MTH 416: Lecture 13

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Lecture Span

- Invertibility & Isomorphisms
- Change of coordinates

Invertibility & Isomorphisms

Calculating Inverse Matrices

Suppose $A, B \in M_{n\times n}(\mathbb{R})$. Note, $B = A^{-1} \iff Ab_i = e_i$. Suppose

$$B = \begin{pmatrix} b_1 & b_2 & b_3 & \dots \end{pmatrix} \tag{1}$$

We expand out $AB \to$

$$AB = \begin{pmatrix} Ab_1 & Ab_2 & \dots \end{pmatrix} \tag{2}$$

Note: $AB = I_n \iff BA = I_n$. How to find A^{-1}: solve $Ab_i = e_i$ which is a LSOE to calculate each b_i. In fact, we can do all of them at the same time.

Example:

Let

$$A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix} \tag{3}$$

Then we augment this matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \tag{4}$$

Row reducing the matrix yields

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & -9 \\ 0 & 1 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -2 & -2 & 7 \end{array}\right) \tag{5}$$

Thus A^{-1} exists and equals the right hand side of this augmented matrix.

Application

If we know A^{-1}, then it becomes easier to solve $LS(A|b)$ because

$$\{\text{solutions to } LS(A|B)\} = \{\text{vectors } x \text{ such that } Ax = b\}$$

Then

$$Ax = b \iff A^{-1}Ax = A^{-1}b \iff x = A^{-1}b \tag{6}$$

So there is a unique solution $x = A^{-1}b$ for any given b for LS.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$. Then $T = L_A$ where $A = [T]_\beta^\beta$ where β is the standard basis.

Question:

How can we tell whether T is invertible?

Answer:

Theorem 0.1. *The following are equivalent*

1. *T is invertible*
2. *A is invertible*
3. *T has rank n*
4. *T has nullity* 0

Proof. If U is a linear transformation, given by a matrix B, then $U = T^{-1} \iff B = A^{-1}$. For 3,4 this is just the rank-nullity theorem. To relate 1 to 3,4: we note

$$\begin{aligned} \text{T is invertible} &\iff \text{T is bijective} \\ &\iff \text{T is injective and surjective} \\ &\iff \text{nullity of T is 0 \& rank of T is } n \end{aligned}$$

Note: an analagous statement is true for $T : V \to W$ where $\dim(V) = \dim(W) = n$ and replacing A with $[T]_\beta^\gamma$ for some basis β of V, and γ of W. □

Consequence:

Theorem 0.2. *If $A, B \in M_{n\times n}(\mathbb{R})$ and $AB = I_n$, then $B = A^{-1}$. In particular, $BA = I_n$.*

Proof. Suppose $A, B \in M_{n\times n}(\mathbb{R})$ and $AB = I$. Then L_A and L_B are linear transformations from $\mathbb{R}^n \to \mathbb{R}^n$ such that

$$L_a \circ L_B = I_{\mathbb{R}^n} \tag{7}$$

In particular, then $L_A \circ L_B$ is invertible. Facts:

1. If $f \circ g$ is injective, then g is injective
2. If $f \circ g$ is surjecitve, then f is surjective

By fact 2, L_A is surjective, in other words, its rank is n. Therefore, L_A is invertible, and thus A is invertible. To prove that $A^{-1} = B$:

$$A^{-1} \iff A^{-1}(AB) \iff B \tag{8}$$

□

Change of coordinates

Motivating Question: Given a vector/linear transformation written in one coordinate system, how can we express it in another coordinate system?

Vectors

Q: Given a vector space V with two basis β, β', how can we calculate some vector $[v]'_\beta$ from $[v]_\beta$.
Idea: Consider $I_V : V_\beta \to V'_\beta$. Then for any $v \in V$, then

$$[v]_{\beta'} = [I_V(v)]_{\beta'} \iff [I_V]_\beta^{\beta'}[v]_\beta \tag{9}$$

This is called change in coordinates. How to calculate $[I_v]_\beta^{\beta'}$? By definition, the columns of $[I_v]_\beta^{\beta'}$ are representations of vectors in β in terms of β'.

Example:

$V = \mathbb{R}^2$ with $\beta = e_1, e_2$. Then let $\beta' = \{v_1 =< 1, 1 >, v_2 =< -1, 1 >\}$. Then

$$e_1 = \frac{v_1 - v_2}{2} \tag{10}$$

and

$$e_2 = \frac{v_1 + v_2}{2} \tag{11}$$

Case of Linear Transformations

General Question

Suppose V, W are vector spaces, and we're given basis β and β' for V and γ, γ' for W. Given $T : V \to W$, how can we calculate $[T]^{\gamma}_{\beta'}$ from $[T]^{\gamma'}_{\beta}$?

Idea

$[T]^{\gamma'}_{\beta'}$ is the matrix that we multiply $[v]'_{\beta}$ by to produce $[T(v)]_{\gamma'}$. To do that, we write

$$[v]_{\beta'} \to [v]_{\beta} \to [T(v)]_{\gamma} \to [T(v)]_{\gamma'} \tag{12}$$

Then formula is

$$[T]^{\gamma'}_{\beta'} = [I_W]^{\gamma'}_{\gamma}[T]^{\gamma}_{\beta}[I_v]^{\beta}_{\beta'} \tag{13}$$

In the special case:

$$[T]_{\beta'} = [I_V]^{\beta'}_{\beta}[T]_{\beta}[I_V]^{\beta}_{\beta'} \tag{14}$$

This is nothing but

$$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q \tag{15}$$