MTH 553: Lecture # 13

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Lecture Span

- HWE in $\mathbb{R}^2$ and $\mathbb{R}^3$

HWE on $\mathbb{R}^2$ and $\mathbb{R}^3$

Consider $h \in C(\mathbb{R}^n)$ for $n \leq 2$. Then define

$$M_h(x,r) = \int_{S^{n-1}} h(x + r\xi)\frac{d\xi}{\omega_n} \tag{1}$$

Where $d\xi$ is a surface area element in S^{n-1} and $\omega_n = |S^{n-1}|$ is the area of S^{n-1}. We note that $\xi \in S^{n-1}$ is a unit sphere and is a unit vector. Moreover, M_h is the mean of the average value of the h over a sphere of radius $|r|$. For $n = 2$, we have that ω_2 is 2π. Then

$$M_h(x,r) = \frac{1}{2\pi}\int_0^{2\pi} h(x + r\xi)d\theta \tag{2}$$

Where $\xi = (\cos\theta, \sin\theta)$. For $n = 3$, then $\omega_3 = 4\pi$.

Observe

1. $M(h, -r) = M(h, r)$
2. If $h \in C^k$, then $M_h \in C^k$.

Lemma 0.1. *(Darboux) For $h \in C^2$ and $r \neq 0$, then*

$$\left(\partial_r^2 + \frac{n-1}{r}\partial_r\right)M_h = M_{\nabla^2 h}(x,r) \tag{3}$$

In other words, the laplacian of the mean is the mean of the laplacian.

Proof. When $n = 2$. Consider $e^{i\theta}$ on the unit circle. For $r > 0$: LHS:

$$\frac{1}{2\pi}\int_0^{2\pi}(g_{rr} + \frac{1}{r}g(-r))d\theta \tag{4}$$

Note that $g(re^{i\theta}) = h(x + re^{i\theta})$. We just took r derivatives.

$$= \frac{1}{2\pi}\int_0^{2\pi}(g_{rr} + \frac{1}{r}g_r + \frac{1}{r^2}g_{\theta\theta})d\theta \tag{5}$$

Since g is periodic and therefore a closed integral is zero. But this is the Laplacian of g

$$\int_0^{2\pi}\nabla^2 g(re^{i\theta})\frac{d\theta}{2\pi} \tag{6}$$

$$\iff \int_0^{2\pi}\nabla^2 h(x + re^{i\theta})\frac{d\theta}{2\pi} \tag{7}$$

$$= M_{\nabla^2 h}(x,r) \tag{8}$$

□

Proof. When $n \geq 2$. Then LHS is

$$= \left(\partial_r + \frac{n-1}{r}\right) \int_{\partial S^n} g_r \frac{d\xi}{\omega_n} \tag{9}$$

Here, the boundary of the ball in S^n is the sphere in S^{n-1}. Then integrate over a sphere of radius r (NOT 1)

$$= \left(\partial_r + \frac{n-1}{r}\right) \int_{\partial B^n(r)} \nabla g(y) \cdot \vec{v} \frac{dS(y)}{r^{n-1}} \tag{10}$$

Then, do a change of variables $y = r\xi$ and $dS(y) = r^{n-1} d\xi$, $\vec{v}$ is the outward normal vector on $\partial B^n(r)$ at $r\xi$. Note that $\nabla g(y) \cdot v = g_r$. Now, we use the divergence theorem

$$\frac{1}{\omega_n}\left(\partial_r + \frac{n-1}{r}\right)\left[\frac{1}{r^{n-1}} \int_{B^n(r)} \nabla \cdot \nabla g(y) dy\right] \tag{11}$$

$$= \frac{1}{\omega_n}\left(\partial_r + \frac{n-1}{r}\right)\left[\frac{1}{r^{n-1}} \int_{B^n(r)} \nabla^2 g(y)\right] \tag{12}$$

In spherical coordinates, this is the same as:

$$= \frac{1}{\omega_n}\left(\partial_r + \frac{n-1}{r}\right)\left[\frac{1}{r^{n-1}} \int_0^r \int_{S^{n-1}} \nabla^2 g(\rho, \xi) d\xi \rho^{n-1} d\rho\right] \tag{13}$$

Next, by integrating in spherical coordinates, firstly the following equation is true:

$$\left(\partial_r + \frac{n-1}{r}\right) \frac{1}{r^{n-1}} = 0 \tag{14}$$

Next,

$$\frac{1}{\omega_n} \frac{1}{r^{n-1}} \int_{S^{n-1}} \nabla^2 g(r, \xi) d\xi r^{n-1} \tag{15}$$

$$= \frac{1}{\omega_n} \int_{S^{n-1}} \nabla^2 g(r, \xi) d\xi = M_{\nabla^2 h}(x, r) \tag{16}$$

□

Homogenous wave equation in $\mathbb{R}^3$

$$u_{tt} - c^2 \nabla^2 u = 0$$
$$u(x, 0) = g(x) \in C^3$$
$$u_t(x, 0) = h(x) \in C^3$$

For $x \in \mathbb{R}^3$.

Kirchoff's Formula

$u \in C^2\left(\mathbb{R}^3 \times [0, \infty)\right)$. Solves HWE in $\mathbb{R}^3$ if

$$u(x, t) = \partial_t \left(t \int_{S^2} g(x + ct\xi) \frac{d\xi}{4\pi}\right) + t \int_{S^2} h(x + ct\xi) \frac{d\xi}{4\pi} \tag{17}$$

The first term is the displacement response. The second term is the response to the initial impulse.

Proof. ($\Longrightarrow$) For all fixed x, $rM_u(x, r)$ satisfies the 1-d wave equation as a function of r and t.

Proof. For $r \neq 0$, we have that $\partial_t^2 \left(rM_u(x, r)\right) = rM_{u_{tt}}(x, r)$ □

$$= c^2 r M_{\nabla^2 u}(x, r) \tag{18}$$

$$= c^2 r \left(\partial_r^2 + \frac{2}{r}\partial_r\right) M_u(x, r) \tag{19}$$

by the wave equation and Darboux. Here, we assumed that u satisfied the wave equation.

$$= c^2 \partial_r^2 \left(r M_u(x, r)\right) \tag{20}$$

By the product rule. Now, by d'Alembert,

$$rM_u(x, r) = \frac{1}{2}\left((r + ct)M_g(x, r + ct) + (r - ct)M_g(x, r - ct)\right) + \frac{1}{2c}\int_{r-ct}^{r+ct} \rho M_h(x, \rho) d\rho \tag{21}$$

$$= \frac{1}{2}\left\{(ct + r)M_g(x, ct + r) + -(ct - r)M_g(x, ct - r)\right\} + \frac{1}{2c}\int_{ct-r}^{ct+r} \rho M_h(x, \rho) d\rho \tag{22}$$

Here, since by oddness,

$$\int_{r-ct}^{ct-r} \rho M_h(x, \rho) d\rho = 0$$

Now, fix t and divide by r and let $r \to 0$. Therefore, $M_u \to u$ and $ct \pm r \to ct$. Therefore, we get:

$$u(x, r) = \partial_\tau \left(\tau M_g(x, \tau)\right)\Big|_{\tau = ct} + \frac{1}{c} \cdot ct M_h(x, ct) \tag{23}$$

Then, use Kirchoff's. $\square$