MTH 416: Lecture 27

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Lecture Span

- Jordan Canonical Form

Theorem 0.1. *If $A \in M_{n\times n}(F)$ is such that char poly(A) splits over F, then A is similar to some B in JCF.*

Application: Calculating the powers of a non-diagonalizable matrix A

A matrix in JCF has a specific formula that we can use to calculate its powers.

Recall: To diagonalize A, we need a basis of eigenvectors.

For JCF, we need generalized eigenvectors.

Definition 0.2.

1. *A vector $v \neq 0$ is a generalized eigenvector if A for an eigenvalue of λ is*

$$(A - \lambda I)^p v = 0 \tag{1}$$

For some integer p.

2. *Given λ, the generalized eigenspace K_λ, is the set of all v (including 0) satisfying the above.*

Lemma 0.3. *This generalized eigenspace K_{λ_i} is a subspace of F^n, containing the normal eigenspace.*

Given a Jordan block of typical form, what is the generalized eigenvectors?

$$Ae_n = \lambda e_n + e_{n-1} \tag{2}$$
$$(A - \lambda I)e_n = e_{n-1} \tag{3}$$
$$(A - \lambda I)e_{n-1} = e_{n-2} \tag{4}$$

etc.

Definition 0.4. *A set of vectors is called a cycle of generalized eigenvectors if it has the form*

$$\beta = \{v_1, \dots, v_p\} \tag{5}$$

Where

$$(A - \lambda I)v_p = v_{p-1} \tag{6}$$
$$(A - \lambda I)v_{p-1} = v_{p-2} \tag{7}$$
$$\cdots \tag{8}$$
$$(A - \lambda I)v_1 = 0 \tag{9}$$

Idea of proof of main theorem:

1. Show that $F^k = \oplus_{i=1}^k K_{\lambda_i}$ (generalized sum of eigenspaces). That is for any $v \in F$ can be written uniquely as

$$v = v_1 + \cdots + v_k \tag{10}$$

for $v_i \in K_{\lambda_i}$ for each v.

2. Each K_{λ_i} has a basis consisting of a disjoint union of cycles of generalized eigenvectors.

Note: The basis vectors for each K_{λ_i} would compose the "diagonalized" matrix.