4-vectors and the Doppler effect

Cliff Sun

April 2, 2024

One Minute Paper 0.1. *Before: The frequency would increase when the truck drives towards me. After: Moving away*

Definition 0.2. *When choosing c = 1, v must be within the range of* $[0, 1]$

To recap, a vector in Newtonian Mechanics transforms the same way as coordinates under rotations. That is, the components of the vector must be conserved when taking the dot-product with itself.

Then, a 4-vector in special relativity is a vector that transforms the same way as coordinates under the Lorentz tranformations. That is:

$$x'^{\mu} = \Lambda^{\mu}_{v} x^{v} \iff v'^{\mu} = \Lambda^{\mu}_{v} v^{v} \tag{1}$$

That is, the components of this transformed vector must be conserved with respect to the relativistic dot-product. Then to take the time-derivative of the 4-vector displacement, we must take the time-derivative with respect to something more absolute. This is the proper time, or expressed by this equation:

$$(d\tau)^2 = dx_{\mu} dx^{\mu} \iff (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{2}$$

There is an implied metric tensor that is operating this relativistic dot-product. This is Lorentz invariant under any Lorentz transformation frame. Then taking the time-derivative of the spatial 4-vector yields the following result:

$$V^{\mu} = (\gamma, \gamma v) \tag{3}$$

Where v is the velocity vector, not the 4-vector. Then, 4-momentum is the following:

$$P^{\mu} = (\gamma m, \gamma m v) \tag{4}$$

In general, the energy of a particle can be reduced to mc^2 in a respective frame. Thus taking the relativistic dot-product, we have that

$$\gamma^2 m^2 c^4 - p^2 = m^2 c^4 \tag{5}$$

Thus, we are able to Lorentz transform the 4-momentum vector because this relativistic dot=product is conserved. This is something that we've already proved in class. Then, since momentum is a 4-vector, we can transform it using a Lorentz transformation, that is:

$$P^{\mu} = \Lambda^{\mu}_{v} P^{\mu} \tag{6}$$

Thus, we see that

$$F^{\mu} = \frac{dP^{\mu}}{d\tau} \iff mA^{\mu} \tag{7}$$

Similarly, the Doppler effect states that the energy of a photon is transformed as the following:

$$E' = E\gamma(1 + \beta) \iff E\sqrt{\frac{1+\beta}{1-\beta}} \tag{8}$$

Such that $\beta \in [0, 1]$.