MTH 447: Lecture 20

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Continuous Functions

Definition 0.1. *$F : A \to \mathbb{R}$. f is* <u>*continuous*</u> *if $\forall x_0 \in A$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $|x - x_0| < \delta$ and $x \in A$, then $|f(x) - f(x_0)| < \epsilon$.*

Definition 0.2. *f is* <u>*uniformly continous*</u> *if $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $|x - y| < \delta$ where $x, y \in A$, then $|f(x) - f(y)| < \epsilon$.*

Main Properties of Uniform Continuity

Theorem 0.3.

1. *If $f : [a, b] \to \mathbb{R}$ and f is continuous, then f is uniformly continuous. (domain is closed bounded interval)*
2. *If $f : A \to \mathbb{R}$ is uniformly continous and x_n is Cauchy, then $f(x_n)$ is Cauchy.*
3. *$f : (a, b) \to \mathbb{R}$, then f is uniformly continous $\iff$ f can be extended to a*

Proof. Proof of (1). Let $f : [a, b] \to \mathbb{R}$, f is continous but not uniformly continuous. We negate the uniform continuity definition: we obtain $\exists \epsilon > 0$, $\forall \delta > 0$, $\exists x, t$ such that $|x - y| < \epsilon$ and $|f(x) - f(y)| \geq \epsilon$. Then $\forall n \in \mathbb{N}$, $\exists x_n, y_n$ such that

$$|x_n - y_n| < \frac{1}{n} \text{ and } |f(x_n) - f(y_n)| \geq \epsilon \tag{1}$$

But $x_n \in [a, b] \implies \exists x_{n_k}$ that converges. Then $x_{n_k} \to x_0$, and

$$|y_{n_k} - x_{n_k}| < \frac{1}{n_k} \leq \frac{1}{k} \implies y_{n_k} \to x_0 \tag{2}$$

Then by continuity,

$$\lim f(x_{n_k}) = \lim f(y_{n_k}) = f(x_0) \tag{3}$$

Then $|f(x_{n_k}) - f(y_{n_k})| \to 0$ which contradicts the negated statement. □

Proof. Proof of (2). Assume x_n is Cauchy. THen □