7.1-7.2 - Relations and Functions as Relations

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Relations

Idea: we're often interested in relationships between two elements of a set. In particular:

1. $a < b$ for $a, b \in \mathbb{R}$
2. $f(a) = b$ where $f : A \to B$
3. $S \subseteq T$ like where $S, T \in P(U)$ where U is the universal set.

Theorem 0.1. *Let A and B be sets, then a relation from A to B is any subset of $A \times B$.*

Theorem 0.2. *A relation on A is a relation from $A \to A$.*

Notation: $(a, b) \in R \iff$ a R b. In this case, R is a noun and a verb.

Example: if $(2, 3) \in R$, then 2 R 3.

Theorem 0.3. *On any set S, we have the relation R = "=" defined as*

$$R = \{(x, x) : x \in S\} \iff x = y \tag{1}$$

That is

$$(x, y) \in R \iff x = y \tag{2}$$

Theorem 0.4. *Given any set S, we have that R = "$\subseteq$" on P(S) defined as*

$$R = \{(A, B) \subseteq P(S) \times P(S) : A \subseteq B\} \tag{3}$$

That is

$$(A, B) \in R \iff A \subseteq B \tag{4}$$

Inverse relationships:

Theorem 0.5. *Given a relationship R from A to B, we have that the inverse relationship is*

$$R^{-1} = \{(x, y) \in B \times A : (y, x) \in R\} \tag{5}$$

A concrete example would be that $<$'s inverse is $>$, etc.

Functions as relations:

Theorem 0.6. *Say $f : A \to B$ is a function, then we have the relation*

$$R = \{(a, f(a)) : a \in A\} \tag{6}$$

This is a relation, but what properties does it have?

Theorem 0.7. *Given a relation R, the domain of R is*

$$R = \{a \in A : (a, b) \in R\} \tag{7}$$

For values of a such that $f(a) = b$.

Now we can redefine functions to be

Theorem 0.8. *A function $f : A \to B$ is a relation f from A to B such that*

$$dom(f) = A \tag{8}$$

and $\forall a \in A, \wedge \forall b_1, b_2 \in B$ are if (a, b_1) and (a, b_2) are in R, then $b_1 = b_2$. THIS IS NOT INJECTIVITY