PHYS 326: Lecture # 18

Cliff Sun

April 1, 2025

Hamiltonian Mechanics

Begin with $L(q_i, \dot{q}_i; t) = T - U$ for $i = 1, \ldots, n$. Then use Euler Lagrangian equations to study mechanics. Define generalized momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{1}$$

Hamilton aims to describe the state of the system in the phase space $(q_i,\, p_i)$. Define the Hamiltonain:

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \iff \text{Jacobi Function, Legendre Transformation} \tag{2}$$

Next, find the equations of motion in terms of p_i, q_i.

$$H(\vec{q}, \vec{p}, t) = \sum_i p_i \dot{q}_i(\vec{q}, \vec{p}; t) - L(\vec{q}, \dot{\vec{q}}(\vec{q}, \vec{p}; t); t) \tag{3}$$

Then

$$\frac{\partial H}{\partial p_j} = \dot{q}_j + \sum_i p_i \frac{\partial \dot{q}_i}{\partial p_j} - \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial p_j} = \dot{q}_j \tag{4}$$

This is because $p_i = \partial L / \partial \dot{q}_i$. Similarly

$$\frac{\partial H}{\partial q_j} = \sum_i p_i \frac{\partial \dot{q}_i}{\partial q_j} - \frac{\partial L}{\partial q_j} - \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial q_i} = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = -\dot{p}_j \tag{5}$$

Conservation Laws

If H does not depend explicitly on t, then H is a constant of motion.

Proof.

$$\frac{dH}{dt} = \sum_i \underbrace{\frac{\partial H}{\partial q_i}}_{=-\dot{p}_i} \dot{q}_i + \sum_i \underbrace{\frac{\partial H}{\partial p_i}}_{=\dot{q}_i} \dot{p}_i + \underbrace{\frac{\partial H}{\partial t}}_{=0} = 0$$

□

If a coordinate q does not appear in H, then its conjugate momentum is conserved.

Proof.

$$\dot{p}_q = -\frac{\partial H}{\partial q} = 0$$

□

Poisson Brackets

Definition 0.1. $\{A, B\}$ *is defined as*

$$\{A, B\} = \sum_i \left[\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i}\right]$$