Before we consider two partial fractions examples, here are three integral formulas that are very easy to verify but because they appear so frequently in problems it might be good to simply memorize them or have on your cheat sheet. Suppose *a* and *b* are constants, then

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

Example. Evaluate $\int \frac{x^2-x+6}{x^3+3x}\,dx$

Solution: Using the method of partial fractions, we have

$$\frac{x^2-x+6}{x^3+3x} = \frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

By the finger method, $A = 2$. Therefore

$$\frac{x^2-x+6}{x^3+3x} = \frac{2}{x} + \frac{Bx+C}{x^2+3}$$

If we set $x = 1$, we get,

$$\frac{3}{2} = 2 + \frac{B+C}{4}$$

Which simplifies to

$$B + C = -2$$

If we set $x = -1$, we get

$$-2 = -2 + \frac{B-C}{4}$$

Which simplifies to

$$B - C = 0$$

Solving simultaneously the two equations for B and C we get, $B = -1$ and $C = -1$. Therefore,

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \frac{2}{x}\,dx + \int \frac{-x-1}{x^2+3}\,dx = \int \frac{2}{x}\,dx - \int \frac{x+1}{x^2+3}\,dx$$

$$\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx = 2\ln|x|$$

To integrate $\int \frac{x+1}{x^2+3}\,dx$, we can distribute it into two parts as follows.

$$\int \frac{x+1}{x^2+3}\,dx = \int \frac{x}{x^2+3}\,dx + \int \frac{1}{x^2+3}\,dx$$

To calculate $\int \frac{x}{x^2+3}\,dx$, we let $u = x^2+3$. Then $du = 2x\,dx$ and by $u-$substitution we get,

$$\int \frac{x}{x^2+3}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+3) + C$$

To calculate $\int \frac{1}{x^2+3}\,dx$, we use the formula $\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ to get

$$\int \frac{1}{x^2+3}\,dx = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

Now substitute all of this back into our original integral, we get

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = 2\ln|x| + \frac{1}{2}\ln(x^2+3) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

Example. Evaluate $\int \frac{x^3+x-1}{x^2-3x-2}\,dx$

Solution: To use the partial fractions method, we need the degree of the numerator to be less than the degree of the denominator, which is not the case with the integrand in this example. When this happens, we long divide the numerator by the denominator. After the long division we get,

$$\frac{x^3+x-1}{x^2-3x+2} = x+3+\frac{8x-7}{x^2-3x+2}$$

The partial fraction decomposition of $\frac{8x-7}{x^2-3x-2}$ has the form

$$\frac{8x-7}{x^2-3x+2} = \frac{8x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

By the finger method, we get $A = 9$ and $B = -1$. Therefore, we get

$$\int \frac{x^3 + x - 1}{x^2 - 3x + 2}\,dx = \int x + 3 + 9\frac{1}{x-2} - \frac{1}{x-1}\,dx$$

$$= \frac{1}{2}x^2 + 3x + 9\ln|x-2| - \ln|x-1| + C$$