(1) First, we compute $f'(x) = \frac{2x+1}{x^2+x+1}$. Then

$$f'(x) = 0 \implies 2x+1 = 0 \implies x = -1/2,$$

and

$$f' \text{ is undefined} \implies x^2+x+1 = 0 \implies x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm i\sqrt{3}}{2},$$

so $x = -1/2$ is the only critical number of f in the interval $(-1, 1)$. Since

$$f(-1) = \ln((-1)^2 - 1 + 1) = \ln(1) = 0,$$
$$f(-1/2) = \ln((-1/2)^2 - 1/2 + 1) = \ln(3/4) \approx -0.2877,$$
$$f(1) = \ln(1^2 + 1 + 1) = \ln(3) \approx 1.0986,$$

the Closed Interval Method implies that $f(-1/2) = \ln(3/4)$ and $f(1) = \ln(3)$ are the absolute minimum and absolute maximum values, respectively, of f on $[-1, 1]$.

(2) (a) By L'Hôpital's Rule, we have

$$\lim_{t\to 0} \frac{e^{3t} - 1}{t} = \lim_{t\to 0} \frac{3e^{3t}}{1} = 3.$$

(b) Let $y = x^{x^2}$, so $\ln y = \ln\left(x^{x^2}\right) = x^2 \ln x$. L'Hôpital's Rule gives

$$\lim_{t\to 0^+} \ln y = \lim_{t\to 0^+} x^2 \ln x = \lim_{t\to 0^+} \frac{\ln x}{1/x^2} = \lim_{t\to 0^+} \frac{1/x}{-2/x^3} = \lim_{t\to 0^+} (-x^2/2) = 0.$$

Thus, $\lim_{t\to 0^+} x^{x^2} = \lim_{t\to 0^+} y = \lim_{t\to 0^+} e^{\ln y} = e^{\lim_{t\to 0^+} \ln y} = e^0 = 1$.

(3) Let x be the side length of the box, and let h be its height. Then the volume and surface area of the box are given by $V = x^2 h$ and $S = 4xh + x^2$. Using the fact that $S = 1200$, we solve the latter equation for $h = \frac{1200 - x^2}{4x}$. Substituting into the the equation for V, we obtain

$$V = f(x) = x^2 \left(\frac{1200 - x^2}{4x}\right) = 300x - \frac{1}{4}x^3.$$

Next, we compute $f'(x) = 300 - \frac{3}{4}x^2$, and

$$f'(x) = 0 \implies 300 - \frac{3}{4}x^2 = 0 \implies \frac{3}{4}x^2 = 300 \implies x^2 = 400 \implies x = \pm 20.$$

Taking the critical number $x = 20$, we compute the corresponding height $h = \frac{1200 - (20)^2}{4(20)} = 10$. Thus, the maximum volume of the box is $V = x^2 h = (20)^2(10) = 4000$ cm^3.

To convince yourself that the critical number $x = 20$ corresponds to a maximum (and not a minimum) of $V = f(x)$, simply notice that $f''(x) = -\frac{3}{2}x$, which is negative for all $x > 0$ (i.e. the volume function is concave downward, so any critical number where the derivative is zero must correspond to an absolute maximum).

(4) We are given the constant acceleration function $a(t) = -16$, so the general formulas for velocity and position are, respectively, $v(t) = -16t + C$ and $s(t) = -8t^2 + Ct + D$ for some constants C and D. Assuming that $s(0) = 0$ (it makes sense to let "position 0" occur at "time 0"), we can simplify the latter equation to $s(t) = -8t^2 + Ct$. Now, using the given information that $v(t) = 0$ occurs when $s(t) = 200$ (i.e. the car's velocity is 0 ft/s when it stops after skidding for 200 ft), we obtain a system of two equations:

$$\begin{cases} -16t + C = 0, \\ -8t^2 + Ct = 200. \end{cases}$$

Solving the first equation for $C = 16t$, and substituting into the second, we find that

$$-8t^2 + t(16t) = 200 \implies -8t^2 + 16t^2 = 200 \implies 8t^2 = 200 \implies t^2 = 25 \implies t = \pm 5.$$

Taking the positive time $t = 5$ s, we find that $C = 16(5) = 80$. Since $v(0) = -16(0) + C = C$, we have found the velocity when the brakes were first applied to be 80 ft/s.

(5) Splitting $[1, 3]$ into the four equal-length subintervals $[1, 3/2]$, $[3/2, 2]$, $[2, 5/2]$, $[5/2, 3]$, the right endpoint approximation for the area under the curve is

$$\begin{aligned} R_4 &= \frac{1}{2} f(3/2) + \frac{1}{2} f(2) + \frac{1}{2} f(5/2) + \frac{1}{2} f(3) \\ &= \frac{1}{2}\left[\frac{2(3/2)}{(3/2)^2+1} + \frac{2(2)}{(2)^2+1} + \frac{2(5/2)}{(5/2)^2+1} + \frac{2(3)}{(3)^2+1}\right] \\ &\approx 1.5064. \end{aligned}$$

From a quick look at the graph of f, or by explicitly computing $\int_1^3 f(x)\,dx$, it is easy to see that this approximation is an underestimate.

(6) (a) The limits of integration are both equal to 1, so $\displaystyle\int_1^{e^0} 2\cos\left(\frac{\sqrt{\sin x}}{\ln(x^6)}\right) e^{12x}\,dx = 0.$

(b) $\displaystyle\int_0^9 [2f(x) - 3g(x)]\,dx = 2\int_0^9 f(x)\,dx - 3\int_0^9 g(x)\,dx = 2(17) - 3(8) = 10.$

(7) (a) By FTC1, $f'(x) = (1 - x^2)e^{x^2}$. Setting $f'(x) = 0$, we find that $1 - x^2 = 0$ or $x = \pm 1$. A sign test of the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$ shows that f' is only positive on $(-1, 1)$. Hence, the function defined by $f(x) = \int_0^x (1 - t^2)e^{t^2}\,dt$ is increasing on the interval $(-1, 1)$.

(b) By FTC2, we have $\displaystyle\int_1^4 f'(x)\,dx = f(4) - f(1) \implies f(4) = \int_1^4 f'(x)\,dx + f(1) = 7 + 12 = 19.$

(c) By FTC1, $F'(x) = \frac{d}{dx}\left[\int_x^\pi \sqrt{1 + \sec t}\,dt\right] = \frac{d}{dx}\left[-\int_\pi^x \sqrt{1 + \sec t}\,dt\right] = -\sqrt{1 + \sec x}.$

(8) (a) Let $u = z^3 + 1$. Then $du = 3z^2\,dz$, or $dz = \frac{1}{3z^2}\,du$, and we have

$$\int \frac{z^2}{z^3+1}\,dz = \int \frac{z^2}{u}\left(\frac{1}{3z^2}\,du\right) = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|z^3 + 1| + C.$$

(b) Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$, and the new limits of integration are $u = \ln(e) = 1$ and $u = \ln(e^4) = 4$. Hence,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_{u=1}^{u=4} \frac{1}{\sqrt{u}}\,du = \int_{u=1}^{u=4} u^{-1/2}\,du = 2u^{1/2}\Big|_{u=1}^{u=4} = 2\left[\sqrt{4} - \sqrt{1}\right] = 2.$$

Bonus. Letting $u = \sin x$, the Chain Rule and FTC1 give

$$\frac{d}{dx}\int_0^{\sin x} \sqrt{1+t^2}\,dt = \frac{d}{dx}\int_0^u \sqrt{1+t^2}\,dt = \frac{d}{du}\left[\int_0^u \sqrt{1+t^2}\,dt\right]\cdot\frac{du}{dx} = \sqrt{1+u^2}\cdot\cos x.$$

That is, $f'(x) = \cos x\sqrt{1 + \sin^2 x}$. By FTC1, we have $g'(y) = f(y)$ on $\mathbb{R}$. Hence,

$$g''(\pi/6) = f'(\pi/6) = \cos(\pi/6)\sqrt{1 + \sin^2(\pi/6)} = \frac{\sqrt{15}}{4} \approx 0.9682.$$