MTH 416: Lecture 21

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Lecture Span

- Criteria for diagonalizability
- Cayley-Hamilton

Continuing from previous Lecture

Theorem 0.1. *Let $A \in M_{n\times n}$ is diagonalizable if and only if*

1. *char poly(A) splits completely over $\mathbb{R}$*
2. *For every eigenvalue λ,*

$$\text{geo multi} = \text{alg multi}$$

This required that

Lemma 0.2. *If $\beta_1, \ldots$ are linearly independent sets in different eigenspaces $E_{\lambda_1}, \ldots$ of A, then*

$$\beta = \beta_1 \cup \ldots \tag{1}$$

is also linearly independent.

Proof. We first prove a special case, that each $\beta_i = \{v_i\}$ where v_i is a non-zero vector. Prove this by induction

Base Case

If $k = 1$, then this is linearly independent.

Induction

Assume that the special case works for some value of k, then we consider $k+1$. Let

$$\beta_1 = \{v_1\}, \ldots, \beta_{k+1} = \{v_{k+1}\} \tag{2}$$

We claim that

$$\beta = \{v_1, \ldots, v_{k+1}\} \tag{3}$$

Suppose that

$$a_1v_1 + \cdots + a_{k+1}v_{k+1} = 0 \tag{4}$$

Then

$$A(a_1v_1) + \cdots + A(a_{k+1}v_{k+1}) = 0 \tag{5}$$

$$a_1\lambda_1v_1 + \cdots + a_{k+1}\lambda_{k+1}v_{k+1} = 0 \tag{6}$$

Subtract (6) - λ_{k+1}(5)

$$a_1(\lambda_1 - \lambda_{k+1})v_1 + \cdots + a_k(\lambda_k - \lambda_{k+1})v_k = 0 \tag{7}$$

We can apply IH, and conclude that

$$a_1 = \cdots = a_k = 0 \tag{8}$$

This implies that

$$a_{k+1} = 0 \tag{9}$$

□

But what about the general case? Given $\beta_1 = \{v_1, v_2\}, \ldots$. We can call

$$a_1 v_1 + a_2 v_2 = w_1 \tag{10}$$

And sum

$$w_1 + w_2 + \cdots = 0 \tag{11}$$

But since the coefficients in front are non-zero, it implies that

$$w_1 = w_2 = \cdots = 0 \tag{12}$$

Thus implies that

$$a_1 = a_2 = \cdots = 0 \tag{13}$$

Cayley Hamilton

Suppose $A \in M_{n\times n}(\mathbb{R})$, then

$$\det(A - tI) = 0 \tag{14}$$

is the characteristic polynomial. But what if we plug in $t =$ matrix?
Warning: the equation

$$\det(A - tI) = 0 \tag{15}$$

is only valid when t is a number, not a matrix. So we're only interested in plugging $t =$ matrix into the characteristic polynomial.

Example:

$$f(t) = t^2 + 1 \tag{16}$$

Then let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{17}$$

Then we claim that

$$f(A) = A^2 + I_2 = 0_{2\times 2} \tag{18}$$

Then

$$A^2 + I_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0_2 \tag{19}$$

So we know that

$$\text{char poly(A)} = t^2 + 1$$

Interesting...

Theorem 0.3. *Cayley-Hamilton Theorem: For any $A \in M_{n\times n}(\mathbb{R})$, A satisfies its own characteristic equation.*

Aside, we can also state this for linear operations

$$T : V \to V \tag{20}$$

Read lecture notes for proof.

Direct Sum Decomposition

Suppose $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable. Let $\lambda_1, \ldots, \lambda_k$ be eigenvalues and E_{λ_i} be eigenspaces.
Fact: If $\beta_1, \ldots, \beta_k$ are basis for $E_{\lambda_1}, \ldots,$ then

$$\beta = \beta_1 \cup \ldots \tag{21}$$

is a basis for $\mathbb{R}^n$.

This fact can be reinterpreted as follows:

Theorem 0.4. *Suppose $W_1, \ldots, W_k$ are subspaces of V. Then the following are equivalent:*

1. *One can form a basis of V by taking the union of a basis of each W_i*
2. *Every vector $v \in V$ can be expressed uniquely in the following form:*

$$w_1 + \cdots + w_k \tag{22}$$

where $w_i \in W_i$.

3. $$\sum_{i=1}^{k} W_i = V \tag{23}$$

For all j, then

$$W_j \cap \sum_{i=1}^{k} W_i = \{0\} \tag{24}$$

Definition 0.5. *Given subspaces $W_i \subseteq V$, then*

$$\sum_i W_i = \{\text{vectors of the form } w_1 + \cdots + w_k\} \tag{25}$$

Definition 0.6. *If $W_i \subseteq V$ satisfies the conditions above, then we can say that V is a direct sum of the W_i, and write*

$$V = W_1' +' \ldots' +' W_k \tag{26}$$

Conclusion

Then if A is diagonalizable, then $\mathbb{R}^n$ decomposes as the direct sum of its eigenspaces.