Subsequences and the Squeeze Theorem

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Last time, we defined some function (x_n) such that

$$x_1 = 1 \tag{1}$$

$$x_{n+1} = \sin(x_n) \quad \forall n \geq 1 \tag{2}$$

Previously, we showed that this function is strictly monotone decreasing and is bounded by 0. As well, by the monotone convergence theorem, this implies that

$$\lim_{n\to\infty} x_n \text{ exists} \tag{3}$$

We introduce some facts:

1. If f is a continuous function and (x_n) is a sequence which converges to some limit x. Then the sequence $(f(x_n))$ converges to $f(x)$.
2. $\sin(x)$ is continuous.

Claim:

$$\lim_{n\to\infty} x_n = 0 \tag{4}$$

Proof. Let $x = \lim_{n\to\infty} x_n$ since we know that it exists. By facts 1 and 2,

$$\lim_{n\to\infty} \sin(x_n) = \sin(x) \tag{5}$$

But

$$\sin(x_n) = x_{n+1} \tag{6}$$

and

$$\sin(x) = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x \tag{7}$$

Thus $x = 0$. $\square$

Subsequences

Recall, last week we found that

$$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0 \tag{8}$$

But what about

$$\lim_{n\to\infty} \frac{\sin(2n)}{2n} \; ? \tag{9}$$

Or

$$\lim_{n\to\infty} \frac{\cos(n)}{n} \; ? \tag{10}$$

Or

$$\lim_{n\to\infty} \frac{\sin(n)}{n} + \frac{1}{2^n} \; ? \tag{11}$$

Definition 0.1. *Let (x_n) be a sequence. Let $(n_i)_{i=1}^{\infty}$ be a strictly increasing sequence of natural numbers. Then we call the sequence*

$$(x_{n_i})_{i=1}^{\infty} \tag{12}$$

a subsequence of (x_n).

Theorem 0.2. *If (x_n) is a sequence that converges to some number x, then every subsequence must converge to the same number.*

Note, if (x_n) diverges, then its subsequences may or may not diverge.

Proof. We prove the subsequence theorem. Let (x_n) be a sequence such that

$$x = \lim_{n\to\infty} x_n \text{ exists} \tag{13}$$

In particular, for all $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $n \geq M$, we have that

$$|x_n - x| < \epsilon \tag{14}$$

Let (n_i) be a sequence of indicies that are strictly increasing. In particular, we can prove that $n_i \geq i$. We claim that

$$\lim_{i\to\infty} x_i = x \tag{15}$$

Let $\epsilon > 0$, choose M to be the same integer as stated in equation 13 with our ϵ. In particular, for all $n \geq M$, we have that

$$|x_n - x| < \epsilon \tag{16}$$

Let $i \geq M$ be arbitrary, since $n_i \geq i$, we have that

$$|x_{n_i} - x| < \epsilon \tag{17}$$

as claimed. □

Squeeze Theorem

Theorem 0.3. *Suppose (a_n), (b_n), and (x_n) are sequences such that*

1. $a_n \leq x_n \leq b_n$
2. $\lim a_n = x = \lim b_n$

Then

$$\lim x_n = x \tag{18}$$

Proof. Suppose (a_n), (b_n), (x_n) are stated. Then for all $\epsilon > 0$, there exists 2 numbers as follows.

1. $M_1 \in \mathbb{N}$ such that for all $n \geq M_1$, we have that

$$|a_n - x| < \epsilon \tag{19}$$

2. $M_2 \in \mathbb{N}$ such that for all $n \geq M_2$, we have that

$$|b_n - x| < \epsilon \tag{20}$$

We claim that

$$\lim x_n = x \tag{21}$$

that is for all $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $n \geq M$, we have that

$$|x_n - x| < \epsilon \tag{22}$$

Let $\epsilon > 0$, plugging ϵ into equations 19 and 20, we get $M_1, M_2 \in \mathbb{N}$. Then we choose $M = \max(M_1, M_2)$. Let $n \geq M$, then $n \geq M_1 \wedge n \geq M_2$. Then in particular

$$x - \epsilon < a_n < x + \epsilon \tag{23}$$

$$x - \epsilon < b_n < x + \epsilon \tag{24}$$

But we have that

$$a_n \leq x_n \leq b_n \tag{25}$$

Then we have that

$$x - \epsilon < a_n \leq x_n \leq b_n < x + \epsilon \tag{26}$$

Thus,

$$x - \epsilon < x_n < x + \epsilon \tag{27}$$

Thus

$$|x_n - x| < \epsilon \tag{28}$$

□