

Cauchy Sequences

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Definition 0.1. A sequence (x_n) is Cauchy if for all $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $n, k \geq M$ we have that

$$|x_n - x_k| < \epsilon \quad (1)$$

Basically if all the terms become really close to x .

Intuitively, a sequence is Cauchy if all the terms in a sequence get close to each other. This is nothing but saying that at most, the distance between terms must be less than ϵ .

Theorem 0.2. For every sequence (x_n) , we have that

$$(x_n) \text{ converges} \iff (x_n) \text{ is Cauchy}$$

Proof. This is a proof of \implies . Suppose that (x_n) is convergent. Then we claim that it is Cauchy. Let $\epsilon > 0$, and let $M \in \mathbb{N}$ with $n, k \geq M$. Because (x_n) converges, we have that there exists $M_1 \in \mathbb{N}$ such that for all $n \geq M$, we have that

$$|x_n - x| < \frac{\epsilon}{2} \quad (2)$$

Suppose we choose this M_1 . Then for all n, k we have that

$$|x_k - x| < \frac{\epsilon}{2} \quad (3)$$

as well. Then we have that $|x_n - x_k| \iff |x_n - x - x_k + x|$ which simplifies down to $|x_n - x_k|$. By the triangle inequality, we have that

$$|x_n - x_k| \leq |x_n - x| + |x_k - x| < \epsilon \quad (4)$$

□

Before we prove the opposite direction, let's first prove a lemma.

Lemma 0.3. If (x_n) is Cauchy, then it is bounded.

Proof. Assume that (x_n) is Cauchy, then we plug in $\epsilon = 1$, then we have that

$$|x_n - x_k| < 1 \quad (5)$$

for all $n, k \in M$ for some $M \in \mathbb{N}$. Then choose $k = M$, we have that

$$|x_n - x_M| < 1 \quad (6)$$

for all $n \geq M$. In particular, we have that

$$|x_n| = |(x_n - x_M) + x_M| \leq |x_n - x_M| + |x_M| < 1 + |x_M| \quad (7)$$

For all $n \geq M$. Then all terms x_n satisfy

$$|x_n| \leq B \quad (8)$$

where

$$B = \max(1 + |x_M|, |x_1|, |x_2|, \dots, |x_{n-1}|) \quad (9)$$

This proves the lemma. □

Proof. Prove of (\Leftarrow) of the Cauchy Convergence Theorem. Let (x_n) be a Cauchy Sequence. Then by the lemma it is bounded. Then we can apply \limsup and \liminf . Let $a = \limsup x_n$ and $b = \liminf x_n$. By a fact that we proved in the previous lecture, this proof will be done by proving that $a = b$. To prove this, we prove that

$$|a - b| < \epsilon \quad (10)$$

for all $\epsilon > 0$. Then we prove this

Proof. Let $\epsilon > 0$, we claim that $|a - b| < \epsilon$. By another fact, there exists subsequences in (x_n) that converges to a and x_{m_i} that converges to b . Because of these two facts, there exists M_1 and M_2 in \mathbb{N} such that for all $i \geq M_1$, we have that

$$|x_{n_i} - a| < \frac{\epsilon}{3} \quad (11)$$

and for all $i \geq M_2$, we have that

$$|x_{m_i} - b| < \frac{\epsilon}{3} \quad (12)$$

Since (x_n) is Cauchy, there exists $M_3 \in \mathbb{N}$ such that

$$|x_{m_i} - x_{n_i}| < \frac{\epsilon}{3} \quad (13)$$

We first choose $i = \max(M_1, M_2, M_3)$. We perform some calculations:

$$|a - b| \iff |a - x_{n_i} - (b - x_{m_i}) + x_{n_i} - x_{m_i}| \leq |a - x_{n_i}| + |b - x_{m_i}| + |x_{n_i} - x_{m_i}| < \epsilon \quad (14)$$

This concludes the proof. □

□