

# MTH 417: Lecture 15

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$$S_3 = \{e, (12), (13), (23), (123), (132)\} \quad (1)$$

All elements in this set partition  $S_3$ .

**Definition 0.1.** Let  $H \leq G$  be a subgroup. Let  $g \in G$ . A subset of the form

$$gH = \{gh | h \in H\} \quad (2)$$

is called a left coset of  $H$  in  $G$ . The right coset is defined as

$$Hg = \{hg | h \in H\} \quad (3)$$

Note, this is also a function of  $g$  where  $g$  is a fixed element of  $G$ .

In general, the left and right cosets are different.

**Proposition 0.2.** Let  $H \leq G$ , and  $a, b \in G$ . The following statements are all equivalent (TFAE):

1.  $a \in bH$
2.  $b \in aH$
3.  $b^{-1}a \in H$
4.  $a^{-1}b \in H$
5.  $aH = bH$

*Proof.* Prove that 1  $\implies$  2. Suppose  $a \in bH$ . This means that  $a = bh \iff ah^{-1} = b \in aH$ .  $\square$

*Proof.* Prove that 2  $\implies$  3. Suppose  $b \in aH \iff h^{-1} = b^{-1}a \in H$ .  $\square$

*Proof.* Prove that 3  $\implies$  4. Suppose  $b^{-1}a = h \iff h^{-1} = a^{-1}b \in H$ .  $\square$

*Proof.* Prove that 4  $\implies$  5. Suppose  $a^{-1}b = h$ . Then let  $\tilde{a}h \in aH$ , then  $a\tilde{h} = bh^{-1}\tilde{h} \in bH$ . A similar exercise for  $bH \subseteq aH$ .  $\square$

*Proof.* Prove that 5  $\implies$  1. Suppose  $aH = bH$ . In particular,  $ae \in aH \iff ae \in bH$ .  $\square$

**Proposition 0.3.**  $H \leq G$ ,  $a, b \in G$ . Then, these are consequences:

1. Either  $aH = bH$  or  $aH \cap bH = \emptyset$
2. The function  $f : aH \rightarrow bH$  such that  $f(x) = ba^{-1}x$ . This is a bijection.
3.  $G = \cup$  Left Cosets

*Proof.* We prove 3. Note that left coset  $\subseteq G$ . Then let  $g \in G$ , then in particular,  $g \in$  some left coset because every left coset has a the element  $ge = g$ .  $\square$

*Proof.* We prove 2. Assume that  $aH \cup bH \neq \emptyset$ . Then  $\exists x \in aH \cup bH$ . Then in particular,  $x \in aH$  means that  $aH = xH$ . Similarly,  $bH = xH$ . The result is trivial.  $\square$

**Theorem 0.4.** Lagrange's Theorem: Let  $G$  be a finite group. Let  $H \leq G$ . Then the order of  $H$  divides the order of  $G$  and the quotient is the number of left cosets of  $H$  in  $G$ .

**Definition 0.5.** For any  $H \leq G$ , the # of left cosets of  $H$  in  $G$  is the index of  $H$  in  $G$ . Denoted  $[G : H]$

Then Lagrange tells us that if  $|G| < \infty$ , then  $|G| = |H|[G : H]$

*Proof.* Suppose  $a_1, \dots, a_k$  be representatives of the left cosets of  $H$  in  $G$ . Then

$$G = \bigcup_i a_i H \quad (4)$$

$$\implies |G| = \sum_i |a_i H| \quad (5)$$

because they are disjoint.

$$\sum_i |a_i H| = \sum_i |H| \quad (6)$$

$$|G| = k|H| \quad (7)$$

This means that the order of  $H$  divides  $G$  and the integer is the number of left cosets.  $\square$