# PHYS 325: Lecture 10

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September 26, 2024

## Lecture Span

- Midterm
- Gravitation

## Midterm

Will cover up to the gravitational forces. On **OCTOBER 10th** or in 2 weeks! There will be a practice midterm, & we will be allowed our own formula sheet.

## Gravitation

Given some mass with density $\rho(\vec{r})$, we have that

$$\vec{F} = -Gm_2 \int \frac{\rho(r_n)}{|r_2 - r_1|^3}(r_2 - r_1)dr_1 \tag{1}$$

**Gravitational field**

$$\vec{g} = \frac{\vec{F}}{m_2} \tag{2}$$

**Gravitational potential $\Phi$**

$$\vec{g} = -\nabla\Phi \tag{3}$$

In general, for a point mass $m_1$, we have that the potential energy is

$$u = \frac{-Gm_1m_2}{r} \tag{4}$$

and the potential is

$$\Phi = \frac{-Gm_1}{r} \tag{5}$$

For an extended mass, we have that

$$\Phi = -G\frac{\int \rho(r_1)}{|r_2 - r_1|}d^3r_1 \tag{6}$$

### Example 1: Finding $\Phi$ of a spherical shell

**Set-up**

1. Uniform Spherical shell
2. Mass $m$, radius $r$, thickness $h$, $\rho = \frac{M}{4\pi R^2 h}$ (Assume thickness is really small)

Let $s(\theta)$ be the line pointing from a point on the shell to the point mass $m_2$.

**Strategy**

$$\Phi = \int d\Phi \implies d\Phi - \frac{-G}{|r - r_1|}dm \tag{7}$$

1. Mass element of a shell

$$dm(r_1) = \rho(r_1)d^3r_1 = \rho r^2 dr \sin\theta d\theta d\phi \tag{8}$$

2. $d\Phi = -\frac{Gdm}{|r-r_1|}$

$$\iff \frac{-G}{s(\theta)}dm \tag{9}$$

3. Now we integrate!

$$\Phi = \int d\Phi = -G\int \frac{dm}{s(\theta)} \tag{10}$$

$$-G\rho hR^2 \int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta}{s(\theta)}d\theta d\phi \tag{11}$$

$$\iff -2\pi G\rho hR^2 \int_0^{\pi} \frac{\sin\theta}{s(\theta)}d\theta \tag{12}$$

We find $s(\theta)$

$$s^2(\theta) = |r - r_1|^2 \iff r^2 - 2r\cdot r_1 + r_1^2 \tag{13}$$

$$x^2 + R^2 - 2Rx\cos\theta \tag{14}$$

We note that

$$\frac{ds^2}{d\theta} \iff 2s\frac{ds}{d\theta} \iff 2s(Rx\sin\theta) \tag{15}$$

$$\frac{\sin\theta}{s}d\theta = \frac{1}{Rx}ds \tag{16}$$

$$\int_0^{\pi}\frac{\sin\theta}{s}d\theta = \int_{s_{min}}^{s_{max}}\frac{1}{Rx}ds \tag{17}$$

Consider 2 sections, $s_{min} = \pm(R - x) > 0$ and $s_{max} = +(R + x) > 0$

$$\Phi = -2\pi G\frac{\rho hR}{x}(s_{max} - s_{min}) \tag{18}$$

**Case 1**

$m_2$ is outside of the shell. That is $x > R$, then $s_{min} = x - R > 0$ and $s_{max} = (R + x)$ Then

$$\Phi = 4\pi G\frac{\rho hR^2}{x} \iff \frac{-Gm}{x} \tag{19}$$

**Case 2:**

Then $s_{max} = R - x > 0$ and $s_{min} = R + x$, then

$$-4\pi G\rho hR = -\frac{GM}{R} = \text{ constant!} \tag{20}$$

**Results**

For a mass that's inside a shell, the potential is constant. But for a mass outside the shell, the potential is the well known $-\frac{GM}{x}$

## Example 2: Finding $\Phi$ of a uniform sphere

### Set-up

1. Solid sphere with radius $R_1$
2. Mass $M_1$, constant density inside the sphere, and 0 everywhere else.

**Goal**

Find $\Phi = \Phi(x)$

**Work**

Let's use previous results, that is relabel $R = r$ and $h = dr$

$$d\Phi = \begin{cases} -4\pi G \frac{\rho r^2}{x} dr : x > r \\ -4\pi G \rho r dr : x < r \end{cases} \tag{21}$$

**Case 1 :** $x > R$

$$\Phi = \int -4\pi G \int_0^R \frac{\rho r^2}{x} dr = \frac{-GM}{r} \tag{22}$$

**Case 2:** $x < R$

$$\Phi = \int_0^x d\Phi_{inner} + \int_x^R d\Phi_{outer} \tag{23}$$

$$\frac{1}{2} GM \frac{3R^2 - x^2}{R^3} \tag{24}$$