Ex. 2.5.8. Evaluate each of the following limits.

(a) $\lim_{x\to\infty} \frac{3x^2+9x+27}{5x^3-7x+6}$.

(b) $\lim_{x\to-\infty} \frac{3x^4+9x+27}{5x^3-7x+6}$.

(c) $\lim_{x\to-\infty} \frac{3x^3+9x+27}{5x^3-7x+6}$.

(d) $\lim_{x\to\infty} \frac{3x^2+9x+27}{\sqrt{16x^4+11x^3-12x+100}}$.

(e) $\lim_{x\to\infty} \left(\sqrt{x^2+4}-x\right)$.

Solution. We will use Proposition 2.5.1 together with Definitions 2.5.3 and 2.5.5.

(a) Both the numerator and denominator appear to approach infinity as x does. The largest power of x that appears in the expression is x^3. Multiplying both the numerator and denominator by $1/x^3$, we obtain

$$\lim_{x\to\infty} \frac{3x^2+9x+27}{5x^3-7x+6} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{3}{x}+\frac{9}{x^2}+\frac{27}{x^3}}{5-\frac{7}{x^2}+\frac{6}{x^3}} = 0.$$

(b) This time we multiply both the numerator and denominator by $1/x^4$, simplify, and conclude that

$$\lim_{x\to-\infty} \frac{3x^4+9x+27}{5x^3-7x+6} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim_{x\to-\infty} \frac{3+\frac{9}{x^3}+\frac{27}{x^4}}{\frac{5}{x}-\frac{7}{x^3}+\frac{6}{x^4}} = \infty.$$

(c) Again multiplying both the numerator and denominator by $1/x^3$, we have

$$\lim_{x\to\infty} \frac{3x^3+9x+27}{5x^3-7x+6} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{3+\frac{9}{x^2}+\frac{27}{x^3}}{5-\frac{7}{x^2}+\frac{6}{x^3}} = \frac{3}{5}.$$

(d) Notice that $x^2 = \sqrt{x^4}$ is the largest power of x that appears in the expression. Multiplying both the numerator and denominator by $1/x^2 = 1/\sqrt{x^4}$ yields

$$\lim_{x\to\infty} \frac{3x^2+9x+27}{\sqrt{16x^4+11x^3-12x+100}} \cdot \frac{\frac{1}{x^2}}{\frac{1}{\sqrt{x^4}}} = \lim_{x\to\infty} \frac{3+\frac{9}{x}+\frac{27}{x^2}}{\sqrt{16+\frac{11}{x}-\frac{12}{x^3}+\frac{100}{x^4}}} = \frac{3}{\sqrt{16}} = \frac{3}{4}.$$

(e) We first multiply by the conjugate of $\left(\sqrt{x^2+4}-x\right)$ over itself, and then multiply both the numerator and denominator of the resulting expression by $1/x = 1/\sqrt{x^2}$, in order to obtain

$$\lim_{x\to\infty} \left(\sqrt{x^2+4}-x\right) \cdot \frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}+x} = \lim_{x\to\infty} \frac{\left(x^2+4\right)-x^2}{\sqrt{x^2+4}+x} \cdot \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{\frac{4}{x}}{\sqrt{1+\frac{4}{x^2}}+1} = 0.$$

Remark 2.5.2 Parts (a) - (c) of Example 2.5.8 illustrate special cases of a very general result. Namely, if $f(x) = P(x)/Q(x)$, where P and Q are polynomial functions of degree m and n with leading coefficients a and b respectively, then f exhibits the one of the three following types of limiting behavior:

1. $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$, if $m < n$.
2. $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$, if $m > n$.
3. $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = a/b$, if $m = n$.