MTH 416: Lecture 9

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Lecture Span

- Midterm Announcement
- Linear transformations and rank-nullity theorem

Midterm

In class next week thursday, HW 5 is due the week after M1. And midterm will cover up to this lecture

Linear transformations

Recall

$$T : V \to W \tag{1}$$

is a linear transformation if

1. $T(x + y) = T(x) + T(y)$
2. $T(cx) = cT(x)$

for all $x, y \in V, c \in \mathbb{R}$. For any linear transformation $T : V \to W$, we can define two important subspaces:

Definition 0.1. *Given a linear transformation: $T : V \to W$:*

1. *The range (or image) of T is the set*

$$R(T) = \{T(v) : v \in V\} \subseteq W \tag{2}$$

2. *The kernel or (nullspace) of T is*

$$N(T) = \{v \in V : T(v) = 0\} \subseteq V \tag{3}$$

Theorem 0.2. 1. *$R(T)$ is a subspace of W*

2. *$N(T)$ is a subspace of V*

Proof. 1. $R(T)$ contains 0_w because $0_w = T(0_v)$ if w_1 and w_2 are in $R(T)$, then we must prove that $w_1 + w_2 \in R(T)$.

Proof. $w_1 = T(v_1)$ and $w_2 = T(v_2)$, for some $v_1, v_2 \in V$. Then $w_1 + w_2 = T(v_1 + v_2)$ which is in $R(T)$. □

Finally, if $w = T(v) \in R(T)$, and $c \in \mathbb{R}$, then

$$cw = c(T(v)) = T(cv) \in R(T) \tag{4}$$

2. $N(T)$ contains 0_v because $T(0_v) = 0_w$. If $v_1, v_2 \in N(T)$, then

$$T(v_1 + v_2) = T(v_1) + T(v_2) = 0 + 0 = 0 \tag{5}$$

if $v \in N(T)$ and $c \in \mathbb{R}$, then

$$T(cv) = c(T(v)) = c(0) = 0 \tag{6}$$

□

Because we know that $R(T)$ and $N(T)$ are subspaces, then we can talk about their dimensions.

Definition 0.3. *The dimension of $R(T)$ is called the* <u>*rank*</u>*, and the dimension of $N(T)$ is called the* <u>*nullity*</u>*.*

Example 1.

$$T : \mathbb{R}^3 \to \mathbb{R}^4 \tag{7}$$

$$T(a, b, c) = (a, b, c, a + b + c) \tag{8}$$

Then

$$R(T) = \{(a, b, c, a + b + c) : a, b, c \in \mathbb{R}\} \tag{9}$$

This is a 3 dimensional subspace of $\mathbb{R}^4$.

$$N(T) = \{(a, b, c) \in \mathbb{R}^3 : (a, b, c, a + b + c) = 0\} \tag{10}$$

So

$$N(T) = \{(0, 0, 0)\} \tag{11}$$

0 dimensional subspace in $\mathbb{R}^3$.

Example 2

$$T : \mathbb{R}^3 \to \mathbb{R}^2 \tag{12}$$

$$T(a, b, c) = (a, b) \tag{13}$$

$$R(T) = \{(a, b) : (a, b, c) \in \mathbb{R}^3\} \iff \mathbb{R}^2 \tag{14}$$

In other words, $R(T)$ is surjective.

$$N(T) = \{(a, b, c) \in \mathbb{R}^3 : (a, b) = 0\} \iff \{(0, 0, c) : c \in \mathbb{R}\} \tag{15}$$

This is the z-axis. So rank = 2, and nullity = 1.

Note

For $T : V \to W$ linear,

1. $0 \leq rank(T) \leq dim(W)$ and
2. $0 \leq nullity(T) \leq dim(V)$

Theorem 0.4. *Suppose $T : V \to W$, and $\beta = \{u_1, \ldots, u_n\}$ is a basis for V. Then*

1. $R(T) = span(\{T(v_1), \ldots, T(v_n)\})$
2. *T is completely determined by what it "does" to $\{u_1, \ldots, u_n\}$*

Proof. For (2), let $v \in V$ be arbitrary, since β is a basis, then v can be uniquely expressed as

$$v = a_1 v_1 + \cdots + a_n v_n \tag{16}$$

Then:

$$T(v) = T(a_1 v_1 + \cdots + a_n v_n) \tag{17}$$

$$\iff a_1 T(v_1) + \ldots a_n T(v_n) \tag{18}$$

This shows that $T(v)$ is completely determined by $T(v_i)$. □

Proof. For (1), $R(T)$ is the set of all possible $T(v)$, which according to the above proof, is

$$span(\{T(v_1), \ldots, T(v_n)\}) \tag{19}$$

□

In fact, generally given any $w_1, \ldots, w_n \in W$, then there is always exactly one linear transformation

$$T : V \to W \tag{20}$$

such that $T(v_i) = w_i$ for each i.

Theorem 0.5. *Given a linear transformation $T : V \to W$, we have that*

$$N(T) = \{0\} \iff T \textit{ is injective, or 1 to 1} \tag{21}$$

Proof. First assume $N(T) = \{0\}$, then we claim that T is injective. Suppose

$$T(v) = T(v') \tag{22}$$

For $v, v' \in V$, then we prove that $v = v'$. Then

$$T(v - v') \iff T(v) - T(v') \iff 0 \tag{23}$$

Thus, $v - v' \in N(T)$, but

$$N(T) = \{0\} \tag{24}$$

then

$$v = v' \tag{25}$$

□

Proof. Now, suppose T is injective, that is

$$T(v) = T(v') \implies v = v' \tag{26}$$

Thus, there is at most one vector such that $T(w) = 0$, namely $w = 0$. So it must be the only one. □

Rank nullity theorem

Theorem 0.6. *Rank-nullity theorem states that suppose T is linear transformation from $V \to W$ where V is finite dimensional. Then*

$$\dim(R(T)) + \dim(N(T)) = \dim(V) \tag{27}$$

Nullity = number of dimensions ”flattened” out, and Rank = number of dimensions left.

Example 1

$$T : \mathbb{R}^{a+b} \to \mathbb{R}^{a+c} \tag{28}$$

for some $a, b, c \geq 0$, then

$$T(x_1, \dots x_a, \dots x_{a+b}) = (x_1, \dots, x_a, 0, \dots, 0) \tag{29}$$

For this T, $R(T) = \{(x_1, \dots, x_a, 0, \dots, 0)\}$, then the dimension of $R(T) = a$. Then

$$N(T) = \{(x_1, \dots, x_{a+b}) \in \mathbb{R}^{a+b} : x_1 = \cdots = x_a = 0\} \tag{30}$$

This is

$$N(T) = \{(0, \dots, 0, x_{a+1}, \dots, x_{a+b})\} \tag{31}$$

So

$$\dim(N(T)) = nullity = b \tag{32}$$

Thus

$$\dim(R(T)) + \dim(N(T)) = a + b = \dim(V) \tag{33}$$

Proof. Let $\dim(V) = n$ and let $\dim(N(T)) = k$. Choose a basis $\{v_1, \dots, v_k\}$ for $N(T)$. By corallary of the replacement theorem: $\{v_1, \dots, v_k\}$ can be extended to a basis $v_1, \dots, v_n$ for V. Then we claim that $\{T(v_{k+1}), \dots, T(v_n)\} = R(T)$.

Note

If this is true, then $\dim(R(T)) = n - k$ so

$$\dim(R(T)) + \dim(N(T)) = n - k + k = n = \dim(V) \tag{34}$$

In other words, we claim that

1. $span(T(v_{k+1}), \dots, T(v_n)) = R(T)$
2. $T(v_{k+1}), \dots, T(v_n)$ are linearly independent.

Proof. This is the proof of (1),

$$R(T) = span(T(v_1), \dots, T(v_n)) \tag{35}$$

But

$$T(v_1), \dots, T(v_k) = 0, \dots, 0 \tag{36}$$

Thus

$$R(T) = span(T(v_{k+1}), \dots T(v_n)) \tag{37}$$

Proof. This is the proof of linearly independence. For the sake of contradiction, then suppose we have a linear dependency, then

$$a_{k_1} T(v_{k+1}) + \cdots + a_n T(v_n) = 0 \tag{38}$$

We claim that

$$a_{k+1}, \dots, a_n = 0 \tag{39}$$

Since T is linear, then we have that

$$0 = T(a_{k+1} v_{k+1}) + \cdots + T(a_n v_n) \tag{40}$$

$$\iff T(a_{k+1} v_{k+1} + \cdots + a_n v_n) \tag{41}$$

Then

$$a_{k+1} v_{k+1} + \cdots + a_n v_n \in N(T) \tag{42}$$

Therefore,

$$a_{k+1} v_{k+1} + \cdots + a_n v_n = b_1 v_1 + \cdots + b_k v_k \tag{43}$$

Then

$$-b_1 v_1 - \cdots - b_k v_k + a_{k+1} v_{k+1} + \cdots + a_n v_n = 0 \tag{44}$$

But because $v_1, \dots, v_n$ are a basis for V, then that means that

$$b_1, \dots b_k, a_{k+1}, \dots a_n = 0 \tag{45}$$

In particular,

$$a_{k+1}, \dots, a_n = 0 \tag{46}$$

Thus all vectors in $R(T)$ are linearly independent. □

□

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