MTH 447: Lecture 14

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Theorem 0.1. *Assume* $z_k \in \Lambda$*,* $z_k \in \mathbb{R}$*,* $\lim_{k\to\infty} = z_0$*, then* $z_0 \in \Lambda$*.*

Definition 0.2. *Let* $C \subseteq \mathbb{R}$*. We say that* C *is sequentially closed if whenever* $x_n \in C$*,* $x_n \to x_0 \implies x_0 \in C$*.*

Proof. $z_k \to z_0$. For all $\epsilon > 0$, $\exists N$ s.t. if $n > N$,

$$|z_0 - z_k| < \epsilon \tag{1}$$

$\exists \delta > 0$ s.t.

$$(z_k - \delta, z_k + \delta) \subseteq (z_0 - \epsilon, z_0 + \epsilon) \tag{2}$$

□

4 different characterizations of lim sup

1. $\lim_{n\to\infty} \sup_{k>n}\{x_k\}$
2. $M = \limsup x_n$ and for any $\epsilon > 0$, $\{n \in \mathbb{N} : |x_n - M| < \epsilon\}$ is infinite.
3. Upper Fuzzy Bounds (see homework)
4. If $M = \limsup x_n$, then
   (a) $\exists$ subseq $x_{n_k} \to M$
   (b) $\nexists$ subseteq $x_{n_k} \to \hat{M}$ for any $\hat{M} > M$

Fun Facts about lim sup

1. If $x_n \to L$, then $\limsup(x_n + y_n) = L + \limsup(y_n)$
2. If $x_n \to L$ and $L > 0$, then $\limsup(x_n y_n) = L \limsup(y_n)$
3. Let x_n be a sequence. THen $x_n \neq 0$ for all n. Then

$$\liminf \left|\frac{x_{n+1}}{x_n}\right| \leq \liminf |x_n|^{\frac{1}{n}} \leq \limsup |x_n|^{\frac{1}{n}} \leq \limsup \left|\frac{x_{n+1}}{x_n}\right| \tag{3}$$