PHYS 326: Lecture 5

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Orthogonality of Eigenmodes

Definition 0.1. *The inner product in*

1. *Cartesian Coordinates:* $\{a, b, c, \dots\} \cdot \{x, y, z, \dots\} = ax + by + cz + \dots$
2. *In vector notation*

$$\vec{v} \cdot \vec{u} = \vec{v}^T \vec{u} = \vec{u}^T \vec{v} \tag{1}$$

We define the *generalized inner product* *as*

$$\vec{v} \cdot \vec{u} = \vec{v}^T \boldsymbol{M} \vec{u} = \vec{u} \cdot \vec{v} \tag{2}$$

Example:

$$\vec{v} \cdot \vec{v} = \vec{v}^T \boldsymbol{M} \vec{v} = 2T \geq 0 \tag{3}$$

Claim: Different eigenmodes $\vec{u}$ are orthogonal.

That is

$$\vec{u}^1 \cdot \vec{u}^2 = \vec{u}^1 \mathbf{M} \vec{u}^2 = 0 \tag{4}$$

Proof. We have that

$$K\vec{u}^1 = \omega^2 M \vec{u}^1 \tag{5}$$

$$K\vec{u}^2 = \omega^2 M \vec{u}^2 \tag{6}$$

We multiply the top part by $\vec{u}_2^T$ and the bottom part with $\vec{u}_1^T$:

$$\vec{u}_2^T K \vec{u}^1 = \omega^2 \vec{u}_2^T M \vec{u}^1 \tag{7}$$

$$\vec{u}_1^T K \vec{u}^2 = \omega^2 \vec{u}_1^T M \vec{u}^2 \tag{8}$$

We take the transpose of equation 8 and minus it from equation 7 to yield

$$0 = (\omega_2^2 - \omega_1^2) u_2^T M \vec{u}_1 \tag{9}$$

□

Normalizing eigenmodes:

$$\vec{u} = \frac{\vec{u}}{(\vec{u}^T M \vec{u})^{\frac{1}{2}}} \tag{10}$$

After normalization

$$\vec{u}_i^T M \vec{u}_j = \delta_{ij} \tag{11}$$

General Initial conditions problem

The general solution is

$$\vec{x}(t) = \sum A_s \vec{u}^s \cos(\omega_s t) + \sum B_s \vec{u}^s \sin(\omega_s t) \tag{12}$$

A_s and B_s are determined by initial conditions. Define

$$\vec{x}_0 = \vec{x}(t=0) = \sum_S A_S \vec{u}^S \tag{13}$$

$$\vec{v}_0 = \vec{v}(t=0) = \sum_S B_S \omega_S \vec{u}^S \tag{14}$$

Then

$$\vec{u}_r^T \mathbf{M} \vec{x}_0 \sum_S A_S \vec{u}_r^T \mathbf{M} \vec{u}^S = A_S \delta_{rS} = A_r \tag{15}$$

$$\vec{u}_r^T \mathbf{M} \vec{v}_0 = \sum_S B_S \omega_S \vec{u}_r^T \mathbf{M} \vec{u}^S = B_S \omega_S \delta_{rS} = B_r \omega_r \tag{16}$$

Thus

$$A_r = \vec{u}_r^T \mathbf{M} \vec{x}_0 \tag{17}$$

$$B_r = \frac{\vec{u}_r^T \mathbf{M} \vec{v}_0}{\omega_r} \tag{18}$$

The Modal Matrix

Definition 0.2. *Define*

$$\cup_{\alpha R} = u_\alpha^R \tag{19}$$

Where R dictates which mode we are at, and α shows which coordinate we are in. This is a matrix of all the modes.

We have that

$$\mathbf{U}^T \mathbf{M} \mathbf{U} = I \implies \mathbf{U}^{-1} = \mathbf{U}^T \tag{20}$$