MTH 416: Lecture 20

Cliff Sun

November 7, 2024

Lecture Span

- Criteria for Diagonalizability

Recall: 2 things can prevent diagonalizability:

1. Not enough eigenvalues
2. Not enough eigenvectors

Theorem 0.1. ***Fundamental Theorem of Algebra****: Every non-zero polynomial with complex number coefficients splits completely over the Complex Numbers. In other words,*

$$f(t) = c(t - a_1)(t - a_2)\dots \tag{1}$$

Note, this is not true of $\mathbb{R}$, e.g. $f(t) = t^2 + 1$

Theorem 0.2. *Let $A \in M_{n\times n}(F)$ where F is some field, then if A is diagonalizable, then the characteristic polynomial of A splits completely in F.*

Proof. Let $A \in M_{n\times n}$ be diagonalizable, then A is similar to some diagonal matrix D. Then

$$\text{char poly}(A) = \text{char poly}(D)$$
$$= \det(D) \iff (\lambda_1 - t)(\lambda_2 - t)\dots \tag{2}$$

This completely splits F. □

Is this an iff statement? NO
For example,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{3}$$

Then

$$char(A) = (t-1)^2 \tag{4}$$

In other to diagonalize the matrix, we need a basis of $\mathbb{R}^2$ eigenvectors, which we don't have.

Definition 0.3. *Suppose λ is an eigenvalue of A*

1. *The <u>algebraic multiplicity</u> of λ is the number of times $(t - \lambda)$ divides the polynomial.*
2. *The <u>geometric multiplicity</u> of λ is the dimension of the Eigenspace associated with λ.*

Main facts about algebraic & geometric multiplicity:

Theorem 0.4. *For any λ,*

$$geo\ multi \le alg\ multi$$

Theorem 0.5. *A matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable iff the follow are true:*

1. *Its characteristic polynomial splits completely*
2. *For every eigenvalue, the geometric multiplicity and algebraic multiplicity are equal.*

Corollary 0.6. *If $A \in M_{n\times n}$ has n distinct real eigenvalues, then it is diagonalizable.*

Note that the converse is false.

Proof. Let $M \in M_{n\times n}(\mathbb{R})$ and let λ be an eigenvalue of A. Then, we prove that

$$\text{geo multi} \leq \text{alg multi}$$

If $\lambda = 1$ has geo multi $= k$, then $\dim(E_\lambda) = k$. Then we choose a basis for E_λ, then we extend this to a basis $\beta = \{v_1, \dots, v_n\}$ for all $\mathbb{R}^n$. Then we write L_A in β coordinates. Then we have that the right hand corner of this matrix is a diagonal line of λ's. Then

$$\text{char poly(A)} = \text{char poly}([L]_\beta)$$

Then char poly($[L]_\beta$) has at least k repititions of λ, thus

$$\text{geo multi} \leq \text{alg multi}$$

□

Proof. We now prove theorem 0.5, then we first proceed in the $\implies$ direction. We've already proved that char poly(A) splits completely. We now note that $A \sim D$ where D is some diagonal matrix. Then for any λ, alg multi $\lambda = \#$ of times λ appears on D. Claim: the geo multi of λ is the same as

$$= \dim(E_\lambda) \tag{5}$$

$$= \dim N(A - \lambda I) \tag{6}$$

$$\dim N(D - \lambda I) \tag{7}$$

$$= \ \#\text{ of times } \lambda \text{ appears on diagonal} \tag{8}$$

$$= \text{alg multi of } \lambda$$

($\impliedby$) Suppose that char poly(A) splits and all λ have geo multi = alg multi.

$$charpoly(A) = (-1)^n(t - \lambda_1)^{m_1}\dots \tag{9}$$

Where λ_i are all distinct and real numbers, and m_i are the algebraic multiplicity of each λ_i. The degree of this polynomial is

$$m_1 + \cdots + m_k = n \tag{10}$$

But since m_i = geo multi of λ_i which is the dimension of E_{λ_i}. Let

$$\beta_i = \text{ a basis for } E_{\lambda_i} \tag{11}$$

This contains m_i vectors. Let

$$\beta = \beta_1 \cup \cdots \cup \beta_k \tag{12}$$

This is a set of n linearly independent vectors. But how do we know this?? To complete the proof, we need the following lemma

Lemma 0.7. *If β_i is a linearly independent set in E_{λ_i} for various distinct eigenvalues $\lambda_1, \dots, \lambda_k$, then*

$$\beta = \beta_1 \cup \cdots \cup \beta_k \tag{13}$$

Is another linearly independent set.

□