MTH 447: Lecture 2

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January 24, 2025

The Algebra of $\mathbb{Z}$:

1. $(x + y) + z = x + (y + z)$
2. $x + y = y + x$
3. $x + 0 = x$ for some $0 \in \mathbb{Z}$
4. $x + (-x) = 0$ for some $-x \in \mathbb{Z}$

This is an abelian group. We then introduce the multiplication rules

1. $(xy)z = x(yz)$
2. $xy = yx$
3. $x \cdot 1 = 1 \cdot x = x$
4. $x(y + z) = xy + xz$

These proporties form a communative ring, but NOT a field because there is no multiplicative inverse. Recall, equivalence relations are

1. Reflexive
2. Symmetric
3. Transitive

The statement $ad = bc$ is an equivalence relation. Where $d, c \neq 0$. In the rational numbers $\mathbb{Q}$, we have a new property that

1. $xx^{-1} = 1$ for some $x^{-1} \in \mathbb{Q}$

Now our set of numbers is a field. Note that

$$\mathbb{Q} = [\mathbb{Z} \times (\mathbb{Z}\ 0)] / \sim \tag{1}$$

Where $\sim$ is the equivalence relation as above. This creates a coordinate system with (p, q) where p is the top part of the fraction and etc. But $\mathbb{Q}$ has a lot of "holes", that we can't use analysis on it. Example: we claim that there is no rational solution for $c^2 = 2$.

Proof. We assume that $x = \frac{p}{q}$ and $x^2 = 2$. Then

$$\frac{p^2}{q^2} = 2 \tag{2}$$

$$p^2 = 2q^2 \tag{3}$$

Thus the right hand side is even. Therefore, p^2 is even. Now that p is even. Thus $p = 2k$. Then

$$4k^2 = 2q^2 \tag{4}$$

$$q^2 = 2k^2 \tag{5}$$

Therefore, q^2 is even, which implies that q is also even. Therefore, there is no rational solution to $x^2 = 2$. $\square$

Theorem 0.1. *Let $x_1 < x_2$ be in $\mathbb{Q}$, then there are infinitely many irrational numbers between x_1 and x_2.*

Lemma 0.2. *Let $x_1 < x_2$ be in $\mathbb{Q}$. Then*

$$x_1 + \frac{x_2 - x_1}{n} \in \mathbb{Q} \tag{6}$$

Is rational. Thus, there are infinitely many rational numbers between x_1 and x_2.

Lemma 0.3. *Given $x_1, x_2 \in \mathbb{Q}$, and $x_1 < x_2$. Then*

$$x_1 + \frac{x_2 - x_1}{\sqrt{2}} \tag{7}$$

is not rational.

Proof. Assume that

$$x_1 + \frac{x_2 - x_1}{\sqrt{2}} = \frac{p}{q} \tag{8}$$

Then we simplfy down to

$$\sqrt{2} = \frac{x_2 - x_1}{\frac{p}{q} - x_1} \in \mathbb{Q} \tag{9}$$

and $\frac{p}{q} \neq x_1$. This is a contradiction. $\square$

Then between 2 rational numbers, there is always at least on irrational number.