# 2.1 Functions and Their Graphs

## Functions

A manufacturer would like to know how his company's profit is related to its production level; a biologist would like to know how the size of the population of a certain culture of bacteria will change over time; a psychologist would like to know the relationship between the learning time of an individual and the length of a vocabulary list; and a chemist would like to know how the initial speed of a chemical reaction is related to the amount of substrate used. In each instance, we are concerned with the same question: How does one quantity depend upon another? The relationship between two quantities is conveniently described in mathematics by using the concept of a function.

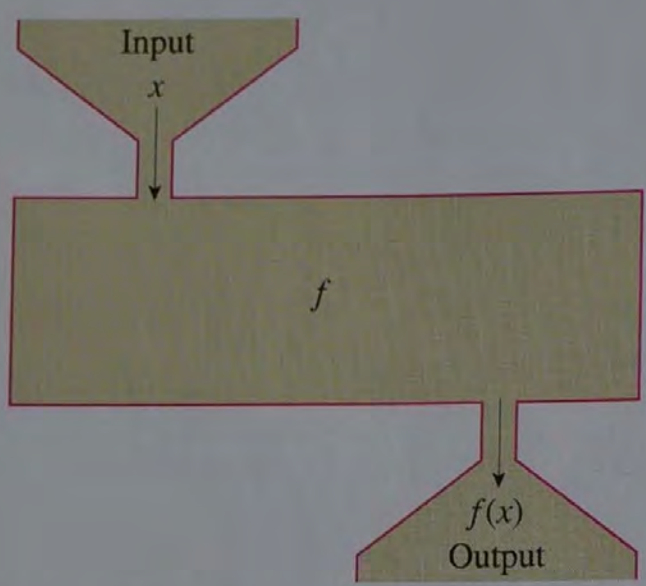


**FIGURE 1**
A function machine

**Function**

A **function** is a rule that assigns to each element in a set $A$ one and only one element in a set $B$.

The set $A$ is called the **domain** of the function. It is customary to denote a function by a letter of the alphabet, such as the letter $f$. If $x$ is an element in the domain of a function $f$, then the element in $B$ that $f$ associates with $x$ is written $f(x)$ (read "$f$ of $x$") and is called the value of $f$ at $x$. The set comprising all the values assumed by $y = f(x)$ as $x$ takes on all possible values in its domain is called the **range** of the function $f$.

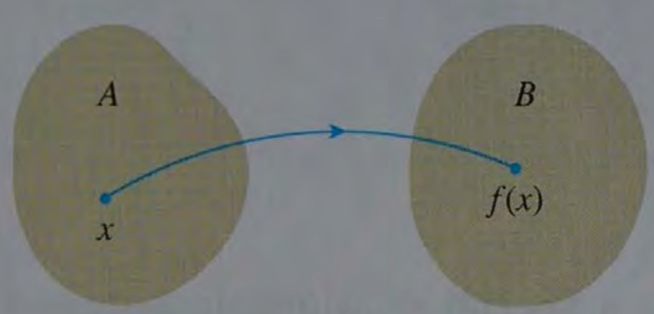


**FIGURE 2**
The function $f$ viewed as a mapping

We can think of a function $f$ as a machine. The domain is the set of inputs (raw material) for the machine, the rule describes how the input is to be processed, and the value(s) of the function are the outputs of the machine (Figure 1).

We can also think of a function $f$ as a mapping in which an element $x$ in the domain of $f$ is mapped onto a unique element $f(x)$ in $B$ (Figure 2).

**Notes**

1. The output $f(x)$ associated with an input $x$ is unique. To appreciate the importance of this uniqueness property, consider a rule that associates with each item $x$ in a department store its selling price $y$. Then, each $x$ must correspond to *one and only one* $y$. Notice, however, that different $x$'s may be associated with the same $y$. In the context of the present example, this says that different items may have the same price.
2. Although the sets $A$ and $B$ that appear in the definition of a function may be quite arbitrary, in this book they will denote sets of real numbers.

An example of a function may be taken from the familiar relationship between the area of a circle and its radius. Letting $x$ and $y$ denote the radius and area of a circle, respectively, we have, from elementary geometry,

$$y = \pi x^2 \tag{1}$$

Equation (1) defines $y$ as a function of $x$ since for each admissible value of $x$ (that is, for each nonnegative number representing the radius of a certain circle) there corresponds precisely one number $y = \pi x^2$ that gives the area of the circle. The rule defining this "area function" may be written as

$$f(x) = \pi x^2 \tag{2}$$

To compute the area of a circle of radius 5 inches, we simply replace $x$ in Equation (2) with the number 5. Thus, the area of the circle is

$$f(5) = \pi 5^2 = 25\pi$$

or $25\pi$ square inches.

In general, to evaluate a function at a specific value of $x$, we replace $x$ with that value, as illustrated in Examples 1 and 2.

VIDEO

**EXAMPLE 1** Let the function $f$ be defined by the rule $f(x) = 2x^2 - x + 1$. Find:

**a.** $f(1)$ **b.** $f(-2)$ **c.** $f(a)$ **d.** $f(a + h)$

**Solution**

**a.** $f(1) = 2(1)^2 - (1) + 1 = 2 - 1 + 1 = 2$
**b.** $f(-2) = 2(-2)^2 - (-2) + 1 = 8 + 2 + 1 = 11$
**c.** $f(a) = 2(a)^2 - (a) + 1 = 2a^2 - a + 1$
**d.** $f(a + h) = 2(a + h)^2 - (a + h) + 1 = 2a^2 + 4ah + 2h^2 - a - h + 1$

**APPLIED EXAMPLE 2 Profit Functions** ThermoMaster manufactures an indoor–outdoor thermometer at its Mexican subsidiary. Management estimates that the profit (in dollars) realizable by ThermoMaster in the manufacture and sale of $x$ thermometers per week is

$$P(x) = -0.001x^2 + 8x - 5000$$

Find ThermoMaster's weekly profit if its level of production is (a) 1000 thermometers per week and (b) 2000 thermometers per week.

**Solution**

**a.** The weekly profit when the level of production is 1000 units per week is found by evaluating the profit function $P$ at $x = 1000$. Thus,

$$P(1000) = -0.001(1000)^2 + 8(1000) - 5000 = 2000$$

or \$2000.

**b.** When the level of production is 2000 units per week, the weekly profit is given by

$$P(2000) = -0.001(2000)^2 + 8(2000) - 5000 = 7000$$

or \$7000.

## Determining the Domain of a Function

Suppose we are given the function $y = f(x)$.* Then, the variable $x$ is called the **independent variable**. The variable $y$, whose value depends on $x$, is called the **dependent variable**.

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*It is customary to refer to a function $f$ as $f(x)$ or by the equation $y = f(x)$ defining it.

To determine the domain of a function, we need to find what restrictions, if any, are to be placed on the independent variable $x$. In many practical applications, the domain of a function is dictated by the nature of the problem, as illustrated in Example 3.

**APPLIED EXAMPLE 3 Packaging** An open box is to be made from a rectangular piece of cardboard 16 inches long and 10 inches wide by cutting away identical squares ($x$ inches by $x$ inches) from each corner and folding up the resulting flaps (Figure 3). Find an expression that gives the volume $V$ of the box as a function of $x$. What is the domain of the function?

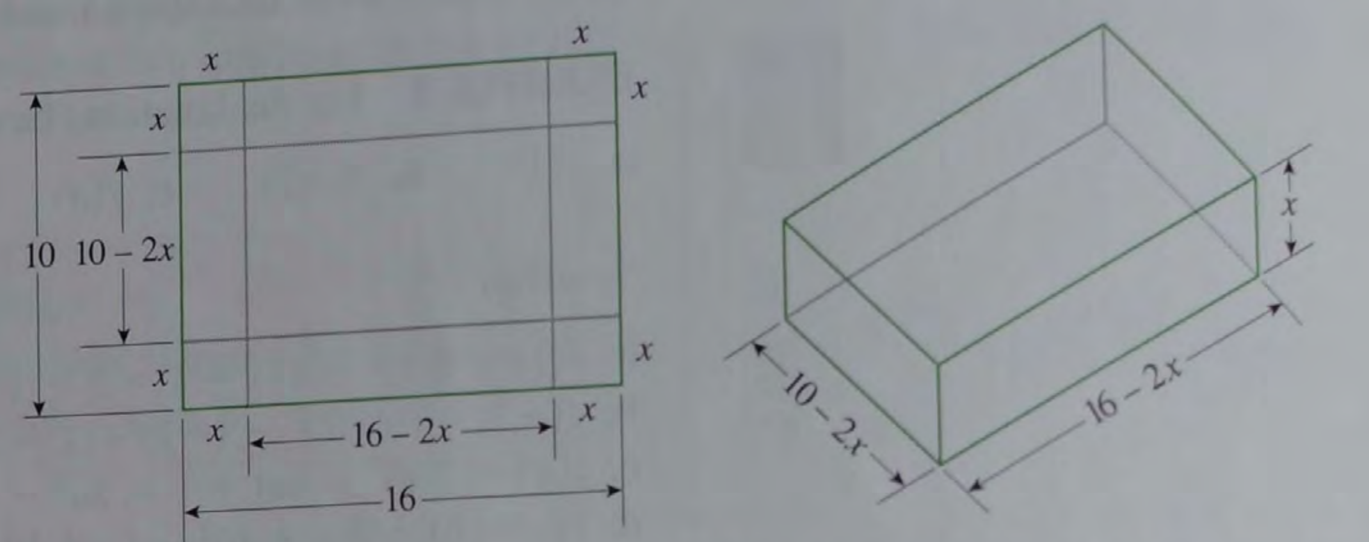



**FIGURE 3**

**(a)** The box is constructed by cutting $x$-by-$x$-inch squares from each corner.

**(b)** The dimensions of the resulting box are $(10 - 2x)''$ by $(16 - 2x)''$ by $x''$.

**Solution** The dimensions of the box are $(16 - 2x)$ inches long, $(10 - 2x)$ inches wide, and $x$ inches high, so its volume (in cubic inches) is given by

$$\begin{aligned} V = f(x) &= (16 - 2x)(10 - 2x)x \qquad \text{Length} \cdot \text{width} \cdot \text{height} \\ &= (160 - 52x + 4x^2)x \\ &= 4x^3 - 52x^2 + 160x \end{aligned}$$

Since the length of each side of the box must be greater than or equal to zero, we see that

$$16 - 2x \geq 0 \qquad 10 - 2x \geq 0 \qquad x \geq 0$$

simultaneously; that is,

$$x \leq 8 \qquad x \leq 5 \qquad x \geq 0$$

All three inequalities are satisfied simultaneously provided that $0 \leq x \leq 5$. Thus, the domain of the function $f$ is the interval $[0, 5]$.

In general, if a function is defined by a rule relating $x$ to $f(x)$ without specific mention of its domain, it is understood that the domain will consist of all values of $x$ for which $f(x)$ is a real number. In this connection, you should keep in mind that (1) division by zero is not permitted and (2) the square root of a negative number is not defined.

**EXAMPLE 4** Find the domain of each function.

**a.** $f(x) = \sqrt{x - 1}$ **b.** $f(x) = \dfrac{1}{x^2 - 4}$ **c.** $f(x) = x^2 + 3$

**Solution**

**a.** Since the square root of a negative number is undefined, it is necessary that $x - 1 \geq 0$. The inequality is satisfied by the set of real numbers $x \geq 1$. Thus, the domain of $f$ is the interval $[1, \infty)$.

**b.** The only restriction on $x$ is that $x^2 - 4$ be different from zero since division by zero is not allowed. But $(x^2 - 4) = (x + 2)(x - 2) = 0$ if $x = -2$ or $x = 2$. Thus, the domain of $f$ in this case consists of the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

**c.** Here, any real number satisfies the equation, so the domain of $f$ is the set of all real numbers.

## Graphs of Functions

If $f$ is a function with domain $A$, then corresponding to each real number $x$ in $A$ there is precisely one real number $f(x)$. We can also express this fact by using **ordered pairs** of real numbers. Write each number $x$ in $A$ as the first member of an ordered pair and each number $f(x)$ corresponding to $x$ as the second member of the ordered pair. This gives exactly one ordered pair $(x, f(x))$ for each $x$ in $A$. This observation leads to an **alternative definition of a function** $f$:

**Function (Alternative Definition)**

A function $f$ with domain $A$ is the set of all ordered pairs $(x, f(x))$ where $x$ belongs to $A$.

Observe that the condition that there be one and only one number $f(x)$ corresponding to each number $x$ in $A$ translates into the requirement that *no two ordered pairs have the same first number.*

Since ordered pairs of real numbers correspond to points in the plane, we have found a way to exhibit a function graphically.

**Graph of a Function of One Variable**

The **graph of a function** $f$ is the set of all points $(x, y)$ in the $xy$-plane such that $x$ is in the domain of $f$ and $y = f(x)$.

Figure 4 shows the graph of a function $f$. Observe that the $y$-coordinate of the point $(x, y)$ on the graph of $f$ gives the height of that point (the distance above the $x$-axis), if $f(x)$ is positive. If $f(x)$ is negative, then $-f(x)$ gives the depth of the point $(x, y)$ (the distance below the $x$-axis). Also, observe that the domain of $f$ is a set of real numbers lying on the $x$-axis, whereas the range of $f$ lies on the $y$-axis.

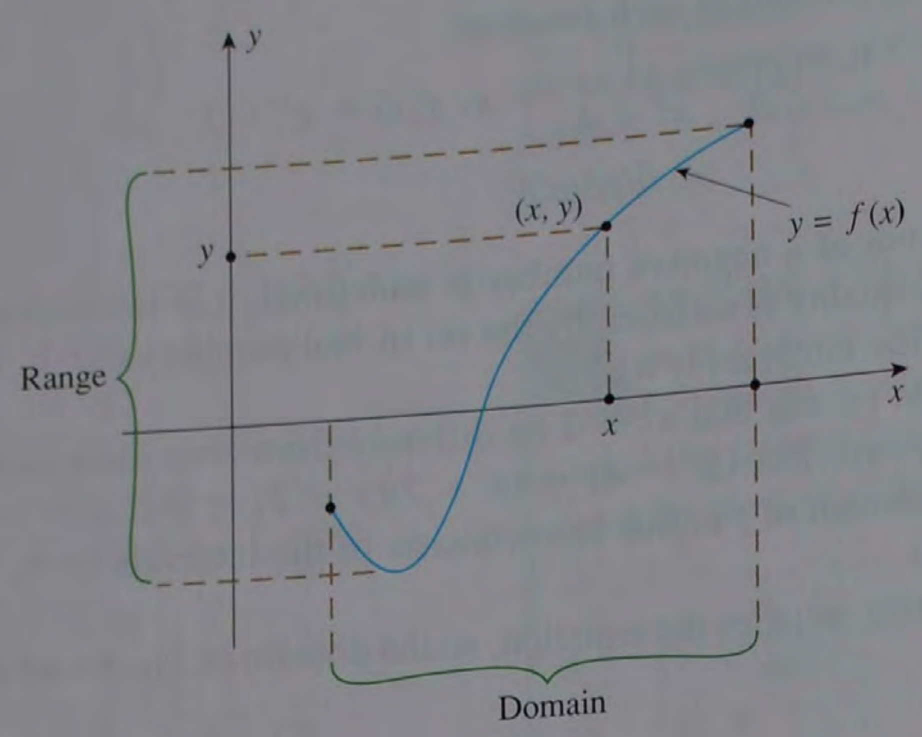



FIGURE 4
The graph of $f$

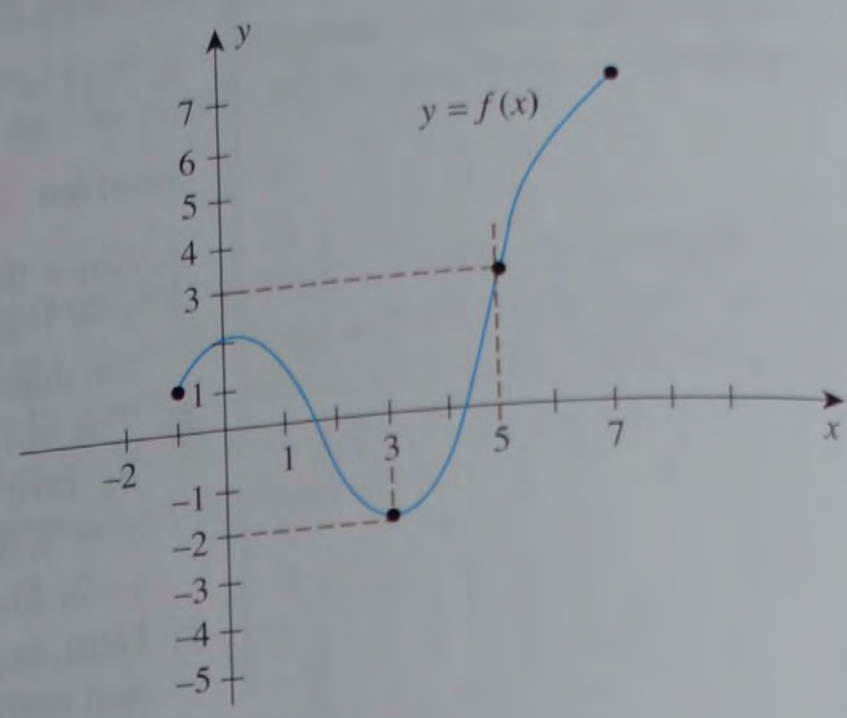



FIGURE 5

**EXAMPLE 5** The graph of a function $f$ is shown in Figure 5.

**a.** What is the value of $f(3)$? The value of $f(5)$?

**b.** What is the height or depth of the point $(3, f(3))$ from the $x$-axis? The point $(5, f(5))$ from the $x$-axis?

**c.** What is the domain of $f$? The range of $f$?

**Solution**

**a.** From the graph of $f$, we see that $y = -2$ when $x = 3$ and conclude that $f(3) = -2$. Similarly, we see that $f(5) = 3$.

**b.** Since the point $(3, -2)$ lies below the $x$-axis, we see that the depth of the point $(3, f(3))$ is $-f(3) = -(-2) = 2$ units below the $x$-axis. The point $(5, f(5))$ lies above the $x$-axis and is located at a height of $f(5)$, or 3 units above the $x$-axis.

**c.** Observe that $x$ may take on all values between $x = -1$ and $x = 7$, inclusive, and so the domain of $f$ is $[-1, 7]$. Next, observe that as $x$ takes on all values in the domain of $f$, $f(x)$ takes on all values between $-2$ and 7, inclusive. (You can easily see this by running your index finger along the $x$-axis from $x = -1$ to $x = 7$ and observing the corresponding values assumed by the $y$-coordinate of each point of the graph of $f$.) Therefore, the range of $f$ is $[-2, 7]$.

Much information about the graph of a function can be gained by plotting a few points on its graph. Later on we will develop more systematic and sophisticated techniques for graphing functions.

**EXAMPLE 6** Sketch the graph of the function defined by the equation $y = x^2 + 1$. What is the range of $f$?

**Solution** The domain of the function is the set of all real numbers. By assigning several values to the variable $x$ and computing the corresponding values for $y$, we obtain the following solutions to the equation $y = x^2 + 1$:

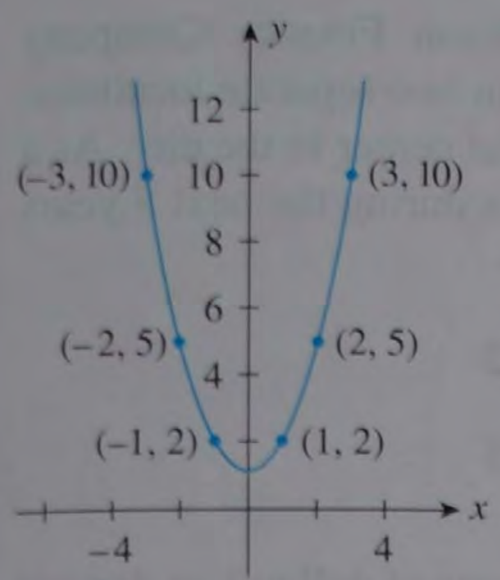



**FIGURE 6**
The graph of $y = x^2 + 1$ is a parabola.

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y$ | 10 | 5 | 2 | 1 | 2 | 5 | 10 |

By plotting these points and then connecting them with a smooth curve, we obtain the graph of $y = f(x)$, which is a parabola (Figure 6). To determine the range of $f$, we observe that $x^2 \geq 0$, if $x$ is any real number and so $x^2 + 1 \geq 1$ for all real numbers $x$. We conclude that the range of $f$ is $[1, \infty)$. The graph of $f$ confirms this result visually. ■

### EXPLORING WITH TECHNOLOGY

Let $f(x) = x^2$.

1. Plot the graphs of $F(x) = x^2 + c$ on the same set of axes for $c = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$.
2. Plot the graphs of $G(x) = (x + c)^2$ on the same set of axes for $c = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$.
3. Plot the graphs of $H(x) = cx^2$ on the same set of axes for $c = -2, -1, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 1, 2$.
4. Study the family of graphs in parts 1–3 and describe the relationship between the graph of a function $f$ and the graphs of the functions defined by (a) $y = f(x) + c$, (b) $y = f(x + c)$, and (c) $y = cf(x)$, where $c$ is a constant.

A function that is defined by more than one rule is called a **piecewise-defined function.**

**EXAMPLE 7** Sketch the graph of the function $f$ defined by

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$$

**Solution** The function $f$ is defined in a piecewise fashion on the set of all real numbers. In the subdomain $(-\infty, 0)$, the rule for $f$ is given by $f(x) = -x$. The equation $y = -x$ is a linear equation in the slope-intercept form (with slope $-1$ and intercept 0). Therefore, the graph of $f$ corresponding to the subdomain $(-\infty, 0)$ is the half line shown in Figure 7. Next, in the subdomain $[0, \infty)$, the rule for $f$ is given by $f(x) = \sqrt{x}$. The values of $f(x)$ corresponding to $x = 0, 1, 2, 3, 4, 9$, and 16 are shown in the following table:

| $x$ | 0 | 1 | 2 | 3 | 4 | 9 | 16 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0 | 1 | $\sqrt{2}$ | $\sqrt{3}$ | 2 | 3 | 4 |

Using these values, we sketch the graph of the function $f$ as shown in Figure 7. ■

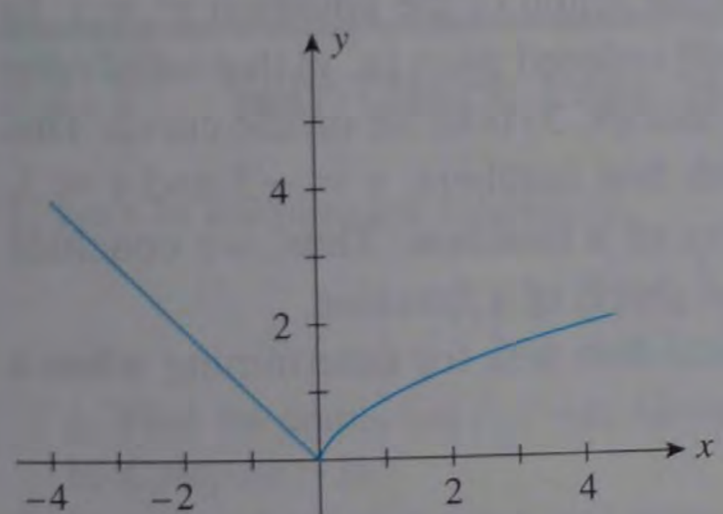



**FIGURE 7**
The graph of $y = f(x)$ is obtained by graphing $y = -x$ over $(-\infty, 0)$ and $y = \sqrt{x}$ over $[0, \infty)$.

VIDEO

**APPLIED EXAMPLE 8 Bank Deposits** Madison Finance Company plans to open two branch offices 2 years from now in two separate locations: an industrial complex and a newly developed commercial center in the city. As a result of these expansion plans, Madison's total deposits during the next 5 years are expected to grow in accordance with the rule

$$f(x) = \begin{cases} \sqrt{2x} + 20 & \text{if } 0 \le x \le 2 \\ \frac{1}{2}x^2 + 20 & \text{if } 2 < x \le 5 \end{cases}$$

where $y = f(x)$ gives the total amount of money (in millions of dollars) on deposit with Madison in year $x$ ($x = 0$ corresponds to the present). Sketch the graph of the function $f$.

**Solution** The function $f$ is defined in a piecewise fashion on the interval $[0, 5]$. In the subdomain $[0, 2]$, the rule for $f$ is given by $f(x) = \sqrt{2x} + 20$. The values of $f(x)$ corresponding to $x = 0$, 1, and 2 may be tabulated as follows:

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(x)$ | 20 | 21.4 | 22 |

Next, in the subdomain $(2, 5]$, the rule for $f$ is given by $f(x) = \frac{1}{2}x^2 + 20$. The values of $f(x)$ corresponding to $x = 3$, 4, and 5 are shown in the following table:

| $x$ | 3 | 4 | 5 |
|---|---|---|---|
| $f(x)$ | 24.5 | 28 | 32.5 |

Using the values of $f(x)$ in this table, we sketch the graph of the function $f$ as shown in Figure 8.

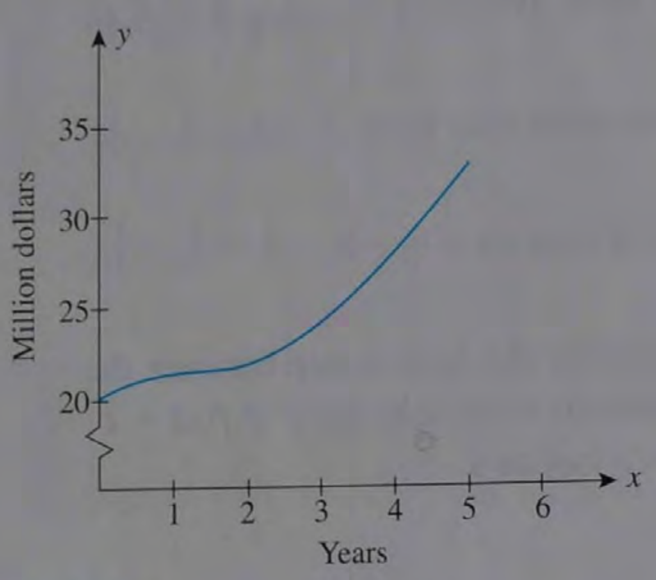



**FIGURE 8**
We obtain the graph of the function $y = f(x)$ by graphing $y = \sqrt{2x} + 20$ over $[0, 2]$ and $y = \frac{1}{2}x^2 + 20$ over $(2, 5]$.

## The Vertical-Line Test

Although it is true that every function $f$ of a variable $x$ has a graph in the $xy$-plane, it is not true that every curve in the $xy$-plane is the graph of a function. For example, consider the curve depicted in Figure 9. This is the graph of the equation $y^2 = x$. In general, the **graph of an equation** is the set of all ordered pairs $(x, y)$ that satisfy the given equation. Observe that the points $(9, -3)$ and $(9, 3)$ both lie on the curve. This implies that the number $x = 9$ is associated with *two* numbers: $y = -3$ and $y = 3$. But this clearly violates the uniqueness property of a function. Thus, we conclude that the curve under consideration cannot be the graph of a function.

This example suggests the following **vertical-line test** for determining when a curve is the graph of a function.

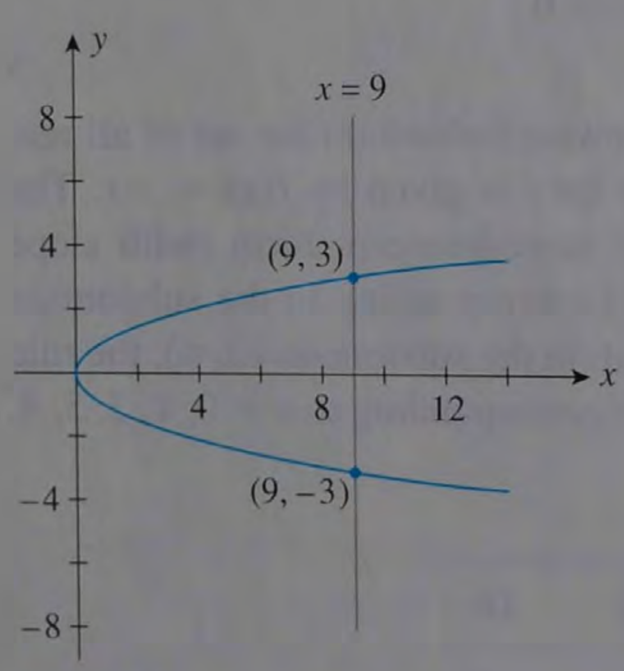



**FIGURE 9**
Since a vertical line passes through the curve at more than one point, we deduce that it is *not* the graph of a function.

**Vertical-Line Test**

A curve in the $xy$-plane is the graph of a function $y = f(x)$ if and only if each vertical line intersects it in at most one point.

**EXAMPLE 9** Determine which of the curves shown in Figure 10 are the graphs of functions of $x$.

**Solution** The curves depicted in Figure 10a, c, and d are graphs of functions because each curve satisfies the requirement that each vertical line intersects the curve in at most one point. Note that the vertical line shown in Figure 10c does *not* intersect the graph because the point on the $x$-axis through which this line passes does not lie in the domain of the function. The curve depicted in Figure 10b is *not* the graph of a function because the vertical line shown there intersects the graph at three points.

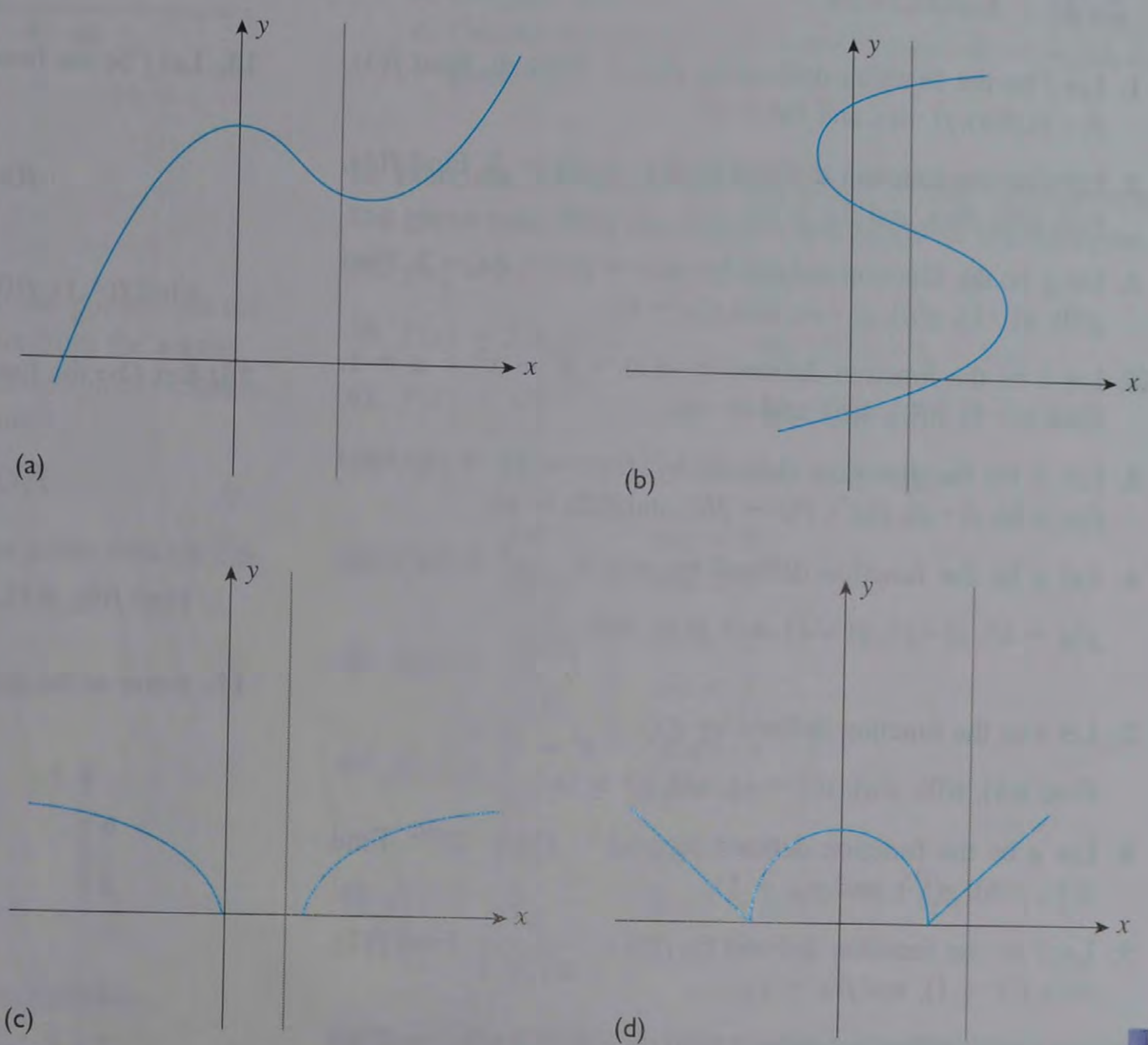



**FIGURE 10**
The vertical-line test can be used to determine which of these curves are graphs of functions.

## 2.1 Self-Check Exercises

1. Let $f$ be the function defined by

$$f(x) = \frac{\sqrt{x+1}}{x}$$

   a. Find the domain of $f$.
   b. Compute $f(3)$.
   c. Compute $f(a + h)$.

2. Statistics show that more and more motorists are pumping their own gas. The following function gives self-serve sales as a percent of all U.S. gas sales:

$$f(t) = \begin{cases} 6t + 17 & \text{if } 0 \le t \le 6 \\ 15.98(t-6)^{1/4} + 53 & \text{if } 6 < t \le 20 \end{cases}$$

   Here $t$ is measured in years, with $t = 0$ corresponding to the beginning of 1974.

   a. Sketch the graph of the function $f$.
   b. What percent of all gas sales at the beginning of 1978 were self-serve? At the beginning of 1994?

   *Source:* Amoco Corporation

3. Let $f(x) = \sqrt{2x+1} + 2$. Determine whether the point $(4, 6)$ lies on the graph of $f$.

*Solutions to Self-Check Exercises 2.1 can be found on page 63.*

## 2.1 Concept Questions

1. a. What is a function?
   b. What is the domain of a function? The range of a function?
   c. What is an independent variable? A dependent variable?

2. a. What is the graph of a function? Use a drawing to illustrate the graph, the domain, and the range of a function.
   b. If you are given a curve in the $xy$-plane, how can you tell if the graph is that of a function $f$ defined by $y = f(x)$?

## 2.1 Exercises

1. Let $f$ be the function defined by $f(x) = 5x + 6$. Find $f(3)$, $f(-3)$, $f(a)$, $f(-a)$, and $f(a + 3)$.

2. Let $f$ be the function defined by $f(x) = 4x - 3$. Find $f(4)$, $f(\frac{1}{4})$, $f(0)$, $f(a)$, and $f(a + 1)$.

3. Let $g$ be the function defined by $g(x) = 3x^2 - 6x - 3$. Find $g(0)$, $g(-1)$, $g(a)$, $g(-a)$, and $g(x + 1)$.

4. Let $h$ be the function defined by $h(x) = x^3 - x^2 + x + 1$. Find $h(-5)$, $h(0)$, $h(a)$, and $h(-a)$.

5. Let $f$ be the function defined by $f(x) = 2x + 5$. Find $f(a + h)$, $f(-a)$, $f(a^2)$, $f(a - 2h)$, and $f(2a - h)$.

6. Let $g$ be the function defined by $g(x) = -x^2 + 2x$. Find $g(a + h)$, $g(-a)$, $g(\sqrt{a})$, $a + g(a)$, and $\dfrac{1}{g(a)}$.

7. Let $s$ be the function defined by $s(t) = \dfrac{2t}{t^2 - 1}$. Find $s(4)$, $s(0)$, $s(a)$, $s(2 + a)$, and $s(t + 1)$.

8. Let $g$ be the function defined by $g(u) = (3u - 2)^{3/2}$. Find $g(1)$, $g(6)$, $g(\frac{11}{3})$, and $g(u + 1)$.

9. Let $f$ be the function defined by $f(t) = \dfrac{2t^2}{\sqrt{t - 1}}$. Find $f(2)$, $f(a)$, $f(x + 1)$, and $f(x - 1)$.

10. Let $f$ be the function defined by $f(x) = 2 + 2\sqrt{5 - x}$. Find $f(-4)$, $f(1)$, $f(\frac{11}{4})$, and $f(x + 5)$.

11. Let $f$ be the function defined by

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \le 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$

Find $f(-2)$, $f(0)$, and $f(1)$.

12. Let $g$ be the function defined by

$$g(x) = \begin{cases} -\frac{1}{2}x + 1 & \text{if } x < 2 \\ \sqrt{x - 2} & \text{if } x \ge 2 \end{cases}$$

Find $g(-2)$, $g(0)$, $g(2)$, and $g(4)$.

13. Let $f$ be the function defined by

$$f(x) = \begin{cases} -\frac{1}{2}x^2 + 3 & \text{if } x < 1 \\ 2x^2 + 1 & \text{if } x \ge 1 \end{cases}$$

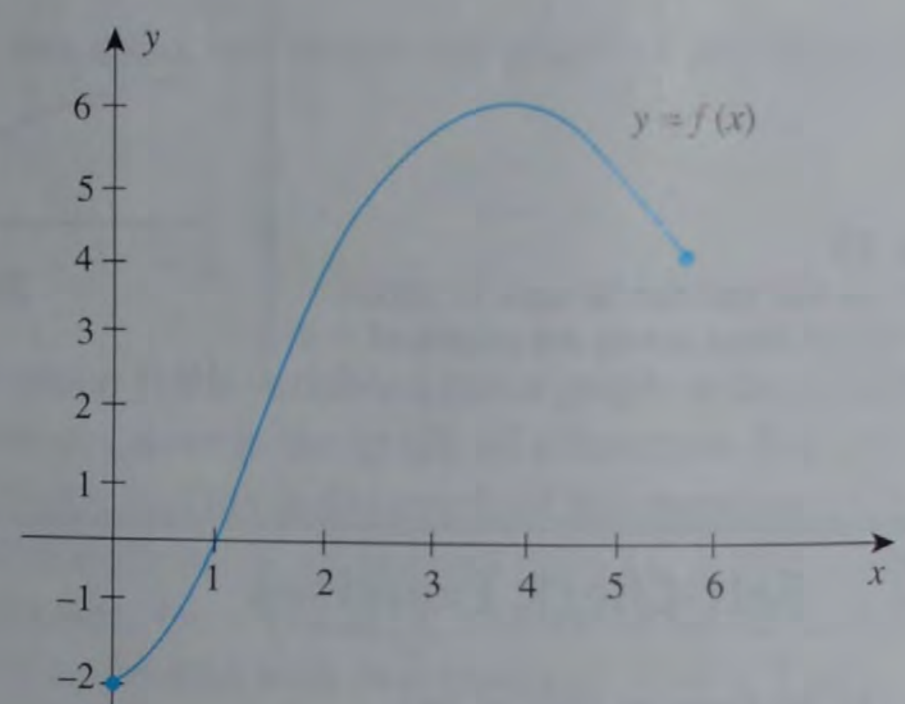
Find $f(-1)$, $f(0)$, $f(1)$, and $f(2)$.

14. Let $f$ be the function defined by

$$f(x) = \begin{cases} 2 + \sqrt{1 - x} & \text{if } x \le 1 \\ \dfrac{1}{1 - x} & \text{if } x > 1 \end{cases}$$

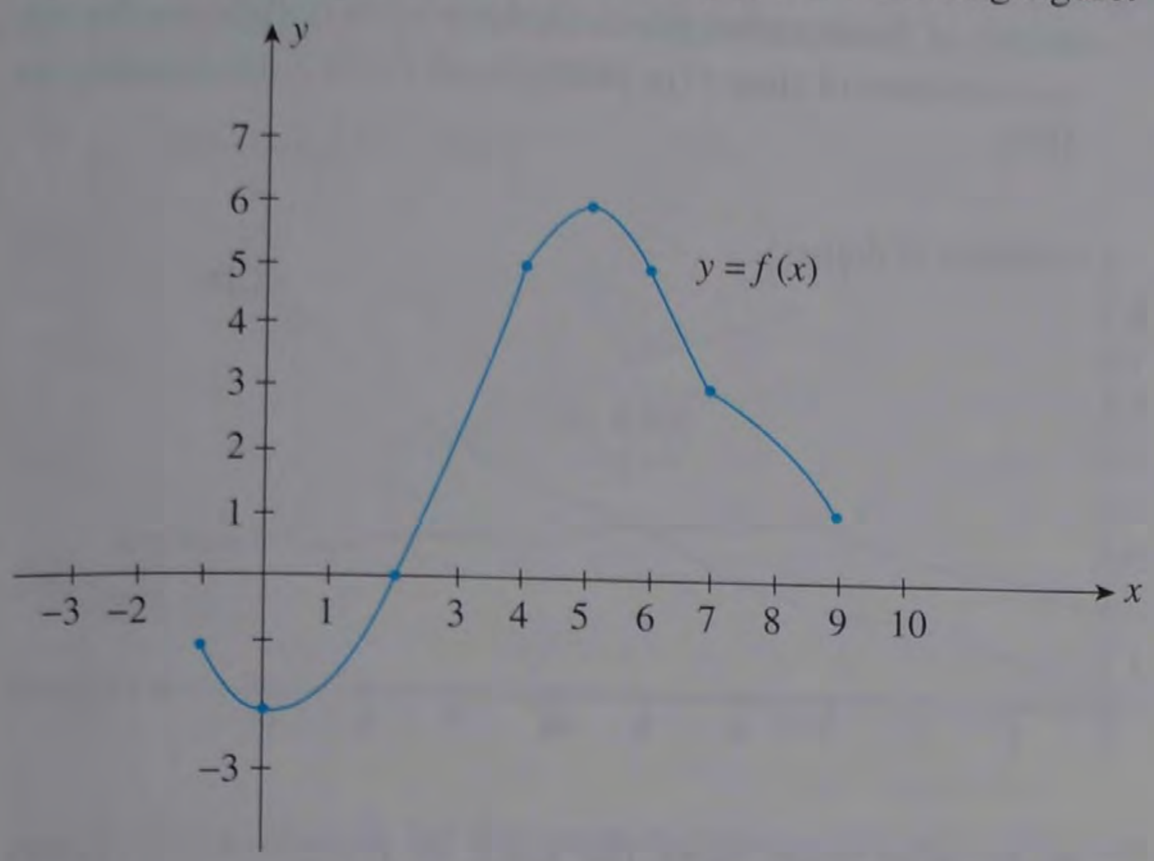
Find $f(0)$, $f(1)$, and $f(2)$.

15. Refer to the graph of the function $f$ in the following figure.




   a. Find the value of $f(0)$.
   b. Find the value of $x$ for which (i) $f(x) = 3$ and (ii) $f(x) = 0$.
   c. Find the domain of $f$.
   d. Find the range of $f$.

16. Refer to the graph of the function $f$ in the following figure.




a. Find the value of $f(7)$.
b. Find the values of $x$ corresponding to the point(s) on the graph of $f$ located at a height of 5 units from the $x$-axis.
c. Find the point on the $x$-axis at which the graph of $f$ crosses it. What is the value of $f(x)$ at this point?
d. Find the domain and range of $f$.

**In Exercises 17–20, determine whether the point lies on the graph of the function.**

17. $(2, \sqrt{3});\ g(x) = \sqrt{x^2 - 1}$

18. $(3, 3);\ f(x) = \dfrac{x + 1}{\sqrt{x^2 + 7}} + 2$

19. $(-2, -3);\ f(t) = \dfrac{|t - 1|}{t + 1}$

20. $\left(-3, -\dfrac{1}{13}\right);\ h(t) = \dfrac{|t + 1|}{t^3 + 1}$

**In Exercises 21–34, find the domain of the function.**

21. $f(x) = x^2 + 3$

22. $f(x) = 7 - x^2$

23. $f(x) = \dfrac{3x + 1}{x^2}$

24. $g(x) = \dfrac{2x + 1}{x - 1}$

25. $f(x) = \sqrt{x^2 + 1}$

26. $f(x) = \sqrt{x - 5}$

27. $f(x) = \sqrt{5 - x}$

28. $g(x) = \sqrt{2x^2 + 3}$

29. $f(x) = \dfrac{x}{x^2 - 1}$

30. $f(x) = \dfrac{1}{x^2 + x - 2}$

31. $f(x) = (x + 3)^{3/2}$

32. $g(x) = 2(x - 1)^{5/2}$

33. $f(x) = \dfrac{\sqrt{1 - x}}{x^2 - 4}$

34. $f(x) = \dfrac{\sqrt{x - 1}}{(x + 2)(x - 3)}$

35. Let $f$ be a function defined by the rule $f(x) = x^2 - x - 6$.
a. Find the domain of $f$.
b. Compute $f(x)$ for $x = -3, -2, -1, 0, \frac{1}{2}, 1, 2, 3$.
c. Use the results obtained in parts (a) and (b) to sketch the graph of $f$.

36. Let $f$ be a function defined by the rule $f(x) = 2x^2 + x - 3$.
a. Find the domain of $f$.
b. Compute $f(x)$ for $x = -3, -2, -1, -\frac{1}{2}, 0, 1, 2, 3$.
c. Use the results obtained in parts (a) and (b) to sketch the graph of $f$.

**In Exercises 37–48, sketch the graph of the function with the given rule. Find the domain and range of the function.**

37. $f(x) = 2x^2 + 1$

38. $f(x) = 9 - x^2$

39. $f(x) = 2 + \sqrt{x}$

40. $g(x) = 4 - \sqrt{x}$

41. $f(x) = \sqrt{1 - x}$

42. $f(x) = \sqrt{x - 1}$

43. $f(x) = |x| - 1$

44. $f(x) = |x| + 1$

45. $f(x) = \begin{cases} x & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$

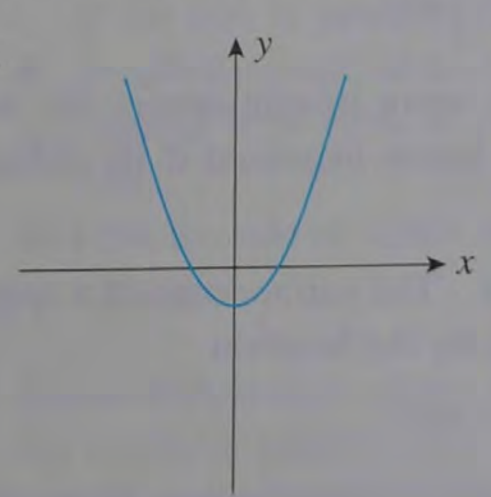
46. $f(x) = \begin{cases} 4 - x & \text{if } x < 2 \\ 2x - 2 & \text{if } x \geq 2 \end{cases}$

47. $f(x) = \begin{cases} -x + 1 & \text{if } x \leq 1 \\ x^2 - 1 & \text{if } x > 1 \end{cases}$

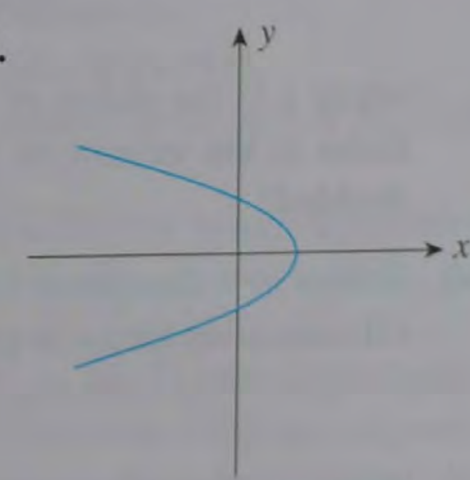
48. $f(x) = \begin{cases} -x - 1 & \text{if } x < -1 \\ 0 & \text{if } -1 \leq x \leq 1 \\ x + 1 & \text{if } x > 1 \end{cases}$

**In Exercises 49–56, use the vertical-line test to determine whether the graph represents $y$ as a function of $x$.**

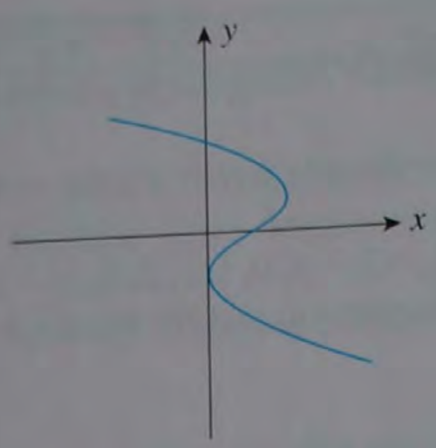
49.



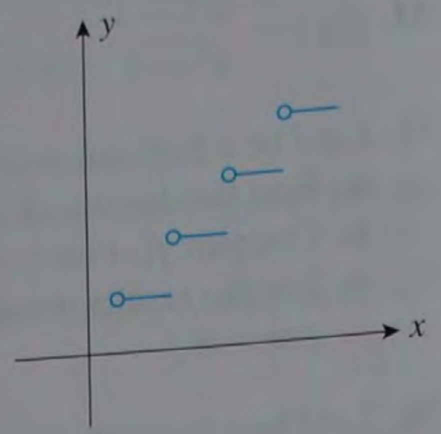

50.



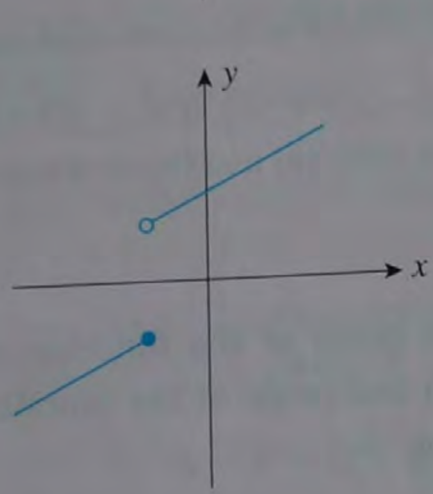
**51.**



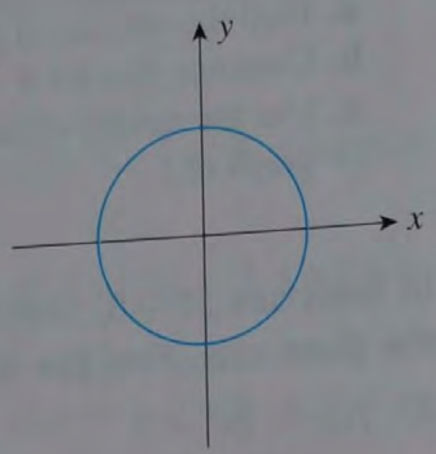

**52.**



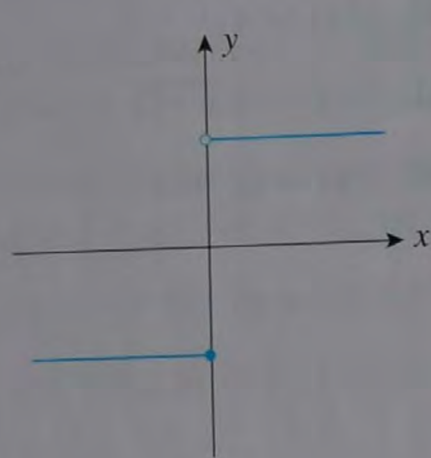

**53.**



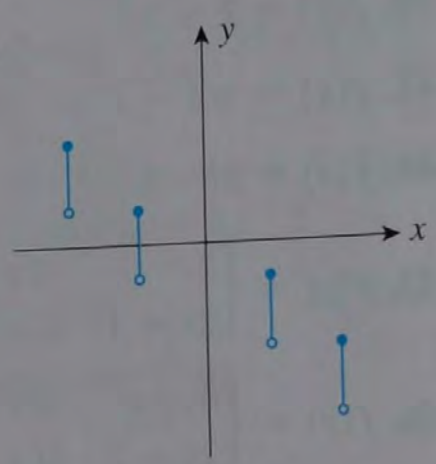

**54.**




**55.**




**56.**




57. The circumference of a circle is given by $C(r) = 2\pi r$, where $r$ is the radius of the circle. What is the circumference of a circle with a 5-in. radius?

58. The volume of a sphere of radius $r$ is given by $V(r) = \frac{4}{3}\pi r^3$. Compute $V(2.1)$ and $V(2)$. What does the quantity $V(2.1) - V(2)$ measure?

59. **Growth of a Cancerous Tumor** The volume of a spherical cancerous tumor is given by the function

$$V(r) = \frac{4}{3}\pi r^3$$

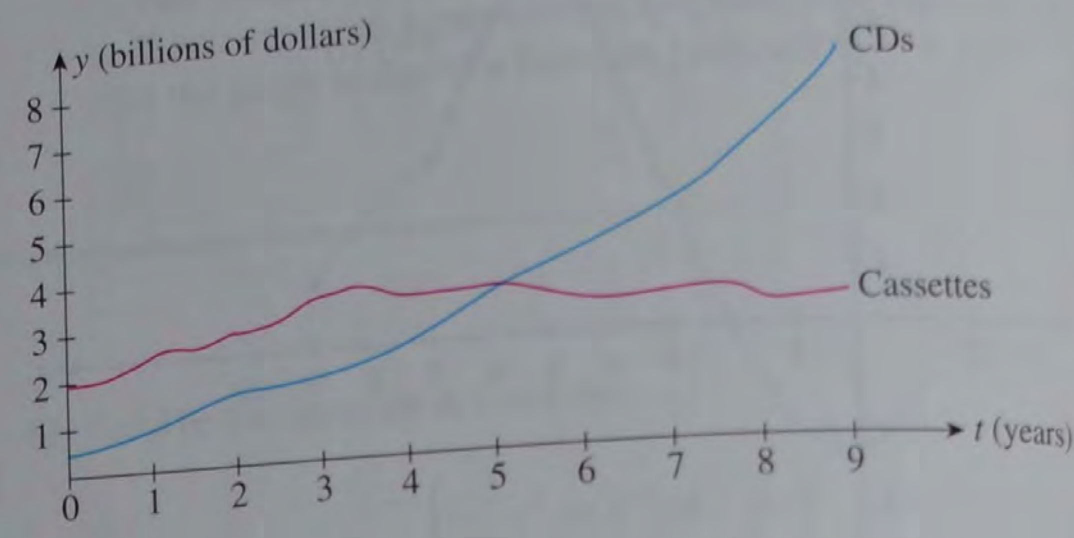
where $r$ is the radius of the tumor in centimeters. By what factor is the volume of the tumor increased if its radius is doubled?

60. **Growth of a Cancerous Tumor** The surface area of a spherical cancerous tumor is given by the function

$$S(r) = 4\pi r^2$$

where $r$ is the radius of the tumor in centimeters. After extensive chemotherapy treatment, the surface area of the tumor is reduced by 75%. What is the radius of the tumor after treatment?

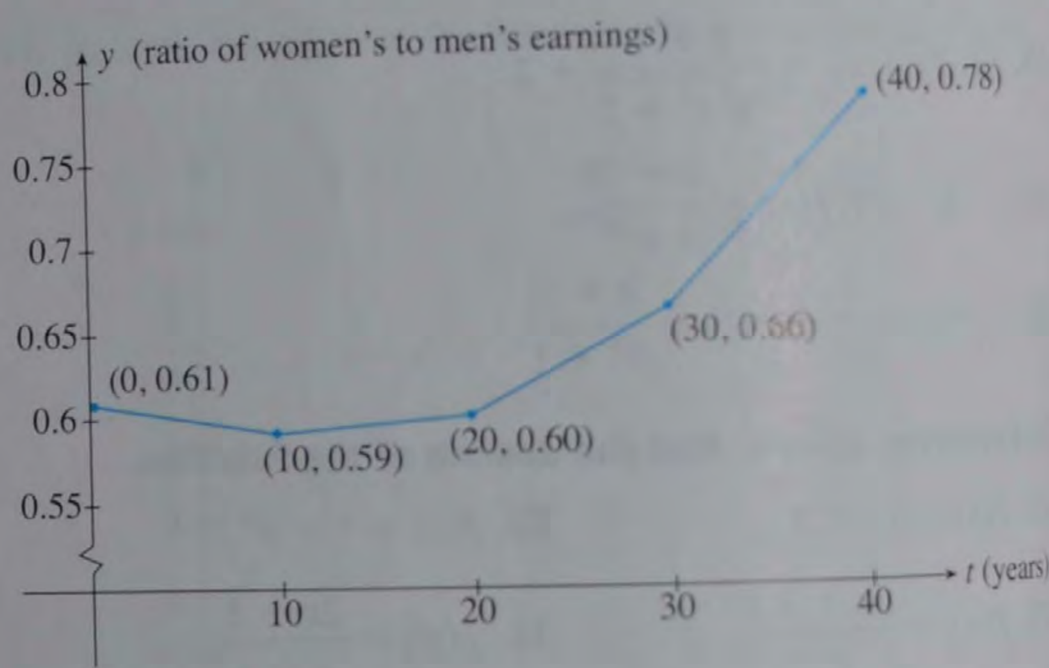
61. **Sales of Prerecorded Music** The following graphs show the sales $y$ of prerecorded music (in billions of dollars) by format as a function of time $t$ (in years), with $t = 0$ corresponding to 1985.




a. In what years were the sales of prerecorded cassettes greater than those of prerecorded CDs?
b. In what years were the sales of prerecorded CDs greater than those of prerecorded cassettes?
c. In what year were the sales of prerecorded cassettes the same as those of prerecorded CDs? Estimate the level of sales in each format at that time.

*Source:* Recording Industry Association of America

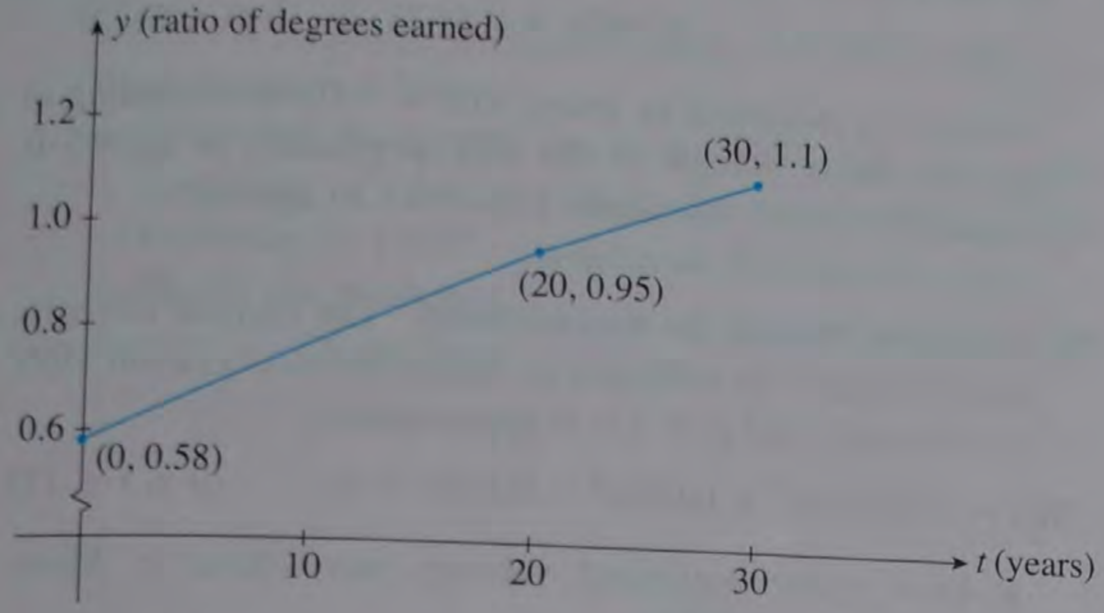
62. **The Gender Gap** The following graph shows the ratio of women's earnings to men's from 1960 through 2000.




a. Write the rule for the function $f$ giving the ratio of women's earnings to men's in year $t$, with $t = 0$ corresponding to 1960.
Hint: The function $f$ is defined piecewise and is linear over each of four subintervals.
b. In what decade(s) was the gender gap expanding? Shrinking?
c. Refer to part (b). How fast was the gender gap (the ratio/year) expanding or shrinking in each of these decades?

*Source:* U.S. Bureau of Labor Statistics

**63. Closing the Gender Gap in Education** The following graph shows the ratio of bachelor's degrees earned by women to men from 1960 through 1990.




a. Write the rule for the function $f$ giving the ratio of bachelor's degrees earned by women to men in year $t$, with $t = 0$ corresponding to 1960.
Hint: The function $f$ is defined piecewise and is linear over each of two subintervals.

b. How fast was the ratio changing in the period from 1960 to 1980? From 1980 to 1990?

c. In what year (approximately) was the number of bachelor's degrees earned by women equal for the first time to that earned by men?

*Source:* Department of Education

**64. Consumption Function** The consumption function in a certain economy is given by the equation

$$C(y) = 0.75y + 6$$

where $C(y)$ is the personal consumption expenditure, $y$ is the disposable personal income, and both $C(y)$ and $y$ are measured in billions of dollars. Find $C(0)$, $C(50)$, and $C(100)$.

**65. Sales Taxes** In a certain state, the sales tax $T$ on the amount of taxable goods is 6% of the value of the goods purchased ($x$), where both $T$ and $x$ are measured in dollars.

a. Express $T$ as a function of $x$.

b. Find $T(200)$ and $T(5.65)$.

**66. Surface Area of a Single-Celled Organism** The surface area $S$ of a single-celled organism may be found by multiplying $4\pi$ times the square of the radius $r$ of the cell. Express $S$ as a function of $r$.

**67. Friend's Rule** Friend's rule, a method for calculating pediatric drug dosages, is based on a child's age. If $a$ denotes the adult dosage (in milligrams) and if $t$ is the age of the child (in years), then the child's dosage is given by

$$D(t) = \frac{2}{25}ta$$

If the adult dose of a substance is 500 mg, how much should a 4-yr-old child receive?

**68. COLAs** Social Security recipients receive an automatic cost-of-living adjustment (COLA) once each year. Their monthly benefit is increased by the amount that consumer prices increased during the preceding year. Suppose that consumer prices increased by 5.3% during the preceding year.

a. Express the adjusted monthly benefit of a Social Security recipient as a function of his or her current monthly benefit.

b. If Harrington's monthly Social Security benefit is now $620, what will be his adjusted monthly benefit?

**69. Broadband Internet Households** The number of U.S. broadband Internet households stood at 20 million at the beginning of 2002 and is projected to grow at the rate of 7.5 million households per year for the next 6 yr.

a. Find a function $f(t)$ giving the projected U.S. broadband Internet households (in millions) in year $t$, where $t = 0$ corresponds to the beginning of 2002.
Hint: The graph of $f$ is a straight line.

b. What is the projected size of U.S. broadband Internet households at the beginning of 2008?

*Source:* Strategy Analytics Inc.

**70. Cost of Renting a Truck** Ace Truck leases its 10-ft box truck at $30/day and $.45/mi, whereas Acme Truck leases a similar truck at $25/day and $.50/mi.

a. Find the daily cost of leasing from each company as a function of the number of miles driven.

b. Sketch the graphs of the two functions on the same set of axes.

c. Which company should a customer rent a truck from for 1 day if she plans to drive at most 70 mi and wishes to minimize her cost?

**71. Linear Depreciation** A new machine was purchased by National Textile for $120,000. For income tax purposes, the machine is depreciated linearly over 10 yr; that is, the book value of the machine decreases at a constant rate, so that at the end of 10 yr the book value is zero.

a. Express the book value of the machine ($V$) as a function of the age, in years, of the machine ($n$).

b. Sketch the graph of the function in part (a).

c. Find the book value of the machine at the end of the sixth year.

d. Find the rate at which the machine is being depreciated each year.

**72. Linear Depreciation** Refer to Exercise 71. An office building worth $1 million when completed in 1986 was depreciated linearly over 50 yr. What was the book value of the building in 2001? What will be the book value in 2005? In 2009? (Assume that the book value of the building will be zero at the end of the 50th year.)

**73. Boyle's Law** As a consequence of Boyle's law, the pressure $P$ of a fixed sample of gas held at a constant temperature is related to the volume $V$ of the gas by the rule

$$P = f(V) = \frac{k}{V}$$

where $k$ is a constant. What is the domain of the function $f$? Sketch the graph of the function $f$.

**74. Poiseuille's Law** According to a law discovered by the 19th-century physician Poiseuille, the velocity (in centimeters/second) of blood $r$ cm from the central axis of an artery is given by

$$v(r) = k(R^2 - r^2)$$

where $k$ is a constant and $R$ is the radius of the artery. Suppose that for a certain artery, $k = 1000$ and $R = 0.2$ so that $v(r) = 1000(0.04 - r^2)$.

**a.** What is the domain of the function $v(r)$?
**b.** Compute $v(0)$, $v(0.1)$, and $v(0.2)$ and interpret your results.
**c.** Sketch the graph of the function $v$ on the interval $[0, 0.2]$.
**d.** What can you say about the velocity of blood as we move away from the central axis toward the artery wall?

**75. Cancer Survivors** The number of living Americans who have had a cancer diagnosis has increased drastically since 1971. In part, this is due to more testing for cancer and better treatment for some cancers. In part, it is because the population is older, and cancer is largely a disease of the elderly. The number of cancer survivors (in millions) between 1975 ($t = 0$) and 2000 ($t = 25$) is approximately

$$N(t) = 0.0031t^2 + 0.16t + 3.6 \qquad (0 \le t \le 25)$$

**a.** How many living Americans had a cancer diagnosis in 1975? In 2000?
**b.** Assuming the trend continued, how many cancer survivors were there in 2005?

*Source:* National Cancer Institute

**76. Worker Efficiency** An efficiency study conducted for Elektra Electronics showed that the number of "Space Commander" walkie-talkies assembled by the average worker $t$ hr after starting work at 8:00 a.m. is given by

$$N(t) = -t^3 + 6t^2 + 15t \qquad (0 \le t \le 4)$$

How many walkie-talkies can an average worker be expected to assemble between 8:00 and 9:00 a.m.? Between 9:00 and 10:00 a.m.?

**77. Politics** Political scientists have discovered the following empirical rule, known as the "cube rule," which gives the relationship between the proportion of seats in the House of Representatives won by Democratic candidates $s(x)$ and the proportion of popular votes $x$ received by the Democratic presidential candidate:

$$s(x) = \frac{x^3}{x^3 + (1 - x)^3} \qquad (0 \le x \le 1)$$

Compute $s(0.6)$ and interpret your result.

**78. Prevalence of Alzheimer's Patients** Based on a study conducted in 1997, the percent of the U.S. population by age afflicted with Alzheimer's disease is given by the function

$$P(x) = 0.0726x^2 + 0.7902x + 4.9623 \qquad (0 \le x \le 25)$$

where $x$ is measured in years, with $x = 0$ corresponding to age 65. What percent of the U.S. population at age 65 is expected to have Alzheimer's disease? At age 90?

*Source:* Alzheimer's Association

**79. Registered Vehicles in Massachusetts** The number of registered vehicles (in millions) in Massachusetts between 1991 ($t = 0$) and 2003 ($t = 12$) is approximately

$$N(t) = -0.0014t^3 + 0.027t^2 - 0.008t + 4.1 \qquad (0 \le t \le 12)$$

**a.** How many registered vehicles were there in Massachusetts in 1991?
**b.** How many registered vehicles where there in Massachusetts in 2003?

*Source:* Mass. Registry of Motor Vehicles

**80. Postal Regulations** In 2006 the postage for first-class mail was raised to 39¢ for the first ounce or fraction thereof and 24¢ for each additional ounce or fraction thereof. Any parcel not exceeding 12 oz may be sent by first-class mail. Letting $x$ denote the weight of a parcel in ounces and $f(x)$ the postage in cents, complete the following description of the "postage function" $f$:

$$f(x) = \begin{cases} 39 & \text{if } 0 < x \le 1 \\ 63 & \text{if } 1 < x \le 2 \\ \cdots & \\ ? & \text{if } 11 < x \le 12 \end{cases}$$

**a.** What is the domain of $f$?
**b.** Sketch the graph of $f$.

**81. Harbor Cleanup** The amount of solids discharged from the MWRA (Massachusetts Water Resources Authority) sewage treatment plant on Deer Island (near Boston Harbor) is given by the function

$$f(t) = \begin{cases} 130 & \text{if } 0 \le t \le 1 \\ -30t + 160 & \text{if } 1 < t \le 2 \\ 100 & \text{if } 2 < t \le 4 \\ -5t^2 + 25t + 80 & \text{if } 4 < t \le 6 \\ 1.25t^2 - 26.25t + 162.5 & \text{if } 6 < t \le 10 \end{cases}$$

where $f(t)$ is measured in tons/day and $t$ is measured in years, with $t = 0$ corresponding to 1989.

**a.** What amount of solids were discharged per day in 1989? In 1992? In 1996?
**b.** Sketch the graph of $f$.

*Source:* Metropolitan District Commission

**82. Rising Median Age** Increased longevity and the aging of the baby boom generation—those born between 1946 and 1965—are the primary reasons for a rising median age. The median age (in years) of the U.S. population from 1900 through 2000 is approximated by the function

$$f(t) = \begin{cases} 1.3t + 22.9 & \text{if } 0 \le t \le 3 \\ -0.7t^2 + 7.2t + 11.5 & \text{if } 3 < t \le 7 \\ 2.6t + 9.4 & \text{if } 7 < t \le 10 \end{cases}$$

where $t$ is measured in decades, with $t = 0$ corresponding to the beginning of 1900.

a. What was the median age of the U.S. population at the beginning of 1900? At the beginning of 1950? At the beginning of 1990?

b. Sketch the graph of $f$.

*Source:* U.S. Census Bureau

**In Exercises 83–86, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.**

**83.** If $a = b$, then $f(a) = f(b)$.

**84.** If $f(a) = f(b)$, then $a = b$.

**85.** If $f$ is a function, then $f(a + b) = f(a) + f(b)$.

**86.** A vertical line must intersect the graph of $y = f(x)$ at exactly one point.

## 2.1 Solutions to Self-Check Exercises

**1. a.** The expression under the radical sign must be nonnegative, so $x + 1 \ge 0$ or $x \ge -1$. Also, $x \ne 0$ because division by zero is not permitted. Therefore, the domain of $f$ is $[-1, 0) \cup (0, \infty)$.

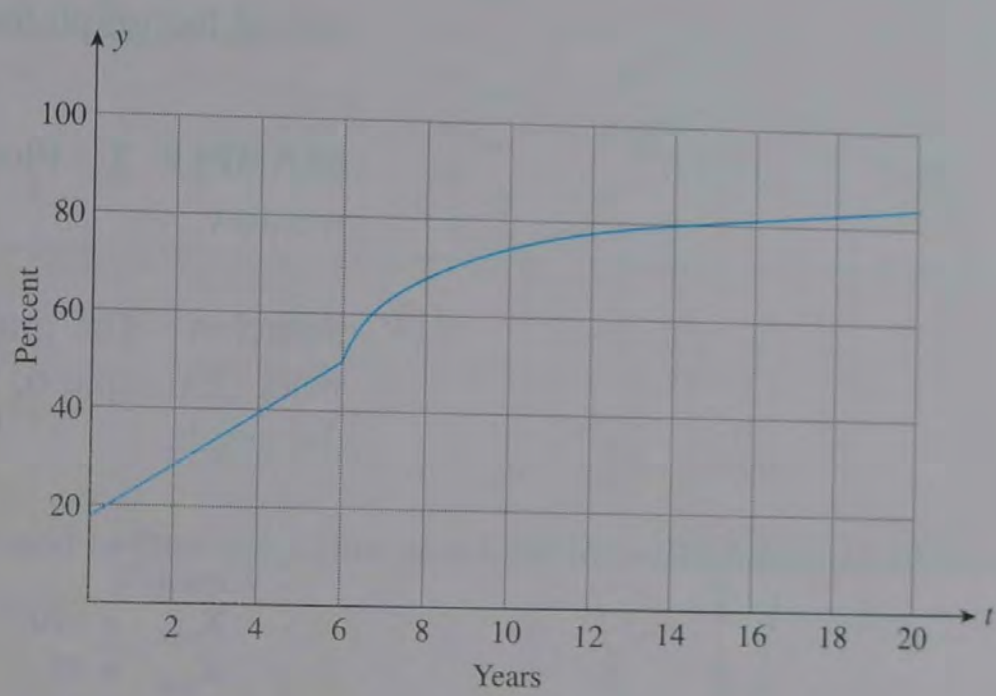
**b.** $f(3) = \dfrac{\sqrt{3+1}}{3} = \dfrac{\sqrt{4}}{3} = \dfrac{2}{3}$

**c.** $f(a + h) = \dfrac{\sqrt{(a+h)+1}}{a+h} = \dfrac{\sqrt{a+h+1}}{a+h}$

**2. a.** For $t$ in the subdomain $[0, 6]$, the rule for $f$ is given by $f(t) = 6t + 17$. The equation $y = 6t + 17$ is a linear equation, so that portion of the graph of $f$ is the line segment joining the points $(0, 17)$ and $(6, 53)$. Next, in the subdomain $(6, 20]$, the rule for $f$ is given by $f(t) = 15.98(t - 6)^{1/4} + 53$. Using a calculator, we construct the following table of values of $f(t)$ for selected values of $t$.

| $t$ | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|
| $f(t)$ | 53 | 72 | 75.6 | 78 | 79.9 | 81.4 | 82.7 | 83.9 |

We have included $t = 6$ in the table, although it does not lie in the subdomain of the function under consideration, in order to help us obtain a better sketch of that portion of the graph of $f$ in the subdomain $(6, 20]$. The graph of $f$ follows:




**b.** The percent of all self-serve gas sales at the beginning of 1978 is found by evaluating $f$ at $t = 4$. Since this point lies in the interval $[0, 6]$, we use the rule $f(t) = 6t + 17$ and find

$$f(4) = 6(4) + 17 = 41$$

giving 41% as the required figure. The percent of all self-serve gas sales at the beginning of 1994 is given by

$$f(20) = 15.98(20 - 6)^{1/4} + 53 \approx 83.9$$

or approximately 83.9%.

**3.** A point $(x, y)$ lies on the graph of the function $f$ if and only if the coordinates satisfy the equation $y = f(x)$. Now,

$$f(4) = \sqrt{2(4) + 1} + 2 = \sqrt{9} + 2 = 5 \ne 6$$

and we conclude that the given point does *not* lie on the graph of $f$.