CALCULUS NOTES

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1. Functions and Models.

1.1. Four Ways to Represent a Function.

Definition 1.1.1. A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of f, and the **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

A function can be thought of as a machine. For example, a machine that makes pies (e.g. if you put in apples, you get an apple pie; you don't also get a peach pie or a cherry pie).

The four possible ways to represent a function are: verbally (by a description in words), numerically (by a table of values), visually (by a graph), and algebraically (by an explicit formula).

Example 1.1.1 (Using function notation)**.** If $f(x) = x^2 + 5x$, evaluate the quotient $\frac{f(a+h)-f(a)}{h}$.

Solution. $\frac{f(a+h)-f(a)}{h} = \frac{[(a+h)^2+5(a+h)]-[a^2+5a]}{h} = \frac{[a^2+2ah+h^2+5a+5h]-[a^2+5a]}{h} = \frac{2ah+h^2+5h}{h} = 2a+h+5.$

Example 1.1.2 (Domain)**.** Identify the domain of the following functions:

(a) $f(x) = 1 - x^7 + 12x^2$.
(b) $g(x) = \sqrt{x+2}$.
(c) $h(x) = \frac{1}{x^2-x}$.

Solution.

(a) The domain of a polynomial is all real numbers, so the domain of f is $(-\infty, \infty)$ or $\mathbb{R}$.
(b) The radicand can't be negative, so $x + 2 \geq 0$ implies that the domain of g is $[-2, \infty)$.
(c) The denominator can't be zero, so $x^2 - x \neq 0$ implies that the domain of h is $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$.

Proposition 1.1.1 (Vertical Line Test)**.** *A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects it more than once.*

Example 1.1.3. Illustrate the use of the Vertical Line Test by comparing the graphs of several functions and non-functions.

A **piecewise defined function** is one that is defined by different formulas over different parts of its domain.

Example 1.1.4. Consider the piecewise function $f(x) = \begin{cases} 4 - x, & x < 1; \\ x^2, & 1 \leq x < 2. \end{cases}$ Find $f(0)$, $f(1)$, $f(3)$.

Solution. $f(0) = 4 - 0 = 4$, $f(1) = 1^2 = 1$, $f(3)$ is not defined because 3 is not in the domain of f.

It is straightforward to graph the function in the example above. Conversely, given a graph of a simple piecewise function, we should be able to recover its explicit formula.

Example 1.1.5. Sketch a simple piecewise function and use its graph to recover its formula.

Definition 1.1.2 (Function symmetry)**.**

- A function f is **even** if $f(-x) = f(x)$ for every x in its domain. If a function is even, its graph is symmetric about the x-axis.
- A function f is **odd** if $f(-x) = -f(x)$ for every x in its domain. If a function is odd, its graph is symmetric about the y-axis.

Example 1.1.6. Determine whether each of the following functions is even, odd, or neither:
(a) $f(x) = x^5 - 7x$.
(b) $g(x) = \frac{6+x^4}{x^2}$.
(c) $h(x) = 1 + x^3 + x^2$.

Solution. (a) odd, (b) even, (c) neither. The symmetry of polynomials is easy to determine!

Definition 1.1.3 (Increasing and decreasing functions)**.**

- A function f is **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$. Graphically, an increasing function "rises" from left to right.
- A function f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$. Graphically, a decreasing function "falls" from left to right.

1.2. **Mathematical Models: A Catalog of Essential Functions.**

Definition 1.2.1. A function P is called a **polynomial** if it has the form $P(x) = a_n x^n + \cdots + a_1 x + a_0$, where n is a nonnegative integer and the numbers $a_0, a_1, \ldots, a_n$ are constants called the **coefficients** of the polynomial. If the leading coefficient a_n is nonzero, then the **degree** of the polynomial is n. The domain of any polynomial is $\mathbb{R}$.

Example 1.2.1. Sketch various simple polynomials, e.g. linear, quadratic, cubic, etc.

A **linear function** is one whose graph is a straight line. Linear functions are commonly written in **slope-intercept form**. For example, if y is a linear function of x, we may write the relationship between the variables as $y = mx + b$, where m is the slope of the line and b is the y-intercept.

The slope of a linear function represents a rate of change of the dependent variable with respect to the independent variable. For example, in a real-life linear model, slope could represent the rate of change of temperature with respect to elevation.

The y-intercept generally represents some sort of initial state of the model. This is especially true when the independent variable represents time, such as in the case of population models or depreciation models.

A polynomial of degree 2 is called a **quadratic function**. Its graph is a parabola, so it can be used to model physical scenarios such as the height of an object with respect to time after it is thrown up in the air.

Definition 1.2.2. A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**.

Example 1.2.2. The general power function $f(x) = x^a$ can take on many forms:

- If $a = n$, where n is a positive integer, the function f is a simple polynomial.
- If $a = 1/n$, where n is a positive integer, then $f(x) = x^{1/n} = \sqrt[n]{x}$, is a **root function**.
- If $a = -1$, then $f(x) = x^{-1} = 1/x$ is the **reciprocal function**.

Definition 1.2.3. A **rational function** f is a ratio of two polynomial functions, e.g. $f(x) = P(x)/Q(x)$.

Example 1.2.3. A simple rational function is the reciprocal function $f(x) = 1/x$. Rational functions arise in physics and chemistry, such as in the case of Boyle's Law: when temperature is constant, the volume V of a gas is inversely proportional to the pressure P; in symbols, $V = C/P$.

Definition 1.2.4. A function f is called an **algebraic function** if it can be constructed using **algebraic operations**, i.e. the four arithmetic operations $+$, $-$, $\times$, $\div$, along with the taking of roots.

Example 1.2.4. An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity v is given by $m = f(v) = \frac{m_0}{1-v^2/c^2}$, where m_0 is the rest mass of the particle and $c = 3.0 \times 10^5$ km/sec is the speed of light in a vacuum.

Trigonometric functions such as sine, cosine, and tangent can also be used in mathematical modeling. The periodic nature of these functions makes them ideal representatives of repetitive phenomena such as vibrating springs and sound waves.

Exponential and logarithmic functions will be reviewed in Sections 1.5 and 1.6 respectively.

Remark 1.2.1. Choosing an appropriate mathematical model is the most difficult part of many practical problems, especially those in which the goal is to make accurate predictions about the future. Graphical tools such as scatter plots can be helpful for visualizing trends in the available data, and in turn these trends can hint at choosing one particular type of model over another. Many calculators offer the capability to analyze scatter plots and produce linear or quadratic regressions (i.e. "best-fit" lines).

1.3. **New Functions from Old Functions.**

Definition 1.3.1. There are ten basic transformations that can be used to modify functions. We can often interpret a seemingly complex function as some variant of a simple "parent function," arrived at by applying one or more of these transformations.

- Vertical and Horizontal Shifts: suppose $c > 0$.
  1. $y = f(x) + c$ is a vertical shift of $y = f(x)$ a distance of c units upward.
  2. $y = f(x) - c$ is a vertical shift of $y = f(x)$ a distance of c units downward.
  3. $y = f(x + c)$ is a horizontal shift of $y = f(x)$ a distance of c units to the left.
  4. $y = f(x - c)$ is a horizontal shift of $y = f(x)$ a distance of c units to the right.
- Vertical and Horizontal Stretching: suppose $c > 1$.
  5. $y = cf(x)$ is a vertical stretch of $y = f(x)$ by a factor of c.
  6. $y = (1/c)f(x)$ is a vertical shrink of $y = f(x)$ by a factor of c.
  7. $y = f(cx)$ is a horizontal shrink of $y = f(x)$ by a factor of c.
  8. $y = f(x/c)$ is a horizontal stretch of $y = f(x)$ by a factor of c.
- Reflecting:
  9. $y = -f(x)$ is a reflection of $y = f(x)$ across the x-axis.
  10. $y = f(-x)$ is a reflection of $y = f(x)$ across the y-axis.

Example 1.3.1. Illustrate the 10 transformations on a simple function such as $y = x^2$ or $y = \sin x$

Definition 1.3.2. To form an **arithmetic combination** of two functions, f and g, we simply apply one of the four arithmetic operations. The rules are exactly as expected:

- Sums: $(f + g)(x) = f(x) + g(x)$.
- Differences: $(f - g)(x) = f(x) - g(x)$.
- Products: $(fg)(x) = f(x)g(x)$.
- Quotients: $(f/g)(x) = f(x)/g(x)$, provided $g(x) \neq 0$.

Example 1.3.2. Let $f(x) = x^2 - 1$ and $g(x) = 2x + 1$. Find (a) $f + g$, (b) $f - g$, (c) fg, (d) f/g, and their domains.

Solution. The functions $f + g$, $f - g$, and fg each have domain $\mathbb{R}$. The domain of f/g is $2x + 1 \neq 0$; that is, $(-\infty, -1/2) \cup (-1/2, \infty)$. The rules for each function are:

(a) $(f + g)(x) = f(x) + g(x) = (x^2 - 1) + (2x + 1) = x^2 + 2x$.

(b) $(f-g)(x) = f(x) - g(x) = (x^2-1) - (2x+1) = x^2 - 2x - 2$.
(c) $(f*g)(x) = f(x)g(x) = (x^2-1)(2x+1) = 2x^3 + x^2 - 2x - 1$.
(d) $(f/g)(x) = f(x)/g(x) = (x^2-1)/(2x+1)$.

Definition 1.3.3 (Function composition)**.** Given two functions, f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by $(f \circ g)(x) = f(g(x))$ for all x in the domain of g such that $g(x)$ is in the domain of f. That is, $(f \circ g)(x)$ is defined if and only if $g(x)$ and $f(g(x))$ are defined.

Example 1.3.3. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. Find the domain and formulas for $f \circ g$ and of $g \circ f$.

Solution. First note that the domain of f is $(-\infty, \infty)$ and the domain of g is $[0, \infty)$. It follows that the domain of $f \circ g$ is $[0, \infty)$ and the domain of $g \circ f$ is $(-\infty, \infty)$. The algebraic formulas are $(f \circ g)(x) = x$ and $(g \circ f)(x) = |x|$.

Example 1.3.4. Use the table below to find: (a) $f(g(0))$, (b) $f(f(2))$, (c) $(g \circ f)(1)$.

x	-2	-1	0	1	2
$f(x)$	6	5	0	-2	-1
$g(x)$	-3	-1	1	3	5

Solution. (a) $f(g(0)) = f(1) = -2$, (b) $f(f(2)) = f(-1) = 5$, (c) $(g \circ f)(1) = g(f(1)) = g(-2) = -3$.

1.4. **Graphing Calculators and Computers.**

There are no notes provided for this section. It essentially discusses some ways in which graphing calculators (and computers) can be useful in the study of calculus. Students in Math 121 are expected to have prior experience with graphing calculators (esp. the TI-83 or TI-84 series), and those that do not may arrange a meeting with the instructor in order to be brought up to speed.

1.5. **Exponential Functions.**

Definition 1.5.1. An **exponential function** has the form $f(x) = a^x$, where a is a positive constant.

Example 1.5.1. Graph and compare the function $f(x) = a^x$ for: (a) $0 < a < 1$, (b) $a = 1$, (c) $a > 1$.

Exponential functions are ideal for modeling populations that grow very rapidly. For example, if a bacterial culture initially contains P_0 bacteria and is known to double in size every half-hour, its growth can be modeled by the exponential function $P(t) = P_0\, 2^t$, where $t \geq 0$ is the elapsed time in half-hours.

Remark 1.5.1.

- The domain of $f(x) = a^x$ is $\mathbb{R}$ for any $a > 0$.
- If $x = n$ is a positive integer, then: $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}}$.
- If $x = 0$, then $a^0 = 1$.
- If $x = -n$, where n is a positive integer, then $a^{-n} = \dfrac{1}{a^n}$.
- If p and q are integers, with $q > 0$, then $a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.

Example 1.5.2. Simplify: (a) $64^{-2/3}$, (b) $(-16)^{3/4}$.

Solution.
(a) $64^{-2/3} = \frac{1}{64^{2/3}} = \frac{1}{\left(\sqrt[3]{64}\right)^2} = \frac{1}{(4)^2} = \frac{1}{16}$.

(b) $(-16)^{3/4} = \left(\sqrt[4]{16}\right)^3 \cdot (-1)^{3/4} = (2)^3 \cdot (-1)^{3/4} = 8 \cdot (-1)^{3/4}$.

Example 1.5.3. What is the meaning of a^x when x is an irrational number? For example, what is $3^{\sqrt{2}}$?

Proposition 1.5.1 (Laws of Exponents)**.** *If a and b are positive numbers and $x, y \in \mathbb{R}$, then:*

$$\textit{1. } a^{x+y} = a^x a^y, \quad \textit{2. } a^{x-y} = \frac{a^x}{a^y}, \quad \textit{3. } (a^x)^y = a^x y, \quad \textit{4. } (ab)^x = a^x b^x.$$

Example 1.5.4. Use the laws of exponents to rewrite and simply the expression $\frac{\left(6y^3\right)^4}{2y^5}$.

Solution. It is straightforward to determine that: $\frac{\left(6y^3\right)^4}{2y^5} = \frac{6^4\left(y^3\right)^4}{2y^5} = \frac{1296y^{12}}{2y^5} = 648y^7$.

Definition 1.5.2. The **natural exponential function**, $f(x) = e^x$, is ubiquitous in mathematics. It is characterized by the **natural base** $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \approx 2.71828$, which is a transcendental number (not a root of a nonzero polynomial with integer coefficients).

Example 1.5.5. The function $f(x) = e^x$ has the property that the slope of its tangent line at the point $(0, 1)$ is exactly 1. On the other hand, the functions $g(x) = 2^x$ and $h(x) = 3^x$ have tangential slopes of about 0.7 and 1.1 respectively at the point $(0, 1)$. Graph all three curves and note how the former lies between the latter two.

1.6. Inverse Functions and Logarithms.

Definition 1.6.1. A function f is called a **one-to-one function** if it never takes on the same value twice; that is: $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Proposition 1.6.1 (Horizontal Line Test)**.** *A function is one-to-one if and only if no horizontal line intersects its graph more than once.*

Example 1.6.1. Illustrate the use of the HLT by comparing the graphs of functions that are/aren't one-to-one.

Definition 1.6.2. Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by $f^{-1}(y) = x \iff f(x) = y$ for any $y \in B$.

We use the notation $f : A \to B$ to denote a function with domain A and range B. Thus, if $f : A \to B$ is a one-to-one function, its inverse is written $f^{-1} : B \to A$.

Remark 1.6.1.

- The function $f : A \to B$ is one-to-one if $f(x) = f(y)$ implies $x = y$ for any $x, y \in A$.
- The graph of f^{-1} is obtained by reflecting the graph of f across the line $y = x$.

Example 1.6.2. It follows from the definition that if the functions $f : A \to B$ and $f^{-1} : B \to A$ are inverses, then $f^{-1}(f(x)) = x$ for every $x \in A$ and $f(f^{-1}(x)) = x$ for every $x \in B$. For example, this property is easily verified in the case of the functions $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$.

In view of the preceding example, and the comment made in Section 1.1 about a function acting like a machine, we can think of an inverse function as a machine that works in reverse, i.e. it disassembles or undoes the result of the original function.

Definition 1.6.3. Let a be a positive constant. It is easy to verify that the exponential function $f(x) = a^x$ is one-to-one, i.e. its inverse exists. The **logarithmic function with base a**, denoted $\log_a$, is uniquely defined by the rule $\log_a x = y \iff a^y = x$.

Remark 1.6.2.
- The domain of $f(x) = \log_a x$ is $(0, \infty)$ for any $a > 0$. The range is $\mathbb{R}$.
- $\log_a (a^x) = x$ for every $x \in \mathbb{R}$.
- $a^{\log_a x} = x$ for every $x > 0$.

Example 1.6.3. Graphically illustrate the inverse relationship between exponential and logarithmic functions.

Proposition 1.6.2 (Laws of Logarithms)**.** *If a, x, and y are positive numbers and $r \in \mathbb{R}$, then:*

$$\textit{1. } \log_a(xy) = \log_a x + \log_a y, \quad \textit{2. } \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y, \quad \textit{3. } \log_a(x^r) = r\log_a x.$$

Example 1.6.4. Simplify $\log_3 36 - \log_3 4$.

Solution. $\log_3 36 - \log_3 4 = \log_3\left(\frac{36}{4}\right) = \log_3 9 = \log_3 3^2 = 2$.

Definition 1.6.4. There are two "special" logarithms that each have a unique name and notation:
1. The logarithm with base 10 is called the **common logarithm**, and is denoted by $\log x$.
2. The logarithm with base e is called the **natural logarithm**, and is denoted by $\ln x$.

Remark 1.6.3.
- By definition, $\ln x = y \iff e^y = x$.
- $\ln(e^x) = x$ for every $x \in \mathbb{R}$.
- $e^{\ln x} = x$ for every $x > 0$.
- From the inverse relationship between the exponential and logarithmic functions it follows that $a^b = e^{\ln\left(a^b\right)} = e^{b\ln a}$ for any numbers a and b, which is a useful tool in some differentiation problems.

Example 1.6.5. Find x if $\ln(x-5) = 3$.

Solution. $\ln(x-5) = 3 \iff x - 5 = e^3 \iff x = 5 + e^3$.

Example 1.6.6. Solve the equation $e^{7+2x} - 6 = 5$.

Solution. $e^{7+2x} - 6 = 5 \iff e^{7+2x} = 11 \iff 7 + 2x = \ln 11 \iff x = (-7 + \ln 11)/2$.

Example 1.6.7. Express $2\ln x - 3\ln y$ as a single logarithm.

Solution. $2\ln x - 3\ln y = \ln x^2 - \ln y^3 = \ln\left(x^2/y^3\right)$.

Proposition 1.6.3 (Change of Base Formula)**.** *For any positive a ($a \neq 1$), we have:* $\log_a x = \ln x/\ln a = \log x/\log a$.

1.7. **Parametric Curves.**

Definition 1.7.1. Supppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations $x = f(t)$ and $y = g(t)$ (called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve**.

Example 1.7.1. Consider the parametric equations $x = t^2 - 2t$ and $y = t + 1$.

(a) Plot points to sketch a graph of the curve.
(b) Eliminate the parameter to find a Cartesian equation of the curve.

Solution.

(a) Plotting the points (x, y) corresponding to $t = -2, -1, 0, 1, 2, 3, 4$, we sketch a horizontal parabola.
(b) First, we solve $y = t + 1$ for t. We then substitute $t = y - 1$ for t in $x = t^2 - 2t$; that is, $x = (y-1)^2 - 2(y-1) = y^2 - 2y + 1 - 2y + 2 = y^2 - 4y + 3$, which supports the result of part (a).

Example 1.7.2. Identify the curve represented by $x = \cos t$ and $y = \sin t$, where $0 \le t \le 2\pi$.

Solution. The ordered pairs $(x, y) = (\cos t, \sin t)$ correspond to points on the graph of a circle. This is easily verified by eliminating the parameter t; in particular, observe that $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. As t increases from 0 to 2π, the point (x, y) moves once around the unit circle in the counterclockwise direction starting from the point $(1, 0)$.

Example 1.7.3. What curve is represented by $x = \cos 2t$ and $y = \sin 2t$, where $0 \le t \le 2\pi$?

Solution. As in the previous example, $x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1$. In this case, however, as t increases from 0 to 2π the point (x, y) starts at $(1, 0)$ and moves twice around the unit circle in the counterclockwise direction.

Example 1.7.4. Find parametric equations for the circle with center (h, k) and radius r?

Solution. We begin with parametric equations for the unit circle, i.e. $x = \cos t$ and $y = \sin t$ for $0 \le t \le 2\pi$. To scale the radius of the circle to r, we multiply both x and y by r. To center the circle at the point (h, k), we add h to x and k to y. Hence, the circle centered at (h, k) with radius r can be represented by the parametric equations $x = h + r\cos t$ and $y = k + r\sin t$, where $0 \le t \le 2\pi$.

Most graphing calculators and computer graphing programs can be used to graph curves defined by parametric equations. In fact, it's instructive to watch a parametric curve being drawn by such a digital graphing tool because the points are plotted in order as the corresponding parameter values increase.

Example 1.7.5. Use a calculator to graph the curve $x = y^4 - 3y^2$.

Solution. If $t = y$ is a parameter, then we have the parametric equations $x = t^4 - 3t^2$ and $y = t$.

The example above illustrates the importance of choosing an appropriate range and increment-value for t. If the range is too small, then the true shape of the curve may not be fully revealed. If the increment-value is too large, the curve may have sharp or square edges. If the range is too large, or the increment-value is too small, the curve will be plotted very slowly as the calculator attempts to display a high level of detail.

Example 1.7.6. Use a calculator to plot complicated curves, such as those illustrated in Exercise 1.7.33.

2. Limits and Derivatives.

2.1. The Tangent and Velocity Problems.

Definition 2.1.1. A **tangent line** at a point P on a curve is a line that passes through P and approximates the curve as perfectly as possible near P.

Consider a nonlinear curve such as a parabola. The more we zoom in on a particular point P on the curve, the more the curve looks like a straight line passing through P.

To approximate the slope of the tangent line at P, we choose a point Q on the curve and calculate the slope of the **secant line** passing through P and Q. The closer Q is to P, the better the slope of the secant line approximates the slope of the tangent line at P. For this reason, we say that the slope of the tangent line to the curve at P is the limit as Q approaches P of the slope of the secant line through P and Q.

Example 2.1.1. Use secant lines to estimate the slope of the tangent line to $f(x) = x^2$ at $P = (1, 1)$.

Solution. We choose x-values closer and closer to $x = 1$, and determine the slope, m_{PQ}, of the secant line through P and $Q = (x, f(x))$ as follows:

$(x, f(x))$	m_{PQ}
$(2, 4)$	$\frac{4-1}{2-1} = 3$
$(1.5, 2.25)$	$\frac{2.25-1}{1.5-1} = 2.5$
$(1.1, 1.21)$	$\frac{1.21-1}{1.1-1} = 2.1$
$(1.01, 1.0201)$	$\frac{1.0201-1}{1.01-1} = 2.01$

It appears that the slope of the tangent line to the graph of $f(x) = x^2$ at the point $(1, 1)$ is $m = 2$. Hence, the equation of the tangent line is $y - 1 = 2(x - 1)$ or $y = 2x - 1$. This is easily verified graphically.

The Velocity Problem. We commonly understand velocity as the rate of change of position with respect to time, but what is the meaning of the **instantaneous velocity** at any particular point in time?

Example 2.1.2. A ball is dropped from 500 feet. Find its (instantaneous) velocity after 5 seconds.

Solution. After t seconds, the ball has dropped a distance of $s(t) = 32t^2$ feet. To approximate the velocity of the ball at $t = 5$ seconds, we can use this formula to calculate the average velocity of the ball over smaller and smaller increments of time starting at $t = 5$, as shown below:

Time Interval	Average Velocity (ft/sec)
$5 \leq t \leq 6$	$\frac{s(6)-s(5)}{6-5} = 352$
$5 \leq t \leq 5.1$	$\frac{s(5.1)-s(5)}{5.1-5} = 323.2$
$5 \leq t \leq 5.01$	$\frac{s(5.01)-s(5)}{5.01-5} = 320.32$
$5 \leq t \leq 5.001$	$\frac{s(5.001)-s(5)}{5.001-5} = 320.032$
$5 \leq t \leq 5.0001$	$\frac{s(5.0001)-s(5)}{5.0001-5} = 320.0032$

Note that as the time interval shrinks, the average velocity approaches 320 ft/sec. Thus, we estimate the instantaneous velocity of the falling ball after 5 seconds to be 320 ft/sec.

A Look Ahead... Notice that both of the preceding examples involve finding the limiting value of the quotient $\frac{f(x)-f(a)}{x-a}$ as x gets closer and closer to a. When this **limit** exists, it is called the **derivative of $f(x)$ at $x = a$** and we denote it by $f'(a)$. This concept will be studied in detail, beginning in Section 2.6.

2.2. **The Limit of a Function.**

Definition 2.2.1. We write $\lim_{x\to a} f(x) = L$ and say, "the limit of $f(x)$, as x approaches a, equals L," if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Example 2.2.1. Let $f(x) = \frac{x^2-9}{x-3}$. What is $f(3)$? What is $\lim_{x\to 3} f(x)$?

Solution. It is easy to see that $f(3)$ is undefined. Next, consider the following tables of values of $f(x)$ for values of x close, but not equal, to 3.

x (from the left)	f(x)
2	5
2.9	5.9
2.99	5.99
2.999	5.999

x (from the right)	$f(x)$
4	7
3.1	6.1
3.01	6.01
3.001	6.001

It appears that $\lim_{x\to 3} f(x) = 6$. Indeed, this can be verified graphically, since the graph of $f(x)$ is identical to the graph of the linear function $g(x) = x + 3$ except at the point $x = 3$.

Example 2.2.2. Explore the limiting behavior of a function that exhibits the various common types of discontinuity, e.g. jumps, holes, one- or two-sided asymptotes.

Example 2.2.3. Estimate $\lim_{x\to 0} \frac{1-\cos x}{x}$.

Solution. From the following table of values, it appears that $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

x	$\frac{1-\cos x}{x}$
± 0.1	± 0.04995
± 0.01	± 0.00410
± 0.001	± 0.00050
± 0.0001	± 0.00005

Indeed, this can be verified graphically, or by using L'Hôpital's Rule as seen in Section 4.5.

Example 2.2.4. Estimate $\lim_{x\to 0} \sin \frac{\pi}{x}$.

Solution. Consider the following two tables of values of $f(x)$ for values of x close to 0.

x	f(x)
± 0.1	0
± 0.01	0
± 0.001	0
± 0.0001	0

x	$f(x)$
± 0.4	1
± 0.016	1
± 0.00064	1
± 0.000256	1

From the table on the left we conclude that $\lim_{x\to 0} \sin \frac{\pi}{x} = 0$, but the table on the right suggests that $\lim_{x\to 0} \sin \frac{\pi}{x} = 1$. Which answer is correct? In fact, neither. Since the value of $\lim_{x\to 0} \sin \frac{\pi}{x}$ is dependent on how x is chosen to approach 0, this limit does not exist.

Definition 2.2.2. We write $\lim_{x\to a^-} f(x) = L$, and say the **left-hand limit of $f(x)$ as x approaches a** is equal to L, if we can make the values of $f(x)$ arbitrarily close to L by choosing x to be less than, and sufficiently close to, a. The **right-hand limit of $f(x)$ as x approaches a** is defined similarly, and is denoted by $\lim_{x\to a^+} f(x) = L$.

Example 2.2.5. Explore the one-sided limits of the functions from Example 2.2.2.

Proposition 2.2.1. *For any function f: $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.*

2.3. **Calculating Limits Using the Limit Laws.**

Proposition 2.3.1 (Limit Laws)**.** *If c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then:*

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
3. $\lim_{x\to a}[cf(x)] = c \lim_{x\to a} f(x)$.
4. $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)] \cdot [\lim_{x\to a} g(x)]$.
5. $\lim_{x\to a}[f(x)/g(x)] = [\lim_{x\to a} f(x)] \,/\, [\lim_{x\to a} g(x)]$, *provided that* $\lim_{x\to a} g(x) \neq 0$.
6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, *where n is a positive integer.*
7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, *where n is a positive integer.*

Some special additional laws are:

8. $\lim_{x\to a} c = c$.
9. $\lim_{x\to a} x = a$.
10. $\lim_{x\to a} x^n = a^n$, *where n is a positive integer (special case of Law 6).*
11. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, *where n is a positive integer (special case of Law 7).*

Example 2.3.1. Calculate $\lim_{x\to 5} 4x^3 - 10x^2 - 225$.

Solution. By Limit Laws 2, 3, 8, 9, 10, we find that $\lim_{x\to 5} 4x^3 - 10x^2 - 225 = 25$.

Example 2.3.2. Calculate $\lim_{x\to -2} \frac{x^3+5x+3}{x-7}$.

Solution. By Limit Laws 1, 2, 3, 5, 8, 9, 10, we find that $\lim_{x\to -2} \frac{x^3+5x+3}{x-7} = \frac{5}{3}$.

The following proposition shows that Examples 2.3.1 and 2.3.2 can actually be solved trivially.

Proposition 2.3.2 (Direct Substitution Property)**.** *If f is a polynomial or a rational function, and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$. This is a consequence of **continuity**, as seen in Section 2.4.*

Example 2.3.3. Calculate $\lim_{x\to 2} \frac{2-x}{4-x^2}$.

Solution. In this case, Proposition 2.3.2 does not apply because 2 is not in the domain of $\frac{2-x}{4-x^2}$. Instead, we apply the following algebraic simplification: $\lim_{x\to 2} \frac{2-x}{4-x^2} = \lim_{x\to 2} \frac{2-x}{(2-x)(2+x)} = \lim_{x\to 2} \frac{1}{2+x} = \frac{1}{4}$.

The following proposition justifies the above solution to Example 2.3.3.

Proposition 2.3.3. *If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limits exist.*

Example 2.3.4. Calculate $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} 2x+1, & \text{if } x \neq 3 \\ 4, & \text{if } x = 3 \end{cases}$.

Solution. By Proposition 2.3.3, it suffices to consider the function $g(x) = 2x+1$. Then by Proposition 2.3.2, since g is a polynomial, we conclude that $\lim_{x\to 3} f(x) = \lim_{x\to 3} g(x) = g(3) = 2(3) + 1 = 7$.

Example 2.3.5. Calculate $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} -x+4, & \text{if } x < 3 \\ x, & \text{if } x \geq 3 \end{cases}$.

Solution. The one-sided limits of f are easily seen to be $\lim_{x\to 3^-} f(x) = 1$ and $\lim_{x\to 3^+} f(x) = 3$. Thus, invoking Proposition 2.2.1, we conclude that $\lim_{x\to 3} f(x)$ does not exist.

Example 2.3.6. Calculate $\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2}$.

Solution. Limit Law 5 does not apply, since 0 is not in the domain of the function. Instead, we will employ the method of **rationalizing the numerator** to determine this limit; that is,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2} &= \lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2} \cdot \frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2} \\
&= \lim_{x\to 0} \frac{(x^2+4)-4}{x^2\left(\sqrt{x^2+4}+2\right)} \\
&= \lim_{x\to 0} \frac{x^2}{x^2\left(\sqrt{x^2+4}+2\right)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x^2+4}+2} \\
&= \frac{1}{\sqrt{\lim_{x\to 0}(x^2+4)}+2} \\
&= \frac{1}{\sqrt{4}+2} \\
&= \frac{1}{2+2} \\
&= \frac{1}{4}.
\end{aligned}$$

Example 2.3.7. Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Solution. As in the case of Example 2.3.5, we compute:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{|x|}{x} &= \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1, \\
\lim_{x\to 0^-} \frac{|x|}{x} &= \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1.
\end{aligned}$$

Thus, again by Proposition 2.2.1, we conclude that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Theorem 2.3.1. *If $f(x) \leq g(x)$ when x is near a (except possible at a) and the limits of f and g both exist as x approaches a, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.*

Theorem 2.3.2 (Squeeze Theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possible at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

Example 2.3.8. Show that $\lim_{x\to 0} x^2 \sin \frac{1}{x} = 0$.

Solution. First, recall that $-1 \le \sin x \le 1$ for any x. Likewise, $-1 \le \sin \frac{1}{x} \le 1$ for any x. Since any non-strict inequality remains true when all sides are multiplied by a non-negative number, and we know that $x^2 \ge 0$ for all x, it follows that $-x^2 \le x^2 \sin \frac{1}{x} \le x^2$. Now, notice that $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$. Thus, by the Squeeze Theorem, we conclude that $\lim_{x\to 0} x^2 \sin \frac{1}{x} = 0$.

2.4. **Continuity.**

Definition 2.4.1. A function f is **continuous at a number *a*** if $\lim_{x\to a} f(x) = f(a)$. If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at *a*** (or f has a **discontinuity** at a) if f is not continuous at a.

Remark 2.4.1. Notice that Definition 2.4.1 implicitly requires three things if f is continuous at a:

1. $f(a)$ exists, **2.** $\lim_{x\to a} f(x)$ exists, **3.** $\lim_{x\to a} f(x) = f(a)$.

Geometrically, we can think of a function that is continuous at every number in an interval as a function whose graph has no breaks in it, i.e. the graph can be drawn without removing pen from paper.

Example 2.4.1. Explore the various types of discontinuity illustrated by the graph in Example 2.2.2. In each case, discuss which of the three properties from Remark 2.4.1 are violated.

Example 2.4.2. Where are each of the following functions discontinuous?

(a) $f(x) = \frac{x^2-x-2}{x-2}$.

(b) $f(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \ne 0; \\ 1, & \text{if } x = 0. \end{cases}$

(c) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2}; & \text{if } x \ne 2 \\ 1, & \text{if } x = 2. \end{cases}$

(d) $f(x) = [[x]]$ (the "greatest integer" function).

Solution.

(a) Since $f(2)$ is not defined, f is discontinuous at 2.

(b) Notice that $f(0) = 1$ is defined, but $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2}$ does not exist (because the function has an asymptote at $x = 0$), so f is discontinuous at 0.

(c) Notice that $f(2) = 1$ is defined and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3$$

exists, but $\lim_{x\to 2} f(x) \ne f(2)$, so f is discontinuous at 2.

(d) The function $f(x) = [[x]]$ is defined by: $[[x]]$ *is the largest integer that is less than or equal to* x. For example, $[[4]] = 4$, $[[4.8]] = 4$, $[[\pi]] = 3$, $\left[\left[\sqrt{2}\right]\right] = 1$, and $[[-1/2]] = -1$. It has discontinuuities at all of the integers, because $\lim_{x\to n}[[x]]$ does not exist if n is an integer.

Definition 2.4.2. A function f is **continuous from the right at a number *a*** if $\lim_{x\to a^+} f(x) = f(a)$, and f is **continuous from the left at *a*** if $\lim_{x\to a^-} f(x) = f(a)$.

Example 2.4.3. At each integer n, the function $f(x) = [[x]]$ is continuous from the right but discontinuous from the left, because

$$\lim_{x\to n^+} f(x) = \lim_{x\to n^+} [[x]] = n = f(n)$$

but

$$\lim_{x\to n^-} f(x) = \lim_{x\to n^-} [[x]] = n - 1 \ne f(n).$$

Definition 2.4.3. A function f is **continuous on an interval** if it is continuous at every number in the interval. If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continous from the left.*

Theorem 2.4.1. *If f and g are continuous at a, and c is a constant, then the following functions are also continuous at a:*

$$\textit{1. } f+g, \quad \textit{2. } f-g, \quad \textit{3. } cf, \quad \textit{4. } fg, \quad \textit{5. } f/g \quad (\text{if } g(a) \neq 0).$$

Theorem 2.4.2. *The following types of functions are continuous at every number in their domains: polynomial functions, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions.*

Example 2.4.4. Where is the function $f(x) = \frac{\ln x + 7}{x^2 - 9}$ continuous?

Solution. By Theorem 2.4.2, we know that the logarithmic function $\ln x$ is continuous on $(0, \infty)$, while the constant function 7 and the polynomial function $x^2 - 9$ are both continuous on $\mathbb{R}$. Thus, by parts 1 and 5 of Theorem 2.4.1, it follows that f is continuous at all positive numbers x such that $x^2 - 9 \neq 0$. That is, f is continuous on $(0, 3) \cup (3, \infty)$.

Theorem 2.4.3. *If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words, $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$.*

Example 2.4.5. Find $\lim_{x\to 1} \ln\left(\frac{1-x}{1-\sqrt{x}}\right)$.

Solution. First, we calculate

$$\begin{aligned}
\lim_{x\to 1} \frac{1-x}{1-\sqrt{x}} &= \lim_{x\to 1} \frac{1-x}{1-\sqrt{x}} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} \\
&= \lim_{x\to 1} \frac{(1-x)\left(1+\sqrt{x}\right)}{1-x} \\
&= \lim_{x\to 1} \left(1+\sqrt{x}\right) \\
&= 1 + \sqrt{\lim_{x\to 1} x} \\
&= 2.
\end{aligned}$$

Since 2 is in the domain of the function $\ln x$, it follows from Theorem 2.4.3 that $\lim_{x\to 1} \ln\left(\frac{1-x}{1-\sqrt{x}}\right) = \ln(2)$.

Theorem 2.4.4. *If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$, given by $(f \circ g)(x) = f(g(x))$ is continuous at a.*

Example 2.4.6. Where are the following functions continuous? (a) $f(x) = \cos x^2$, (b) $g(x) = \ln\left(\sin^2 x\right)$.

Solution.

(a) Since $\cos x$ and x^2 are both continuous on $\mathbb{R}$, their composition $f(x) = \cos x^2$ is also continuous on $\mathbb{R}$ by Theorem 2.4.4.

(b) Note that $\sin^2 x$ is continuous on $\mathbb{R}$, and $\ln(x)$ is continuous on its domain, so by Theorem 2.4.4 the function $\ln\left(\sin^2 x\right)$ is continuous wherever it is defined; that is, for all values of x such that $\sin^2 x > 0$. In fact, $\sin^2 x = (\sin x)^2 \geq 0$ for all $x \in \mathbb{R}$, while $\sin x = 0$ if and only if $x = n\pi$ where n is an integer. Thus, $\ln\left(\sin^2 x\right)$ is discontinuous at every integer multiple of π and is continuous on the intervals between these values.

Theorem 2.4.5 (Intermediate Value Theorem)**.** *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.*

Example 2.4.7. Show that the polynomial $f(x) = 3x^4 - 4x^3 + 2x^2 - x - 10$ has a root between 1 and 2.

Solution. Since f is a polynomial, it is continuous at all real numbers; in particular, it is continuous on the interval $(1, 2)$. Since 0 is a number between $f(1) = -10$ and $f(2) = 12$, the Intermediate Value Theorem guarantees the existence of a number $c \in (1, 2)$ such that $f(c) = 0$. That is, the polynomial f has a root between 1 and 2.

2.5. **Limits Involving Infinity.**

Definition 2.5.1 (Infinite Limits)**.** The notation $\lim_{x\to a} f(x) = \infty$ means that the values of $f(x)$ an be made arbitrarily large (as large as we please) by taking x sufficiently close to a (on either side of a) but not equal to a. An analogous statement defines the expression $\lim_{x\to a} f(x) = -\infty$, and likewise the following one-sided infinite limits: $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^-} f(x) = -\infty$, $\lim_{x\to a^+} f(x) = -\infty$.

Remark 2.5.1. It is important to note that in Definition 2.5.1, we do not regard ∞ as a number, nor does it mean that the limit exists. It simply denotes a particular way in which a limit can fail to exist. The expression $\lim_{x\to a} f(x) = \infty$ is often read as "the limit of $f(x)$, as x approaches a, is infinity," or "$f(x)$ becomes infinite as x approaches a," or "$f(x)$ increases without bound as x approaches a."

Example 2.5.1. Find $\lim_{x\to 0} \frac{1}{x^2}$.

Solution. The values of $\frac{1}{x^2}$ are inversely proportional to (the square of) the values of x. In other words, very small values of x correspond to very large values of $\frac{1}{x^2}$. This is seen explicitly from a graphical analysis, or from the following table.

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.01	10,000
± 0.001	1,000,000

Thus, we conclude that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Example 2.5.2. Find $\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$.

Solution. If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number, and $2x$ is close to 6. Thus, the quotient $\frac{2x}{x-3}$ is a large positive number, and we intuitively conclude that $\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$. Similarly, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative number, while $2x$ is still a positive number (close to 6), and hence $\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$.

Definition 2.5.2. The line $x = a$ is called a **vertical asymptote** (V.A.) of the curve $y = f(x)$ if any limit (or one-sided limit) of $f(x)$ as x approaches a is either ∞ or $-\infty$.

Example 2.5.3. Explore the vertical asymptotes of familiar functions such as $f(x) = \ln x$ (V.A. at $x = 0$) and $g(x) = \tan x$ (infinite number of V.A.'s, one at $x = (2n + 1)\pi/2$ for every integer n).

Definition 2.5.3 (Limits at Infinity)**.** Let f be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large. An analogous statement defines the expression $\lim_{x\to-\infty} f(x) = L$.

Example 2.5.4. Note how $\lim_{x\to\infty} \frac{1}{x^2} = 0$, while $\lim_{x\to\infty} \cos x$ does not exist.

Definition 2.5.4. The line $y = L$ is called a **horizontal asymptote** (H.A.) of the curve $y = f(x)$ if the limit of $f(x)$ as x approaches either ∞ or $-\infty$ is equal to L.

Example 2.5.5. Explore the horizontal asymptotes of familiar functions such as $f(x) = e^x$ (H.A. at $y = 0$) and $g(x) = \arctan x$ (H.A.'s at $y = -\pi/2$ and $y = \pi/2$).

Definition 2.5.5 (Infinite Limits at Infinity)**.** The notation $\lim_{x\to\infty} f(x) = \infty$ is used to indicate that the values of $f(x)$ become arbitrarily large as x becomes arbitrarily large. Similar meanings are attached to each of the following expressions: $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = -\infty$.

Example 2.5.6. Many familiar functions exhibit infinite limiting behavior at infinity, e.g.

$$\lim_{x\to\infty} e^x = \infty, \ \lim_{x\to-\infty} x^2 = \infty, \ \lim_{x\to-\infty} x^3 = -\infty.$$

Example 2.5.7. Find $\lim_{x\to0^-} e^{1/x}$ and $\lim_{x\to0^+} e^{1/x}$.

Solution. Following Example 2.5.1, it is easy to verify that $\lim_{x\to0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to0^+} \frac{1}{x} = \infty$. Letting $t = 1/x$ and using the results of Examples 2.5.5 and 2.5.6 respectively, it follows that:

$$\lim_{x\to0^-} e^{1/x} = \lim_{t\to-\infty} e^t = 0 \ \text{ and } \ \lim_{x\to0^+} e^{1/x} = \lim_{t\to\infty} e^t = \infty.$$

Proposition 2.5.1. *If n is a positive integer, then* $\lim_{x\to-\infty} \frac{1}{x^n} = 0$ *and* $\lim_{x\to\infty} \frac{1}{x^n} = 0$.

Example 2.5.8. Evaluate each of the following limits.

(a) $\lim_{x\to\infty} \frac{3x^2+9x+27}{5x^3-7x+6}$.

(b) $\lim_{x\to-\infty} \frac{3x^4+9x+27}{5x^3-7x+6}$.

(c) $\lim_{x\to-\infty} \frac{3x^3+9x+27}{5x^3-7x+6}$.

(d) $\lim_{x\to\infty} \frac{3x^2+9x+27}{\sqrt{16x^4+11x^3-12x+100}}$.

(e) $\lim_{x\to\infty} \left(\sqrt{x^2+4} - x\right)$.

Solution. We will use Proposition 2.5.1 together with Definitions 2.5.3 and 2.5.5.

(a) Both the numerator and denominator appear to approach infinity as x does. The largest power of x that appears in the expression is x^3. Multiplying both the numerator and denominator by $1/x^3$, we obtain

$$\lim_{x\to\infty} \frac{3x^2+9x+27}{5x^3-7x+6} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{3}{x} + \frac{9}{x^2} + \frac{27}{x^3}}{5 - \frac{7}{x^2} + \frac{6}{x^3}} = 0.$$

(b) This time we multiply both the numerator and denominator by $1/x^4$, simplify, and conclude that

$$\lim_{x\to-\infty} \frac{3x^4+9x+27}{5x^3-7x+6} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim_{x\to-\infty} \frac{3+\frac{9}{x^3}+\frac{27}{x^4}}{\frac{5}{x}-\frac{7}{x^3}+\frac{6}{x^4}} = \infty.$$

(c) Again multiplying both the numerator and denominator by $1/x^3$, we have

$$\lim_{x\to\infty} \frac{3x^3+9x+27}{5x^3-7x+6} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{3+\frac{9}{x^2}+\frac{27}{x^3}}{5-\frac{7}{x^2}+\frac{6}{x^3}} = \frac{3}{5}.$$

(d) Notice that $x^2 = \sqrt{x^4}$ is the largest power of x that appears in the expression. Multiplying both the numerator and denominator by $1/x^2 = 1/\sqrt{x^4}$ yields

$$\lim_{x\to\infty} \frac{3x^2+9x+27}{\sqrt{16x^4+11x^3-12x+100}} \cdot \frac{\frac{1}{x^2}}{\frac{1}{\sqrt{x^4}}} = \lim_{x\to\infty} \frac{3+\frac{9}{x}+\frac{27}{x^2}}{\sqrt{16+\frac{11}{x}-\frac{12}{x^3}+\frac{100}{x^4}}} = \frac{3}{\sqrt{16}} = \frac{3}{4}.$$

(e) We first multiply by the conjugate of $\left(\sqrt{x^2+4}-x\right)$ over itself, and then multiply both the numerator and denominator of the resulting expression by $1/x = 1/\sqrt{x^2}$, in order to obtain

$$\lim_{x\to\infty} \left(\sqrt{x^2+4}-x\right) \cdot \frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}+x} = \lim_{x\to\infty} \frac{\left(x^2+4\right)-x^2}{\sqrt{x^2+4}+x} \cdot \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{\frac{4}{x}}{\sqrt{1+\frac{4}{x^2}}+1} = 0.$$

Remark 2.5.2. Parts (a) - (c) of Example 2.5.8 illustrate special cases of a very general result. Namely, if $f(x) = P(x)/Q(x)$, where P and Q are polynomial functions of degree m and n with leading coefficients a and b respectively, then f exhibits the one of the three following types of limiting behavior:

1. $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$, if $m < n$.
2. $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$, if $m > n$.
3. $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = a/b$, if $m = n$.

2.6. **Derivatives and Rates of Change.**

Definition 2.6.1. The **derivative of a function f at a number a**, denoted by $f'(a)$, is defined by $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$, provided that the limit exists.

Remark 2.6.1. As alluded to in Section 2.1, the derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$. In particular, $f'(a)$ represents:

1. The **slope of the tangent line** to the graph of $y = f(x)$ at the point $(a, f(a))$ on the curve.
2. The **instantaneous velocity** at time a, when $f(x)$ is a position function.

Example 2.6.1. Find the equation of the tangent line to the curve $f(x) = \frac{2}{x+1}$ at the point $(0, 2)$.

Solution. The slope of the tangent line to $f(x)$ at $(0, 2)$ is given by the derivative

$$f'(0) = \lim_{x\to0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0} \frac{\frac{2}{x+1}-2}{x} = \lim_{x\to0} \frac{\frac{2-2(x+1)}{x+1}}{x} = \lim_{x\to0} \frac{-2x}{x(x+1)} = \lim_{x\to0} \frac{-2}{x+1} = -2.$$

Hence, the equation of the tangent line at the point $(0, 2)$ is given by $y-2 = -2(x-0)$, or $y = -2x+2$.

Example 2.6.2. The displacement of a particle moving in a straight line is given by $s(t) = \sqrt{t+16}-4$. Find the velocity of the particle at $t = 9$, $t = 20$, and $t = 33$.

Solution. First, we find a general formula for $s'(a)$.

$$\begin{aligned}
s'(a) &= \lim_{t\to a} \frac{s(t)-s(a)}{t-a} \\
&= \lim_{t\to a} \frac{\left(\sqrt{t+16}-4\right)-\left(\sqrt{a+16}-4\right)}{t-a} \\
&= \lim_{t\to a} \frac{\sqrt{t+16}-\sqrt{a+16}}{t-a} \cdot \frac{\sqrt{t+16}+\sqrt{a+16}}{\sqrt{t+16}+\sqrt{a+16}} \\
&= \lim_{t\to a} \frac{(t+16)-(a+16)}{(t-a)\left(\sqrt{t+16}+\sqrt{a+16}\right)} \\
&= \lim_{t\to a} \frac{(t-a)}{(t-a)\left(\sqrt{t+16}+\sqrt{a+16}\right)} \\
&= \lim_{t\to a} \frac{1}{\sqrt{t+16}+\sqrt{a+16}} \\
&= \frac{1}{2\sqrt{a+16}}.
\end{aligned}$$

Then it is easy to see that $s'(9) = 1/10$, $s'(20) = 1/12$, and $s'(33) = 1/14$.

2.7. **The Derivative as a Function.**

In Section 2.6, we considered the derivative of a function f at a fixed number a, defined by

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

If we let $x = a+h$ and substitute above for x, we obtain the alternative formula

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Since in this discussion a is arbitrary, it is reasonable to suppose that a is actually a variable quantity. In fact, this is a particularly useful generalization; it allows us to regard the derivative f' as a new function, derived from f by one of the equivalent limiting operations above. In general, with a replaced by the variable x, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Example 2.7.1. Let $f(x) = x^3 - x$. Calculate $f'(x)$ and determine the slope of the tangent line to the curve f when $x = 1$ and $x = 2$.

Solution. By the formula introduced above, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\left((x+h)^3-(x+h)\right)-\left(x^3-x\right)}{h} \\
&= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x-h-x^3+x}{h} \\
&= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3-h}{h} \\
&= \lim_{h\to 0} 3x^2+3xh+h^2-1 \\
&= 3x^2-1.
\end{aligned}$$

Then $f'(1) = 2$ and $f'(2) = 11$ represent the slope of the tangent line to f at $x = 1$ and $x = 2$ respectively.

Example 2.7.2. Let $f(x) = 1/x$. Find a formula for $f'(x)$.

Solution. As in the previous example, we carefully compute $f'(x)$ using the limit definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

If we use $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x).$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of differentiation. The symbol dy/dx, introduced by Leibniz, is especially useful and will be substituted frequently for the "prime" notation. If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a, we write $dy/dx|_{x=a}$, which is synonymous with $f'(a)$.

Definition 2.7.1. A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Theorem 2.7.1. *If f is differentiable at a, then f is continuous at a.*

Example 2.7.3. Illustrate the common graphical features that cause a function not to be differentiable, e.g. corners, discontinuities, and vertical tangents.

Remark 2.7.1. Example 2.7.3 motivates the vital observation that the converse of Theorem 2.7.1 is false! That is, a function can be continuous but not differentiable at a point. In fact, there exist functions defined on $\mathbb{R}$ that everywhere continuous and yet nowhere differentiable.

Definition 2.7.2 (Higher Derivatives)**.** Suppose $y = f(x)$ is a differentiable function. Since f' is also a function, it may have a derivative of its own. This new function, denoted by f'', is called the **second derivative** of f because it is the derivative of the derivative of f. In Leibniz notation, we write the second derivative of f as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

The **third derivative**, denoted by f''' or $\frac{d^3y}{dx^3}$, is the derivative of the second derivative, and so on. For $n \geq 4$, as the prime notation becomes impractical, we generally denote the nth derivative by $f^{(n)}$ or $\frac{d^ny}{dx^n}$.

Remark 2.7.2. We have already observed that if $s(t)$ is a position function, then $v(t) = s'(t)$ represents velocity. Furthermore, $a(t) = v'(t) = s''(t)$ represents acceleration, i.e. the rate of change of velocity.

2.8. What Does f' Say About f?

Proposition 2.8.1. *Suppose that f is a differentiable function. The sign of its derivative, f', provides the following information about the behavior of the graph of f:*

- *If $f'(x) > 0$ on an interval, then f is increasing on that interval.*
- *If $f'(x) < 0$ on an interval, then f is decreasing on that interval.*

An important issue that is not addressed by Proposition 2.8.1 is what happens when $f'(x) = 0$. In this case, the tangent line to the graph of f through the point $(x, f(x))$ has a slope of zero. In particular, the function f has a local minimum or a local maximum at that point, as we will discuss further in Section 4.2.

Proposition 2.8.2. *Suppose that f is a twice-differentiable function. The sign of its second derivative, f'', provides the following information about the behavior of the graph of f:*

- *If $f''(x) > 0$ on an interval, then f is **concave upward** on that interval.*
- *If $f''(x) < 0$ on an interval, then f is **concave downward** on that interval.*

Definition 2.8.1. An **inflection point** is a point where the curve changes its direction of concavity.

Example 2.8.1. Apply Propositions 2.8.1 and 2.8.2 to simple a polynomial function. Compare the shape of the curve with the tangential slope at various points. Identify intervals where the function is increasing and decreasing, local minima and maxima, intervals of concavity, and inflection points.

Example 2.8.2. Sketch a possible graph of a function f that satisfies the following conditions:

1. $f'(x) > 0$ on $(-\infty, 1)$; $f'(x) < 0$ on $(1, \infty)$.
2. $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$; $f''(x) < 0$ on $(-2, 2)$.
3. $\lim_{x\to-\infty} f(x) = -2$; $\lim_{x\to\infty} f(x) = 0$.

When given a function f, it may be useful to find a function F whose derivative is f; that is, $F'(x) = f(x)$. If such a function F exists, we call it an *antiderivative* of f. The concept of antiderivatives is of fundamental importance in connecting the ideas of differential and integral Calculus, as seen in Section 5.4.

Example 2.8.3. Consider a graph representing some derivative $y = f'(x)$.

(a) On what intervals is f increasing? On what intervals is f decreasing?
(b) At what values of x does f have a local maximum or minimum?
(c) On what intervals is f concave upward? On what intervals is f concave downward?
(d) What are the (x-coordinates of the) points of inflection of f?
(e) Assuming that $f(0) = 0$, sketch a possible graph of f.

3. Differentiation Rules.

3.1. **Derivatives of Polynomials and Exponential Functions.**

We have already informally discussed, or are aware of, the derivatives of the most basic polynomial functions. The constant function $f(x) = c$ is graphically represented by a horizontal line, so its slope is zero everywhere and hence $f'(x) = \frac{d}{dx}(c) = 0$ for all $x \in \mathbb{R}$. Of equal simplicity is the case of the linear function $g(x) = x$; its slope is one everywhere and hence $g'(x) = \frac{d}{dx}(x) = 1$ for all $x \in \mathbb{R}$. Now, if $f(x) = x^n$, for some integer $n \geq 2$, it is useful to apply a limit definition of the derivative in order to determine f'.

Example 3.1.1. Differentiate $f(x) = x^5$.

Solution. By the limit definition of the derivative from Section 2.7, we obtain

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^5 - x^5}{h}.$$

Now, to expand the term $(x+h)^5$, we can use the **binomial coefficients** given by **Pascal's Triangle**. In particular, the sixth row of the triangle contains the coefficients $1, 5, 10, 10, 5, 1$, which implies that

$$(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5.$$

Thus, the derivative of $f(x) = x^5$ is given by

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^5 - x^5}{h} \\ &= \lim_{h\to 0} \frac{(x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5) - x^5}{h} \\ &= \lim_{h\to 0} \frac{5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5}{h} \\ &= \lim_{h\to 0} (5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4) \\ &= 5x^4. \end{aligned}$$

Theorem 3.1.1 (Power Rule)**.** *If n is any real number, then* $\frac{d}{dx}(x^n) = nx^{n-1}$.

Proof. We proceed much as in Example 3.1.1, but it is necessary to apply the general form of the Binomial Theorem in order to expand the term $(x+h)^n$. □

Example 3.1.2. Differentiate each of the following: (a) $f(x) = x^{10}$, (b) $g(x) = \sqrt[5]{x^2}$, (c) $h(x) = \frac{1}{x^4}$.

Solution. By the Power Rule, we have:

(a) $f'(x) = \frac{d}{dx}\left(x^{10}\right) = 10x^9$.
(b) $g'(x) = \frac{d}{dx}\left(x^{2/5}\right) = \frac{2}{5}x^{-3/5}$.
(c) $h'(x) = \frac{d}{dx}\left(x^{-4}\right) = -4x^{-5}$.

Definition 3.1.1. The **normal line** to a curve at the point P is the line through P that is perpendicular to the tangent line at P.

Example 3.1.3. Find the equation of the normal line to the curve $y = x\sqrt{x}$ at the point $(4, 8)$.

Solution. Let $y = f(x) = x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$. Then by the Power Rule

$$f'(x) = \frac{d}{dx}\left(x^{3/2}\right) = \frac{3}{2}x^{1/2}.$$

In particular, the slope of the tangent line at $(4, 8)$ is given by $f'(4) = 3$. The slope of any line perpendicular to the tangent is the negative reciprocal of 3; that is, $-\frac{1}{3}$. Thus, the equation of the normal line to the curve $y = x\sqrt{x}$ at the point $(4, 8)$ is $y - 8 = -\frac{1}{3}(x - 4)$, or $y = -\frac{1}{3}x + \frac{28}{3}$.

The following theorems combine with the Power Rule to allow us to easily differentiate any polynomial.

Theorem 3.1.2 (Constant Multiple Rule). *If c is a constant and f is a differentiable function, then*
$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right].$$

Proof. Let $g(x) = cf(x)$. Then
$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] \\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= cf'(x). \end{aligned}$$
$\square$

Theorem 3.1.3 (Sum Rule). *If f and g are differentiable functions, then*
$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right].$$

Proof. Let $F(x) = f(x) + g(x)$. Then
$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{h(x+h)-h(x)}{h} \\ &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right] \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= f'(x)+g'(x). \end{aligned}$$
$\square$

Theorem 3.1.4 (Difference Rule). *If f and g are differentiable, then*
$$\frac{d}{dx}\left[f(x)-g(x)\right] = \frac{d}{dx}\left[f(x)\right] - \frac{d}{dx}\left[g(x)\right].$$

Proof. The argument is the same as in the case of the Sum Rule, except that we must use the Difference Law for limits in place of the Sum Law. $\square$

Example 3.1.4. Find $\frac{d}{dx}\left(x^7 - 6x^6 + 3x^5 + 7x^3 - 8x + 11\right)$.

Solution. By the preceding theorems, we find that:
$$\begin{aligned} \frac{d}{dx}\left(x^7 - 6x^6 + 3x^5 + 7x^3 - 8x + 11\right) &= \frac{d}{dx}\left(x^7\right) - 6\frac{d}{dx}\left(x^6\right) + 3\frac{d}{dx}\left(x^5\right) + 7\frac{d}{dx}\left(x^3\right) - 8\frac{d}{dx}(x) + \frac{d}{dx}(11) \\ &= 7x^6 - 6\cdot 6x^5 + 3\cdot 5x^4 + 7\cdot 3x^2 + 8 \\ &= 7x^6 - 36x^5 + 15x^4 + 21x^2 - 8. \end{aligned}$$

Example 3.1.5. Find the points on the curve $y = x^4 - 2x^2$ where the tangent line is horizontal.

Solution. A horizontal tangent line occurs wherever the slope of the derivative is zero. Note that

$$y' = 4x^3 - 4x = 0 \implies 4x\left(x^2-1\right) = 0 \implies 4x(x-1)(x+1) = 0 \implies x = 0,\, x = 1,\, x = -1.$$

Thus, the curve $y = x^4 - 2x^2$ has horizontal tangent lines at the points $(-1,-1)$, $(0,0)$, and $(1,-1)$.

Recall from Ex. 1.5.5 that the function $f(x) = e^x$ has the property that the slope of its tangent line at the point $(0,1)$ is exactly 1. By the limit definition of the derivative, it follows that e is the unique number satisfying

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1,$$

which yields the important differentiation rule:

$$\frac{d}{dx}\left(e^x\right) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x\left(e^h - 1\right)}{h} = e^x \lim_{h\to 0} \frac{e^h-1}{h} = e^x.$$

Example 3.1.6. If $f(x) = 3e^x - 4x^2$, find f' and f''.

Solution. Combining the rule above with the Difference, Constant Multiple, and Power Rules, we have:

$$f'(x) = \frac{d}{dx}\left(3e^x - 4x^2\right) = 3\frac{d}{dx}\left(e^x\right) - 4\frac{d}{dx}\left(x^2\right) = 3\left(e^x\right) - 4(2x) = 3e^x - 8x,$$
$$f''(x) = \frac{d}{dx}\left(3e^x - 8x\right) = 3\frac{d}{dx}\left(e^x\right) - 8\frac{d}{dx}(x) = 3\left(e^x\right)) - 8(1) = 3e^x - 8.$$

Example 3.1.7. At what point on the curve $y = 4e^x - 6$ is the tangent line parallel to $y = 4x$?

Solution. Let $x = a$ be the x-coordinate of the point in question. As in the previous example

$$f'(x) = \frac{d}{dx}\left(4e^x - 6\right) = 4e^x.$$

The slope of the tangent line to the curve $y = 4e^x - 6$ at the point $x = a$ is thus $4e^a$, while the slope of $y = 4x$ at any point is 4. Setting $4e^a = 4$ and solving for a, we have $e^a = 1$ or simply $a = 0$. Hence, the required point is $(0,4)$.

3.2. **The Product and Quotient Rules.**

By analogy with the Sum and Difference Rules, one might suppose that similar rules exist for the products and quotients of functions. This is not the case, however, as illustrated by the following simple example.

Example 3.2.1. Let $f(x) = \frac{x}{3}$ and $g(x) = x^2$, and show that $(fg)'(x) \neq f'(x)g'(x)$.

Solution. Clearly $f'(x) = \frac{1}{3}$ and $g'(x) = 2x$, so that $f'(x)g'(x) = \frac{2}{3}x$. On the other hand, $(fg)(x) = \frac{x^3}{3}$, which has the derivative $(fg)'(x) = x^2$.

To develop a general rule for correctly differentiating products of functions, we return to the limit definition of the derivative.

Theorem 3.2.1 (Product Rule)**.** *If f and g are differentiable functions, then*

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right].$$

Proof. Let $F(x) = f(x)g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{h(x+h)-h(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)g(x+h)-f(x)g(x+h)]+[f(x)g(x+h)-f(x)g(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{g(x+h)\,[f(x+h)-f(x)]}{h} + \frac{f(x)\,[g(x+h)-g(x)]}{h}\right] \\
&= \lim_{h\to 0} g(x+h)\cdot \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x)\cdot \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= g(x)f'(x) + f(x)g'(x).
\end{aligned}$$

□

Example 3.2.2. If $f(x) = x^2e^x$, find $f'(x)$.

Solution. By the Product Rule, we have

$$f'(x) = \frac{d}{dx}\left(x^2e^x\right) = x^2\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(x^2\right) = x^2e^x + 2xe^x = xe^x(x+2).$$

Example 3.2.3. Differentiate $f(t) = \left(\sqrt{t}+1\right)\left(t^2+t+1\right)$.

Solution. Again by the Product Rule, we have

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\left(\sqrt{t}+1\right)\left(t^2+t+1\right)\right] \\
&= \left(\sqrt{t}+1\right)\frac{d}{dx}\left(t^2+t+1\right) + \left(t^2+t+1\right)\frac{d}{dx}\left(\sqrt{t}+1\right) \\
&= \left(\sqrt{t}+1\right)(2t+1) + \left(t^2+t+1\right)\left(\frac{1}{2}t^{-1/2}\right).
\end{aligned}$$

Example 3.2.4. If $f(x) = (\sqrt{x}+1)\,g(x)$, where $g(9) = -6$ and $g'(9) = \frac{1}{2}$, find $f'(9)$.

Solution. First, notice that

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\left(\sqrt{x}+1\right)g(x)\right] \\
&= \left(\sqrt{x}+1\right)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left(\sqrt{x}+1\right) \\
&= \left(\sqrt{x}+1\right)g'(x) + g(x)\left(\frac{1}{2}x^{-1/2}\right).
\end{aligned}$$

Thus,

$$f'(9) = \left(\sqrt{9}+1\right)g'(9) + g(9)\left(\frac{1}{2}(9)^{-1/2}\right) = (4)\left(\frac{1}{2}\right) + (-6)\left(\frac{1}{6}\right) = 1.$$

As with the Product Rule, the Quotient Rule involves a somewhat complicated expression.

Theorem 3.2.2 (Quotient Rule)**.** *If f and g are differentiable functions, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}\left[f(x)\right] - f(x)\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}.$$

Example 3.2.5. If $y = \frac{x^3+6x-6}{x^2+4}$, find y'.

Solution. By the Quotient Rule, we have that

$$\begin{aligned} y' &= \frac{d}{dx}\left[\frac{x^3+6x-6}{x^2+4}\right] \\ &= \frac{\left(x^2+4\right)\frac{d}{dx}\left(x^3+6x-6\right)-\left(x^3+6x-6\right)\frac{d}{dx}\left(x^2+4\right)}{\left(x^2+4\right)^2} \\ &= \frac{\left(x^2+4\right)\left(3x^2+6\right)-\left(x^3+6x-6\right)(2x)}{\left(x^2+4\right)^2}. \end{aligned}$$

Example 3.2.6. Find the equation of the tangent line to the curve $y = \frac{e^x}{e^x+1}$ at the point $(0, 1/2)$.

Solution. Again by the Quotient Rule, we have that

$$\begin{aligned} y' &= \frac{d}{dx}\left[\frac{e^x}{e^x+1}\right] \\ &= \frac{\left(e^x+1\right)\frac{d}{dx}\left(e^x\right)-\left(e^x\right)\frac{d}{dx}\left(e^x+1\right)}{\left(e^x+1\right)^2} \\ &= \frac{\left(e^x+1\right)\left(e^x\right)-\left(e^x\right)\left(e^x\right)}{\left(e^x+1\right)^2} \\ &= \frac{e^x}{\left(e^x+1\right)^2}. \end{aligned}$$

Thus, the slope of the tangent line to the curve when $x = 0$ is given by $y'|_{x=0} = \frac{e^0}{(e^0+1)^2} = \frac{1}{2^2} = \frac{1}{4}$, and the equation of the tangent line to the curve at the point $(0, 1/2)$ is $y = \frac{1}{4}x + \frac{1}{2}$.

3.3. **Derivatives of Trigonometric Functions.**

Example 3.3.1. Estimate $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$.

Solution. Consider the following table.

θ	$\frac{\sin\theta}{\theta}$
± 0.1	0.99833416
± 0.01	0.99998333
± 0.001	0.99999983
± 0.0001	0.99999999

From this, we conclude that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

Similarly, we can determine that $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$. We are now equipped to establish a formula for the derivative of the sine function.

Theorem 3.3.1. *$\frac{d}{dx}(\sin x) = \cos x$.*

Proof. Let $f(x) = \sin x$. By the definition of the derivative and a well-known trigonometric identity, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x\cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \left[\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x \sin h}{h}\right] \\
&= \lim_{h\to 0} \left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right] \\
&= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0}\left(\frac{\sin h}{h}\right) \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x.
\end{aligned}$$

□

By an analogous proof, we obtain a similar formula for the derivative of the cosine function.

Theorem 3.3.2. $\frac{d}{dx}(\cos x) = -\sin x$.

Remark 3.3.1. The derivatives of sine and cosine can be recalled with the aid of a simple circular diagram.

Since the tangent function is a quotient of the sine and cosine functions, its derivative can now be determined.

Theorem 3.3.3. $\frac{d}{dx}(\tan x) = \sec^2 x$.

Proof. By the Quotient Rule, we obtain

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x.
\end{aligned}$$

□

We can further apply the Quotient Rule to establish formulas for the derivative of each of the remaining basic trigonometric functions.

Theorem 3.3.4. *1.* $\frac{d}{dx}(\csc x) = -\csc x \cot x$, *2.* $\frac{d}{dx}(\sec x) = \sec x \tan x$, *3.* $\frac{d}{dx}(\cot x) = -\csc^2 x$.

Example 3.3.2. For what values of x does the graph of $f(x) = \frac{\sec x}{\tan x + 1}$ have a horizontal tangent line?

Solution. Recalling the trigonometric identity $\tan^2 x + 1 = \sec^2 x$, the Quotient Rule gives

$$\begin{aligned} f'(x) &= \frac{(1+\tan x)\cdot \frac{d}{dx}(\sec x) - \sec x \cdot \frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \\ &= \frac{(1+\tan x)\cdot \sec x \tan x - \sec x \cdot \sec^2 x}{(1+\tan x)^2} \\ &= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} \\ &= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2} \end{aligned}$$

Since $\sec x$ is never equal to 0, we see that $f'(x) = 0$ when $\tan x = 1$, which is for $x = n\pi + \pi/4$, where n is any integer. At these points, the tangent line to the graph of f is horizontal.

Example 3.3.3. A 12-foot ladder rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/4$?

Solution. By the geometric definition of the sine function, we may write $\sin\theta = \frac{x}{12}$. That is, $x = 12\sin\theta$. The rate of change of x with respect to θ is then given by $\frac{dx}{d\theta} = \frac{d}{d\theta}(12\sin\theta) = 12\cos\theta$. Thus, when $\theta = \pi/4$, the rate of change of x with respect to θ is simply $12\cos(\pi/4) = 12\cdot\sqrt{2}/2 = 6\sqrt{2} \approx 8.5$ ft/rad.

3.4. **The Chain Rule.**

Theorem 3.4.1 (Chain Rule)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f\circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product*

$$F'(x) = f'(g(x))\cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example 3.4.1. Find the derivative of $f(x) = \sqrt[3]{x^5 + x}$.

Solution. Rewriting the function f, we have by the Chain Rule that

$$f'(x) = \frac{d}{dx}(x^5 + x)^{1/3} = (1/3)(x^5 + x)^{-2/3}\cdot(5x + 1).$$

Example 3.4.2. Find the derivative of $f(x) = \sin\left(x^4\right)$ and $g(x) = \sin^4 x$.

Solution. By the Chain Rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\sin(x^4) = \cos(x^4)\cdot 4x. \\ g'(x) &= \frac{d}{dx}(\sin x)^4 = 4(\sin x)^3\cdot\cos x. \end{aligned}$$

Corollary 3.4.2 (Power Rule combined with Chain Rule)**.** *If n is any real number and $u = g(x)$ is differentiable, then*

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}\cdot g'(x).$$

Example 3.4.3. Differentiate $h(x) = \frac{1}{\sqrt{x^2+5x+10}}$.

Solution. Rewriting the function h and applying the Chain Rule, we obtain

$$h'(x) = \frac{d}{dx}(x^2+5x+10)^{-1/2} = (-1/2)(x^2+5x+10)^{-3/2} \cdot (2x+5).$$

Example 3.4.4. Find y' if $y = (x^2+3x+2)^7(3x-8)^4$.

Solution. Combining the Chain Rule with the Product Rule, we find

$$y' = \frac{d}{dx}\left[(x^2+3x+2)^7(3x-8)^4\right] = (x^2+3x+2)^7 \cdot 4(3x-8)^3 \cdot 3 + (3x-8)^4 \cdot 7(x^2+3x+2)^6 \cdot (2x+3).$$

Example 3.4.5. Find the derivative of $y = e^{\tan x}$.

Solution. By the Chain Rule, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\tan x}\right) = e^{\tan x} \cdot \frac{d}{dx}(\tan x) = e^{\tan x} \cdot \sec^2 x.$$

Example 3.4.6. Let $a > 0$ be a constant. Find $\frac{d}{dx}(a^x)$.

Solution. Recall from Section 1.6 that $a = e^{\ln a}$ for any $a > 0$. In particular, $a^x = \left(e^{\ln a}\right)^x = e^{(\ln a)x}$. It follows by the Chain Rule that

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{(\ln a)x}\right) = e^{(\ln a)x} \cdot \frac{d}{dx}((\ln a)x) = e^{(\ln a)x} \cdot \ln a = a^x \ln a.$$

Example 3.4.7. If $f(x) = \cos(\sin(e^x))$, find $f'(x)$.

Solution. Applying the Chain Rule twice, we obtain

$$f'(x) = \frac{d}{dx}(\cos(\sin(e^x))) = -\sin(\sin(e^x)) \cdot \frac{d}{dx}(\sin(e^x)) = -\sin(\sin(e^x)) \cdot \cos(e^x) \cdot e^x.$$

Example 3.4.8. If $f(x) = e^{\sec(3x)}$, find $f'(x)$.

Solution. Again using the Chain Rule twice, we have

$$f'(x) = \frac{d}{dx}\left(e^{\sec(3x)}\right) = e^{\sec(3x)} \cdot \frac{d}{dx}(\sec(3x)) = e^{\sec(3x)} \cdot \sec(3x)\tan(3x) \cdot \frac{d}{dx}(3x) = 3\sec(3x)\tan(3x)e^{\sec(3x)}.$$

Remark 3.4.1. The Chain Rule gives us a way of finding the slope of tangent lines to parametric curves. In particular, suppose that $x = f(t)$ and $y = g(t)$ are differentiable functions that define a parametric curve. We wish to find the tangent line at a point on the curve where y is also a differentiable function of x. By the Chain Rule, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$. Assuming that $\frac{dx}{dt} \neq 0$, it follows that $\frac{dy}{dx} = \left(\frac{dy}{dt}\right)/\left(\frac{dx}{dt}\right)$.

Example 3.4.9. Find an equation of the tangent line to the parametric curve $x = 2\sin(2t)$ and $y = 2\sin t$ at the point $(\sqrt{3}, 1)$. Where does this curve have horizontal or vertical tangents?

Solution. At the point with parameter value t, the slope of the tangent line to the curve is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(2\sin t)}{\frac{d}{dt}(2\sin(2t))} = \frac{2\cos t}{2\cos(2t) \cdot 2} = \frac{\cos t}{2\cos(2t)}.$$

The point $(\sqrt{3}, 1)$ corresponds to the parameter value $t = \pi/6$, so the slope of the tangent at that point is

$$\frac{dy}{dx}\Big|_{t=\pi/6} = \frac{\cos(\pi/6)}{2\cos(\pi/3)} = \frac{\sqrt{3}/2}{2(1/2)} = \frac{\sqrt{3}}{2}.$$

An equation of the tangent line is therefore: $y - 1 = \frac{\sqrt{3}}{2}\left(x - \sqrt{3}\right)$.

The tangent line is horizontal when $\frac{dy}{dx} = 0$. This occurs when $\cos t = 0$ (and $\cos(2t) \neq 0$); that is, when $t = \pi/2$ or $t = 3\pi/2$ (note that the entire curve is given by the parameter values $0 \leq t \leq 2\pi$). Thus, the curve has horizontal tangents at the points $(0, 2)$ and $(0, -2)$.

The tangent line is vertical when $\frac{dx}{dt} = 4\cos(2t) = 0$ (and $\cos t \neq 0$); that is, when $t = \pi/4$, $3\pi/4$, $5\pi/4$, or $7\pi/4$. The corresponding four points on the curve are $\left(\pm 2, \pm\sqrt{2}\right)$.

3.5. **Implicit Differentiation.**

The functions we have dealt with up until now have been described by expressing one variable explicitly in terms of another, e.g. $y = e^x$ or $y = x\cos x$. Some functions, however, are defined implicitly by a relationship between x and y such as $x^2 + y^2 = 25$ or $x^3 + y^3 = 3xy$.

In some cases, it is possible to solve such an equation for y as an explicit function (or functions) of x. For instance, the equation $x^2 + y^2 = 25$ can be written equivalently as the two equations $y = \pm\sqrt{25 - x^2}$. On the other hand, it's not easy to solve $x^3 + y^3 = 3xy$ for either variable explicitly. Fortunately, the method of **implicit differentiation** allows us to find the derivative of y without solving for y as a function of x.

The general approach to implicit differentation is as follows. Given an equation in x and y that defines y implicitly as a differentiable function of x, we can find y' by differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

Example 3.5.1. Find the equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(-3, 4)$.

Solution. First, we differentiate both sides of the equation with respect to x

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \implies \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0.$$

Since y is a function of x, the Chain Rule implies that $\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$. Thus, we have the equation

$$2x + 2y\frac{dy}{dx} = 0 \implies 2y\frac{dy}{dx} = -2x \implies \frac{dy}{dx} = -\frac{x}{y}.$$

At the point $(-3, 4)$, we have $x = -3$ and $y = 4$, so

$$\frac{dy}{dx}\Big|_{(x,y)=(-3,4)} = \frac{3}{4}.$$

An equation of the tangent line to the circle at the point $(-3, 4)$ is therefore $y - 4 = \frac{3}{4}(x + 3)$.

Example 3.5.2. Find y' if $x^3 + y^3 = 3xy$. At what point in the first quadrant does this curve have a horizontal tangent line?

Solution. Regarding y as a function of x, and differentiating both sides of the equation with respect to x via the Chain and Product Rules, we have

$$\begin{aligned} \frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(3xy) \quad &\Longrightarrow \quad \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3\left(x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right) \\ &\Longrightarrow \quad 3x^2 + 3y^2y' = 3(xy'+y) \\ &\Longrightarrow \quad y^2y' - xy' = y - x^2 \\ &\Longrightarrow \quad y'(y^2-x) = y - x^2 \\ &\Longrightarrow \quad y' = \frac{y-x^2}{y^2-x}. \end{aligned}$$

Now, the tangent line is horizontal if and only if $y' = 0$. By the formula above, this happens whenever $y - x^2 = 0$ (provided that $y^2 - x \neq 0$). Substituting $y = x^2$ into the original equation, we obtain

$$x^3 + (x^2)^3 = 3x(x^2) \quad \Longrightarrow \quad x^3 + x^6 = 3x^3 \quad \Longrightarrow \quad x^6 - 2x^3 = 0 \quad \Longrightarrow \quad x^3(x^3-2) = 0.$$

Since $x \neq 0$ in the first quadrant, this implies $x^3 - 2 = 0$ or $x = 2^{1/3}$. At this x-value, we have $y = (2^{1/3})^2 = 2^{2/3}$. Thus, the tangent line is horizontal at the point $(2^{1/3}, 2^{2/3})$ in the first quadrant.

Example 3.5.3. Find y' if $\sin(x+y) = y\cos x$.

Solution. Again using the Chain and Product Rules, we have

$$\begin{aligned} \frac{d}{dx}(\sin(x+y)) = \frac{d}{dx}(y\cos x) \quad &\Longrightarrow \quad \cos(x+y)\frac{d}{dx}(x+y) = y\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(y) \\ &\Longrightarrow \quad \cos(x+y)\cdot(1+y') = -y\sin x + \cos x \cdot y' \\ &\Longrightarrow \quad y'(\cos x - \cos(x+y)) = y\sin x + \cos(x+y) \\ &\Longrightarrow \quad y' = \frac{y\sin x + \cos(x+y)}{\cos x - \cos(x+y)}. \end{aligned}$$

3.6. **Inverse Trigonometric Functions and Their Derivatives.**

Definition 3.6.1. The **inverse trigonometric functions** are defined as follows:

- $\sin^{-1} x = y$ or $\arcsin x = y \iff \sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- $\cos^{-1} x = y$ or $\arccos x = y \iff \cos y = x$ and $0 \leq y \leq \pi$
- $\tan^{-1} x = y$ or $\arctan x = y \iff \tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Remark 3.6.1. The inverse sine or **arcsine** function, $\sin^{-1} x$, represents the angle y between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x. Similar intuition describes the other inverse trigonometric functions. Furthermore,

- The domain of both $\sin^{-1} x$ and $\cos^{-1} x$ is $[-1,1]$.
- The domain of $\tan^{-1} x$ is $(-\infty,\infty)$, and $\lim_{x\to\pm\infty} \tan^{-1} x = \pm\frac{\pi}{2}$ as discussed in Ex 2.5.3.

Example 3.6.1. Explore the inverse trigonometric functions (arcsine, arccosine, arctangent) and their graphs in comparison to the ordinary trigonometric functions (sine, cosine, tangent).

Example 3.6.2. Note that the cancelation equations for inverse functions hold for arcsine, arccosine, and arctangent. For example,

$$\sin^{-1}(\sin x) = x \text{ for } -\tfrac{\pi}{2} \leq x \leq \tfrac{\pi}{2} \quad \text{and} \quad \sin(\sin^{-1} x) = x \text{ for } -1 \leq x \leq 1.$$

In particular, $\sin^{-1}(1/2) = \pi/6$, $\sin^{-1}(-\sqrt{2}/2) = -\pi/4$, $\tan^{-1}(1) = \pi/4$, and $\tan^{-1}(0) = 0$.

Example 3.6.3. Simplify the expression $\cos(\tan^{-1} x)$.

Solution (1)**.** Let $y = \tan^{-1} x$. Then $\tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$. We wish to determine $\cos y$, but it is easier to find $\sec y$ first using the Pythagorean identity $\sec^2 y = 1 + \tan^2 y = 1 + x^2$. Indeed, we have $\sec y = \sqrt{1+x^2}$ $\left(\text{since } \sec y > 0 \text{ for } -\frac{\pi}{2} < y < \frac{\pi}{2}\right)$. Thus, $\cos(\tan^{-1} x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1+x^2}}$.

Solution (2)**.** Alternatively, let $y = \tan^{-1} x$ and construct a right triangle such that y is one of its acute angles. Then the opposite and adjacent side-lengths are x and 1 respectively, and hence the hypotenuse has length $\sqrt{1+x^2}$. It follows that $\cos(\tan^{-1} x) = \cos y = \frac{1}{\sqrt{1+x^2}}$.

Example 3.6.4. Evaluate $\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right)$.

Solution. If we let $t = \frac{1}{x-2}$, we know that $t \to \infty$ as $x \to 2^+$. Therefore, by the H.A. properties of the arctangent function, we have

$$\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right) = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}.$$

Using the implicit differentiation method from Section 3.5 we are now prepared to establish the derivative of the arcsine function.

Theorem 3.6.1. *$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$.*

Proof. Let $y = \sin^{-1} x$. Then $\sin y = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Differentating implicitly with respect to x yields

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x) \implies \cos y \cdot \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\cos y}.$$

Since $\cos y \ge 0$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we have by the Pythagorean identity that

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1-x^2}.$$

Hence, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$, as desired. □

By similar proofs, we can also obtain formulas for the derivatives of the arccosine and arctangent functions.

Theorem 3.6.2. *$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$.*

Theorem 3.6.3. *$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.*

Example 3.6.5. Differentiate $f(x) = \sin^{-1}(\sqrt{x})$.

Solution. Combining the Chain Rule with Theorem 3.6.1 above, we obtain

$$f'(x) = \frac{d}{dx}\sin^{-1}\left(\sqrt{x}\right) = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(x^{1/2}) = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x(1-x)}}.$$

3.7. **Derivatives of Logarithmic Functions.**

In this section, we use implicit differentiation to find the derivatives of logarithmic functions $y = \log_a x$, and in particular of the natural logarithmic function $y = \ln x$.

Theorem 3.7.1. *$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$.*

Proof. Let $y = \log_a x$. Then $a^y = x$. Differentiating implicitly with respect to x yields

$$\frac{d}{dx}(a^y) = \frac{d}{dx}(x) \implies a^y \ln a \cdot \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{a^y \ln a} \implies \frac{dy}{dx} = \frac{1}{x \ln a}.$$

□

In the special case of the natural logarithm function, the substitution $a = e$ yields the following simple formula.

Corollary 3.7.2. $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

As a consequence of the Chain Rule, we also have a very useful generalized formula for the derivative of the natural logarithm function composed with some other differentiable function.

Corollary 3.7.3. *If $u = g(x)$ is a differentiable function of x, then $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ or $\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$.*

Example 3.7.1. Differentiate $y = \ln(x^4 + 3x^2 + 5x)$.

Solution. By Corollary 3.7.3 we easily obtain

$$y' = \frac{\frac{d}{dx}(x^4 + 3x^2 + 5x)}{x^4 + 3x^2 + 5x} = \frac{4x^3 + 6x + 5}{x^4 + 3x^2 + 5x}.$$

Example 3.7.2. Find $\frac{d}{dx}(\ln(\cos x))$.

Solution. Again it is very easy to compute that

$$\frac{d}{dx}(\ln(\cos x)) = \frac{\frac{d}{dx}(\cos x)}{\cos x} = -\frac{\sin x}{\cos x}.$$

Example 3.7.3. Differentiate $y = \sqrt{\ln x}$.

Solution. Using the Chain Rule, we have

$$y' = \frac{d}{dx}(\ln x)^{1/2} = \frac{1}{2}(\ln x)^{-1/2} \cdot \frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}.$$

Example 3.7.4. Differentiate $f(x) = \ln\left[\frac{\sqrt{x+5}}{(x-7)^2}\right]$.

Solution. By the Quotient and Chain Rules, we find that

$$\begin{aligned}
f'(x) &= \frac{1}{\left[\frac{\sqrt{x+5}}{(x-7)^2}\right]} \cdot \frac{d}{dx}\left(\frac{(x+5)^{1/2}}{(x-7)^2}\right) \\
&= \frac{(x-7)^2}{\sqrt{x+5}} \cdot \frac{(x-7)^2 \frac{d}{dx}(x+5)^{1/2} - (x+5)^{1/2} \frac{d}{dx}(x-7)^2}{(x-7)^4} \\
&= \frac{(x-7)^2}{\sqrt{x+5}} \cdot \frac{(x-7)^2 \cdot (1/2)(x+5)^{-1/2} - (x+5)^{1/2} \cdot 2(x-7)}{(x-7)^4} \\
&= \frac{(1/2)(x-7)(x+5)^{-1/2} - 2(x+5)^{1/2}}{(x-7)\sqrt{x+5}} \\
&= \frac{1}{2(x+5)} - 2(x-7).
\end{aligned}$$

Example 3.7.5. Show that $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x \neq 0$.

Solution. Let $f(x) = \ln|x|$, and notice that we can write

$$f(x) = \begin{cases} \ln x, & \text{if } x > 0; \\ \ln(-x), & \text{if } x < 0. \end{cases}$$

By Corollary 3.7.3, it follows that

$$f'(x) = \begin{cases} \frac{1}{x}, & \text{if } x > 0; \\ \frac{1}{-x}(-1) = \frac{1}{x}, & \text{if } x < 0. \end{cases}$$

Hence, $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x \neq 0$.

The calculation of derivatives of complicated functions involving products, quotients, and/or powers can often be simplified by taking logarithms. This method is called **logarithmic differentation**, and it generally consists of the following three steps:

1. Take the natural logarithm of both sides of an equation $y = f(x)$ and simplify using the Laws of Logarithms,
2. Differentiate implicitly with respect to x,
3. Solve the resulting for y'.

Example 3.7.6. Logarithmic differentation allows us to easily prove the generalized Power Rule from Section 3.1. Indeed, let n be any real number and set $y = x^n$. Then $\ln|y| = \ln|x|^n = n\ln|x|$, for $x \neq 0$ (if $x = 0$, it is obvious that $\frac{d}{dx}(x^n) = \frac{d}{dx}(0) = 0$). Implicitly differentiating both sides, we obtain

$$\frac{y'}{y} = \frac{n}{x} \implies y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}.$$

Example 3.7.7. Differentiate $y = \frac{(2x+3)^{2/5}\sqrt{x^4+3x+1}}{(3x+4)^6}$.

Solution. First, we take the natural logarithm of both sides of the equation and simplify

$$\ln y = \ln\left[\frac{(2x+3)^{2/5}\sqrt{x^4+3x+1}}{(3x+4)^6}\right] \implies \ln y = \frac{2}{5}\ln(2x+3) + \frac{1}{2}\ln(x^4+3x+1) - 6\ln(3x+4).$$

Now, by implicit differentiation we have

$$\frac{y'}{y} = \frac{2}{5}\cdot\frac{2}{2x+3} + \frac{1}{2}\cdot\frac{4x^2+3}{x^4+3x+1} - 6\cdot\frac{3}{3x+4}.$$

Hence

$$y' = \frac{(2x+3)^{2/5}\sqrt{x^4+3x+1}}{(3x+4)^6}\left[\frac{4}{5(2x+3)} + \frac{4x^2+3}{2(x^4+3x+1)} - \frac{18}{3x+4}\right].$$

Example 3.7.8. Differentiate $y = x^{\sqrt{x}}$.

Solution. Taking the natural logarithm of both sides of the equation yields $\ln y = \ln\left(x^{\sqrt{x}}\right) = \sqrt{x}\ln x$. Now, by implicit differentiation and the Product Rule, we have

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + \ln x\cdot\frac{1}{2\sqrt{x}} \implies y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right).$$

Remark 3.7.1. It is important to distinguish carefully between the various exponential-type expressions when differentiating. In general, there are four cases for exponents and bases:

1. (constant base, constant exponent) use the fact that the derivative of a constant is zero: $\frac{d}{dx}(a^b) = 0$.
2. (variable base, constant exponent) use the Power Rule: $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$.
3. (constant base, variable exponent) use the diff. rule for exp. functions: $\frac{d}{dx}[a^{g(x)}] = a^{g(x)}\ln(a)g'(x)$.
4. (variable base, variable exponent) use logarithmic differentiation, as in Ex 3.7.8.

3.8. **Rates of Change in the Natural and Social Sciences.**

We know that if $y = f(x)$, then the derivative $\frac{dy}{dx}$ can be interpreted as the rate of change of y with respect to x. Now, with a myriad of differentiation rules at our disposal, we explore some of the applications of this concept to physics, economics, and other sciences.

Example 3.8.1 (Physics: Velocity and Acceleration)**.** The position of a particle is given by the equation $s(t) = t^3 - 9t^2 + 24t$, where s is measured in meters and t is in seconds.

(a) Find the velocity of the particle at time t.
(b) What is the velocity of the particle after 1 s? After 3 s?
(c) When is the particle at rest?
(d) When is the particle moving forward (that is, in the positive direction)?
(e) Draw a diagram to represent the motion of the particle.
(f) Find the total distance traveled by the particle during the first five seconds.
(g) Find the acceleration of the particle at time t after after 4 s.
(h) When is the particle speeding up? When is it slowing down?

Solution. We use the fact that velocity is the instantaneous rate of change of position ($v(t) = s'(t)$) and acceleration is the instantaneous rate of change of velocity ($a(t) = v'(t) = s''(t)$).

(a) $v(t) = s'(t) = 3t^2 - 18t + 24$.
(b) $v(1) = 9$ m/s, $v(3) = -3$ m/s.
(c) The particle is at rest when $v(t) = 0$. That is, when $3t^2 - 18t + 24 = 0 \implies 3(t-2)(t-4) = 0$, so at $t = 2$ and $t = 4$ s.
(d) The particle moves in the positive direction when $v(t) > 0$. That is, when $(t-2)(t-4) > 0$. This inequality is true when both factors are positive ($t > 4$) or when both factors are negative ($t < 2$). Consequently, the particle is moving forward for $t < 2$ and $t > 4$ and backward for $2 < t < 4$.
(e) Using the information from part (d), we can sketch the motion of the particle along the s-axis.
(f) From parts (d) and (e), the particle travels $|s(2) - s(0)| = 20$ meters from $t = 0$ to $t = 2$, $|s(4) - s(2)| = 4$ meters from $t = 2$ to $t = 4$, and $|s(5) - s(4)| = 4$ meters from $t = 4$ to $t = 5$. Thus, the total distance traveled by the particle during the first five seconds is $20 + 4 + 4 = 28$ m.
(g) $a(t) = v'(t) = 6t - 18$, so $a(4) = 6$ m/s^2.
(h) The particle speeds up when $a(t) > 0$ and slows down when $a(t) < 0$. Since $a(t)$ is a linear function, it is easy to see these inequalities hold when $t > 3$ and when $t < 3$ respectively.

Definition 3.8.1. If $C(x)$ represents the cost of producing x items, then $C'(x)$ is the instantaneous rate of change of the cost function and is referred to as the **marginal cost**.

Remark 3.8.1. The marginal cost of producing x items is approximately equal to the cost of producing one more item if x items have already been produced. Indeed, $C'(x) \approx \frac{C(x+1)-C(x)}{1} = C(x+1) - C(x)$.

Example 3.8.2 (Economics: Marginal Cost)**.** A company estimates that the cost (in dollars) of producing x items is given by $C(x) = 15000 + 6x + 0.02x^2$. Find the marginal cost of producing 500 items and interpret its meaning.

Solution. Clearly $C'(x) = 6 + 0.04x$, and it follows that $C'(500) = \$26$/item. This gives the rate at which the production cost is increasing when $x = 500$, and predicts the additional cost of producing the 501st item. Note that the actual cost of producing the 501st item is $C(501) - C(500) = \$26.02$.

In general, given an equation relating two or more variables, we can use a derivative to compute the instantaneous rate of change of one variable with respect to another.

Example 3.8.3. The volume of a spherical cell is $V(r) = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1\mu\text{m} = 10^{-6}\text{m}$). Find the instantaneous rate of change of V with respect to r when $r = 5\mu\text{m}$.

Solution. It is easy to compute that $V'(r) = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$, and hence $V'(5) = 100\pi\ \mu\text{m}^2$. Notice that the formula for V' is identical to the formula for the surface area of a sphere (even the units are correct!). In fact, this observation makes a lot of sense if you think about how an infinitesimal change in radius affects the volume of the sphere.

3.9. **Linear Approximations and Differentials.**

There are no notes provided for this section. It describes the use of tangent lines in linearly approximating function values. The related notion of differentials is also introduced.

4. Applications of Differentiation.

4.1. **Related Rates.**

In a related rates problem, the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differeniate both sides with respect to time.

Example 4.1.1. If $z^2 = x^2 + y^2$, and we know that $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 3$, find $\frac{dz}{dt}$ when $x = 5$ and $y = 12$.

Solution. Differentiating both sides of the equation with respect to t using the Chain Rule, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \implies 2\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

When $x = 5$ and $y = 12$, the original equation implies that $z = 13$. Thus,

$$13\frac{dz}{dt} = 5 \cdot 2 + 12 \cdot 3 \implies \frac{dz}{dt} = \frac{46}{13}.$$

The general strategy for solving related rates application problems consists of the following steps.

1. Read the problem carefully.
2. Draw a diagram, if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use geometry to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Example 4.1.2. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

Solution. Let S be the snowball's surface area, let r be its radius, and let D be its diameter. Then $D = 2r$, so $S = 4\pi r^2 = 4\pi\left(\frac{D}{2}\right)^2 = \pi D^2$. We are given that $\frac{dS}{dt} = -1$, and we wish to find $\frac{dD}{dt}\big|_{D=10}$. Differentiating the equation $S = \pi D^2$ with respect to t using the Chain Rule gives

$$\frac{d}{dt}(S) = \frac{d}{dt}(\pi D^2) \implies \frac{dS}{dt} = 2\pi D\frac{dD}{dt} \implies \frac{dD}{dt} = -\frac{1}{2\pi D}.$$

Thus,

$$\frac{dD}{dt}\Big|_{D=10} = -\frac{1}{2\pi(10)} = -\frac{1}{20\pi} \text{ cm/min.}$$

Example 4.1.3. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 p.m.?

Solution. At time t hours after noon, let $x = 35t$ be the distance that ship A has traveled east, let $y = 25t$ be the distance that ship B has traveled north, and let D be the distance between the ships. We are given that $\frac{dx}{dt} = 35$ and $\frac{dy}{dt} = 25km$, and we wish to find $\frac{dD}{dt}$ when $t = 4$. From a diagram of the positions and movements of the two ships, we see that x, y, and D are related by $D^2 = (150 - x)^2 + y^2$. Differentiating this equation with respect to t, we have

$$\frac{d}{dt}(D^2) = \frac{d}{dt}\left((150 - x)^2 + y^2\right) \implies 2D\frac{dD}{dt} = -2(150-x)\frac{dx}{dt} + 2y\frac{dy}{dt} \implies D\frac{dD}{dt} = (x-150)\frac{dx}{dt} + y\frac{dy}{dt}.$$

When $t = 4$, we have $x = 140$ and $y = 100$, and hence $D = \sqrt{(150 - 140)^2 + 100^2} = 100.5$. In this case, it follows that

$$100.5\frac{dD}{dt} = (140 - 150) \cdot 35 + 100 \cdot 25 \implies \frac{dD}{dt} = 21.39 \text{ km/h.}$$

Example 4.1.4. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 in deep?

Solution. Let h be the height of the water in the trough, and let V be its volume. We are given $\frac{dV}{dt} = 12$ and we wish to find $\frac{dh}{dt}\big|_{h=1/2}$. Since the trough is 10 ft long, the volume V is equal to ten times the area of the iscosceles triangle of height h. From a simple diagram, we can use similar triangles to establish that this area must be $\frac{3}{2}h^2$. Thus, $V = 10 \cdot \frac{3}{2}h^2 = 15h^2$. Differentiating both sides of this equation with respect to t yields

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(15h^2\right) \implies \frac{dV}{dt} = 30h\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{1}{30h}\frac{dV}{dt}.$$

Hence,

$$\frac{dh}{dt}\Big|_{h=1/2} = \frac{1}{30(1/2)} \cdot 12 = \frac{4}{5} \text{ ft/min.}$$

Example 4.1.5. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

Solution. Let x be the distance from the man to the point on the path closest to the searchlight, and let θ be the angle between the beam of the searchlight and the perpendicular to the path. We are given that $\frac{dx}{dt} = 4$, and we wish to find $\frac{d\theta}{dt}$ when $x = 15$. The equation relating x and θ can be written as $\frac{x}{20} = \tan\theta$. Differentiating both sides of this equation with respect to t, we obtain

$$\frac{d}{dt}\left(\frac{x}{20}\right) = \frac{d}{dt}(\tan\theta) \implies \frac{1}{20}\frac{dx}{dt} = \sec^2\theta\frac{d\theta}{dt} \implies \frac{d\theta}{dt} = \frac{\cos^2\theta}{20}\frac{dx}{dt} = \frac{\cos^2\theta}{20}\cdot 4 = \frac{\cos^2\theta}{5}.$$

When $x = 15$, the length of the beam is 25, so $\cos\theta = \frac{20}{25} = \frac{4}{5}$. It follows that at this point the searchlight is rotating at a rate of

$$\frac{d\theta}{dt} = \frac{(4/5)^2}{5} = \frac{16}{125} = 0.128 \text{ rad/s.}$$

4.2. **Maximum and Minimum Values.**

Definition 4.2.1. Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all $x \in D$.
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all $x \in D$.

Definition 4.2.2. The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ for all x in an open interval containing the point c.
- **local minimum** value of f if $f(c) \leq f(x)$ for all x in an open interval containing the point c.

The maxima and minima of a function are commonly referred to as **extrema** or **extreme values**. It is thus common to discuss the absolute or global extrema and the local or relative extrema.

Example 4.2.1. Graphically illustrate the differences between absolute and relative extrema.

Example 4.2.2. Find and categorize all extrema of: (a) $f(x) = \sin x$, (b) $g(x) = x^2$, (c) $h(x) = x^3$.

Example 4.2.3. Use a calculator to graph the function $f(x) = 3x^4 - 16x^3 + 18x^2$, defined on $-1 \leq x \leq 4$, and conclude that absolute extrema can occur at the endpoints of an interval.

Theorem 4.2.1 (Extreme Value Theorem)**.** *If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c, d \in [a,b]$.*

Example 4.2.4. Note that if a function is not continuous on its domain, or if its domain is not a closed interval, then the Extreme Value Theorem does not guarantee the existence of any absolute extrema. Explore this concept by graphing different functions that fail to exhibit absolute extrema.

Theorem 4.2.2 (Fermat's Theorem)**.** *If f has a local maximum or local minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.*

Remark 4.2.1. When $f'(c) = 0$, it is not necessarily true that f has a local maximum or minimum at c (e.g. $f(x) = x^3$ at the origin). Furthermore, a function may have a local extreme value even at a point where its derivative does not exist (e.g. $f(x) = |x|$ at the origin). In other words, although Fermat's Theorem is useful, it is important to avoid reading too much into it.

Definition 4.2.3. A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

In terms of critical numbers, Fermat's Theorem can be rephrased as follows.

Theorem 4.2.3. *If f has a local maximum or minimum at c, then c is a ciritcal number of f.*

To find the absolute extrema of a continuous function on a closed interval, the following three-step procedure always works.

Proposition 4.2.1 (Closed Interval Method)**.** *To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a,b]$:*

1. *Find the values of f at the critical numbers of f in (a,b).*
2. *Find the values of f at the endpoints of the interval.*
3. *The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.*

Example 4.2.5. On the interval $[-1,4]$, find the values of x where each of the following functions has an absolute maximum and absolute minimum: (a) $f(x) = x^{3/5}(4-x)$, (b) $g(x) = 3x^4 - 16x^3 + 18x^2$.

Solution. For each function, we follow the steps of the Closed Interval Method.

(a) The derivative of f is given by $f'(x) = x^{3/5} \cdot (-1) + (4-x) \cdot \frac{3}{5}x^{-2/5} = \frac{-5x+12-3x}{5x^{2/5}} = \frac{12-8x}{5x^{2/5}}$. Clearly $f'(x) = 0 \implies x = 3/2$, and $f'(x)$ is undefined for $x = 0$. Comparing $f(-1) = -5$, $f(0) = 0$, $f(3/2) = 3.1886$, and $f(4) = 0$, we conclude that f has an absolute minimum at $(-1,-5)$ and an absolute maximum at $(3/2, 3.1886)$.

(b) The derivative of g is given by $g'(x) = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$. Thus, $g'(x)$ is defined everywhere, and $g'(x) = 0 \implies x = 0$, $x = 1$, $x = 3$. Comparing $g(-1) = 37$, $g(0) = 0$, $g(1) = 5$, $g(3) = -27$, and $g(4) = 32$, we conclude that g has an absolute minimum at $(3,-27)$ and an absolute maximum at $(-1, 37)$.

4.3. **Derivatives and the Shapes of Curves.**

Consider an object moving in a straight line according to the differentiable position function $s = f(t)$. The following theorem implies that between any two times $t = a$ and $t = b$ there is a time $t = c$ at which the instantaneous velocity, $f'(c)$, is equal to the average velocity, $(f(b) - f(a))/(b-a)$, over the interval $[a,b]$.

Theorem 4.3.1 (Mean Value Theorem). *If f is a differentiable function on the interval $[a, b]$, then there exists a number c between a and b such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

or, equivalently, $f(b) - f(a) = f'(c)(b - a)$.

Graphically, the Mean Value Theorem states that between $x = a$ and $x = b$ there is at least one point on the curve $y = f(x)$ where the tangent line is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Recall the Increasing/Decreasing Test, which can be proved using the Mean Value Theorem.

Example 4.3.1. Where is the function $f(x) = 3x^4 - 4x^3 - 36x^2 - 35$ is increasing? decreasing?.

Solution. First, we compute that $f'(x) = 12x^3 - 12x^2 - 72x = 12x(x - 3)(x + 2)$. To use the Increasing/Decreasing Test, we need to identify where $f'(x) > 0$ and where $f'(x) < 0$. Dividing the real line into four intervals whose endpoints are the critical numbers -2, 0, 3, we perform a sign test on the factors of $f'(x)$. For example, on the interval $(-\infty, -2)$, the factors $12x$, $x-3$, and $x+2$ are all negative, implying that $f'(x) < 0$. A similar analysis of the remaining intervals leads to the conclusion that the function f is decreasing on $(-\infty, -2)$ and $(0, 3)$, and increasing on $(-2, 0)$ and $(3, \infty)$. A graph confirms this result.

As a consequence of the Increasing/Decreasing Test, we have the following useful fact.

Proposition 4.3.1 (First Derivative Test). *Suppose that c is a critical number of a continuous function f.*

(a) If f' changes from positive to negative at c, then f has a local maximum at c.
(b) If f' changes from negative to positive at c, then f has a local minimum at c.
(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

Example 4.3.2. Use the First Derivative Test to identify the local extrema of the function from Ex 4.3.1.

Solution. Since $f'(x)$ changes from negative to positive at $x = -2$, it follows that $f(-2) = -99$ is a local minimum of f. Similarly, $f(0) = -35$ is a local maximum and $f(3) = -224$ is a local minimum.

In view of the Concavity Test, we have an alternate derivative-based test for identifying the local extreme values of a function.

Proposition 4.3.2 (Second Derivative Test). *Suppose f'' is continuous around c.*

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
(c) If $f''(c) = 0$, the test is inconclusive, so $f(c)$ may be a maximum, a minimum, or neither.

Example 4.3.3. Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Solution. If $f(x) = x^4 - 4x^3$, then $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ and $f''(x) = 12x^2 - 24x = 12x(x-2)$. Setting $f'(x) = 0$ gives the critical numbers $x = 0$ and $x = 3$, while setting $f''(x) = 0$ yields $x = 0$ and $x = 2$.

Since $f''(0) = 0$ and $f''(3) = 36 > 0$, the Second Derivative Test implies that $f(3) = -27$ is a local minimum but reveals nothing about $f(0)$. However, since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test implies that f does not have a local maximum or minimum at 0.

Dividing the real line into three intervals whose endpoints are 0 and 2, we use a sign test of the second derivative to determine that f is concave upward on $(-\infty, 0)$ and $(2, \infty)$, concave downward on $(0, 2)$, and hence has inflection points at $(0, 0)$ and $(2, -16)$. We are now equipped to roughly sketch the graph of f.

Example 4.3.4. Use the First and Second Derivative Tests to sketch a graph of the function $f(x) = x^{2/3}(6-x)^{1/3}$.

Solution. First, we compute $f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}}$ and $f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}$. Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4, 6.

A sign test of the factors of f', together with the First Derivative Test, reveals that $f(0) = 0$ is a local minimum, $f(4) = 2^{5/3}$ is a local maximum, and $f(6) = 0$ is neither.

Interval	$4-x$	$x^{1/3}$	$(6-x)^{2/3}$	$f'(x)$	f
$x < 0$	+	-	+	-	decreasing on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increasing on $(0, 4)$
$4 < x < 6$	-	+	+	-	decreasing on $(4, 6)$
$x > 6$	-	+	+	-	decreasing on $(6, \infty)$

Furthermore, since $|f'(x)| \to \infty$ as $x \to 0$ and $x \to 6$, it follows that f has vertical tangents at $(0, 0)$ and $(0, 6)$. Since $x^{4/3} \geq 0$ for all x, it is easy to see that $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$, and that $f''(x) > 0$ for $x > 6$. Thus, f is concave downward on $(-\infty, 0)$ and $(0, 6)$, concave upward on $(6, \infty)$, and hence has an inflection point at $(6, 0)$. Finally, we combine this information to sketch a graph of the function f.

Example 4.3.5. Discuss the curve $f(t) = t + \cos t$, defined on $-2\pi \leq t \leq 2\pi$, with respect to intervals of increase and intervals of decrease, local maxima and minima, concavity, and points of inflection. Use this information to sketch the curve.

Solution. We first obtain $f'(t) = 1 - \sin t$ and $f''(t) = -\cos t$, which are both defined on $-2\pi \leq t \leq 2\pi$. Since $f'(t) = 0$ if and only if $\sin t = 1$, it is clear that $t = -\frac{3\pi}{2}$ and $t = \frac{\pi}{2}$ are the only critical numbers of f on its domain.

Notice that $f'(t) \geq 0$ for all t. In particular, $f'(t) > 0$ on the intervals between the critical numbers. This implies that f is increasing on $\left(-2\pi, -\frac{3\pi}{2}\right)$, $\left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{\pi}{2}, 2\pi\right)$, and it follows that f has no local maxima nor local minima.

A sign test of the second derivative, together with the Second Derivative Test, shows that f is concave downward on $\left(-2\pi, -\frac{3\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$, concave upward on $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and hence has inflection points at $\left(-\frac{3\pi}{2}, -\frac{3\pi}{2}\right)$, $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$. Now we sketch a graph of f.

4.4. **Graphing with Calculus and Calculators.**

There are no notes provided for this section. It describes the intricacies of digital curve-plotting via graphing calculators, and in particular the necessity of selecting an appropriate window in order to graphically discover as much information as possible about a function and its derivatives.

4.5. **Indeterminate Forms and l'Hopital's Rule.**

The function $f(x) = \frac{\sqrt{x}-2}{x-4}$ is not defined at $x = 4$, and both the numerator and denominator are zero when the $x = 4$. Nevertheless, we can determine the exact value of $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ by factoring the denominator. Indeed

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}.$$

Likewise, the function $g(x) = \frac{\cos x}{x-\pi/2}$ is not defined at $x = \frac{\pi}{2}$, but in this case there is no obvious way to simplify the expression in order to evaluate $\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x-\pi/2}$. The next result provides a method for determining such limits.

Theorem 4.5.1 (L'Hôpital's Rule)**.** *Let f and g be differentiable functions, with $g'(x) \neq 0$ near a (except possibly at a). Suppose that*

$$\lim_{x\to a} f(x) = 0 \qquad \textit{and} \qquad \lim_{x\to a} g(x) = 0,$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \textit{and} \qquad \lim_{x\to a} g(x) = \pm\infty$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right-hand side exists (or is $\pm\infty$).

Remark 4.5.1. L'Hôpital's Rule is also valid for one-sided limits and for limits at infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

Remark 4.5.2. Before using L'Hôpital's Rule, always be sure to check that the numerator and denominator either both converge to 0 or both converge to $\pm\infty$.

Example 4.5.1 (Indeterminate form of type 0/0)**.** Find $\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x-\pi/2}$.

Solution. As we have already observed, $\lim_{x\to\frac{\pi}{2}} \cos x = 0$ and $\lim_{x\to\frac{\pi}{2}}(x - \pi/2) = 0$. Applying L'Hôpital's Rule, we find that

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x-\pi/2} = \lim_{x\to\frac{\pi}{2}} \frac{-\sin x}{1} = \lim_{x\to\frac{\pi}{2}} (-\sin x) = -\sin\left(\frac{\pi}{2}\right) = -1.$$

Example 4.5.2. Find $\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$.

Solution. A naively application of L'Hôpital's Rule yields

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \lim_{x\to\pi^-} \frac{\cos x}{\sin x} = \lim_{x\to\pi^-} \cot x = -\infty.$$

This is **wrong**! Indeed, although the numerator approaches zero as $x \to \pi^-$, the denominator does not, so L'Hôpital's Rule cannot be applied. Instead, since the function $\frac{\sin x}{1-\cos x}$ is continuous at $x = \pi$, this limit can be easily determined by direct substitution as follows:

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \frac{\sin\pi}{1-\cos\pi} = \frac{0}{1-(-1)} = 0.$$

Example 4.5.3. Find $\lim_{x\to\left(\frac{\pi}{2}\right)^+} \frac{\cos x}{1-\sin x}$.

Solution. Since $\lim_{x\to\left(\frac{\pi}{2}\right)^+} \cos x = 0$ and $\lim_{x\to\left(\frac{\pi}{2}\right)^+}(1-\sin x) = 0$, we may apply L'Hôpital's Rule to obtain

$$\lim_{x\to\left(\frac{\pi}{2}\right)^+} \frac{\cos x}{1-\sin x} = \lim_{x\to\left(\frac{\pi}{2}\right)^+} \frac{-\sin x}{-\cos x} = \lim_{x\to\left(\frac{\pi}{2}\right)^+} \tan x = -\infty.$$

Example 4.5.4 (Indeterminate form of type ∞/∞)**.** Find $\lim_{x\to\infty} \frac{\ln(\ln x)}{x}$.

Solution. Notice that $\lim_{x\to\infty} \ln(\ln x) = \infty$ and $\lim_{x\to\infty} x = \infty$. By L'Hôpital's Rule, it follows that

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{x} = \lim_{x\to\infty} \frac{\left(\frac{1/x}{\ln x}\right)}{1} = \lim_{x\to\infty} \frac{1}{x\ln x} = 0.$$

Example 4.5.5 (Using L'Hôpital's Rule multiple times)**.** Find $\lim_{x\to 0} \frac{e^x-1-x}{x^2}$.

Solution. We see that $\lim_{x\to 0}(e^x - 1 - x) = 0$ and $\lim_{x\to 0} x^2 = 0$. Hence, L'Hôpital's Rule gives

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x}.$$

This is yet another indeterminate form, with $\lim_{x\to 0}(e^x - 1) = 0$ and $\lim_{x\to 0} 2x = 0$, so a second application of L'Hôpital's Rule yields

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x\to a} f(x)g(x)$ will be, if it even exists. However, we can evalute this type of limit explicitly by writing the product fg as a quotient, either $fg = \frac{f}{1/g}$ or $fg = \frac{g}{1/f}$, which converts the limit into an indeterminate form of type $0/0$ or ∞/∞ so that we can use L'Hôpital's Rule.

Example 4.5.6 (Indeterminate form of type $0 \cdot \infty$)**.** Find $\lim_{x\to -\infty} x^2 e^x$.

Solution. Notice that $\lim_{x\to -\infty} x^2 = \infty$, whereas $\lim_{x\to -\infty} e^x = 0$. Rewriting the product x^2e^x as the quotient $\frac{x^2}{e^{-x}}$, we may apply L'Hôpital's Rule twice in order to obtain

$$\lim_{x\to -\infty} x^2 e^x = \lim_{x\to -\infty} \frac{x^2}{e^{-x}} = \lim_{x\to -\infty} \frac{2x}{-e^{-x}} = \lim_{x\to -\infty} \frac{2}{e^{-x}} = \lim_{x\to -\infty} 2e^x = 0.$$

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit $\lim_{x\to a}[f(x) - g(x)]$ may be either $\pm\infty$ or some finite number. To find out which it is, we convert this indeterminate difference into an indeterminate quotient (for instance, by using a common denominator, rationalization, or factoring out a common factor) in order to apply L'Hôpital's Rule.

Example 4.5.7 (Indeterminate form of type $\infty - \infty$)**.** Find $\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

Solution. Since $\lim_{x\to 1^+} \frac{1}{\ln x} = \infty$ and $\lim_{x\to\infty} \frac{1}{x-1} = \infty$, we are dealing with an indeterminate difference. By establishing a common demoninator between the terms, we can combine the difference into a single quotient: $\frac{1}{\ln x} - \frac{1}{x-1} = \frac{x-1-\ln x}{(x-1)(\ln x)}$. Then $\lim_{x\to 1^+} \frac{x-1-\ln x}{(x-1)(\ln x)}$ is an indeterminate form of type $0/0$, so by two applications of L'Hôpital's Rule we have

$$\begin{aligned}
\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) &= \lim_{x\to 1^+} \frac{x-1-\ln x}{(x-1)(\ln x)} \\
&= \lim_{x\to 1^+} \frac{1 - \frac{1}{x}}{(x-1)\cdot \frac{1}{x} + \ln x \cdot 1} \\
&= \lim_{x\to 1^+} \frac{\frac{x-1}{x}}{\frac{x-1+x\ln x}{x}} \\
&= \lim_{x\to 1^+} \frac{x-1}{x-1+x\ln x} \\
&= \lim_{x\to 1^+} \frac{1}{1 + x\cdot\frac{1}{x} + \ln x \cdot 1} \\
&= \frac{1}{2}.
\end{aligned}$$

Example 4.5.8. Find $\lim_{x\to\infty}(xe^{1/x} - x)$.

Solution. Notice that $\lim_{x\to\infty} xe^{1/x} = \infty$ and $\lim_{x\to\infty} x = \infty$, and that $xe^{1/x} - x = x(e^{1/x} - 1)$ can then be rewritten as the quotient $\frac{x}{(e^{1/x}-1)^{-1}}$. Then using L'Hôpital's Rule (three times!) we obtain

$$\begin{aligned}
\lim_{x\to\infty}(xe^{1/x} - x) &= \lim_{x\to\infty} \frac{x}{(e^{1/x}-1)^{-1}} \\
&= \lim_{x\to\infty} \frac{1}{-1(e^{1/x}-1)^{-2} \cdot e^{1/x} \cdot (-x^{-2})} \\
&= \lim_{x\to\infty} \frac{(e^{1/x}-1)^2}{x^{-2}e^{1/x}} \\
&= \lim_{x\to\infty} \frac{2(e^{1/x}-1) \cdot e^{1/x} \cdot (-x^{-2})}{x^{-2} \cdot e^{1/x} \cdot (-x^{-2}) + e^{1/x} \cdot (-2x^{-3})} \\
&= \lim_{x\to\infty} \frac{2(e^{1/x}-1)}{x^{-2} + 2x^{-1}} \\
&= \lim_{x\to\infty} \frac{2e^{1/x} \cdot (-x^2)}{-2x^{-3} - 2x^{-2}} \\
&= \lim_{x\to\infty} \frac{e^{1/x}}{x^{-1} + 1} \\
&= 1.
\end{aligned}$$

Several indeterminate forms arise from the limit $\lim_{x\to a}[f(x)]^{g(x)}$, namely

1. Type 0^0, if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.
2. Type ∞^0, if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = 0$.
3. Type 1^∞, if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$.

Each of these cases can be treated either by taking the natural logarithm

$$y = [f(x)]^{g(x)} \implies \ln y = g(x)\ln[f(x)]$$

or by writing the function as an exponential

$$[f(x)]^{g(x)} = e^{g(x)\ln[f(x)]}.$$

Example 4.5.9 (Indeterminate form of type 1^∞)**.** Find $\lim_{x\to 0}(1-2x)^{1/x}$.

Solution. Since $\lim_{x\to 0}(1-2x) = 1$, and both $\lim_{x\to 0^-}(1/x)$ and $\lim_{x\to 0^+}(1/x)$ are infinite, we are dealing with an indeterminate power. Letting $y = (1-2x)^{1/x}$, and taking the natural logarithm of both sides, we obtain $\ln y = \frac{\ln(1-2x)}{x}$. Notice that $\lim_{x\to 0} \frac{\ln(1-2x)}{x}$ is an indeterminate quotient of type $0/0$, so by L'Hôpital's Rule it follows that

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1-2x)}{x} = \lim_{x\to 0} \frac{\left(\frac{-2}{1-2x}\right)}{1} = \lim_{x\to 0} \frac{-2}{1-2x} = -2.$$

Now, using the fact that $y = e^{\ln y}$, we obtain the solution

$$\lim_{x\to 0}(1-2x)^{1/x} = \lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{-2}.$$

4.6. **Optimization Problems.**

The methods we have learned in this chapter for finding extreme values can be practically applied to the class of so-called **optimization problems**, which consist of maximizing or minimizing some variable quantity. The general strategy for solving optimization problems consists of the following steps.

1. Read the problem carefully.
2. Draw a diagram.
3. Introduce notation. Assign a symbol, say Q, to the quantitiy that is to be optimized. Also select letters, such as $a, b, c, \ldots, x, y$, to represent other unknown quantities, and label the diagram with as much information as possible.
4. Express Q in terms of some of the other symbols from Step 3.
5. Use other relationships between the unknown quantities to eliminate all but one of the variables in the expression for Q. Thus, we are left with a function of the form $Q = f(x)$.

6. Use the methods of Section 4.2 and Section 4.3 to find the absolute maximum or minimum value of f. In particular, if the domain of f is a closed interval, then use the Closed Interval Method.

Notice the similarity between the steps above and those presented in Section 4.1 for solving related rate problems. In fact, many of these techniques are ubiquitous in the problem-solving strategies for all types of mathematical applications.

Example 4.6.1. A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four identical side-by-side pens. What is the largest possible total area of the pens?

Solution. Let A, l, and w be the area, length, and width respectively of the rectangular area. Then $A = lw$, and assuming that the area is divided into fourths along its length we also have $2l + 5w = 750$. Solving the latter equation for $l = \frac{750-5w}{2}$, and substituting into the former equation, we obtain $A = \left(\frac{750-5w}{2}\right) w = 375w - \frac{5}{2}w^2$. Having expressed $A = f(w)$ in terms of one variable, we now compute $f'(w) = 375 - 5w$. It follows that $w = 75$ is the only critical number of f. Since the domain of f is easily observed to be $[0, 150]$, and $f(0) = f(150) = 0$, the Closed Interval Method implies that $f(75) = 14062.5$ ft^2 is the largest possible total area of the pens.

Example 4.6.2. A box with a square base and open top must have a volume of 32000 cm^3. Find the dimensions of the box that minimize the amount of material used to construct it.

Solution. Let x and h be the side length and height respectively of the box, and let S be the amount of material used to construct it. Then $S = x^2 + 4xh$ and $x^2h = 32000$. Solving the latter equation for $h = \frac{32000}{x^2}$, and substituting into the former equation, we obtain $S = x^2 + 4x\left(\frac{32000}{x^2}\right) = x^2 + 128000x^{-1}$. Having expressed $S = f(x)$ in terms of one variable, we now compute $f'(x) = 2x - 128000x^{-2} = \frac{2x^3-128000}{x^2}$. It follows that $x = 0$ and $x = 40$ are the critical numbers of f. In this case, $x > 0$ is the only obvious domain restriction for f, so we apply the First Derivative Test. Since $f'(x) < 0$ for $x < 40$ and $f'(x) > 0$ for $x > 40$, we conclude that $x = 40$ is a local minimum, and is hence the absolute minimum, of f. Therefore, the dimensions $40 \times 40 \times 20$ cm minimize the amount of material needed to construct the box.

Expanding on the preceding example, wherein it was argued that a local minimum of the function f was actually the absolute minimum, we have the following general result based on the First Derivative Test.

Proposition 4.6.1 (First Derivative Test for Absolute Extreme Values)**.** *Suppose that c is a critical number of a continuous function f defined on an interval.*

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.
(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Example 4.6.3. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Solution. A point (x, y) on the parabola satisfies $x = \frac{1}{2}y^2$. Thus, we wish to minimize the distance

$$d = \sqrt{(x-1)^2 + (y-4)^2} = \sqrt{\left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2}.$$

Equivalently, we can minimize $f(y) = d^2$, since it is easier to work with. Differentiating, we obtain

$$f'(y) = 2\left(\frac{1}{2}y^2 - 1\right) y + 2(y-4) = y^3 - 8,$$

so $f'(y) = 0$ when $y = 2$. Observe that $f'(y) < 0$ when $y < 2$ and $f'(y) > 0$ when $y > 2$, so by the First Derivative Test for Absolute Extreme Values, the absolute minimum occurs when $y = 2$. The corresponding x-value is $x = \frac{1}{2}(2)^2 = 2$. Thus, the point on the parabola that is closest to $(1, 4)$ is $(2, 2)$.

Example 4.6.4. Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution. Let (x, y) be the vertex of the rectangle that lies on the ellipse in the first quadrant. Then the area of the rectangle is $A = (2x)(2y) = 4xy$. Solving the equation of the ellipse for $y = \frac{b}{a}\sqrt{a^2 - x^2}$, and substituting into the expression for A, we obtain $A = 4x\frac{b}{a}\sqrt{a^2 - x^2}$. Having expressed $A = f(x)$ as a function of one variable, with domain $[0, a]$, we now compute

$$f'(x) = 4\frac{b}{a}x\left[\frac{1}{2}\left(a^2 - x^2\right)^{-1/2} \cdot (-2x)\right] + 4\frac{b}{a}\left(a^2 - x^2\right)^{1/2} = \frac{4\frac{b}{a}\left(a^2 - 2x^2\right)}{\sqrt{a^2 - x^2}}.$$

It follows that

$$f'(x) = 0 \implies 4\frac{b}{a}\left(a^2 - 2x^2\right) = 0 \implies x^2 = \frac{a^2}{2} \implies x = \frac{a}{\sqrt{2}}.$$

Since $f(0) = f(a) = 0$, the Closed Interval Method implies that $f\left(a/\sqrt{2}\right) = 2ab$ is the area of the largest rectangle that can be inscribed in the ellipse.

Example 4.6.5. An 8 ft tall fence runs parallel to a building at a distance of 4 ft from its base. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building.

Solution. Let L, x, and y be the length of the ladder, the distance from the base of the building to the base of the latter, and the distance from the base of the building to the top of the ladder respectively. Then $L = \sqrt{x^2 + y^2}$ and, by similar triangles, $\frac{y}{x} = \frac{8}{x-4}$. Solving the latter equation for $y = \frac{8x}{x-4}$, and substituting into the former equation, we obtain

$$L = \sqrt{x^2 + \left(\frac{8x}{x-4}\right)^2}.$$

Rather than working with L directly, we will minimize the simpler function $L^2 = f(x)$. Differentiating with respect to x yields

$$f'(x) = 2x + 2\left(\frac{8x}{x-4}\right) \cdot \left(\frac{8(x-4) - 8x}{(x-4)^2}\right) = \frac{2x((x-4)^3 - 256)}{(x-4)^3}.$$

It follows that $f'(x) = 0 \implies x = 4 + 256^{1/3} \approx 10.35$ (note that $x = 0$ and $x = 4$ are not critical numbers because the domain of f is $(4, \infty)$). Since $f'(x) > 0$ for $x < 10.35$ and $f'(x) < 0$ for $x > 10.35$, the First Derivative Test for Absolute Extreme Values implies that $f(10.35)$ is the absolute minimum value of f. The corresponding y-value is $y = \frac{8(10.35)}{10.35-4} = 13.04$, so the shortest the ladder can be is $L = 16.65$ ft.

Recall that if the $C(x)$ represents is the cost of producing x units of a certain product, then the **marginal cost** is given by the derivative $C'(x)$. The function $c(x) = C(x)/x$ represents the **average cost** of producing x units. Note that average cost is minimized when average cost equals marginal cost. Indeed

$$c'(x) = 0 \implies \frac{xC'(x) - C(x)}{x^2} = 0 \implies \frac{C(x)}{x} = C'(x).$$

If $p(x)$ is the price per unit that a company can charge if it sells x units, then p is called the **demand function** or **price function**. If x units are sold at price $p(x)$, then the total revenue is $R(x) = xp(x)$ and the total profit is $P(x) = R(x) - C(x)$. Then the derivatives R' and P' are respectively called the **marginal revenue function** and **marginal profit function**. Note that profit is maximized when marginal revenue equals marginal cost, since $P'(x) = 0 \implies R'(x) - C'(x) = 0 \implies R'(x) = C'(x)$.

Example 4.6.6. Assume that the cost function and demand function for a certain item are $C(x) = 680 + 4x + 0.01x^2$ and $p(x) = 12 - \frac{x}{500}$ respectively. Find the production level that will maximize profit.

Solution. As remarked above, maximum profit occurs when marginal revenue equals marginal cost. In this case, $C'(x) = 4 + 0.02x$, and

$$R(x) = xp(x) = 12x - \frac{x^2}{500} \implies R'(x) = 12 - \frac{x}{250}.$$

Setting $R'(x) = C'(x)$, we have

$$12 - \frac{x}{250} = 4 + 0.02x \implies 3000 - x = 1000 + 5x \implies 2000 = 6x \implies x \approx 333.33.$$

Thus, a production level of about $x = 333$ items will maximize profit.

Example 4.6.7. A company has been selling 1000 television sets a week at \$450 each. A survey indicates that for each \$10 rebate offered to the buyer, the number of sets sold will increase by 100 per week.

(a) Find the demand function.
(b) How large a rebate should the company offer the buyer in order to maximize revenue.
(c) If the weekly cost function is $C(x) = 68000 + 150x$, what rebate amount will maximize profit?

Solution.

(a) If x is the number of television sets sold per week, then the weekly increase in sales is $x - 1000$. For each increase of 100 units sold the price is decreased by \$10, so for each additional unit sold the decrease in price will be $\frac{1}{100} \times 10$. Thus, the demand function is given by
$$p(x) = 450 - \frac{10}{100}(x - 1000) = 550 - \frac{1}{10}x.$$
(b) The revenue function is $R(x) = xp(x) = 550x - \frac{1}{10}x^2$. It follows that
$$R'(x) = 0 \implies 550 - \frac{1}{5}x = 0 \implies 550 = \frac{1}{5}x \implies x = 2750.$$
Since $R'(x) < 0$ for $x < 2750$ and $R'(x) > 0$ for $x > 2750$, the First Derivative Test for Absolute Extreme Values implies that maximum revenue is achieved when $x = 2750$. The corresponding price is $p(2750) = 275$ and the rebate is 450 - 275 = 175. Therefore, to maximize revenue, the manufacturer should offer a rebate of \$175.
(c) Setting $R'(x) = C'(x)$, we have
$$550 - \frac{1}{5}x = 150 \implies 400 = \frac{1}{5}x \implies x = 2000.$$
The corresponding price is $p(2000) = 350$ and the rebate is 450 - 350 = 100. Therefore, to maximize profit, the manufacturer should offer a rebate of \$100.

4.7. **Newton's Method.**

There are no notes provided for this section. It introduces an iterative numerical scheme, called Newton's Method, that is particularly useful for approximating the roots of complicated equations.

4.8. **Antiderivatives.**

Definition 4.8.1. A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Theorem 4.8.1. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.*

Example 4.8.1. Find the most general antiderivative of: (a) $f(x) = \sec^2 x$, (b) $g(x) = e^x$, (c) $h(x) = x^6$.

Solution.

(a) Since $\frac{d}{dx}(\tan x) = \sec^2 x$, the most general antiderivative of f is $F(x) = \tan x + C$.
(b) Since $\frac{d}{dx}(e^x) = e^x$, the most general antiderivative of g is $G(x) = e^x + C$.
(c) Since $\frac{d}{dx}\left(\frac{1}{7}x^7\right) = x^6$, the most general antiderivative of h is $H(x) = \frac{1}{7}x^7 + C$.

The following table gives particular antidifferentiation formulas for many common functions.

Function	Particular antiderivative
$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
x^n $(n \neq -1)$	$(x^{n+1})/(n+1)$
$1/x$	$\ln\|x\|$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$1/\sqrt{1-x^2}$	$\sin^{-1} x$
$1/(1+x^2)$	$\tan^{-1} x$

In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function. The second formula says that the antiderivative of a sum is the sum of the antiderivatives. We assume the convention $F' = f$ and $G' = g$.

Example 4.8.2. Find all functions f such that $f'(x) = 5\sec^2 x + \frac{2\sqrt{x}+4}{x}$.

Solution. With $f'(x) = 5\sec^2 x + 2x^{-1/2} + 4x^{-1}$, we see that the general antiderivative is given by

$$f(x) = 5\tan x + 2\left(\frac{x^{1/2}}{1/2}\right) + 4\ln|x| + C = 5\tan x + 4\sqrt{x} + 4\ln|x| + C.$$

Example 4.8.3. Find f if $f'(x) = 7e^x + 5\sin x$ and $f(0) = 1$.

Solution. The general antiderivative of f' is $f(x) = 7e^x - 5\cos x + C$, and

$$f(0) = 1 \implies 7e^0 - 5\cos(0) + C = 1 \implies C = -1.$$

Hence, $f(x) = 7e^x - 5\cos x - 1$.

Example 4.8.4. Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 3$, and $f'(1) = 5$.

Solution. The general antiderivative of f'' is $f'(x) = 4x^3 + 3x^2 - 4x + C$, and

$$f'(1) = 5 \implies 4(1)^3 + 3(1)^2 - 4(1) + C = 5 \implies C = 2.$$

That is, $f'(x) = 4x^3 + 3x^2 - 4x + 2$. Next, the general antiderivative of f' is $f(x) = x^4 + x^3 - 2x^2 + 2x + D$, and

$$f(0) = 3 \implies (0)^4 + (0)^3 - 2(0)^2 + 2(0) + D = 3 \implies D = 3.$$

Hence, $f(x) = x^4 + x^3 - 2x^2 + 2x + 3$.

Example 4.8.5. A stone is dropped off a cliff and hits the ground with a speed of 120 ft/s. Determine the height of the cliff.

Solution. The stone accelerates due to gravity at a constant rate given by $a(t) = -32$ ft/s^2. With s and v representing position and velocity respectively, we know that $a(t) = v'(t)$ and $v(t) = s'(t)$. It follows that the general antiderivative of a is $v(t) = -32t + C$, and

$$v(0) = 0 \implies -32(0) + C = 0 \implies C = 0.$$

That is, $v(t) = -32t$. Next, the general antiderivative of v is $s(t) = -16t^2 + D$. Notice that $s(0) = D$ is the unknown height of the cliff, and we are given that $v(t) = -120$ when $s(t) = 0$. Solving $v(t) = -32t = -120$ for $t = 3.75$ s, the time at which the stone hits the ground, we then have that

$$s(3.75) = 0 \implies -16(3.75)^2 + D = 0 \implies D = 225 \text{ ft}.$$

5. Integrals.

5.1. Areas and Distances.

We begin by attempting to solve the **area problem**; that is, the problem of finding the area under a curve $y = f(x)$ between two numbers $x = a$ and $x = b$. The area of a rectangle is easily computed, and we apply this concept in a limiting sense to determine the area of more general regions with curved edges.

Example 5.1.1. Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

Solution. We first notice that the area, A, of the region in question must be less than 1 (and greater than 0) since it is contained within a square of side length one. To improve this estimate, we divide the unit interval $[0, 1]$ into four equal subintervals: $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$, $[3/4, 1]$. Now, using the values of $y = x^2$ at either the left or right endpoints of these subintervals, we can construct four rectangles with combined area approximately equal to A. In particular, the left endpoint approximation is

$$L_4 = \frac{1}{4}(0)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.2188,$$

while the right endpoint approximation is

$$R_4 = \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}(1)^2 = \frac{15}{32} = 0.4688.$$

Thus, we can conclude that $0.2188 < A < 0.4688$. We can obtain better estimates by increasing the number of subintervals, as shown in the following table.

n	L_n	R_n
10	0.2850000	0.3850000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

From this, it appears that a good estimate for the area under the curve is $A = 1/3$.

Example 5.1.2. Continuing the example above, show that $\lim_{n\to\infty} L_n = 1/3$ and $\lim_{n\to\infty} R_n = 1/3$.

Solution. Dividing the unit interval $[0, 1]$ into n equal subintervals $[0, 1/n]$, $[1/n, 2/n]$, …, $[(n-1)/n, 1]$, the right endpoint approximation of A is given by

$$\begin{aligned} R_n &= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\ &= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \\ &= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\ &= \frac{2n^2 + 3n + 1}{6n^2}. \end{aligned}$$

Since the numerator and denominator of this rational expression have the same degree, it follows that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2n^2 + 3n + 1}{6n^2} = \frac{2}{6} = \frac{1}{3}.$$

A similar calculation shows that $\lim_{n\to\infty} L_n = 1/3$. This confirms that $A = 1/3$ is indeed the area under the parabola $y = x^2$ from 0 to 1.

Definition 5.1.1. The **area** A of the region that lies under the graph of the continuous function f on some interval $[a, b]$ is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $x_0 \,(= a)$, x_1, …, $x_n \,(= b)$ partition the interval $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$, and the **sigma notation**, $\sum_{i=1}^{n}$, is used to compactly denote a sum of many terms.

Remark 5.1.1. It can be proved that the limit in Definition 5.1.1 always exists, since we are assuming that f is continuous. Furthermore, the left endpoint approximation yields the same value in the limit:

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

In fact, instead of using left or right endpoints, we could take the height of the ith rectangle to be the value of f at any number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. We call the numbers x_1^*, x_2^*, …, x_n^* the **sample points**, and a more general expression for the area A is:

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Example 5.1.3. Estimate the area, A, of the region under the curve $f(x) = e^{-x}$ between $x = 0$ and $x = 2$ by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

Solution. With $n = 4$, the subintervals of equal width $\Delta x = 0.5$ are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of these subintervals are $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$, and $x_4^* = 1.75$ respectively, and the sum of the areas of the four approximating rectangles is

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(x_i^*)\Delta x \\ &= f(0.25)\Delta x + f(0.75)\Delta x + f(1.25)\Delta x + f(1.75)\Delta x \\ &= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5) \\ &\approx 0.8557. \end{aligned}$$

Thus, an estimate for the area is $A \approx 0.8557$. With $n = 10$, the subintervals are $[0, 0.2]$, $[0.2, 0.4]$, …, $[1.8, 2]$, and the midpoints are $x_1^* = 0.1$, $x_2^* = 0.3$, …, $x_{10}^* = 1.9$. Thus, a better approximation for the area is

$$A \approx M_{10} = \sum_{i=1}^{10} f(x_i^*)\Delta x = 0.2\left(e^{-0.1} + e^{-0.3} + \cdots + e^{-1.9}\right) \approx 0.8632.$$

5.2. **The Definite Integral.**

Definition 5.2.1. If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0\,(= a)$, x_1, …, $x_n\,(= b)$ be the endpoints of these subintervals, and we let x_1^*, x_2^*, …, x_n^* be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided that this limit exists. If it does exist, we say that f is **integrable** on $[a, b]$.

Remark 5.2.1. Along with the introduction of the definite integral comes new notation and terminology.

- The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums.
- In the notation $\int_a^b f(x)\,dx$, the function $f(x)$ is called the **integrand**. The numbers a and b are called the **limits of integration**; in particular, a is the **lower limit** and b is the **upper limit**.
- The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a **Riemann sum**.

Remark 5.2.2. The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral, i.e. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(s)\,ds$.

We have defined the definite integral for an integrable function. It turns out that not all functions are integrable, but the following theorem shows that the most commonly occurring functions are in fact integrable.

Theorem 5.2.1. *If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.*

If f is integrable on $[a, b]$, then the limit in Definition 5.2.1 exists and gives the same value no matter how we choose the sample points x_i^*. To simplify the calculation of the integral we often take the sample points to be right endpoints. Then $x_i^* = x_i$, and the definition of the integral simplifies as follows.

Theorem 5.2.2. *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 1, 2, \ldots, n$.

Although right endpoint sample points are convenient for calculating limits, a better approximation of an integral using a finite number of rectangles is generally achieved by choosing "midpoint" sample points. Recall that this is precisely the approach that was used in Example 5.1.3.

Theorem 5.2.3 (Midpoint Rule)**.** *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x\left[f(\bar{x}_1) + \cdots + f(\bar{x}_n)\right],$$

where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ for $i = 1, 2, \ldots, n$.

When we use a limit to evaluate a definite integral, we need to know how to work with sums. The following formulas are often useful.

Proposition 5.2.1. *For any positive integer n, the following properties hold.*

1. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
2. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
3. $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.
4. $\sum_{i=1}^{n} c = nc$.
5. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$.
6. $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$.
7. $\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$.

When we know something about the definite integral of one or more functions, the following proposition provides formulas for useful generalizations.

Proposition 5.2.2. *For real numbers a, b, c, and integrable functions f and g, the following properties hold.*

1. $\int_a^a f(x)\,dx = 0$.
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.
3. $\int_a^b c\,dx = c(b-a)$.
4. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.
5. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

6. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x) - \int_a^b g(x)\,dx$.

7. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$, *assuming that* $a < b < c$.

Additionally, we have the following comparison properties.

8. *If* $f(x) \geq 0$ *for* $a \leq x \leq b$, *then* $\int_a^b f(x)\,dx \geq 0$.

9. *If* $f(x) \geq g(x)$ *for* $a \leq x \leq b$, *then* $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

10. *If* $m \leq f(x) \leq M$ *for* $a \leq x \leq b$, *then* $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Example 5.2.1. Discuss positive, negative, and zero area "under" a curve in the context of Proposition 5.2.2 (specifically, properties 1, 2, and 7). Compute the exact area under a curve comprised of straight line segments and circular arcs.

5.3. **Evaluating Definite Integrals.**

In Section 5.2 we saw that computing definite integrals from the definition of a limit of Riemann sums is often a long and tedious process. The following important result, which forms one half of the Fundamental Theorem of Calculus (see Section 5.4), provides a useful shortcut for evaluating definite integrals when we happen to know an antiderivative of the **integrand**.

Theorem 5.3.1 (Evaluation Theorem)**.** *If* f *is continuous on the interval* $[a, b]$, *then*

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F *is any antiderivative of* f, *i.e.* $F' = f$.

Example 5.3.1. Use the Evaluation Theorem to exactly determine $\int_1^6 \frac{1}{x}\,dx$.

Solution. We have previously seen that $F(x) = \ln|x|$ is a particular antiderivative of $f(x) = \frac{1}{x}$. Therefore,

$$\int_1^6 \frac{1}{x}\,dx = \int_1^6 f(x)\,dx = F(6) - F(1) = \ln|6| - \ln|1| = \ln(6) \approx 1.7918.$$

Definition 5.3.1. If $F'(x) = f(x)$, then the **indefinite integral of $f(x)$**, is given by

$$\int f(x)\,dx = F(x) + C,$$

where C is any constant. In other words, $\int f(x)\,dx$ is the general antiderivative of f.

Remark 5.3.1. It is important to distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a number, whereas an indefinite integral $\int f(x)\,dx$ is a function (or family of functions).

From Definition 5.3.1, we see that the table of antiderivative formulas given in Section 4.9 is equivalently a table of indefinite integrals.

Example 5.3.2. Find the indefinite integral $\int \left(5 \sec x \tan x - 6\sqrt{x}\right)\,dx$.

Solution. Since $\frac{d}{dx}(\sec x)\,dx = \sec x \tan x$ and $\frac{d}{dx}\left(\frac{2}{3}x^{3/2}\right) = x^{1/2} = \sqrt{x}$, we find that

$$\int \left(5 \sec x \tan x - 6\sqrt{x}\right)\,dx = 5 \sec x - 4x^{3/2} + C.$$

Example 5.3.3. Evaluate $\int_0^{\pi/4}(\sin x + \cos x)\, dx$.

Solution. Since $\int(\sin x + \cos x)\, dx = -\cos x + \sin x + C$, we have

$$\int_0^{\pi/4} (\sin x + \cos x)\, dx = (-\cos x + \sin x)\Big|_0^{\pi/4} = (-\cos(\pi/4) + \sin(\pi/4)) - (-\cos(0) + \sin(0)) = 1.$$

Example 5.3.4. Evaluate $\int_1^9 \left(\frac{y^2 - y^2\sqrt{y} + 1}{y^2}\right) dy$.

Solution. Rewriting the integrand as $1 - y^{1/2} + y^{-2}$, we compute

$$\begin{aligned}\int_1^9 \left(\frac{y^2 - y^2\sqrt{y} + 1}{y^2}\right) dy &= \int_1^9 \left(1 - y^{1/2} + y^{-2}\right) dy \\ &= \left(y - \frac{2}{3}y^{3/2} - y^{-1}\right)\Bigg|_1^9 \\ &= \left(9 - \frac{2}{3}(9^{3/2}) - 9^{-1}\right) - \left(1 - \frac{2}{3}(1^{3/2}) - 1^{-1}\right) \\ &= -76/9 \approx -8.4444.\end{aligned}$$

Corollary 5.3.2 (Net Change Theorem)**.** *The integral of a rate of change is the net change:*

$$\int_a^b F'(x)\, dx = F(b) - F(a).$$

Example 5.3.5. A particle moves along a straight line with velocity at time t given by $v(t) = t^2 - 8t + 12$.

(a) Find the displacement of the particle from $t = 0$ to $t = 9$.
(b) Find the total distance traveled during this time period.

Solution.

(a) If position is given by s, where $s' = v$, then the displacement from $t = 0$ to $t = 9$ is

$$s(9) - s(0) = \int_0^9 (t^2 - 8t + 12)\, dt = \left(\frac{1}{3}t^3 - 4t^2 + 12t\right)\Bigg|_0^9 = \left(\frac{1}{3}(9)^3 - 4(9)^2 + 12(9)\right) - (0) = 27.$$

(b) To calculate total distance, we need to pay attention to when the particle moves in the positive direction ($v(t) > 0$) and when it moves in the negative direction ($v(t) < 0$). In both cases, the distance traveled is $\int |v(t)|\, dt$. Since $v(t) = t^2 - 8t + 12 = (t-6)(t-2)$, we have $v(t) > 0$ when $0 < t < 2$ and $6 < t < 9$, and $v(t) < 0$ when $2 < t < 6$. Thus

$$\begin{aligned}\int_0^9 |v(t)|\, dt &= \int_0^2 v(t)\, dt - \int_2^6 v(t)\, dt + \int_6^9 v(t)\, dt \\ &= \left(\frac{1}{3}t^3 - 4t^2 + 12t\right)\Bigg|_0^2 - \left(\frac{1}{3}t^3 - 4t^2 + 12t\right)\Bigg|_2^6 + \left(\frac{1}{3}t^3 - 4t^2 + 12t\right)\Bigg|_6^9 \\ &= 32/2 - (-32/3) + 27 \\ &= 145/3 \approx 48.3333.\end{aligned}$$

5.4. **The Fundamental Theorem of Calculus.**

Theorem 5.4.1 (Fundamental Theorem of Calculus)**.** *Suppose f is continuous on an interval I.*

1. *If $g(x) = \int_a^x f(t)\, dt$ for any fixed $a \in I$, then $g'(x) = f(x)$ for all $x \in I$.*

2. *If $[a, b] \subset I$, then $\int_a^b f(x)\, dx = F(b) - F(a)$, where F is any antiderivative of f, i.e. $F' = f$.*

Proof. If f has an antiderivative, say F, in closed form, then the first part of the theorem is easily verified. Indeed,

$$g(x) = \int_a^x f(t)\,dt = F(t)\big|_a^x = F(x) - F(a),$$

which implies that

$$g'(x) = \frac{d}{dx}(F(x) - F(a)) = F'(x) = f(x).$$

Now suppose that f does not have an antiderivative in closed form (e.g. $f(t) = e^{t^2}$ or $f(t) = \frac{\sin t}{t}$). We will not give a formal proof for this more difficult case, but rather an intuitive justification. Let x and $x+h$ be in the open interval (a,b), and observe that $g(x+h) - g(x)$ is the area underneath the curve $y = f(t)$ between $t = x$ and $t = x+h$. This narrow strip of area is closely approximated by a rectangle of height $f(x)$ and width h. That is, $g(x+h) - g(x) \approx f(x) \cdot h$, or

$$\frac{g(x+h) - g(x)}{h} \approx \frac{f(x) \cdot h}{h} = f(x).$$

As h approaches 0, this approximation becomes better and better, and we conclude that

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} f(x) = f(x).$$

To prove the second part of the theorem, let $g(x) = \int_a^x f(t)\,dt$. Then $g'(x) = f(x)$ (by the first part of the Theorem) and $F'(x) = f(x)$ (because F is assumed to be an antiderivative of f), so $g(x) = F(x) + C$ for some constant C. To find C, we substitute $x = a$ into the above equation and obtain $g(a) = F(a) + C$. Since

$$g(a) = \int_a^a f(t)\,dt = 0,$$

it follows that $0 = F(a) + C$ or $C = -F(a)$. Therefore, $g(x) = F(x) - F(a)$. Substituting $x = b$ into this equation yields $g(b) = F(b) - F(a)$, but $g(b) = \int_a^b f(t)\,dt$ implies that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

□

The second part of the Fundamental Theorem of Calculus (FTC2) is the same as Theorem 5.3.1 (the Evaluation Theorem), which was illustrated in Examples 5.3.2, 5.3.3, and 5.3.4. We now turn our attention to the following examples of the first part of the Fundamental Theorem of Calculus (FTC1).

Example 5.4.1. If $g(x) = \int_0^x \sqrt{1+2t}\,dt$, find $g'(x)$.

Solution. Since $f(t) = \sqrt{1+2t}$ is continuous on $(-1/2, \infty)$, FTC1 gives $g'(x) = \sqrt{1+2x}$ for $x > -1/2$.

Example 5.4.2. If $g(x) = \int_x^{10} e^{t^2}\,dt$, find $g'(x)$.

Solution. Notice that

$$g(x) = \int_x^{10} e^{t^2}\,dt = -\int_{10}^x e^{t^2}\,dt = \int_{10}^x \left(-e^{t^2}\right)\,dt.$$

Since $f(t) = -e^{t^2}$ is continuous on $\mathbb{R}$, FTC1 gives $g'(x) = -e^{x^2}$ for $x \in \mathbb{R}$.

Example 5.4.3. If $g(x) = \int_{e^x}^0 \sin^3 t\,dt$, find $g'(x)$.

Solution. First, we rewrite g as

$$g(x) = \int_{e^x}^0 \sin^3 t\,dt = -\int_0^{e^x} \sin^3 t\,dt = \int_0^{e^x} \left(-\sin^3 t\right)\,dt.$$

Now, let $u = e^x$. Given that $f(t) = -\sin^3 t$ is continuous on $\mathbb{R}$, the Chain Rule and FTC1 imply that

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left[\int_0^{e^x} \left(-\sin^3 t\right)\, dt\right] \\ &= \frac{d}{dx}\left[\int_0^{u} \left(-\sin^3 t\right)\, dt\right] \\ &= \frac{d}{du}\left[\int_0^{u} \left(-\sin^3 t\right)\, dt\right] \cdot \frac{du}{dx} \\ &= -\sin^3 u \cdot \frac{d}{dx}(e^x) \\ &= -e^x \sin^3(e^x), \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

5.5. **The Substitution Rule.**

Definition 5.5.1. If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number. The differential dy is then defined in terms of x and dx by the equation $dy = f'(y)\, dy$.

Example 5.5.1. If $y = \sin x$, then $dy = \cos x\, dx$. If $y = 4x^3$, then $dy = 12x^2\, dx$.

Proposition 5.5.1 (Substitution Rule)**.** *If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then*

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du.$$

Example 5.5.2. Find $\int x^2\sqrt{x^3+1}\, dx$.

Solution. Let $u = x^3 + 1$. Then $du = 3x^2\, dx$, or $dx = \frac{1}{3x^2}\, du$, so we have

$$\int x^2\sqrt{x^3+1}\, dx = \int x^2\sqrt{u}\left(\frac{1}{3x^2}\, du\right) = \frac{1}{3}\int u^{1/2}\, du = \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C = \frac{2}{9}(x^3+1)^{3/2} + C.$$

Example 5.5.3. Find $\int e^{\cos t}\sin t\, dt$.

Solution. Let $u = \cos t$. Then $du = -\sin t\, dt$, or $dt = -\frac{1}{\sin t}\, du$, so we have

$$\int e^{\cos t}\sin t\, dt = \int e^u \sin t\left(-\frac{1}{\sin t}\, du\right) = -\int e^u\, du = -e^u + C = -e^{\cos t} + C.$$

Example 5.5.4. Find $\int \frac{dx}{5-3x}$.

Solution. Let $u = 5 - 3x$. Then $du = -3\, dx$, or $dx = -\frac{1}{3}\, du$, so we have

$$\int \frac{dx}{5-3x} = \int \frac{-\frac{1}{3}\, du}{u} = -\frac{1}{3}\int \frac{du}{u} = -\frac{1}{3}\cdot \ln|u| + C = -\frac{1}{3}\ln|5-3x| + C.$$

Example 5.5.5. Find $\int \sin w \cos^6 w\, dw$.

Solution. Let $u = \cos w$. Then $du = -\sin w\, dw$, or $dw = -\frac{1}{\sin w}\, du$, so we have

$$\int \sin w\cos^6 w\, dw = \int \sin w \cdot u^6\left(-\frac{1}{\sin w}\, dw\right) = -\int u^6\, du = -\frac{1}{7}u^7 + C = -\frac{1}{7}\cos^7 x + C.$$

Example 5.5.6. Find $\int \frac{x}{1+x^4}\,dx$.

Solution. Let $u = x^2$. Then $du = 2x\,dx$, or $dx = \frac{1}{2x}\,du$, so we have
$$\int \frac{x}{1+x^4}\,dx = \int \frac{x}{1+u^2}\left(\frac{1}{2x}\,du\right) = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}(x^2) + C.$$

Proposition 5.5.2 (Substitution Rule for Definite Integrals)**.** *If g' is continuous on $[a, b]$, and f is continuous on the range of $u = g(x)$, then*
$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof. Let F be an antiderivative of f. Observe that $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$, since by the Chain Rule we have
$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x) = f'(g(x))g'(x).$$
Thus, FTC2 gives
$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\big|_a^b = F(g(b)) - F(g(a)).$$
On the other hand, FTC2 also gives
$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)),$$
and this completes the proof. □

Example 5.5.7. Find $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$.

Solution (1)**.** Let $u = x^2$. Then $du = 2x\,dx$, or $dx = \frac{1}{2x}\,du$. Moreover, the new limits of integration are given by $u = (0)^2 = 0$ and $u = (\sqrt{\pi})^2 = \pi$, so we have
$$\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \int_{u=0}^{u=\pi} x\cos u\left(\frac{1}{2x}\,du\right) = \frac{1}{2}\int_{u=0}^{u=\pi}\cos u\,du = \frac{1}{2}\cdot\sin u\big|_{u=0}^{u=\pi} = \frac{1}{2}(0-0) = 0.$$

Solution (2)**.** An alternate approach is to compute an indefinite integral and then use the original limits of integration. Let $u = x^2$. Then $du = 2x\,dx$, or $dx = \frac{1}{2x}\,du$, so we have
$$\int x\cos(x^2)\,dx = \int x\cos u\left(\frac{1}{2x}\,du\right) = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2) + C.$$
Hence, $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\big|_0^{\sqrt{\pi}} = 0$.

Example 5.5.8. Find $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Solution. Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$, or $dx = 2\sqrt{x}\,du = 2u\,du$. Moreover, the new limits of integration are given by $u = \sqrt{1} = 1$ and $u = \sqrt{4} = 2$, so we have
$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_{u=1}^{u=2}\frac{e^u}{u}\cdot 2u\,du = 2\int_{u=1}^{u=2} e^u\,du = 2\cdot e^u\big|_{u=1}^{u=2} = 2(e^2 - e) = 2e(e-1).$$

Example 5.5.9. Find $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$.

Solution. Let $u = \sqrt{1+2x}$. Then $du = \frac{2}{2\sqrt{1+2x}}\,dx$, or $dx = \sqrt{1+2x}\,du = u\,du$. Moreover, the new limits of integration are given by $u = \sqrt{1+2(0)} = 1$ and $u = \sqrt{1+2(4)} = 9$, so we have
$$\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx = \int_{u=1}^{u=3}\frac{x}{u}\cdot u\,du = \int_{u=1}^{u=3} x\,du.$$

Now, since $u = \sqrt{1+2x} \implies u^2 = 1+2x \implies u^2 - 1 = 2x \implies \frac{1}{2}(u^2-1) = x$, it follows that

$$\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx = \frac{1}{2}\int_{u=1}^{u=3} \left(u^2-1\right)\,du = \frac{1}{2}\cdot\left(\frac{u^3}{3}-u\right)\Bigg|_{u=1}^{u=3} = \frac{1}{2}\left(6-\left(-\frac{2}{3}\right)\right) = \frac{10}{3}.$$

Proposition 5.5.3 (Integrals of Symmetric Functions)**.** *Suppose f is continuous on $[-a,a]$.*

(a) If f is even (i.e. $f(-x) = f(x)$), then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd (i.e. $f(-x) = -f(x)$), then $\int_{-a}^a f(x)\,dx = 0$.

Example 5.5.10. Find $\int_{-2}^2 (x^4-1)\,dx$.

Solution. Let $f(x) = x^4 - 1$. Then

$$f(-x) = (-x)^4 - 1 = x^4 - 1,$$

so f is an even function. By Proposition 5.5.3, it follows that

$$\int_{-2}^2 (x^4-1)\,dx = 2\int_0^2 (x^4-1)\,dx = 2\left(\frac{1}{5}x^5 - x\right)\Bigg|_0^2 = 2\left(\frac{32}{5}-2\right) = \frac{44}{5} = 8.8.$$

Example 5.5.11. Find $\int_{-1}^1 \frac{\tan x}{1+x^2+x^4}\,dx$.

Solution. Let $f(x) = \frac{\tan x}{1+x^2+x^4}$. Then

$$f(-x) = \frac{\tan(-x)}{1+(-x)^2+(-x)^4} = \frac{-\tan x}{1+x^2+x^4} = -f(x),$$

so f is an odd function. Consequently, we have

$$\int_{-1}^1 \frac{\tan x}{1+x^2+x^4}\,dx = 0.$$

5.6. **Integration by Parts.**

Proposition 5.6.1 (Integration by Parts)**.** *If f and g are differentiable functions, then we have the formula*

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

Equivalently, letting $u = f(x)$ and $v = g(x)$, we may write $\int u\,dv = uv - \int v\,du$.

Proof. The Product Rule states that

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

In the notation for indefinite integrals, this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x),$$

or

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x).$$

Rearranging the terms, we obtain the desired formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

Now, with $u = f(x)$ and $v = g(x)$, the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so by the Substitution Rule we have $\int u\,dv = uv - \int v\,du$. ☐

Example 5.6.1. Find $\int x\sin x\,dx$

Solution. Let $u = x$ and $dv = \sin x\, dx$. Then $du = dx$ and $v = -\cos x$, and we find that

$$\int x \sin x\, dx = -x\cos x - \int (-\cos x)\, dx = -x\cos x + \sin x + C.$$

Example 5.6.2. Find $\int \ln x\, dx$

Solution. Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\, dx$ and $v = x$, and we find that

$$\int \ln x\, dx = x\ln x - \int x \cdot \frac{1}{x}\, dx = x\ln x - \int 1\, dx = x\ln x - x + C.$$

Remark 5.6.1. LIATE is a mnemonic device for recalling which functions have priority to be selected as u when integrating by parts. It stands for: logarithmic functions, inverse trigonometric functions, algebraic functions, trigonometric functions, exponential functions.

Example 5.6.3. Find $\int x^2 e^x\, dx$

Solution (1)**.** Let $u = x^2$ and $dv = e^x\, dx$. Then $du = 2x\, dx$ and $v = e^x$, and we find that

$$\int x^2 e^x\, dx = x^2 e^x - \int 2xe^x\, dx.$$

It appears that should now use integration by parts a second time. Letting $u = x$ and $dv = e^x\, dx$, so $du = dx$ and $v = e^x$, we obtain

$$\int x^2 e^x\, dx = x^2 e^x - 2\left(xe^x - \int e^x\, dx\right) = x^2e^x - 2xe^x + 2e^x + C.$$

Solution (2)**.** When integrating by parts multiple times, the following "tabular" method can be helpful.

	derivatives of u	antiderivatives of v
$+$	x^2	e^x
$-$	$2x$	e^x
$+$	2	e^x
$-$	0	e^x

From the table, we find the formula for the indefinite integral to be $\int x^2 e^x\, dx = x^2e^x - 2xe^x + 2e^x + C$.

Example 5.6.4. Find $\int \cos x\, e^x\, dx$

Solution. Letting $u = \cos x$ and $dv = e^x$, we construct the following table:

	derivatives of u	antiderivatives of v
$+$	$\cos x$	e^x
$-$	$-\sin x$	e^x
$+$	$-\cos x$	e^x
$-$	$\sin x$	e^x

Although it is clear that differentiating u will never yield zero, we can write

$$\int \cos x\, e^x\, dx = \cos x\, e^x + \sin x\, e^x - \int \cos x\, e^x\, dx.$$

It follows that $2\int \cos x\, e^x\, dx = \cos x\, e^x + \sin x\, e^x$, and hence $\int \cos x\, e^x\, dx = \frac{1}{2}e^x(\cos x + \sin x) + C$.

Proposition 5.6.2 (Integration by Parts for Definite Integrals)**.** *If f and g are differentiable functions on the interval $[a, b]$, then we have the formula*

$$\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\, dx$$

Equivalently, letting $u = f(x)$ and $v = g(x)$, we may write $\int_a^b u\, dv = uv\big|_a^b - \int_a^b v\, du$.

Example 5.6.5. Find $\int_0^1 \tan^{-1} x\,dx$

Solution. Let $u = \tan^{-1} x$ and $dv = dx$. Then $du = \frac{1}{1+x^2}\,dx$ and $v = x$, and we find that

$$\int_0^1 \tan^{-1} x\,dx = x\tan^{-1} x\Big|_0^1 - \int \frac{x}{1+x^2}\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx.$$

We now apply the method of substitution to the remaining integral term. Let $w = 1 + x^2$ (since u is already in use). Then $dw = 2x\,dx$, or $dx = \frac{1}{2x}\,dw$. Moreover, the new limits of integration are $w = 1 + (0)^2 = 1$ and $w = 1 + 1^2 = 2$, and so we have

$$\int_0^1 \frac{x}{1+x^2}\,dx = \int_{w=1}^{w=2} \frac{x}{w}\cdot\frac{1}{2x}\,dw = \frac{1}{2}\int_{w=1}^{w=2}\frac{1}{w}\,dw = \frac{1}{2}\cdot\ln|w|\Big|_{w=1}^{w=2} = \frac{1}{2}\ln 2.$$

Hence, $\int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \frac{1}{2}\ln 2$.

5.7. **Additional Techniques of Integration.**

Trigonometric Integrals: we can use trigonometric identities to integrate certain combinations of trigonometric functions.

Example 5.7.1. Find $\int \sin^3 x\cos^2 x\,dx$.

Solution. Using the Pythagorean identity and the substitution $u = \cos x$, we have

$$\begin{aligned}\int \sin^3 x\cos^2 x\,dx &= \int (1-\cos^2 x)\sin x\cos^2 x\,dx\\ &= \int (\cos^2 x - \cos^4 x)\sin x\,dx\\ &= \int (u^4 - u^2)\,du\\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C\\ &= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.\end{aligned}$$

In general, we try to write an integrand involving powers of sine and cosine in such a way that it only contains one sine factor (and the remainder of the expression in terms of cosine) or one cosine factor (and the remainder of the expression in terms of sine). If the integrand contains only even powers of both sine and cosine, however, this strategy fails, and it is useful to recall the half-angle identities.

Example 5.7.2. Find $\int \cos^2 x\,dx$.

Solution. If we write $\cos^2 x = 1 - \sin^2 x$, the integral is no simpler to evaluate. Using the half-angle formula $\cos^2 x = \frac{1+\cos(2x)}{2}$, however, we have

$$\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos(2x))\,dx = \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.$$

Trigonometric Substitutions: when dealing with integrals involving expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, it is often effective to make a trigonometric substitution that eliminates the radical.

Example 5.7.3. Find $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Solution. Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\, d\theta$, so

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\,(3\cos\theta\,d\theta) = \int \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta}\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta.$$

Now, using the identity $1+\cot^2\theta = \csc^2\theta$, and the fact that $\int \csc^2\theta\,d\theta = -\cot\theta + C$, we have

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int (\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C.$$

The above example suggests that if an integrand contains a factor of the form $\sqrt{a^2-x^2}$, then a trigonometric substitution of $x = a\sin\theta$ or $x = a\cos\theta$ may be effective. However, this is not always the best method. To evaluate $\int x\sqrt{a^2-x^2}\,dx$, for instance, a simpler substitution is $u = a^2 - x^2$, because then $du = -2x\,dx$.

When an integrand contains an expression of the form $\sqrt{a^2+x^2}$, the substitution $x = a\tan\theta$ should be considered because the identity $1+\tan^2\theta = \sec^2\theta$ eliminates the root sign. Similarly, if the factor $\sqrt{x^2-a^2}$ occurs, the substitution $x = a\sec\theta$ is effective.

Example 5.7.4. Find $\int \frac{dx}{x^2\sqrt{x^2+4}}$.

Solution. Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$, and we find that

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} = \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\tan^2\theta+1}}\,d\theta = \frac{1}{4}\int \csc\theta\cot\theta\,d\theta.$$

Using the fact that $\int \csc\theta\cot\theta\,d\theta = -\csc\theta + C$, we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{1}{4}\csc\theta + C = -\frac{\sqrt{x^2+4}}{4x} + C.$$

Partial Fractions: when integrating rational functions (ratios of polynomials), it is sometimes convenient to express them as sums of simpler quotients that we already know how to integrate.

Example 5.7.5. Find $\int \frac{5x-4}{2x^2+x-1}\,dx$.

Solution. Notice that $\frac{5x-4}{2x^2+x-1} = \frac{5x-4}{(x+1)(2x-1)}$. In a case like this, where the numerator has a smaller degree than the denominator, we can write the given rational function as a sum of **partial fractions**:

$$\frac{5x-4}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1}.$$

To find the values of A and B, we multiply both sides of the above equation by $(x+1)(2x-1)$, obtaining

$$5x-4 = A(2x-1) + B(x+1) \implies 5x-4 = (2A+B)x + (-A+B).$$

The coefficients of x on either side of the equation must be equal, and likewise the constant terms must be equal. This yields a system of linear equations:

$$\begin{cases} 2A+B &= 5, \\ -A+B &= -4. \end{cases}$$

Solving for $A = 3$ and $B = -1$, it follows that

$$\frac{5x-4}{(x+1)(2x-1)} = \frac{3}{x+1} - \frac{1}{2x-1},$$

and each of the partial fractions on the right-hand side is easy to integrate. Indeed, using the substitutions $u = x+1$ and $u = 2x-1$, respectively, we have

$$\int \frac{5x-4}{2x^2+x-1}\,dx = \int \frac{3}{x+1}\,dx - \int \frac{1}{2x-1}\,dx = 3\ln|x+1| - \frac{1}{2}\ln|2x-1| + C.$$

Remark 5.7.1. The method of partial fractions can be extended to more complicated rational functions.

(a) If the degree of the numerator is greater than or equal to the degree of the denominator, we must first take the preliminary step of performing long division. For instance,
$$\frac{2x^3-11x^2-2x+2}{2x^2+x-1}=x-6+\frac{5x-4}{(x+1)(2x-1)}.$$
(b) If the denominator has more than two linear factors, we need to include a term corresponding to each factor. For example,
$$\frac{x+6}{x(x-3)(4x+5)}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{4x+5}.$$
(c) If a linear factor is repeated, we need to include extra terms in the partial fraction expansion. Here's an example:
$$\frac{x}{(x+2)^2(x-1)}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{x-1}.$$
(d) When we factor a denominator as far as possible, it might happen that we obtain an irreducible quadratic factor ax^2+bx+c, where the discriminant b^2-4ac is negative. Then the corresponding partial fraction is of the form
$$\frac{Ax+B}{ax^2+bx+c}.$$
To integrate this type of expression, we consider the two terms $\frac{Ax}{ax^2+bx+c}$ and $\frac{B}{ax^2+bx+c}$ separately. The former can often be integrated using the Substitution Rule, while for the latter it may be helpful to complete the square in the denominator and then apply the formula $\int\frac{du}{u^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$.

Example 5.7.6. Find $\int\frac{x^2+2}{x^3-4x}\,dx$.

Solution. Since $x^3-4x=x(x-2)(x+2)$, we write
$$\frac{x^2+2}{x(x-2)(x+2)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}.$$
Multiplying by the common denominator yields
$$\begin{aligned}x^2+2&=A(x-2)(x+2)+B(x)(x+2)+C(x)(x-2)\\&=A(x^2-4)+B(x^2+2x)+C(x^2-2x)\\&=(A+B+C)x^2+(2B-2C)x-4A.\end{aligned}$$
Equating the coefficients of like-powered terms, we obtain the linear system
$$\begin{cases}A+B+C&=&1,\\2B-2C&=&0,\\-4A&=&2.\end{cases}$$
Solving for $A=-1/2$ and $B=C=3/4$, it follows that
$$\frac{x^2+2}{x^3-4x}=-\frac{1}{2x}+\frac{3}{4(x-2)}+\frac{3}{4(x+2)},$$
and we are able to evaluate the integral
$$\int\frac{x^2+2}{x^3-4x}\,dx=-\frac{1}{2}\ln|x|+\frac{3}{4}\ln|x-2|+\frac{3}{4}\ln|x+2|+C.$$

5.8. Integration Using Tables and Computer Algebra Systems.

There are no notes provided for this section. It discusses computational strategies for evaluating complicated integrals.

5.9. Approximate Integration.

There are no notes provided for this section. It introduces two methods (the Trapezoidal Rule and Simpson's Rule) for accurately approximating the area under a curve when it is impossible to find the exact value of a definite integral.

5.10. **Improper Integrals.**

Definition 5.10.1 (Improper Integral of Type 1)**.**

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \ge a$, then
$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx,$$
provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \le b$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx,$$
provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are said to be **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent for some $a \in \mathbb{R}$, then we define
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

Example 5.10.1. Calculate $\int_1^\infty \frac{1}{x^2}\,dx$.

Solution. Let $t \ge 1$. Then
$$\int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^t = -\frac{1}{t} - \left(-\frac{1}{1}\right) = 1 - \frac{1}{t}.$$
Since $\lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1$, we conclude that
$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{t^2}\,dx = \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1.$$

Example 5.10.2. Show that $\int_1^\infty \frac{1}{x}\,dx$ is divergent.

Solution. Let $t \ge 1$. Then
$$\int_1^t \frac{1}{x}\,dx = \ln|x|\big|_1^t = \ln|t| - \ln|1| = \ln t.$$
Since $\lim_{t\to\infty}\ln t = \infty$, we conclude that $\int_1^\infty \frac{1}{x}\,dx$ is divergent.

The results of Examples 5.10.1 and 5.10.2 are generalized in the following proposition.

Proposition 5.10.1. *The improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \le 1$.*

Example 5.10.3. Calculate $\int_{-\infty}^0 x^2 e^{x^3}\,dx$.

Solution. Let $t \le 0$. Then, substituting $u = x^3$, we have
$$\int_t^0 x^2 e^{x^3}\,dx = \frac{1}{3}\int_{t^3}^0 e^u\,du = \frac{1}{3}e^u\bigg|_{t^3}^0 = \frac{1}{3} - \frac{1}{3}e^{t^3}.$$
Since $\lim_{t\to-\infty}\left(\frac{1}{3} - \frac{1}{3}e^{t^3}\right) = \frac{1}{3}$, we conclude that
$$\int_{-\infty}^0 x^2 e^{x^3}\,dx = \lim_{t\to-\infty}\int_t^0 x^2 e^{x^3}\,dx = \lim_{t\to-\infty}\left(\frac{1}{3} - \frac{1}{3}e^{t^3}\right) = \frac{1}{3}.$$

Example 5.10.4. Calculate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$.

Solution. First, notice that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\, dx + \int_{0}^{\infty} \frac{1}{1+x^2}\, dx,$$

provided that the improper integrals on the right-hand side are convergent. Since

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\, dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\, dx = \lim_{t\to-\infty} \tan^{-1} x\big|_t^0 = \lim_{t\to-\infty} (-\tan^{-1} t) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2},$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\, dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\, dx = \lim_{t\to\infty} \tan^{-1} x\big|_0^t = \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2},$$

we conclude that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Definition 5.10.2 (Improper Integral of Type 2)**.**

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\, dx = \lim_{t\to b^-} \int_a^t f(x)\, dx,$$

provided this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\, dx = \lim_{t\to a^+} \int_t^b f(x)\, dx,$$

provided this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\, dx$ is said to be **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ are convergent, then we define

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Example 5.10.5. Find $\int_1^5 \frac{1}{\sqrt{x-1}}\, dx$.

Solution. This is an improper integral of Type 2, since $f(x) = \frac{1}{\sqrt{x-1}}$ is discontinuous at $x = 1$. Thus,

$$\int_1^5 \frac{1}{\sqrt{x-1}}\, dx = \lim_{t\to1^+} \int_t^5 \frac{1}{\sqrt{x-1}}\, dx = \lim_{t\to1^+} \left(2\sqrt{x-1}\big|_t^5\right) = \lim_{t\to1^+} 2\left(2-\sqrt{t-1}\right) = 4.$$

Example 5.10.6. Find $\int_0^{\pi/2} \tan\theta\, d\theta$.

Solution. Again, this is an improper integral of Type 2, since $f(x) = \tan\theta$ is discontinuous at $\theta = \pi/2$. Recall that $\frac{d}{d\theta}(\sec\theta) = \sec\theta\tan\theta$. Since

$$\int \tan\theta\, d\theta = \int \tan\theta \cdot \frac{\sec\theta}{\sec\theta}\, d\theta = \int \frac{\frac{d}{d\theta}(\sec\theta)}{\sec\theta}\, d\theta = \ln|\sec\theta| + C,$$

it follows that

$$\int_0^{\pi/2} \tan\theta\, d\theta = \lim_{t\to\pi/2^-} \int_0^t \tan\theta\, d\theta = \lim_{t\to\pi/2^-} \ln|\sec\theta|\big|_0^t = \lim_{t\to\pi/2^-} (\ln|\sec t| - \ln|\sec(0)|) = \infty,$$

since $\lim_{t\to\pi/2^-} \sec t = \infty$, and hence $\int_0^{\pi/2} \tan\theta\, d\theta$ must be divergent.

Example 5.10.7. Find $\int_1^4 \frac{1}{x-2}\,dx$.

Solution. Since the line $x = 2$ is a vertical asymptote for the integrand, we begin by writing

$$\int_1^4 \frac{1}{x-2}\,dx = \int_1^2 \frac{1}{x-2}\,dx + \int_2^4 \frac{1}{x-2}\,dx.$$

Next, notice that

$$\int_1^2 \frac{1}{x-2}\,dx = \lim_{t\to 2^-} \int_1^t \frac{1}{x-2}\,dx = \lim_{t\to 2^-} \ln|x-2|\big|_1^t = \lim_{t\to 2^-} (\ln|t-2| - \ln|-1|) = \lim_{t\to 2^-} \ln(1-t) = -\infty.$$

Thus, $\int_1^2 \frac{1}{x-2}\,dx$ diverges, so even without computing $\int_2^4 \frac{1}{x-2}\,dx$ we can conclude that $\int_1^4 \frac{1}{x-2}\,dx$ diverges.

Theorem 5.10.1 (Comparison Theorem for Improper Integrals)**.** *Let $a \in \mathbb{R}$, and suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for all $x \geq a$.*

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Example 5.10.8. Show that $\int_0^\infty e^{-x^2}\,dx$ converges.

Solution. First, we write

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx.$$

Since e^{-x^2} is continuous everywhere, the first term on the right-hand side is simply an ordinary definite integral; that is, it is equal to some finite number, and does not impact the convergence of $\int_0^\infty e^{-x^2}\,dx$. On the other hand, $x \geq 1 \implies x^2 \geq x \implies -x \geq -x^2 \implies e^{-x} \geq e^{-x^2} \geq 0$, and since

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx = \lim_{t\to\infty} \left(-e^{-x}\big|_1^t\right) = \lim_{t\to\infty} \left(e^{-1} - e^{-t}\right) = e^{-1} < \infty,$$

we conclude by Theorem 5.10.1 that $\int_0^\infty e^{-x^2}\,dx$ converges.

6. Applications of Integration.

6.1. More About Areas.

Theorem 6.1.1. *The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all $x \in [a, b]$, is*

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Remark 6.1.1. When $f(x) \geq g(x) \geq 0$ for all $x \in [a, b]$ in the above theorem, a quick sketch easily justifies the result. If f and g are not necessarily nonnegative, we can apply a vertical shift to both of them in such a way that the transformed functions are nonnegative and the area between them remains unchanged, and then the first case applies.

Remark 6.1.2. A more fundamental approach to establishing Theorem 6.1.1 is to consider the limit as n approaches infinity of the Riemann sum $\sum_{i=1}^{n}[f(x_i^*) - g(x_i^*)]\Delta x$, which represents an n-rectangle approximation of the area between the curves f and g.

Example 6.1.1. Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = -\pi/4$, and $x = \pi/4$.

Solution. A quick sketch shows that $\cos x \geq \sin x$ for $-\pi/4 \leq x \leq \pi/4$. Hence, the area of the region in question is

$$A = \int_{-\pi/4}^{\pi/4} (\cos x - \sin x)\, dx = (\sin x + \cos x)\Big|_{-\pi/4}^{\pi/4} = (\sin(\pi/4) + \cos(\pi/4)) - (\sin(-\pi/4) + \cos(-\pi/4)) = \sqrt{2}.$$

Example 6.1.2. Find the area of the region enclosed by the curves $f(x) = 1 + 3x - x^2$ and $g(x) = x^2 + x + 1$.

Solution. We first find the points of intersection of these two parabolas by solving their equations simultaneously:

$$f(x) = g(x) \implies 1 + 3x - x^2 = x^2 + x + 1 \implies 2x^2 - 2x = 0 \implies 2x(x-1) = 0 \implies x = 0,\ x = 1.$$

Thus, the points of intersection are $(0, 1)$ and $(1, 3)$. Notice that $f(x) \geq g(x)$ on the interval $[0, 1]$. Hence, the area of the region enclosed by the curves f and g is given by

$$\begin{aligned} A &= \int_0^1 (f(x) - g(x))\ dx \\ &= \int_0^1 \big(1 + 3x - x^2 - (x^2 + x + 1)\big)\ dx \\ &= \int_0^1 \big(-2x^2 + 2x\big)\ dx \\ &= \left(-\frac{2}{3}x^3 + x^2\right)\Big|_0^1 \\ &= \frac{1}{3}. \end{aligned}$$

Example 6.1.3. Find the area of the region enclosed by the curves $4x + y^2 = 12$ and $x = y$.

Solution. First, we rewrite the first equation in terms of x as a function of y, i.e. $x = 3 - \frac{1}{4}y^2$. Then the y-values of the points of intersection of $x = 3 - \frac{1}{4}y^2$ and $x = y$ are given by

$$3 - \frac{1}{4}y^2 = y \implies y^2 + 4y - 12 = 0 \implies (y+6)(y-2) = 0 \implies y = -6,\ y = 2.$$

Since $3 - \frac{1}{4}y^2 \geq y$ on the interval $[0,1]$, it follows that the area of the region enclosed by the curves $4x + y^2 = 12$ and $x = y$ is given by

$$A = \int_{-6}^{2} \left(\left(3 - \frac{1}{4}y^2\right) - y \right) dy = \left(3y - \frac{1}{12}y^3 - \frac{1}{2}y^2\right)\Big|_{-6}^{2} = \frac{64}{3} \approx 21.3333.$$

6.2. **Volumes.**

Definition 6.2.1. Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\, dx.$$

Example 6.2.1. Find the volume of a sphere of radius r.

Solution. If we place the sphere so that its center is at the origin, then its intersection with the plane P_x is a circle whose radius (by the Pythagorean Theorem) is $y = \sqrt{r^2 - x^2}$ for $-r \leq x \leq r$. Thus, the cross-sectional area of the sphere in P_x is given by

$$A = \pi y^2 = \pi(r^2 - x^2).$$

Using the definition of volume, with $a = -r$ and $b = r$, and the fact that $A(x)$ is even, we then have

$$V = \int_{-r}^{r} A(x)\, dx = \int_{-r}^{r} \pi(r^2 - x^2)\, dx = 2\pi \int_0^r (r^2 - x^2)\, dx = 2\pi \left(r^2 x - \frac{1}{3}x^3\right)\Big|_0^r = \frac{4}{3}\pi r^3.$$

Example 6.2.2. Find the volume of a pyramid whose base is a square with side-length L and whose height is h

Solution. If we place the vertex of the pyramid at the origin, and its central axis along the x-axis, then it intersects the plane P_x in a square of side length s for $0 \leq x \leq h$. We can express s in terms of x by observing from similar triangles that

$$\frac{x}{h} = \frac{s/2}{L/2} = \frac{s}{L} \implies s = \frac{L}{h}x.$$

Thus, the cross-sectional area of the pyramid in P_x is $A(x) = \frac{L^2}{h^2}x^2$, and is volume is

$$V = \frac{L^2}{h^2} \int_0^h x^2\, dx = \frac{L^2}{h^2} \left(\frac{1}{3}x^3\right)\Big|_0^h = \frac{1}{3}L^2 h.$$

Proposition 6.2.1. *If f is continuous on $[a,b]$, and $f(x) \geq 0$ on $[a,b]$, then the volume of the solid obtained by rotating the region under the curve $y = f(x)$ from a to b about the x-axis is given by $V = \int_a^b \pi(f(x))^2\, dx$.*

Proof. For each x between a and b, the intersection of the solid with the plane P_x is a circle of radius $f(x)$. Thus, the cross-sectional area is $A(x) = \pi(f(x))^2$, and the volume of the solid of revolution is given by

$$V = \int_a^b \pi(f(x))^2\, dx.$$

□

Example 6.2.3. Find the volume of the solid obtained by rotating the region under the curve $y = e^x$ from $x = 0$ to $x = \ln 4$ about the x-axis.

Solution. By Proposition 6.2.1, we have

$$V = \int_0^{\ln 4} \pi(e^x)^2\, dx = \pi \int_0^{\ln 4} e^{2x}\, dx = \frac{\pi}{2}\, e^{2x}\Big|_0^{\ln 4} = \frac{\pi}{2}\left(e^{2\ln 4} - e^0\right) = \frac{15\pi}{2} \approx 23.5619.$$

Example 6.2.4. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 4$, and $x = 0$ about the y-axis.

Solution. First, we rewrite $y = \sqrt{x}$ as $x = y^2$ for $y \geq 0$. Then, replacing x by y in Proposition 6.2.1, we obtain

$$V = \int_0^4 \pi(y^2)^2\, dy = \pi \int_0^4 y^4\, dy = \frac{\pi}{5}y^5\Big|_0^4 = \frac{1024\pi}{5} \approx 643.3982.$$

Example 6.2.5. Find the volume of the solid obtained by rotating the region under the curve $y = e^x$ from $x = 0$ to $x = \ln 4$ about the line $y = -1$.

Solution. In this case, we cannot conveniently apply Proposition 6.2.1. It is easy to see, however, that the cross-sectional areas of the solid are circles of radius e^x+1. Thus, $A(x) = \pi(e^x+1)^2 = \pi(e^{2x}+2e^x+1)$ for $0 \leq x \leq \ln 4$, and the volume of the solid is given by

$$V = \int_0^{\ln 4} \pi(e^{2x} + 2e^x + 1)\, dx = \pi\left(\frac{1}{2}e^{2x} + 2e^x + x\right)\Big|_0^{\ln 4} = \pi\left(\frac{27}{2} + \ln 4\right) \approx 47.7667.$$

Example 6.2.6. The region enclosed by the curves $y = \sqrt{x}$ and $y = x$ is rotated about the x-axis. Find the volume of the resulting solid.

Solution. Notice that, for $0 \leq x \leq 1$, the intersection of the specified solid with the plane P_x is a "washer" with inner radius x and outer radius $\sqrt{x}$. Thus, the area is $A(x) = \pi(\sqrt{x})^2 - \pi x^2 = \pi(x - x^2)$, and the volume of the solid is

$$V = \int_0^1 \pi(x - x^2)\, dx = \pi\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6} \approx 0.5236.$$

The preceding examples illustrate two important approaches for computing the volume of a solid of revolution – the **disk method** and the **washer method** – which are distinguished in the following remark.

Remark 6.2.1. In general, we calculate the volume of a solid of revolution using the basic defining formula $V = \int_a^b A(x)\, dx$ or $V = \int_c^d A(y)\, dy$, and we find the cross-sectional area $A(x)$ or $A(y)$ in one of two ways:

- If the cross-section is a circle or disk (as in Examples 6.2.3, 6.2.4, and 6.2.5), we find its radius, r, and use $A = \pi r^2$.
- If the cross-section is a washer (as in Example 6.3.2), we find its inner radius, r_{in}, and its outer radius, r_{out}, and use $A = \pi(r_{\text{out}}^2 - r_{\text{in}}^2)$.

6.3. **Volumes by Cylindrical Shells.**

Some problems involving solids of revolution are very difficult, if not impossible, to solve by the disk and washer methods described in Section Section 6.2. For instance, consider the problem of finding the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis. Slicing this solid perpendicular to the y-axis, we obtain a washer, but to determine its inner and outer radius we would need to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y, a daunting prospect. Instead, we now consider the so-called **method of cylindrical shells**.

Proposition 6.3.1. *If f is continuous on $[a, b]$, and $f(x) \geq 0$ on $[a, b]$, then the volume of the solid obtained by rotating the region under the curve $y = f(x)$ from a to b about the y-axis is given by $V = \int_a^b (2\pi x)f(x)\, dx$.*

Example 6.3.1. Find the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis.

Solution. Notice that the specified region is the same as the region under the curve $y = 2x^2 - x^3$ between $x = 0$ and $x = 2$. Thus, by Proposition (6.3.1), we have

$$V = \int_0^2 (2\pi x)(2x^2 - x^3)\,dx = 2\pi \int_0^2 (2x^3 - x^4)\,dx = 2\pi \left(\frac{1}{2}x^4 - \frac{1}{5}x^5\right)\Big|_0^2 = 2\pi\left(8 - \frac{32}{5}\right) = \frac{16\pi}{5} \approx 10.0531.$$

Example 6.3.2. Find the volume of the solid obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ about the y-axis.

Solution. Clearly, the enclosed region is contained between $x = 0$ and $x = 1$, and a typical shell has radius x, circumference $2\pi x$, and height $x - x^2$. Thus, following the logic of Proposition (6.3.1), we obtain

$$V = \int_0^1 (2\pi x)(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx = 2\pi \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6} \approx 0.5236.$$

6.4. **Arc Length.**

Suppose that a curve C is given by the parametric equations $x = f(t)$ and $y = g(t)$, where $a \le t \le b$, and assume that C is **smooth** in the sense that the derivatives $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero for $a < t < b$ (this condition ensures that C has no sudden change in direction). To calculate the length of C, we first divide the interval $[a, b]$ into n subintervals of equal length $\Delta t = \frac{b-a}{n}$. If $t_0, t_1, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of the points $P_i(x_i, y_i)$ that lie on C. The length, L, of C is approximated by the sum of the lengths of the distances between successive points P_i, and the approximation improves as n increases. Thus, we set

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|.$$

For computational purposes, we need a more convenient expression for L. If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, then the length of the ith approximating line segment is

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}.$$

Then, by the definition of the derivative, we have $f'(t_i) \approx \frac{\Delta x_i}{\Delta t}$ and $g'(t_i) \approx \frac{\Delta y_i}{\Delta t}$ for Δt small. Equivalently, $\Delta x_i \approx f'(t_i)\Delta t$ and $\Delta y_i \approx g'(t_i)\Delta t$, so

$$|P_{i-1}P_i| \approx \sqrt{[f'(t_i)\Delta t]^2 + [g'(t_i)\Delta t]^2} \approx \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2} \cdot \Delta t.$$

Thus,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2} \cdot \Delta t = \int_a^b \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2}\,dt.$$

We now formalize the concept of arc length in the following definition.

Definition 6.4.1 (Arc Length Formula)**.** If a smooth curve defined by the parametric equations $x = f(t)$ and $y = g(t)$, where $a \le t \le b$, is traversed exactly once as t increases from a to b, then its **arc length** is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Remark 6.4.1. If we are given a curve with equation $y = f(x)$ for $a \le x \le b$, then regarding x as a parameter yields the parametric equations $x = x$ and $y = f(x)$, and the arc length formula becomes

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

Similarly, if $x = f(y)$ for $a \le y \le b$, then we can regard y as a parameter and obtain the arc length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dx.$$

Example 6.4.1. Find the length of the curve defined by $x = \cos t$ and $y = \sin t$, for $-\pi/2 \le t \le \pi$.

Solution. It is easy to see that $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t$. Thus, by the definition, we have

$$L = \int_{-\pi/2}^{\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_{-\pi/2}^{\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt = \int_{-\pi/2}^{\pi} 1\, dt = \pi - (-\pi/2) = \frac{3\pi}{2}.$$

This conclusion can be verified by plotting the parametric curve. Indeed, its graph traces out three-fourths of the unit circle in the Cartesian plane, which has known circumference (i.e. arc length) 2π.

Example 6.4.2. Find the length of the curve defined by $x = t^3$ and $y = t^2$, for $0 \le t \le 1$.

Solution. Since $\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 2t$, we can use the Substitution Rule to obtain

$$L = \int_0^1 \sqrt{\left(3t^2\right)^2 + (2t)^2}\, dt = \int_0^1 \sqrt{9t^4 + 4t^2}\, dt = \int_0^1 t\sqrt{9t^2 + 4}\, dt = \frac{1}{27}\left(13^{3/2} - 8\right) \approx 1.4397.$$

6.5. **Average Value of a Function.**

Suppose that we would like to calculate the average value of a function, f, on an interval $[a, b]$. Dividing $[a, b]$ into n equal subintervals, each with length $\Delta x = \frac{b-a}{n}$, we can choose arbitrary points $x_1^*, \ldots, x_n^*$ in successive subintervals, and compute the average of the numbers $f(x_1^*), \ldots, f(x_n^*)$ to be

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{n} = \frac{f(x_1^*) + \cdots + f(x_n^*)}{\left(\frac{b-a}{\Delta x}\right)} = (f(x_1^*) + \cdots + f(x_n^*)) \cdot \frac{\Delta x}{b-a} = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

For large n, the above expression represents the average of a large number of closely-spaced values of the function f on the interval $[a, b]$. Thus, the fact that

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\, dx$$

motivates the following definition.

Definition 6.5.1. The **average value** of a function f on an interval $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

Example 6.5.1. Find the average value of $f(x) = \cos x$ on the interval $[0, \pi/2]$.

Solution. By the definition, we have

$$f_{\text{ave}} = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} \cos x\, dx = \frac{2}{\pi}\left(\sin x\Big|_0^{\pi/2}\right) = \frac{2}{\pi}\left(\sin(\pi/2) - \sin(0)\right) = \frac{2}{\pi}.$$

6.6. **Applications to Physics and Engineering.**

Definition 6.6.1. If a constant force of magnitude F is acting on a rigid body that is moving translationally in the direction of the force, then the **work** W done by this force along a path of length d is given by $W = Fd$.

A natural question is: how do we calculate work if an object is being acted on by a variable force?

Suppose that the object moves along the x-axis from $x = a$ to $x = b$, and at each point x between a and b a continuous force $f(x)$ acts on the object. We divide the interval $[a, b]$ into n equal subintervals of equal length $\Delta x = \frac{b-a}{n}$, and select a sample point x_i^* from each subinterval.

If n is large, then Δx is small, and since f is continuous it stays nearly constant on each subinterval. Thus, the work done in moving the object across the ith subinterval is approximately $f(x_i^*)\Delta x$, and the total work done in moving the object from a to b is appoximately given by $\sum_{i=1}^{n} f(x_i^*)\Delta x$.

As n approaches infinity, Δx approaches 0, and the approximation $W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$ becomes more and more accurate. Therefore, we define the work done in moving the object from a to b as

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx.$$

Example 6.6.1. When a particle is located at a distance of x ft from the origin, a force of $f(x) = x^2+2x$ lb acts on it. How much work is done in moving the particle from $x = 1$ to $x = 3$?

Solution. Following the discussion above, we conclude that the work done is given by

$$W = \int_1^3 (x^2 + 2x)\,dx = \left(\frac{1}{3}x^3 + x^2\right)\Big|_1^3 = \left(\frac{1}{3}(3)^3 + 3^2\right) - \left(\frac{1}{3}(1)^3 - 1^2\right) = \frac{50}{3} \approx 16.6667 \text{ ft-lb.}$$

Example 6.6.2. A force of 10 N is required to hold a spring that has been stretched 5 cm beyond its natural length. How much work is done in stretching the spring an additional 5 cm beyond its natural length?

Solution. According to Hooke's Law, the force required to hold a spring that has been stretched x meters beyond its natural length is given by $f(x) = kx$, where $k > 0$ is called the **spring constant**. In this case, it follows that

$$10 = 0.05k \implies k = 200.$$

Thus, $f(x) = 200x$ N and the work done in stretching the spring an additional 5 cm (i.e. from 0.05 m beyond its natural length to 0.10 m beyond its natural length) is

$$W = \int_{0.05}^{0.10} 200x\,dx = 100x^2\Big|_{0.05}^{0.10} = 100\left((0.10)^2 - (0.05)^2\right) = 0.75 \text{ J.}$$

Example 6.6.3. A 10-ft chain weighs 25 lb and hangs from a ceiling. Find the work done in lifting the lower end of the chain to the ceiling so that it is level with the upper end.

Solution. In this case, the upper half of the chain never moves, but a point that is $0 \le x \le 5$ ft from the bottom of the chain must be raised a distance of $10 - 2x$ ft. Since the weight of the chain is a constant 2.5 lb/ft, the force required to lift any point on the lower half of it is given by $f(x) = 2.5(10 - 2x)$ lb. Hence, the work done in lifting the lower end of the chain to the ceiling is

$$W = \int_0^5 2.5(10 - 2x)\,dx = 2.5\left(10x - x^2\right)\Big|_0^5 = 2.5\left(\left(10(5) - (5)^2\right) - 0\right) = 62.5 \text{ ft-lb.}$$

Example 6.6.4. A cylindrical tank with a height of 4 m and a diameter of 2 m is filled with water. Find the work required to empty the tank by pumping all the water out over the top.

Solution. We begin by dividing the interval $[0, 4]$ into n identical subintervals and choosing a sample point, x_i^*, from each. These subintervals divide the water in the tank into n layers, each of which has a circular cross-section with radius $r = 1$. The approximate volume of the ith layer of water is

$$V_i = \pi r^2 \Delta x \approx \pi \Delta x \text{ m}^3.$$

Using the fact that the density of water is $\rho = 1000$ kg/m^3, it follows that the mass of the ith layer is

$$m_i = \rho V_i \approx 1000\pi\Delta x \text{ kg.}$$

Recall Newton's Second Law of Motion, which states that force equals mass times acceleration. In this case, the acceleration is due to gravity: $a = 9.8$ m/s^2. Thus, the force required to raise the ith layer of water is

$$F_i = m_i a \approx 9800\pi\Delta x \text{ N}$$

and the work done in raising it to the top of the tank is

$$W_i = F_i x_i^* \approx 9800\pi x_i^* \Delta x \text{ J.}$$

Hence, the work done in emptying the tank by pumping all of the water out over the top is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} W_i = \int_0^4 9800\pi x\, dx = 4900\pi x^2 \Big|_0^4 = 4900\pi \left((4)^2 - 0\right) = 78400\pi \approx 2.46 \times 10^5 \text{ J.}$$

6.7. **Applications to Economics and Biology.**

6.8. **Probability.**

8. Infinite Sequences and Series.

8.1. **Sequences.**

A **sequence** can be thought of as a list of numbers written in a particular order:

$$a_1, a_2, a_3, \dots, a_n, \dots$$

We will consider **infinite sequences**, so the nth term, a_n, is always followed by another term, a_{n+1}.

Remark 8.1.1. Notations such as $\{a_1, a_2, a_3, \dots\}$, $\{a_n\}$, and $\{a_n\}_{n=1}^{\infty}$ are often used interchangeably.

Example 8.1.1. Illustrate the most common ways of describing sequences: listing out the terms explicitly versus writing a formula for the nth term.

Example 8.1.2. Find a formula for the general term, a_n, of the sequence $\left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \dots\right\}$.

Solution. We are given that $a_1 = -\frac{1}{4}$, $a_2 = \frac{2}{9}$, $a_3 = -\frac{3}{16}$, $a_4 = \frac{4}{25}$. Observe that the numerator of each term corresponds to its index, while the denominators are successive perfect squares starting from $4 = 2^2$. Moreover, the terms of this sequence alternate in sign. Therefore, we conclude that

$$a_n = (-1)^n \frac{n}{(n+1)^2}.$$

Example 8.1.3. Some examples of sequences that don't have simple defining expressions are as follows:

(a) The sequence $\{s_n\}$, where s_n is the total number of students enrolled at KU at the start of the nth academic year, with $n = 1$ corresponding to 1866.
(b) If we let a_n be the digit in the nth decimal place of the irrational number π, then $\{a_n\}$ is a well-defined sequence whose first few terms are $\{1, 4, 1, 5, 9, \dots\}$.
(c) The famous **Fibonacci sequence** $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1, \qquad f_2 = 1, \qquad f_n = f_{n-1} + f_{n-2},\ n \geq 3.$$

That is, each term is the sum of the two preceding terms, so $\{f_n\} = \{1, 1, 2, 3, 5, \dots\}$.

Definition 8.1.1. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n = \infty$. In this case, the sequence $\{a_n\}$ is divergent, but in particular we say that $\{a_n\}$ diverges to infinity.

Theorem 8.1.1. *If* $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$ *when* n *is a positive integer, then* $\lim_{n\to\infty} a_n = L$.

Example 8.1.4. From Proposition 2.5.1, we know that $\lim_{x\to\infty} \frac{1}{x^r} = 0$ when $r > 0$ and $x \in \mathbb{R}$. Therefore, by Theorem 8.1.1, it follows that $\lim_{n\to\infty} \frac{1}{n^r}$ when $r > 0$ and n is a positive integer.

Example 8.1.5. Show that $\left\{\frac{\ln n}{n}\right\}$ converges.

Solution. Both the numerator and denominator of $a_n = \frac{\ln n}{n}$ approach infinity as $n \to \infty$. Applying L'Hôpital's Rule to the real-valued function $f(x) = \frac{\ln x}{x}$, we find that

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Hence, by Theorem 8.1.1, we conclude that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

As an immediate consequence of Theorem 8.1.1, we obtain Limit Laws for sequences that are analogous to those given for real-valued functions in Section 2.3.

Proposition 8.1.1 (Limit Laws)**.** *If $\{a_n\}$ and $\{b_n\}$ are sequences and c is a constant, then:*

1. $\lim_{n\to\infty}[a_n + b_n] = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.
2. $\lim_{n\to\infty}[a_n - b_n] = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty}[ca_n] = c\lim_{n\to\infty} a_n$.
4. $\lim_{n\to\infty}[a_n b_n] = [\lim_{n\to\infty} a_n] \cdot [\lim_{n\to\infty} b_n]$.
5. $\lim_{n\to\infty}[a_n/b_n] = [\lim_{n\to\infty} a_n] \,/\, [\lim_{n\to\infty} b_n]$, *provided that* $\lim_{n\to\infty} b_n \neq 0$.
6. $\lim_{n\to\infty}[a_n]^p = [\lim_{n\to\infty} a_n]^p$, *where* $a_n > 0$ *and* $p > 0$.

Example 8.1.6. Determine whether the sequence $\left\{\frac{n^3}{n+1}\right\}$ converges or diverges.

Solution. We apply a method similar to that used in Section 2.5. Dividing the numerator and denominator of $a_n = \frac{n^3}{n+1}$ by the highest power of n that appears in the entire expression and then using Limit Laws, we obtain

$$\lim_{n\to\infty} \frac{n^3}{n+1} = \lim_{n\to\infty} \frac{1}{\frac{1}{n^2} + \frac{1}{n^3}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} \frac{1}{n^2} + \lim_{n\to\infty} \frac{1}{n^3}} = \frac{1}{0+0} = \infty.$$

The Squeeze Theorem can also be adapted for sequences as follows.

Theorem 8.1.2 (Squeeze Theorem for Sequences)**.** *If $a_n \leq b_n \leq c_n$ for $n \geq N$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Corollary 8.1.3. *Since $-|a_n| \leq a_n \leq |a_n|$, it follows from Theorem 8.1.2 that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Example 8.1.7. Since $\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$ for every positive integer n and $\lim_{n\to\infty} \frac{1}{n} = 0$, we conclude by Corollary 8.1.3 that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Theorem 8.1.4. *If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.*

Example 8.1.8. Since the cosine function is continuous at 0, we find by Theorem 8.1.4 that

$$\lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{2}{n}\right) = \cos(0) = 1.$$

Proposition 8.1.2. *The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r. In particular:*

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } -1 < r < 1, \\ 1, & \text{if } r = 1, \\ \text{DNE}, & \text{otherwise.} \end{cases}$$

Definition 8.1.2. A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$; that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$; that is, $a_1 > a_2 > a_3 > \cdots$. A sequence is **monotonic** if it is either increasing or decreasing.

Example 8.1.9. The sequences $\{e^n\}$ and $\{e^{-n}\}$ are monotonic (increasing and decreasing, respectively), while the alternating sequence $\{(-1)^n\}$ is not.

Definition 8.1.3. A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$. It is **bounded below** if there is a number m such that $m \leq a_m$ for all $n \geq 1$. If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Example 8.1.10. The sequences $\{e^{-n}\}$ and $\{(-1)^n\}$ are bounded, while the sequence $\{e^n\}$ is bounded below but not above.

Theorem 8.1.5 (Monotonic Sequence Theorem)**.** *Every bounded, monotonic sequence is convergent.*

Example 8.1.11. Discuss Theorem 8.1.5 in the context of Examples 8.1.9 and 8.1.10.

8.2. **Series.**

If we add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$, we obtain an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

This is called an **infinite series**, and is denoted by $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$.

The sum of infinitely many terms may or may not exist as a finite real number. For example,

$$1 + 2 + 3 + \cdots + n + \cdots = \infty,$$

whereas

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1.$$

Definition 8.2.1. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth **partial sum**:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is said to be **convergent** and we write

$$\sum_{n=1}^{\infty} a_n = s.$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series $\sum a_n$ is said to be **divergent**.

Remark 8.2.1. A finite number of terms does not affect the convergence or divergence of a series. That is, if there exists a positive integer N such that the series $\sum_{n=N}^{\infty} a_n$, then the series $\sum_{n=1}^{\infty} a_n$ must also converge (observe, however, that the sum $\sum_{n=N}^{\infty} a_n$ is not necessarily equal to the sum $\sum_{n=1}^{\infty} a_n$). Likewise, if $\sum_{n=N}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

Definition 8.2.2. A **geometric series** is defined by

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots ,$$

where r is called **common ratio** and we assume that $a \neq 0$.

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$. Since this sequence of partial sums diverges to infinity, the geometric series also diverges in this case.

If $r \neq 1$, then

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

Subtracting these two equations from one another, we obtain

$$s_n - rs_n = a - ar^n \implies s_n = \frac{a(1-r^n)}{1-r}.$$

By Proposition 8.1.2, it follows that if $-1 < r < 1$, then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \lim_{n\to\infty} r^n = \frac{a}{1-r},$$

while if $r \leq -1$ or $r > 1$, then the sequence $\{r^n\}$ is divergent so $\lim_{n\to\infty} s_n$ does not exist.

In summary, we have just proved the following result.

Proposition 8.2.1. *The geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ *and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$, *then the geometric series is divergent.*

Example 8.2.1. Determine whether each of the following geometric series converges or diverges.

(a) $1 + 0.4 + 0.16 + 0.064 + \cdots$.

(b) $\sum_{n=1}^{\infty} \frac{\pi^n}{3^{n+1}}$.

Solution.

(a) We may rewrite this geometric series as

$$1 + \frac{4}{10} + \frac{16}{100} + \frac{64}{1000} + \cdots = 1 + \frac{4}{10} + \left(\frac{4}{10}\right)^2 + \left(\frac{4}{10}\right)^3 + \cdots.$$

Therefore, the common ratio is $r = \frac{4}{10} < 1$ and we conclude that the series converges to a sum of

$$\frac{a}{1-r} = \frac{1}{1-\frac{4}{10}} = \frac{10}{6}.$$

(b) Since

$$\sum_{n=1}^{\infty} \frac{\pi^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{\pi}{3}\right)^n,$$

the common ratio of this geometric series is $r = \frac{\pi}{3} > 1$ and we conclude that the series diverges.

Example 8.2.2 (Telescoping Sum)**.** Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Solution. Since this is not a geometric series, we begin by analyzing its partial sums:

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}.$$

Recalling the method of Partial Fractions from Section 5.7, we may write

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}.$$

It follows that

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{i(i-1)} \\ &= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}, \end{aligned}$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

Example 8.2.3 (Harmonic Series)**.** Show that the **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\cdots$ diverges.

Solution. Here, we consider the partial sums $s_2, s_4, s_8, s_{16}, s_{32}, \ldots$.

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}.$$

Similarly, $s_{16} > 1 + \frac{4}{2}$, $s_{32} > 1 + \frac{5}{2}$, and in general $s_{2^n} > 1 + \frac{n}{2}$. It follows that $s_{2^n} \to \infty$ as $n \to \infty$. Hence, the sequence $\{s_n\}$ diverges, which implies that the harmonic series diverges.

Theorem 8.2.1. *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Remark 8.2.2. The converse of Theorem 8.2.1 is not true in general. That is, if $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\sum a_n$ is convergent. Indeed, observe that $1/n \to 0$ as $n \to \infty$, and yet we have seen that the harmonic series, $\sum 1/n$, is divergent.

Proposition 8.2.2 (Divergence Test)**.** *If* $\lim_{n\to\infty} \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Example 8.2.4. Since $\lim\limits_{n\to\infty} \dfrac{n}{\sqrt{n^2+4}} = 1 \neq 0$, we conclude that the series $\sum\limits_{n=1}^{\infty} \dfrac{n}{\sqrt{n^2+4}}$ diverges.

Theorem 8.2.2. *If* $\sum a_n$ *and* $\sum b_n$ *are convergent series and* c *is a constant, then*

1. $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$. *2.* $\displaystyle\sum_{n=1}^{\infty} [a_n + b_n] = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. *3.* $\displaystyle\sum_{n=1}^{\infty} [a_n - b_n] = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

8.3. **The Integral and Comparison Tests; Estimating Sums.**

When there exists a simple formula for the nth partial sum of a series, as in the geometric and telescoping cases, computing $\lim_{n\to\infty} s_n$ is one of the easiest ways to determine whether or not the series converges. However, for more general series, we must often rely on tests that enable us to identify convergence by estimating the value of a sum rather than computing it exactly.

Proposition 8.3.1 (Integral Test)**.** *Suppose that f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)\,dx$ is convergent. In other words:*

(a) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
(b) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 8.3.1. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converge or diverge.

Solution. Since

$$\int_1^\infty \frac{1}{x^2} = \lim_{t\to\infty} \int_1^t \frac{1}{x^2} = \lim_{t\to\infty} \left(-\frac{1}{x}\bigg|_1^t \right) = \lim_{t\to\infty} \left(-\frac{1}{t} + 1 \right) = 1$$

and

$$\int_1^\infty \frac{1}{\sqrt{x}} = \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}} = \lim_{t\to\infty} \left(2\sqrt{x}\bigg|_1^t \right) = \lim_{t\to\infty} \left(2\sqrt{t} - 2 \right) = \infty,$$

we conclude by Proposition 8.3.1 that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges while $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. In either case, a geometric argument supports our answer. Observe that, as a rough approximation, $1 < \sum_{n=1}^{\infty} \frac{1}{n^2} \le 2$. It turns out that the exact value of this sum is $\pi^2/6 \approx 1.6449$, but proving this fact requires sophisticated mathematical methods that are beyond the scope of this course.

Expanding on the preceding example, would like to identify *all* values of p for which the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$, while if $p = 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$. In either case, Proposition 8.2.2 implies that the corresponding series diverges.

On the other hand, if $p > 0$, then the function $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1,\infty)$, so by Propositions 5.10.1 and 8.3.1 we conclude that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \le 1$.

We have thus established the following result.

Proposition 8.3.2 (p-series)**.** *The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.*

Often, it may be useful to compare a given series to a similar series that is known to converge or diverge.

Proposition 8.3.3 (Comparison Test)**.** *Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.*

(a) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.
(b) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

Example 8.3.2. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converges or diverges.

Solution. We observe that

$$0 < \frac{n}{2n^3+1} < \frac{n}{2n^3} = \frac{1}{2n^2} < \frac{1}{n^2}, \qquad \text{for all } n.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, it follows by Proposition 8.3.3 that $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converges.

Example 8.3.3. Determine whether the series $\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$ converges or diverges.

Solution. We observe that

$$\frac{4+3^n}{2^n} > \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n > 0, \qquad \text{for all } n.$$

Since $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ is a divergent geometric series, it follows by Proposition 8.3.3 that $\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$ diverges.

Proposition 8.3.4 (Limit Comparison Test)**.** *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms. If* $\lim_{n\to\infty} \frac{a_n}{b_n} = c$*, where* $c > 0$ *is a finite number, then either both series converge or both diverge.*

Example 8.3.4. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges or diverges.

Solution. Making a comparison to the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ appears to be the natural choice, except that $\frac{1}{2^n-1} > \frac{1}{2^n}$ for all n. Instead, with $a_n = \frac{1}{2^n-1} > 0$ and $b_n = \frac{1}{2^n} > 0$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n-1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = 1 > 0.$$

Since this limit exists as a finite, positive number, it follows by Proposition 8.3.4 that $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges.

Proposition 8.3.5 (Remainder Estimate for the Integral Test)**.** *Suppose* $\sum a_k$ *converges, where* $f(k) = a_k$ *for all* k *and* f *is a continuous, positive, decreasing function for* $x \geq n$*. If* $R_n = s - s_n$*, then*

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

Example 8.3.5. Approximate the sum of the series $\sum \frac{1}{n^3}$ by using the sum of the first 10 terms and estimate the error involved in this approximation. How many terms are needed to ensure that the sum is accurate to within 0.0005?

Solution. Since $f(x) = \frac{1}{x^3}$ satisfies the conditions Proposition 8.3.5, we begin by computing

$$\int_n^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_n^t \frac{1}{x^3}\,dx = \lim_{t\to\infty} \left(-\frac{1}{2x^2}\bigg|_n^t \right) = \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2n^2} \right) = \frac{1}{2n^2}.$$

Approximating the sum of the series by the 10th partial sum, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975,$$

and since

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2(10)^2} = \frac{1}{200},$$

we conclude that the size of the error is at most 0.005.

Accuracy to within 0.0005 means that $R_n \leq 0.0005$. Since

$$R_n \leq \int_n^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2},$$

it suffices to find n such that

$$\frac{1}{2n^2} < 0.0005.$$

Solving this inequality yields $n > \sqrt{1000} \approx 31.6$, so we must add up the first 32 terms of the series in order to guarantee an approximation of the sum that is accurate to within 0.0005.

Remark 8.3.1. An improved estimate for s can be obtained from Proposition 8.3.5 by adding s_n to all three sides of the inequality. Since $R_n = s - s_n$, this yields

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_n^{\infty} f(x)\,dx.$$

8.4. **Other Convergence Tests.**

The convergence tests we have seen so far apply only to series with positive terms. In this section, we explore other tests that apply to series whose terms are not necessarily positive.

An **alternating series** is a series whose terms are alternately positive and negative. In general, the nth term of an alternating series is of the form $a_n = (-1)^{n-1}b_n$ or $a_n = (-1)^n b_n$, where $b_n = |a_n|$ is a positive number.

Proposition 8.4.1 (Alternating Series Test). *If the alternating series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

satisfies (i) $b_n > 0$ for all n, (ii) $b_{n+1} \le b_n$ for all n, and (iii) $\lim_{n\to\infty} b_n = 0$, then the series is convergent.

Example 8.4.1. The **alternating harmonic series**

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges because it satisfies the conditions of Proposition 8.4.1.

Example 8.4.2. The series $\sum_{n=1}^{\infty}(-1)^n\frac{2n}{5n+1}$ is alternating, but

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{2n}{5n+1} = \frac{2}{5} \neq 0.$$

Hence, Proposition 8.4.1 does not apply. Instead, we observe that $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n\frac{2n}{5n+1}$ does not exist, so the series diverges by Proposition 8.2.2.

Example 8.4.3. Test the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ for convergence or divergence.

Solution. Clearly, $b_n = \frac{n^2}{n^3+1} > 0$ for all n, and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n^2}{n^3+1} = 0,$$

but it is not obvious that the sequence $\{b_n\}$ is decreasing. Let $f(x) = \frac{x^2}{x^3+1}$. Then

$$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2},$$

and considering only positive x values, we find that $f'(x) < 0$ if $2 - x^3 < 0$. That is, f is decreasing on the interval $(\sqrt[3]{2}, \infty)$. It follows that $f(n+1) < f(n)$ and therefore $b_{n+1} < b_n$ when $n \ge 2$. Hence, all three hypotheses of Proposition 8.4.1 hold and we conclude that the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ converges.

Theorem 8.4.1 (Alternating Series Estimation Theorem). *If $s = \sum(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies the conditions of Proposition 8.4.1, then $|R_n| = |s - s_n| \le b_{n+1}$.*

Definition 8.4.1. A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Theorem 8.4.2. *If a series $\sum a_n$ is absolutely convergent, then it is convergent.*

Example 8.4.4. The series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent by Theorem 8.4.2 because

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series. On the other hand, we have seen that the alternating harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent, but it is not absolutely convergent because

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is the (divergent) harmonic series.

Example 8.4.5. Determine whether the series $\sum_{n=1}^{\infty}\frac{\sin n}{n^3}$ is convergent or divergent.

Solution. This series has both positive and negative terms, but it is not an alternating series because its terms change sign irregularly. Consider the related series

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^3}\right| = \sum_{n=1}^{\infty}\frac{|\sin n|}{n^3}.$$

Since $|\sin n| \le 1$ for all n, we have

$$\frac{|\sin n|}{n^3} \le \frac{1}{n^3},$$

and it follows by Proposition that the series $\sum_{n=1}^{\infty}\frac{\sin n}{n^3}$ is absolutely convergent and therefore convergent.

Proposition 8.4.2 (Ratio Test)**.**

(i) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (and hence convergent).*

(ii) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *or* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(iii) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, *then the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of* $\sum a_n$.

Example 8.4.6. Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{2^n}$ is convergent or divergent.

Solution. With $a_n = (-1)^n\frac{n^2}{2^n}$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}\frac{(n+1)^2}{2^{n+1}}}{(-1)^n\frac{n^2}{2^n}}\right| = \frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \longrightarrow \frac{1}{2} < 1.$$

Thus, by Proposition 8.4.2, the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{2^n}$ is absolutely convergent and therefore convergent.

Example 8.4.7. Determine whether the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ is convergent or divergent.

Solution. With $a_n = \frac{n^n}{n!}$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right| = \frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} = \frac{(n+1)^n}{n!}\cdot\frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \longrightarrow e > 1.$$

Thus, by Proposition 8.4.2, the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ is divergent.

Remark 8.4.1. In the preceding example, we used the Ratio Test to establish the divergence of $\sum_{n=1}^{\infty}\frac{n^n}{n!}$. Another viable method by which to arrive at the same conclusion is to use the Divergence Test. Indeed,

since
$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \geq n,$$
we find that $\lim_{n\to\infty} a_n = \infty$ and therefore the series diverges by Proposition 8.2.2.

8.5. **Power Series.**

A **power series** is a series of the form
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$
where x is a variable and the c_n's are constants that are called the **coefficients** of the series. A power series may converge for some values of x and diverge for others. Hence, its sum is a function
$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$
whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial, except that it is comprised of infinitely many terms.

More generally, a series of the form
$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots,$$
is called a **power series in $(x-a)$** or a **power series centered at a** or a **power series about a**.

Theorem 8.5.1. *For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:*

(i) The series converges only when $x = a$.
(ii) The series converges for all x.
(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in case (iii) above is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii).

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (i), the interval consists of the single point $\{a\}$. In case (ii), the interval is $(-\infty, \infty)$. In case (iii), the inequality $|x-a| < R$ may be rewritten as $a - R < x < a + R$ and when x is an endpoint of this interval anything can happen; that is, the interval has one of the following forms:
$$(a-R, a+R), \qquad (a-R, a+R], \qquad [a-R, a+R), \qquad [a-R, a+R].$$

Example 8.5.1. Find the radius of convergence and the interval of convergence of the series $\sum_{n=0}^{\infty} x^n$.

Solution. With $a_n = x^n$, we have
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x|.$$
By Proposition 8.4.2, it follows that the series converges when $|x| < 1$, so the radius of convergence is $R = 1$. Next, we check convergence for $x = -1$ and $x = 1$. Since $\sum_{n=0}^{\infty}(-1)^n$ and $\sum_{n=0}^{\infty} 1^n$ both diverge (e.g. by Proposition 8.2.2), we conclude that the interval of convergence is $(-1, 1)$.

Example 8.5.2. Find the radius of convergence and the interval of convergence of the series $\sum_{n=0}^{\infty} n!x^n$.

Solution. With $a_n = n!x^n$, we have
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!x^{n+1}}{nx^n}\right| = (n+1)|x| \longrightarrow \begin{cases} \infty, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$
By Proposition 8.4.2, it follows that the series converges when $x = 0$ and diverges otherwise. Hence, the radius of convergence is $R = 1$ and, by convention, the interval of convergence is $\{0\}$.

Example 8.5.3. Find the radius of convergence and the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution. With $a_n = \frac{x^n}{n!}$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \frac{|x|}{n+1} \longrightarrow 0.$$

By Proposition 8.4.2, it follows that the series converges for all x. Hence, the radius of convergence is $R = \infty$ and, by convention, the interval of convergence is $(-\infty, \infty)$.

Example 8.5.4. Find the radius of convergence and the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n\sqrt{n}}$.

Solution. With $a_n = \frac{1}{\sqrt{n}}\left(\frac{x-2}{2}\right)^n$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{\sqrt{n+1}}\left(\frac{x-2}{2}\right)^{n+1}}{\frac{1}{\sqrt{n}}\left(\frac{x-2}{2}\right)^n}\right| = \frac{\sqrt{n}}{\sqrt{n+1}}\frac{|x-2|}{2} \longrightarrow \frac{|x-2|}{2}.$$

By Proposition 8.4.2, it follows that the series converges when $\frac{|x-2|}{2} < 1$; that is, when $|x-2| < 2$, so the radius of convergence is $R = 2$. Next, we check convergence for $x = 0$ and $x = 4$. Since

$$\sum_{n=0}^{\infty} \frac{(0-2)^n}{2^n\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}\left(\frac{-2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges (e.g. by Proposition 8.4.1) and

$$\sum_{n=0}^{\infty} \frac{(-4-2)^n}{2^n\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}\left(\frac{-6}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-3)^n}{\sqrt{n}}$$

diverges (e.g. by Proposition 8.2.2), we conclude that the interval of convergence is $[0, 4)$.

8.6. **Representations of Functions as Power Series.**

Since the sum of a power series, when it converges, is actually a function, we may use power series in circumstances where working with closed-form formulas for functions is impractical or impossible.

Example 8.6.1. Since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ when $|x| < 1$, we say that the power series representation of the function $f(x) = \frac{1}{1-x}$ is

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots.$$

Example 8.6.2. Express $\frac{1}{1+x^2}$ as a power series.

Solution. By the previous example, we find that

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

for $|-x^2| < 1$. The radius of convergence of this power series is $R = 1$, and checking endpoints shows that its interval of convergence is $(-1, 1)$.

Example 8.6.3. Find a power series representation for $\frac{1}{2+x}$.

Solution. By way of some clever factoring, we conclude that

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2\left(1-\left(-\frac{x}{2}\right)\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

for $|-x/2| < 1$. The radius of convergence and interval of convergence of this power series are $R = 2$ and $(-2, 2)$, respectively.

Example 8.6.4. Find a power series representation for $\frac{x^2}{2+x}$.

Solution. It follows from the preceding example that

$$\frac{x^2}{2+x} = x^2 \cdot \frac{1}{2+x} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{2^{n-1}} x^n$$

for $|-x/2| < 1$. Once again, the radius of convergence and interval of convergence are $R = 2$ and $(-2, 2)$, respectively.

Theorem 8.6.1. *If the power series* $\sum c_n(x-a)^n$ *has radius of convergence* $R > 0$, *then the function* f *defined by*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$.

The radii of convergence of the power series in (i) and (ii) are both R.

Example 8.6.5. Express $\frac{1}{(1-x)^2}$ as a power series.

Solution. By Theorem 8.6.1, we have that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n.$$

The radius of convergence of this power series is the same as that of $\sum_{n=0}^{\infty} x^n$, namely $R = 1$, and checking endpoints shows that its interval of convergence is $(-1, 1)$.

Example 8.6.6. Find a power series representation for $\ln(2+x)$.

Solution. First, observe that

$$\frac{d}{dx}\left(\ln(2+x)\right) = \frac{1}{2+x}.$$

By Example 8.6.3 and Theorem 8.6.1, it follows that

$$\ln(2+x) = \int \frac{1}{2+x}\,dx = \int \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n\right] dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}(n+1)} x^{n+1} = C + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n n} x^n.$$

To find C, we substitute $x = 0$ and obtain

$$\ln(2+0) = C + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n n}(0)^n \implies C = \ln(2).$$

Hence, the power series representation is

$$\ln(2+x) = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n n} x^n,$$

which has radius of convergence $R = 2$ and interval of convergence $(-2, 2]$.

8.7. **Taylor and Maclaurin Series.**

Suppose that f is a function defined by the power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots, \qquad |x-a| < R.$$

By Theorem 8.6.1, we may differentiate this expression term-by-term, yielding

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots, && |x-a| < R,\\ f''(x) &= 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + \cdots, && |x-a| < R,\\ f'''(x) &= 6c_3 + 24c_4(x-a) + \cdots, && |x-a| < R,\\ &\vdots \end{aligned}$$

Substituting $x = a$ into the expressions for $f, f', f'', f''', \ldots$, we obtain

$$f(a) = c_0,\ f'(a) = c_1,\ f''(a) = 2c_2,\ f'''(a) = 6c_3, \ldots.$$

In general, the pattern is that $f^{(n)}(a) = n!c_n$, so the nth coefficient of the power series is given by

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

This formula holds for all $n \geq 0$, provided that we assume the conventions $0! = 1$ and $f^{(0)} = f$. Thus, we have established the following important result.

Theorem 8.7.1. *If f has a power series representation (expansion) at a, i.e.*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad |x-a| < R,$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

By Theorem 8.7.1, we see that if f has a power series expansion at a, then it must be of the following form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

This series is called the **Taylor series of the function f at a** (or **about a** or **centered at a**).

In the special case when $a = 0$, the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

and is called the **Maclaurin series of the function f**.

Example 8.7.1. Find the Maclaurin series for the function $f(x) = e^x$.

Solution. If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all n. Therefore, the Maclaurin series for f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

By Example 8.5.3, we already know that the radius of convergence and interval of convergence of this power series are $R = \infty$ and $(-\infty, \infty)$, respectively.

From the preceding example, we can only conclude that *if* e^x has a power series expansion at 0, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The partial sums of the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ are given by

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Observe that T_n is a polynomial of degree n called the ***n*th-degree Taylor polynomial of f at a**. In general, as the following theorem asserts, a function f is equal to the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x).$$

Theorem 8.7.2. *If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.*

In trying to show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f, it is often useful to apply the following remainder estimate.

Theorem 8.7.3 (Taylor's Inequality)**.** *If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality*

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \le d.$$

Example 8.7.2. Find the Taylor series for the function $f(x) = e^x$ at $a = 2$.

Solution. As before, $f^{(n)}(x) = e^x$, so $f^{(n)}(2) = e^2$ for all n. Therefore, the Taylor series for f at $a = 2$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n.$$

It can be verified that the radius of convergence and interval of convergence of this power series are $R = \infty$ and $(-\infty, \infty)$, respectively. Moreover, $\lim_{n\to\infty} R_n(x) = 0$, so

$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n \qquad \text{for all } x.$$

We have found two different power series expansions for e^x: the Maclaurin series from Example 8.7.1 and the Taylor series from Example 8.7.2. Both are useful for approximating the value of e^x, but the former is better when $x \approx 0$ and the latter is better when $x \approx 2$.

Example 8.7.3. Find the Maclaurin series for the function $f(x) = \sin x$.

Solution. Observe that

$$\begin{array}{rcl c rcl}
f(x) &=& \sin x & \Longrightarrow & f(0) &=& 0,\\
f'(x) &=& \cos x & \Longrightarrow & f'(0) &=& 1,\\
f''(x) &=& -\sin x & \Longrightarrow & f''(0) &=& 0,\\
f'''(x) &=& -\cos x & \Longrightarrow & f'''(0) &=& -1,\\
f^{(4)}(x) &=& \sin x & \Longrightarrow & f^{(4)}(0) &=& 0.\\
 & & & \vdots & & &
\end{array}$$

By the repeating nature of the derivatives, we conclude that

$$\begin{aligned}
\sin x &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.
\end{aligned}$$

It is straightforward to verify that the interval of convergence of this power series is $(-\infty, \infty)$.

Example 8.7.4. Find the Maclaurin series for the function $f(x) = \cos x$.

Solution. Rather than explicitly computing $f^{(n)}(0)$ for all n, we simply applying Theorem 8.6.1 to obtain
$$\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$
for all x.

The following table summarizes the Maclaurin series we have encountered thus far and introduces some new ones.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad R = 1.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad R = \infty.$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \qquad R = \infty.$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \qquad R = \infty.$$

$$\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad R = 1.$$

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad R = 1.$$

$$(1+x)^k = \sum_{n=1}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots, \qquad R = 1.$$

8.8. **Applications of Taylor Polynomials.**

There are no notes provided for this section. It recalls the definition of Taylor polynomials as partial sums of Taylor series and highlights their use in the approximation of other functions.

9. Vectors and the Geometry of Space.

9.1. **Three-Dimensional Coordinate Systems.**

The two-dimensional Cartesian (or rectangular) coordinate system can be extended very naturally to three (or more) dimensions. In addition to the x-axis and y-axis, we now have the z-axis, whose direction is determined by the **right-hand rule**, i.e. if you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis. The three axes are co-perpendicular, intersect at a single point called the **origin**, and divide three-dimensional space into eight **octants** (analogous to the four **quadrants** in 2D).

A point in the three-dimensional Cartesian coordinate system is denoted by the ordered triple $P = (a, b, c)$, where the **coordinates** a, b, and c are the directed distances from the yz-plane, xz-plane, and xy-plane, respectively, to the point P.

Example 9.1.1. Illustrate the equations and graphs of simple curves and surfaces (such as lines, circles, planes, and cylinders) in three dimensions.

The formula for the distance between two points in a plane can be extended to the following three-dimensional formula.

Proposition 9.1.1 (Distance Formula in Three Dimensions)**.** *The distance between the points* $P_1 = (x_1, y_1, z_1)$ *and* $P_2 = (x_2, y_2, z_2)$ *is given by*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example 9.1.2. Find the distance between the points $(1, -1, 7)$ and $(0, 4, 7)$.

Solution. By Proposition 9.1.1, we find the distance between these points to be

$$\sqrt{(0 - 1)^2 + (4 - (-1))^2 + (7 - 7)^2} = \sqrt{1 + 25 + 0} = \sqrt{26} \approx 5.0990.$$

Example 9.1.3. Find the shortest distance between the point $(1, -1, 7)$ and the xy-plane.

Solution. It suffices to consider only the z-dimensional distance between them, which is 7.

Next, we infer the equation of a sphere from that of its two-dimensional analog.

Proposition 9.1.2 (Equation of a Sphere)**.** *The equation of a sphere with center* $C = (h, k, \ell)$ *and radius* r *is*

$$(x - h)^2 + (y - k)^2 + (z - \ell)^2 = r^2.$$

In particular, if the center is the origin, then the equation reduces to

$$x^2 + y^2 + z^2 = r^2.$$

Example 9.1.4. Determine the center and radius of the sphere whose equation is $x^2 + y^2 + z^2 - 6x + 4y - 2z = 12$.

Solution. Completing squares, we find that

$$\begin{aligned} x^2 + y^2 + z^2 - 6x + 4y = 12 &\implies (x^2 - 6x + 9) + (y^2 + 4y + 4) + (z^2) = 25 \\ &\implies (x - 3)^2 + (y + 2)^2 + (z)^2 = 5^2, \end{aligned}$$

so the sphere has radius 5 and is centered at the point $(3, -2, 0)$.

9.2. **Vectors.**

The term **vector** is used to indicate a quantity (such as a displacement, velocity, or force) that has both magnitude and direction. Thus, a vector $\vec{v}$ is often represented geometrically by an arrow pointing from an **initial point** to a **terminal point**.

Definition 9.2.1. If $\vec{u}$ and $\vec{v}$ are vectors positioned so the initial point of $\vec{v}$ is at the terminal point of $\vec{u}$, then the **sum** $\vec{u}+\vec{v}$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$.

An illustration of Definition 9.2.1 explains why vector addition can be remembered as the **Triangle Law**. The **Parallelogram Law** shows that vector addition is commutative, i.e. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$.

Definition 9.2.2. If c is a scalar and $\vec{v}$ is a vector, then the **scalar multiple** $c\vec{v}$ is the vector whose length is $|c|$ times the length of $\vec{v}$ and whose direction is the same as $\vec{v}$ if $c>0$ and is opposite to $\vec{v}$ if $c<0$. If $c=0$ or $\vec{v}=\vec{0}$, then $c\vec{v}=\vec{0}$.

Observe that two nonzero vectors are **parallel** if and only if they are scalar multiples of one another. In particular, the vector $-\vec{v}=(-1)\vec{v}$ has the same length as $\vec{v}$ but points in the opposite direction; this is called the **negative** of $\vec{v}$.

The **difference** of two vectors $\vec{u}$ and $\vec{v}$ is defined as $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$.

Given two points $A=(x_1,y_1,z_1)$ and $B=(x_2,y_2,z_2)$, the vector $\vec{a}$ from A to B is defined by

$$\vec{a}=\langle x_2-x_1, y_2-y_1, z_2-z_1\rangle.$$

Definition 9.2.3. The **magnitude** or **length** of a vector $\vec{a}$ is given by

$$|\vec{a}|=\sqrt{a_1^2+a_2^2} \quad \text{if } \vec{a}=\langle a_1,a_2\rangle \text{ is two-dimensional}$$

and

$$|\vec{a}|=\sqrt{a_1^2+a_2^2+a_3^3} \quad \text{if } \vec{a}=\langle a_1,a_2,a_3\rangle \text{ is three-dimensional.}$$

Proposition 9.2.1. *If $\vec{a}=\langle a_1,a_2\rangle$ and $\vec{b}=\langle b_1,b_2\rangle$, then*

$$\vec{a}+\vec{b}=\langle a_1+b_1,a_2+b_2\rangle, \qquad \vec{a}-\vec{b}=\langle a_1-b_1,a_2-b_2\rangle, \qquad c\vec{a}=\langle ca_1,ca_2\rangle.$$

These formulas extend analogously to three or more dimensions.

Proposition 9.2.2 (Properties of Vectors)**.** *If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors, and c and d are scalars, then*

1. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ *(additive commutativity).*
2. $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$ *(additive associativity).*
3. $\vec{a}+\vec{0}=\vec{a}$ *(additive identity).*
4. $\vec{a}+(-\vec{a})=\vec{0}$ *(additive inverse).*
5. $c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$ *(multiplicative distributivity 1).*
6. $(c+d)\vec{a}=c\vec{a}+d\vec{a}$ *(multiplicative distributivity 2).*
7. $(cd)\vec{a}=c(d\vec{a})$ *(multiplicative associativity).*
8. $1\vec{a}=\vec{a}$ *(multiplicative identity).*

Definition 9.2.4. The **standard basis vectors** in three dimensions are denoted by $\hat{\imath}=\langle 1,0,0\rangle$, $\hat{\jmath}=\langle 0,1,0\rangle$, and $\hat{k}=\langle 0,0,1\rangle$.

Any vector in three-dimensions can be expressed in terms of the standard basis vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. For example, if $\vec{a}=\langle a_1,a_2,a_3\rangle$, then

$$\vec{a}=\langle a_1,a_2,a_3\rangle=\langle a_1,0,0\rangle+\langle 0,a_2,0\rangle+\langle 0,0,a_3\rangle=a_1\langle 1,0,0\rangle+a_2\langle 0,1,0\rangle+a_3\langle 0,0,1\rangle=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}.$$

Definition 9.2.5. A **unit vector** is a vector whose magnitude is 1.

Observe that every standard basis vector is a unit vector. In general, if $\vec{a} \neq 0$, then the unit vector that has the same direction as $\vec{a}$ is

$$\hat{u} = \frac{\vec{a}}{|\vec{a}|}.$$

Vectors are very useful in physics and engineering. In the following examples, we see how vectors can be used to represent velocities and forces.

Example 9.2.1. A ball is thrown upward with a $30°$ angle of elevation and a speed of 50 ft/sec. Find the horizontal and vertical components of the velocity vector.

Solution. The speed of 50 ft/sec represents the magnitude of the velocity vector. Its horizontal and vertical components are therefore $50\cos(30) \approx 43.3$ ft/sec and $50\sin(30) = 25$ ft/sec, respectively.

Example 9.2.2. A 10-kg mass hangs from two wires that make angles of $50°$ and $32°$ with the horizontal. Find the tensions (forces) $\vec{T_1}$ and $\vec{T_2}$ in the two wires and the magnitudes of these tensions.

Solution. We begin by observing that the gravitational force acting on the mass is given by

$$\vec{g} = 10\left(-9.8\hat{\jmath}\right) = -98\hat{\jmath}.$$

Next, we express $\vec{T_1}$ and $\vec{T_2}$ in terms of their horizontal and vertical components:

$$\begin{aligned}\vec{T_1} &= -|\vec{T_1}|\cos(50°)\hat{\imath} + |\vec{T_1}|\sin(50°)\hat{\jmath},\\ \vec{T_2} &= |\vec{T_2}|\cos(32°)\hat{\imath} + |\vec{T_2}|\sin(32°)\hat{\jmath}.\end{aligned}$$

Since the **resultant force** on the mass must be $\vec{0}$ in order for the system to maintain equilibrium, we have

$$\vec{T_1} + \vec{T_2} = -\vec{g} \implies \left(-|\vec{T_1}|\cos(50°)\hat{\imath} + |\vec{T_1}|\sin(50°)\hat{\jmath}\right) + \left(|\vec{T_2}|\cos(32°)\hat{\imath} + |\vec{T_2}|\sin(32°)\hat{\jmath}\right) = 98\hat{\jmath}.$$

Equating like components, it follows that

$$\begin{aligned}-|\vec{T_1}|\cos(50°) + |\vec{T_2}|\cos(32°) &= 0,\\ |\vec{T_1}|\sin(50°) + |\vec{T_2}|\sin(32°) &= 98.\end{aligned}$$

Solving this system of equations yields $|\vec{T_1}| \approx 83.93$ and $|\vec{T_2}| \approx 63.61$. Hence, the tension vectors are

$$\begin{aligned}\vec{T_1} &\approx -53.95\hat{\imath} + 64.29\hat{\jmath},\\ \vec{T_2} &\approx 53.94\hat{\imath} + 33.71\hat{\jmath}.\end{aligned}$$

9.3. **The Dot Product.**

Recall from Section 6.6 that the work done by a constant force F in moving an object through a distance d is given by $W = Fd$, provided that the force is directed along the line of motion of the object. We now consider a more general scenario in which the constant force $\vec{F}$ moves an object through a displacement $\vec{D}$, where the vectors $\vec{F}$ and $\vec{D}$ are not necessarily parallel and the angle between them is denoted by θ. In this case, the work done by $\vec{F}$ is defined to be

$$W = |\vec{F}||\vec{D}|\cos\theta.$$

This example motivates the following definition of one important type of vector multiplication.

Definition 9.3.1. The **dot product** of two nonzero vectors $\vec{a}$ and $\vec{b}$ is the number

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta,$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \leq \theta \leq \pi$.

Example 9.3.1. A sled is dragged across flat ground by a rope that is held at an angle of 60° above the horizontal. How much work is done by a force of 50 N in moving the sled over a distance of 10 m?

Solution. By the definition of the dot product, the work done is

$$W = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos(60^\circ) = (50)(10)(0.5) = 250\,\text{J}.$$

Remark 9.3.1.
- If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then we define $\vec{a} \cdot \vec{b} = 0$.
- Two vectors $\vec{a}$ and $\vec{b}$ are **orthogonal** or **perpendicular** if and only if $\vec{a} \cdot \vec{b} = 0$.

In the example of finding the work done by a force $\vec{F}$ in moving an object through a displacement $\vec{D}$ by calculating $W = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos\theta$, it makes sense to assume that $0 \le \theta \le \pi/2$. However, in the general definition of the dot product we allow for $0 \le \theta \le \pi$. If $\pi/2 < \theta \le \pi$, then $\cos\theta < 0$ and we interpret the negative value of W as representing the fact that, when the $\vec{F}$ and $\vec{D}$ point away from each other, the force contributes negatively to the work done in moving the object.

Proposition 9.3.1 (Properties of the Dot Product)**.** *If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors, and c is a scalar, then*

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$,
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$,
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$,
4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = a \cdot (c\vec{b})$,
5. $\vec{0} \cdot \vec{a} = 0$.

In component form, the dot product of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is given by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Example 9.3.2. Find the dot product of the vectors $\langle 2, 5, 1 \rangle$ and $\langle -1, 0, 3 \rangle$.

Solution. By the formula above, we find that

$$\langle 2, 5, 1 \rangle \cdot \langle -1, 0, 3 \rangle = (2)(-1) + (5)(0) + (1)(3) = 1.$$

Example 9.3.3. Show that the vectors $3\hat{\imath} - 4\hat{\jmath}$ and $8\hat{\imath} + 6\hat{\jmath}$ are orthogonal.

Solution. It suffices to show that their dot product equals zero:

$$(3\hat{\imath} - 4\hat{\jmath}) \cdot (8\hat{\imath} + 6\hat{\jmath}) = (3)(8) + (-4)(6) = 0.$$

Example 9.3.4. Find the angle between the vectors $\vec{a} = \langle 1, 1, 1 \rangle$ and $\vec{b} = \langle 0, 2, -5 \rangle$.

Solution. Since

$$\vec{a} \cdot \vec{b} = \langle 1, 1, 1 \rangle \cdot \langle 0, 2, -5 \rangle = (1)(0) + (1)(2) + (1)(-5) = -3$$

and

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = \sqrt{3}\sqrt{29} \cos\theta = \sqrt{87} \cos\theta,$$

it follows that

$$-3 = \sqrt{87} \cos\theta \implies \cos\theta = \frac{-3}{\sqrt{87}} \implies \theta = \cos^{-1}\left(\frac{-3}{\sqrt{87}}\right) \approx 1.8983 \text{ (or about } 108.8^\circ).$$

Definition 9.3.2 (Vector Projections)**.**

- The **scalar projection** of $\vec{b}$ onto $\vec{a}$ is: $\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$.

- The **vector projection** of $\vec{b}$ onto $\vec{a}$ is: $\text{proj}_{\vec{a}}\vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\dfrac{\vec{a}}{|\vec{a}|} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}.$

Example 9.3.5. Find the scalar projection and vector projection of $\vec{b} = \langle 5, 0\rangle$ onto $\vec{a} = \langle 3, -4\rangle$.

Solution. Since $|a| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$, the scalar projection of $\vec{b}$ onto $\vec{a}$ is

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{(3)(5) + (-4)(0)}{5} = 3$$

and the vector projection of $\vec{b}$ onto $\vec{a}$ is

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} = \frac{3}{5}\vec{a} = \left\langle \frac{9}{5}, -\frac{12}{5}\right\rangle.$$

9.4. **The Cross Product.**

If we tighten a bolt by applying a force to a wrench, we produce a turning effect called **torque**. The magnitude of this torque $\vec{\tau}$ depends on two things:

- The distance $|\vec{r}|$ from the axis of the bolt to the point where the force is applied.
- The scalar component of the force $\vec{F}$ in the direction perpendicular to $\vec{r}$. This is simply $|\vec{F}|\sin\theta$, where θ is the angle between the vectors $\vec{r}$ and $\vec{F}$.

In particular, the magnitude of the torque is given by $|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$. The direction of the torque vector is along the axis of rotation of the bolt, so if $\hat{n}$ is a unit vector that points in the direction of motion of a right-threaded bolt, we let

$$\vec{\tau} = \left(|\vec{r}||\vec{F}|\sin\theta\right)\hat{n}.$$

This example motivates the following definition of a second type of vector multiplication.

Definition 9.4.1. If $\vec{a}$ and $\vec{b}$ are nonzero three-dimensional vectors, then the **cross product** $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a}\times\vec{b} = \left(|\vec{a}||\vec{b}|\sin\theta\right)\hat{n},$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \le \theta \le \pi$, and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ whose direction is given by the **right-hand rule**, i.e. if the fingers of your right hand curl through the angle θ from $\vec{a}$ to $\vec{b}$, then your thumb points in the direction of $\hat{n}$.

Example 9.4.1. A bolt is tightened by applying a force of 50 N to a wrench that is 20 cm long. If the angle between the wrench and the direction of the force is 30°, find the magnitude of the torque about the center of the bolt.

Solution. The magnitude of the torque vector is given by

$$|\vec{\tau}| = |\vec{r}\times\vec{F}| = |\vec{r}||\vec{F}|\sin(30^\circ)|\vec{n}| = (0.2)(50)(0.5) = 5\,\text{N}\cdot\text{m}.$$

Remark 9.4.1.

- If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then we define $\vec{a}\times\vec{b} = \vec{0}$.
- Two vectors $\vec{a}$ and $\vec{b}$ are **parallel** if and only if $\vec{a}\times\vec{b} = \vec{0}$.
- Since $\vec{a}\times\vec{b}$ is a scalar multiple of $\hat{n}$, it is orthogonal to both $\vec{a}$ and $\vec{b}$.
- The length of $\vec{a}\times\vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Example 9.4.2. Compute the cross products of various pairs of the standard basis vectors in order to illustrate a lack of commutativity.

Proposition 9.4.1 (Properties of the Cross Product)**.** *If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors, and c is a scalar, then*

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$,
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = a \times (c\vec{b})$,
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$,
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$.

In component form, the cross product of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is given by

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \hat{\imath} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\
&= \hat{\imath}(a_2b_3 - a_3b_2) - \hat{\jmath}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1) \\
&= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle.
\end{aligned}$$

Here, we are using the notation of **determinants**. Specifically, a **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

while a **determinant of order 3** can be defined in terms of second-order determinants as

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Example 9.4.3. Find the cross product $\vec{a} \times \vec{b}$, where $\vec{a} = \langle 1, 3, -2 \rangle$ and $\vec{b} = \langle -1, 0, 5 \rangle$.

Solution. By the formula above, we have

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 5 \end{vmatrix} \\
&= \hat{\imath} \begin{vmatrix} 3 & -2 \\ 0 & 5 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} \\
&= \hat{\imath}(15 - 0) - \hat{\jmath}(5 - 2) + \hat{k}(0 - (-3)) \\
&= \langle 15, -3, 3 \rangle.
\end{aligned}$$

It is easy to check that $\langle 1, 3, -2 \rangle \cdot \langle 15, -3, 3 \rangle = 0$ and $\langle -1, 0, 5 \rangle \cdot \langle 15, -3, 3 \rangle = 0$, which confirms that $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Example 9.4.4. Consider a plane that passes through the points $P = (1, 4, 6)$, $Q = (-2, 5, -1)$, and $R = (1, -1, 1)$. Find a vector perpendicular to this plane.

Solution. The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore perpendicular to the plane through P, Q, and R. From Section 9.2, we know that

$$\begin{aligned}
\overrightarrow{PQ} &= (-2 - 1)\hat{\imath} + (5 - 4)\hat{\jmath} + (-1 - 6)\hat{k} = -3\hat{\imath} + \hat{\jmath} - 7\hat{k}, \\
\overrightarrow{PR} &= (1 - 1)\hat{\imath} + (-1 - 4)\hat{\jmath} + (1 - 6)\hat{k} = -5\hat{\jmath} - 5\hat{k}.
\end{aligned}$$

Hence, the vector

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \hat{\imath}(-5 - 35) - \hat{\jmath}(15 - 0) + \hat{k}(15 - 0) = \langle -40, -15, 15 \rangle.$$

is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as $\langle -8, -3, 3 \rangle$, is also perpendicular to the plane.

Proposition 9.4.2. *The volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the magnitude of their* ***scalar triple product****, i.e.* $V = \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$.

Example 9.4.5. Use the scalar triple product to show that the vectors $\vec{a} = \langle 1, 5, -2\rangle$, $\vec{b} = \langle 3, -1, 0\rangle$, and $\vec{c} = \langle 5, 9, -4\rangle$ are coplanar.

Solution. Using the component form of the dot and cross products, we find that

$$\begin{aligned}\vec{a}\cdot(\vec{b}\times\vec{c}) &= \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} \\ &= 1\begin{vmatrix} -1 & 0 \\ 9 & -4 \end{vmatrix} - 5\begin{vmatrix} 3 & 0 \\ 5 & -4 \end{vmatrix} + (-2)\begin{vmatrix} 3 & -1 \\ 5 & 9 \end{vmatrix} \\ &= (4-0) - 5(-12-0) - 2(27-(-5)) \\ &= 0.\end{aligned}$$

Hence, the volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$ is zero, so all three vectors must lie in the same plane.

The **vector triple product** is defined by $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.

9.5. Equations of Lines and Planes.

A line in two dimensions is uniquely determined by its slope and a point that it passes through. Similarly, a line L in three dimensions is uniquely determined by a point on L and the direction of L. Since the direction of L can be described by a vector, we have the following definition.

Definition 9.5.1. The **vector equation** of a line L through the point $P_0 = (x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle a, b, c\rangle$ is given by $\vec{r} = \vec{r}_0 + t\vec{v}$, where $\vec{r}_0$ is the position vector of P_0 and t is a parameter that is allowed to range over all real numbers. Each value of t gives the position vector $\vec{r}$ of a point on L.

If $\vec{v} = \langle a, b, c\rangle$, then we have $t\vec{v} = \langle ta, tb, tc\rangle$. Writing $\vec{r} = \langle x, y, z\rangle$ and $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$ yields the following equivalent definition of a line in three dimensions.

Definition 9.5.2. The **parametric equations** of the line L through the point $P_0 = (x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle a, b, c\rangle$ are given by $x = x_0 + at$, $y = y_0 + bt$, and $z = z_0 + ct$.

Yet another definition of a line in three dimensions is obtained by eliminating the parameter t. If none of a, b, or c is zero, we can solve each of the three parametric equations for t and equate the results.

Definition 9.5.3. The **symmetric equations** of the line L through the point $P_0 = (x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle a, b, c\rangle$ are given by

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

Example 9.5.1.

(a) Find a vector equation and parametric equations for the line that passes through the point $(6, -5, 2)$ and is parallel to the vector $\langle 3, 9, -2\rangle$.
(b) Find two other points on this line.

Solution.

(a) Since $\vec{r}_0 = \langle 6, -5, 2\rangle$ and $\vec{v} = \langle 3, 9, -2\rangle$, we have the vector equation

$$\vec{r} = \vec{r}_0 + t\vec{v} = \langle 6, -5, 2\rangle + t\langle 3, 9, -2\rangle = \langle 6+3t, -5+9t, 2-2t\rangle,$$

and therefore the parametric equations are

$$x = 6 + 3t, \qquad y = -5 + 9t, \qquad z = 2 - 2t.$$

(b) Choosing $t = 1$ and $t = -1$ gives $(9, 4, 0)$ and $(3, -14, 4)$, respectively, as other points on the line.

Example 9.5.2.

(a) Find parametric equations and symmetric equations for the line that passes through the points $A = (6, 1, -3)$ and $B = (2, 4, 5)$.

(b) At what point does this line intersect the xy-plane?

Solution.

(a) The vector

$$\overrightarrow{AB} = \langle 2-6, 4-1, 5-(-3) \rangle = \langle -4, 3, 8 \rangle$$

passes through the points A and B and is therefore parallel to the line in question. Taking the point A as P_0, we obtain the parametric equations

$$x = 6 - 4t, \qquad y = 1 + 3t, \qquad z = -3 + 8t.$$

and the symmetric equations

$$\frac{x-6}{-4} = \frac{y-1}{3} = \frac{z+3}{8}.$$

(b) The line intersects the xy-plane when $z = 0$. Substituting $z = 0$ into the symmetric equations yields

$$\frac{x-6}{-4} = \frac{y-1}{3} = \frac{3}{8},$$

which can be solved for $x = 9/2$ and $y = 17/8$. Therefore, the line intersects the xy-plane at the point $(9/2, 17/8, 0)$.

In general, the vector equation of a line through (tip of the) vector $\vec{r}_0$ in the direction of a vector $\vec{v}$ is $\vec{r} = \vec{r}_0 + t\vec{v}$. If the line also passes through the (tip of the) vector $\vec{r}_1$, then we can take $\vec{v} = \vec{r}_1 - \vec{r}_0$. In this case, the vector equation becomes

$$\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0) = (1-t)\vec{r}_0 + t\vec{r}_1,$$

which motivates the following definition.

Definition 9.5.4. The **line segment** of from $\vec{r}_0$ to $\vec{r}_1$ is given by the vector equation

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1, \qquad 0 \le t \le 1.$$

Example 9.5.3. Show that the lines

$$L_1 : \ \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} \qquad \text{and} \qquad L_2 : \ \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

Solution. The lines are not parallel because the corresponding vectors, $\langle 1, 2, 3 \rangle$ and $\langle -4, -3, 2 \rangle$, are not parallel. If L_1 and L_2 had a point of intersection, there would be values of $t = t_1$ and $t = t_2$ such that

$$\begin{aligned} t_1 &= 3 - 4t_2, \\ 1 + 2t_1 &= 2 - 3t_2, \\ 2 + 3t_1 &= 1 + 2t_2. \end{aligned}$$

However, solving the first two equations yields $t_1 = -1$ and $t_2 = 1$, and these values do not satisfy the third equation. Hence, we conclude that L_1 and L_2 are skew lines.

To describe a plane in three dimensions, it suffices to specify a point that lies in the plane and a vector that is orthogonal to it. This orthogonal vector, generally denoted by $\vec{n}$, is called a **normal vector**. Let $P = (x, y, z)$ be an arbitrary point in the plane, and let $\vec{r}$ and $\vec{r}_0$ be the position vectors of P and P_0, respectively. Then the vector $\overrightarrow{P_0P} = \vec{r} - \vec{r}_0$ lies in the plane Since $\vec{n}$ is orthogonal to the plane (and therefore orthogonal to every vector in the plane), we have

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0,$$

which motivates the following definition.

Definition 9.5.5. The **vector equation** of the plane through the point $P_0 = (x_0, y_0, z_0)$ with normal vector $\vec{n}$ is given by $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$, where $\vec{r}_0$ is the position vector of P_0.

Writing $\vec{n} = \langle a, b, c\rangle$, $\vec{r} = \langle x, y, z\rangle$, and $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$ yields the following equivalent definition.

Definition 9.5.6. The **scalar equation** of the plane through the point $P_0 = (x_0, y_0, z_0)$ with normal vector $\vec{n}$ is given by $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Finally, letting $d = -(ax_0 + by_0 + cz_0)$, we obtain yet another representing of a plane in three dimensions.

Definition 9.5.7. The **linear equation** of the plane through the point $P_0 = (x_0, y_0, z_0)$ with normal vector $\vec{n}$ is given by $ax + by + cz + d = 0$.

Example 9.5.4. Find an equation of the plane through the point $(6, 3, 2)$ and perpendicular to the vector $\langle -2, 1, 5\rangle$. Then find the intercepts and sketch the plane.

Solution. Let $a = -2$, $b = 1$, $c = 5$, $x_0 = 6$, $y_0 = 3$, and $z_0 = 2$, we compute that

$$d = -(ax_0 + by_0 + cz_0) = -((-2)(6) + (1)(3) + (5)(2)) = -1.$$

Hence, the linear equation of the plane is

$$-2x + y + 5z - 1 = 0.$$

Setting $y = z = 0$, we find that the x-intercept of the plane is $x = -1/2$. Similarly, we obtain $y = 1$ and $z = 1/5$ as the y- and z-intercepts, respectively. This allows us to easily sketch the plane in the three-dimensional Cartesian coordinate system.

Example 9.5.5. Find an equation of the plane that passes through the points $P = (1, 4, 6)$, $Q = (-2, 5, -1)$, and $R = (1, -1, 1)$.

Solution. In Example 9.4.4, we computed what we now understand to be a normal vector for the plane in question. Therefore, using the point P and the normal vector $\vec{n} = \langle -8, -3, 3\rangle$, we find the equation of the plane to be

$$-8(x - 1) - 3(y - 4) + 3(z - 6) = 0 \implies 8x + 3y - 3z = 2.$$

Example 9.5.6. Find the point at which the line $x = 3 - t$, $y = 2 + t$, $z = 5t$ intersects the plane $x - y + 2z = 9$.

Solution. Substituting the parametric equations for x, y, and z into the equation of the plane, we find that

$$(3 - t) - (2 + t) + 2(5t) = 9 \implies 1 + 8t = 9 \implies t = 1.$$

Therefore, the line intersects the plane when the parameter value is $t = 1$. This corresponds to the point with coordinates $x = 3 - 1 = 2$, $y = 2 + 1 = 3$, and $z = 5(1) = 5$.

Two planes are parallel if their normal vectors are parallel. More generally, the angle between two planes is the same as the angle between their normal vectors, which can be found using the methods of Section 9.3.

Example 9.5.7.

(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
(b) Find the symmetric equations for the line of intersection L of these two planes.

Solution.

(a) The normal vectors of these two planes are $\vec{n}_1 = \langle 1, 1, 1\rangle$ and $\vec{n}_2 = \langle 1, -2, 3\rangle$. Letting θ denote the angle between the planes, we find that

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1+1+1}\sqrt{1+4+9}} = \frac{2}{\sqrt{42}} \implies \theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ.$$

(b) Since L lies in both planes, it is perpendicular to both $\vec{n}_1$ and $\vec{n}_2$, and therefore parallel to their cross product

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \hat{\imath}(3 - (-2)) - \hat{\jmath}(3-1) + \hat{k}(-2-1) = \langle 5, -2, -3\rangle.$$

On the other hand, a point on L can be found by setting $z = 0$ in the equations of both planes. Indeed, doing so yields the system of equations $x + y = 1$ and $x - 2y = 1$, whose solution is $x = 1$ and $y = 0$, so the point $(1, 0, 0)$ lies on L. Thus, we conclude that L is represented by the symmetric equations

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}.$$

Suppose that we would like to know the distance D from the point $P_1 = (x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$. Letting $P_0 = (x_0, y_0, z_0)$ be any point in the plane, it suffices to compute the scalar projection of the vector

$$\vec{b} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle.$$

onto the normal vector $\vec{n} = \langle a, b, c\rangle$. Thus, we find that

$$\begin{aligned} D &= \left|\text{comp}_{\vec{n}}\vec{b}\right| \\ &= \left|\frac{\vec{n}\cdot\vec{b}}{|\vec{n}|}\right| \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}. \end{aligned}$$

Example 9.5.8. Find the distance between the parallel planes $2x - 3y + z = 4$ and $4x - 6y + 2z = 3$.

Solution. First, we verify that the planes are indeed parallel, since their normal vectors, $\langle 2, -3, 1\rangle$ and $\langle 4, -6, 2\rangle$, are parallel. Therefore, to find the distance between the planes, we must simply choose a point on one plane and calculate its distance to the other plane. Letting $x = y = 0$ in the first equation, we find that $z = 4$, so $(0, 0, 4)$ is a point on the first plane. By the distance formula, it follows that the distance between the planes is

$$D = \frac{|4(0) - 6(0) + 2(4) - 3|}{\sqrt{4^2 + (-6)^2 + 2^2}} = \frac{5}{\sqrt{56}} \approx 0.6682.$$

9.6. **Functions and Surfaces.**

Many physical quantities are naturally expressed as functions of two or more variables. These so-called **multivariate** functions are similar to their **univariate** counterparts in that they have a domain and range, and can be represented graphically.

Definition 9.6.1. A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, i.e. $\{f(x, y) : (x, y) \in D\}$.

We often write $z = f(x, y)$, and x and y are said to be **independent variables** while z is the **dependent variable**. The domain of f is a subset of the xy-plane.

Example 9.6.1. Describe the domain and range of the function $f(x,y) = x^2 + 2y^2$.

Solution. Any ordered pair (x, y) produces a valid output from this function, so the domain is $\mathbb{R}^2$, the entire xy-plane. The range is the set $[0, \infty)$, since $x^2 \geq 0$ and $y^2 \geq 0$ for all x and y.

Example 9.6.2. Find and sketch the domain of each of the following functions:

(a) $f(x,y) = \dfrac{\sqrt{y-x-1}}{x+1}$.

(b) $g(x,y) = x^2 \ln(x^2 - y)$.

Solution.

(a) The expression for f makes sense if the denominator is nonzero and the radiant is nonnegative. Therefore, the domain of f is $D = \{(x,y) : y - x - 1 \geq 0,\ x \neq -1\}$.

(b) For g, the expression within the natural logarithm must be strictly positive, which implies that the domain is $D = \{(x,y) : x^2 - y > 0\}$.

Definition 9.6.2. If f is a function of two variables with domain D, then the **graph** of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ atnd $(x, y) \in D$.

In Example 9.5.4, we saw how to use intercepts to sketch the graph of a plane. We now explore the graphs of some other three-dimensional surfaces described by multivariate functions.

Example 9.6.3. Sketch the graph of the function $f(x, y) = y^2 - 1$.

Solution. Regardless of the value of x, the value of $z = f(x, y)$ will always be $y^2 - 1$. This means that any vertical plane parallel to the yz-plane will intersect the graph of f in a parabolic curve. Therefore, the graph of f forms the shape of a **parabolic cylinder**, made up up infinitely many shifted copies of the same parabola.

In sketching the graph of a multivariate function, it is often useful to start by determining the shapes of some of its cross-sections. For example, fixing $x = k$ and letting y vary, we reduce the bivariate function $z = f(x, y)$ to the univariate function $z = f(k, y)$, whose graph is a curve that results from intersecting the surface $z = f(x, y)$ with the vertical plane $x = k$. Similarly, we may slice the surface with the vertical plane $y = k$ or the horizontal plane $z = k$. All three types of curves resulting from these slices are called **traces** of the surface $z = f(x, y)$.

Example 9.6.4. Use traces to sketch the graph of each of the following functions:

(a) $f(x, y) = x^2 + 2y^2$.

(b) $g(x, y) = y^2 - x^2$.

Solution.

(a) The traces in the horizontal planes $x = k$ and $y = k$ are the parabolas $z = k^2 + 2y^2$ and $z = x^2 + 2k^2$, respectively. On the other hand, the trace in the horizontal plane $z = k$ is the ellipses $x^2 + 2y^2 = k$. Therefore, the graph of f forms the shape of an **elliptic paraboloid**.

(b) The traces in the vertical planes $x = k$ and $y = k$ are the parabolas $z = y^2 - k^2$ and $z = x^2 - k^2$, respectively. The traces in the horizontal plane $z = k$ is the hyperbola $y^2 - x^2 = k$. Therefore, the graph of f forms the shape of a **hyperbolic paraboloid**.

The graph of a second-degree equation in three variables x, y, and z, is called a **quadric surface**. The previous example illustrated two types of quadric surfaces: an elliptic paraboloid and a hyperbolic paraboloid. Other types include **ellipsoids**, **cones**, and **hyperboloids**, which are all described in the table below.

Quadric Surface	General Equation	Properties
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	• All traces are ellipses. • If $a=b=c$, the ellipsoid is a sphere.
Cone	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	• Horizontal traces are ellipses. • Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$, but are pairs of lines if $k=0$.
Elliptic Paraboloid	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	• Horizontal traces are ellipses. • Vertical traces are parabolas. • The variable raised to the first power indicates the axis of the paraboloid.
Hyperbolic Paraboloid	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$	• Horizontal traces are ellipses. • Vertical traces are parabolas. • The orientation depends on the sign of c.
Hyperboloid of One Sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	• Horizontal traces are ellipses. • Vertical traces are hyperbolas. • The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperboloid of Two Sheets	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	• Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. • Vertical traces are hyperbolas. • The two minus signs indicate two sheets.

9.7. **Cylindrical and Spherical Coordinates.**

In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.

To convert from cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

On the other hand, to convert from rectangular to cylindrical coordinates, we use

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \qquad z = z.$$

Example 9.7.1.

(a) Plot the point with cylindrical coordinates $(4, -\pi/3, 5)$ and find its rectangular coordinates.
(b) Find cylindrical coordinates of the point with rectangular coordinates $(1, -1, 4)$.

Solution.

(a) For $(r, \theta, z) = (4, -\pi/3, 5)$, we find that

$$\begin{aligned} x &= 4\cos(-\pi/3) = 4(1/2) = 2, \\ y &= 4\sin(-\pi/3) = 4(-\sqrt{3}/2) = -2\sqrt{3}, \\ z &= 5. \end{aligned}$$

Therefore, the rectangular coordinates are $(2, -2\sqrt{3}, 5)$.

(b) For $(x, y, z) = (1, -1, 4)$, we have

$$\begin{aligned} r &= \sqrt{1^2 + (-1)^2} = \sqrt{2}, \\ \theta &= \tan^{-1}(-1/1) = -\pi/4 + 2n\pi, \\ z &= 4. \end{aligned}$$

Therefore, one set of cylindrical coordinates is $(\sqrt{2}, -\pi/4, 4)$, while another is $(\sqrt{2}, 7\pi/4, 4)$. As with polar coordinates, there are infinitely-many possibilities.

Example 9.7.2. Describe the surface whose equation in cylindrical coordinates is $z = r$.

Solution. The z-value of each point on the surface is the same as the distance from that point to the z-axis. Since the equation does not depend on θ, we find that any horizontal trace in the plane $z = k > 0$ is a circle of radius k. Therefore, we suspect that this surface is a cone. Indeed, converting to rectangular coordinates, we obtain $z^2 = x^2 + y^2$, which is the equation of a circular cone whose axis is the z-axis.

Example 9.7.3. Find an equation in cylindrical coordinates for the ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1$.

Solution. Since $r^2 = x^2 + y^2$, we have

$$\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1 \implies z^2 = 1 - \frac{r^2}{4}.$$

In the **spherical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (ρ, θ, ϕ), where ρ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line that passes through both P and the origin.

To convert from spherical to rectangular coordinates, we use the equations

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho\cos\phi.$$

On the other hand, the distance formula shows that

$$\rho^2 = x^2 + y^2 + z^2,$$

which can be used to convert from rectangular to spherical coordinates.

Example 9.7.4.

(a) Plot the point with spherical coordinates $(2, \pi/3, \pi/4)$ and find its rectangular coordinates.
(b) Find spherical coordinates of the point with rectangular coordinates $(0, -1, -1)$.

Solution.

(a) For $(\rho, \theta, \phi) = (2, \pi/3, \pi/4)$, we find that

$$\begin{aligned} x &= 2\sin(\pi/4)\cos(\pi/3) = 2(\sqrt{2}/2)(1/2) = \sqrt{2}/2, \\ y &= 2\sin(\pi/4)\sin(\pi/3) = 2(\sqrt{2}/2)(\sqrt{3}/2) = \sqrt{6}/2, \\ z &= 2\cos(\pi/4) = 2(\sqrt{2}/2) = \sqrt{2}. \end{aligned}$$

Therefore, the rectangular coordinates are $(\sqrt{2}/2, \sqrt{6}/2, \sqrt{2})$.

(b) For $(x, y, z) = (0, -1, -1)$, we have

$$\rho = \sqrt{0^2 + (-1)^2 + (-1)^2} = \sqrt{2}.$$

Therefore,

$$\begin{aligned} \cos\phi &= \frac{z}{\rho} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \implies \phi = \frac{3\pi}{4}, \\ \cos\theta &= \frac{x}{\rho\sin\phi} = \frac{0}{\sqrt{2}\cos(3\pi/4)} = 0 \implies \theta = \frac{3\pi}{2} + 2n\pi, \end{aligned}$$

and we conclude that one set of spherical coordinates is $(\sqrt{2}, 3\pi/2, 3\pi/4)$.

Example 9.7.5. Find an equation in spherical coordinates for the hyperboloid of one sheet given by the equation $x^2 + y^2 - z^2 = 1$.

Solution. Performing the substitutions $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$, we obtain the equation

$$\begin{aligned} x^2 + y^2 - z^2 = 1 \quad &\Longrightarrow \quad \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta - \rho^2 \cos^2\phi = 1 \\ &\Longrightarrow \quad \rho^2 \left[\sin^2\phi \left(\cos^2\theta + \sin^2\theta\right) - \cos^2\phi\right] = 1 \\ &\Longrightarrow \quad \rho^2 \left[\sin^2\phi - \cos^2\phi\right] = 1. \end{aligned}$$

Example 9.7.6. Find a rectangular equation for the surface whose spherical equation is $\rho = \sin\phi\cos\theta$.

Solution. We find that

$$x^2 + y^2 + z^2 = \rho^2 = \rho\left(\sin\phi\cos\theta\right) = x.$$

It follows that

$$\left(x - \frac{1}{2}\right)^2 + y^2 + z^2 = \frac{1}{4},$$

which is the equation of a sphere with center $\left(\frac{1}{2}, 0, 0\right)$ and radius $\frac{1}{2}$.

10. Vector Functions.

10.1. **Vector Functions and Space Curves.**

Recall that, in general, a function is a rule that assigns to each element in the domain an element in the range. A **vector-valued function**, or simply **vector function**, is one whose domain is a set of real numbers and whose range is a set of vectors.

If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\vec{r}(t)$, then f, g, and h are called the **component functions** of $\vec{r}$ and we can write

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}.$$

Example 10.1.1. If $\vec{r}(t) = \left\langle \sqrt{4-t^2}, e^{-3t}, \ln(t+1)\right\rangle$, then the component functions are $f(t) = \sqrt{4-t^2}$, $g(t) = e^{-3t}$, and $h(t) = \ln(t+1)$. The domain of $\vec{r}$ consists of all values of t for which the expression for $\vec{r}(t)$ is defined. Since the functions f, g, and h are all defined when $4 - t^2 \geq 0$ and $t + 1 > 0$, we conclude that the domain of $\vec{r}$ is the interval $(-1, 2]$.

Definition 10.1.1. If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle,$$

provided that the limits of the component functions exist.

Example 10.1.2. Find $\lim_{t\to\infty} \vec{r}(t)$, where $\vec{r}(t) = \left\langle \frac{4+t^2}{4-t^2}, \tan^{-1} t, \frac{2-e^{-2t}}{2t}\right\rangle$.

Solution. Recalling the long-term behavior of the component functions $f(t) = \frac{4+t^2}{4-t^2}$, $g(t) = \tan^{-1} t$, and $h(t) = \frac{2-e^{-2t}}{2t}$, we find that

$$\lim_{t\to\infty} \vec{r}(t) = \left[\lim_{t\to\infty} \frac{4+t^2}{4-t^2}\right]\hat{\imath} + \left[\lim_{t\to\infty} \tan^{-1} t\right]\hat{\jmath} + \left[\lim_{t\to\infty} \frac{2-e^{-2t}}{2t}\right]\hat{k} = \hat{\imath} + \frac{\pi}{2}\hat{\jmath}.$$

Definition 10.1.2. A vector function $\vec{r}$ is **continuous at a** if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$; that is, if its component functions are all continuous at a.

Definition 10.1.3. The set of all points (x, y, z) in space, where $x = f(t)$, $y = g(t)$, and $z = h(t)$ for continuous real-valued functions f, g, and h, is called a **space curve**.

The equations $x = f(t)$, $y = g(t)$, and $z = h(t)$ are called the **parametric equations** of a curve C with parameter t. We can think of C as being traced out by a moving particle whose position at time t is given by the Cartesian coordinate triple $(f(t), g(t), h(t))$. Letting $\vec{r}(t) = \langle f(t), g(t), r(t)\rangle$ denote the position vector corresponding to this point, we find that the space curve C is completely determined by the continuous vector function $\vec{r}$. Conversely, any continuous vector function $\vec{r}$ defines a space curve that is traced out by the tip of the moving vector $\vec{r}(t)$.

The simplest type of space curve is a line, and in Section 9.5 we have already seen the relationship between the vector equation of a line and the parametric equations of a line.

Example 10.1.3. Sketch the curve whose vector equation is $\vec{r}(t) = (t\cos t)\,\hat{\imath} + t\hat{\jmath} + (t\sin t)\,\hat{k}$ for $t \geq 0$.

Solution. The parametric equations for this curve are $x = t\cos t$, $y = t$, and $z = t\sin t$. Since

$$x^2 + z^2 = t^2\cos^2 t + t^2\sin^2 t = t^2,$$

we know that at time t the curve passes through a point on a circle of radius t in the plane $y = t$. Moreover, as t increases, the curve spirals away from the origin in a clockwise manner around the y-axis.

Example 10.1.4. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x + z = 1$.

Solution. Let C denote the curve in question. The projection of C onto the xy-plane is the circle $x^2 + y^2 = 4$, $z = 0$, which has the parametric representation $x = 2\cos t$, $y = 2\sin t$, $0 \leq t \leq 2\pi$. From the equation of the plane, we have

$$x + z = 1 \implies z = 1 - x \implies z = 1 - 2\cos t.$$

Therefore, the vector function for C is $\vec{r}(t) = (2\cos t)\hat{\imath} + (2\sin t)\hat{\jmath} + (1 - 2\cos t)\hat{k}$, $0 \leq t \leq 2\pi$.

10.2. **Derivatives and Integrals of Vector Functions.**

Definition 10.2.1. The **derivative** of a vector function $\vec{r}$ is defined by

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

The vector $\vec{r}\,'(t)$ is called the **tangent vector** to the curve defined by $\vec{r}$ at the point $P = \vec{r}(t)$, provided that $\vec{r}\,'(t)$ exists and $\vec{r}\,'(t) \neq \vec{0}$. The **tangent line** to the curve at the point P is the line passing through P that is parallel to the tangent vector $\vec{r}\,'(t)$.

It is sometimes also useful to consider the **unit tangent vector**, which is given by $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

Theorem 10.2.1. *If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$, where f, g, and h are differentiable functions, then*

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\hat{\imath} + g'(t)\hat{\jmath} + h'(t)\hat{k}.$$

Example 10.2.1. Find the unit tangent vector to the curve defined by $\vec{r}(t) = \langle te^{-t}, 2\tan^{-1} t, 2e^t \rangle$ at the point where $t = 0$.

Solution. First, we find the derivative of the vector function $\vec{r}$ to be

$$\vec{r}\,'(t) = \left\langle e^{-t}(1-t), \frac{1}{1+t^2}, 2e^t \right\rangle.$$

Since $\vec{r}(0) = \langle 0, 0, 2 \rangle$ and $\vec{r}\,'(0) = \langle 1, 1, 2 \rangle$, it follows that the unit tangent vector at the point $(0, 0, 2)$ is

$$\vec{T}(0) = \frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \frac{\langle 1, 1, 2 \rangle}{\sqrt{1^2 + 1^2 + 2^2}} = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle.$$

Example 10.2.2. Find the tangent line to the curve $x = 1 + 2\sqrt{t}$, $y = t^3 - t$, $z^3 = t^3 + t$ at the point $(3, 0, 2)$.

Solution. The vector equation for the curve is $\vec{r}(t) = \langle 1 + 2\sqrt{t}, t^3 - t, t^3 + t \rangle$ and the parameter value corresponding to the point $(3, 0, 2)$ is $t = 1$. The derivative of $\vec{r}$ is given by

$$\vec{r}\,'(t) = \left\langle \frac{1}{\sqrt{t}}, 3t^2 - 1, 3t^2 + 1 \right\rangle.$$

Therefore, the tangent vector to the curve at the point $(3, 0, 2)$ is $\vec{r}\,'(1) = \langle 1, 2, 4 \rangle$ and the equation of the tangent line to the curve at that point is

$$x = 3 + t, \qquad y = 2t, \qquad z = 2 + 4t.$$

Theorem 10.2.2 (Vector Function Differentiation Rules)**.** *If $\vec{u}$ and $\vec{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function, then the following properties hold.*

1. $\dfrac{d}{dt}\left[\vec{u}(t) \pm \vec{v}(t)\right] = \vec{u}\,'(t) \pm \vec{v}\,'(t)$.

2. $\dfrac{d}{dt}\left[c\vec{u}(t)\right] = c\vec{u}\,'(t)$.

3. $\dfrac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$.

4. $\dfrac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$.

5. $\dfrac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$.

6. $\dfrac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\vec{u}\,'(f(t))$.

Definition 10.2.2. If $\vec{r}$ is a continuous function defined for $a \leq t \leq b$, then the **definite integral of $\vec{r}$ from a to b** is

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\hat{\imath} + \left(\int_a^b g(t)\,dt\right)\hat{\jmath} + \left(\int_a^b h(t)\,dt\right)\hat{k}.$$

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a),$$

where $\vec{R}$ is an antiderivative of $\vec{r}$, i.e. $\vec{R}\,'(t) = \vec{r}(t)$.

Example 10.2.3. If $\vec{r}(t) = 16t\hat{\imath} - 9t^2\hat{\jmath} + 2e^t\hat{k}$, then

$$\int \vec{r}(t)\,dt = \left(\int 16t\,dt\right)\hat{\imath} - \left(\int 9t^2\,dt\right)\hat{\jmath} + \left(\int 2e^t\,dt\right)\hat{k} = 8t^2\hat{\imath} - 3t^3\hat{\jmath} + 2e^t\hat{k} + \vec{C},$$

where $\vec{C}$ is a vector constant of integration, and

$$\int_0^1 \vec{r}(t)\,dt = \left(8t^2\hat{\imath} - 3t^3\hat{\jmath} + 2e^t\hat{k}\right)\Big|_0^1 = 8\hat{\imath} - 3\hat{\jmath} + 2(e-1)\hat{k}.$$

10.3. **Arc Length and Curvature.**

Recall from Section 6.4 that the arc length of the parametrically defined planar curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, can be found as the limit of the sum of the lengths of inscribed line segments; in particular, this leads to the formula $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$ when f' and g' are continuous. In a similar manner, we now define an arc length formula for space curves.

Definition 10.3.1. Suppose that the space curve C is defined by the vector equation $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f', g', and h' exist and are continuous for $a \leq t \leq b$. If C is traversed exactly once as t increases from a to b, then its length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

Assuming that $\vec{r}(t) = \langle f(t), g(t)\rangle$ represents a general planar curve, while $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ represents a general space curve, we see that the arc length of either can be compactly written as $L = \int_a^b |\vec{r}\,'(t)|\,dt$.

Example 10.3.1. Find the length of the arc of the circular helix with vector equation $\vec{r}(t) = \langle \cos t, t, \sin t\rangle$ from the point $(1, 0, 0)$ to the point $(1, 2\pi, 0)$.

Solution. First, we observe that $\vec{r}\,'(t) = \langle -\sin t, 1, \cos t\rangle$ is continuous and

$$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (1)^2 + (\cos t)^2} = \sqrt{2}.$$

Since the arc from $(1, 0, 0)$ to $(1, 2\pi, 0)$ corresponds to the parameter interval $0 \leq t \leq 2\pi$, it follows that the arc length in question is given by

$$L = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2} \cdot 2\pi \approx 8.8858.$$

A space curve can be represented by more than one vector function. For example, the curve defined by $\vec{r}_1(t) = \langle 1, t, t^2 \rangle$, $1 \leq t \leq 2$, is equivalent to the curve defined by $\vec{r}_2(u) = \langle 1, e^u, e^{2u} \rangle$, $0 \leq u \leq \ln 2$, where the connection between the parameters t and u is given by the equation $t = e^u$. Such representations are called **parameterizations**. In general, the arc length formula will yield a result that is independent of the parametrization that is used to represent a curve.

Definition 10.3.2. Suppose that the space curve C is defined by the vector equation $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$, where $\vec{r}\,'$ is continuous and C is traversed exactly once as t increases from a to b. Then its **arc length function** s is defined by

$$s(t) = \int_a^t |\vec{r}\,'(u)|\; du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\; du.$$

Thus, $s(t)$ represents the length of the part of C that lies between $\vec{r}(a)$ and $\vec{r}(t)$. Notice that, by the first part of the Fundamental Theorem of Calculus, differentiating both sides of the formula for s yields $\frac{ds}{dt} = |\vec{r}\,'(t)|$.

It can be useful to parametrize a curve with respect to its arc length, since arc length is intrinsically related to the shape of the curve and does not depend on a particular coordinate system. If a curve is given in terms of a parameter t, and $s(t)$ is its arc length function, then we may be able to solve for t as a function of s and reparametrize the curve in terms of s by substituting for t; that is, $t = t(s) \implies \vec{r}(t) = \vec{r}(t(s))$. Thus, for example, $\vec{r}(t(2))$ would be the position vector of the point that is 2 units of length along the curve from its starting point.

Example 10.3.2. Reparametrize the helix $\vec{r}(t) = \langle \cos t, t, \sin t \rangle$ with respect to its arc length, measured from $(1, 0, 0)$ in the direction that t increases.

Solution. The initial point $(1, 0, 0)$ corresponds to the parameter value $t = 0$. From Example 10.3.1, we have that

$$\frac{ds}{dt} = |\vec{r}\,'(t)| = \sqrt{2},$$

and so

$$s = s(t) = \int_0^t |\vec{r}\,'(u)|\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t.$$

Therefore, $t = s/\sqrt{2}$, and the desired reparametrization is given by

$$\vec{r}(t(s)) = \left\langle \cos\left(s/\sqrt{2}\right), \, s/\sqrt{2}, \, \sin\left(s/\sqrt{2}\right) \right\rangle.$$

A parametrization $\vec{r}(t)$ is called **smooth** on an interval I if $\vec{r}\,'$ is continuous and nonzero on I. A curve is called **smooth** if it has a smooth parametrization. Intuitively, a smooth curve can be thought of as having no sharp corners or cusps. If C is a smooth curve defined by the vector function $\vec{r}$, recall that the unit tangent vector is defined by

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

and indicates the direction of the curve at time t. The curvature of C at a given point is a measure of how quickly the curve is changing direction at that point. Naturally, this concept is defined in terms of the unit tangent vector, as follows.

Definition 10.3.3. The **curvature** of a curve is $\kappa = \left|\dfrac{d\vec{T}}{ds}\right|$, where $\vec{T}$ is the unit tangent vector.

By the Chain Rule and the fact that $\frac{ds}{dt} = |\vec{r}\,'(t)|$, we have

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{dt}\cdot\frac{dt}{ds}\right| = \left|\frac{d\vec{T}/dt}{ds/dt}\right| = \frac{\left|\vec{T}\,'(t)\right|}{|\vec{r}\,'(t)|}.$$

Example 10.3.3. Using the formula $\kappa = |\vec{T}\,'(t)|/|\vec{r}\,'(t)|$, it is straightforward to show that the curvature of a circle of radius a has constant curvature $1/a$. Indeed, with the parametrization $\vec{r}(t) = \langle a\cos t, a\sin t\rangle$, we have

$$\vec{r}\,'(t) = \langle -a\sin t, a\cos t\rangle \implies |\vec{r}\,'(t)| = a$$

and

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\sin t, \cos t\rangle \implies \vec{T}\,'(t) = \langle -\cos t, -\sin t\rangle \implies |\vec{T}\,'(t)| = 1.$$

This result supports the intuition that large circles have small curvature and small circles have large curvature.

Using the Product Rule for vector functions and the definition of the cross product, it is possible to prove the following alternate formula for curvature.

Theorem 10.3.1. *The curvature of the curve given by the vector function $\vec{r}$ is* $\kappa(t) = \dfrac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$.

Since $|\vec{T}(t)| = 1$ for all t, we have

$$1 = |\vec{T}(t)|^2 = \vec{T}(t)\cdot\vec{T}(t) \implies 0 = \frac{d}{dt}\left[\vec{T}(t)\cdot\vec{T}(t)\right] = \vec{T}\,'(t)\cdot\vec{T}(t) + \vec{T}(t)\cdot\vec{T}\,'(t) = 2\vec{T}\,'(t)\cdot\vec{T}(t).$$

That is, $\vec{T}\,'(t)\cdot\vec{T}(t) = 0$, so the unit tangent vector is orthogonal to its derivative. At any point where $\kappa \neq 0$, we define the **principal unit normal vector** (or simply **unit normal vector**) by

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}.$$

The unit vector $\vec{B}(t) = \vec{T}(t)\times\vec{N}(t)$, which is orthogonal to both $\vec{T}(t)$ and $\vec{N}(t)$, is called the **binormal vector**.

The plane determined by the normal and binormal vectors at a point P on a curve C is called the **normal plane** of C at P, and consists of all lines that are orthogonal to the tangent vector. The plane determined by the tangent and normal vectors is called the **osculating plane** of C at P, and is the plane that comes closest to containing the part of the curve near P. Note that for a planar curve, the osculating plane is simply the plane that contains the curve.

The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which the normal vector points), and has radius $\rho = 1/\kappa$ is called the **osculating circle** (or the **circle of curvature**) of C at P. It is the circle that best describes how C behaves near P, as it shares the same tangent vector, normal vector, and curvature at P.

Example 10.3.4. Find equations for the normal plane and osculating plane of $\vec{r}(t) = \langle \cos t, t, \sin t\rangle$ at the point $P = (0, \pi/2, 1)$.

Solution. We begin by computing the unit tangent, normal, and binormal vectors to the curve. From Examples 10.3.1 and 10.3.2, we have $\vec{r}\,'(t) = \langle -\sin t, 1, \cos t\rangle$ and $|\vec{r}\,'(t)| = \sqrt{2}$. It follows that

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{\sqrt{2}}\langle -\sin t, 1, \cos t\rangle,$$

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \langle -\cos t, 0, -\sin t\rangle,$$

$$\vec{B}(t) = \vec{T}(t)\times\vec{N}(t) = \frac{1}{\sqrt{2}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & 1 & \cos t \\ -\cos t & 0 & -\sin t\end{vmatrix} = \frac{1}{\sqrt{2}}\langle -\sin t, -1, \cos t\rangle.$$

Observe that the point P corresponds to the parameter value $t = \pi/2$. Therefore, the normal plane at P has normal vector $\vec{r}\,'(\pi/2) = \langle -1, 1, 0\rangle$, and its equation is

$$-1(x-0) + 1(y-\pi/2) + 0(z-1) = 0 \implies y = x + \frac{\pi}{2}.$$

The osculating plane at P has normal vector $\vec{B}(\pi/2) = \frac{1}{\sqrt{2}}\langle -1, -1, 0\rangle$, so its equation is

$$-1(x-0) - 1(y-\pi/2) + 0(z-1) = 0 \implies y = -x + \frac{\pi}{2}.$$

10.4. **Motion in Space: Velocity and Acceleration.**

Suppose that a particle moves through space so that its position vector at time t is $\vec{r}(t)$. Based on earlier analysis, it is natural to conclude that the **velocity vector** $\vec{v}(t)$ of the particle at time t is the same as the tangent vector $\vec{r}\,'(t)$. The **speed** of the particle at time t is equal to the magnitude of the velocity vector, which makes sense because

$$|\vec{v}(t)| = |\vec{r}\,'(t)| = \frac{ds}{dt} = \text{ rate of change of distance with respect to time.}$$

Moreover, the **acceleration** of the particle is defined as the derivative of velocity: $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

Example 10.4.1. Find the velocity, speed, and acceleration at time $t = 1$ of a particle with the position function $\vec{r}(t) = \langle \sqrt{2}t, e^t, e^{-t}\rangle$.

Solution. In general, we have

$$\begin{aligned} \vec{v}(t) &= \vec{r}\,'(t) = \langle \sqrt{2}, e^t, -e^{-t}\rangle, \\ |\vec{v}(t)| &= \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}}, \\ \vec{a}(t) &= \vec{v}\,'(t) = \langle 0, e^t, e^{-t}\rangle. \end{aligned}$$

Therefore, at time $t = 1$, the velocity, speed, and acceleration of the particle are $\vec{v}(1) = \langle \sqrt{2}, e, -1/e\rangle$, $|\vec{v}(1)| = \sqrt{2 + e^2 + 1/e^2}$, and $\vec{a}(1) = \langle 0, e, 1/e\rangle$, respectively.

Example 10.4.2. Find the position at time t of a particle with initial position $\vec{r}(0) = \langle 1, 0, 0\rangle$, initial velocity $\vec{v}(0) = \langle 0, 0, 1\rangle$, and acceleration function $\vec{a}(t) = \langle 1, 2, 0\rangle$.

Solution. Since $\vec{a}(t) = \vec{v}\,'(t)$, we have

$$\vec{v} = \int \vec{a}(t)\,dt = \int \langle 1, 2, 0\rangle\,dt = \langle t, 2t, 0\rangle + \vec{C},$$

and $\vec{v}(0) = \langle 0, 0, 1\rangle$ implies that $\vec{C} = \langle 0, 0, 1\rangle$. Next, since $\vec{v}(t) = \vec{r}\,'(t)$, we have

$$\vec{r}(t) = \int \vec{v}(t) = \int \langle t, 2t, 1\rangle\,dt = \left\langle \frac{t^2}{2}, t^2, t\right\rangle + \vec{D},$$

and $\vec{r}(0) = \langle 1, 0, 0\rangle$ implies that $\vec{D} = \langle 1, 0, 0\rangle$. Therefore, the position of the particle at time t is given by

$$\vec{r}(t) = \left\langle \frac{t^2}{2} + 1, t^2, t\right\rangle.$$

In general, we can use vector integrals to recover velocity when acceleration is known and position when velocity is known:

$$\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} \vec{a}(u)\,du \qquad \text{and} \qquad \vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{v}(u)\,du.$$

Example 10.4.3. A projectile is fired with an angle of elevation α and initial velocity $\vec{v}_0$. Assuming that air resistance is negligible and the only external force is due to gravity, find the position function of the projectile. What value of α maximizes the range (the horizontal distance traveled)?

Solution. Suppose that the projectile starts at the origin, so that its initial position is $\vec{r}(0) = \vec{0}$. Since the only force is due to gravity, the vector version of Newton's Second Law of Motion gives

$$\vec{F} = m\vec{a} = -mg\hat{\jmath},$$

where $g = |\vec{a}| = 9.8$ m/s^2. In other words, $\vec{a} = -g\hat{\jmath}$, and it follows that

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\,du = \vec{v}_0 - gt\hat{\jmath}.$$

Integrating again yields

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\,du = \vec{v}_0 - \frac{1}{2}gt^2\hat{\jmath}.$$

Letting $v_0 = |\vec{v}_0|$ denote the initial speed of the projectile, we may write

$$\vec{v}_0 = (v_0\cos\alpha)\hat{\imath} + (v_0\sin\alpha)\hat{\jmath}.$$

Hence,

$$\vec{r}(t) = [(v_0\cos\alpha)t]\,\hat{\imath} + \left[(v_0\sin\alpha)t - \frac{1}{2}gt^2\right]\hat{\jmath}.$$

Equivalently, the parametric equations of the trajectory are

$$x = (v_0\cos\alpha)t \qquad \text{and} \qquad y = (v_0\sin\alpha)t - \frac{1}{2}gt^2.$$

The horizontal distance d traveled by the projectile before it hits the ground is equal to the value of x when $y = 0$. Setting $y = 0$, we obtain $t = 0$ or $t = (2v_0\sin\alpha)/g$. The latter value of t corresponds to

$$d = x = (v_0\cos\alpha)\frac{2v_0\sin\alpha}{g} = \frac{v_0^2(2\sin\alpha\cos\alpha)}{g} = \frac{v_0^2\sin(2\alpha)}{g}.$$

Since $|\sin(2\alpha)| \leq 1$, we conclude that d is maximized when $\sin(2\alpha) = 1$; that is, when $\alpha = \pi/4$.

When studying the motion of a particle, it is sometimes useful to resolve its acceleration into two components: one in the direction of the tangent and the other in the direction of the normal. Letting $v = |\vec{v}|$ denote the speed of the particle, we have

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{v}(t)}{v}.$$

Therefore, $\vec{v} = v\vec{T}$. Differentiating both sides of this equation respect to t yields

$$\vec{a} = \vec{v}\,' = v'\vec{T} + v\vec{T}\,'.$$

On the other hand, we have

$$\kappa = \frac{|\vec{T}\,'|}{|\vec{r}\,'|} = \frac{|\vec{T}\,'|}{v} \implies |\vec{T}\,'| = \kappa v$$

Since the unit normal vector is defined as $\vec{N} = \vec{T}\,'/|\vec{T}\,'|$, it follows that

$$\vec{T}\,' = |\vec{T}\,'|\,\vec{N} = \kappa v\vec{N}.$$

Hence, acceleration can be written as

$$\vec{a} = v'\vec{T} + \kappa v^2\vec{N}.$$

Equivalently, letting $a_T = v'$ and $a_N = \kappa v^2$ denote the tangential and normal components, respectively, we have

$$\vec{a} = a_T\vec{T} + a_N\vec{N}.$$

It is convenient to express a_T and a_N in terms of $\vec{r}$, $\vec{r}\,'$, and $\vec{r}\,''$. This can be accomplished by observing that

$$\vec{v}\cdot\vec{a} = v\vec{T}\cdot(v'\vec{T} + \kappa v^2\vec{N}) = vv'\vec{T}\cdot\vec{T} + \kappa v^3\vec{T}\cdot\vec{N} = vv',$$

since $\vec{T}\cdot\vec{T} = |\vec{T}|^2 = 1^2 = 1$ and $\vec{T}\cdot\vec{N} = 0$. Therefore

$$a_T = v' = \frac{\vec{v}\cdot\vec{a}}{v} = \frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{|\vec{r}\,'(t)|}$$

and

$$a_N = \kappa v^2 = \frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}|\vec{r}\,'(t)|^2 = \frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|}.$$

Example 10.4.4. Find the tangential and normal components of the acceleration of a particle with position function $\vec{r}(t) = \langle 1, t, t^2 \rangle$.

Solution. First, we compute

$$\begin{aligned} \vec{r}(t) &= \langle 1, t, t^2 \rangle, \\ \vec{r}\,'(t) &= \langle 0, 1, 2t \rangle, \\ \vec{r}\,''(t) &= \langle 0, 0, 2 \rangle. \end{aligned}$$

It follows that

$$\vec{r}\,'(t) \cdot \vec{r}\,''(t) = \langle 0, 1, 2t \rangle \cdot \langle 0, 0, 2 \rangle = 4t$$

and

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = \langle 2, 0, 0 \rangle.$$

Therefore, the tangential and normal components of the acceleration are

$$a_T = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{|\vec{r}\,'(t)|} = \frac{4t}{\sqrt{1+4t^2}}$$

and

$$a_N = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|} = \frac{2}{\sqrt{1+4t^2}}.$$

10.5. **Parametric Surfaces.**

In much the same way that we describe a space curve by a vector function $\vec{r}(t)$ of single parameter t, we can describe a surface by a vector function $\vec{r}(u, v)$ of two parameters u and v. Suppose that

$$\vec{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$$

is a vector-valued function defined on a region D in the uv-plane. As $(u, v) \in D$ varies, the tip of the position vector $\vec{r}(u, v)$ traces out a **parametric surface** S. The surface may also be described as the set of all points $(x, y, z) \in \mathbb{R}^3$ that satisfy the parametric equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, where $(u, v) \in D$.

11. Partial Derivatives.

11.1. **Functions of Several Variables.**

In this section, we consider functions of two or more variables, which may be defined in any of four ways described in Section 1.1: verbally, numerically, algebraically, and visually. While the first three of these ways are nearly identical to what we are familiar with in the univariate case, the fourth requires some new techniques to account for the additional variable(s).

Definition 11.1.1. The **level curves** of a function f of two variables are the curves with equations $f(x,y)=k$, where k is a constant (in the range of f).

Level curves are commonly used in constructing contour plots, such as topographic and isothermal maps. They are a special case of the traces described in Section 9.6.

Example 11.1.1. Illustrate the level curves of several bivariate functions. Discuss the estimation of function values using contour plots.

Functions of three variables are difficult to visualize, since their graphs lie in four-dimensional space. However, some insight as to the behavior of such functions can be gleaned from their so-called **level surfaces**, which are defined by the equations $f(x,y,z)=k$, where k is a constant.

Functions of four or more variables can also be considered, and indeed are extremely useful in applications. For example, many of the functions used by engineers, economists, and financial experts depend on hundreds of variables. Although impossible to represent graphically, such functions can still be described verbally, numerically, and algebraically.

11.2. **Limits and Continuity.**

As in the univariate case, we often wish to understand the behavior of a multivariate function near, but not at, a particular point.

Definition 11.2.1. We write

$$\lim_{(x,y)\to(a,b)} f(x,y)=L,$$

and we say that the **limit of $\boldsymbol{f(x,y)}$ as $\boldsymbol{(x,y)}$ approaches $\boldsymbol{(a,b)}$** is L, if we can make the values of $f(x,y)$ as close to L as we like by taking the point (x,y) sufficiently close to the point (a,b), but not equal to (a,b).

Notice that, by definition, the existence of the limit L depends only on whether the *distance* between $f(x,y)$ and L can be made arbitrarily small by taking the *distance* between (x,y) and (a,b) to be sufficiently small. In particular, the direction of approach does not matter; if the limit exists, then $f(x,y)$ must always approach L, regardless of the path along which (x,y) approaches (a,b). The contrapositive of this statement is summarized in the following result and is often used to prove the nonexistence of a limit.

Proposition 11.2.1. *If $f(x,y)\to L_1$ as $(x,y)\to(a,b)$ along a path C_1 and $f(x,y)\to L_2$ as $(x,y)\to(a,b)$ along a path C_2, where $L_1\neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.*

On the other hand, to prove that a limit does exist, it can be helpful to generalize the results of Section 2.3. In particular, the Limit Laws remain valid for multivariate functions: the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on. Moreover, we have

$$\lim_{(x,y)\to(a,b)} x=a, \qquad \lim_{(x,y)\to(a,b)} y=b, \qquad \lim_{(x,y)\to(a,b)} c=c,$$

and an analogue of the Squeeze Theorem also holds.

Example 11.2.1. Determine whether $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ exists.

Solution. If we approach $(0,0)$ along the x-axis by setting $y = 0$, we have $f(x,0) = x^2/x^2 = 1$ for all $x \neq 0$. On the other hand, if we approach $(0,0)$ along the y-axis by setting $x = 0$, we have $f(0,y) = -y^2/y^2 = -1$ for all $y \neq 0$. Therefore $f(x,y) \to 1$ as $(x,y) \to (0,0)$ along the x-axis and $f(x,y) \to -1$ as $(x,y) \to (0,0)$ along the y-axis. By Proposition 11.2.1, it follows that the limit does not exist.

Example 11.2.2. Determine whether $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exists.

Solution. If $y = 0$, then $f(x,0) = 0/x^2 = 0$ for all $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Likewise, if $x = 0$, then $f(0,y) = 0/y^2 = 0$ for al $y \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the y-axis. However, this does not show that the limit exists. Indeed, $f(x,x) = x^2/(x^2+x^2) = 1/2$ for all $x \neq 0$, so $f(x,y) \to 1/2$ as $(x,y) \to (0,0)$ along the line $y = x$. Therefore, again by Proposition 11.2.1, we conclude that the limit does not exist.

Example 11.2.3. Determine whether $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$ exists.

Solution. If, as in the preceding examples, we find the limit of f as $(x,y) \to (0,0)$ along various paths, we may begin to suspect that the limit of this function actually does exist and is equal to 0. To prove it, we consider the distance from $f(x,y)$ to 0, which is given by

$$\left|\frac{x^2y}{x^2+y^2} - 0\right| = \left|\frac{x^2y}{x^2+y^2}\right| = \frac{x^2|y|}{x^2+y^2}.$$

Since $x^2 \leq x^2 + y^2$, we have

$$0 \leq \frac{x^2|y|}{x^2+y^2} \leq |y|.$$

Moreover, we observe that

$$\lim_{(x,y)\to(0,0)} 0 = 0 \qquad \text{and} \qquad \lim_{(x,y)\to(0,0)} |y| = 0,$$

so, by the Squeeze Theorem, we conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0.$$

Definition 11.2.2. A function f of two variables is called **continuous at (a,b)** if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say f is **continuous on S** if f is continuous at every point $(a,b) \in S$.

A **polynomial function of two variables** (or simply **polynomial**) is a sum of terms of the form cx^my^n, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of polynomials. For example, the bivariate function

$$f(x,y) = x^3 + 3x^2y + 3xy^2 + y^3$$

is a polynomial, whereas

$$g(x,y) = \frac{xy-1}{x+y+1}$$

is a rational function. As in the univariate case, every bivariate polynomial is continuous on $\mathbb{R}^2$, while every bivariate rational function is continuous on its domain.

Example 11.2.4. Evaluate $\lim_{(x,y)\to(1,0)} \left(x^3 + 3x^2y + 3xy^2 + y^3\right)$.

Solution. Since $f(x,y) = x^3 + 3x^2y + 3xy^2 + y^3$ is a polynomial, it is continuous everywhere, so its limit at any point can be computed by direct substitution. In particular, we find that

$$\lim_{(x,y)\to(1,0)} \left(x^3 + 3x^2y + 3xy^2 + y^3\right) = (1)^3 + 3(1)^2(0) + 3(1)(0)^2 + (0)^3 = 1.$$

Example 11.2.5. The function

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

is discontinuous at the origin, since $f(0,0)$ is undefined. The function

$$g(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

is also discontinuous at the origin, since $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist.

Example 11.2.6. The function

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

is continuous everywhere. Indeed, f is continuous for $(x,y) \neq (0,0)$ since it is equal to a well-defined rational function there. Moreover, f is also continuous at $(x,y) = (0,0)$, since

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0 = f(0,0).$$

Everything we have discussed in this section can be extended to functions of three or more variables. For example, the notation

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$$

means that the values of $f(x,y,z)$ approach the number L as the point (x,y,z) approaches the point (a,b,c) along any path in the domain of f, and the function f is continuous at (a,b,c) if and only if

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c).$$

11.3. **Partial Derivatives.**

Suppose that f is a function of two variables, x and y, and we let x vary while fixing $y = b$, where b is a constant. Then it is as if we are dealing the univariate function $g(x) = f(x,b)$. If g has a derivative at a, then we call it the **partial derivative of f with respect to x at (a,b)**, and denote it by $f_x(a,b)$. Therefore, we have

$$f_x(a,b) = g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

Similarly, the **partial derivative of f with respect to y at (a,b)** is defined by

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

Definition 11.3.1. If f is a function of two variables, then its **partial derivatives** are the functions f_x and f_y, defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad \text{and} \qquad f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

In view of the definition, a good rule of thumb for finding the partial derivatives of $z = f(x,y)$ is as follows:

- to find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
- to find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Remark 11.3.1. There are many alternative notations for partial derivatives. Indeed, if $z = f(x, y)$, then

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f,$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f.$$

Example 11.3.1. If $f(x, y) = e^{-x} \sin(\pi y)$, find $f_x(0, 1/2)$ and $f_y(0, 1/2)$.

Solution. Keeping y constant, and differentiating with respect to x, we find that

$$f_x(x,y) = -e^{-x}\sin(\pi y) \implies f_x(0,1/2) = -e^0 \sin(\pi/2) = -1.$$

On the other hand, keeping x constant, and differentiating with respect to y, we obtain

$$f_y(x,y) = \pi e^{-x}\cos(\pi y) \implies f_y(0,1/2) = \pi e^0 \cos(\pi/2) = 0.$$

The partial derivatives of bivariate functions can be interpreted geometrically as the slopes of tangent lines. In particular, $f_x(a, b)$ and $f_y(a, b)$ represent the slopes of the tangent lines at point $P = (a, b, c)$ to the traces of the surface $z = f(x, y)$ in the planes $y = b$ and $y = a$, respectively.

Example 11.3.2. If $f(x, y) = \ln\left(\frac{x}{1+y}\right)$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial x}$.

Solution. Using the Chain Rule for univariate functions, we have

$$\frac{\partial f}{\partial x} = \ln\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \frac{1+y}{x} \cdot \frac{1}{1+y} = \frac{1}{x},$$
$$\frac{\partial f}{\partial x} = \ln\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = \frac{1+y}{x} \cdot \frac{-x}{(1+y)^2} = \frac{-1}{1+y}.$$

Example 11.3.3. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial x}$ if z is defined implicitly as a function of x and y by the equation $x^2 + y^2 + z^2 = 3xyz$.

Solution. To find $\frac{\partial z}{\partial x}$, we differentiate implicitly with respect to x, while treating y as a constant, which gives

$$2x + 2z\frac{\partial z}{\partial x} = 3yz + 3xy\frac{\partial z}{\partial x} \implies \frac{\partial z}{\partial x} = \frac{2x - 3yz}{3xy - 2z}.$$

Similarly, implicit differentiation with respect to y yields

$$\frac{\partial z}{\partial y} = \frac{2y - 3xz}{3xy - 2z}.$$

Partial derivatives can be defined analogously for functions of three or more variables and computing them remains very straightforward. For example, if $f(x, y, z) = x \cos y \sin z$, then $f_x(x, y, z) = \cos y \sin z$, $f_y(x, y, z) = -x \sin y \sin z$, and $f_z(x, y, z) = x \cos y \cos z$.

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f. If $z = f(x, y)$, then we may write

$$(f_x)_x(x,y) = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2},$$
$$(f_x)_y(x,y) = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x},$$
$$(f_y)_x(x,y) = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y},$$
$$(f_y)_y(x,y) = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}.$$

Higher order partial derivatives can be defined analogously.

Example 11.3.4. Find the second partial derivatives of $f(x, y) = e^{-x}\sin(\pi y)$.

Solution. By Example 11.3.1, we have $f_x(x, y) = -e^{-x}\sin(\pi y)$ and $f_y(x, y) = \pi e^{-x}\cos(\pi y)$. Therefore

$$f_{xx} = \frac{\partial}{\partial x}\left(-e^{-x}\sin(\pi y)\right) = e^{-x}\sin(\pi y), \qquad f_{xy} = \frac{\partial}{\partial y}\left(-e^{-x}\sin(\pi y)\right) = -\pi e^{-x}\cos(\pi y),$$
$$f_{yx} = \frac{\partial}{\partial x}\left(\pi e^{-x}\cos(\pi y)\right) = -\pi e^{-x}\cos(\pi y), \quad f_{yy} = \frac{\partial}{\partial y}\left(\pi e^{-x}\cos(\pi y)\right) = \pi^2 e^{-x}\cos(\pi y).$$

In the preceding example, we found that $f_{xy} = f_{yx}$. This is not a coincidence, as the following result asserts.

Theorem 11.3.1 (Clairaut's Theorem). *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.*

11.4. **Tangent Planes and Linear Approximations.**

No notes are provided for this section. It covers the use of tangent planes in linearly approximating multivariate function values. The related notion of differentials is also recalled from Section 3.9.

11.5. **The Chain Rule.**

Recall that the Chain Rule for univariate functions gives a way of differentiating composite functions: if $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is also a differentiable function (of t) and $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$. This idea can be extended to the case of multivariate functions, for which there are several versions of the Chain Rule.

Proposition 11.5.1 (Chain Rule: Case 1). *Suppose that $z = f(x, y)$ is a differentiable function x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Example 11.5.1. If $z = x^2 + y^2 + xy$, where $x = \sin t$ and $y = e^t$, then find $\frac{dz}{dt}$ when $t = 0$.

Solution. By the multivariate Chain Rule (Case 1), we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x + y)(\cos t) + (2y + x)\left(e^t\right).$$

Since $x = \sin(0) = 0$ and $y = e^0 = 1$ when $t = 0$, it follows that

$$\left.\frac{dz}{dt}\right|_{t=0} = (2(0) + 1)(\cos(0)) + (2(1) + 0)\left(e^0\right) = 3.$$

Proposition 11.5.2 (Chain Rule: Case 2). *Suppose that $z = f(x, y)$ is a differentiable function x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \textit{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Example 11.5.2. If $z = x^2y^3$, where $x = s\cos t$ and $y = s\sin t$, then find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution. By the multivariate Chain Rule (Case 2), we have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2xy^3)(\cos t) + (3x^2y^2)(\sin t) = 5s^4\cos^2 t\sin^3 t,$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2xy^3)(-s\sin t) + (3x^2y^2)(s\cos t) = -2s^5\cos t\sin^4 t + 3s^5\cos^3 t\sin^2 t.$$

Proposition 11.5.3 (Chain Rule: General Version)**.** *Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$, and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and*

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt_i} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt_i}.$$

for each $i = 1, 2, \ldots, m$.

The general version of the Chain Rule contains three types of variables: $t_1, \ldots, t_m$ are the **independent** variables, $x_1, \ldots, x_n$ are the **intermediate** variables, and u is the **dependent** variable. To remember the Chain Rule formula, it is helpful to draw a **tree diagram** with branches connecting the dependent variable to all of the intermediate variables and connecting each intermediate variable to all of the dependent variables.

Example 11.5.3. Illustrate several flavors of the General Version of the Chain Rule via tree diagrams.

Using the Chain Rule, we can develop a convenient formula for implicit partial differentiation. If $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then differentiating both sides of this equation yields

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Similarly, if the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of both x and y, then we can obtain

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Example 11.5.4. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^2 + y^2 + z^2 = 3xyz$. Compare with Example 11.3.3.

Solution. Let $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz$. Then it is straightforward to compute that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x - 3yz}{2z - 3xy} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y - 3xz}{2z - 3xy}.$$

11.6. **Directional Derivatives and the Gradient Vector.**

Recall that, if $z = f(x, y)$, then the partial derivatives f_x and f_y represent the rates of change of z in the x- and y-directions; that is, in the directions of the standard basis vectors $\hat{\imath}$ and $\hat{\jmath}$. Suppose that we wish to find the rate of change of z at the point (x_0, y_0) in the direction of an arbitrary unit vector $\hat{u} = \langle a, b \rangle$. Let S denote the surface defined by $z = f(x, y)$ and let $z_0 = f(x_0, y_0)$ so that $P = (x_0, y_0, z_0)$ is a point on S. Then, the vertical plane that passes through P in the direction of $\hat{u}$ intersects S in a curve C, and the slope of the tangent line T to C at the point P is the desired rate of change of z in the direction of $\hat{u}$.

Definition 11.6.1. The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\hat{u} = \langle a, b \rangle$ is

$$D_{\hat{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h},$$

provided that the limit exists.

Notice that $D_{\hat{\imath}} f = f_x$ and $D_{\hat{\jmath}} f = f_y$, so partial derivatives are special cases of the directional derivative.

Theorem 11.6.1. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\hat{u} = \langle a, b \rangle$ and*

$$D_{\hat{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

If the unit vector $\hat{u}$ makes an angle θ with the positive x-axis, then we can write $\hat{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula for the directional derivative becomes $D_{\hat{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$.

Example 11.6.1. Find the directional derivative of $f(x,y) = ye^{-x}$ at the point $(0,4)$ in the direction of the unit vector given by the angle $\theta = 2\pi/3$.

Solution. By the formula above, we have

$$D_{\hat{u}}f(x,y) = -ye^{-x}\cos(2\pi/3) + e^{-x}\sin(2\pi/3) = \frac{1}{2}ye^{-x} + \frac{\sqrt{3}}{2}e^{-x} = \frac{e^{-x}}{2}\left(y+\sqrt{3}\right).$$

Therefore

$$D_{\hat{u}}f(0,4) = \frac{e^0}{2}\left(4+\sqrt{3}\right) = \frac{4+\sqrt{3}}{2}.$$

Definition 11.6.2. If f is a function of two variables x and y, then the **gradient** of f is the vector function $\vec{\nabla} f$ defined by

$$\vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}.$$

We may express the directional derivative in the direction of a unit vector $\hat{u}$ as the scalar projection of the gradient vector onto $\hat{u}$, i.e. $D_{\hat{u}}f(x,y) = \vec{\nabla} f(x,y)\cdot\hat{u}$.

Example 11.6.2. Find the directional derivative of $f(x,y) = 1 + 2x\sqrt{y}$ at the point $(3,4)$ in the direction of the vector $\vec{v} = \langle 4,-3\rangle$.

Solution. We first compute the gradient vector at $(3,4)$ to be

$$\vec{\nabla} f(x,y) = \langle 2\sqrt{y}, x/\sqrt{y}\rangle \implies \vec{\nabla} f(3,4) = \langle 2\sqrt{4}, 3/\sqrt{4}\rangle = \langle 4, 3/2\rangle.$$

Next, since $\vec{v}$ is not a unit vector, we normalize it in order to obtain

$$\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \langle 4/5, -3/5\rangle.$$

Therefore

$$D_{\hat{u}}f(3,4) = \vec{\nabla} f(3,4)\cdot\hat{u} = \langle 4,3/2\rangle\cdot\langle 4/5,-3/5\rangle = 16/5 - 9/10 = 23/10.$$

It is straightforward to extend the notion of directional derivatives and gradients to functions of three variables.

Definition 11.6.3. The **directional derivative** of f at (x_0,y_0,z_0) in the direction of a unit vector $\hat{u} = \langle a,b,c\rangle$ is

$$D_{\hat{u}}f(x_0,y_0,z_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h},$$

provided that the limit exists.

Using vector notation, we can compactly express the limits in Definitions 11.6.1 and 11.6.3 as

$$D_{\hat{u}}f(\vec{x}_0) = \lim_{h\to 0}\frac{f(\vec{x}_0 + h\hat{u}) - f(\vec{x}_0)}{h}.$$

Definition 11.6.4. If f is a function of three variables x, y, and z, then the **gradient** of f is the vector function $\vec{\nabla} f$ defined by

$$\vec{\nabla} f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial y}\hat{k}.$$

Just as with functions of two variables, we can compute the directional derivative as

$$D_{\hat{u}}f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c = \vec{\nabla} f(x,y,z)\cdot\hat{u}.$$

Example 11.6.3. If $f(x,y,z) = (x+2y+3z)^{3/2}$, find the gradient of f and the directional derivative of f at $(1,1,2)$ in the direction of $\vec{v} = \langle 0,2,-1\rangle$.

Solution. The gradient of f is

$$\vec{\nabla} f(x,y,z) = \frac{3}{2}\langle (x+2y+3z)^{1/2}, 2(x+2y+3z)^{1/2}, 3(x+2y+3z)^{1/2}\rangle$$

At $(1,1,2)$, we have $\vec{\nabla} f(1,1,2) = \langle 9/2, 18/2, 27/2\rangle$. The unit vector in the direction of $\vec{v}$ is

$$\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \langle 0, 2/\sqrt{5}, -1/\sqrt{5}\rangle.$$

Therefore

$$D_{\hat{u}} f(1,1,2) = \vec{\nabla} f(1,1,2)\cdot \hat{u} = \langle 9/2, 18/2, 27/2\rangle \cdot \langle 0, 2/\sqrt{5}, -1/\sqrt{5}\rangle = 0 + 18/\sqrt{5} - 27/2\sqrt{5} = 9/2\sqrt{5}.$$

Theorem 11.6.2. *Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\hat{u}} f(\vec{x})$ is $|\vec{\nabla} f(\vec{x})|$ and it occurs when $\hat{u}$ has the same direction as the gradient vector $\vec{\nabla} f(\vec{x})$.*

Example 11.6.4. In what direction does $f(x,y) = \sin(xy)$ have the maximum rate of change at the point $(1,0)$? What is this maximum rate of change?

Solution. The gradient of f at $(1,0)$ is

$$\vec{\nabla} f(x,y) = \langle y\cos(xy), x\cos(xy)\rangle \implies \vec{\nabla} f(1,0) = \langle 0, 1\rangle.$$

Therefore, the maximum rate of change of f at the point $(1,0)$ is

$$|\vec{\nabla} f(1,0)| = \sqrt{0^2+1^2} = 1$$

and it occurs in the direction given by the vector $\langle 0,1\rangle$.

Suppose that S is a level surface of the function $f(x,y,z)$; that is, S is defined by the equation $f(x,y,z) = k$ for some constant k. Let $P = (x_0,y_0,z_0)$ be a point on S and let C be any curve that lies on S and passes through P. If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is the continuous vector function that describes C, then any point $(x(t), y(t), z(t))$ on C must satisfy

$$f(x(t), y(t), z(t)) = k.$$

Assuming that x, y, z, are differentiable functions of t, and f is a differentiable function of x, y, z, it follows by the Chain Rule that

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0.$$

Using the fact that $\vec{\nabla} f = \langle f_x, f_y, f_z\rangle$ and $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t)\rangle$, it is equivalent to write

$$\vec{\nabla} f \cdot \vec{r}\,'(t) = 0.$$

In particular, if t_0 is the parameter value corresponding to the point P, then we have

$$\vec{\nabla} f(x_0,y_0,z_0)\cdot \vec{r}\,'(t_0) = 0.$$

This shows that the gradient vector at P is perpendicular to the tangent vector to any curve C on S that passes through P. If $\vec{\nabla} f(x_0,y_0,z_0) \neq \vec{0}$, the **tangent plane** to the level surface $f(x,y,z) = k$ at the point $P = (x_0,y_0,z_0)$ is the plane that passes through P and has normal vector $\vec{\nabla} f(x_0,y_0,z_0)$. We can write the scalar equation of this plane as

$$f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0.$$

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. Since the gradient vector is normal to the tangent plane, the symmetric equations of the normal line are

$$\frac{x-x_0}{f_x(x_0,y_0,z_0)} = \frac{y-y_0}{f_y(x_0,y_0,z_0)} = \frac{z-z_0}{f_z(x_0,y_0,z_0)}.$$

Example 11.6.5. Find the equations of the tangent plane and the normal line to the surface $x^2 - 2y^2 + z^2 + yz = 2$ at the point $(2,1,-1)$.

Solution. Since $x^2 - 2y^2 + z^2 + yz = 2$ is the level surface (with $k = 2$) of the function $f(x, y, z) = x^2 - 2y^2 + z^2 + yz$, we have

$$\begin{aligned} f_x(x,y,z) &= 2x & &\implies & f_x(2,1,-1) &= 4, \\ f_y(x,y,z) &= -4y+z & &\implies & f_y(2,1,-1) &= -5, \\ f_z(x,y,z) &= 2z+y & &\implies & f_z(2,1,-1) &= -1. \end{aligned}$$

Therefore, the scalar equation of the tangent plane at $(2, 1, -1)$ is

$$4(x-2) - 5(y-1) - 1(z+1) = 0$$

and the symmetric equations of the normal line are

$$\frac{x-2}{4} = \frac{y-1}{-5} = \frac{z+1}{-1}.$$

11.7. **Maximum and Minimum Values.**

Definition 11.7.1. Let f be a function of two variables, x and y. If $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b), we say that f has a **local maximum** at (a, b). Similarly, if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b).

If the inequalities in the above definition hold for all points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b). Collectively, maximum and minimum values are often called **extreme values** or simply **extrema**. As in the univariate case, the word **relative** may be used interchangeably with **local** and the word **global** may be used interchangeably with **absolute** when discussing extrema.

Theorem 11.7.1 (Fermat's Theorem for Bivariate Functions)**.** *If f has a local extremum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.*

If $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or either these partial derivative does not exist, then (a, b) is called a **critical point** (or **stationary point**) of f. Hence, Fermat's Theorem asserts that if f has a local extremum at (a, b), then (a, b) is a critical point of f. As in the univariate case, the converse is false; that is, at a critical point, a function could have a local maximum, a local minimum, or neither.

Example 11.7.1. Find and classify the extrema of the function $f(x, y) = -x^2 + 4x - y^2 + 6y - 12$.

Solution. The partial derivatives of f are $f_x(x, y) = -2x + 4$ and $f_y(x, y) = -2y + 6$. Equating them to zero yields the critical point $(2, 3)$. By completing the square, we may rewrite f as

$$f(x,y) = -x^2+4x-y^2+6y-12 = -(x^2-4x+4)-(y^2-6y+9)-(12-4-9) = -(x-2)^2-(y-3)^2+1.$$

Since $-(x - 2)^2 \leq 0$ and $-(y - 3)^2 \leq 0$, we have $f(x, y) \leq 1$ for all values of x and y. Therefore, $f(1, 3) = 1$ is a local maximum (in fact, it is an absolute minimum) of f. This can be verified geometrically from the graph of f, which is a downward-opening elliptic paraboloid with vertex $(2, 3, 1)$.

Example 11.7.2. Find and classify the extrema of the function $f(x, y) = x^2/2 - y^2/2$.

Solution. Since $f_x(x, y) = x$ and $f_y(x, y) = -y$, the only critical point is $(0, 0)$. If $y = 0$ and $x \neq 0$, we have $f(x, y) = x^2/2 > 0$. On the other hand, if $x = 0$ and $y \neq 0$, we have $f(x, y) = -y^2/2 < 0$. It follows that every disk centered at $(0, 0)$ contains points where f is positive and points where f is negative. Hence, $f(0, 0) = 0$ is not an extreme value, so we conclude that f does not have any extrema.

Proposition 11.7.1 (Second Derivatives Test)**.** *Suppose that (a, b) is a critical point of f and that the second partial derivatives of f are continuous on a disk centered at (a, b). Let D be the determinant of the* ***Hessian matrix****, defined by*

$$D = D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

(a) *If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.*
(b) *If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.*
(c) *If $D < 0$, then $f(a,b)$ is a* ***saddle point****, which is neither a local a maximum nor a local minimum.*
(d) *If $D = 0$, the test is inconclusive, so $f(a,b)$ could be a local maximum, a local minimum, or a saddle.*

Example 11.7.3. Find and classify the extrema of the function $f(x,y) = x^4 + y^4 - 4xy + 2$.

Solution. Equating the partial derivatives $f_x(x,y) = 4x^3 - 4y$ and $f_y(x,y) = 4y^3 - 4x$ to zero yields a system of equations with the solutions $(0,0)$, $(-1,-1)$, and $(1,1)$. The second partial derivatives are

$$\begin{aligned} f_{xx}(x,y) &= 12x^2, & f_{xy}(x,y) &= -4, \\ f_{yx}(x,y) &= -4, & f_{yy}(x,y) &= 12y^2, \end{aligned}$$

so we compute the determinant of the Hessian matrix of f to be

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (12x^2)(12y^2) - (-4)^2 = 144x^2y^2 - 16.$$

Since $D(0,0) = -16 < 0$, we conclude that the origin is a saddle point of f. On the other hand, $D(-1,-1) = D(1,1) = 128 > 0$ and $f_{xx}(-1,-1) = f_{xx}(1,1) = 12 > 0$, so the $f(-1,-1) = 0$ and $f(1,1) = 0$ are both local minima of f.

For a univariate function f, the Extreme Value Theorem states that if f is continuous on a closed interval $[a,b]$, then it has both an absolute maximum and an absolute minimum on $[a,b]$. Using the Closed Interval Method, we found these absolute extrema by evaluating f not only at its critical numbers but also at the endpoints a and b. We now introduce similar results for bivariate functions.

Theorem 11.7.2 (Extreme Value Theorem for Functions of Two Variables)**.** *If f is continuous on a closed, bounded set $S \in \mathbb{R}^2$, then f attains an absolute maximum value $M = f(x_1, y_1)$ and an absolute minimum value $m = f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.*

Proposition 11.7.2. *To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set S:*

1. *Find the values of f at the critical points of f in S.*
2. *Find the extreme values of f on the boundary of S.*
3. *The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.*

Example 11.7.4. Find the absolute extrema of the function $f(x,y) = x^2 + y^2 + x^2y + 4$ on the closed, bounded set $S = \{(x,y) : |x| \le 1, |y| \le 1\}$.

Solution. Since f is a polynomial, it is continuous on S, so by Theorem 11.7.2 it attains both an absolute maximum and an absolute minimum. Equating the partial derivatives of f to zero yields the system of equations

$$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases} \implies \begin{cases} 2x + 2xy = 0 \\ 2y + x^2 = 0 \end{cases} \implies \begin{cases} x(1+y) = 0 \\ y = -x^2/2. \end{cases}$$

Substituting the second equation into the first, we find that

$$x(1 - x^2/2) = 0 \implies x = 0 \text{ or } x = \pm\sqrt{2}$$

and it follows that $(0,0)$, $(-\sqrt{2},-1)$ and $(\sqrt{2},-1)$ are the critical points of f. At these points, we have the function values $f(0,0) = 4$, $f(-\sqrt{2},-1) = 5$, and $f(\sqrt{2},-1) = 5$. Next, we look at the values of f on the boundary of the rectangle S, which consists of the four line segments

$$\begin{aligned} L_1 &= \{y = 1 : -1 \le x \le 1\}, \\ L_2 &= \{y = -1 : -1 \le x \le 1\}, \\ L_3 &= \{x = 1 : -1 \le y \le 1\}, \\ L_4 &= \{x = -1 : -1 \le y \le 1\}. \end{aligned}$$

On L_1, we set $y = 1$, so $f(x, y) = 2x^2 + 5$, which has a minimum value of $f(0, 1) = 5$ and a maximum value of $f(-1, 1) = f(1, 1) = 7$ on the interval $-1 \leq x \leq 1$.

On L_2, we set $y = -1$, so $f(x, y) = 5$ is constant on the interval $-1 \leq x \leq 1$.

On L_3, we set $x = 1$, so $f(x, y) = y^2 + y + 5$, which has a minimum value of $f(1, -1/2) = 19/4$ and a maximum value of $f(1, 1) = 7$ on the interval $-1 \leq y \leq 1$.

On L_4, we set $x = -1$, so $f(x, y) = y^2 + y + 5$, which has a minimum value of $f(-1, -1/2) = 19/4$ and a maximum value of $f(-1, 1) = 7$ on the interval $-1 \leq y \leq 1$.

Combining all of this information, we conclude that f has the absolute maximum value $f(-1, 1) = f(1, 1) = 7$ and the absolute minimum value $f(0, 0) = 4$ on the set S.

11.8. **Lagrange Multipliers.**

Consider the problem of finding the extrema of a bivariate function $f(x, y)$ that is subject to a constraint of the form $g(x, y) = k$. This is accomplished by identifying the largest and smallest values of c such that the level curve $f(x, y) = c$ intersects the curve $g(x, y) = k$. These extreme values of c occur at points where the curves $f(x, y) = c$ and $g(x, y) = k$ just touch each other; that is, where they have a common tangent line. At such a point, say (x_0, y_0), the normal lines to the curves $f(x, y) = c$ and $g(x, y) = k$ are identical. In other words, the gradient vectors $\vec{\nabla} f(x_0, y_0)$ and $\vec{\nabla} g(x_0, y_0)$ are parallel, so there exists a scalar λ such that $\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$.

Therefore, to find the extrema of f subject to the constraint $g(x, y) = k$, it suffices compare the values $f(x, y)$ at all points (x, y) such that $\vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y)$ for some scalar λ. This procedure, known as the method of Lagrange multipliers, is generalized to functions of three variables and stated precisely as follows.

Proposition 11.8.1. *To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, assuming that these extreme values exist and $\vec{\nabla} g \neq \vec{0}$ on the surface $g(x, y, z) = k$:*

(a) Find all values of x, y, z, and λ such that $\vec{\nabla} f(x, y, z) = \lambda \vec{\nabla} g(x, y, z)$ and $g(x, y, z) = k$.
(b) Evaluate f at all the points (x, y, z) that result form step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Example 11.8.1. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 y$, subject to the constraint $x^2 + 2y^2 = 6$.

Solution. Let $g(x, y) = x^2 + 2y^2$. The equations $\vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y)$ and $g(x, y) = 6$ may be rewritten as the system

$$\left\{ \begin{array}{rcl} f_x(x,y) &=& \lambda g_x(x,y) \\ f_y(x,y) &=& \lambda g_y(x,y) \\ g(x,y) &=& 6 \end{array} \right. \qquad \Longrightarrow \qquad \left\{ \begin{array}{rcl} 2xy &=& 2x\lambda \\ x^2 &=& 4y\lambda \\ x^2 + 2y^2 &=& 6. \end{array} \right.$$

From the first equation, we see that $x = 0$ or $y = \lambda$. If $x = 0$, then the third equation gives $y = \pm\sqrt{3}$. If $y = \lambda$, then the second equation gives $x^2 = 4y^2$, so by the third equation we have $y = \pm 1$. In total, we have found six solutions to the system: $(0, \pm\sqrt{3})$ and $(\pm 2, \pm 1)$. Since

$$f(0, \pm\sqrt{3}) = 0,$$
$$f(\pm 2, 1) = 4,$$
$$f(\pm 2, -1) = -4,$$

we conclude that the maximum of the function is $f(\pm 2, 1) = 4$ and the minimum is $f(\pm 2, -1) = -4$.

Example 11.8.2. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

Solution. One way to solve this problem is by using the Second Derivatives Test, but instead we will apply the method of Lagrange multipliers. The distance from any point $(x, y, z) \in \mathbb{R}^3$ to the point

$(4, 2, 0)$ is given by

$$d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}.$$

However, for simplicity, we will define the function f to be the square of the distance:

$$f(x, y, z) = d^2 = (x-4)^2 + (y-2)^2 + z^2.$$

The constraint is that the point (x, y, z) satisfies the equation $z^2 = x^2 + y^2$; that is,

$$g(x, y, z) = z^2 - x^2 - y^2 = 0.$$

The equations $\vec{\nabla} f(x, y, z) = \lambda \vec{\nabla} g(x, y, z)$ and $g(x, y, z) = 0$ may be rewritten as

$$\left\{\begin{array}{rcl} f_x(x,y,z) &=& \lambda g_x(x,y,z) \\ f_y(x,y,z) &=& \lambda g_y(x,y,z) \\ f_z(x,y,z) &=& \lambda g_z(x,y,z) \\ g(x,y,z) &=& 0 \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{rcl} 2(x-4) &=& -2x\lambda \\ 2(y-2) &=& -2y\lambda \\ 2z &=& 2z\lambda \\ z^2 - x^2 - y^2 &=& 0. \end{array}\right.$$

The first two equations give

$$2(x-4) = -2x\lambda \implies x = \frac{4}{1+\lambda},$$

$$2(y-2) = -2y\lambda \implies y = \frac{2}{1+\lambda},$$

while, for $\lambda \neq 1$, the third equation gives

$$2z = 2z\lambda \implies z = 0.$$

Substituting these values into the fourth equation, we have

$$z^2 - x^2 - y^2 = 0 \implies -\left(\frac{4}{1+\lambda}\right)^2 - \left(\frac{2}{1+\lambda}\right)^2 = 0 \implies \frac{20}{(1+\lambda)^2} = 0,$$

which has no solution. Therefore, we conclude that $\lambda = 1$, which corresponds to $x = 2$ and $y = 1$. It follows that the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$ are $(2, 1, \pm\sqrt{5})$.

Suppose that we wish to maximize and/or minimize a function $f(x, y, z)$ that is subject to two constraints of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. In this case, the method of Lagrange multipliers involves finding numbers x, y, z, λ, and μ that simultaneously satisfy the vector equation

$$\vec{\nabla} f(x, y, z) = \lambda \vec{\nabla} g(x, y, z) + \mu \vec{\nabla} h(x, y, z)$$

and the scalar equations $g(x, y, z) = k$ and $h(x, y, z) = c$.

Example 11.8.3. Find the maximum and minimum values of the function $f(x, y, z) = x + 2y$ subject to the constraints $x + y + z = 1$ and $y^2 + z^2 = 4$.

Solution. Let $g(x, y, z) = x + y + z$ and $h(x, y, z) = y^2 + z^2$. The Lagrange conditions may be written as

$$\left\{\begin{array}{rcl} f_x(x,y,z) &=& \lambda g_x(x,y,z) + \mu h_x(x,y,z) \\ f_y(x,y,z) &=& \lambda g_y(x,y,z) + \mu h_y(x,y,z) \\ f_z(x,y,z) &=& \lambda g_z(x,y,z) + \mu h_z(x,y,z) \\ g(x,y,z) &=& 1 \\ h(x,y,z) &=& 4 \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{rcl} 1 &=& \lambda \\ 2 &=& \lambda + 2y\mu \\ 0 &=& \lambda + 2z\mu \\ x + y + z &=& 1 \\ y^2 + z^2 &=& 4. \end{array}\right.$$

Substituting $\lambda = 1$ into the second and third equations, we find that $y = 1/(2\mu)$ and $z = -1/(2\mu)$. Substituting these values into the fifth equation yields $\mu = \pm\sqrt{1/8}$. Therefore, $y = \pm\sqrt{2}$, $z = \pm\sqrt{2}$, and it follows from the fourth equation that the solutions to the system are $(1, \sqrt{2}, -\sqrt{2})$, $(1, -\sqrt{2}, \sqrt{2})$, $(1-2\sqrt{2}, \sqrt{2}, \sqrt{2})$, and $(1+2\sqrt{2}, -\sqrt{2}, -\sqrt{2})$. Checking the values of f at these four points, we conclude that the maximum is $f(1, \sqrt{2}, -\sqrt{2}) = 1 + 2\sqrt{2}$ and the minimum is $f(1, -\sqrt{2}, \sqrt{2}) = 1 - 2\sqrt{2}$.

12. Multiple Integrals.

12.1. **Double Integrals over Rectangles.**

Recall that in Sections 5.1 and 5.2, we defined the definite integral of f from a to b as a limit of Riemann sums and interpreted it as the area under the curve $y = f(x)$ between $x = a$ and $x = b$. We now introduce the double integral of a bivariate function f as a way of calculating volume under the surface $z = f(x, y)$.

Suppose that f is a bivariate function defined on the closed rectangle
$$R = [a, b] \times [c, d] = \left\{(x, y) \in \mathbb{R}^2 : a \le x \le b,\ c \le y \le d\right\}.$$
The graph of f is a surface with the equation $z = f(x, y)$, and for now we assume that $f(x, y) \ge 0$. Let S be the solid that lies above R and under the graph of f; that is,
$$S = \left\{(x, y, z) \in \mathbb{R}^3 : 0 \le z \le f(x, y),\ (x, y) \in R\right\}.$$
The first step in determining the volume of S is to divide R into subrectangles. We divide $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and divide $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$. These subintervals collectively form a grid of subrectangles
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) : x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\},$$
each with an area of $\Delta A = \Delta x \Delta y$. If we choose a sample point (x^*_{ij}, y^*_{ij}) in each R_{ij}, we can approximate the part of S that lies above R_{ij} by a rectangular box with volume $f(x^*_{ij}, y^*_{ij})\Delta A$. Summing the volumes of these boxes over all R_{ij}, we obtain an approximation for the total volume of S:
$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A.$$
Intuitively, as m and n become larger and larger, the approximation of V by this so-called **double Riemann sum** becomes better and better. Therefore, to compute the exact volume of S, we simply take the limit as $n, m \to \infty$. This discussion motivates the following definition.

Definition 12.1.1. The **double integral** of f over the rectangle R is
$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A,$$
provided that the limit exists. If $f(x, y) \ge 0$, then $\iint_R f(x, y)\, dA$ represents the volume of the solid that lies above R and below the surface $z = f(x, y)$.

The Midpoint Rule for single integrals has a double integral counterpart.

Theorem 12.1.1 (Midpoint Rule). *If f is integrable over the rectangle R, then*
$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A,$$
where $\bar{x}_i$ and $\bar{y}_j$ are the midpoints of $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$, respectively.

Example 12.1.1. Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R \sin(x + y)\, dA$, where $R = [0, \pi] \times [0, \pi]$. First, take the sample points to be the lower left corners. Then use the Midpoint Rule.

Solution. With $m = n = 2$, the four subrectangles of R are $[0, \pi/2] \times [0, \pi/2]$, $[0, \pi/2] \times [\pi/2, \pi]$, $[\pi/2, \pi] \times [0, \pi/2]$, and $[\pi/2, \pi] \times [\pi/2, \pi]$. The area of each subrectangle is $\Delta A = (\pi/2)(\pi/2) = \pi^2/4$.

Taking the sample points to be the lower left corners, we find that

$$\begin{aligned}\iint\limits_R \sin(x+y)\,dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= \frac{\pi^2}{4}\left[\sin(0+0) + \sin\left(0+\frac{\pi}{2}\right) + \left(\frac{\pi}{2}+0\right) + \left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right] \\ &= \frac{\pi^2}{2} \\ &\approx 4.9348.\end{aligned}$$

On the other hand, taking the sample points to be the midpoints of each subrectangle, we have

$$\begin{aligned}\iint\limits_R \sin(x+y)\,dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_{ij}, \bar{y}_{ij})\Delta A \\ &= \frac{\pi^2}{4}\left[\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}+\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{4}+\frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{4}+\frac{3\pi}{4}\right)\right] \\ &= 0.\end{aligned}$$

In Section 12.2, we will see that $\iint_R \sin(x+y)\,dA = 0$, so in this case the approximation given by the Midpoint Rule turns out to be exact.

Recall from Section 6.5 that the average value of a univariate function f on an interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In a similar manner, we define the average value of a bivariate function f on a rectangle R to be

$$f_{\text{ave}} = \frac{1}{A(R)}\iint\limits_R f(x,y)\,dA,$$

where $A(R)$ is the area of R.

To conclude this section, we list a few basic properties of double integrals.

Proposition 12.1.1. *If f and g are integrable functions and c is a constant, then:*

1. $\displaystyle\iint\limits_R [f(x,y) \pm g(x,y)]\,dA = \iint\limits_R f(x,y)\,dA \pm \iint\limits_R g(x,y)\,dA.$
2. $\displaystyle\iint\limits_R cf(x,y)\,dA = c\iint\limits_R f(x,y)\,dA.$
3. *If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$, then* $\displaystyle\iint\limits_R f(x,y)\,dA \geq \iint\limits_R g(x,y)\,dA.$

12.2. **Iterated Integrals.**

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$. The notation $\int_c^d f(x,y)\,dy$ implies that x is held fixed while $f(x,y)$ is integrated with respect to y over the interval $[c, d]$. This procedure is called partial integration with respect to y. Letting $A(x) = \int_c^d f(x,y)\,dy$, and integrating the function A with respect to x over the interval $[a, b]$, we have

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx.$$

The integral on the right hand side of the above equation is called an **iterated integral**.

Theorem 12.2.1 (Fubini's Theorem)**.** *If f is continuous on the rectangle $R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$, then*

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 12.2.1. Show that $\iint_R \sin(x+y)\,dA = 0$, where $R = [0,\pi] \times [0,\pi]$.

Solution. Since $f(x,y) = \sin(x+y)$ is continuous on $\mathbb{R}^2$, it is certainly continuous on R. By Fubini's Theorem, it follows that

$$\begin{aligned}\iint\limits_R \sin(x+y)\,dA &= \int_0^\pi \int_0^\pi \sin(x+y)\,dy\,dx \\ &= \int_0^\pi \left[-\cos(x+y)\Big|_{y=0}^{y=\pi}\right] dx \\ &= \int_0^\pi \left[-\cos(x+\pi) - \cos(x)\right] dx \\ &= \left[-\sin(x+\pi) - \sin(x)\right]\Big|_{x=0}^{x=\pi} \\ &= \left[-\sin(2\pi) - \sin(\pi)\right] - \left[-\sin(\pi) - \sin(0)\right] \\ &= 0.\end{aligned}$$

Example 12.2.2. Compute $\iint_R xe^{xy}\,dA$, where $R = [0,1/2] \times [0,2]$.

Solution. If we integrate with respect to y first and then with respect to x, we easily find that

$$\begin{aligned}\iint\limits_R xe^{xy}\,dA &= \int_0^{1/2} \int_0^2 xe^{xy}\,dy\,dx \\ &= \int_0^{1/2} \left[e^{xy}\Big|_{y=0}^{y=2}\right] dx \\ &= \int_0^{1/2} \left[e^{2x} - 1\right] dx \\ &= \left[\frac{1}{2}e^{2x} - x\right]\Bigg|_{x=0}^{x=1/2} \\ &= \left[\frac{1}{2}e - \frac{1}{2}\right] - \left[\frac{1}{2} - 0\right] \\ &= \frac{e}{2} - 1.\end{aligned}$$

On the other hand, integrating with respect to x first and then with respect to y, we encounter a problem that is difficult to solve by hand. Indeed, with $u = x$ and $dv = e^{xy}$, integration by parts yields

$$\begin{aligned}\iint\limits_R xe^{xy}\,dA &= \int_0^2 \int_0^{1/2} xe^{xy}\,dx\,dy \\ &= \int_0^2 \left[\frac{x}{y}e^{xy}\Big|_{x=0}^{x=1/2} - \int_0^{1/2} \frac{1}{y}e^{xy}\,dx\right] dy \\ &= \int_0^2 \left[\frac{1}{2y}e^{y/2} - \left(\frac{1}{y^2}e^{xy}\Big|_{x=0}^{x=1/2}\right)\right] dy \\ &= \int_0^2 \left[\frac{1}{2y}e^{y/2} - \frac{1}{y^2}e^{y/2} + \frac{1}{y^2}\right] dy \qquad = \cdots \qquad = \frac{e}{2} - 1.\end{aligned}$$

Therefore, we see that the order of integration can make a difference, not in the result but rather in the amount of effort that may be required to obtain it.

In the special case where $f(x,y)$ can be factored as the product of a function of x and a function of y, the double integral of f can be written in a simple form. In particular, if $f(x,y) = g(x)h(y)$ and $R = [a,b]\times[c,d]$, then

$$\iint\limits_R f(x,y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy.$$

Example 12.2.3. If $R = [0,2] \times [0,\pi/2]$, then
$$\iint\limits_R x \sin y \, dA = \int_0^2 x\,dx \int_0^{\pi/2} \sin y \, dy = \left(\frac{x^2}{2}\Big|_0^2 \right) \left(-\cos y \Big|_0^{\pi/2} \right) = (2-0)\,(0-(-1)) = 2.$$

12.3. **Double Integrals over General Regions.**

For single integrals, the region of integration is always an interval. However, for double integrals we may wish to integrate not only over rectangles but also over more general regions. Suppose that D is a bounded region, meaning that it can be enclosed within a rectangular region R, and define a new function F with domain R by
$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D; \\ 0, & \text{if } (x,y) \in R \backslash D. \end{cases}$$
Then, if F is integrable over R, we define the **double integral of f over D** by
$$\iint\limits_D f(x,y)\,dA = \iint\limits_R F(x,y)\,dA.$$

Proposition 12.3.1. *If f is continuous on a **type I** region D such that*
$$D = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\},$$
then
$$\iint\limits_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

Example 12.3.1. Evaluate $\iint_D (1+2y)\,dA$, where $D = \{(x,y) : 0 \le x \le 1, x^2 \le y \le x\}$.

Solution. Since D is defined as a type I region, we find that
$$\begin{aligned} \iint\limits_D (1+2y)\,dA &= \int_0^1 \int_{x^2}^{x} (1+2y)\,dy\,dx \\ &= \int_0^1 \left[\left(y+y^2\right) \Big|_{y=x^2}^{y=x} \right] dx \\ &= \int_0^1 \left[\left(x+x^2\right) - \left(x^2+x^4\right) \right] dx \\ &= \int_0^1 \left[x - x^4 \right] dx \\ &= \left[\frac{x^2}{2} - \frac{x^5}{5} \right] \Bigg|_{x=0}^{x=1} \\ &= \frac{1}{2} - \frac{1}{5} \\ &= \frac{3}{10}. \end{aligned}$$

Proposition 12.3.2. *If f is continuous on a **type II** region D such that*
$$D = \{(x,y) : c \le y \le d, h_1(2) \le x \le h_2(y)\},$$
then
$$\iint\limits_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Example 12.3.2. Evaluate $\iint_D y^2\,dA$, where $D = \{(x,y) : -1 \le y \le 1, -y-2 \le x \le y\}$.

Solution. Since D is defined as a type II region, we find that

$$\begin{aligned}\iint_D y^2\,dA &= \int_{-1}^{1}\int_{-y-2}^{y} y^2\,dx\,dy \\ &= \int_{-1}^{1}\left[xy^2\Big|_{x=-y-2}^{x=y}\right]dy \\ &= \int_{-1}^{1}\left[y^3-(-y-2)y^2\right]dy \\ &= \int_{-1}^{1}\left[2y^3+2y^2\right]dy \\ &= \left[\frac{2y^4}{4}+\frac{2y^3}{3}\right]\Bigg|_{y=-1}^{y=1} \\ &= \left[\frac{(1)^4}{2}+\frac{2(1)^3}{3}\right]-\left[\frac{(-1)^4}{2}+\frac{2(-1)^3}{3}\right] \\ &= \frac{4}{3}.\end{aligned}$$

In some cases, it is desirable to change a region from type I to type II, or vice versa, as illustrated by the following example.

Example 12.3.3. Evaluate the iterated integral $\int_0^1\int_y^1 e^{x^2}\,dx\,dy$.

Solution. As it stands, this integral cannot be evaluated exactly by hand, since there is no closed-form antiderivative for the function e^{y^2}. However, if we rewrite the region $D=\{(x,y):0\le y\le 1, y\le x\le 1\}$ as $D=\{(x,y):0\le x\le 1, 0\le y\le x\}$, we easily obtain

$$\begin{aligned}\int_0^1\int_y^1 e^{x^2}\,dx\,dy &= \int_0^1\int_0^x e^{x^2}\,dy\,dx \\ &= \int_0^1\left[ye^{x^2}\Big|_{y=0}^{y=x}\right]dy \\ &= \int_0^1 xe^{x^2}\,dy \\ &= \frac{1}{2}e^{x^2}\Big|_{x=0}^{x=1} \\ &= \frac{1}{2}(e-1) \\ &\approx 0.8591.\end{aligned}$$

Not only does Proposition 12.1.1 hold for a general region D, but we also have several other properties.

Proposition 12.3.3. *If f and g are integrable functions, c is a constant, and $A(D)$ represents the area of the region D, then:*

1. $\iint_D 1\,dA = A(D)$.
2. *If $D=D_1\cup D_2$, where D_1 and D_2 do not overlap, except possibly on their boundaries, then*
$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$
3. *If $m\le f(x,y)\le M$ for all $(x,y)\in D$, then $mA(D)\le \iint_D f(x,y)\,dA \le MA(D)$.*

12.4. **Double Integrals in Polar Coordinates.**

Proposition 12.4.1. *If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then*

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Example 12.4.1. Evaluate $\iint_R \sqrt{1-x^2-y^2}\,dA$, where $R = \big\{(x,y) : x^2+y^2 \le 1, x \le 0\big\}$.

Solution. The region of integration can be expressed as $R = \{(r,\theta) : 0 \le r \le 1, \pi/2 \le \theta \le 3\pi/2\}$, which allows us to rewrite the integral in polar form as

$$\iint_R \sqrt{1-x^2-y^2}\,dA = \int_{\pi/2}^{3\pi/2} \int_0^1 \sqrt{1-r^2}\;r\,dr\,d\theta.$$

Hence, using the Substitution Rule and the fact that the integrand does not depend on θ, we have

$$\begin{aligned}\iint_R \sqrt{1-x^2-y^2}\,dA &= \int_{\pi/2}^{3\pi/2} 1\,d\theta \cdot \int_0^1 \sqrt{1-r^2}\;r\,dr \\ &= \left(\frac{3\pi}{2} - \frac{\pi}{2}\right) \int_0^1 \frac{\sqrt{u}}{2}\,du \\ &= \frac{\pi}{2} \int_0^1 u^{1/2}\,du \\ &= \frac{\pi}{2}\left(\frac{2}{3}u^{3/2}\Big|_0^1\right) \\ &= \frac{\pi}{3}.\end{aligned}$$

As in the Cartesian case, double integration in polar coordinates can be extended to more general regions.

Proposition 12.4.2. *If f is continuous on a polar rectangle of the form*

$$D = \{(r,\theta) : \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\},$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Polar coordinates can also be used to evaluate certain types of single integrals, as illustrated by the following example.

Example 12.4.2. Show that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1$.

Solution. The integrand $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is of particular importance in the study of probability, as it represents the *probability density function* of the standard normal distribution. The graph of this function is classically referred to as the "standard bell curve." We wish to prove the well-known fact that the area under this curve is equal to one, but unfortunately the function f does not have a closed-form antiderivative. Therefore, we introduce a clever integration technique that involves polar coordinates. Let

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx,$$

and notice that we may also write

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy.$$

It follows that

$$I^2 = I \cdot I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy.$$

Combining this product into an iterated double integral, converting to polar coordinates, using the fact that the new integrand does not depend on θ, and applying the Substitution Rule, we find that

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-(x^2+y^2)/2}\, dx\, dy \\ &= \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} e^{-r^2/2}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2\pi}\, d\theta \cdot \int_0^{\infty} e^{-r^2/2}\, r\, dr \\ &= \left(\frac{2\pi}{2\pi} - \frac{0}{2\pi}\right)\int_{-\infty}^{0} e^u\, du \\ &= e^u\Big|_{-\infty}^{0} \\ &= 1. \end{aligned}$$

Hence, we conclude that $I = \sqrt{I^2} = 1$, as desired.

12.5. **Applications of Double Integrals.**

Whereas in Section 6.6 we computed the total mass and center of mass of a lamina with constant density, we now consider a lamina with variable density. Suppose the lamina occupies a region D and has continuous density function $\rho(x, y)$. Let R be a rectangle containing D and suppose that $\rho(x, y) = 0$ for $(x, y) \in R\backslash D$. If we divide R into subrectangles R_{ij} and choose sample points $(x_{ij}^*, y_{ij}^*) \in R_{ij}$, then the mass of the part of the lamina that occupies R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\Delta A$, where ΔA is the area of R_{ij}. Adding up all such masses and then letting the number of subrectangles approach infinity, we find the **total mass** of the lamina to be

$$m = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D \rho(x, y)\, dA.$$

Recall Section 6.6 that the moment of a particle about an axis is defined as the product of its mass and its directed distance from the axis. Therefore, the **moment about the x-axis** of the entire lamina is

$$M_x = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^*\rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y\rho(x, y)\, dA,$$

while the **moment about the y-axis** is

$$M_y = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^*\rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x\rho(x, y)\, dA.$$

Since the **center of mass** $(\bar{x}, \bar{y})$ of the lamina is defined so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$, we arrive at the following result.

Proposition 12.5.1. *The coordinates of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are*

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)\, dA \qquad \textit{and} \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)\, dA,$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\, dA.$$

Example 12.5.1. Find the mass and center of mass of the lamina that occupies the region bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$, with density $\rho(x, y) = y$.

Solution. Since the region D occupied by the lamina is described by $0 \le x \le 1$ and $0 \le y \le e^x$, we find the total mass to be

$$m = \iint_D \rho(x, y)\, dA = \int_0^1\int_0^{e^x} y\, dy\, dx = \int_0^1 \left[\frac{y^2}{2}\Bigg|_{y=0}^{y=e^x}\right] dx = \int_0^1 \frac{e^{2x}}{2}\, dx = \frac{e^{2x}}{4}\Bigg|_{x=0}^{x=1} = \frac{e^2 - 1}{4}.$$

Similarly, we compute

$$\begin{aligned}
M_x &= \iint\limits_D y\rho(x,y)\,dA \\
&= \int_0^1 \int_0^{e^x} y^2\,dy\,dx \\
&= \int_0^1 \left[\frac{y^3}{3} \Bigg|_{y=0}^{y=e^x} \right] dx \\
&= \int_0^1 \frac{e^{3x}}{3}\,dx \\
&= \frac{e^{3x}}{9} \Bigg|_{x=0}^{x=1} \\
&= \frac{e^3-1}{9}
\end{aligned}$$

and

$$\begin{aligned}
M_y &= \iint\limits_D x\rho(x,y)\,dA \\
&= \int_0^1 \int_0^{e^x} xy\,dy\,dx \\
&= \int_0^1 \left[\frac{xy^2}{2} \Bigg|_{y=0}^{y=e^x} \right] dx \\
&= \int_0^1 \frac{xe^{2x}}{2}\,dx \\
&= \frac{e^{2x}}{8}(2x-1) \Bigg|_{x=0}^{x=1} \\
&= \frac{e^2+1}{8}.
\end{aligned}$$

Hence, the center of mass of the lamina is

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right) = \left(\frac{e^2+1}{2(e^2-1)}, \frac{4(e^3-1)}{9(e^2-1)} \right).$$

Example 12.5.2. A lamina occupies the part of the disk $x^2+y^2 \le 1$ that lies in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the y-axis.

Solution. Since the distance from a point (x,y) to the origin is simply x, the density of the lamina is given by $\rho(x,y) = Kx$, where K is some constant. Since the region D occupied by the lamina is naturally described in polar coordinates by $0 \le \theta \le \pi/2$ and $0 \le r \le 1$, we find the total mass to be

$$\begin{aligned}
m &= \iint\limits_D \rho(x,y)\,dA \\
&= \int_0^{\pi/2} \int_0^1 (Kr\cos\theta)\; r\,dr\,d\theta \\
&= K \int_0^{\pi/2} \cos\theta\,d\theta \cdot \int_0^1 r^2\,dr \\
&= K \left(\sin\theta \Big|_{\theta=0}^{\theta=\pi/2} \right) \left(\frac{r^3}{3} \Bigg|_{r=0}^{r=1} \right) \\
&= \frac{K}{3}.
\end{aligned}$$

Similarly, we compute

$$\begin{aligned}
M_x &= \iint_D y\rho(x,y)\,dA \\
&= \int_0^{\pi/2}\int_0^1 (r\sin\theta)(Kr\cos\theta)\,r\,dr\,d\theta \\
&= K\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta \cdot \int_0^1 r^3\,dr \\
&= K\int_0^1 u\,du \cdot \int_0^1 r^3\,dr \\
&= K\left(\frac{u^2}{2}\bigg|_{u=0}^{u=1}\right)\left(\frac{r^4}{4}\bigg|_{r=0}^{r=1}\right) \\
&= \frac{K}{8}
\end{aligned}$$

and

$$\begin{aligned}
M_y &= \iint_D x\rho(x,y)\,dA \\
&= \int_0^{\pi/2}\int_0^1 (r\cos\theta)(Kr\cos\theta)\,r\,dr\,d\theta \\
&= K\int_0^{\pi/2} \cos^2\theta\,d\theta \cdot \int_0^1 r^3\,dr \\
&= K\int_0^{\pi/2} \left[\frac{1}{2}+\cos(2\theta)\right]\,d\theta \cdot \int_0^1 r^3\,dr \\
&= K\left(\left[\frac{\theta}{2}+\frac{\sin(2\theta)}{2}\right]\bigg|_{\theta=0}^{\theta=\pi/2}\right)\left(\frac{r^4}{4}\bigg|_{r=0}^{r=1}\right) \\
&= \frac{K\pi}{16}.
\end{aligned}$$

Hence, the center of mass of the lamina is

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{m},\frac{M_x}{m}\right) = \left(\frac{3}{8},\frac{3\pi}{16}\right).$$

The **moment of inertia** (also called the **second moment**) of a particle with mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. If we again consider a lamina occupying the region D with density $\rho(x,y)$, then proceeding as we did for ordinary moments we find that the **moment of inertia about the x-axis** of the entire lamina is

$$I_x = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij}^*)^2\rho(x_{ij}^*,y_{ij}^*)\Delta A = \iint_D y^2\rho(x,y)\,dA,$$

while the **moment of inertia about the y-axis** is

$$I_y = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij}^*)^2\rho(x_{ij}^*,y_{ij}^*)\Delta A = \iint_D x^2\rho(x,y)\,dA.$$

A third moment of inertia, called the **moment of inertia about the origin** or the **polar moment of inertia**, is given by

$$I_0 = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}^*)^2+(y_{ij}^*)^2\right]\rho(x_{ij}^*,y_{ij}^*)\Delta A = \iint_D (x^2+y^2)\rho(x,y)\,dA.$$

Example 12.5.3. Find the moments of inertia I_x, I_y, and I_0 of a disk D with density $\rho(x,y)=\rho$ that has radius a and is centered at the origin.

Solution. In polar coordinates, D is described by $0 \le \theta \le 2\pi$ and $0 \le r \le a$. We begin by computing the moment of inertia about the origin to be

$$I_0 = \iint_D (x^2+y^2)\rho(x,y)\,dA = \int_0^{2\pi}\int_0^a (r^2)\rho\, r\,dr\,d\theta = \int_0^{2\pi}\rho\,d\theta \cdot \int_0^a r^3\,dr = \rho(2\pi-0)\left(\frac{r^4}{4}\bigg|_{r=0}^{r=a}\right) = \frac{\pi\rho a^4}{2}.$$

Instead of computing the other moments of inertia directly, we observe that $I_0 = I_x + I_y$ (always true) and $I_x = I_y$ (by symmetry). Thus, we have

$$I_x = I_y = \frac{I_0}{2} = \frac{\pi\rho a^4}{4}.$$

12.6. **Surface Area.**

Definition 12.6.1. If a smooth parametric surface S is given by the equation

$$\vec{r}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}, \qquad (u,v) \in D$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\,dA,$$

where $\vec{r}_u = x_u\hat{\imath} + y_u\hat{\jmath} + z_u\hat{k}$ and $\vec{r}_v = x_v\hat{\imath} + y_v\hat{\jmath} + z_v\hat{k}$.

Example 12.6.1. Find the surface area of the surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, where $0 \le u \le 1$ and $0 \le v \le 2$.

Solution. Letting $\vec{r}(u,v) = u^2\hat{\imath} + uv\hat{\jmath} + \frac{1}{2}v^2\hat{k}$, we have $\vec{r}_u(u,v) = 2u\hat{\imath} + v\hat{\jmath}$ and $\vec{r}_v(u,v) = u\hat{\jmath} + v\hat{k}$. Therefore

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = v^2\hat{\imath} - 2uv\hat{\jmath} + 2u^2\hat{k},$$

and it follows that

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{(v^2+2u^2)^2} = v^2 + 2u^2.$$

Hence, the surface area is

$$\begin{aligned} A(S) &= \iint_D |\vec{r}_u \times \vec{r}_v|\,dA \\ &= \int_0^1\int_0^2 (v^2 + 2u^2)\,dv\,du \\ &= \int_0^1 \left[\left(\frac{1}{3}v^3 + 2u^2v\right)\bigg|_{v=0}^{v=2}\right] du \\ &= \int_0^1 \left[\frac{8}{3} + 4u^2\right] du \\ &= \left[\frac{8}{3}u + \frac{4}{3}u^3\right]\bigg|_{u=0}^{u=1} \\ &= \left[\frac{8}{3} + \frac{4}{3}\right] \\ &= 4. \end{aligned}$$

For the special case of a surface S with equation $z = f(x,y)$, where $(x,y) \in D$ and f has continuous partial derivatives, we take the parametric equations to be $x = x$, $y = y$, and $z = f(x,y)$. It follows that

$$\vec{r}_x = \hat{\imath} + f_x\hat{k}, \quad \vec{r}_x = \hat{\imath} + f_x\hat{k} \quad \Longrightarrow \quad \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\hat{\imath} - f_y\hat{\jmath} + \hat{k},$$

and hence the surface area formula becomes

$$A(S) = \iint\limits_D \sqrt{1 + f_x^2 + f_y^2}\, dA.$$

Example 12.6.2. Find the surface area of the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$.

Solution. Letting $f(x, y) = \sqrt{1 - x^2 - y^2}$, we have $f_x(x, y) = \frac{x}{\sqrt{1-x^2-y^2}}$ and $f_y(x, y) = \frac{y}{\sqrt{1-x^2-y^2}}$. Therefore, the integrand of the surface area formula is

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \left(\frac{x}{\sqrt{1 - x^2 - y^2}}\right)^2 + \left(\frac{y}{\sqrt{1 - x^2 - y^2}}\right)^2} = \sqrt{1 + \frac{x^2 + y^2}{1 - x^2 - y^2}}.$$

Since this expression is naturally represented in polar coordinates, we find that

$$\begin{aligned} A(S) &= \iint\limits_D \sqrt{1 + f_x^2 + f_y^2}\, dA \\ &= \int_0^{2\pi} \int_0^1 \sqrt{1 + \frac{r^2}{1 - r^2}}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} 1\, d\theta \cdot \int_0^1 \frac{r}{\sqrt{1 - r^2}}\, dr \\ &= (2\pi - 0) \int_0^1 \frac{1}{2\sqrt{u}}\, du \\ &= \pi \int_0^1 u^{-1/2}\, du \\ &= \pi \left(2u^{1/2} \Big|_0^1 \right) \\ &= 2\pi. \end{aligned}$$

12.7. **Triple Integrals.**

Just as defined single integrals for univariate functions and double integrals for bivariate functions, we can now define triple integrals for trivariate functions. We begin with the simplest case, where f is defined on a rectangular box:

$$B = \{(x, y, z) : a \le x \le b, c \le y \le d, r \le z \le s\}.$$

By dividing $[a, b]$ into l subintervals of width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz, we are able to decompose B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k],$$

each with volume $\Delta V = \Delta x \Delta y \Delta z$. We then form the **triple Riemann sum**

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V,$$

where $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \in B_{ijk}$ is a sample point. Taking l, m, and n to infinity, and recalling the results from Sections 5.2 and 12.1, we arrive at the following definition.

Definition 12.7.1. The **triple integral** of f over the box B is

$$\iiint\limits_B f(x, y, z)\, dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V,$$

provided that the limit exists.

Just as for double integrals, the practical way of evaluating triple integrals is to iterate.

Theorem 12.7.1 (Fubini's Theorem)**.** *If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then*

$$\iiint\limits_{B} f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

Example 12.7.1. Evaluate $\iiint_B (xy + z^2)\, dV$, where $B = \{(x, y, z) : 0 \le x \le 1, -1 \le y \le 1, 0 \le z \le 2\}$.

Solution. Since $f(x, y, z) = xy + z^2$ is continuous on $\mathbb{R}^3$, it is certainly continuous on B. By Fubini's Theorem, it follows that

$$\begin{aligned}
\iiint\limits_{B} (xy + z^3)\, dV &= \int_0^2 \int_{-1}^1 \int_0^1 (xy + z^3)\, dx\, dy\, dz \\
&= \int_0^2 \int_{-1}^1 \left[\left(\frac{x^2 y}{2} + xz^3 \right) \Bigg|_{x=0}^{x=1} \right] dy\, dz \\
&= \int_0^2 \int_{-1}^1 \left[\frac{y}{2} + z^3 \right] dy\, dz \\
&= \int_0^2 \left[\left(\frac{y^2}{4} + yz^2 3 \right) \Bigg|_{y=-1}^{y=1} \right] dz \\
&= \int_0^2 \left[\left(\frac{1}{4} + z^3 \right) - \left(\frac{1}{4} - z^3 \right) \right] dz \\
&= \int_0^2 2z^3\, dz \\
&= \frac{z^4}{2} \Bigg|_{z=0}^{z=2} \\
&= 8.
\end{aligned}$$

To define a triple integral of the form $\iiint_E f(x, y, z)\, dV$, where E is a more general bounded region, we begin by enclosing E in a box B and defining a function F that agrees with f on E but is zero for points in B that are outside of E. Then, provided that f is continuous and the boundary of E is "reasonably smooth," we have

$$\iiint\limits_{E} f(x, y, z)\, dV = \iiint\limits_{B} F(x, y, z)\, dV.$$

We restrict our attention to continuous functions and to certain simple types of regions.

A solid region E is said to be of **type 1** if it has the form

$$E = \{(x, y, z) : (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\},$$

where D is a projection of E onto the xy-plane. If E is a type 1 region, then

$$\iiint\limits_{E} f(x, y, z)\, dV = \iint\limits_{D} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA.$$

A solid region E is said to be of **type 2** if it has the form

$$E = \{(x, y, z) : (y, z) \in D, u_1(y, z) \le z \le u_2(y, z)\},$$

where D is a projection of E onto the yz-plane. If E is a type 2 region, then

$$\iiint\limits_{E} f(x, y, z)\, dV = \iint\limits_{D} \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA.$$

A solid region E is said to be of **type 3** if it has the form

$$E = \{(x, y, z) : (x, z) \in D, u_1(x, z) \le z \le u_2(x, z)\},$$

where D is a projection of E onto the xz-plane. If E is a type 3 region, then

$$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA.$$

In any of these three cases, the planar region D may be of type I or type II, as defined in Section 12.3. For example, if E is a type 1 solid region and its projection onto the xy-plane is a type I planar region, then

$$E = \{(x,y,z) : a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\}$$

and

$$\iiint\limits_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

We see that there are six possible permutations of the differentials dx, dy, and dz. Thus, there are six possible orders of integration for triple integrals in Cartesian coordinates. Rather than memorize formulas for all of these cases, one should apply the following rule of thumb when setting up iterated integrals: any integral whose limits of integration depend on multiple variables should be evaluated first, followed by any integral(s) whose limits of integration depend on a single variable, followed by any integral(s) whose limits of integration are constant.

Example 12.7.2. Evaluate $\iiint_E 6xz\,dV$, where $E = \{(x,y,z) : 0 \le x \le z, 0 \le y \le x+z, 0 \le z \le 1\}$.

Solution. Since z is bounded by constants, x is bounded by functions of one variable, and y is bounded by functions of two variables, we find that

$$\begin{aligned}
\iiint\limits_E 6xz\,dV &= \int_0^1 \int_0^z \int_0^{x+z} 6xz\,dy\,dx\,dz \\
&= \int_0^1 \int_0^z \left[6xyz\Big|_{y=0}^{y=x+z}\right] dx\,dz \\
&= \int_0^1 \int_0^z \left[6x^2z + 6xz^2\right] dx\,dz \\
&= \int_0^1 \left[\left(2x^3z + 3x^2z^2\right)\Big|_{x=0}^{x=z}\right] dz \\
&= \int_0^1 \left[2z^4 + 3z^4\right] dz \\
&= \int_0^1 5z^4\,dz \\
&= z^5\Big|_{z=0}^{z=1} \\
&= 1.
\end{aligned}$$

Example 12.7.3. Rewrite the iterated integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$ in five other ways.

Solution. We begin by sketching the solid region $E = \{(x,y,z) : 0 \le x \le 1, \sqrt{x} \le y \le 1, 0 \le z \le 1-y\}$ and each of its projections onto the coordinate planes. From this, we find that

$$\begin{aligned}
\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx &= \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\,dz\,dx\,dy \\
&= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\,dx\,dz\,dy \\
&= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\,dx\,dy\,dz \\
&= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx \\
&= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz.
\end{aligned}$$

12.8. **Triple Integrals in Cylindrical and Spherical Coordinates.**

Combining the results from Sections 12.4 and 12.7, we arrive at the following formula for triple integration in cylindrical coordinates.

Proposition 12.8.1. *If f is continuous on a region*

$$E = \{(x,y,z) : (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\},$$

where

$$D = \{(r,\theta) : \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\},$$

then

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta.$$

Example 12.8.1. Evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the region that lies inside the cylinder $x^2+y^2=16$ and between the planes $z=-5$ and $z=4$.

Solution. The region of integration is naturally described in cylindrical coordinates as

$$E = \{(r,\theta,z) : 0 \le r \le 4, 0 \le \theta \le 2\pi, -5 \le z \le 4\}.$$

Therefore, we have

$$\begin{aligned}
\iiint_E \sqrt{x^2+y^2}\,dV &= \int_{-5}^4 \int_0^{2\pi} \int_0^4 \sqrt{r^2}\, r\,dr\,d\theta\,dz \\
&= \int_{-5}^4 1\,dz \cdot \int_0^{2\pi} 1\,d\theta \cdot \int_0^4 r^2\,dr \\
&= (4-(-5))\,(2\pi - 0)\left(\frac{r^3}{3}\bigg|_{r=0}^{r=4}\right) \\
&= 384\pi.
\end{aligned}$$

A somewhat more complicated formula exists for triple integrals in spherical coordinates.

Proposition 12.8.2. *If f is continuous on a spherical wedge*

$$E = \{(\rho,\theta,\phi) : a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\},$$

then

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

Example 12.8.2. Evaluate $\iiint_E z\,dV$, where E lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the first octant.

Solution. The region of integration is naturally described in spherical coordinates as

$$E = \{(\rho,\theta,\phi) : 1 \le \rho \le 2, 0 \le \theta \le \pi/2, 0 \le \phi \le \pi/2\}.$$

Therefore, recalling the conversion formula $z = \rho\cos\phi$, we have

$$\begin{aligned}
\iiint\limits_E z\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\cos\phi)\ \rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&= \int_0^{\pi/2} 1\,d\theta\cdot\int_1^2 \rho^3\,d\rho\cdot\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi \\
&\left(\frac{\pi}{2}-0\right)\left(\left.\frac{\rho^4}{4}\right|_{\rho=1}^{\rho=2}\right)\int_0^1 u\,du \\
&= \frac{\pi}{2}\left(\frac{2^4}{4}-\frac{1^4}{4}\right)\left(\left.\frac{u^2}{2}\right|_{u=0}^{u=1}\right) \\
&= \frac{15\pi}{16}.
\end{aligned}$$

12.9. **Change of Variables in Multiple Integrals.**

There are no notes provided for this section. It describes a multidimensional analogue to Proposition 5.5.1 that is related to the important probabilistic technique of random variable transformations.