MTH 553: Lecture # 2

Cliff Sun

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Lecture Span

- First order equations
- Cauchy problem for quasilinear 1st order equations (coefficients can depend on u)1

ODE background

Suppose autonomous equation (no t variable on the RHS)

$$\frac{dx}{dt} = a(x, y, z) \tag{1}$$
$$\frac{dy}{dt} = b(x, y, z) \tag{2}$$
$$\frac{dz}{dt} = c(x, y, z) \tag{3}$$

Suppose a curve modeled by (x, y, z) with the tangent vector $\langle a, b, c \rangle$. With an autonomous system, you don't need to know the time, just where you are. Suppsoe $x = x_0$, and etc at $t = 0$.

$$\frac{dx}{dt} = a \tag{4}$$
$$\frac{dy}{dt} = 1 \tag{5}$$
$$\frac{dz}{dt} = 0 \tag{6}$$

The solution is

$$x = at + x_0 \tag{7}$$
$$y = t + y_0 \tag{8}$$
$$z = z_0 \tag{9}$$

Special case, if $y_0 = 0$, then the path is simply a straight line $x = a(y) + x_0$ and $z = z_0$.

Cauchy Problem for quasilinear first order PDE

Given a curve $\Gamma \in \mathbb{R}^3$, find $u(x, y)$ solving

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{10}$$

Where a, b are coefficients. With $\Gamma \subset \{\text{graph of } u\}$. Note, Γ represents a curve of initial boundary conditions, which only a part of the "graph of u".

Example - IVP (initial value problem) for transport equation

$$au_x + u_y = 0$$
$$u(x, 0) = k(x)$$

Therefore,

$$\Gamma : \begin{cases} z = k(x) \\ y = 0 \end{cases} \tag{11}$$

Note, $z \iff u(x, y)$. This PDE has an associated autonomous system (1). For each point $(x_0, 0, z_0) \in \Gamma$, we can sketch the solution path and the resulting surface is the graph of the solution u.

Proof.

$$u(x, y) = k(x_0) \tag{12}$$
$$u(x, y) = k(x - ay) \text{ (traveling wave)} \tag{13}$$

i.e. transport initial height to every point along the path. Note that we are defining an initial 1d curve $z = k(x)$, and then we can move this curve. This is a traveling wave. □

Solution to Cauchy Problem

Given an autonomous 3-d system. Suppose an initial curve $\Gamma \in \mathbb{R}^3$. Then, we guess that Γ would be moved somehow across $\mathbb{R}^3$. For each $(x_0, y_0, z_0) \in \Gamma$, we sketch the characeristic curve. We claim that the resulting surface $u(x, y)$ is a solution to the Cauchy problem. i.e, define $u(x, y)$ by $u(x(t), y(t)) = z(t)$ for all t and for all characeristic curves starting at Γ.

Check:

Along each characeristic curve,

$$\begin{aligned} c(x(t), y(t), z(t)) &= c = dz/dt \quad (14)\\ &= u_x(x, y, z)dx/dt + u_x(x, y, z)dy/dt \quad (15)\\ &= u_x a + u_y b \quad (16) \end{aligned}$$

But what can go wrong?: (1) characeristics may not exist $\forall t$, like blow-up for example like $dz/dt = z^2$ which blows up in finite time.

(2) Surface may fold, which then no longer becomes a graph (like a carpet that's been rolled up).

(3) Γ itself may be a characeristic. In this case, you do not get a surface.

"Local" Theorem

Our construction does solve the Cauchy problem near Γ if the projection of Γ to the xy-plane is never parallel to $< a, b >$. In other words, given

$$\Gamma = \begin{cases} x = f(s) \\ y = g(s) \\ z = h(s) \end{cases} \tag{17}$$

We want $< f'(s), g'(s) >$ not parallel to $< a, b >$ (coefficients of the Cauchy problem). The point, the characeristics will truly flow away from Γ and generate a surface for at least small t.

Proof. Be intimidated. □