PHYS 486 Notes

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Lecture 9: 9/23

Recap:

1. $|\psi\rangle = \sum_k c_k |\alpha_k\rangle$
2. $|\psi\rangle = \int dx \psi(x)|x\rangle$
3. Observables $A = A^\dagger$, real eigenvalues
4. Commutator (for hermitian operators):
$$[\hat{A}, \hat{B}] = AB - BA \tag{1}$$
If $[A, B] = 0$, then a simulataneous eigenbasis exists. This means that $\exists\, |j\rangle$ such that
$$A|j\rangle = \alpha_j |j\rangle \tag{2}$$
$$B|j\rangle = \beta_j |j\rangle \tag{3}$$

Finding eigenvalues and eigenvectors

Let
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
We then diagonalize this, and find the eigenvalues. In the continuous case,
$$\hat{p}\psi = p\psi$$
$$-i\hbar\partial_x\psi = p\psi$$
$$\psi = A\exp\left(i\frac{p}{\hbar}x\right)$$
These eigenvectors are never physical since they cannot be normalizable. Finding eigenstates of energy yields
$$-\frac{\hbar^2}{2m}\partial_x^2\psi = E\psi$$
$$(\partial_x - \frac{i\sqrt{2mE}}{\hbar})(\partial_x + \frac{i\sqrt{2mE}}{\hbar})\psi = 0$$
$$\psi = A\exp(ikx) + B\exp(-ikx)$$
This means that if we perform a perfect measurement of energy. Then, we know that our particle moves left or right, and we do not know which.

Basis Transformations

Suppose that we know ψ in some basis ($|\alpha_k\rangle$). We want to know it in $|\beta_k\rangle$. We know that

$$c_k = \langle \alpha_k | \psi \rangle$$

and vice versa. Then

$$d_n = \langle \beta_n | \sum_k c_k |\alpha_k\rangle$$

$$1 = \sum_k c_k \langle \beta_n | \alpha_k \rangle$$

Therefore,

$$|\psi\rangle = \sum_{k,n} c_k \langle \beta_n | \alpha_k \rangle |\beta_n\rangle$$

In the continuous case, for example

$$\hat{p}|p\rangle = p|p\rangle$$

Then

$$\int dx \, |x\rangle \langle x| \, \hat{p} \, |p\rangle = p \int dx \, |x\rangle \langle x|p|x|p\rangle$$

Note that

$$|p\rangle = |x\rangle \langle x|p\rangle$$

Then on the right hand side, we expand out the momentum basis as

$$\int dx dx' \, |x\rangle \langle x| \, \hat{p} \, |p\rangle = \int dx \, |x\rangle \langle x| \, \hat{p} \, |x'\rangle \langle x'|p\rangle$$

Note that $\langle x|p\rangle = f_p(x)$, then we analyze

$$\langle x| \, \hat{p} \, |x'\rangle = \text{matrix element of p at x,x'}$$

Let

$$\langle x| \, \hat{p} \, |x'\rangle = -i\hbar \partial_x \delta(x - x')$$

then we can reobtain the momentum operator.

Back to the infinite square well. Then

1. Energy Basis: $\langle n|\psi\rangle = c_n$
2. Position Basis: $\langle x|\psi_n\rangle = \psi_n(x) = \sqrt{\frac{2}{a}} \sin(\dots)$
3. Momentum Basis: $\langle p|\psi_n\rangle = \psi_n(p)$

Example: Two level system (qubit)

$$H = \hbar\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \iff \hbar\omega\sigma_z$$

Note that σ_z is a Pauli Matrix operator.

General state of the qubit is

$$\alpha|0\rangle + \beta|1\rangle$$

Consider an operator

$$\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hbar\sigma_x$$

Lecture 16: 9/25

Midterm: Up to Lecture 9, Homework 3, and Discussion 3. We discuss the two-level system (TLS). Simple observable:

$$a\sigma_z = a \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenvectors $|0\rangle$ and $|1\rangle$. Another observable:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle \langle 1| + |1\rangle \langle 0|$$

Double Well

Consider an infinite well with a finite wall in its middle. If the particle is on the left side, then it is in $|0\rangle$, and else $|1\rangle$. Suppose we measure this particle with σ_x. Then, suppose we have a tunneling operator $\hat{T}$ such that:

$$\hat{T}|0\rangle = |1\rangle$$
$$\hat{T}|1\rangle = |0\rangle$$

We can compute the matrix elements from here:

$$T = \sigma_x$$

Uncertainty Principle

The high level: measuring $\hat{A}$ disrupts the possible measurements of the other observable $\hat{B}$. We note that

$$\sigma_A^2 = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle$$

$$= \langle \psi | (\hat{A} - \langle A \rangle)^2 | \psi \rangle \tag{4}$$
$$= \langle (\hat{A} - \langle A \rangle)\psi | (\hat{A} - \langle A \rangle)\psi \rangle \tag{5}$$
$$= \langle f | f \rangle \tag{6}$$

For the second operator:

$$\sigma_B^2 = \langle g | g \rangle$$

with similar definition. From linear algebra, we know that

$$|\langle f|g\rangle|^2 \leq \langle f|f\rangle \langle g|g\rangle$$

We get that

$$\sigma_A^2 \sigma_B^2 \geq |\langle f|g\rangle|^2$$

For any complex number $z = z' + iz''$, then

$$|z|^2 \geq z''^2 = [\frac{1}{2i}(z - z^*)]^2$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}(\langle f|g\rangle - \langle g|f\rangle) \right)^2$$

Note,

$$\langle f|g\rangle = \langle \psi | (\hat{A} - \langle A \rangle)(\hat{B} - \langle B \rangle) | \psi \rangle$$
$$= \langle \psi | (\hat{A}\hat{B} - \hat{A}\langle B \rangle - \hat{B}\langle A \rangle + \langle A \rangle \langle B \rangle | \psi \rangle$$