MTH 416: Lecture 5

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Lecture Span

- Reduced row echelon form (RREF)
- Linear independence

Recap:

Elementary row operations

1. Switch 2 rows
2. Scalar a row by a const $\neq 0$
3. Add multiple of one row to another row

RREF

1. All rows of 0's are at the bottom
2. First non-zero entry per row is a 1
3. Each leading 1 is the only non-zero entry in its column
4. Leading 1's go Northwest to Southeast.

Goal: Put any matrix in RREF using elementary row operations. The algorithm that establishes this is Gaussian Elimination.

Notation

1. M is a $m \times n$ matrix (m rows & n columns)
2. R_i = i-th row
3. C_j = j-th column

Algorithm

1. Set $r = 0$ and $j = 0$
2. If $j \geq n$, then stop and return current matrix, otherwise increment j.
3. If C_j has all 0's, below the r-th row, then go to step 1.
4. Find a non-zero among rows $r + 1$ of C_j, by switching rows put it on $r + 1$, then set $r = r + 1$.
5. Scale row r by a constant to make its j-th entry = 1.
6. Subtract multiples of R_r from every other row to zero out all of column j except for R_r.
7. Go to step 1.

Proof of Gaussian Elimination

Proof. First, the algorithm must stop eventually because when we reach step 1 n times, then $j = n$ which stops the algorithm.

To prove that this algorithm returns a matrix in RREF, we claim that everytime we get to step 1, the matrix has the form

$$\begin{pmatrix} RREF & ? \\ 0 & ? \end{pmatrix}$$

That is the RREF form is j columns wide and r rows long. We will prove this by induction on j, or the column index. This means that once $j = n$, we will have the above matrix be in completely RREF form, which proves our claim.

Base Case

The first time we reach step 1, $j = 0$. But this means that any $m \times n$ matrix has the form of the above matrix when $j = 0$.

Induction Step

Suppose that the matrix is currently in the form of the above matrix, then we must prove that the next time we go to step 1 that the matrix remains in the form of the above matrix, but with the known RREF section expanded. There are 2 cases:

1. If we leave step 1, and proceed to step 2 and immediately go back to step 1, then j has been incremented and the matrix has been unchanged. In this case, the known RREF section has increased a column, but this section is still in RREF form.

2. We return to step 1 from step 6. In this case, r and j have incremented, and the matrix has changed. In particular, the RREF section has increased by 1 in both its rows and columns. But we claim that this matrix is still in RREF. In step 3, we found some non-zero number and brought it to $r + 1$ (note the RREF matrix has not been changed) Then in step 4, we scale the column by a constant such that the value at $M_{r,j} = 1$, then we take this row and subtract multiples of this row from other rows to "zero-out" all other entires in C_j. Note that the entire left side of the matrix has been left untouched. To prove that M is still in the form

   $$\begin{pmatrix} RREF & ? \\ 0 & ? \end{pmatrix}$$

   Note that the top left corner by adding a pivot plus a row of 0's (0's in front of pivot) and a column of 0's. Therefore, by induction, we have shown that the final matrix will be in RREF.

□

Linear Independence

Recall: one powerful method to construct subspaces W of vector space V is through spans.

$$span(u_1, \cdots, u_k) = \{a_1 u_1 + \cdots + a_k u_k\} \tag{1}$$

1. If $W = span(u_1, \cdots, u_k)$, are all these vectors necessary?

2. Or can W be written as

   $$span(u_1, \cdots, u_k) \tag{2}$$

   And how many vectors do I need?

3. Given $W \subseteq V$, then how "big" is W. e.g Subspaces of $\mathbb{R}^3$ include $\{0\}$, a line through 0, a plane through 0, or $\mathbb{R}^3$. Can we mathematically measure its "dimensionality"?

Definition 0.1. *Let V be a vector space and $u_1, \cdots, u_k \in V$, then*

1. *The vectors are <u>linearly dependent</u> if there exist scalars $a_1, \cdots, a_k$ not zero such that*

   $$a_1 u_1 + \cdots + a_k u_k = 0 \tag{3}$$

   All scalars equaling zero is the "trivial solution".

2. *Otherwise, the vectors are a <u>linearly independent</u> set.*