PHYS 325: Lecture 5

Cliff Sun

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Lecture Span

- Non-linear Drag
- Time varying mass

Non-linear Drag

$$F(v) = -mg + cv^2 \iff -mg + \frac{c}{m}mv^2 \wedge \sigma = \frac{c}{m} \tag{1}$$

Thus, this simplifies to

$$F(v) = m(g - \sigma)v^2 \iff m\dot{v} = m(g - \sigma v^2) \iff \boxed{\dot{v} = g - \sigma v^2} \tag{2}$$

$$\frac{\dot{v}}{g - \sigma v^2} = 1 \tag{3}$$

$$\int_{v_0}^{v} \frac{\dot{v}}{g - \sigma v^2} dt = \int_{t_0}^{t} dt \tag{4}$$

$$\frac{-1}{\sqrt{g\sigma}} \tanh^{-1}\left(v\sqrt{\frac{\sigma}{g}}\right) = t \tag{5}$$

$$v(t) = -\sqrt{\frac{g}{\sigma}} \tanh\left(\sqrt{g\sigma}t\right) \tag{6}$$

$$t \to \infty \implies v(t) \to -\sqrt{\frac{g}{\sigma}} \tag{7}$$

In general, a special case of ODEs would be if

$$F(v) = f(v)g(t) \tag{8}$$

Then you can split the terms and achieve the following:

$$m\frac{dv}{dt}\frac{1}{f(v)} = g(t) \tag{9}$$

Then you're able to integrate both sides. Then what about

$$F = f(v)h(x) \tag{10}$$

Then we say

$$m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt}\frac{1}{f(v)} \iff m\frac{dv}{dx}v\frac{1}{f(v)} = g(x) \tag{11}$$

$$\int m\frac{v}{f(v)}dv = \int g(x) \tag{12}$$

Time Varying Mass, $M(t)$

Given this rocket with mass $= M(t)$ and velocity $= v(t)$ at time t, then say a mass dm is thrown out of the rocket at $t + \Delta t$ at some speed u relative to the rocket. We say that $v_f = v_i + dv$ and $M_f = M_i - dm$. Then

$$P_f = dm(v-u) + M_f v_f \iff dm(v-u) + (M_i + dm)(v_i + dv) \tag{13}$$

So expanding this out yields

$$P_f = dm(-u) + M_i v_i - dmdv + Mdv = M_i v_i \tag{14}$$

We can assume that

$$-dmdv \sim 0 \text{ due to the 2nd order pertubation} \tag{15}$$

So we get

$$Mdv = udm \tag{16}$$

We know that

$$dm = -dM \text{ change of the mass of the ejected propellant} = \text{change of the mass of the rocket} \tag{17}$$

$$Mdv = -udM \tag{18}$$

$$dv = -u\frac{dM}{M} \tag{19}$$

$$v = v_0 - u\ln(\frac{M}{M_0}) \tag{20}$$

Where u is the speed of the propellant, we assume that this is constant. Then assume $M = 0.1M_0$, at some time t with $v_0 = 0$, then we get that

$$v_f = -u\ln(0.1) \sim 2.3u \tag{21}$$

Which means that about 90% of the rocket must be mass in order to achieve $2.3u$, which isn't alot of velocity!

Time varying mass in gravity

Now we can't assume that

$$P_i = P_f \tag{22}$$

But rather

$$P_f - P_i = dp = -Mgdt \tag{23}$$

$$Mdv + udM = -Mgdt \tag{24}$$

$$\frac{dv}{dt}M + u\frac{dM}{dt} = -Mg \tag{25}$$

$$\frac{dv}{dt} = -\frac{u}{M}\frac{dM}{dt} - g \tag{26}$$

We have to specify $\frac{dM}{dt}$ because it is a degree of freedom, and can vary from rocket to rocket. For our cases, assume that

$$\frac{dM}{dt} = c \implies \dot{M} = \alpha \tag{27}$$

Then

$$M = M_0 - \alpha t \tag{28}$$

$$\frac{dv}{dt} = -g - u\frac{\alpha}{M_0 - \alpha t} \tag{29}$$

$$v(t) = v_0 - gt - u\ln\left(\frac{M_0 - \alpha t}{M_0}\right) \tag{30}$$

In general,

$$\vec{F}(\vec{r}, \dot{\vec{r}}, t) = m\vec{a} = m\ddot{\vec{r}} \tag{31}$$

In cartesian, we obtain a system of ODEs with respect to $m\ddot{x}$, $m\ddot{y}$, and $m\ddot{z}$. For example, assume that a particle lives in the B field, with a magnetic field that goes in the z direction. We have that

$$\vec{F} = q\vec{v} \times \vec{B} \tag{32}$$

The particle then spirals in a helix shape, with a radius dependent on its velocity.

$$= q\left[\vec{v} \times \vec{B}\right] \tag{33}$$

We take the determinant of the following matrix:

$$\begin{bmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{bmatrix}$$

This simplifies down to

$$qB(v_y\vec{e_x} - v_x\vec{e_y}) = \vec{F} \tag{34}$$

$$F_x = qBv_y = m\ddot{x} \tag{35}$$

$$F_y = -qBv_x = m\ddot{y} \tag{36}$$

$$F_z = 0 = m\ddot{z} \implies \dot{z} = c \implies z = cx + b \tag{37}$$

$$qB\dot{y} = m\ddot{x} \tag{38}$$

$$-qB\dot{x} = m\ddot{y} \tag{39}$$

$$qB\ddot{y} = m\dddot{x} \tag{40}$$

$$qB(-\frac{qB}{m})\dot{x} = m\dddot{x} \tag{41}$$

$$-(\frac{qB}{m})^2 v_x = \ddot{v_x} \tag{42}$$

Then,

$$\ddot{v_x} = -\omega^2 v_x \wedge \omega^2 = (\frac{qB}{m})^2 \tag{43}$$

Then the solution

$$v_x = A\sin(\omega t + \phi) \tag{44}$$

The velocity is going in a circle! Now let's try using complex numbers!

$$v_x + iv_y = \eta \tag{45}$$

Then

$$Re(\eta) = v_x \tag{46}$$

$$Im(\eta) = v_y \tag{47}$$

Then we have

$$\ddot{v_x} = -\omega^2 v_x \tag{48}$$

$$\ddot{v_y} = -\omega^2 v_y \tag{49}$$

$$\dot{\eta} = \dot{v_x} + i\dot{v_y} \tag{50}$$

$$= -i\omega(iv_y + v_x) \tag{51}$$

$$\dot{\eta} = -i\omega\eta \tag{52}$$

$$\eta = e^{-i\omega t} \tag{53}$$