MTH 416: Lecture 7

Cliff Sun

September 17, 2024

Lecture Span

- Bases & dimensions

Theorem 0.1. *If V has a finite spanning set, then it has a finite basis.*

Proof. Suppose that V is spanned by $S = \{u_1, \ldots, u_k\}$, then by a theorem from last class, there exists some linearly independent subset of S, β, such that

$$span(\beta) = span(S) = V \tag{1}$$

□

Fact: Every vector space has a basis

Theorem 0.2. *Let V be a vector space containing*

1. *Spanning set G consisting of n vectors*
2. *A linearly independent set L containing of m vectors*

Then:

1. *$m \leq n$, and*
2. *There exists a subset $H \subseteq G$ that contains $m - n$ vectors such that $H \cup L$ spans V.*

Ideas:

1. *$m \leq dim(V) \leq n$*
2. *Any linearly independent set can be substituted into any spanning set the result still spanning the vector space.*

Corollary 0.3. *If V has a finite basis, all basis of V are finite & exactly the same size.*

Proof. Suppose V has a finite basis $\beta = \{u_1, \ldots, u_n\}$. And some other basis γ. We need to prove that γ is finite and γ has the smae size as β.

Finite

Suppose by contradiction that γ is infinite. Choose some $\gamma' \subset \gamma$ consisting of $n + 1$ vectors. By homework 3, γ' is also linearly independent. Applying the replacement theorem $G = \beta$ and $L = \gamma'$. Then $m \leq n$ and $m = n + 1 \leq n$ which is a contradiction.

$\#(\gamma) = \#(\beta)$

Applying the replacement theorem, we let $G = \beta$ and $L = \gamma$, then $m \leq n$ but similar $G = \gamma$ and $L = \beta$, then $n \leq m$. Thus $n = m$. Thus this proves the corallary. □

Therefore, this let's us define

1. V is <u>finite-dimensional</u> if it has a finite basis (if and only if it has a finite spanning set)
2. If V is finite dimensional, then we define its # of dimensions to be the number of vectors in any basis.

Theorem 0.4. *Suppose that W is a subspace of V, where V is finite dimensional. Then*

1. *$dim(W) \leq dim(V)$*
2. $\dim(W) = dim(V) \iff W = V$

Proof. Let $\beta_v = \{v_1, \ldots, v_k\}$ be a basis of V. Assume that W has a finite basis $\beta_w = \{w_1, \ldots, w_m\}$. Applying the replacement theorem to $G = \beta_v$ and $L = \beta_w$, we're given that $m \leq n \iff dim(W) \leq dim(V)$, if $m = n$, then part 2 of the replacement theorem says that some set $H = \emptyset$ union with β_w spans V. In particular, that means that β_w spans V. Then $W = span(\beta_w) = V$. $\square$

So how do we know that W has a finite basis?

Proof. Procedure to build one:

1. Start with $S = \emptyset$
2. As long as the $span(S) \neq W$, add S one vector in W which is not already in $span(S)$.

One can verify that the set S is linearly independent at each step. By the replacement theorem, this must stop before S contains $n+1$ vectors. It stops when $span(S) = W$ and S is linearly dependent. In other words, S is a basis and is finite dimensional. $\square$

Some more collaries from the replacement theorem: if $dim(V) = n$, then

1. Every spanning set of V contains $\geq n$ vectors, and has a basis as a subset
2. Every linearly independent set contains at most n vectors, and can be enlarged to a basis.
3. Given a set of n vectors, we claim that it is linearly independent iff it spans V.

Let's prove the replacement theorem:

Proof. Induction on m, namely starting at $m = 0$.

Base case

If $m = 0$, then $L = \emptyset$, clearly $0 \leq n$. Then choose $H = G$, then $H \cup L = H$ and $H = G$, and the theorem assumed that G was a spanning set. Thus we have proved the base case.

Inductive step

Assume that the theorem is true for some value of m, then we claim it to be true for $m+1$. Given a spanning set $G = \{u_1, \ldots, u_n\}$ and $L = \{v_1, \ldots, v_{m+1}\}$. Set $L' = \{v_1, \ldots, v_m\} \subset L$. By HW 3, we have that L' is linearly independent. Plugging G and L' into the inductive hypothesis, $m \leq n$ and we can substitute L' into G to get a spanning set. That is after relabeling the vectors $u_i's$ $\{v_1, \ldots, v_m, u_1, \ldots, u_{n-m}\} = V$. It follows that $v_{m+1} = a_1v_1 + a_mv_m + b_1u_1 + b_{n-m}u_{n-m}$ for some constants $a_i, b_i \in \mathbb{R}$. Because L is linearly independent, then v_{m+1} is not a linear combination of $v_1, \ldots, v_m$. Thus, at least one $b_j \neq 0$. WLOG let $b_1 \neq 0$. In particular, $n - m > 0$, thus $n > m$ which proves statement 1. For 2, I claim that

$$span(v_1, \ldots, v_{m+1}, u_2, \ldots, u_{n-m}) = V \tag{2}$$

We claim that u_1 is also in this span. Since you can solve for u_1 using

$$v_{m+1} = a_1v_1 + a_mv_m + b_1u_1 + b_{n-m}u_{n-m} \tag{3}$$

Thus V is spanned by these vectors. This concludes the proof. $\square$