More properties of sup/inf, bounded functions, triangle inequality

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Suppose that $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, recall:

1. $\sup(x + A) = x + \sup(A)$
2. $\inf(x + A) = x + \inf(A)$
3. $\sup(xA) = x \sup(A)$ if $x > 0$
4. $\inf(xA) = x \inf(A)$ if $x > 0$
5. $\sup(xA) = x \inf(A)$ if $x < 0$
6. $\inf(xA) = x \sup(A)$ if $x < 0$

We first prove (1):

Proof. For any $b \in \mathbb{R}$, b is an upper-bound of A

$$b \geq a \quad \forall a \in A \tag{1}$$

$$b + x \geq a + x \quad \forall a \in A \tag{2}$$

$$b + x \text{ is an upperbound of A} \tag{3}$$

Choosing $b = \sup(A)$, this says that $b + x$ is an upperbound of $x + A$. Let c be an upperbound of $A + x$, we claim that $b + x$ is the least upperbound of c. Since c is an upperbound of $x + A$, we have that

$$c - x \geq a \quad \forall a \in A \tag{4}$$

Since b is the least upperbound of A, it follows that

$$c - x \geq b \tag{5}$$

This implies that

$$c \geq b + x \tag{6}$$

Thus, the least upperbound of A is $\sup(A) + x$. □

Conjecture 0.1. *Suppose that A, B are non-empty subsets of $\mathbb{R}$, such that $\forall a \in A$ and $\forall b \in B$, we have that*

$$a \leq b \tag{7}$$

We claim that

1. *A is bounded above*
2. *B is bounded below*
3. $\sup(A) \leq \inf(B)$

Proof. To prove (1), we choose any element in B. □

Proof. Similarly, to prove (2), we choose any element in A. □

These proofs imply that A and B have a supremum and an infimum, respectively.

Proof. To prove (3), we proceed with contradiction. Suppose that $\inf(B) < \sup(A)$, then we get a contradiction because we have that a value in B is less than a value in A. Thus it follows that $\sup(A) \leq \inf(B)$. □

Theorem 0.2. *If $S \subseteq \mathbb{R}$ is an non-empty set which is bounded above, then for all $\epsilon > 0$, there exists an element $x \in S$ such that*

$$\sup(S) - \epsilon < x \leq \sup(S) \tag{8}$$

Definition 0.3. *For $x \in \mathbb{R}$, we define $|x|$ to be the usual definition.*

Conjecture 0.4. *Triangle Inequality: For all real numbers x, y we have the following:*

$$|x + y| \leq |x| + |y| \tag{9}$$

Proof. Let x, y be arbitrary real numbers, we have that

$$-|x| \leq x \leq |x| \tag{10}$$

$$-|y| \leq y \leq |y| \tag{11}$$

Adding these yields

$$-|x| - |y| \leq x + y \leq |x| + |y| \tag{12}$$

But this is equivalent to saying

$$|x + y| \leq |x| + |y| \tag{13}$$

This concludes the proof. □

Corollary 0.5. *The following are true for all $x, y, x_1, \ldots, x_n \in \mathbb{R}$,*

1. $|x - y| \leq |x| + |y|$
2. $|(|x| - |y|)| \leq |x - y|$
3. $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$

Definition 0.6. *Let $f : D \to \mathbb{R}$ be a function where D is any set. We define 3 things*

1. *f is a <u>bounded</u> function is $\exists M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in D$.*
2. $\sup(f)$ *is the supremum of the image of f*
3. $\inf(f)$ *is the infimum of the image of f*