Monotone Sequences and Tails of Sequences

Cliff Sun

April 19, 2024

Definition 0.1. *Let (x_n) be a sequence:*

1. *(x_n) is (monotone) increasing if $x_{n+1} \geq x_n$ for all n*
2. *(x_n) is (monotone) decreasing if $x_{n+1} \leq x_n$ for all n*
3. *(x_n) is monotone if (x_n) is either monotone increasing or monotone decreasing*

Theorem 0.2. *Let (x_n) be a montone sequence.*

1. *(x_n) converges if and only if it is bounded.*
2. *If (x_n) is increasing and bounded, then*
$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\} \tag{1}$$
3. *If (x_n) is decreasing and bounded, then*
$$\lim_{n\to\infty} x_n = \inf(x_n : n \in \mathbb{N}) \tag{2}$$

Note:

1. If (x_n) is increasing, then (x_n) is bounded below by x_1.
2. If (x_n) is decreasing, then (x_n) is bounded above x_1
3. The only interesting question is if (x_n) is bounded above/below if it's increasing/decreasing.

Corollary 0.3. *The comparison test for infinite series says that if Σa_n and Σb_n are infinite series and $a_n, b_n \geq 0$, and $a_n \leq b_n$ for all n. Then if Σb_n converges, then Σa_n converges.*

Definition 0.4. *The sum of some series a_n is the limit as $n \to \infty$ of the sequence s_n where*
$$s_n = a_1 + a_2 + \cdots + a_n \tag{3}$$

Proof. We already know the " $\Longrightarrow$ " of (1), we will prove (2). By replacing $x_n = -x_n$, we will prove (3). And combining (2) and (3), we will have proved " $\Longleftarrow$ " of (1). So suppose that (x_n) is increasing and bounded above. Since x_n is nonempty and bounded above, we have that it must have a least upper bound, in other words it has a supremum. So we must prove that

For all $\epsilon > 0$, there exists some $M \in \mathbb{N}$ such that for all $n \geq M$, $|x_n - x| < \epsilon$ where $x = \sup(x_n)$.

Recall from homework 11, for any nonempty, bounded above set S, and for any $\epsilon > 0$, there exists an $x \in S$ such that
$$\sup(S) - \epsilon < x \leq \sup(S) \tag{4}$$
In particular, given any $\epsilon > 0$, there exists some $M \in \mathbb{N}$ such that $\sup(x_n) - \epsilon < x_M \leq \sup(x_n)$. Choose that M, then we have that
$$x - \epsilon < x_M \leq x_n \leq x < x + \epsilon \tag{5}$$
But in particular
$$|x_n - x| < \epsilon \tag{6}$$
This concludes the proof. $\square$

Tails of a sequence

Definition 0.5. *Let (x_n) be a sequence, and let $k \in \mathbb{N}$. Then the k-Tail of (x_n) is the sequence*

$$(x_{n+k})_{n=1}^{\infty} = (x_{k+1}, x_{k+2}, \cdots) \tag{7}$$

In other words, you drop the first k terms.

Theorem 0.6. *If (x_n) is any sequence, $x \in \mathbb{R}$, and $k \in \mathbb{N}$, then*

$$\lim_{n\to\infty}(x_n) = x \iff \lim_{n\to\infty}(x_{n+k}) = x \tag{8}$$

In particular

1. If (x_n) converges to x, then all of its k-tails must also converge to the same number x.
2. If (x_n) diverges, then all of its k-tails also diverge.