MTH 416: Lecture 2

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Lecture Span

- Vector Spaces
- Subspaces

8 Axioms

1. $\forall x, y \in V, \quad x + y \iff y + x$
2. $\forall x, y, z \in V, \quad (x + y) + z \iff x + (y + z)$
3. $\exists\, 0 \in V$ such that for all vectors $x \in V$, we have that $0 + x = x$
4. $\forall x \in V, \exists y \in V$ such that $x + y = 0$. In other words, $y = -x$
5. For each $x \in V$, $1 \cdot x = x$
6. For each $a, b \in \mathbb{R}$ and $x \in V$, $(ab)x = a(bx)$
7. For each $a \in \mathbb{R}$ and $x, y \in V$, $a(x + y) = ax + ay$
8. For each $a, b \in \mathbb{R}$ and $x \in V$, we have that $(a + b)x = ax + bx$

Vector Spaces

We will be discussing Vector Spaces over other fields besides $\mathbb{R}^n$. So what is a field?

Definition 0.1. *A field is some set that is equipped as operations addition, subtraction, multiplication, & division. All satisfying the usual properties. Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, etc. Note, $\mathbb{Z}$ is not a field since dividing an integer by an integer does not necessarily yield an integer.*

To define a vector space over $\mathbb{C}$, replace all previous instances of $\mathbb{R}$ with $\mathbb{C}$.

Generally, linear algebra works well over any field. But when one introduces a vector space, specifiying the field of scalars is part of defining a vector space. But usually, the field of interest is implied.

For example, $\mathbb{C}$ is a vector space over $\mathbb{C}$. But $\mathbb{C}$ is also a vector space over $\mathbb{R}$. But both instances of $\mathbb{C}$ are not the same! If $\mathbb{C}$ is a vector space over $\mathbb{C}$, then $\mathbb{C}$ is a 1 dimensional vector space. But if $\mathbb{C}$ is a vector space over $\mathbb{R}$, then this instance of $\mathbb{C}$ is a 2 dimensional vector space.

Non-Examples of vector spaces

1. $V = \mathbb{R}^2$ with coordinate wise scalar multiplication, but $(x_1, y_1) + (x_2, y_2) \iff (x_1 + x_2, y_1 - y_2)$. Defining this addition operation as this means that Axiom 1 & 2 fail!
2. $V = \mathbb{R}^2$ with coordinate wise addition, but scalar multiplication is defined as $c(x, y) \iff (cx, y)$. Axiom 8 breaks with this definition of a vector space.

Proofs using Axioms

Cancellation Theorem for addition

Theorem 0.2. *If u, v, w are vectors in V, and $u + w = v + w$, then $u = v$.*

Proof. Suppose $u + w = v + w$. By Axiom 4, w has an additive inverse y. In particular, $w + y = 0$. Thus, if we add y to both sides, then we get

$$u + w = v + w \tag{1}$$

$$(u + w) + y = (v + w) + y \tag{2}$$

$$u + (w + y) = v + (w + y) \tag{3}$$

$$u = v \tag{4}$$

This concludes the proof. (Note: We have to specify the parenthesis in equation 2 since the original equation involved just adding $u + w$ and y was not in the picture.) □

Corollary 0.3. *The 0 vector is unique. As in suppose 0 and $0'$ both satisfy Axiom 3, then $0 = 0'$.*

Corollary 0.4. *Additive inverses are unique. That is, suppose y and y' satisfy Axiom 4, then $y = y'$.*

Corollary 0.5. *If v is a vector in V, then $0 \cdot v = 0$.*

Proof. Suppose

$$v = v \tag{5}$$

We claim that

$$v + 0 \cdot v = v + 0 \tag{6}$$

Indeed,

$$v + 0 \cdot v = 1 \cdot v + 0 \cdot v \tag{7}$$

$$= (1 + 0) \cdot v \tag{8}$$

$$= 1 \cdot v \tag{9}$$

$$= v \tag{10}$$

$$= v + 0 \tag{11}$$

Thus,

$$v + 0 \cdot v = v + 0 \tag{12}$$

Using the cancellation theorem, we have that

$$0 \cdot v = 0 \tag{13}$$

□

Subspaces

Suppose $V = \mathbb{R}^3$. Suppose some plane W lives in V. If we give V the same operations as V, is W a vector space? Claim: Yes!
We claim that W is closed over addition and multiplication. That is, $(\forall w_1, w_2 \in W, w_1 + w_2 \in W)$ and $(cw \in W)$.
Suppose that W is closed over addition and multiplication, we still have to check all 8 axioms. Answer: Yes, this is true even for all $x, y \in V \supseteq W$

Definition 0.6. *Given a vector space V, a <u>subspace</u> of V is the subset $W \subseteq V$ such that*

1. $0 \in W$
2. *W is closed under addition ($w_1, w_2 \in W \implies w_1 + w_2 \in W$)*
3. *W is closed under scalar multiplication ($c \in F$, $\forall w \in W \implies c \cdot w \in W$)*

Note, Subspace and Vector Space are <u>NOT</u> the same thing.

Theorem 0.7. *Let V be a vector space and W be a subset of V. Then W is a <u>subspace</u> of V if and only if W is a vector space when given the same operations as V.*

Proof. This is a proof of the $\implies$ direction. Suppose that W is a subspace, then we claim that it is a vector space when given the same operations as V. Because of properties $2, 3$, addition and scalar multiplication produce outputs in W. So we must check the 8 Axioms. Axioms $1, 2, 5, 6, 7, 8$ are true within V, thus must be true in W. For 3, we know that from property 1. But for 4, let $w \in W$, then we know that there exists $y = -w \in V$ because V is a vector space. But is $y \in W$? But because W is closed under scalar multiplication, we multiply by -1 to achieve $-w$. Thus it follows that $-w \in W$.

Lemma 0.8. $-1 \cdot w \iff -w$

Proof. We know that

$$
\begin{aligned}
w + -1 \cdot w &= 1 \cdot w + -1 \cdot w && (14)\\
&= (1 + -1) \cdot w && (15)\\
&= 0 \cdot w && (16)\\
&= 0 && (17)\\
&= w + -w && (18)
\end{aligned}
$$

□

□

Examples:

1. For any V, V and 0 are subspaces.
2. If $V = \mathbb{R}^3$, then any line/plane containing 0 that lives within V is a subspace.
3. If $V = M_{n \times n}(\mathbb{R})$, then the following are subspaces: Diagonal Matrices (all entries are 0 except for the diagonal), upper triangular matrices (all entries are 0 except for the upper triangular section of the matrix), and symmetrical matrices ($A^T = A$)
4. If $V = F(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$, then the following are subspaces:
   (a) Polynomials
   (b) Continuous Functions
   (c) Differential Functions
   (d) Functions such that $f(7) = 0$. The only constraint is that $f(7) = 0$, the rest of the function is unmonitored.

Non-Examples in $\mathbb{R}^2$:

1. $\{(x, y) : x, y \geq 0\}$, multiplying by a negative scalar doesn't work.
2. $\{(x, y) : x = 0 \vee y = 0\}$, addition fails