PHYS 325: Lecture 9

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Lecture Span

- Central forces

Central Forces

Let

$$\vec{L} = \vec{r} \times m\vec{v} \tag{1}$$

We try to show that $\vec{L}$ is constant. Or

$$D_t(L) = 0 \tag{2}$$

We expand this out

$$D_t(L) = D_t(\vec{r} \times m\vec{v}) \tag{3}$$

$$\iff \dot{\vec{r}} \times m\vec{v} + \vec{r} \times m\vec{a} \tag{4}$$

$$\iff \vec{r} \times \vec{F} \tag{5}$$

But since $\vec{F}$ is in the direction of the $\hat{r}$ component, we have that

$$\vec{r} \times F\hat{r} \iff 0 \tag{6}$$

Thus

$$D_t(L) = 0 \iff L = c \tag{7}$$

Because $\vec{L}$ is constant, then we have that

1. motion remains in the $\vec{v} - \vec{r}$ plane
2. Then we can rotate the coordinate plane such that $\theta = \frac{\pi}{2} \iff \dot{\theta} = 0$
3. $\vec{r} = r(t)\hat{r}$ $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}$ which simplifies to

$$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \tag{8}$$

4. Thus $\vec{L} = \vec{r} \times m\vec{v}$

$$= mr\hat{r} \times m(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}) \tag{9}$$

$$= -mr^2\dot{\phi}\hat{\theta} \tag{10}$$

$$= mr^2\dot{\phi}\hat{z} \tag{11}$$

Conservative Central Force

Given

$$\vec{F} = f\hat{r} \iff -\vec{\nabla}u \tag{12}$$

Because there is no dependence of θ, ϕ, we have that

$$U = \int \vec{f} \cdot d\vec{r} \tag{13}$$

We write

$$L = mr^2\dot{\phi} \tag{14}$$

We write this as

$$\dot{\phi} = \frac{L}{mr^2} \tag{15}$$

$$E = T + U \tag{16}$$

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + U \tag{17}$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + U \tag{18}$$

Given

$$U = \frac{L^2}{2mr^2} - \frac{k}{r} \tag{19}$$

Case 1

$$E > 0 \tag{20}$$

$$\frac{L^2}{2mr^2} \tag{21}$$

dominates.

Case 2

$$E < 0 \tag{22}$$

$$\frac{k}{r} \tag{23}$$

dominates.

Gravitational Force

Consider 2 masses of m_1, m_2, gravitational force between the 2 masses would be

$$-\frac{Gm_1m_2}{r^2}\hat{r} \tag{24}$$

Consider an extended object with mass m_1, then

$$m_1 = \int_V \rho(r)dr \tag{25}$$

Then,

$$F = -Gm_1\frac{\int_V \rho(r)}{(r_2 - r_1)^3}(r_2 - r_1)dr_1 \tag{26}$$