MTH 416: Lecture 19

Cliff Sun

October 31, 2024

Lecture Span

- Diagonalizability
- Eigenvalues, Eigenvectors

Theorem 0.1. *Suppose $A \in M_{n\times n}$*

1. *λ is an eigenvalue $\iff$*
$$\det(A - \lambda I) = 0 \tag{1}$$

2. *Given some $0 \neq v \in \mathbb{R}^n$, then v is an eigenvector with an eigenvector $\lambda \iff$*
$$v \in N(A - \lambda I) \tag{2}$$

Proof. We first prove 2.. Given $v \neq 0$, then v is an eigenvector with eigenvalue $\lambda \iff$

$$Av = \lambda v \tag{3}$$
$$\iff Av = (\lambda I)v \tag{4}$$
$$\iff (A - \lambda I)v = 0 \tag{5}$$
$$v \in N(A - \lambda I) \tag{6}$$

We now prove statement 1., now λ is an eigenvalue of $A \iff$

$$\exists v \neq 0 \text{ such that } v \in N(A - \lambda I) \tag{7}$$
$$\iff (A - \lambda I) > 0 \tag{8}$$
$$\iff A - \lambda I \text{ is not invertible.} \tag{9}$$
$$\iff \det(A - \lambda I) = 0 \tag{10}$$

□

Note 1:

$N(A - \lambda I)$ is called the Eigenspace of A with eigenvalue λ, denoted E_λ.

$$E_\lambda = \{\text{ eigenvalues with eigenvalue } \lambda \text{ }\} \cup \{0\}$$

Note 2:

The function $f(t) = \det(A - tI)$ is polynomial of degree n, called the characteristic polynomial of A. Its leading coefficient is always ± 1, specifically $(-1)^n$.

Corollary 0.2. *Any $n \times n$ matrix has at most n eigenvalues.*

Proof. Any polynomial of degree n has at most n roots. □

Aside

It also makes sense to talk about eigenstuff of an operator T on an infinite dimensional vector space V.

Ex:

Let $V = \{$ functions $f : \mathbb{R} \to \mathbb{R}$ that are infinitely differentiable. $\}$ Then we claim that every $\lambda \in \mathbb{R}$ is an eigenvalue of T.

Proof. Given λ, choose $e^{\lambda x}$. Trivial $\square$

Example

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix} \tag{11}$$

Goal: Calculate all eigenstuff; try to diagonolize the A if possible.

Step 1:

$$\det(A - tI) \tag{12}$$

$$= \det \begin{pmatrix} -t & 0 & 1 \\ 0 & 2-t & 0 \\ -2 & 0 & 3-t \end{pmatrix} \tag{13}$$

Cofactor expansion on the middle row:

$$= (2-t)\det \begin{pmatrix} -t & 1 \\ -2 & 3-t \end{pmatrix} \tag{14}$$

$$= (2-t)(-3t + t^2 + 2) \tag{15}$$

$$= (2-t)(t-1)(t-2) \tag{16}$$

Thus: $\lambda = 1, 2$

Step 2:

Eigenvectors for $\lambda = 1$,

$$A - I = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \end{pmatrix} \tag{17}$$

So

$$E_1 = N(A - I) = \text{ solution set to LS } \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 \end{array}\right) \tag{18}$$

We row reduce

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 \end{array}\right) \tag{19}$$

$$R_3 = R_3 - 2R_1 \tag{20}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{21}$$

Thus we have that

$$a_1 = a_3 \tag{22}$$

$$a_2 = 0 \tag{23}$$

Thus,

$$E_1 = \{a_3, 0, a_3\}\ a_3 \in \mathbb{R} \tag{24}$$

$$= span(\{(1,0,1)\}) \tag{25}$$

Eigenvectors for $\lambda = 2$.

$$E_2 = N(A - 2I) \tag{26}$$

Row reduce

$$\left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 \end{array}\right) \tag{27}$$

$$R_3 = R_3 - R_1 \tag{28}$$

$$\left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{29}$$

Thus

$$a_1 = \frac{1}{2}a_3 \tag{30}$$

Thus our solution set is

$$E_2 = \{(\frac{1}{2}a_3, a_2, a_3)\} \tag{31}$$

$$= span(\{(\frac{1}{2}, 0, 1), (0,1,0)\}) \tag{32}$$

Step 3:

Recall from last time,

$$A \text{ is diagonolize } \iff \exists \text{ basis of } \mathbb{R}^3 \text{ consisting of eigenvectors of A.}$$

$$\text{Try } \beta = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

This is a basis. So

$$[L_A]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{33}$$

Note that

$$L_A(v_1) = v_1 \tag{34}$$

$$L_A(v_2) = 2v_2 \tag{35}$$

$$L_A(v_3) = 2v_3 \tag{36}$$

Note that

$$[L_A]_\beta = Q^{-1}AQ \tag{37}$$

Such that Q is the change of basis matrix

$$Q = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{38}$$

Yes, A is diagonolizable, and

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{39}$$

Application

For the same matrix A, what is A^n? We know that

$$Q^{-1}AQ = D \tag{40}$$

Thus

$$A = QDQ^{-1} \tag{41}$$

Thus

$$A^n = (QDQ^{-1})^n \iff QD^nQ^{-1} \tag{42}$$

In general, there are 2 things that can stop us from diagonalizing a matrix:

1. Not enough eigenvalues
2. Not enough eigenvectors

Definition 0.3. *If $f(t)$ is a polynomial, with coefficients in a field $F(=\mathbb{R}, \mathbb{C}, \dots)$. Then we say that f splits over the field F if*

$$f(t) = c(t-a_1)(t-a_2)\dots(t-a_d) \tag{43}$$

Where d is the degree of the polynomial.