MTH 553: Lecture # 9

Cliff Sun

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Lecture Span

- Non homogenous wave equationm (NHWE)

Duhamel's Principle

Here, we solve the NHWE with homogenous initial conditions. Assume $f \in C^1(\mathbb{R} \times [0, \infty))$ where $x \in \mathbb{R}$ and $t \in [0, \infty)$. Let $Z(x, t; s)$ solve the problem (for each $s \in [0, t)$)

$$\begin{aligned} Z_{tt} - c^2 Z_{xx} &= 0 \\ Z(x, 0; s) &= 0 \\ Z_t(x, 0; s) &= f(x, s) \end{aligned}$$

Here, s is a fixed time. Therefore, $f(x, s)$ is the initial velocity, is the original driving force related to the NHWE. Then let

$$z(x, t) = \int_0^t Z(x, t - s; s) ds$$

Here, this is the sum of responses of impulses at time $s \in [0, t]$. Here, this is a general initial condition where we can set of which f we can use as the initial condition. Here, the $; s$ encodes which initial condition we are choosing. We claim that z solves our NHWE.

Proof. We can solve for Z with d'Alembert. Here,

$$Z(x, t; s) = \frac{1}{2c} \int_{x-ct}^{x+ct} f(\xi, s) d\xi \tag{1}$$

Then, we use the fundamental theorem of Calculus:

$$\begin{aligned} z_t &= \int_0^t Z_t(x, t - s; s) ds + \underbrace{Z(x, t - t; t)}_{=0} \\ z_{tt} &= \int_0^t Z_{tt}(x, t - s; s) ds + \underbrace{Z_t(x, t - t; t)}_{=f(x,t)} \\ &= c^2 z_{xx}(x, t) + f(x, t) \end{aligned}$$

This line was because

$$Z_{tt} = c^2 Z_{xx} \implies z_{tt} = c^2 z_{xx} \tag{2}$$

And we now check the initial conditions, clearly

$$z(x, 0; s) = 0 \tag{3}$$

$$z_t(x, 0; s) = 0 \tag{4}$$

$\square$

IBVP for NHWE on an interval

$$u_{tt} - c^2 u_{xx} = f(x,t)$$
$$u(x,0) = g_1(x)$$
$$u_t(x,0) = h_1(x)$$

Here, the boundary conditions are

$$au(0,t) - bu_x(0,t) = \alpha(t)$$
$$c^* u(l,t) + du_x(l,t) = \beta(t)$$

Where a, b, c^*, d are all constants with

$$|a| + |b| > 0$$
$$|c^*| + |d| > 0$$

Step 1, we reduce to homogenous boundary conditions. Find any function $B(x,t)$ that satisfies the boundary conditions. E.g.

$$B(x,t) = \frac{(x-L)^2}{aL^2 + 2bL}\alpha(t) + \frac{x^2}{c^* L^2 + 2dL}\beta(t) \tag{5}$$

Decompose this problem, $u = v + B$ where B solves boundary conditions and v solves the problem. Then $v = u - B$, where

$$v_{tt} - c^2 v_{xx} = f_2(x,t) = f_1(x,t) - B_{tt} + c^2 B_{xx}$$

With initial conditions

$$v(x,0) = g_2(x) = g_1(x) - B(x,0)$$
$$v_t(x,0) = h_2(x) = h_1(x) - B_t(x,0)$$

And boundary conditions

$$av(0,t) - bv_x(0,t) = 0$$
$$c^* v(L,t) + dv_x(L,t) = 0$$

Step 2, decompose $v = w + z$ where w satisfies the non homogenous initial conditions and z satisfies the non homogenous wave equation. Then

System 2a

$$z_{tt} - c^2 z_{xx} = f_2(x,t)$$
$$z(x,0) = 0$$
$$z_t(x,0) = 0$$

With boundary conditions

$$az(0,t) - bz_x(0,t) = 0$$
$$c^* z(L,t) + dz_x(L,t) = 0$$

Moreover, we can find the governing equations of w:

System 2b

$$
\begin{aligned}
w_{tt} - c^2 w_{xx} &= 0 \\
z(x,0) &= g_2 \\
z_t(x,0) &= h_2
\end{aligned}
$$

With boundary conditions

$$
\begin{aligned}
az(0,t) - bz_x(0,t) &= 0 \\
c^* z(L,t) + dz_x(L,t) &= 0
\end{aligned}
$$

Step 3a: Solve (2a) using (2b). For all $s \leq 0$. That is, find $Z(x,t;s)$

$$
\begin{aligned}
Z_{tt} - c^2 Z_{xx} &= 0 \\
Z(x,0;s) &= 0 \\
Z_t(x,0;s) &= f_2(x,s)
\end{aligned}
$$

Along with boundary conditions

$$
\begin{aligned}
aZ(0,t;s) - bZ_x(0,t;s) &= 0 \\
c^* Z(L,t;s) + dZ_x(L,t;s) &= 0
\end{aligned}
$$

Proof. Find Z by using solution for (2b). □

Step 3b: solve (3b) by Fourier Series. Decompose one more time to reduce to have either the initial condition with $g_2 = 0$ or initial condition with $h_2 = 0$. First assume $g_2 = 0$. Let

$$
w(x,t) = \sum_{n=0}^{\infty} [a_n \sin(\lambda_n x) + b_n \cos(\lambda_n x)] \sin(\lambda_n ct) \tag{6}
$$

Ignore convergence issues. lmao