PHYS 487: Lecture # 4

Cliff Sun

January 29, 2026

Recap

- Spatial operations: $T(a)$, Π, $R_n(\varphi)$
- Consequences of symmetry
  1. T implies conservation of p or q
  2. Π
  3. R implies conservation of L

Lecture Span

1. symmetry $\iff$ degeneracy
2. Time evolution as transformation

Degeneracy

We've seen this before in the spectrum of the hydrogen atom. We aim to show that these degeneracies are due to symmetries.

We first postulate that

$$[H, Q] = 0 \implies \text{ can have degeneracy, or multiple SS with the same E.} \tag{1}$$

Then if $|\psi_n\rangle$ is an S.S., then $|\psi_n'\rangle = Q|\psi_n\rangle$ is too. WE then compute

$$\begin{aligned} H|\psi_n'\rangle &= H(Q|\psi_n\rangle) \\ &= QH|\psi_n\rangle \\ &= QE_n|\psi_n\rangle \\ &= E_n|\psi_n'\rangle \end{aligned}$$

If $Q \neq I$, then $|\psi_n'\rangle$ can be a new state with the same energy as $|\psi_n\rangle$, therefore, we have degeneracy. Next, we consider the QHO. Here,

$$[H, \Pi] = 0 \tag{2}$$

Since the potential well of the QHO is parity symmetric. Then

$$\Pi|\psi_n\rangle = \pm|\psi_n\rangle \tag{3}$$

But this not a new state, since this $\pm$ is really just a phase factor of $e^{i\pi}$ corresponding to -1 and etc. Such phase factors are not physical. This is an example of which a symmetric hamiltonian is not degenerate.

Here, we propose a way of finding if degeneracy occurs. Namely, degeneracy must exist if

$$[H, A] = 0 = [H, B] \text{ but } [A, B] \neq 0 \tag{4}$$

This means that one dimensional Hamiltonians cannot have degeneracy.

Let $H|\psi\rangle = E|\psi\rangle$ and $A|\psi\rangle = a_n|\psi\rangle$. Then $B|\psi\rangle$ is also a S.S. of H.

Multiple, non-commuting operators will produce degeneracy

Example: Angular momentum

Let $V = V(r)$, then consider SS: $|\psi_{nlm}\rangle$ with $H|\psi_{nlm}\rangle = E_n |\psi_{nlm}\rangle$. Then we know that

$$[H, L_i] \to [H, L_\pm] \tag{5}$$

But $[L_i, L_j] \neq 0$ for $i \neq j$. Consider

$$\begin{aligned} [H, L_\pm] |\psi_{nlm}\rangle &= 0 \\ HL_\pm |\psi_{nlm}\rangle &= L_\pm H |\psi_{nlm}\rangle \\ H|\psi_{nlm\pm1}\rangle &= E_n |\psi_{nlm\pm1}\rangle \end{aligned}$$

Here, this means that we have degeneracy.

Symmetry in the 1/r potential

We consider the vector

$$\vec{M} = \frac{p \times L - L \times p}{2m} + V(r)r \tag{6}$$

$$[H, M_i] \neq 0 \quad \text{but } [L_i, M_j] \neq 0 \tag{7}$$

Time as a transformation

Recall SWE:

$$i\hbar\partial_t \psi = (-i\hbar\partial_x)^2 \frac{1}{2m}\psi \tag{8}$$

Looking for u (time translation operator; "unitary evolution") such that

$$u(t)\psi(x, 0) = \psi(x, t) \tag{9}$$

Here,

$$u(t)|\psi(0)\rangle = |\psi(t)\rangle = \sum_k \frac{1}{k!}\frac{d^k}{dt^k}|\psi(t)\rangle t^k \tag{10}$$

Insert Hamiltonian, then we get

$$u(t) = \exp\left(-\frac{it}{\hbar}\hat{H}\right) \tag{11}$$

This is called the time translation operator. Note, we assume that the Hamiltonian is constant with respect to time. We validate this result:

$$\begin{aligned} |\psi(t)\rangle &= u(t)|\psi(0)\rangle \\ &= u(t)\sum_n c_n |\psi_n\rangle \\ &= \sum_n c_n e^{-iHt/\hbar} |\psi_n\rangle \\ &= \sum_n c_n e^{-iE_n t/\hbar} |\psi_n\rangle \end{aligned}$$

Example: spin and magnetic field. We have a Hamiltonian $H = \hbar\gamma B_0 S_z = (\hbar/2)\omega_L \sigma_z$. Then

$$T = \frac{1}{2}\frac{2\pi}{\omega_L}$$

Then we can construct

$$u(T) = \exp\left(-\frac{i}{\hbar}T\frac{\hbar}{2}\omega_L\sigma_z\right) = \exp\Big(-i\frac{\pi}{2}\sigma_z\Big) \tag{12}$$

Because σ_z is diagonal, then

$$u(T) = \begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix} \equiv \sigma_z \tag{13}$$

Heisenberg Picture

$$\hat{O}_H(t) = u^\dagger(t)\hat{O}u(t) \tag{14}$$

Schrödinger:

$$|\psi(t)\rangle = u(t)\,|\psi(0)\rangle \tag{15}$$

$$\langle O(t)\rangle = \langle\psi|\underbrace{u^\dagger O u}_{O_H(t)}|\psi\rangle \tag{16}$$

Spin-1/2

$$H = \frac{\hbar\omega_L}{2}\sigma_z \tag{17}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \tag{18}$$

$$u(t) = \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\omega_L t/2) \end{pmatrix} \tag{19}$$

In dirac notation:

$$u(t) = |\uparrow\rangle\langle\uparrow| + e^{i\omega_L t}|\downarrow\rangle\langle\downarrow| \tag{20}$$

$$u^\dagger(t) = |\uparrow\rangle\langle\uparrow| + e^{-i\omega_L t}|\downarrow\rangle\langle\downarrow| \tag{21}$$

$$\sigma_x = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| \tag{22}$$

How do we find $\hbar/2\langle\sigma_x(t)\rangle$? We calculate

$$\sigma_x^H = u^\dagger\sigma_x u = |\uparrow\rangle\langle\downarrow| e^{-i\omega_L t} + e^{i\omega_L t}|\downarrow\rangle\langle\uparrow| \tag{23}$$

We also calculate

$$\langle\sigma_x\rangle = \langle\psi|\sigma_x^H|\psi\rangle \tag{24}$$

$$= \frac{\hbar}{2}\frac{1}{2}\left(e^{-i\omega_L t} + e^{i\omega_L t}\right) = \frac{\hbar}{2}\cos(\omega_L t) \tag{25}$$

Note, this was done as a ”passive transformation”.