PHYS 325: Lecture 3

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Lecture Span

- Equations of Motion
- Simple Harmonic Oscillator

Equations of Motion (EOM)

Goal: Derive EOM and solve for the particle's trajectory $\vec{r}(t)$

Definition 0.1. *An equation of motion is a differential equation for $\vec{r}(t)$. An example is*

$$\vec{F} = m\vec{a} \iff m\ddot{\vec{r}}(t) \tag{1}$$

or

$$\vec{F} = \dot{\vec{p}} \tag{2}$$

Strategy

1. Choose reference frame & coordinate system
2. Identify relevant (external) forces, that is not including any forces within the system (molecular forces, etc.)
3. Determine equation of motion
$$\vec{F} = m\ddot{\vec{r}} \tag{3}$$
4. Integrate EOM for a given $\vec{F}(\vec{r}, \dot{\vec{r}}, t)$ to find $\vec{r}(t)$
5. Fix integration constants using initial or boundary conditions

Zero forces

From Newton's 2nd law (N2L),

$$0 = \vec{F} = m\vec{a} = m\dot{\vec{v}} \implies \vec{v} = \vec{v_0} \tag{4}$$

Thus

$$\vec{v} = \frac{d\vec{r}}{dt} \implies \vec{r}(t) = \vec{v_0}t + \vec{k_0} \tag{5}$$

Constant Force

Derive EOM:

$$\vec{F} = m\vec{a} = c \tag{6}$$

$$\vec{a} = \frac{\vec{F}}{m} = \ddot{\vec{r}} \tag{7}$$

$$\vec{v}(t) = \frac{\vec{F}}{m}t + \vec{v_0} \tag{8}$$

$$\vec{r}(t) = \frac{\vec{F}}{2m}t^2 + \vec{v_0}t + \vec{r_0} \tag{9}$$

Where

$$\vec{a} = \frac{\vec{F}}{m} \tag{10}$$

Time dependent force

1. Derive EOM from N2L
$$\vec{a} = \frac{\vec{F}(t)}{m} \tag{11}$$
2. Integrate twice to get $\vec{r}(t)$ with the appropriate constants of integration

Example: Forced Harmonic Oscillator

Setup: Particle of mass m that moves along the x-axis $-\infty < x < \infty$ and is subject to a force $\vec{F} = F_0 \cos(\alpha t)$. Find the equation of motion. The particle starts at $t = 0, x = 0, v = 0$

$$\vec{a} = \frac{F_0}{m} \cos(\alpha t) \tag{12}$$
$$\vec{v} = \frac{F_0}{\alpha m} \sin(\alpha t) \tag{13}$$
$$\vec{r} = -\frac{F_0}{\alpha^2 m} \cos(\alpha t) + r_0 \tag{14}$$

but
$$\vec{r}(0) = -\frac{F_0}{\alpha^2 m} + r_0 = 0 \tag{15}$$

Thus
$$r_0 = \frac{F_0}{\alpha^2 m} \tag{16}$$

So
$$\vec{r} = \frac{F_0}{\alpha^2 m}(1 - \cos(\alpha t)) \tag{17}$$

Position Dependent Force

$$ma = F(x) = m\dot{v} \tag{18}$$
$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} \iff v \cdot \frac{dv}{dx} \tag{19}$$
$$mv\frac{dv}{dx} = F(x) \tag{20}$$
$$\frac{1}{2}m(v^2 - v_0^2) = \int_{x_0}^{x} F(x)dx \tag{21}$$

If Force is conservative, then it could be written as a potential gradient.

- Path independent
- $\vec{F} = -\vec{\nabla} U$
- Conservative means there musn't be friction, or be time dependent(?)

Assuming that Force is conservative, then we call $F = -\frac{dU}{dx}$, then

$$\int_{x_0}^{x} -\frac{dU}{dx} = -[U(x) - U(x_0)] \tag{22}$$

Then,
$$E = KE + U(x) = KE_0 + U(x_0) \tag{23}$$

Conservation of mechanical energy!! Then

$$v(x) = \pm\sqrt{\frac{2}{m}(E - U(x))} \tag{24}$$