MTH 447: Lecture # 16

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Ratio and Root Tests

Suppose a_n is a sequence. Define

$$r_n = \left| \frac{a_{n+1}}{a_n} \right| \tag{1}$$

And

$$\rho_n = |a_n|^{\frac{1}{n}} \tag{2}$$

Theorem 0.1. *Consider* $\sum a_n$

1. *If* $\limsup r_n < 1$*, then series absolutely converges*
2. *If* $\liminf r_n > 1$*, then series diverges*
3. *If* $\liminf r_n \leq 1 \leq \limsup r_n$*, then we can't say anything about the series*
4. *If* $\limsup \rho_n < 1$*, then series absolutely converges*
5. *If* $\limsup \rho_n > 1$*, then series diverges.*
6. *If* $\limsup \rho_n = 1$*, then we can't anything about the series.*

Proof. Proof of (4). Assume $\limsup \rho_n < 1$. Then $\limsup \rho_n = \alpha$. Thus, $\forall \epsilon > 0$, $\exists N$ such that $n > N \implies$

$$\alpha - \epsilon \leq \sup_{k>n} \rho_k \leq \alpha + \epsilon \tag{3}$$

That is

$$\rho_k \leq \alpha + \epsilon \tag{4}$$

$$\iff |a_k|^{\frac{1}{k}} \leq \alpha + \epsilon \tag{5}$$

$$|a_k| \leq (\alpha + \epsilon)^k \tag{6}$$

That is

$$\sum_{k=N+1}^{\infty} |a_k| \leq \sum_{k=N+1}^{\infty} (\alpha + \epsilon)^k \tag{7}$$

This absolutely converges if $|\alpha + \epsilon| < 1$. ☐

Proof. Proof of (5). If $\limsup \rho_n = \alpha > 1$. Then there exists a subsequence that converges to $\alpha > 1$. There exists infinitely many n such that

$$|a_n| > 1 \tag{8}$$

Thus this shows that $\sum a_n$ diverges. ☐

Proof. Proof of (1), if $\limsup r_n < 1 \iff \limsup \rho_n < 1 \implies$ convergence. ☐

Proof. Same argument for (2). ☐

Alternating Series Test

Theorem 0.2. *Assume a_n is a non-negative sequence and $a_n \to 0$. Consider*

$$\sum_{n=1}^{\infty}(-1)^n a_n \tag{9}$$

or

$$\sum_{n=1}^{\infty}(-1)^{n+1} a_n \tag{10}$$

Both series converge.