Example. A particle is moving along the curve $y = \sqrt{x^2 + x + 4} - 2$. As it passes through the point (3,2), its x-coordinate increases at a rate of 5 cm/sec. How fast is the distance from the particle to the point $(1, -2)$ changing at this instant.

Solution: Let D be the distance from the particle to the point $(1, -2)$. We are told that when the particle passes through (3,2), $\frac{dx}{dt} = 5$ and we are asked to find $\frac{dD}{dt}$ at that instant. So we need an equation that relates D to x when the particle is at point (x, y) on the curve. By the distance formula, we have

$$D = \sqrt{(x-1)^2 + (y+2)^2}$$

Substitute y in terms of x into this equation, we get,

$$D = \sqrt{(x-1)^2 + \left(\sqrt{x^2 + x + 4} - 2 + 2\right)^2} = \sqrt{(x-1)^2 + x^2 + x + 4}$$

Therefore,

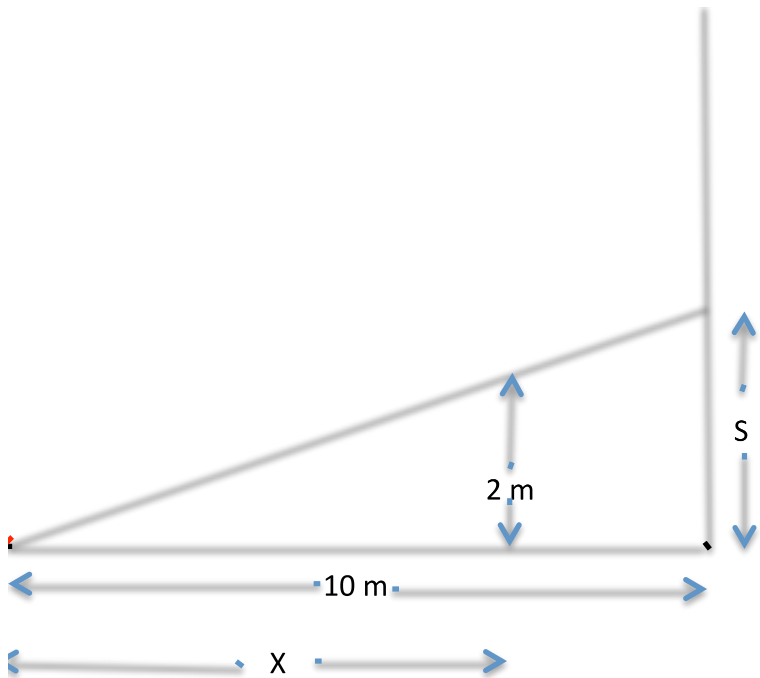
$$D = \sqrt{2x^2 - x + 5}$$

Then at the instant in question,

$$\frac{dD}{dt} = \frac{1}{2}(2 \cdot 3^2 - 3 + 5)^{-1/2}(4 \cdot 3 - 1) \cdot 5 = \frac{55}{2\sqrt{20}} \text{ cm/sec.}$$

Example. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

Solution: Draw a diagram.



X represents the distance the man walks from the spotlight and S represents the length of his shadow. The arrow of height 2 m represents the man. We are told $\frac{dX}{dt} = 1.6$ and we are asked to find $\frac{dS}{dt}$ when $X = 6$. By similar triangles, we have $\frac{S}{10} = \frac{2}{X}$ or $S = \frac{20}{X} = 20X^{-1}$. Then,

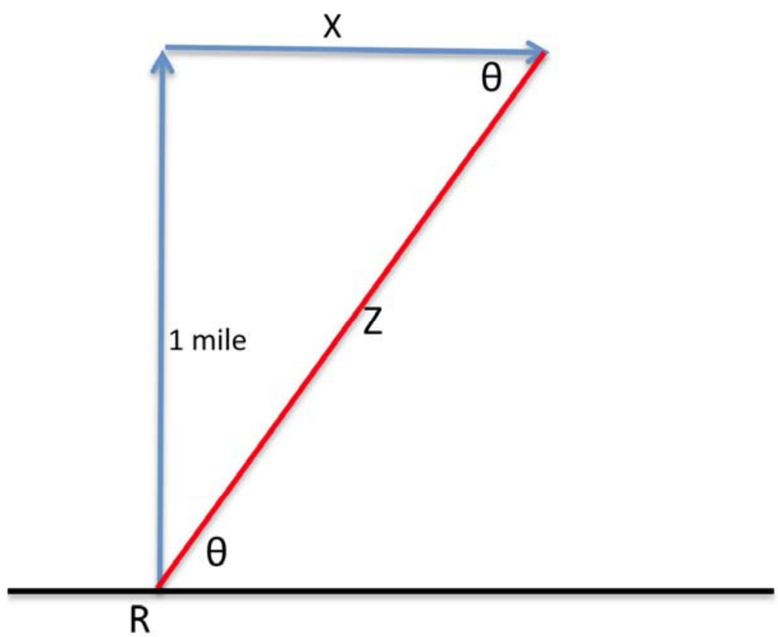
$$\frac{dS}{dt} = -20X^{-2} \cdot \frac{dX}{dt}$$

At the moment in question,

$$\frac{dS}{dt} = -20 \cdot 6^{-2} \cdot (1.6) = -\frac{8}{9} \text{ m/s}.$$

Example. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. (a) Find the rate at which the distance from the plane to the station is increasing when the plane is 2 mi horizontally from the station. (b) At what rate is the angle from the ground to the ray from the station to the plane decreasing at this time?

Solution: Draw a Diagram



R stands for the radar station, X is the horizontal distance the plane is from the station, Z is the absolute distance from the radar station to the plane and θ is the angle. Then $\frac{dX}{dt} = 500$.

In (a) we want $\frac{dZ}{dx}$ when $X = 2$. Since $Z^2 = X^2 + 1$, we get,

$$2Z\frac{dZ}{dt} = 2X\frac{dX}{dt}$$

That is,

$$\frac{dZ}{dt} = \frac{X}{Z}\frac{dX}{dt}$$

Note by the Pythagorean theorem that $Z = \sqrt{5}$ when $X = 2$.

Substituting $\frac{dX}{dt} = 500$, $X = 2$ and $Z = \sqrt{5}$, we get, $\frac{dZ}{dt} = \frac{2}{\sqrt{5}} \cdot 500 = 447.21$ mi/h at the point in question.

In (b) we want $\frac{d\theta}{dt}$. So we need to relate θ to either X or Z or both. It seems $\tan\theta$ is the simplest way to go. We have,

$$\tan\theta = \frac{1}{X}$$

Differentiate with respect to t, we get,

$$\sec^2\theta \cdot \frac{d\theta}{dt} = -X^{-2} \cdot \frac{dX}{dt}$$

Then,

$$\frac{d\theta}{dt} = -\frac{\cos^2\theta}{X^2} \cdot \frac{dX}{dt}$$

As in part (a), we have $Z = \sqrt{5}$ when $X = 2$, which implies $\cos\theta = \frac{2}{\sqrt{5}}$. Substituting this information and $\frac{dX}{dt} = 500$ into the above equation, we get

$$\frac{d\theta}{dt} = -\frac{\frac{4}{5}}{4} \cdot 500 = -100 \text{ radians/sec.}$$

Example. If $z^2 = x^2 + y^2$, $dx/dt = 2$, and $dy/dt = 3$, find dz/dt when $x = 5$ and $y = 12$.

Solution: Differentiate both sides of the equation with respect to t using the chain rule. Then

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

or

$$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

When $x = 5$ and $y = 12$, $z = 13$. So we get

$$13\frac{dz}{dt} = 5 \cdot 2 + 12 \cdot 3 = 46$$

or

$$\frac{dz}{dt} = \frac{46}{13}$$

Example. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which its diameter decreases when the diameter is 10 cm.

Solution: Let S be the snowball's surface area, r be it's radius and D be its diameter. Then $S = 4\pi r^2$. Then $\frac{dS}{dx} = -1$. Since the diameter is twice the radius, we only need to find $\frac{dr}{dt}$ when $r = 5$ and double it. If we differentiate both sides of $S = 4\pi r^2$ with respect to t, we get

$$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

Then, when the diameter is 10, we have

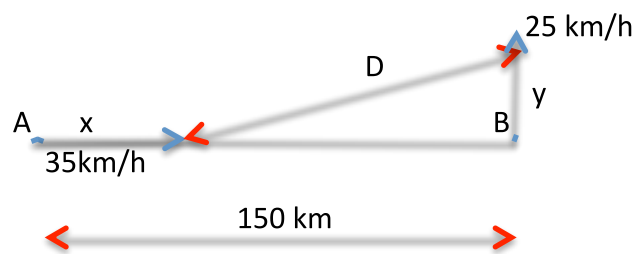
$$-1 = 8\pi \cdot 5\frac{dr}{dt}$$

So that, $\frac{dr}{dt} = -\frac{1}{40\pi}$ cm/min and $\frac{dD}{dt} = -\frac{1}{20\pi}$.

Example. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 P.M.?

Solution: Let x be the distance ship A travels, y the distance ship B travels after noon, and D the distance between at time t hours after noon. Then $\frac{dx}{dt} = 35$, $\frac{dy}{dt} = 25$, and we want $\frac{dD}{dt}$ when $t = 4$. From the diagram below we get,

$$D^2 = (150 - x)^2 + y^2$$



Differentiating with respect to t, we get

$$2D \cdot \frac{d(D)}{dt} = -2(150 - x) \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$$

or

$$D \cdot \frac{d(D)}{dt} = (x - 150) \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt}$$

When $t = 4, x = 140, y = 100$, and $D = \sqrt{10^2 + 100^2} = 100.5$. Therefore,

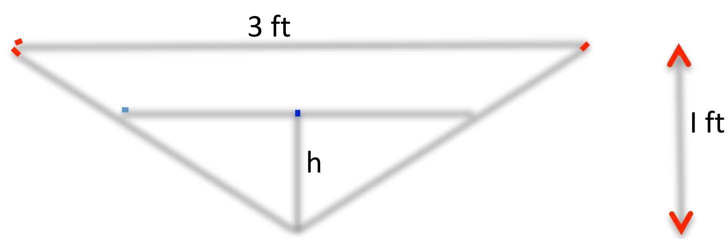
$$100.5\frac{dD}{dt} = -10 \cdot 35 + 100 \cdot 25$$

Hence,

$$\frac{dD}{dt} = 21.39 \text{ km/h}$$

Example. A trough is 10 ft. long and its ends have the shape of isosceles triangles that are 3 feet across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 inches deep?

Solution: Let h be the height of the trough and V its volume. Then we have $\frac{dV}{dt} = 12$ and we want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$.



The volume of water in the trough is 10 times the area of the triangle of height h above. Note that the isosceles triangle of height h and that of height 1 above are similar, with proportionality constant $\frac{h}{1} = h$. Therefore, the triangle of height h has area h^2 times the area of the triangle of height 1. Therefore, the area of the triangle of height h is $\frac{3}{2}h^2$, which implies

$$V = \frac{3}{2}h^2 \cdot 10 = 15h^2$$

Differentiating both sides with respect to t gives

$$\frac{dV}{dt} = 30h\frac{dh}{dt}$$

Substituting, we get

$$12 = 30 \cdot \left(\frac{1}{2}\right)\frac{dh}{dt}$$

Hence,

$$\frac{dh}{dt} = \frac{4}{5}\ \text{ft/min}$$