

# MTH 553: Lecture # 4

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## Last time

- Inviscid Burger's Equation

$$uu_x + u_y = 0 \quad (1)$$

$$u(x, 0) = k(x) \quad (2)$$

## Lecture Span

1. Method of general solutions

## Method of general solutions

**Objective:** Find two expressions of  $x, y$  that are constant along each characteristic. Consider an arbitrary function of these two expressions. Find  $\Gamma$  to determine the solution function.

### Example

$$uu_x + u_y = x \quad (3)$$

$$u(x, 1) = 2x \quad (4)$$

Here, the characteristic system is

$$\frac{dx}{dt} = z \quad (5)$$

$$\frac{dy}{dt} = y \quad (6)$$

$$\frac{dz}{dt} = x \quad (7)$$

Here, we can find

$$\frac{dz}{dx} = \frac{x}{z} \iff zdz = xdx \iff \frac{1}{2}z^2 = \frac{1}{x^2} + C_1(s) \quad (8)$$

Here, we can then define a constant function (dependent on where on  $\Gamma$  you start on)

$$\phi(x, y, z) = z^2 - x^2 \quad (9)$$

Lets try adding (5) + (7), then we obtain

$$\frac{d(x+z)}{dt} = (x+z) \quad (10)$$

Then consider

$$\frac{d(x+z)}{dy} = \frac{x+z}{y} \iff \frac{dy}{y} = \frac{d(x+z)}{x+z} \quad (11)$$

$$\iff x + z = C_2(s)y \quad (12)$$

We can then define another constant function (dependent on starting point on  $\Gamma$ ) as

$$\psi(x, y, z) = \frac{x + z}{y} \quad (13)$$

Then a general solution is

$$F(u^2 - x^2, \frac{x + u}{y}) = 0 \quad (14)$$

Where  $F$  depicts a relationship between these values that originates on  $\Gamma$ .

We can determine  $F$  from  $\Gamma$ , that is  $u(x, 1) = 2x$ . Let  $u = 2x$ , then we obtain

$$F(3x^2, 3x) = 0 \quad (15)$$

Satisfied by  $F(a, b) = a - 1/3b^2$ , therefore, we obtain

$$u - x^2 - \frac{1}{3} \left( \frac{x + u}{y} \right)^2 = 0 \quad (16)$$

Solving for  $u$ , we obtain

$$u = \frac{x \pm 3xy^2}{3y^2 - 1} \quad (17)$$

To satisfy the initial condition, we choose  $+$ . Therefore, the final solution is

$$u = \frac{x + 3xy^2}{3y^2 - 1} \quad (18)$$

The solution exists for  $y > 1/\sqrt{3}$  and  $\forall x$ . **NOTE:** There is no uniqueness theory (yet) for non-linear equations.

## Weak Solutions

### Jump condition

This determines where "the shock goes". Given some smooth function  $G(z)$ , then define

$$G(u)_x + u_y = 0 \quad (19)$$

Similarly,

$$G'(u)u_x + u_y = 0 \quad (20)$$

For the conservation law, consider

$$G(z) = \frac{1}{2}z^2 \implies G'(z) = z \quad (21)$$

Then this is the burger's equation. We call  $u$  a weak solution of (19) if it satisfies the x-integrated version of (19). That is,  $u$  satisfies:

$$G(u(b, y)) - G(u(a, y)) + \frac{d}{dy} \int_a^b u(x, y) dx = 0 \quad (22)$$

for all  $a < b$  and all  $y$ . But what's so cool about this?

You can have discontinuities w.r.t  $x$

If  $u$  is a smooth solution that satisfies (19), then it satisfies (22), by using the fundamental theorem of Calculus. But the converse is false, because  $u$  can have discontinuities and still be a weak solution.

### Example

Specifically, suppose  $x = \xi(y)$  as our shock curve, which defines where this discontinuity occur. Let  $u = u_l(y)$  be to the left of the shock curve and  $u = u_r(y)$  be similarly defined for the right. Now, suppose  $u(x, y)$  jumps across a  $C^1$ -smooth curve  $x = \xi(y)$ . But everywhere else,  $u$  is a smooth solution. Then suppose  $u$  solves the conservation law to the left and right of  $x = \xi(y)$ . Then, we plug this equation into (22).

$$G(u(b, y)) - G(u(a, y)) + \frac{d}{dy} \left[ \int_a^{\xi(y)} u_l(x, y) dx + \int_{\xi(y)}^b u_r(x, y) dx \right] \quad (23)$$

$\forall a, b$  with  $a < \xi(y) < b$  for all  $y$ . Need not check intervals of  $[a, b]$  that don't contain  $\xi(y)$ .

$$\iff 0 = G(u(b, y)) - G(u(a, y)) + \int_a^{\xi(y)} u_y(x, y) dx + \int_{\xi(y)}^b u_y(x, y) dx + \xi'(y) u_l - \xi'(y) u_r \quad (24)$$

By fundamental theorem of Calculus and chain rule. Where

$$u_l = \lim_{x \rightarrow \xi^-} u$$

and  $u_r$  defined similarly. Moreover,  $u_y = -G(u)_x$

$$\iff 0 = -G(u_l) + G(u_r) + \xi'(y) (u_l - u_r) \quad (25)$$

By evaluating the integrals. Then we obtain

$$\xi'(y) = \frac{G(u_l) - G(u_r)}{u_l - u_r} \quad (26)$$