Convergence

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Definition 0.1. *Let $I \subseteq \mathbb{R}$ be an interval, $f(x)$ be a function on I, and $f_n(x)$ be a sequence of functions on I. Then we say that $f_n(x)$ converges to $f(X)$ pointwisely on I if and only if for any $x_0 \in I$*

$$\lim_{m\to\infty} f_m(x_0) = f(x_0) \tag{1}$$

Applying to the Fourier Series, we see that

Definition 0.2. $\sum_{nodd} \frac{4}{n\pi}\sin(\frac{n\pi x}{l})$ *converges to 1 on (0,l) pointwisely*

However, it's difficult to check pointwise convergence for this function, since you'd have to check an infinite amount of points. Thus, we propose this new method of check convergence:

Definition 0.3. *We define $f_m \to f \iff \lim_{m\to\infty} |f_m - f| = 0$*

However, in order to this, we must have a notion of distance to measure the distance between f_m and f. For this problem, we define something called the sup norm distance.

Definition 0.4. *Let $I \subseteq R$ be an interval, and $f(x)$ and $g(x)$ be functions on I. Then the sup norm distance between f and g on I is given by the following:*

$$sup_{x\in I}|f(x) - g(x)| \tag{2}$$

In this case, the sup represents the maximum, so this represents the maximum of the distance between the functions.

Using this definition, choosing some $\epsilon \in \mathbb{R}$, and that $sup|f(x) - g(x)| < \epsilon$, then that implies that g(x) is trapped in an interval of $[f(x) - \epsilon, f(x) + \epsilon]$ for x in I. Then we say that

Definition 0.5. *f_m uniformly converges to $f(x)$ on $I \iff \lim_{m\to\infty} sup_{x\in I}|f_m(x) - f(x)| = 0$ for all $x \in I$*