MTH 416: Lecture 24

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Lecture Span

- Gram-Schmidt

Idea: Given a basis for V, can we adjust it to make it orthonormal?
Given that $\{w_1, w_2\}$ is a basis in $\mathbb{R}^2$, how to make them orthonormal?

Solution

We let $v_1 = w_1$, then we can decompose w_2 to have a portion that is orthogonal to w_1, (call it v_2), and a portion that is along the lines of w_1, (call it cv_1). How do we find c? We use the inner product:

$$\langle w_2, v_1 \rangle = \langle v_2, v_1 \rangle + c\langle v_1, v_1 \rangle \tag{1}$$

$$\implies c = \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \in \mathbb{R} \tag{2}$$

Given that

$$w_2 = cv_1 + v_2 \tag{3}$$

So

$$\{v_1 = w_1, v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1\} \tag{4}$$

Example in $\mathbb{R}^3$

Given 2 orthogonal vectors v_1, v_2 and a 3rd non-orthogonal vector w_3, how do we find the 3rd orthogonal vector?

Solution

We can write

$$w_3 = c_1 v_1 + c_2 v_2 + v_3 \tag{5}$$

WIth v_3 perpendicular to v_1, v_2. How can we find c_1, c_2?

$$\langle w_3, v_1 \rangle = c_1 \langle v_1, v_1 \rangle + 0 + 0 \tag{6}$$

$$c_1 = \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} \tag{7}$$

c_2 can be calculated in a similar way. Therefore,

$$v_3 = w_3 - c_2 v_2 - c_1 v_1 \tag{8}$$

Theorem 0.1. *(<u>Gram-Schmidt</u>): Let V be any inner product space, and let $\{w_1, \dots, w_n\}$ be a linearly independent set. Then define $v_1 = w_1$ and for $v_k > 1$,*

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\langle v_j, v_j \rangle} v_j \tag{9}$$

Then we claim

1. *$span(v_1, \dots, v_n) = span(w_1, \dots, w_n)$*
2. *$\{v_1, \dots, v_n\}$ is an orthogonal set*

Proof. Induct on n, base case is $n = 1$. This is trivial. For the inductive step, we consider this to be true for some n, and we now consider $n+1$. Given $\{w_1, \dots, w_{n+1}\}$, then $\{v_1, \dots, v_n, v_{n+1}\}$. Then $v_{n+1} = w_{n+1} - c_1 v_1 - \dots$. By IH,

$$span(v_1, \dots, v_n) = span(w_1, \dots, w_n) \tag{10}$$

But

$$v_{n+1} \in span(w_{n+1}, v_1, \dots, v_n) \tag{11}$$

$$= span(w_1, \dots, w_{n+1}) \tag{12}$$

So

$$span(v_i) \subseteq span(w_i) \tag{13}$$

To prove the opposite direction, we perform

$$v_{n+1} = w_{n+1} - c_1 v_1 - \dots \tag{14}$$

$$\implies v_{n+1} + c_1 v_1 + \dots = w_{n+1} \tag{15}$$

This implies that

$$span(v_1, \dots, v_{n+1}) = w_{n+1} \tag{16}$$

Since $span(v_1, \dots, v_n) = span(w_1, \dots, w_n)$, we have that

$$span(v_1, \dots, v_{n+1}) = span(w_1, \dots, w_{n+1}) \tag{17}$$

□

We now prove 2,

Proof. Induct on n, the base case $n = 1$ is vacuous (mindless, useless). For the inductive step, suppose it's true for n, and consider $n+1$. We must prove that

$$v_{n+1} = w_{n+1} - \sum_{j=1}^{n} \frac{\langle w_{n+1}, v_j \rangle}{\langle v_j, v_j \rangle} v_j \tag{18}$$

is orthogonal to $v_1, \dots, v_n$. (Note: $\{v_1, \dots, v_j\}$ are linearly independent, thus $\langle v_j, v_j \rangle \neq 0$). Fix $i \in \{1, \dots, n\}$ and calculate

$$\langle v_{n+1}, v_i \rangle = \langle w_{n+1}, v_i \rangle - \sum_{j=1}^{n} \frac{\langle w_{n+1}, v_j \rangle}{\langle v_j, v_j \rangle} \langle v_j, v_i \rangle \tag{19}$$

Considering $\langle v_j, v_i \rangle$, this evaluates to $0 \iff i \neq j$.

$$= \langle w_{n+1}, v_j \rangle - \frac{\langle w_{n+1}, v_j \rangle}{\langle v_j, v_j \rangle} \langle v_j, v_j \rangle \tag{20}$$

$$= 0 \tag{21}$$

By induction, this proves the theorem. □

Corollary 0.2. *Every finite dimensional inner product space has an orthonormal basis.*

Proof. Let V be some finite dimensional inner product space. Let $\{w_1, \dots, w_n\}$ be a basis for V. Applying Gram-Schmidt gives a new orthogonal set $\{v_1, \dots, v_n\}$ with the same span. So $\{v_1, \dots, v_n\}$ is an orthogonal basis, and $\{u_1, \dots, u_n\} = \left\{ \frac{v_1}{||v_1||}, \dots \right\}$ is an orthonormal basis. $\square$

Definition 0.3. *Let V be an inner product space and S some nonempty subset. Then the* *orthogonal compliment* *of S is*

$$S^{\perp} = \{x \in V, \langle x, y \rangle = 0 \text{ for all } y \in S\} \tag{22}$$

Note:

1. *$S^{\perp}$ is a subspace of V.*
2. *$S^{\perp}$ is $span(S)^{\perp}$*

Theorem 0.4. *Suppose V is a $n-$dimensional inner product space, and W is a subspace. Then*

1. *Any orthonormal basis $\{v_1, \dots, v_k\}$ for W can be extended to an orthonormal basis for V.*
2. *If we extend, the new vectors $\{v_{k+1}, \dots, v_n\}$ is a basis for $W^{\perp}$.*
3. $\dim(W) + \dim(W^{\perp}) = \dim(V)$

Proof. Extend $\{v_1, \dots, v_k\}$ to some basis $\{v_1, \dots, v_k, w_{k+1}, \dots, w_n\}$ of V. We can turn this into an orthonormal using Gram-Schmidt + normalize the vectors. $\square$

We prove 3,

Proof. $\dim(W) = k$, $\dim(W^{\perp}) = n - k$, thus $\dim(W) + \dim(W^{\perp}) = n \iff \dim(V)$. $\square$