MTH 447: Lecture # 12

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February 21, 2025

Definition 0.1. *Given a sequence (x_n), a sequence is called* <u>*Cauchy*</u> *if*

$$\forall \epsilon > 0, \exists N \text{ s.t. } n, m > N \implies |x_n - x_m| < \epsilon \tag{1}$$

That is, x_n get close to each other.

Theorem 0.2. *Convergent $\implies$ Cauchy*

Proof. Let $\lim x_n = L$, then

$$\forall \epsilon > 0, \ \exists N \text{ such that } |x_n - L| < \epsilon$$

Pick $n, m > N$, then

$$|x_n - x_m| = |x_n - L + L - x_m| \leq |x_n - L| + |x_m - L| < \epsilon \tag{2}$$

□

Theorem 0.3. *Cauchy $\implies$ bounded*

Proof. Pick $\epsilon = 1$, then

$$\exists N \text{ such that } n, m > N, \ |x_n - x_m| < 1$$

Then

$$\forall n > N \implies |x_n - x_{N+1}| < 1$$

Then

$$|x_n| \leq |x_{N+1}| + 1 \tag{3}$$

For all $n > N$. Then choose $M = \max(|x_1|, |x_2|, \dots)$, then

$$|x_n| \leq M \tag{4}$$

For all n. □

Theorem 0.4. *If x_n is a sequence of real numbers, and x_n is Cauchy. Then x_n converges.*

Proof. $\forall \epsilon > 0$, $\exists \tilde{N}$ such that $n, m > \tilde{N}$, then $|x_n - x_m| < \epsilon$. Then choose $m = \tilde{N} + 1$, then

$$|x_n - x_{\tilde{N}+1}| < \epsilon \ \ \forall n > \tilde{N} \tag{5}$$

Then

$$x_n \leq x_{\tilde{N}+1} + \epsilon \tag{6}$$

and

$$x_n \geq x_{\tilde{N}+1} - \epsilon \tag{7}$$

Then

$$\limsup x_n = \lim_{\tilde{N} \to \infty} \sup_{k > \tilde{N}} \{x_k\} \tag{8}$$

$$= \lim_{\tilde{N} \to \infty} b_{\tilde{N}} \tag{9}$$

Namely,

$$b_{\tilde{N}} \leq x_{\tilde{N}+1} + \epsilon \tag{10}$$

Then

$$\limsup x_n \leq b_{\tilde{N}} \leq x_{\tilde{N}+1} \tag{11}$$

Because $\limsup$ is a decreasing function. See lecture notes for completion of proof. □

Subsequences

Let (x_n) be a sequence. Choose a sequence of natural numbers

$$1 \leq n_1, n_2, \dots \tag{12}$$

Where $n_k < n_{k+1}$, then we define a new subsequence

$$y_n = x_{n_k} \tag{13}$$

Definition 0.5. *Let (x_n) be a sequence. Let $P(x)$ be a boolean that is true or false for all x. Then a subsequenc of (x_n) satisfying P is*

$$n_1 = \min_k \{P(x_k) \text{ true }\} \tag{14}$$

$$n_2 = \min_{k > n_1} \{P(x_k) \text{ true }\} \tag{15}$$

$$n_3 = \min_{k > n_2} \{P(x_k)\} \tag{16}$$