Bounded Functions and Sequences & Limits

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Proposition 0.1. *The sum of any two bounded functions on the same domain D is bounded.*

Proof. Suppose that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded. Then $|f(x)| \leq M$ and $|g(x)| \leq N$ for all $x \in D$. We then add them together

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M + N \tag{1}$$

Thus the function sum of both $f(x)$ and $g(x)$ is bounded. □

Definition 0.2. *A sequence of numbers is a function from $f : \mathbb{N} \to \mathbb{R}$*

Note:

1. All sequences are infinite
2. We write $x_1, x_2, \cdots$ instead of $f(1), f(2), \cdots$
3. The sequences as a whole is denoted as $(x_n)_{n=1}^{\infty}$ or (x_n) for short.
4. An example would be $x_n = n$ or $(n)_{n=1}^{\infty} = (1, 2, 3, \cdots)$

Boundedness, sup/inf all apply to sequences, in particular to the $\text{Im}(f) = \{x_n : x \in \mathbb{N}\}$

Definition 0.3. *Let (x_n) be a sequence and $x \in \mathbb{R}$:*

1. *We say that $\lim_{n\to\infty} x_n = x$ if there exists some $\epsilon > 0$, there exists some $M \in \mathbb{N}$ such that for all $n \geq M$ such that $|x_n - n| < \epsilon$*
2. *A sequence converges if $\lim_{n\to\infty} x_n = x$ for some x, other it diverges.*

Suppose that $x_n = \frac{\sin(n)}{n}$, claim is that it converges to 0.

Proof. Let $\epsilon > 0$ be arbitrary, let $n \geq M$ be arbitrary, we must prove that

$$|\frac{\sin(n)}{n}| < \epsilon \tag{2}$$

We choose $M = \frac{1}{\epsilon} + 1$, and using the fact that

$$|\frac{\sin(n)}{n}| \leq \frac{1}{n} \tag{3}$$

We can prove this. □

Proposition 0.4. *If (x_n) is a convergent sequence, then it is bounded.*

Proof. Let (x_n) be a covergent sequence, we claim that it is bounded. In other words, there exists some $B \in \mathbb{R}$ such that

$$|x_n| \leq B \quad \forall n \in \mathbb{N} \tag{4}$$

Let $\epsilon = 1$ since this a convergent sequence, we see that there exists some $M \in \mathbb{R}$ such that for all $n \geq M$,

$$|x_n - x| < 1 \tag{5}$$

We see that

$$\begin{aligned} |x_n| &= |(x_n - x) + x| && (6) \\ &\leq |x_n - x| + |x| && (7) \\ &< 1 + |x| && (8) \end{aligned}$$

We now know that all x_n are bounded between this value, except for $x_1, x_2, x_3, \ldots$. Then we say that

$$B = \max(1 + |x|, |x_1|, |x_2|, |x_3|, \ldots, |x_{M-1}|) \tag{9}$$

□

Proposition 0.5. *A sequence can have, at most, 1 limit.*

Proof. Suppose that (x_n) is a sequence such that $x_n \to x$ and $x_n \to y$, then we claim that $x = y$. To do this, let $\epsilon > 0$ be arbitrary, we claim that $|x - y| < \epsilon$. Plugging in $\frac{\epsilon}{2}$ to the definition, we see that there exists some M_1, there exists some $M_1 \in \mathbb{N}$ such that for all $n \geq M_1$, we have that

$$|x_n - x| < \frac{\epsilon}{2} \tag{10}$$

Similary, we have that for some $M_2 \in \mathbb{N}$, we have that

$$|x_n - x| < \frac{\epsilon}{2} \tag{11}$$

Choose $n = \max(M_1, M_2)$, then $n \geq M_1$ and $n \geq M_2$. Then,

$$|x - y| = |(x_n - y) - (x_n - x)| \leq |x_n - y| + |x_n - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{12}$$

□