpha|=2} a_\alpha D^\alpha u = a_{2,0} u_{x_1,x_1} + a_{1,1} u_{x_1,x_2} + a_{0,2} u_{x_2,x_2} \tag{10}$$

Is a linear combination of all partial derivatives. Consequence, let $m \leq 1$, then

$$Lu = \sum_{\alpha:|\alpha|\leq m} a_\alpha(x) D^\alpha u \tag{11}$$

Where m denotes the maximum number of partial derivatives you have to take in any direction. Then

$$\int_\Omega (Lu) v dx = \int_\Omega u (L' v) dx \tag{12}$$

Where L' is the adjoint operator such that

$$L' v = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha(x) v) \tag{13}$$

Proof, use the integration by parts formuala, but let $v = a_\alpha v$. Note that $v \in c_0^m(\Omega)$ where v vanishes near $\partial\Omega$.

Examples

1. $Lu = u_{x_1,x_1} \implies L'v = v_{x_1,x_1}$. Here, $L = L'$, so it is "formally self-adjoint".
2. $\nabla^2 u = u_{x_1,x_1} + u_{x_2,x_2}$. It is clear that $\nabla^2 = \nabla'^2$
3. $Lu = u_{x_1}$, then $L'v = v_{x_1}$. So $L \neq L'$. (because $(-1)^{|\alpha|} = -1$ since $\alpha = (1,0)$)

Definition 0.2. *Let $f \in L'(\Omega)$ be an integrable function. (integral of f is finite over Ω). Then we call u a classical solution of $Lu = f$ if*

$$u \in C^m(\Omega) \text{ and } Lu = f \tag{14}$$

Call u a weak solution of $Lu = f$ if

$$u \in L^1(\Omega) \text{ and } \int_\Omega u(L'v) = \int_\Omega f v dx \tag{15}$$

for all $v \in C_0^m(\Omega)$.

Property: Classical $\implies$ weak. i.e. if $Lu = f$ classically then $Lu = f$ weakly.

Proof. Substitute $Lu = f$ and use the integration by parts theorem. $\square$

Property (2): Weak + smooth function $\implies$ classical solution. i.e. if u is a weak solution and is also differentiable enough times, then it is also a classical solution.

Proof. Use integration by parts and also rearrange terms. $\square$