MATH 121, Calculus I — Sample Exam I (Fall 2013)

Name: ___________________________________

KU ID No.: ___________________________________

Lab Instructor: ___________________________________

This exam has total value 200 points. It consists of two parts. The first part contains 14 multiple-choice questions, each worth 10 points. The second part contains 3 long-answer problems, each worth 20 points. There are 17 problems in total to be solved. Additionally the last page of the exam contains an extra-credit problem that is worth 20 points. This is strictly a closed-book exam and the use of calculators is prohibited.

Score

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Problem 7	
Problem 8	
Problem 9	
Problem 10	
Problem 11	
Problem 12	
Problem 13	
Problem 14	
Problem 15	
Problem 16	
Problem 17	
Extra Credit	
Total	

Multiple-Choice Questions

Instructions: Place the appropriate letter for your answer for each problem in the blank box that is provided. Correct answers do not require work to receive full credit. However, partial credit can be awarded to incorrect answers based on work shown in the adjacent blank space. Hence, you are strongly advised to show work for each problem.

(1) [**10 points**] Find the value of $\lim_{x\to 0} \dfrac{\sin^2 x}{2x}$.

(A) 1.

(B) 2.

(C) 1/2.

(D) 0.

Answer: D

Solution 1. Recall that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Therefore, $\lim_{x\to 0} \frac{\sin^2 x}{2x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{2}$

$= 1 \cdot 0$

$= \boxed{0.}$

Solution 2. Use L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\sin^2 x \to 0}{2x \to 0} = \lim_{x\to 0} \frac{2\sin x \cos x}{2} = \boxed{0.}$$

(2) [**10 points**] Let $f(x) = xe^x$. Find the point $(x, f(x))$ at which the second derivative $f''(x)$ equals zero.

(A) $(2, 2e^2)$.

(B) $(0, 0)$.

(C) $(-2, -2e^{-2})$.

(D) $(-4, -4e^{-4})$.

Answer: C

Solution. $f(x) = xe^x$

$\Rightarrow f'(x) = xe^x + e^x$ (Product Rule)

$\Rightarrow f''(x) = (xe^x + e^x) + e^x$

$= e^x(x+2)$ (Product Rule again)

We want to find when $f''(x) = 0$.

Since e^x is never zero, we only have to check $x+2=0 \Rightarrow x=-2$.

This gives the point $(x, f(x)) = \boxed{(-2, -2e^{-2}).}$

(3) **[10 points]** If $y = \tan^{-1}(x^2)$, determine $\dfrac{dy}{dx}$.

(A) $\dfrac{1}{1+x^4}$.

(B) $\dfrac{2x}{1+x^4}$.

(C) $2x \cdot \sec^2(x^2)$.

(D) $2x \cdot \tan^{-1}(x^2)$.

Answer: B.

Solution. Recall that $\dfrac{d}{dx}(\tan^{-1}(x)) = \dfrac{1}{1+x^2}$.

Therefore, by the Chain Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}(x^2)\right) = \frac{1}{1+(x^2)^2} \cdot 2x = \boxed{\frac{2x}{1+x^4}.}$$

(4) **[10 points]** What is the slope of the tangent line to the curve

$$x + y = xy$$

at the point $(2, 2)$?

(A) -1.

(B) -2.

(C) -3.

(D) -4.

Answer: A

Solution. By Implicit Differentiation:

$$\frac{d}{dx}(x+y) = \frac{d}{dx}(xy) \Rightarrow 1 + \frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow \frac{dy}{dx}(1-x) = y-1$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-1}{1-x}.$$

Therefore, the slope of the tangent line to the curve at $(2,2)$ is:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,2)} = \frac{2-1}{1-2} = \boxed{-1.}$$

(5) [**10 points**] Which of the following statements are true? (Since there may be more than one correct answer, determine all correct answers.)

(A) If $\lim_{x \to a} f(x)$ exists, then f is continuous at a.

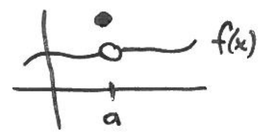
(B) If f is continuous at a, then f is differentiable at a.

(C) If f is continuous at a, then $\lim_{x \to a} f(x)$ exists.

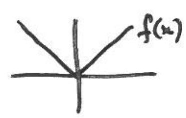
(D) If f is differentiable at a, then f is continuous at a.

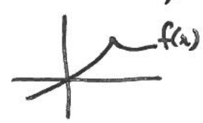
Answer: $\boxed{\text{C, D}}$

(A) is false, because there could be a hole:



(B) is also false, because there could be a corner, a cusp, or a vertical tangent line:





(6) [**10 points**] Let f and g be functions that are differentiable everywhere. Suppose that

$$f(2) = 2, \qquad g(2) = 1, \qquad f'(1) = 3, \qquad g'(2) = -2.$$

Use this information to find the value of $(f \circ g)'(2)$.

(A) 6.

(B) 3.

(C) -3.

(D) -6.

Answer: $\boxed{\text{D}}$

Solution. Recall the Chain Rule:

$$\frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Therefore,

$$\begin{aligned}(f \circ g)'(2) &= \frac{d}{dx}\big(f(g(x))\big)\Big|_{x=2} \\ &= f'(g(2)) \cdot g'(2) \\ &= f'(1) \cdot (-2) \\ &= 3 \cdot (-2) \\ &= \boxed{-6.}\end{aligned}$$

(7) [**10 points**] Suppose that the function g satisfies the following inequality

$$1 + x \leq g(x) \leq e^x$$

for all values of x. Find the value of $\lim_{x \to 0} g(x)$.

(A) 1.

(B) 2.

(C) 3.

(D) None of the above.

Answer: A

Solution. Notice that:

$$\lim_{x \to 0} (1+x) = 1 + 0 = 1.$$

$$\lim_{x \to 0} e^x = e^0 = 1.$$

Therefore, by the Squeeze Theorem:

$$\lim_{x \to 0} g(x) = \boxed{1.}$$

(8) [**10 points**] Find the values of c that make the following piecewise-defined function continuous everywhere:

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2; \\ (c^2 - c)x - 8 & \text{if } x \geq 2. \end{cases}$$

(A) 2 and 4.

(B) -2 and 3.

(C) 0 and 8.

(D) -4 and -8.

Answer: B

Solution. We just have to make sure that the two pieces of the function match up at $x = 2$.

Since $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = \lim_{x \to 2}(x+2) = 4$,

we need to choose 'c' so that

$$(c^2 - c)x - 8 = 4 \quad \text{when } x = 2.$$

$$\Rightarrow (c^2 - c)(2) - 8 = 4 \Rightarrow c^2 - c - 4 = 2$$

$$\Rightarrow c^2 - c - 6 = 0$$

$$\Rightarrow (c-3)(c+2) = 0$$

$$\Rightarrow \boxed{c = 3, \ c = -2.}$$

(9) [**10 points**] Find the point on the curve $y = 3x^2 - 4x + 5$ where the tangent line is parallel to the line $y = -22x + 7$.

(A) $(0, 5)$.

(B) $(-1, 12)$.

(C) $(-2, 25)$.

(D) $(-3, 44)$.

Answer: $\boxed{D}$

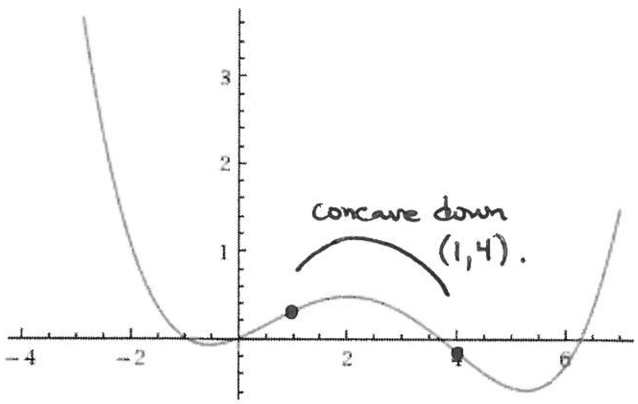
<u>Solution.</u> $y' = 6x - 4$.

We want to find when $y' = -22$.

$$y' = -22 \Rightarrow 6x - 4 = -22$$
$$\Rightarrow 6x = -18$$
$$\Rightarrow x = -3.$$

So the point is $\boxed{(-3, 44)}$.

(10) [**10 points**] Given the graph of the function f below, find the open interval(s) where the second derivative $f''(x) < 0$.



(A) $(4, \infty)$.

(B) $(-\infty, 1)$.

(C) $(1, 4)$.

(D) $(-\infty, 1)$ and $(4, \infty)$.

$f''(x) < 0 \iff$ concave down.

Answer: $\boxed{C}$

(11) [**10 points**] Let $g(t) = \frac{1}{9}(2t+1)^3$. Find an equation of the tangent line to the graph of g at $t = 1$.

(A) $y = 6$.

(B) $y = 6t - 3$.

(C) $y = 3t$.

(D) $y = 6t - 6$.

Answer: $\boxed{B}$

<u>Solution.</u> $g(t) = \frac{1}{9}(2t+1)^3$

$$\Rightarrow g'(t) = \frac{3}{9}(2t+1)^2 \cdot 2 = \frac{2}{3}(2t+1)^2.$$

$$\Rightarrow g'(1) = \frac{2}{3}(2(1)+1)^2 = 6.$$

The slope of the tangent line at $t=1$ is 6.

Since $g(1) = \frac{1}{9}(2(1)+1)^3 = 3$, the equation of the tangent line at $(1,3)$ is:

$$y - 3 = 6(t-1) \Rightarrow \boxed{y = 6t - 3.}$$

(12) [**10 points**] Determine the second derivative of the function $f(x) = x^2 \cdot \ln(2x)$.

(A) $2 \cdot \ln(2x) + 3$.

(B) $2 \cdot \ln(2x) + \frac{3}{2}$.

(C) 0.

(D) $2 + \frac{1}{2x}$.

Answer: $\boxed{A}$

<u>Solution.</u> By the Product Rule:

$$f'(x) = x^2 \cdot \frac{2}{2x} + 2x\ln(2x) = x + 2x\ln(2x)$$

$$\Rightarrow f''(x) = 1 + 2x \cdot \frac{2}{2x} + 2\ln(2x) = \boxed{3 + 2\ln(2x).}$$

(13) [**10 points**] Let $y = \sqrt{x}(x+1)^5 e^{x^2}$. Determine $\dfrac{dy}{dx}$.

(A) $\sqrt{x}(x+1)^5 e^{x^2}\left(\dfrac{1}{2x} + 5x + e^{x^2}\right)$.

(B) $\sqrt{x}(x+1)^5 e^{x^2}\left(\dfrac{1}{2\sqrt{x}} + \dfrac{5}{x+1} + 2x\right)$.

(C) $\sqrt{x}(x+1)^5 e^{x^2}\left(\dfrac{5(x+1)^4}{2\sqrt{x}} + xe^{x^2}\right)$.

(D) $\sqrt{x}(x+1)^5 e^{x^2}\left(\dfrac{1}{2x} + \dfrac{5}{x+1} + 2x\right)$.

Answer: D

Solution. Using Logarithmic Differentiation:

$$y = \sqrt{x}(x+1)^5 e^{x^2}$$

$$\Rightarrow \ln(y) = \tfrac{1}{2}\ln x + 5\ln(x+1) + x^2$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2x} + \frac{5}{x+1} + 2x$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \sqrt{x}(x+1)^5 e^{x^2}\left(\frac{1}{2x} + \frac{5}{x+1} + 2x\right).}$$

(14) [**10 points**] Find the value of $\displaystyle\lim_{x\to\infty} \frac{3x^2+1}{\sqrt{4x^4+2x^2+1}}$.

(A) 0.

(B) 3.

(C) $\dfrac{3}{2}$.

(D) ∞.

Answer: C

Solution.

$$\lim_{x\to\infty} \frac{3x^2+1}{\sqrt{4x^4+2x^2+1}} = \lim_{x\to\infty} \frac{3x^2+1}{\sqrt{4x^4+2x^2+1}}\cdot\frac{(1/x^2)}{(1/\sqrt{x^4})}$$

$$= \lim_{x\to\infty} \frac{3 + 1/x^2}{\sqrt{4 + 2/x^2 + 1/x^4}}$$

$$= \frac{3}{\sqrt{4}}$$

$$= \boxed{\frac{3}{2}.}$$

Long-Answer Problems

Instructions: Please show all necessary work and provide full justification for each answer. Place a box around each answer.

(15) (a) [**5 points**] Use the definition of the derivative to express $g'(3)$ as the limit of the difference quotient of $g(x)$ evaluated at $x = 3$.

(b) [**15 points**] Letting $g(x) = \sqrt{x+1}$, find the value of $g'(3)$ by explicitly evaluating the limit from Part (a).

Caution: Do <u>not</u> use the Power Chain Rule to solve Part (b).

Solution. (a) $\boxed{g'(3) = \lim_{x\to 3} \frac{g(x) - g(3)}{x-3}}$ or $\boxed{g'(3) = \lim_{h\to 0} \frac{g(3+h) - g(3)}{h}}$.

(b) $g(x) = \sqrt{x+1}$.

$$\Rightarrow g'(3) = \lim_{x\to 3} \frac{\sqrt{x+1} - \sqrt{3+1}}{x-3}$$

$$= \lim_{x\to 3} \frac{(\sqrt{x+1} - 2)}{(x-3)} \cdot \frac{(\sqrt{x+1} + 2)}{(\sqrt{x+1} + 2)}$$

$$= \lim_{x\to 3} \frac{(x+1) - 4}{(x-3)(\sqrt{x+1} + 2)}$$

$$= \lim_{x\to 3} \frac{\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{x+1} + 2)}$$

$$= \lim_{x\to 3} \frac{1}{\sqrt{x+1} + 2}$$

$$= \frac{1}{\sqrt{3+1} + 2}$$

$$= \boxed{\frac{1}{4}}.$$

(16) [**20 points**] The position function of a particle on the real line is given, in meters, by

$$s(t) = at^2 + bt + 3.$$

At time $t = 1$, the particle's velocity is 8 m/s and its acceleration is 6 m/s^2. Use this information to find the values of a and b.

Solution. We are given that $v(1) = 8$ and $a(1) = 6$.

By taking derivatives, we obtain the general formulas for velocity and acceleration:

$$s(t) = at^2 + bt + 3$$

$$\Rightarrow \quad v(t) = s'(t) = 2at + b \qquad ①$$

$$\Rightarrow \quad a(t) = v'(t) = s''(t) = 2a. \qquad ②$$

Using $a(1) = 6$ in ②, we find that:

$$2a = 6 \Rightarrow \boxed{a = 3}$$

Then, using $a = 3$ and $v(1) = 8$ in ①, we get:

$$2(3)(1) + b = 8 \Rightarrow \boxed{b = 2}$$

(17) [**20 points**] Determine when the function $f(x) = x^3 - 3x + 1$ is increasing or decreasing.

Solution.

$f(x) = x^3 - 3x + 1 \Rightarrow f'(x) = 3x^2 - 3.$

Setting $f'(x) = 0$, we find that:

$$3x^2 - 3 = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow (x-1)(x+1) = 0 \Rightarrow x = \pm 1.$$

To find when $f'(x) < 0$ and $f'(x) > 0$, we perform a "sign test" on the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

interval	test point 'c'	sign of $f'(c)$	$f(x)$ is...
$(-\infty, -1)$	-2	$+$	increasing
$(-1, 1)$	0	$-$	decreasing
$(1, \infty)$	2	$+$	increasing

Therefore, $f(x)$ is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$.

(Extra Credit) [**20 points**] Choose **one** of the following problems.

(i) A problem to be selected from proposed problems by students.

(ii) A spherical balloon has air pumped into it at the rate of 18 in^3 per second. What is the rate at which the radius of the balloon is expanding when its volume is 36π in^3?

Note: The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$.

(ii) <u>Solution.</u> We are given $\frac{dV}{dt} = 18$ in^3/sec and we want to find $\frac{dr}{dt}$ when $V = 36\pi$ in^3. Since $V = \frac{4}{3}\pi r^3$, Implicit Differentiation gives:

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Therefore, $\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt} = \frac{1}{4\pi r^2} \cdot 18.$

Next, we need to know 'r' when $V = 36\pi$. This is easily computed from the formula:

$$V = \frac{4}{3}\pi r^3 \Rightarrow 36\pi = \frac{4}{3}\pi r^3 \Rightarrow 27 = r^3 \Rightarrow r = 3.$$

Thus, the rate of change of the radius with respect to time when the volume is 36π in^3 is given by:

$$\left.\frac{dr}{dt}\right|_{V=36\pi} = \left.\frac{dr}{dt}\right|_{r=3} = \frac{1}{4\pi(3)^2} \cdot 18 = \boxed{\frac{1}{2\pi} \text{ in/sec.}}$$