MTH 416: Lecture 10

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Lecture Span

- Rank-nullity Theorem
- Matrix Representation of Linear Transformations

Rank-nullity Theorem

Theorem 0.1. *Suppose $T :\to W$, linear, V is finite dimensional, then:*

$$\dim R(T) + \dim N(T) = \dim V \tag{1}$$

Example 1

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and

$$T(x, y) = (y, x + y) \tag{2}$$

Then

$$N(T) = \{(x, y) \in \mathbb{R}^2, (y, x + y = 0)\} \iff \{0\} \tag{3}$$

Thus

$$\dim R(T) + 0 = 2 \iff \dim R(T) = 2 \tag{4}$$

This means that $R(T)$ is surjective, and covers all of $\mathbb{R}^2$, thus T is surjective.

Example 2

$$T : P_n(\mathbb{R}) \to P_n(|\mathbb{R}) \tag{5}$$

$$T = f' \tag{6}$$

$N(T) = \{c\} : c \in \mathbb{R} \iff P_0(\mathbb{R}) \to P_n(\mathbb{R})$
So $\dim N(T) = 1$. By rank nullity, we have that

$$\dim R(T) + \dim N(T) = \dim P_n(\mathbb{R}) \tag{7}$$

Then $R(T) = P_{n-1}(\mathbb{R})$ and has dimension n.

Corollary 0.2. *Suppose $T : V \to W$, where $\dim V = m$ and $\dim W = n$ then*

1. *If T is injective, then $n \geq m$*
2. *If T is surjective, then $m \geq n$*
3. *If $n = m$, then T is injective iff it's surjective.*

Let's prove (3).

Proof. Suppose that $\dim V = \dim W$, then we know that

$$T \text{ injective } \iff N(T) = \{0\} \tag{8}$$

But

$$N(T) = \{0\} \iff \dim N(T) = 0 \tag{9}$$

Thus

$$\dim R(T) + \dim N(T) = n \iff \dim R(T) = n \tag{10}$$

$$R(T) = W \tag{11}$$

This is equivalent to saying that $R(T)$ is surjective. □

Matrix Representation of Linear Transformations

Suppose V is a vector space, with basis $\beta = \{v_1, \dots, v_n\}$. Then say $v \in V$ can be written uniquely such that

$$v = a_1 v_1 + \dots + a_n v_n \tag{12}$$

Definition 0.3. *The coordinate vector of v with respect to the basis β is*

$$[v]_\beta \iff < a_1, a_2, \dots, a_n > \in \mathbb{R}^n \tag{13}$$

Example 1

Suppose $V = P_2(\mathbb{R})$ with basis $\{1, x, x^2\}$ and $v = x^2 + 2x + 3$. Then

$$[v]_\beta =< 3, 2, 1 > \tag{14}$$

Note, the order of the basis matters. If $\beta' =< x^2, x, 1 >$, then the coordinate vector would be

$$[v]_\beta =< 1, 2, 3 > \tag{15}$$

In other words, coordinate vectors depend on an ordered basis.

Example 2

Let $V = \mathbb{R}^n$ with the standard ordered basis, that is

$$\beta =< e_1, e_2, \dots, e_n >= \{\{1, 0, \dots, 0\}, \{0, 1, \dots, 0\}, \dots, \} \tag{16}$$

Then for any $v =< v_1, \dots, v_n > \in \mathbb{R}^n$, we have that

$$[v]_\beta = v \tag{17}$$

Theorem 0.4. *If V has an ordered basis $\beta = \{v_1, \dots, v_n\}$, then the function*

$$T : V \to \mathbb{R}^n \tag{18}$$

and $T(v) = [v]_\beta$. This is a bijective linear transformation. That is, each vector is mapped to its coordinate vector. This should be bijective because each vector in V can be expressed uniquely as the sum of its basis vectors.

Let V and W be vector spaces, and suppose we have ordered basis

$$\beta = \{v_1, \dots, v_n\} \in V \tag{19}$$

$$\gamma = \{w_1, \dots, w_m\} \in W \tag{20}$$

Suppose $T \to W$ is linear. Recall T is uniquely determined by $T(v_1), \dots, T(v_n)$.

Definition 0.5. *A matrix of T in the ordered basis β, γ is*

$$[T]_\beta^\gamma = \begin{pmatrix} [T(v_1)]_\gamma & \dots & [T(v_n)]_\gamma \end{pmatrix} \tag{21}$$

This is an $m \times n$ matrix. Explicitly, if the entry in position (i, j) is a_{ij}, then

$$T(v_j) = a_{1j} w_1 + \dots + a_{mj} w_m \tag{22}$$

Example 1

$T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T(x, y) = (x + y, y) \tag{23}$$

$$\beta = \{e_1, e_2\} \tag{24}$$

$$[T]_\beta^\gamma = \begin{pmatrix} [T(v_1)]_\gamma & \dots & [T(v_n)]_\gamma \end{pmatrix} \tag{25}$$

or

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{26}$$

Because

$$T(< 1, 0 >) =< x + y, y > \iff < 1, 0 > \tag{27}$$

and

$$T(<0,1>) = <x+y, y> \iff <0+1, 1> \iff <1,1> \tag{28}$$

Let

$$\gamma = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\} \tag{29}$$

Fact: Once we choose the ordered basis β for V, and γ for W, then

$$T \iff [T]_\beta^\gamma \tag{30}$$

is a one-to-one correspondence between linear Transformations and $m \times n$ matrices. Suppose

$$A \in M_{m\times n}(\mathbb{R}) \tag{31}$$

and

$$x = \begin{pmatrix}x_1\\ \dots \\ x_n\end{pmatrix} \in \mathbb{R}^n \tag{32}$$

Recall Ax is the vector $\gamma \in \mathbb{R}^m$ such that

$$y_i = \sum_{j=i}^{n} A_{ij}x_j \tag{33}$$

Note: if $u_1, \dots, u_n$ are columns of A, then

$$Ax = x_1u_1 + x_2u_2 + \dots + x_nu_n \tag{34}$$

Note: $LS(A,b) \iff Ax = b$ such that x is a vector of variables.

Theorem 0.6. *Suppose $T : V \to W$ is linear, and β and γ are ordered basis for V and W respectively. Then for any $v \in V$, we have that*

$$[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta \tag{35}$$

That is, every vector in $T(v)$ can be written as a matrix multiplication of $[v]_\beta$. Idk why this notation is a thing, since it's basically saying

$$w = Av : \forall v \in V \wedge w \in W \tag{36}$$