Cauchy Sequences

Cliff Sun

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Definition 0.1. *A sequence (x_n) is* <u>*Cauchy*</u> *if for all $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $n, k \geq M$ we have that*

$$|x_n - x_k| < \epsilon \tag{1}$$

Basically if all the terms become really close to x.

Intuitively, a sequence is Cauchy if all the terms in a sequence get close to each other. This is nothing but saying that at most, the distance between terms must be less than ϵ.

Theorem 0.2. *For every sequence (x_n), we have that*

$$(x_n) \text{ converges} \iff (x_n) \text{ is Cauchy}$$

Proof. This is a proof of $\implies$. Suppose that (x_n) is convergent. Then we claim that it is Cauchy. Let $\epsilon > 0$, and let $M \in \mathbb{N}$ with $n, k \geq M$. Because (x_n) converges, we have that there exists $M_1 \in \mathbb{N}$ such that for all $n \geq M$, we have that

$$|x_n - x| < \frac{\epsilon}{2} \tag{2}$$

Suppose we choose this M_1. Then for all n, k we have that

$$|x_k - x| < \frac{\epsilon}{2} \tag{3}$$

as well. Then we have that $|x_n - x_k| \iff |x_n - x - x_k + x|$ which simplifies down to $|x_n - x_k$. By the triangle inequality, we have that

$$|x_n - x_k| \leq |x_n - x| + |x_k - x| < \epsilon \tag{4}$$

$\square$

Before we prove the opposite direction, let's first prove a lemma.

Lemma 0.3. *If (x_n) is Cauchy, then it is bounded.*

Proof. Assume that (x_n) is Cauchy, then we plug in $\epsilon = 1$, then we have that

$$|x_n - x_k| < 1 \tag{5}$$

for all $n, k \in M$ for some $M \in \mathbb{N}$. Then choose $k = M$, we have that

$$|x_n - x_M| < 1 \tag{6}$$

for all $n \geq M$. In particular, we have that

$$|x_n| = |(x_n - x_M) + x_M| \leq |x_n - x_M| + |x_M| < 1 + |x_M| \tag{7}$$

For all $n \geq M$. Then all terms x_n satisfy

$$|x_n| \leq B \tag{8}$$

where

$$B = \max(1 + |x_M|, |x_1|, |x_2|, \ldots, |x_{n-1}|) \tag{9}$$

This proves the lemma. $\square$

Proof. Prove of ($\Longleftarrow$) of the Cauchy Convergence Theorem. Let (x_n) be a Cauchy Sequence. Then by the lemma it is bounded. Then we can apply lim sup and lim inf. Let $a = \limsup x_n$ and $b = \liminf x_n$. By a fact that we proved in the previous lecture, this proof will be done by proving that $a = b$. To prove this, we prove that

$$|a - b| < \epsilon \tag{10}$$

for all $\epsilon > 0$. Then we prove this

Proof. Let $\epsilon > 0$, we claim that $|a - b| < \epsilon$. By another fact, there exists subsequences in (x_{n_i}) that converges to a and x_{m_i} that converges to b. Because of these two facts, there exists M_1 and M_2 in $\mathbb{N}$ such that for all $i \geq M_1$, we have that

$$|x_{n_i} - a| < \frac{\epsilon}{3} \tag{11}$$

and for all $i \geq M_2$, we have that

$$|x_{m_i} - b| < \frac{\epsilon}{3} \tag{12}$$

Since (x_n) is Cauchy, there exists $M_3 \in \mathbb{N}$ such that

$$|x_{m_i} - x_{n_i}| < \frac{\epsilon}{3} \tag{13}$$

We first choose $i = \max(M_1, M_2, M_3)$. We perform some calculations:

$$|a - b| \iff |a - x_{n_i} - (b - x_{m_i}) + x_{n_i} - x_{m_i}| \leq |a - x_{n_i}| + |b - x_{m_i}| + |x_{n_i} - x_{m_i}| < \epsilon \tag{14}$$

This concludes the proof. $\square$

$\square$