PHYS 487: Lecture # 10

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Lecture Span

- Angular Momentum

Recall

The perturbation to the Hamiltonian is

$$H' \propto J_1 \cdot J_2$$

Where J_i is the angular momentum of a particle. Moreover,

$$[H', J_1] \neq 0 \neq [H', J_2]$$

So, then we found that we want to understand this Hamiltonian from the basis of the total angular momentum such that $J = J_1 + J_2$. For individual A.M, we have that

$$|j_1, m_1, j_2, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle \tag{1}$$

This means that the total number of states is

$$j_1(j_1+1) \cdot j_2(j_2+1) \to (2j_1+1) \cdot (2j_2+1)$$

Kronecker Products

Suppose two vectors

$$|\alpha\rangle \otimes |\beta\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \end{pmatrix}$$

For example, suppose $S_1 \cdot S_2$. Clearly, S_1 only acts on the first particle and so on. Both live in different subspaces. Formally, this is the same as

$$S_1 \cdot S_2 = \vec{S_1} \otimes \vec{\mathbb{I}} \cdot \vec{\mathbb{I}} \otimes S_2 \tag{2}$$

$$= \begin{pmatrix} S_{1,x} \otimes \mathbb{I} \\ S_{1,y} \otimes \mathbb{I} \\ S_{1,z} \otimes \mathbb{I} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{I} \otimes S_{2,x} \\ \mathbb{I} \otimes S_{2,y} \\ \mathbb{I} \otimes S_{2,z} \end{pmatrix} \tag{3}$$

Correct quantum numbers

Commuting operators:

$$[J^2, J_z] = 0 \tag{4}$$

But,

$$[J_{1,z}, J^2] \neq 0 \tag{5}$$

$$[J^2, J_{1\vee 2}^2] = 0 \tag{6}$$

But we said that

$$J_1 \cdot J_2 \propto (J^2 - J_1^2 - J_2^2) \tag{7}$$

Now, now there is more quantum numbers, so define a state

$$|j, m_1, j_1, j_2\rangle \tag{8}$$

We now conduct perturbation theory with these states. Formally, this is a basis change. In other words:

$$\begin{aligned} |j, m_j, j_1, j_2\rangle &= \sum_{m_1, m_2} |j_1, m_1, j_2, m_2\rangle \langle j_1, m_1, j_2, m_2 | j, m_j, j_1, j_2\rangle \\ &= \sum_{m_1, m_2} \langle j_1, m_1, j_2, m_2 | j, m_j, j_1, j_2\rangle |j_1, m_1, j_2, m_2\rangle \end{aligned}$$

These are the Clebsch-Gordon coefficients. We note several things. Firstly, j_1, j_2 are always fixed (like charge of a particle fixed). Therefore, for brevity

$$|j, m_j, j_1, j_2\rangle \to |j, m_j\rangle \tag{9}$$

For each j, we have that $m_j = -j, -j+1, \ldots, +j$. Moreover, j (total angular momentum) can take on values of $|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2$. From this, we note that m in general is not unique, therefore, the Clebsch gordon coefficient tables become disgusting. Moreover:

$$J_z |j_1, m_1, j_2, m_2\rangle = (J_{1,z} + J_{2,z}) |j_1, m_1, j_2, m_2\rangle = \hbar(m_1 + m_2) |j_1, m_1, j_2, m_2\rangle \tag{10}$$

So now, the question is how can I move the basis from $|j, m_j, j_1, j_2\rangle$ to $|j_1, m_1, j_2, m_2\rangle$. The procedure is the following:

1. $j = j_1 + j_2$ and $m_j = -j, \ldots, +j$
2. $|(j_1 + j_2), j_1 + j_2\rangle = |j_1, j_1, j_2, j_2\rangle$ [!] (basis transformatin)
3. Apply ladder operators $J_- = J_{1,-} + J_{2,-}$ (iteratively) Recall that

   $$J_\pm |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}\hbar |j, m \pm 1\rangle$$

   Then, find all m_1, m_2 such that $m_1 + m_2 = m$.
4. Next, find $j = j_1 + j_2 - 1$

Example, two spin 1/2's (electrons)

Let $\vec{S} = \vec{S_1} + \vec{S_2}$. Now, consider

$$|s = 1, m_s = 1\rangle = |s_1 = 1/2, m_1 = 1/2, s_2 = 1/2, m_2 = 1/2\rangle \tag{11}$$

Now, use the ladder operator. Consider

$$\begin{aligned} S_- |11\rangle &= (S_{1,-} + S_{2,-}) \left(\left| \frac{1}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle \right) \\ &= \hbar \left(\left| \frac{1}{2}, -\frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \left| \frac{1}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right) \end{aligned}$$

Or equivalently,

$$|10\rangle = \frac{1}{\sqrt{2}} (|\downarrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle) \tag{12}$$

Then,

$$S_- \left|10\right\rangle = \left|1, -1\right\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \tag{13}$$

Finally, the starting point $\left|00\right\rangle$, orthogonal to $\left|10\right\rangle$. This has to be

$$\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, \frac{1}{2}\right\rangle - \left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right) \tag{14}$$