PHYS 325: Lecture 23

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Dynamics in rotating frames

$$F_{\text{true}} = ma_{oa} - F_{\text{Fict}} \tag{1}$$

$$\iff ma_{\text{app}} = F_{\text{true}} + F_{\text{Fict}} \tag{2}$$

$$\iff ma = F_{\text{true}} - ma_{OO'} - m\omega \times (\omega \times r_{o'a}) - 2m\omega \times v_{\text{app}} - m\dot{\omega} \times r_{ra'} \tag{3}$$

$$ma_{oo'} = \text{ elevator force} \tag{4}$$

$$2m\omega \times v_{\text{app}} = \text{ coriolis force} \tag{5}$$

$$m\dot{\omega} \times r_{o'a} = \text{euler force} \tag{6}$$

Example: Puck on an icy disk

- Puck launched from perimeter towards B
- $x(t = 0) = 0$, $\dot{x}(t = 0) = 0$. $y(t = 0) = -R$, $\dot{y}(t = 0) = u$.
- Find the trajectory of r in the rotating frame.

Calculation

1. EOM
   - $F_{\text{true}} = 0$
   - $a_{oo'} = 0$ (doesn't move)
   - $\dot{\omega} \times r_{o'a} = 0$, constant ω
2. centrifugal force:

$$\omega \times (\omega \times r_{o'a}) \tag{7}$$

$$\omega = \omega e_z \tag{8}$$

$$r_{o'a} = xe_x + ye_y \tag{9}$$

$$= -\omega^2(xe_x + ye_y) \tag{10}$$

3. Coriolis force

$$\omega \times v_{app} = \omega \times (\dot{x}e_x + \dot{y}e_y) \tag{11}$$

$$= \omega\dot{x}e_y - \omega\dot{y}e_x \tag{12}$$

4. EOM is
$$m(\ddot{x}e_x + \ddot{y}e_y) = ma_{app} = m\omega^2(xe_x + ye_y) - 2m\omega(\dot{x}e_y + \dot{y}e_x) \tag{13}$$
Then
$$e_x : \ddot{x} = \omega^2 x + 2\omega\dot{y} \tag{14}$$
$$e_y : \ddot{y} = \omega^2 y - 2\omega\dot{x} \tag{15}$$
Trick: use complex function $\Psi = x(t) + iy(t) \in \mathbb{C}$. Inserting it into the equation yields
$$\ddot{\Psi} + 2i\omega\dot{\Psi} - \omega^2\Psi = 0 \tag{16}$$
Let ansatz: $\Psi = e^{\lambda t}$, then
$$\lambda^2 + 2i\omega\lambda - \omega^2 = 0 \tag{17}$$
then
$$\lambda = -\frac{2i\omega}{2} \pm \sqrt{-\omega^2 + \omega^2} = -i\omega \tag{18}$$
Thus
$$\Psi(t) = Ae^{-\omega t} + Bte^{-i\omega t} \tag{19}$$
Using the initial conditions
$$\Psi(t=0) = x(t=0) + iy(t=0) = -iR = A \tag{20}$$
$$\dot{\Psi}(t=0) = \dot{x}(t=0) + i\dot{y}(t=0) = iu = -i\omega A + B \tag{21}$$
$$B = iu + R\omega \tag{22}$$
$$\Psi(t) = -iRe^{-i\omega t} + (iu + R\omega)te^{-i\omega t} \tag{23}$$
Then
$$x(t) = Re(\Psi(t)) \tag{24}$$
$$y(t) = Im(\Psi(t)) \tag{25}$$

Motion on rotating earth

We define O to be in the middle of Earth, and O' to be on the surface of Earth. Then we also use spherical coordinates. Then
$$R_e = 6400km \tag{26}$$
$$\omega = \frac{2\pi}{\text{day}} \tag{27}$$
$$v_{lab} = 1600km/h \tag{28}$$
$$\dot{\omega} = 17m/s \text{ per century } \approx 0 \tag{29}$$

$$a_{oo'} = \omega \times (\omega \times R) \tag{30}$$
$$= \omega^2 R \cos\theta \tag{31}$$
For $\theta = 0$, we have that
$$\omega^2 R \approx 0.034 << |g| \tag{32}$$
$$\omega \times (\omega \times r_{o'a}) \approx 0 \tag{33}$$
Then
$$ma_{\text{app}} = F_{\text{true}} - 2m\omega \times v_{\text{app}} \tag{34}$$

Coriolis Force on Earth

We define

1. $e_x = e = \text{east}$
2. $e_y = n = \text{north}$
3. $e_z = u = \text{up}$
4. $\theta = \text{latitude}$
5. $p = ncos\theta + u\sin\theta$
6. $\omega = Tp$

Then

$$m\ddot{r} = F_{\text{true}} - 2m\omega \times r \tag{35}$$

b

$$F_{\text{true}} = -mgu \tag{36}$$

$$= -mgu - 2m|\omega|p \times (-nsin\theta + ucos\theta) \tag{37}$$