8.1 - Cardinalities of Infinite Sets

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Recall:

Theorem 0.1. *Let A and B be finite sets, then*

$$|A| = |B| \iff \exists F : A \to B \quad \textit{Bijective} \tag{1}$$

In addition:

$$|A| <= |B| \iff \exists f : A \to B \quad \textit{Injective} \tag{2}$$

Similarly:

$$|A| >= |B| \iff \exists f : A \to B \quad \textit{Surjective} \tag{3}$$

Definition 0.2. *Let A and B be arbitrary sets, perhaps infinite. Then the following is true:*

1. *$|A| = |B|$ if there exists a bijection from A to B*
2. *$|A| <= |B|$ if there exists an injective function from A to B*
3. *$|A| < |B|$ if there exists only an injective function.*

But $|A|$ hasn't been explicitly stated yet. And it's not a number or ∞. Then we claim that

$$|\mathbb{N}| = |2\mathbb{N}| \tag{4}$$

Or that the cardinality of the natural numbers is the same as that of the even natural numbers.

Proof. To prove this claim, we must give a bijective function from $\mathbb{N} \to 2\mathbb{N}$. Then choose a function as the following:

$$f : \{\mathbb{N} \to 2\mathbb{N} : f(x) = 2x\} \tag{5}$$

This is bijective, thus we have proved the claim. □

Claim: $|\mathbb{Z}| <= |\mathbb{R}|$

Proof. THe function $f : \mathbb{Z} \to \mathbb{R}$ such that $f(n) = n$ is injective. □

Conjecture 0.3. *On any collection of sets, the relation*

$$A \sim B \iff |A| = |B| \tag{6}$$

is an equivalence relation. That is that this relation is reflexive, symmetric, and transitive.

Conjecture 0.4. $|A| = |B| \implies |A| <= |B|$

Conjecture 0.5. $|A| <= |B| <= |C| \implies |A| <= |C|$

Conjecture 0.6. $|A| <= |B| <= |A| \implies |A| = |B|$

We first prove the first conjecture.

Proof. To prove that this relation is in-fact an equivalence relation, we must first prove its reflexivity. That it for any set A, there exists a bijection from A to A. To choose such a function, we choose the identity funtion, that it

$$f : A \to A \text{ such that } f(n) = n \tag{7}$$

To prove symmetry, if $f : A \to B$ is a bijection, then we choose f^{-1} as the bijective function from B to A. Thus this relation is symmetric. Next to prove that this relation is transitive. First assume that

$$f : A \to B \text{ is bijective} \tag{8}$$

and that $|B| = |C|$, that is

$$g : B \to C \text{ is bijective,} \tag{9}$$

Then we choose a function $g \circ f$ is bijective. □

Definition 0.7. *If A is a set, then its cardinality $|A|$ is defined to be its equivalence class under the previous relation.*

Countably Infinite Sets

Definition 0.8.

1. *$|\mathbb{N}$ is called $\aleph_0$*
2. *We call a set A countably infinite (denumerable) $\iff$ $|A| = \aleph_0$.*
3. *We call a set countable if $|A| <= \aleph_0$*