Names: ______________________________

Work in groups of five to complete the following problems. Each problem should be solved on a separate sheet of paper and each sheet of paper should have at least one group member's name on it. Algebraic support must be shown to receive full credit (i.e. show work!). Answers should be exact unless otherwise specified.

1. (10 pts.) A lamina occupies the part of the disk $x^2+y^2 \leq 1$ that lies in the **first and second quadrants**. The density at any point is proportional to the **square of its distance** from the y-axis.

(a) Find the moments M_x and M_y.

(b) Find the total mass m and center of mass $(\bar{x}, \bar{y})$.

Solution.

(a) Since the distance from a point (x, y) to the y-axis is simply x, the density of the lamina is given by $\rho(x, y) = Kx^2$, where K is some constant. The region D occupied by the lamina is naturally described in polar coordinates by $0 \leq \theta \leq \pi$ and $0 \leq r \leq 1$. By symmetry, $M_y = 0$. On the other hand, we have

$$\begin{aligned}
M_x &= \iint_D y\rho(x, y)\, dA \\
&= \int_0^\pi \int_0^1 (r\sin\theta)(Kr^2\cos^2\theta)\, r\, dr\, d\theta \\
&= K\int_0^\pi \sin\theta\cos^2\theta\, d\theta \cdot \int_0^1 r^4\, dr \\
&= K\int_{-1}^1 u^2\, du \cdot \int_0^1 r^4\, dr \\
&= K\left(\frac{u^3}{3}\bigg|_{u=-1}^{u=1}\right)\left(\frac{r^5}{5}\bigg|_{r=0}^{r=1}\right) \\
&= \frac{2K}{15}.
\end{aligned}$$

(b) The total mass is

$$\begin{aligned}
m &= \iint_D \rho(x, y)\, dA \\
&= \int_0^\pi \int_0^1 \left(Kr^2\cos^2\theta\right) r\, dr\, d\theta \\
&= K\int_0^\pi \cos^2\theta\, d\theta \cdot \int_0^1 r^3\, dr \\
&= K\int_0^\pi \left[\frac{1}{2} + \cos(2\theta)\right] d\theta \cdot \int_0^1 r^3\, dr \\
&= K\left(\left[\frac{\theta}{2} + \frac{\sin(2\theta)}{2}\right]\bigg|_{\theta=0}^{\theta=\pi}\right)\left(\frac{r^4}{4}\bigg|_{r=0}^{r=1}\right) \\
&= \frac{K\pi}{8}.
\end{aligned}$$

Hence, the center of mass of the lamina is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(0, \frac{16}{15\pi}\right).$$

2. (5 pts.) Evaluate $\displaystyle\iiint_E yz\cos(x^5)\,dV$, where $E = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le x, x \le z \le 2x\}$.

Solution.

$$\begin{aligned}
\iiint_E yz\cos(x^5)\,dV &= \int_0^1 \int_0^x \int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx \\
&= \int_0^1 \int_0^x \left[\frac{1}{2} yz^2 \cos(x^5) \Big|_{z=x}^{z=2x} \right] dy\,dx \\
&= \int_0^1 \int_0^x \frac{3}{2} x^2 y \cos(x^5)\,dy\,dx \\
&= \int_0^1 \left[\frac{3}{4} x^2 y^2 \cos(x^5) \Big|_{y=0}^{y=x} \right] dx \\
&= \int_0^1 \frac{3}{4} x^4 \cos(x^5)\,dx \\
&= \int_{x=0}^{x=1} \frac{3}{20} \cos u\,du \\
&= \frac{3}{20} \sin u \Big|_{x=0}^{x=1} \\
&= \frac{3}{20} \sin(x^5) \Big|_{x=0}^{x=1} \\
&= \frac{3}{20} \sin(1) \\
&\approx 0.1262
\end{aligned}$$

3. (5 pts.) Evaluate $\displaystyle\iiint_E 1\,dV$, where $E = \{(r,\theta,z) : 0 \le r \le 4, 0 \le \theta \le 2\pi, r \le z \le 4\}$.

Solution.

$$\begin{aligned}
\iiint_E 1\,dV &= \int_0^{2\pi} \int_0^4 \int_r^4 r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^4 \left[rz\Big|_{z=r}^{z=4} \right] dr\,d\theta \\
&= \int_0^{2\pi} \int_0^4 \left[4r - r^2 \right] dr\,d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^4 \left[4r - r^2 \right] dr \\
&= (2\pi - 0) \left(\left[2r^2 - \frac{r^3}{3} \right] \Big|_{r=0}^{r=4} \right) \\
&= 2\pi \left(32 - \frac{64}{3} \right) \\
&= \frac{64\pi}{3}.
\end{aligned}$$

4. (5 pts.) Evaluate $\displaystyle\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV$, where E is enclosed by $x^2+y^2+z^2=9$ in the first octant.

Solution.

$$\begin{aligned}
\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 e^\rho\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{\pi/2} d\theta\cdot\int_0^{\pi/2}\sin\phi\,d\phi\cdot\int_0^3 \rho^2 e^\rho\,d\rho \\
&= \left(\frac{\pi}{2}-0\right)\left(-\cos\phi\Big|_{\phi=0}^{\phi=\pi/2}\right)\left(\left[\rho^2e^\rho-2\rho e^\rho+2e^\rho\right]\Bigg|_{\rho=0}^{\rho=3}\right) \\
&= \frac{\pi}{2}\left(\left[9e^3-6e^3+2e^3\right]-\left[2e^0\right]\right) \\
&= \frac{\pi}{2}\left(5e^3-2\right).
\end{aligned}$$

(Note that the integral with respect to $d\rho$ requires Integration by Parts.)