MTH 553: Lecture # 7

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Lecture Span

- Uniqueness of entropy solution

Theorem 0.1. *If G is uniformly convex or uniformly concave, then there exists at most one weak solution of the IVP (initial value problem)*

$$
\begin{aligned}
G(u)_x + u_y &= 0 \\
u(x, 0) &= k(x)
\end{aligned}
$$

that satisfies the entropy condition. In particular, the entropy condition singles out a physically meaningful solution.

Example - Riemann's problem

Assume that G is uniformly convex, and u_l and u_r are real numbers. Consider

$$
\begin{aligned}
G(u)_x + u_y &= 0 \\
u(x, 0) &= \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}
\end{aligned}
$$

Then the unique entropy solution is

Case 1

If $u_l > u_r$ (shock curve, not a fan), then

$$
u(x, y) = \begin{cases} u_l & x < \sigma y \\ u_r & x > \sigma y \end{cases} \tag{1}
$$

Where

$$
\sigma = \frac{G(u_l) - G(u_r)}{u_l - u_r} \tag{2}
$$

Case 2

If $u_l < u_r$, then this is a fan. Then

$$
u(x, t) = \begin{cases} u_l & x < G'(u_l)y \\ G'^{-1}\left(\frac{x}{y}\right) & G'(u_l)y < x < G'(u_r)y \\ u_r & x > G'(u_r)y \end{cases} \tag{3}
$$

Proof. We now prove each limiting case. For part 1, we check the entropy condition, which states

$$G'(u_l) > \xi'(y) > G'(u_r) \tag{4}$$

This is true for general uniformly convex function. For part 2, we check u is continuous at the fan boundary $x = G'(u_l)y$, $u(x, y) = G'^{-1}(x/y) = G'^{-1}(G(u_l)) = u_l$. Note, u is a smooth solution in the fan and outside of the fan since $u(x, y) = v(x/y)$ where $v = G'^{-1}$. Check

$$\begin{aligned} &G(u)_x + u_y \\ &\quad = G'(v(x/y))v'(x/y)/y + v'(x/y)(-x/y^2) \\ &\quad = 0 \end{aligned}$$

Since $G'(v(x/y)) = x/y$. We don't have to worry about the entropy condition since this is not a shock. □

Example

Burger's equation

$$\left(\frac{1}{2}u^2\right)_x + u_y = 0 \tag{5}$$

Consider

$$u = \begin{cases} 0 & x < -1 \\ -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases} \tag{6}$$

This is symmetrical, so only consider $x > 0$. For $x > 0$, $0 < y \leq 2$, we have a shock on

$$x = 1 + \frac{1}{2}y \tag{7}$$

We also get a fan $|x| < y$, with

$$u = G'^{-1}(x/y) = \frac{x}{y} \tag{8}$$

But at $y = 2$, the shock hits the fan. So then what is the shock curve?

$$\xi'(y) = \frac{1}{2}(u_l + u_r) = \frac{x}{2y} = \frac{dx}{dy} \tag{9}$$

Solving this ODE yields

$$\frac{dx}{x} = \frac{dy}{2y} \tag{10}$$

$$x = C\sqrt{y} \tag{11}$$

Where at $y = 2$, $x = 2$, so then

$$2 = C\sqrt{2} \implies C = \sqrt{2} \tag{12}$$

Therefore,

$$\xi(y) = \sqrt{2y} \tag{13}$$

Now, we brief fully non-linear PDEs.

Not examinable fully non-linear equations

$$F(x, y, u, u_x, u_y) = 0 \tag{14}$$

Associated characteristic ODE system

$$\begin{aligned}
dx/dt &= F_p(x, y, z, p, q) \\
dy/dt &= F_q \\
du/dt &= pF_p + qF_q \\
dp/dt &= -F_x - pF_z \\
dq/dt &= -F_y - qF_q
\end{aligned}$$