Names: ____________________

Work in groups of five to complete the following problems. Each problem should be solved on a separate sheet of paper and each sheet of paper should have at least one group member's name on it. Algebraic support must be shown to receive full credit (i.e. show work!). Answers should be exact unless otherwise specified.

1. (5 pts.) Find all critical points of the function $f(x, y) = x^2 + xy + y^2 + y$ and determine whether each one corresponds to a maximum, a minimum, or a saddle point of f.

Solution. First, we compute the partial derivatives

$$f_x(x, y) = 2x + y$$

and

$$f_y(x, y) = x + 2y + 1.$$

Solving the system of equations $f_x = 0$ and $f_y = 0$ yields the single critical point $(x, y) = (1/3, -2/3)$. Next, we compute the second partial derivatives $f_{xx}(x, y) = f_{yy}(x, y) = 2$ and $f_{xy}(x, y) = f_{yx}(x, y) = 1$. It follows that the determinant of the Hessian matrix of f is

$$D = f_{xx}f_{yy} - [f_{xy}]^2 = 2^2 - 1^2 = 3 > 0.$$

Since $f_{xx}(1/3, -2/3) = 2 > 0$ and $D(1/3, -2/3) = 3 > 0$, we conclude that f has a min at $(1/3, -2/3)$.

2. (5 pts.) Use Lagrange multipliers to find and classify the extrema of the function $f(x, y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 1$.

Solution. Let $g(x, y) = x^2 + y^2$. Then $\vec{\nabla} f = \lambda \vec{\nabla} g$ implies that

$$f_x = \lambda g_x \implies 2x = 2x\lambda$$

and

$$f_y = \lambda g_y \implies 4y = 2y\lambda.$$

From the first equation, we have $x = 0$ or $\lambda = 1$. If $x = 0$, then the constraint equation $g(x, y) = 1$ gives $y = \pm 1$. On the other hand, if $\lambda = 1$, then the second equation gives $y = 0$ and hence the constraint equation gives $x = \pm 1$. Therefore, f has possible extreme values at $(0, 1)$, $(0, -1)$, $(1, 0)$, $(-1, 0)$. Evaluating f at these points, we find that the maximum value of f on $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$.

3. (5 pts.) Evaluate the integral $\iint_R (6x^2y^3 - 5y^4)\,dA$, where $R = \{(x,y) : 0 \le x \le 3, 0 \le y \le 1\}$.

Solution.

$$\begin{aligned}
\iint\limits_R (6x^2y^3 - 5y^4)\,dA &= \int_0^3 \int_0^1 (6x^2y^3 - 5y^4)\,dy\,dx \\
&= \int_0^3 \left[\left(\frac{3}{2}x^2y^4 - y^5 \right) \Big|_{y=0}^{y=1} \right] dx \\
&= \int_0^3 \left[\frac{3}{2}x^2 - 1 \right] dx \\
&= \left[\frac{1}{2}x^3 - x \right] \Big|_{x=0}^{x=3} \\
&= \frac{21}{2}.
\end{aligned}$$

4. (5 pts.) Evaluate the integral $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\,dx\,dy$ by reversing the order of integration.

Solution.

$$\begin{aligned}
\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\,dx\,dy &= \int_0^{\sqrt{\pi}} \int_0^x \cos(x^2)\,dy\,dx \\
&= \int_0^{\sqrt{\pi}} \left[\cos(x^2)\, y \,\Big|_{y=0}^{y=x} \right] dx \\
&= \int_0^{\sqrt{\pi}} \cos(x^2)\, x\,dx \\
&= \int_0^{\pi} \frac{1}{2} \cos u\,du \\
&= \frac{1}{2} \sin u \,\Big|_0^{\pi} \\
&= 0.
\end{aligned}$$

5. (5 pts.) Evaluate the integral $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$ by converting to polar coordinates.

Solution.

$$\begin{aligned}
\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx &= \int_0^{\pi} \int_0^3 \sin(r^2)\, r\,dr\,d\theta \\
&= \int_0^{\pi} 1\,d\theta \cdot \int_0^3 \sin(r^2)\, r\,dr \\
&= (\pi - 0) \int_0^9 \frac{1}{2} \sin u\,du \\
&= \frac{\pi}{2}(-\cos u)\Big|_0^9 \\
&= \frac{\pi}{2}(-\cos(9) - (-\cos(0)) \\
&= \frac{\pi}{2}(1 - \cos(9))
\end{aligned}$$