MTH 416: Lecture 6

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Lecture Span

- Linear (in)dependence
- Bases & Dimensions

Linear (in)dependence

Recall: $\{u_1, \ldots, u_k\} \subseteq V$ is linearly dependent if there exists some non-trivial solution to the equation

$$a_1 u_1 + \cdots + a_k u_k = 0 \tag{1}$$

if not then they are linearly independent.

Example:

In $\mathbb{R}^2$, given $u_1 = (x_1, y_1)$, $u_2 = (x_2, y_2)$, and $u_3 = (x_3, y_3)$, then we look for solutions

$$a_1 u_1 + a_2 u_2 + a_3 u_3 = 0 \tag{2}$$

Then make the augmented matrix:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & 0 \\ y_1 & y_2 & y_3 & 0 \end{array}\right]$$

Then we row reduce it using Gaussian elimination to

$$\left[\begin{array}{ccc|c} 1 & 0 & ? & 0 \\ 0 & 1 & ? & 0 \end{array}\right]$$

Then we see that they are linearly dependent, as there are infinitely many solutions with a_3 being a free variable. That means that there must exist a non-zero solution.

Generally...

Given vectors $u_1, \ldots, u_k \in \mathbb{R}^n$, then write the columns as a matrix and put it in RREF. If there is at least one free variable, then the vectors are linearly dependent. Then no free variables imply linearly independence. Since that would imply that $a_k = 0 \ \forall k \in \mathbb{N}$.

Observation

Vectors $\{u_1, \ldots, u_k\}$ are lin. dep. if and only iff one of them is a linear combination of another.

Proof. $\implies$, suppose $u_1, \ldots, u_K$ are linearly dependent. That is there exists non-trivial solutions to

$$a_1 u_1 + \cdots + a_k u_k = 0 \tag{3}$$

Choose some $i < k$, such that $a_i \neq 0$, then move it to the other side. Then divide by a_i. Then

$$u_i = \frac{\sum (a_j u_j)}{a_i} \tag{4}$$

$\impliedby$, suppose that $u_i = a_1' u_1 + \ldots$, then subtracting u_i from both sides yields a linear dependence. $\square$

Edge cases

1. Any set containing the 0 vector is linearly dependent.

$$a_1 \cdot 0 + \cdots = 0 \; \forall a_1 \in \mathbb{R} \tag{5}$$

2. Any set consisting of one non-zero vector is linearly independent
3. The empty set is linearly independent.

We can also talk about linear independence of infinite sets $S \subseteq V$. Linear combinations of infinitely many vectors is defined to be linear combinations of finite subsets. So S is linearly dependent if and only if $u_1, \ldots, u_k$ are a finite amount of vectors in S.

Theorem 0.1. *Suppose $u_1, \ldots, u_k$ are vectors in V, then $W = span(u_1, \ldots, u_k)$. Then there is a subset of these vectors which is linearly independent and span $= W$.*

Proof. If $u_1, \ldots, u_k$ are already linearly independent, then choose the subset to be all of them. Else, if $u_1, \ldots, u_k$ are linearly dependent, then by definition at least 1 u_i is a linear combination of the rest, and thus removing u_i wouldn't change the span since we can reproduce u_i using the other vectors. We can repeat this until the set is linearly independent. □

Bases & Dimensions

Definition 0.2. *1. A set $S \in V$ is called a spanning set if $span(S) = V$.*

2. S is a basis if it is a linearly independent spanning set.

Claim: $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis of $\mathbb{R}^3$

Proof. Linearly dependent if $a_1u_1 + a_2u_2 + a_3u_3 = 0$, then $(a_1, a_2, a_3) = \vec{0}$.
Spanning set, given $(a_1, a_2, a_3) \in \mathbb{R}^3$, we can write it as $a_1u_1 + a_2u_2 + a_3u_3$. □

Example: In $\mathbb{R}^n$, let $e_i = \{0, \ldots, 1, \ldots, 0\}$ where 1 is in the i-th position. Then the set $\{e_1, \ldots\}$ is a basis of $\mathbb{R}^n$, then this is called the standard basis.

Example: The set $\{1, x, \ldots\}$ is a basis of $P_n(\mathbb{R})$.

Example: The set $\{1, x, x^2, \ldots\}$ is a basis of $P(\mathbb{R})$. Note this is an infinite set.

Note: We will define the dimensionality to be the number of basis vectors that span the vector space. But that gives arise to some issues

1. We don't know that every vector space has a basis
2. We haven't proved all basis of V have the same size.

Theorem 0.3. *Suppose β is the candidate basis of V. Then β is a basis if and only if every single vector in V can be write uniquely as a linear combination of*

$$a_1u_1 + a_2u_2 + \ldots \tag{6}$$

By uniquely, that means that there is only one set of a_i's that satisfies the above equation.

Proof. ($\implies$), Assume that β is a basis, choose that there is one and only one way to write every single vector in V for a given choice of scalars $(a_1, \ldots)$. Since β spans the vector space V, the constants $a_1, \ldots$ do exist. Suppose $v = a_1u_1 + \cdots = b_1u_1 + \ldots$. Then we prove that $a_1 = b_1, \ldots$. Subtracting $a - b$ yields

$$(a_1 - b_1)u_1 + \cdots = 0 \tag{7}$$

But since β is linearly independent, then we have that all constants $c_i = a_i - b_i$ are all 0, thus $a_i = b_i$. This means that

$$v = \sum a_i u_i \tag{8}$$

is unique. □