Name: ______________________________

Complete the following problems to the best of your ability. Clearly number each question and write your name on each sheet of paper you turn in. Algebraic support must be shown to receive full credit (i.e. show work!). Answers should be exact unless otherwise specified.

1. (10 pts.) Show that the curve $x = t\cos t$, $y = t\sin t$, $z = t$ lies on the cone $z^2 = x^2 + y^2$.

Solution. Since

$$x^2 + y^2 = (t\cos t)^2 + (t\sin t)^2 = t^2(\cos^2 t + \sin^2 t) = t^2 = z^2,$$

we conclude that the curve lies on the cone.

2. (10 pts.) Find the tangent line to the curve $x = \ln t$, $y = 2\sqrt{t}$, $z = t^2$ at the point $(0, 2, 1)$.

Solution. Let $\vec{r}(t) = \langle \ln t, 2\sqrt{t}, t^2\rangle$ be the vector function for the curve. Then

$$\vec{r}\,'(t) = \langle 1/t, 1/\sqrt{t}, 2t\rangle.$$

We observe that the point $(0, 2, 1)$ corresponds to the parameter value $t = 1$, and $\vec{r}\,'(1) = \langle 1, 1, 2\rangle$. Therefore, the parametric equations for the tangent line are $x = t$, $y = 2 + t$, $z = 1 + 2t$.

3. (15 pts.) Let $\vec{r}$ be the vector function defined by $\vec{r}(t) = \langle 2\sin t, 5t, 2\cos t\rangle$.

(a) Find the unit tangent vector $\vec{T}(t)$.

(b) Find the curvature $\kappa(t)$.

Solution.

(a) First, we compute

$$\vec{r}\,'(t) = \langle 2\cos t, 5, -2\sin t\rangle.$$

It follows that

$$|\vec{r}\,'(t)| = \sqrt{(2\cos t)^2 + 5^2 + (-2\sin t)^2} = \sqrt{4\cos^2 t + 25 + 4\sin^2 t} + \sqrt{29}.$$

Therefore, the unit tangent vector is

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{\sqrt{29}}\langle 2\cos t, 5, -2\sin t\rangle.$$

(b) First, we compute

$$\vec{T}\,'(t) = \frac{1}{\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle.$$

It follows that

$$|\vec{T}\,'(t)| = \frac{1}{\sqrt{29}}\sqrt{(-2\sin t)^2 + 0^2 + (-2\cos t)^2} = \frac{1}{\sqrt{29}}\sqrt{4\sin^2 t + 4\cos^2 t} = \frac{2}{\sqrt{29}}.$$

Therefore, the curvature is

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{2/\sqrt{29}}{\sqrt{29}} = \frac{2}{29}.$$

4. (10 pts.) Find the position at time t of a particle with initial position $\vec{r}(0) = \langle 1, 0, 0\rangle$ and velocity function $\vec{v}(t) = \langle t, e^t, e^{-t}\rangle$.

Solution. We begin by anti-differentiating velocity to find the position:

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle t^2/2, e^t, -e^{-t}\rangle + \vec{C}.$$

To determine the vector-valued constant $\vec{C}$, we use the initial position information:

$$\vec{r}(0) = \langle 1, 0, 0\rangle \implies \langle 0, 1, -1\rangle + \vec{C} = \langle 1, 0, 0\rangle \implies \vec{C} = \langle 1, -1, 1\rangle.$$

Therefore, the position of the particle at time t is given by

$$\vec{r}(t) = \langle t^2/2, e^t, -e^{-t}\rangle + \langle 1, -1, 1\rangle = \langle t^2/2 + 1, e^t - 1, -e^{-t} + 1\rangle.$$

5. (20 pts.) Determine whether or not each limit exists. If it does, state its value. Justify your answers.

(a) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} x^2y + e^{x+z} - yz + \cos y.$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2ye^y}{x^4 + 4y^2}.$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2 + y^2}.$

Solution.

(a) The function $f(x, y, z) = x^2y + e^{x+z} - yz + \cos y$ is continuous, so by direct substitution we find that

$$\lim_{(x,y,z)\to(0,0,0)} x^2y + e^{x+z} - yz + \cos y = 0 + e^0 - 0 + \cos(0) = 2.$$

(b) If we fix $y = 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2ye^y}{x^4 + 4y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = 0,$$

whereas if we fix $y = x^2$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2ye^y}{x^4 + 4y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2(x^2)e^{x^2}}{x^4 + 4(x^2)^2} = \lim_{(x,y)\to(0,0)} \frac{x^4e^{x^2}}{5x^4} = \lim_{(x,y)\to(0,0)} \frac{e^{x^2}}{5} = \frac{1}{5} \neq 0.$$

Since different paths yield different values, we conclude that the limit **does not exist**.

(c) (Taken from the notes.) If we find the limit of $f(x, y) = \frac{x^2y}{x^2+y^2}$ as $(x, y) \to (0, 0)$ along a few different paths, we may begin to suspect that the limit of this function actually does exist and is equal to 0. To prove it, we consider the distance from $f(x, y)$ to 0, which is given by

$$\left|\frac{x^2y}{x^2 + y^2} - 0\right| = \left|\frac{x^2y}{x^2 + y^2}\right| = \frac{x^2|y|}{x^2 + y^2}.$$

Since $x^2 \le x^2 + y^2$, we have

$$0 \le \frac{x^2|y|}{x^2 + y^2} \le |y|.$$

Moreover, we observe that

$$\lim_{(x,y)\to(0,0)} 0 = 0 \qquad \text{and} \qquad \lim_{(x,y)\to(0,0)} |y| = 0,$$

so, by the Squeeze Theorem, we conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2 + y^2} = 0.$$

6. (15 pts.) Let $u = \sqrt{r^2 + s^2}$, where $r = y + x\cos t$ and $s = x + y\sin t$.

(a) Sketch and label a Chain Rule tree diagram for the function u.

(b) Use the Chain Rule to find the partial derivative $\frac{\partial u}{\partial x}$ when $x = 1$, $y = 2$, and $t = 0$.

Solution.

(a) (See examples in the textbook.)

(b) By the Chain Rule, we have

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} \\ &= \left(\frac{1}{2}(r^2+s^2)^{-1/2}\cdot 2r\right)(\cos t) + \left(\frac{1}{2}(r^2+s^2)^{-1/2}\cdot 2s\right)(1) \\ &= \frac{r\cos t + s}{\sqrt{r^2+s^2}}.\end{aligned}$$

Since $r\big|_{\{x=1,y=2,t=0\}} = 2 + 1\cos(0) = 3$ and $s\big|_{\{x=1,y=2,t=0\}} = 1 + 2\sin(0) = 1$, it follows that

$$\left.\frac{\partial u}{\partial x}\right|_{\{x=1,y=2,t=0\}} = \frac{3\cos(0)+1}{\sqrt{3^2+1^2}} = \frac{4}{\sqrt{10}}.$$

7. (20 pts.) Let f be the trivariate function defined by $f(x, y, z) = \sqrt{xyz}$.

(a) Find the gradient vector of f as a function of x, y, and z.

(b) Evaluate the gradient at the point $(3, 2, 6)$. What does this vector represent?

(c) Find the directional derivative of f at $(3, 2, 6)$ in the direction of the vector $\vec{v} = \langle -1, -2, 2\rangle$.

Solution.

(a) The gradient of f, as a function of x, y, and z, is given by

$$\vec{\nabla} f(x,y,z) = \langle f_x, f_y, f_z\rangle = \left\langle \frac{yz}{2\sqrt{xyz}}, \frac{xz}{2\sqrt{xyz}}, \frac{xy}{2\sqrt{xyz}}\right\rangle.$$

(b) At the point $(3, 2, 6)$, we have

$$\vec{\nabla} f(3,2,6) = \left\langle \frac{(2)(6)}{2\sqrt{(3)(2)(6)}}, \frac{(3)(6)}{2\sqrt{(3)(2)(6)}}, \frac{(2)(3)}{2\sqrt{(3)(2)(6)}}\right\rangle = \langle 1, 3/2, 1/2\rangle,$$

which represents the direction of the maximum rate of change, at the point $(3, 2, 6)$, of the surface defined by the function f.

(c) The directional derivative of f at $(3, 2, 6)$ in the direction of $\vec{v} = \langle -1, -2, 2\rangle$ is

$$\begin{aligned}D_{\hat{u}} f(3,2,6) &= \vec{\nabla} f(3,2,6)\cdot\hat{u} \\ &= \langle 1, 3/2, 1/2\rangle \cdot \langle -1/3, -2/3, 2/3\rangle \\ &= (1)(-1/3) + (3/2)(-2/3) + (1/2)(2/3) \\ &= -1,\end{aligned}$$

where $\hat{u} = \vec{v}/|\vec{v}| = \langle -1/3, -2/3, 2/3\rangle$.

E.C. (10 pts.) Find and classify all extrema of the function $f(x, y) = y^2 - 2y \cos x$, where $-1 \le x \le 7$.

Solution. The partial derivatives of f are $f_x(x, y) = 2y \sin x$ and $f_y = 2y - 2\cos x$. Setting these expressions equal to zero yields the system of equations

$$\begin{cases} 2y \sin x = 0, \\ 2y - 2\cos x = 0. \end{cases}$$

The second equation is equivalent to $y = \cos x$. Substituting this into the first equation, we find that $\cos x \sin x = 0$. Given the domain restriction $-1 \le x \le 7$, it follows that $x = 0,\ \pi/2,\ \pi,\ 3\pi/2,\ 2\pi$. Therefore, the five critical points of f are $(0, 1)$, $(\pi/2, 0)$, $(\pi, -1)$, $(3\pi/2, 0)$, and $(2\pi, 1)$.

Next, we compute the determinant of the Hessian matrix to be

$$D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2y\cos x & 2\sin x \\ 2\sin x & 2 \end{vmatrix} = 4y\cos x - 4\sin^2 x.$$

Since

$$D(\pi/2, 0) = D(3\pi/2, 0) = -4 < 0,$$

we immediately conclude that the function f has saddle points at $(\pi/2, 0)$ and $(\pi/2, 0)$. On the other hand, we find that

$$D(0, 1) = D(\pi, -1) = D(2\pi, 1) = 4 > 0$$

and

$$f_{xx}(0, 1) = f_{xx}(\pi, -1) = f_{xx}(2\pi, 1) = 1 > 0.$$

Therefore, the function f has local minima at the points $(0, 1)$, $(\pi, -1)$, and $(2\pi, 1)$