PHYS 486: Lecture # 23

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Solving the radial equation

Let

$$R = \frac{u}{r} \tag{1}$$

Then

$$\frac{dR}{dr} = (r\frac{du}{dr} - u)\frac{1}{r^2} \tag{2}$$

Then, our radial equation becomes

$$-\frac{\hbar}{2m}\frac{d^2u}{dr^2} + [V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}]u = E_n \tag{3}$$

As a toy system, we examine the infinite well in 3-D in spherical coordinates.

$$V(r) = \begin{cases} 0 & r \leq a \\ \infty & r > a \end{cases} \tag{4}$$

Inside the well,

$$\frac{d^2u}{dr^2} = [\frac{l(l+1)}{r^2} - k^2]u \tag{5}$$

For $l = 0$, then

$$u(r) = A\sin(kr) + B\cos(kr) \tag{6}$$

Since the WF $= u/r$, then $B = 0$. with BC: $\sin(ka) = 0$, then $ka = N\pi$. Thus,

$$E = \frac{N^2\pi^2\hbar^2}{2ma^2} \tag{7}$$

For $l > 0$, generally:

$$u(r) = Arj_l(kr) + Bru_l(kr) \tag{8}$$

$$j_l(x) = (-x)^l(\frac{1}{x}\frac{d}{dx})^l\frac{\sin(x)}{x} \tag{9}$$

Then,

$$R(r) = Aj_l(kr) \wedge j_l(ka) = 0 \tag{10}$$

The wave function is then

$$\psi = AR(r)Y(\phi, \theta) \tag{11}$$

The Hydrogen Atom

Given a Hydrogen Atom with an immobile proton and an electron that is free to move around. Then

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} \tag{12}$$

Thus, the radial equation is

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m_e}\frac{l(l+1)}{r^2}\right]u = E_u \tag{13}$$