Fourier Transform

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The fourier transform is written as follows

$$\hat{\phi(p)} = \int_{-\infty}^{\infty} \phi(x)e^{-ipx}dx \tag{1}$$

$\phi(x)$	$\hat{\phi(x)}$
$e^{-\alpha\lvert x\rvert} \quad \alpha > 0$	$\frac{2\alpha}{\alpha^2+p^2}$
$e^{-\alpha x^2} \quad \alpha > 0$	$\sqrt{\frac{\pi}{\alpha}}e^{\frac{-\alpha p^2}{4}}$
$\delta(x)$	1
1	$2\pi\delta(p)$
$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\pi\delta(x) + \frac{1}{ip}$

Properties

Linearity

$$\widehat{af + bg} = a\widehat{f} + b\widehat{g} \tag{2}$$

Where for notation definitions, $\widehat{f}$ means to take the fourier transform of f.

Fourier transform of derivatives

Let $f(x)$ be a function on $\mathbb{R}$, and

$$h(x) = f'(x) \tag{3}$$

Then

$$\widehat{h(p)} = ip \cdot \widehat{f(p)} \tag{4}$$

Convolution

Let $f(x), g(x)$ be functions

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy \text{ is a convolution of } f, g \tag{5}$$

Notation:

$$h = f * g \tag{6}$$

Then

$$\widehat{h(p)} = \widehat{f(p)} \cdot \widehat{g(p)} \tag{7}$$

Heat Equation

Let $\phi(x)$ be a function on $\mathbb{R}$, consider

$$\begin{cases} u_t - ku_{xx} - 0 & t > 0, x \in \mathbb{R} \\ u(x,0) = \phi(x) \end{cases}$$

Let $\widehat{u(p,t)}$ be the Fourier transform of $u(x,t)$, Then

$$\begin{cases} \widehat{u_t} - k(ip)^2\widehat{u_{xx}} - 0 & t > 0, x \in \mathbb{R} \\ \widehat{u(x,0)} = \widehat{\phi(x)} \end{cases}$$

This becomes an ode with a fixed p, thus we have that the solution must be in the form of

$$\widehat{u}(p,t) = h(p)e^{-kp^2t} \tag{8}$$

Applying the initial condition yields

$$\widehat{u}(p,t) = \widehat{\phi}(p)e^{-kp^2t} \tag{9}$$

Heat Equation

$$u_t - ku_{xx} = 0 \tag{10}$$

$$u(x,o) = \phi(x) \tag{11}$$

Let $\widehat{u}(p,t)$ be the fourier transform of $u(x,t)$, then equation 10 turns into

$$\widehat{u_t} + kp^2\widehat{u}(p,t) = 0 \tag{12}$$

with

$$\widehat{u}(p,t) = \widehat{\phi}(p) \tag{13}$$

Combining both equations yields

$$\widehat{u}(p,t) = \widehat{\phi}(p)e^{-kp^2t} \tag{14}$$

But since this function is the multiplication of two functions, then

$$u(x,t) = f * g \tag{15}$$

We match $\phi(x) = e^{-ax^2}$ to $\widehat{\phi}(p) = \sqrt{\frac{\pi}{\alpha}}e^{-\frac{p^2}{4a}}$ with $a > 0$. So we want

$$-kp^2t = -\frac{p^2}{4a} \implies a = \frac{1}{4kt} \tag{16}$$

So

$$\phi(x) = e^{-\frac{x^2}{4kt}} \tag{17}$$

Solve PDE by Fourier transform

1. Rewrite the equation in terms of $\widehat{u}$
2. Solve the transformed ODE equation
3. Inverse transform to get u

Wave Equation

$$u_{tt} - c^2 u_{xx} = 0 \tag{18}$$
$$u(x,0) = \phi(x) \tag{19}$$
$$u_t(x,0) = \psi(x) \tag{20}$$

We first must transform the equation, so equation 18 becomes

$$\widehat{u_{tt}} - c^2 (ip)^2 \widehat{u} = 0 \tag{21}$$

With all the initial conditions being trivially transformed. We solve the equations to yields

$$\widehat{u}(p,t) = c_1(p)\cos(cpt) + c_2(p)\sin(cpt) \tag{22}$$

$$c_1(p) = \widehat{\phi}(p) \tag{23}$$

$$\widehat{\psi}(p) = cp \cdot c_2(p) \implies c_2(p) = \frac{1}{cp}\widehat{\psi}(p) \tag{24}$$

$$\widehat{u}(p,t) = \widehat{\psi}(p)\cos(cpt) + \frac{1}{cp}\widehat{\psi}(p)\sin(cpt) \tag{25}$$

Find u from $\widehat{u}$. But we skip, and we see that this will indeed yield the wave solution

$$u(x,t) = \frac{1}{2}[\psi(x-ct) + \psi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \tag{26}$$

Laplace equation

$$u_{xx} + u_{yy} = 0 \tag{27}$$
$$u(x,0) = \phi(x) \tag{28}$$

We transform the equation to be the following:

$$(ip)^2 \widehat{u}(p,y) + \widehat{u}(p,y) = 0 \tag{29}$$

With IVP being transformed trivially. We solve the equation to yield:

$$u(p,x) = c_1(p)e^{py} + c_2(p)e^{-py} \tag{30}$$

We need a extra condition, we will require $\widehat{u}(p,y)$ to be bounded when $y \to \infty$. Thus, e^{py} is not allowed if $p > 0$. Thus

$$\widehat{u}(p,y) = c_2(p)e^{-py} \tag{31}$$

else if $p < 0$

$$\widehat{u}(p,y) = c_2(p)e^{py} \tag{32}$$

To solve this issue, we rewrite the transformed function to be

$$\widehat{u}(p,y) = c_3(p)e^{-|p|y} \tag{33}$$

Applying the initial condition yields

$$\widehat{u}(p,y) = \widehat{\phi}(p)e^{-|p|y} \tag{34}$$

We notice that it's a multiplication, thus the function must be a convolution, we have that

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-z)^2 + y^2}\phi(z)dz \tag{35}$$