Hopefully Helpful Examples (Section 3.1)

General Strategy: So far, up to this section, we have three basic derivative formulas. They involve the derivatives of c (i.e. a constant), x^n and e^x.

$$\frac{dc}{dx} = 0, \qquad \frac{d(x^n)}{dx} = nx^{n-1}, \qquad \frac{d(e^x)}{dx} = e^x$$

In order to apply them, we will need to use algebra to write each function as a sum or difference of these types of functions.

1. Differentiate $f(x) = (x-2)(2x+3)$.

Solution: Expand the product to rewrite $f(x)$. We get $f(x) = 2x^2 - x - 6$. Then differentiate term by term to get $f'(x) = 2(2x) - 1 = 4x - 1$.

2. Differentiate $f(x) = 5\sqrt{x} - 2e^x$.

Solution: Write $\sqrt{x}$ as a power of x. We get $f(x) = 5x^{1/2} - 2e^x$. Now apply the above formulas. We get $f'(x) = 5\left(\frac{1}{2}x^{-1/2}\right) - 2e^x = \frac{5}{2}x^{-\frac{1}{2}} - 2e^x$.

3. Differentiate $f(r) = \left(\frac{1}{2}r^2\right)^5$.

Solution: By the laws of exponents, $f(r) = \left(\frac{1}{2}\right)^5 (r^2)^5 = \frac{1}{32}r^{10}$.
Now use the power formula to get $f'(x) = \frac{10}{32}r^9 = \frac{5}{16}r^9$.

4. Differentiate $f(x) = \pi^5$

Solution: A typical mistake is to apply the power formula to get $f'(x) = 5\pi^4$. But π, and hence π^5, are constants. So the correct formula to use here is $\frac{dc}{dx} = 0$. Therefore, $f'(x) = 0$.

5. Differentiate $f(x) = \frac{3x^2-6x+5}{\sqrt[3]{x}}$.

Solution: First, replace $\sqrt[3]{x}$ with $x^{1/3}$. We get $f(x) = \frac{3x^2-6x+5}{x^{1/3}}$. Now divide each term in the numerator by the denominator. We get $f(x) = \frac{3x^2}{x^{1/3}} - \frac{6x}{x^{1/3}} + \frac{5}{x^{1/3}}$. By the laws of exponents $f(x) = 3x^{2-\frac{1}{3}} - 6x^{1-\frac{1}{3}} + 5x^{-\frac{1}{3}}$. Simplify further to get $f(x) = 3x^{\frac{5}{3}} - 6x^{\frac{2}{3}} +$

$5x^{-\frac{1}{3}}$. Everything we did so far, was just to rewrite $f(x)$ as a sum of powers of x. Now, we can use the formula $\frac{d(x^n)}{dx} = nx^{n-1}$ to differentiate $f(x)$.
We get $f'(x) = 3 \cdot \frac{5}{3} x^{\left(\frac{5}{3}-1\right)} - 6 \cdot \frac{2}{3} x^{\left(\frac{2}{3}-1\right)} + 5 \cdot \left(-\frac{1}{3} x^{\left(-\frac{1}{3}-1\right)}\right)$. Simply to get $f'(x) = 5x^{\frac{2}{3}} - 4x^{-\frac{1}{3}} - \frac{5}{3} x^{-\frac{4}{3}}$.

6. Find an equation of the tangent line to $f(x) = x - \frac{1}{x}$ at the point $\left(2, \frac{3}{2}\right)$.

Solution: The derivative gives a formula for the slope of the tangent line. First, rewrite $f(x)$ as $f(x) = x - x^{-1}$. Then use the formula $\frac{d(x^n)}{dx} = nx^{n-1}$ to get $f'(x) = 1 - (-1x^{-2}) = 1 + x^{-2}$. Therefore, the slope of the tangent at $\left(2, \frac{3}{2}\right)$ is $f'(2) = 1 + 2^{-2} = \frac{5}{4}$. By the point-slope formula, the equation of the tangent line is $y - \frac{3}{2} = \frac{5}{4}(x - 2)$, which simplifies to, $y = \frac{5}{4}x - 1$.

7. Find the first and second derivatives of $f(r) = r^4 - 3r^2 + 16r$.

Solution: Use the formula $\frac{d(x^n)}{dx} = nx^{n-1}$ to get $f'(r) = 4r^3 - 3 \cdot 2r^1 + 16$. So $f'(r) = 4r^3 - 6r + 16$. To get the second derivative, we differentiate $f'(r)$. So $f''(r) = 4 \cdot 3r^2 - 6$. So $f''(r) = 12r^2 - 6$.

8. On what interval is the function $f(x) = 5x^2 - 10x - 25$ increasing?

Solution: A function increases when its derivative is positive and decreases when its derivative is negative. So $f(x)$ increases when $f'(x) > 0$. Differentiate $f(x)$, we get $f'(x) = 10x - 10$. So $f(x)$ increases when $f'(x) = 10x - 10 > 0$, i.e. $10x > 10$, i.e. $x > 1$. So $f(x) = 5x^2 - 10x - 25$ increases on the interval $(1, \infty)$.

9. On what interval is the graph of $f(x) = 2e^x - 3x^2 + 11$ is concave down?

Solution: The graph of $f(x)$ is concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$. By the above derivative formulas, $f'(x) = 2e^x - 6x$. Now differentiate $f'(x)$ to get $f''(x) = 2e^x - 6$. Then $f(x)$ is concave down when $f''(x) < 0$, i.e. $2e^x - 6 < 0$, i.e. $2e^x < 6$, i.e. $e^x < 3$, i.e. $\ln e^x < \ln 3$, i.e. $x < \ln 3$. Therefore, $f(x) = 2e^x - 3x^2 + 11$ is concave down on the interval $(-\infty, \ln 3)$.

10. Find the points on the curve $y = x^3 - 3x^2 - 9x + 3$ where the tangent line is horizontal.

Solution: The tangent line is horizontal when its slope is 0. Since the derivative, y', is a formula for the slope of the tangent line for points (x, y) on the curve, we are looking for points whose x- coordinates satisfy the equation $y' = 0$. Since in this case $y' = 3x^2 - 6x - 9$, we are looking for points (x, y) on the curve whose x-coordinates satisfy $3x^2 - 6x - 9 = 0$, i.e. $x^2 - 2x - 3 = 0$, i.e. $(x - 3)(x + 1) = 0$, i.e. $x = 3$ or $x = -1$. To find the points on the curve $y = x^3 - 3x^2 - 9x + 3$ with these x-coordinates, we just substitute $x = 3$ and $x = -1$ into this equation. When $x = 3, y = -24$ and when $x = -1, y = 8$. So the points on the curve $y = x^3 - 3x^2 - 9x + 3$ where the tangent line is horizontal are $(3, -24)$ and $(-1, 8)$.

11. Find a second degree polynomial $f(x)$ such that $f(3) = 13, f'(3) = 12, f''(3) = 6$.

Solution: If $f(x)$ is a second degree polynomial, it has the form $f(x) = ax^2 + bx + c$ for constants a, b, c. Then $f'(x) = 2ax + b$ and $f''(x) = 2a$. Since $f''(3) = 6$, we get $2a = 6$, i.e. $a = 3$. Since $f'(3) = 12$, we get $6a + b = 12$, which implies $18 + b = 12$ since $a = 3$. Hence, $b = -6$. Since $f(3) = 13$, we get $9a + 3b + c = 13$. Since $a = 3$ and $b = -6$, we get $9 + c = 13$, i.e. $c = 4$. Therefore, $f(x) = 3x^2 - 6x + 4$.

12. The equation of motion of a particle is $s(t) = 2t^3 - 7t^2 + 8t + 1$. (a) Find the velocity and acceleration as a function of t; (b) find the acceleration when the velocity is 0.

Solution: (a) The velocity is the rate of change of position, i.e. the velocity is given by $v(t) = s'(t) = 6t^2 - 14t + 8$. The acceleration is the rate of change of velocity, i.e. the acceleration is given by $a(t) = v'(t) = s''(t) = 12t - 14$. (b) The velocity is 0 when $v(t) = 0$, i.e. $6t^2 - 14t + 8 = 0$, i.e. $3t^2 - 7t + 4 = 0$, i.e. $(3t - 4)(t - 1) = 0$. Therefore, the velocity is 0 when $t = 1$ and when $t = \frac{4}{3}$. Substituting these values into $a(t) = 12t - 14$, we get the acceleration is $a = -2$ when $t = 1$ and $a = 2$ when $t = \frac{4}{3}$. These are the accelerations when the velocity is 0.