Generic Title

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Delta Distribution

Denoted as $\delta(t-a)$. Note thta

$$f(a) = \int_{-\infty}^{\infty} f(t)\delta(t-a)dt \tag{1}$$

For functions continuous at $t = a$. Note that $\delta(\pm\infty) = 0$. Note that

Dirac Delta $\neq$ Kronecker Delta

Example:

$$\int_{\mathbb{R}} \sin(t)\delta(t - \frac{3}{2}\pi)dt = \sin(\frac{3}{2}\pi) = -1 \tag{2}$$

Note: because the delta distribution is defined within the integral, any delta distribution that isn't in its standard form, that is

$$\delta(f(t)) \neq \delta(t-a) \tag{3}$$

Then performing u substitution is absolutely necessary.

Impulse Forces

Consider force $\Delta t << \frac{1}{\omega_n}$, then

$$F(t) = I\delta(t) \tag{4}$$

A response to an impulse is called a Green's Function. Then a response to an impulse function at $t = 0$, we have that

$$G(t) = \begin{cases} 0 & t < 0 \\ \exp(-\frac{c}{2m}t)\frac{\sin(\omega_d t)}{m\omega_d} & t > 0 \end{cases} \tag{5}$$

This is the response for

$$m\ddot{x} + c\dot{x} + kx = \delta(t-a) \tag{6}$$

Heavyside ("step") function

$$H(y - y_0) = \begin{cases} 1 & y > y_0 \\ 0 & y < y_0 \end{cases} \tag{7}$$

You can rewrite the Green's function using this Heavyside function.
In general,

$$L[G(t-a)] = \delta(t-a) \tag{8}$$

Arbitrary Forcing & Convolution

Force with 2 impulses, $I_1(t = t_1)$, $I_2(t = t_2)$.

$$F(t) = I_1\delta(t - t_1) + I_2\delta(t - t_2) \tag{9}$$

Then

$$x_p(t) = I_1 G(t - t_1) + I_2 G(t - t_2) \tag{10}$$

Then a force with N impulses @ t_q

$$F(t) = \sum I_q \delta(t - t_q) \tag{11}$$

Now

$$x_p(t) = \sum I_q H(t - t_q) G(t - t_q) \tag{12}$$

In the limit that $\Delta t \to 0$, we have that

$$F(t) = \int_{-\infty}^{\infty} F(s)\delta(t - s) ds \tag{13}$$

Response $x_p(t)$:

$$x_p(t) = \int_{-\infty}^{\infty} F(s) G(t - s) ds \tag{14}$$

This is the general solution for a damped oscillator for an arbitrary driving force!