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1. Functions and Models.

1.1. Four Ways to Represent a Function.

Definition 1.1.1. A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of f, and the **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

A function can be thought of as a machine. For example, a machine that makes pies (e.g. if you put in apples, you get an apple pie; you don't also get a peach pie or a cherry pie).

The four possible ways to represent a function are: verbally (by a description in words), numerically (by a table of values), visually (by a graph), and algebraically (by an explicit formula).

Example 1.1.1 (Using function notation)**.** If $f(x) = x^2 + 5x$, evaluate the quotient $\frac{f(a+h)-f(a)}{h}$.

Solution. $\frac{f(a+h)-f(a)}{h} = \frac{[(a+h)^2+5(a+h)]-[a^2+5a]}{h} = \frac{[a^2+2ah+h^2+5a+5h]-[a^2+5a]}{h} = \frac{2ah+h^2+5h}{h} = 2a + h + 5.$

Example 1.1.2 (Domain)**.** Identify the domain of the following functions:

(a) $f(x) = 1 - x^7 + 12x^2$.
(b) $g(x) = \sqrt{x+2}$.
(c) $h(x) = \frac{1}{x^2-x}$.

Solution.

(a) The domain of a polynomial is all real numbers, so the domain of f is $(-\infty, \infty)$ or $\mathbb{R}$.
(b) The radicand can't be negative, so $x + 2 \geq 0$ implies that the domain of g is $[-2, \infty)$.
(c) The denominator can't be zero, so $x^2-x \neq 0$ implies that the domain of h is $(-\infty, 0)\cup(0, 1)\cup(1, \infty)$.

Proposition 1.1.1 (Vertical Line Test)**.** *A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects it more than once.*

Example 1.1.3. Illustrate the use of the Vertical Line Test by comparing the graphs of several functions and non-functions.

A **piecewise defined function** is one that is defined by different formulas over different parts of its domain.

Example 1.1.4. Consider the piecewise function $f(x) = \begin{cases} 4-x, & x < 1; \\ x^2, & 1 \leq x < 2. \end{cases}$ Find $f(0)$, $f(1)$, $f(3)$.

Solution. $f(0) = 4 - 0 = 4$, $f(1) = 1^2 = 1$, $f(3)$ is not defined because 3 is not in the domain of f.

It is straightforward to graph the function in the example above. Conversely, given a graph of a simple piecewise function, we should be able to recover its explicit formula.

Example 1.1.5. Sketch a simple piecewise function and use its graph to recover its formula.

Definition 1.1.2 (Function symmetry)**.**

- A function f is **even** if $f(-x) = f(x)$ for every x in its domain. If a function is even, its graph is symmetric about the x-axis.
- A function f is **odd** if $f(-x) = -f(x)$ for every x in its domain. If a function is odd, its graph is symmetric about the y-axis.

Example 1.1.6. Determine whether each of the following functions is even, odd, or neither:
(a) $f(x) = x^5 - 7x$.
(b) $g(x) = \frac{6+x^4}{x^2}$.
(c) $h(x) = 1 + x^3 + x^2$.

Solution. (a) odd, (b) even, (c) neither. The symmetry of polynomials is easy to determine!

Definition 1.1.3 (Increasing and decreasing functions)**.**
- A function f is **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$. Graphically, an increasing function "rises" from left to right.
- A function f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$. Graphically, a decreasing function "falls" from left to right.

1.2. **Mathematical Models: A Catalog of Essential Functions.**

Definition 1.2.1. A function P is called a **polynomial** if it has the form $P(x) = a_n x^n + \cdots + a_1 x + a_0$, where n is a nonnegative integer and the numbers $a_0, a_1, \ldots, a_n$ are constants called the **coefficients** of the polynomial. If the leading coefficient a_n is nonzero, then the **degree** of the polynomial is n. The domain of any polynomial is $\mathbb{R}$.

Example 1.2.1. Sketch various simple polynomials, e.g. linear, quadratic, cubic, etc.

A **linear function** is one whose graph is a straight line. Linear functions are commonly written in **slope-intercept form**. For example, if y is a linear function of x, we may write the relationship between the variables as $y = mx + b$, where m is the slope of the line and b is the y-intercept.

The slope of a linear function represents a rate of change of the dependent variable with respect to the independent variable. For example, in a real-life linear model, slope could represent the rate of change of temperature with respect to elevation.

The y-intercept generally represents some sort of initial state of the model. This is especially true when the independent variable represents time, such as in the case of population models or depreciation models.

A polynomial of degree 2 is called a **quadratic function**. Its graph is a parabola, so it can be used to model physical scenarios such as the height of an object with respect to time after it is thrown up in the air.

Definition 1.2.2. A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**.

Example 1.2.2. The general power function $f(x) = x^a$ can take on many forms:
- If $a = n$, where n is a positive integer, the function f is a simple polynomial.
- If $a = 1/n$, where n is a positive integer, then $f(x) = x^{1/n} = \sqrt[n]{x}$, is a **root function**.
- If $a = -1$, then $f(x) = x^{-1} = 1/x$ is the **reciprocal function**.

Definition 1.2.3. A **rational function** f is a ratio of two polynomial functions, e.g. $f(x) = P(x)/Q(x)$.

Example 1.2.3. A simple rational function is the reciprocal function $f(x) = 1/x$. Rational functions arise in physics and chemistry, such as in the case of Boyle's Law: when temperature is constant, the volume V of a gas is inversely proportional to the pressure P; in symbols, $V = C/P$.

Definition 1.2.4. A function f is called an **algebraic function** if it can be constructed using **algebraic operations**, i.e. the four arithmetic operations $+$, $-$, $\times$, $\div$, along with the taking of roots.

Example 1.2.4. An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity v is given by $m = f(v) = \frac{m_0}{1-v^2/c^2}$, where m_0 is the rest mass of the particle and $c = 3.0 \times 10^5$ km/sec is the speed of light in a vacuum.

Trigonometric functions such as sine, cosine, and tangent can also be used in mathematical modeling. The periodic nature of these functions makes them ideal representatives of repetitive phenomena such as vibrating springs and sound waves.

Exponential and logarithmic functions will be reviewed in Sections 1.5 and 1.6 respectively.

Remark 1.2.1. Choosing an appropriate mathematical model is the most difficult part of many practical problems, especially those in which the goal is to make accurate predictions about the future. Graphical tools such as scatter plots can be helpful for visualizing trends in the available data, and in turn these trends can hint at choosing one particular type of model over another. Many calculators offer the capability to analyze scatter plots and produce linear or quadratic regressions (i.e. "best-fit" lines).

1.3. **New Functions from Old Functions.**

Definition 1.3.1. There are ten basic transformations that can be used to modify functions. We can often interpret a seemingly complex function as some variant of a simple "parent function," arrived at by applying one or more of these transformations.

- Vertical and Horizontal Shifts: suppose $c > 0$.
  1. $y = f(x) + c$ is a vertical shift of $y = f(x)$ a distance of c units upward.
  2. $y = f(x) - c$ is a vertical shift of $y = f(x)$ a distance of c units downward.
  3. $y = f(x + c)$ is a horizontal shift of $y = f(x)$ a distance of c units to the left.
  4. $y = f(x - c)$ is a horizontal shift of $y = f(x)$ a distance of c units to the right.
- Vertical and Horizontal Stretching: suppose $c > 1$.
  5. $y = cf(x)$ is a vertical stretch of $y = f(x)$ by a factor of c.
  6. $y = (1/c)f(x)$ is a vertical shrink of $y = f(x)$ by a factor of c.
  7. $y = f(cx)$ is a horizontal shrink of $y = f(x)$ by a factor of c.
  8. $y = f(x/c)$ is a horizontal stretch of $y = f(x)$ by a factor of c.
- Reflecting:
  9. $y = -f(x)$ is a reflection of $y = f(x)$ across the x-axis.
  10. $y = f(-x)$ is a reflection of $y = f(x)$ across the y-axis.

Example 1.3.1. Illustrate the 10 transformations on a simple function such as $y = x^2$ or $y = \sin x$

Definition 1.3.2. To form an **arithmetic combination** of two functions, f and g, we simply apply one of the four arithmetic operations. The rules are exactly as expected:

- Sums: $(f + g)(x) = f(x) + g(x)$.
- Differences: $(f - g)(x) = f(x) - g(x)$.
- Products: $(fg)(x) = f(x)g(x)$.
- Quotients: $(f/g)(x) = f(x)/g(x)$, provided $g(x) \neq 0$.

Example 1.3.2. Let $f(x) = x^2 - 1$ and $g(x) = 2x + 1$. Find (a) $f + g$, (b) $f - g$, (c) fg, (d) f/g, and their domains.

Solution. The functions $f + g$, $f - g$, and fg each have domain $\mathbb{R}$. The domain of f/g is $2x + 1 \neq 0$; that is, $(-\infty, -1/2) \cup (-1/2, \infty)$. The rules for each function are:

(a) $(f + g)(x) = f(x) + g(x) = (x^2 - 1) + (2x + 1) = x^2 + 2x$.
(b) $(f - g)(x) = f(x) - g(x) = (x^2 - 1) - (2x + 1) = x^2 - 2x - 2$.
(c) $(f * g)(x) = f(x)g(x) = (x^2 - 1)(2x + 1) = 2x^3 + x^2 - 2x - 1$.
(d) $(f/g)(x) = f(x)/g(x) = (x^2 - 1)/(2x + 1)$.

Definition 1.3.3 (Function composition)**.** Given two functions, f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by $(f \circ g)(x) = f(g(x))$ for all x in the domain of g such that $g(x)$ is in the domain of f. That is, $(f \circ g)(x)$ is defined if and only if $g(x)$ and $f(g(x))$ are defined.

Example 1.3.3. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. Find the domain and formulas for $f \circ g$ and of $g \circ f$.

Solution. First note that the domain of f is $(-\infty, \infty)$ and the domain of g is $[0, \infty)$. It follows that the domain of $f \circ g$ is $[0, \infty)$ and the domain of $g \circ f$ is $(-\infty, \infty)$. It is easy to see that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, but the are not the same function because each has a different domain.

Example 1.3.4. Use the table below to find: (a) $f(g(0))$, (b) $f(f(2))$, (c) $(g \circ f)(1)$.

x	-2	-1	0	1	2
$f(x)$	6	5	0	-2	-1
$g(x)$	-3	-1	1	3	5

Solution. (a) $f(g(0)) = f(1) = -2$, (b) $f(f(2)) = f(-1) = 5$, (c) $(g \circ f)(1) = g(f(1)) = g(-2) = -3$.

1.4. **Graphing Calculators and Computers.**

This section is omitted. It essentially discusses some ways in which graphing calculators (and computers) can be useful in the study of calculus. Students in Math 121 are expected to have prior experience with graphing calculators (esp. the TI-83 or TI-84 series), and those that do not may arrange a meeting with the instructor in order to be brought up to speed.

1.5. **Exponential Functions.**

Definition 1.5.1. An **exponential function** has the form $f(x) = a^x$, where a is a positive constant.

Example 1.5.1. Graph and compare the function $f(x) = a^x$ for: (a) $0 < a < 1$, (b) $a = 1$, (c) $a > 1$.

Exponential functions are ideal for modeling populations that grow very rapidly. For example, if a bacterial culture initially contains P_0 bacteria and is known to double in size every half-hour, its growth can be modeled by the exponential function $P(t) = P_0\, 2^t$, where $t \geq 0$ is the elapsed time in half-hours.

Remark 1.5.1.

- The domain of $f(x) = a^x$ is $\mathbb{R}$ for any $a > 0$.
- If $x = n$ is a positive integer, then: $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}}$.
- If $x = 0$, then $a^0 = 1$.
- If $x = -n$, where n is a positive integer, then $a^{-n} = \dfrac{1}{a^n}$.
- If p and q are integers, with $q > 0$, then $a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.

Example 1.5.2. Simplify: (a) $64^{-2/3}$, (b) $(-16)^{3/4}$.

Solution.

(a) $64^{-2/3} = \frac{1}{64^{2/3}} = \frac{1}{(\sqrt[3]{64})^2} = \frac{1}{(4)^2} = \frac{1}{16}$.

(b) $(-16)^{3/4} = \left(\sqrt[4]{16}\right)^3 \cdot (-1)^{3/4} = (2)^3 \cdot (-1)^{3/4} = 8 \cdot (-1)^{3/4}$.

Example 1.5.3. What is the meaning of a^x when x is an irrational number? For example, what is $3^{\sqrt{2}}$?

Proposition 1.5.1 (Laws of Exponents)**.** *If a and b are positive numbers and $x, y \in \mathbb{R}$, then:*

$$\mathbf{1.}\ a^{x+y} = a^x a^y,\ \ \mathbf{2.}\ a^{x-y} = \frac{a^x}{a^y},\ \ \mathbf{3.}\ (a^x)^y = a^x y,\ \ \mathbf{4.}\ (ab)^x = a^x b^x.$$

Example 1.5.4. Use the laws of exponents to rewrite and simply the expression $\frac{\left(6y^3\right)^4}{2y^5}$.

Solution. It is straightforward to determine that: $\frac{\left(6y^3\right)^4}{2y^5} = \frac{6^4\left(y^3\right)^4}{2y^5} = \frac{1296y^{12}}{2y^5} = 648y^7$.

Definition 1.5.2. The **natural exponential function**, $f(x) = e^x$, is ubiquitous in mathematics. It is characterized by the **natural base** $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828$, which is a transcendental number (not a root of a nonzero polynomial with integer coefficients).

Example 1.5.5. The function $f(x) = e^x$ has the property that the slope of its tangent line at the point $(0, 1)$ is exactly 1. On the other hand, the functions $g(x) = 2^x$ and $h(x) = 3^x$ have tangential slopes of about 0.7 and 1.1 respectively at the point $(0, 1)$. Graph all three curves and note how the former lies between the latter two.

1.6. **Inverse Functions and Logarithms.**

Definition 1.6.1. A function f is called a **one-to-one function** if it never takes on the same value twice; that is: $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Proposition 1.6.1 (Horizontal Line Test)**.** *A function is one-to-one if and only if no horizontal line intersects its graph more than once.*

Example 1.6.1. Illustrate the use of the HLT by comparing the graphs of functions that are/aren't one-to-one.

Definition 1.6.2. Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by $f^{-1}(y) = x \iff f(x) = y$ for any $y \in B$.

We use the notation $f : A \to B$ to denote a function with domain A and range B. Thus, if $f : A \to B$ is a one-to-one function, its inverse is written $f^{-1} : B \to A$.

Remark 1.6.1.
- The function $f : A \to B$ is one-to-one if $f(x) = f(y)$ implies $x = y$ for any $x, y \in A$.
- The graph of f^{-1} is obtained by reflecting the graph of f across the line $y = x$.

Example 1.6.2. It follows from the definition that if the functions $f : A \to B$ and $f^{-1} : B \to A$ are inverses, then $f^{-1}(f(x)) = x$ for every $x \in A$ and $f(f^{-1}(x)) = x$ for every $x \in B$. For example, this property is easily verified in the case of the functions $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$.

In view of the preceding example, and the comment made in Section 1.1 about a function acting like a machine, we can think of an inverse function as a machine that works in reverse, i.e. it disassembles or undoes the result of the original function.

Definition 1.6.3. Let a be a positive constant. It is easy to verify that the exponential function $f(x) = a^x$ is one-to-one, i.e. its inverse exists. The **logarithmic function with base *a***, denoted $\log_a$, is uniquely defined by the rule $\log_a x = y \iff a^y = x$.

Remark 1.6.2.
- The domain of $f(x) = \log_a x$ is $(0, \infty)$ for any $a > 0$. The range is $\mathbb{R}$.

- $\log_a(a^x) = x$ for every $x \in \mathbb{R}$.
- $a^{\log_a x} = x$ for every $x > 0$.

Example 1.6.3. Graphically illustrate the inverse relationship between exponential and logarithmic functions.

Proposition 1.6.2 (Laws of Logarithms). *If a, x, and y are positive numbers and $r \in \mathbb{R}$, then:*

$$\textbf{1. } \log_a(xy) = \log_a x + \log_a y, \quad \textbf{2. } \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y, \quad \textbf{3. } \log_a(x^r) = r\log_a x.$$

Example 1.6.4. Simplify $\log_3 36 - \log_3 4$.

Solution. $\log_3 36 - \log_3 4 = \log_3\left(\frac{36}{4}\right) = \log_3 9 = \log_3 3^2 = 2$.

Definition 1.6.4. There are two "special" logarithms that each have a unique name and notation:

1. The logarithm with base 10 is called the **common logarithm**, and is denoted by $\log x$.
2. The logarithm with base e is called the **natural logarithm**, and is denoted by $\ln x$.

Remark 1.6.3.

- By definition, $\ln x = y \iff e^y = x$.
- $\ln(e^x) = x$ for every $x \in \mathbb{R}$.
- $e^{\ln x} = x$ for every $x > 0$.
- From the inverse relationship between the exponential and logarithmic functions it follows that $a^b = e^{\ln(a^b)} = e^{b\ln a}$ for any numbers a and b, which is a useful tool in some differentiation problems.

Example 1.6.5. Find x if $\ln(x-5) = 3$.

Solution. $\ln(x-5) = 3 \iff x - 5 = e^3 \iff x = 5 + e^3$.

Example 1.6.6. Solve the equation $e^{7+2x} - 6 = 5$.

Solution. $e^{7+2x} - 6 = 5 \iff e^{7+2x} = 11 \iff 7 + 2x = \ln 11 \iff x = (-7 + \ln 11)/2$.

Example 1.6.7. Express $2\ln x - 3\ln y$ as a single logarithm.

Solution. $2\ln x - 3\ln y = \ln x^2 - \ln y^3 = \ln\left(x^2/y^3\right)$.

Proposition 1.6.3 (Change of Base Formula). *For any positive a ($a \neq 1$), we have:* $\log_a x = \ln x/\ln a = \log x/\log a$.

1.7. **Parametric Curves.**

Definition 1.7.1. Supppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations $x = f(t)$ and $y = g(t)$ (called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve**.

Example 1.7.1. Consider the parametric equations $x = t^2 - 2t$ and $y = t + 1$.

(a) Plot points to sketch a graph of the curve.
(b) Eliminate the parameter to find a Cartesian equation of the curve.

Solution.

(a) Plotting the points (x, y) corresponding to $t = -2, -1, 0, 1, 2, 3, 4$, we sketch a horizontal parabola.
(b) First, we solve $y = t + 1$ for t. We then substitute $t = y - 1$ for t in $x = t^2 - 2t$; that is, $x = (y-1)^2 - 2(y-1) = y^2 - 2y + 1 - 2y + 2 = y^2 - 4y + 3$, which supports the result of part (a).

Example 1.7.2. Identify the curve represented by $x = \cos t$ and $y = \sin t$, where $0 \le t \le 2\pi$.

Solution. The ordered pairs $(x, y) = (\cos t, \sin t)$ correspond to points on the graph of a circle. This is easily verified by eliminating the parameter t; in particular, observe that $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. As t increases from 0 to 2π, the point (x, y) moves once around the unit circle in the counterclockwise direction starting from the point $(1, 0)$.

Example 1.7.3. What curve is represented by $x = \cos 2t$ and $y = \sin 2t$, where $0 \le t \le 2\pi$?

Solution. As in the previous example, $x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1$. In this case, however, as t increases from 0 to 2π the point (x, y) starts at $(1, 0)$ and moves twice around the unit circle in the counterclockwise direction.

Example 1.7.4. Find parametric equations for the circle with center (h, k) and radius r?

Solution. We begin with parametric equations for the unit circle, i.e. $x = \cos t$ and $y = \sin t$ for $0 \le t \le 2\pi$. To scale the radius of the circle to r, we multiply both x and y by r. To center the circle at the point (h, k), we add h to x and k to y. Hence, the circle centered at (h, k) with radius r can be represented by the parametric equations $x = h + r \cos t$ and $y = k + r \sin t$, where $0 \le t \le 2\pi$.

Most graphing calculators and computer graphing programs can be used to graph curves defined by parametric equations. In fact, it's instructive to watch a parametric curve being drawn by such a digital graphing tool because the points are plotted in order as the corresponding parameter values increase.

Example 1.7.5. Use a calculator to graph the curve $x = y^4 - 3y^2$.

Solution. If $t = y$ is a parameter, then we have the parametric equations $x = t^4 - 3t^2$ and $y = t$.

The example above illustrates the importance of choosing an appropriate range and increment-value for t. If the range is too small, then the true shape of the curve may not be fully revealed. If the increment-value is too large, the curve may have sharp or square edges. If the range is too large, or the increment-value is too small, the curve will be plotted very slowly as the calculator attempts to display a high level of detail.

Example 1.7.6. Use a calculator to plot complicated curves, such as those illustrated in Exercise 1.7.33.

2. Limits and Derivatives.

2.1. **The Tangent and Velocity Problems.**

Definition 2.1.1. A **tangent line** at a point P on a curve is a line that passes through P and approximates the curve as perfectly as possible near P.

Consider a nonlinear curve such as a parabola. The more we zoom in on a particular point P on the curve, the more the curve looks like a straight line passing through P.

To approximate the slope of the tangent line at P, we choose a point Q on the curve and calculate the slope of the **secant line** passing through P and Q. The closer Q is to P, the better the slope of the secant line approximates the slope of the tangent line at P. For this reason, we say that the slope of the tangent line to the curve at P is the limit as Q approaches P of the slope of the secant line through P and Q.

Example 2.1.1. Use secant lines to estimate the slope of the tangent line to $f(x) = x^2$ at $P = (1, 1)$.

Solution. We choose x-values closer and closer to $x = 1$, and determine the slope, m_{PQ}, of the secant line through P and $Q = (x, f(x))$ as follows:

$(x, f(x))$	m_{PQ}
$(2, 4)$	$\frac{4-1}{2-1} = 3$
$(1.5, 2.25)$	$\frac{2.25-1}{1.5-1} = 2.5$
$(1.1, 1.21)$	$\frac{1.21-1}{1.1-1} = 2.1$
$(1.01, 1.0201)$	$\frac{1.0201-1}{1.01-1} = 2.01$

It appears that the slope of the tangent line to the graph of $f(x) = x^2$ at the point $(1, 1)$ is $m = 2$. Hence, the equation of the tangent line is $y - 1 = 2(x - 1)$ or $y = 2x - 1$. This is easily verified graphically.

The Velocity Problem. We commonly understand velocity as the rate of change of position with respect to time, but what is the meaning of the **instantaneous velocity** at any particular point in time?

Example 2.1.2. A ball is dropped from 500 feet. Find its (instantaneous) velocity after 5 seconds.

Solution. After t seconds, the ball has dropped a distance of $s(t) = 32t^2$ feet. To approximate the velocity of the ball at $t = 5$ seconds, we can use this formula to calculate the average velocity of the ball over smaller and smaller increments of time starting at $t = 5$, as shown below:

Time Interval	Average Velocity (ft/sec)
$5 \le t \le 6$	$\frac{s(6)-s(5)}{6-5} = 352$
$5 \le t \le 5.1$	$\frac{s(5.1)-s(5)}{5.1-5} = 323.2$
$5 \le t \le 5.01$	$\frac{s(5.01)-s(5)}{5.01-5} = 320.32$
$5 \le t \le 5.001$	$\frac{s(5.001)-s(5)}{5.001-5} = 320.032$
$5 \le t \le 5.0001$	$\frac{s(5.0001)-s(5)}{5.0001-5} = 320.0032$

Note that as the time interval shrinks, the average velocity approaches 320 ft/sec. Thus, we estimate the instantaneous velocity of the falling ball after 5 seconds to be 320 ft/sec.

A Look Ahead... Notice that both of the preceding examples involve finding the limiting value of the quotient $\frac{f(x)-f(a)}{x-a}$ as x gets closer and closer to a. When this **limit** exists, it is called the **derivative of** $\boldsymbol{f(x)}$ **at** $\boldsymbol{x = a}$ and we denote it by $f'(a)$. This concept will be studied in detail, beginning in Section 2.6.

2.2. **The Limit of a Function.**

Definition 2.2.1. We write $\lim_{x\to a} f(x) = L$ and say, "the limit of $f(x)$, as x approaches a, equals L," if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Example 2.2.1. Let $f(x) = \frac{x^2-9}{x-3}$. What is $f(3)$? What is $\lim_{x\to 3} f(x)$?

Solution. It is easy to see that $f(3)$ is undefined. Next, consider the following tables of values of $f(x)$ for values of x close, but not equal, to 3.

x (from the left)	f(x)
2	5
2.9	5.9
2.99	5.99
2.999	5.999

x (from the right)	$f(x)$
4	7
3.1	6.1
3.01	6.01
3.001	6.001

It appears that $\lim_{x\to 3} f(x) = 6$. Indeed, this can be verified graphically, since the graph of $f(x)$ is identical to the graph of the linear function $g(x) = x + 3$ except at the point $x = 3$.

Example 2.2.2. Explore the limiting behavior of a function that exhibits the various common types of discontinuity, e.g. jumps, holes, one- or two-sided asymptotes.

Example 2.2.3. Estimate $\lim_{x\to 0} \frac{1-\cos x}{x}$.

Solution. From the following table of values, it appears that $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

x	$\frac{1-\cos x}{x}$
± 0.1	± 0.04995
± 0.01	± 0.00410
± 0.001	± 0.00050
± 0.0001	± 0.00005

Indeed, this can be verified graphically, or by using L'Hôpital's Rule as seen in Section 4.5.

Example 2.2.4. Estimate $\lim_{x\to 0} \sin \frac{\pi}{x}$.

Solution. Consider the following two tables of values of $f(x)$ for values of x close to 0.

x	f(x)
± 0.1	0
± 0.01	0
± 0.001	0
± 0.0001	0

x	$f(x)$
± 0.4	1
± 0.016	1
± 0.00064	1
± 0.000256	1

From the table on the left we conclude that $\lim_{x\to 0} \sin \frac{\pi}{x} = 0$, but the table on the right suggests that $\lim_{x\to 0} \sin \frac{\pi}{x} = 1$. Which answer is correct? In fact, neither. Since the value of $\lim_{x\to 0} \sin \frac{\pi}{x}$ is dependent on how x is chosen to approach 0, this limit does not exist.

Definition 2.2.2. We write $\lim_{x\to a^-} f(x) = L$, and say the **left-hand limit of $f(x)$ as x approaches a** is equal to L, if we can make the values of $f(x)$ arbitrarily close to L by choosing x to be less than, and sufficiently close to, a. The **right-hand limit of $f(x)$ as x approaches a** is defined similarly, and is denoted by $\lim_{x\to a^+} f(x) = L$.

Example 2.2.5. Explore the one-sided limits of the functions from Example 2.2.2.

Proposition 2.2.1. *For any function f:* $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L$ *and* $\lim_{x\to a^+} f(x) = L$.

2.3. Calculating Limits Using the Limit Laws.

Proposition 2.3.1 (Limit Laws). *If c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then:*

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
3. $\lim_{x\to a}[cf(x)] = c \lim_{x\to a} f(x)$.
4. $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)] \cdot [\lim_{x\to a} g(x)]$.
5. $\lim_{x\to a}[f(x)/g(x)] = [\lim_{x\to a} f(x)] \,/\, [\lim_{x\to a} g(x)]$, *provided that* $\lim_{x\to a} g(x) \neq 0$.
6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, *where n is a positive integer.*
7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, *where n is a positive integer.*

Some special additional laws are:

8. $\lim_{x\to a} c = c$.
9. $\lim_{x\to a} x = a$.
10. $\lim_{x\to a} x^n = a^n$, *where n is a positive integer (special case of Law 6).*
11. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, *where n is a positive integer (special case of Law 7).*

Example 2.3.1. Calculate $\lim_{x\to 5} 4x^3 - 10x^2 - 225$.

Solution. By Limit Laws 2, 3, 8, 9, 10, we find that $\lim_{x\to 5} 4x^3 - 10x^2 - 225 = 25$.

Example 2.3.2. Calculate $\lim_{x\to -2} \frac{x^3+5x+3}{x-7}$.

Solution. By Limit Laws 1, 2, 3, 5, 8, 9, 10, we find that $\lim_{x\to -2} \frac{x^3+5x+3}{x-7} = \frac{5}{3}$.

The following proposition shows that Examples 2.3.1 and 2.3.2 can actually be solved trivially.

Proposition 2.3.2 (Direct Substitution Property). *If f is a polynomial or a rational function, and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$. This is a consequence of* **continuity**, *as seen in Section 2.4.*

Example 2.3.3. Calculate $\lim_{x\to 2} \frac{2-x}{4-x^2}$.

Solution. In this case, Proposition 2.3.2 does not apply because 2 is not in the domain of $\frac{2-x}{4-x^2}$. Instead, we apply the following algebraic simplification: $\lim_{x\to 2} \frac{2-x}{4-x^2} = \lim_{x\to 2} \frac{2-x}{(2-x)(2+x)} = \lim_{x\to 2} \frac{1}{2+x} = \frac{1}{4}$.

The following proposition justifies the above solution to Example 2.3.3.

Proposition 2.3.3. *If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limits exist.*

Example 2.3.4. Calculate $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} 2x+1, & \text{if } x \neq 3 \\ 4, & \text{if } x = 3 \end{cases}$.

Solution. By Proposition 2.3.3, it suffices to consider the function $g(x) = 2x + 1$. Then by Proposition 2.3.2, since g is a polynomial, we conclude that $\lim_{x\to 3} f(x) = \lim_{x\to 3} g(x) = g(3) = 2(3) + 1 = 7$.

Example 2.3.5. Calculate $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} -x+4, & \text{if } x < 3 \\ x, & \text{if } x \geq 3 \end{cases}$.

Solution. The one-sided limits of f are easily seen to be $\lim_{x\to 3^-} f(x) = 1$ and $\lim_{x\to 3^+} f(x) = 3$. Thus, invoking Proposition 2.2.1, we conclude that $\lim_{x\to 3} f(x)$ does not exist.

Example 2.3.6. Calculate $\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2}$.

Solution. Limit Law 5 does not apply, since 0 is not in the domain of the function. Instead, we will employ the method of **rationalizing the numerator** to determine this limit; that is,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2} &= \lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2} \cdot \frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2} \\
&= \lim_{x\to 0} \frac{(x^2+4)-4}{x^2\left(\sqrt{x^2+4}+2\right)} \\
&= \lim_{x\to 0} \frac{x^2}{x^2\left(\sqrt{x^2+4}+2\right)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x^2+4}+2} \\
&= \frac{1}{\sqrt{\lim_{x\to 0}(x^2+4)}+2} \\
&= \frac{1}{\sqrt{4}+2} \\
&= \frac{1}{2+2} \\
&= \frac{1}{4}.
\end{aligned}$$

Example 2.3.7. Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Solution. As in the case of Example 2.3.5, we compute:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{|x|}{x} &= \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1, \\
\lim_{x\to 0^-} \frac{|x|}{x} &= \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1.
\end{aligned}$$

Thus, again by Proposition 2.2.1, we conclude that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Theorem 2.3.1. *If $f(x) \leq g(x)$ when x is near a (except possible at a) and the limits of f and g both exist as x approaches a, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.*

Theorem 2.3.2 (Squeeze Theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possible at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

Example 2.3.8. Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

Solution. First, recall that $-1 \leq \sin x \leq 1$ for any x. Likewise, $-1 \leq \sin\frac{1}{x} \leq 1$ for any x. Since any non-strict inequality remains true when all sides are multiplied by a non-negative number, and we know that $x^2 \geq 0$ for all x, it follows that $-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$. Now, notice that $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$. Thus, by the Squeeze Theorem, we conclude that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

2.4. **Continuity.**

Definition 2.4.1. A function f is **continuous at a number a** if $\lim_{x\to a} f(x) = f(a)$. If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at a** (or f has a **discontinuity** at a) if f is not continuous at a.

Remark 2.4.1. Notice that Definition 2.4.1 implicitly requires three things if f is continuous at a:

$$\textbf{1. } f(a) \text{ exists}, \quad \textbf{2. } \lim_{x\to a} f(x) \text{ exists}, \quad \textbf{3. } \lim_{x\to a} f(x) = f(a).$$

Geometrically, we can think of a function that is continuous at every number in an interval as a function whose graph has no breaks in it, i.e. the graph can be drawn without removing pen from paper.

Example 2.4.1. Explore the various types of discontinuity illustrated by the graph in Example 2.2.2. In each case, discuss which of the three properties from Remark 2.4.1 are violated.

Example 2.4.2. Where are each of the following functions discontinuous?

(a) $f(x) = \frac{x^2-x-2}{x-2}$.

(b) $f(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$

(c) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2}; & \text{if } x \neq 2 \\ 1, & \text{if } x = 2. \end{cases}$

(d) $f(x) = [[x]]$ (the "greatest integer" function).

Solution.

(a) Since $f(2)$ is not defined, f is discontinuous at 2.

(b) Notice that $f(0) = 1$ is defined, but $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2}$ does not exist (because the function has an asymptote at $x = 0$), so f is discontinuous at 0.

(c) Notice that $f(2) = 1$ is defined and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3$$

exists, but $\lim_{x\to 2} f(x) \neq f(2)$, so f is discontinuous at 2.

(d) The function $f(x) = [[x]]$ is defined by: $[[x]]$ *is the largest integer that is less than or equal to* x. For example, $[[4]] = 4$, $[[4.8]] = 4$, $[[\pi]] = 3$, $\left[\left[\sqrt{2}\right]\right] = 1$, and $[[-1/2]] = -1$. It has discontinuuities at all of the integers, because $\lim_{x\to n}[[x]]$ does not exist if n is an integer.

Definition 2.4.2. A function f is **continuous from the right at a number a** if $\lim_{x\to a^+} f(x) = f(a)$, and f is **continuous from the left at a** if $\lim_{x\to a^-} f(x) = f(a)$.

Example 2.4.3. At each integer n, the function $f(x) = [[x]]$ is continuous from the right but discontinuous from the left, because

$$\lim_{x\to n^+} f(x) = \lim_{x\to n^+} [[x]] = n = f(n)$$

but

$$\lim_{x\to n^-} f(x) = \lim_{x\to n^-} [[x]] = n - 1 \neq f(n).$$

Definition 2.4.3. A function f is **continuous on an interval** if it is continuous at every number in the interval. If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continous from the left.*

Theorem 2.4.1. *If f and g are continuous at a, and c is a constant, then the following functions are also continuous at a:*

$$\textbf{1. } f+g, \quad \textbf{2. } f-g, \quad \textbf{3. } cf, \quad \textbf{4. } fg, \quad \textbf{5. } f/g \quad (\textit{if } g(a) \neq 0).$$

Theorem 2.4.2. *The following types of functions are continuous at every number in their domains: polynomial functions, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions.*

Example 2.4.4. Where is the function $f(x) = \frac{\ln x + 7}{x^2 - 9}$ continuous?

Solution. By Theorem 2.4.2, we know that the logarithmic function $\ln x$ is continuous on $(0, \infty)$, while the constant function 7 and the polynomial function $x^2 - 9$ are both continuous on $\mathbb{R}$. Thus, by parts 1 and 5 of Theorem 2.4.1, it follows that f is continuous at all positive numbers x such that $x^2 - 9 = 0$. That is, f is continuous on $(0, 3) \cup (3, \infty)$.

Theorem 2.4.3. *If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words, $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$.*

Example 2.4.5. Find $\lim_{x \to 1} \ln\left(\frac{1-x}{1-\sqrt{x}}\right)$.

Solution. First, we calculate

$$\begin{aligned}
\lim_{x \to 1} \frac{1-x}{1-\sqrt{x}} &= \lim_{x \to 1} \frac{1-x}{1-\sqrt{x}} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} \\
&= \lim_{x \to 1} \frac{(1-x)\left(1+\sqrt{x}\right)}{1-x} \\
&= \lim_{x \to 1} \left(1+\sqrt{x}\right) \\
&= 1 + \sqrt{\lim_{x \to 1} x} \\
&= 2.
\end{aligned}$$

Since 2 is in the domain of the function $\ln x$, it follows from Theorem 2.4.3 that $\lim_{x \to 1} \ln\left(\frac{1-x}{1-\sqrt{x}}\right) = \ln(2)$.

Theorem 2.4.4. *If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$, given by $(f \circ g)(x) = f(g(x))$ is continuous at a.*

Example 2.4.6. Where are the following functions continuous? (a) $f(x) = \cos x^2$, (b) $g(x) = \ln\left(\sin^2 x\right)$.

Solution.

(a) Since $\cos x$ and x^2 are both continuous on $\mathbb{R}$, their composition $f(x) = \cos x^2$ is also continuous on $\mathbb{R}$ by Theorem 2.4.4.

(b) Note that $\sin^2 x$ is continuous on $\mathbb{R}$, and $\ln(x)$ is continuous on its domain, so by Theorem 2.4.4 the function $\ln\left(\sin^2 x\right)$ is continuous wherever it is defined; that is, for all values of x such that $\sin^2 x > 0$. In fact, $\sin^2 x = (\sin x)^2 \geq 0$ for all $x \in \mathbb{R}$, while $\sin x = 0$ if and only if $x = n\pi$ where n is an integer. Thus, $\ln\left(\sin^2 x\right)$ is discontinuous at every integer multiple of π and is continuous on the intervals between these values.

Theorem 2.4.5 (Intermediate Value Theorem)**.** *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.*

Example 2.4.7. Show that the polynomial $f(x) = 3x^4 - 4x^3 + 2x^2 - x - 10$ has a root between 1 and 2.

Solution. Since f is a polynomial, it is continuous at all real numbers; in particular, it is continuous on the interval $(1, 2)$. Since 0 is a number between $f(1) = -10$ and $f(2) = 12$, the Intermediate Value Theorem guarantees the existence of a number $c \in (1, 2)$ such that $f(c) = 0$. That is, the polynomial f has a root between 1 and 2.

2.5. **Limits Involving Infinity.**

Definition 2.5.1 (Infinite Limits)**.** The notation $\lim_{x\to a} f(x) = \infty$ means that the values of $f(x)$ an be made arbitrarily large (as large as we please) by taking x sufficiently close to a (on either side of a) but not equal to a. An analogous statement defines the expression $\lim_{x\to a} f(x) = -\infty$, and likewise the following one-sided infinite limits: $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^-} f(x) = -\infty$, $\lim_{x\to a^+} f(x) = -\infty$.

Remark 2.5.1. It is important to note that in Definition 2.5.1, we do not regard ∞ as a number, nor does it mean that the limit exists. It simply denotes a particular way in which a limit can fail to exist. The expression $\lim_{x\to a} f(x) = \infty$ is often read as "the limit of $f(x)$, as x approaches a, is infinity," or "$f(x)$ becomes infinite as x approaches a," or "$f(x)$ increases without bound as x approaches a."

Example 2.5.1. Find $\lim_{x\to 0} \frac{1}{x^2}$.

Solution. The values of $\frac{1}{x^2}$ are inversely proportional to (the square of) the values of x. In other words, very small values of x correspond to very large values of $\frac{1}{x^2}$. This is seen explicitly from a graphical analysis, or from the following table.

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.01	$10,000$
± 0.001	$1,000,000$

Thus, we conclude that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Example 2.5.2. Find $\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$.

Solution. If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number, and $2x$ is close to 6. Thus, the quotient $\frac{2x}{x-3}$ is a large positive number, and we intuitively conclude that $\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$. Similarly, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative number, while $2x$ is still a positive number (close to 6), and hence $\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$.

Definition 2.5.2. The line $x = a$ is called a **vertical asymptote** (V.A.) of the curve $y = f(x)$ if any limit (or one-sided limit) of $f(x)$ as x approaches a is either ∞ or $-\infty$.

Example 2.5.3. Explore the vertical asymptotes of familiar functions such as $f(x) = \ln x$ (V.A. at $x = 0$) and $g(x) = \tan x$ (infinite number of V.A.'s, one at $x = (2n + 1)\pi/2$ for every integer n).

Definition 2.5.3 (Limits at Infinity)**.** Let f be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large. An analogous statement defines the expression $\lim_{x\to -\infty} f(x) = L$.

Example 2.5.4. Note how $\lim_{x\to\infty} \frac{1}{x^2} = 0$, while $\lim_{x\to\infty} \cos x$ does not exist.

Definition 2.5.4. The line $y = L$ is called a **horizontal asymptote** (H.A.) of the curve $y = f(x)$ if the limit of $f(x)$ as x approaches either ∞ or $-\infty$ is equal to L.

Example 2.5.5. Explore the horizontal asymptotes of familiar functions such as $f(x) = e^x$ (H.A. at $y = 0$) and $g(x) = \arctan x$ (H.A.'s at $y = -\pi/2$ and $y = \pi/2$).

Definition 2.5.5 (Infinite Limits at Infinity)**.** The notation $\lim_{x\to\infty} f(x) = \infty$ is used to indicate that the values of $f(x)$ become arbitrarily large as x becomes arbitrarily large. Similar meanings are attached to each of the following expressions: $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = -\infty$.

Example 2.5.6. Many familiar functions exhibit infinite limiting behavior at infinity, e.g.

$$\lim_{x\to\infty} e^x = \infty, \ \lim_{x\to-\infty} x^2 = \infty, \ \lim_{x\to-\infty} x^3 = -\infty.$$

Example 2.5.7. Find $\lim_{x\to0^-} e^{1/x}$ and $\lim_{x\to0^+} e^{1/x}$.

Solution. Following Example 2.5.1, it is easy to verify that $\lim_{x\to0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to0^+} \frac{1}{x} = \infty$. Letting $t = 1/x$ and using the results of Examples 2.5.5 and 2.5.6 respectively, it follows that:

$$\lim_{x\to0^-} e^{1/x} = \lim_{t\to-\infty} e^t = 0 \ \text{ and } \ \lim_{x\to0^+} e^{1/x} = \lim_{t\to\infty} e^t = \infty.$$

Proposition 2.5.1. *If n is a positive integer, then* $\lim_{x\to-\infty} \frac{1}{x^n} = 0$ *and* $\lim_{x\to\infty} \frac{1}{x^n} = 0$.

Example 2.5.8. Evaluate each of the following limits.

(a) $\lim_{x\to\infty} \frac{3x^2+9x+27}{5x^3-7x+6}$.

(b) $\lim_{x\to-\infty} \frac{3x^4+9x+27}{5x^3-7x+6}$.

(c) $\lim_{x\to-\infty} \frac{3x^3+9x+27}{5x^3-7x+6}$.

(d) $\lim_{x\to\infty} \frac{3x^2+9x+27}{\sqrt{16x^4+11x^3-12x+100}}$.

(e) $\lim_{x\to\infty} \left(\sqrt{x^2+4} - x\right)$.

Solution. We will use Proposition 2.5.1 together with Definitions 2.5.3 and 2.5.5.

(a) Both the numerator and denominator appear to approach infinity as x does. The largest power of x that appears in the expression is x^3. Multiplying both the numerator and denominator by $1/x^3$, we obtain

$$\lim_{x\to\infty} \frac{3x^2+9x+27}{5x^3-7x+6} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{3}{x} + \frac{9}{x^2} + \frac{27}{x^3}}{5 - \frac{7}{x^2} + \frac{6}{x^3}} = 0.$$

(b) This time we multiply both the numerator and denominator by $1/x^4$, simplify, and conclude that

$$\lim_{x\to-\infty} \frac{3x^4+9x+27}{5x^3-7x+6} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim_{x\to-\infty} \frac{3 + \frac{9}{x^3} + \frac{27}{x^4}}{\frac{5}{x} - \frac{7}{x^3} + \frac{6}{x^4}} = \infty.$$

(c) Again multiplying both the numerator and denominator by $1/x^3$, we have

$$\lim_{x\to\infty} \frac{3x^3+9x+27}{5x^3-7x+6} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{3 + \frac{9}{x^2} + \frac{27}{x^3}}{5 - \frac{7}{x^2} + \frac{6}{x^3}} = \frac{3}{5}.$$

(d) Notice that $x^2 = \sqrt{x^4}$ is the largest power of x that appears in the expression. Multiplying both the numerator and denominator by $1/x^2 = 1/\sqrt{x^4}$ yields

$$\lim_{x\to\infty} \frac{3x^2+9x+27}{\sqrt{16x^4+11x^3-12x+100}} \cdot \frac{\frac{1}{x^2}}{\frac{1}{\sqrt{x^4}}} = \lim_{x\to\infty} \frac{3 + \frac{9}{x} + \frac{27}{x^2}}{\sqrt{16 + \frac{11}{x} - \frac{12}{x^3} + \frac{100}{x^4}}} = \frac{3}{\sqrt{16}} = \frac{3}{4}.$$

(e) We first multiply by the conjugate of $\left(\sqrt{x^2+4} - x\right)$ over itself, and then multiply both the numerator and denominator of the resulting expression by $1/x = 1/\sqrt{x^2}$, in order to obtain

$$\lim_{x\to\infty} \left(\sqrt{x^2+4} - x\right) \cdot \frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}+x} = \lim_{x\to\infty} \frac{\left(x^2+4\right) - x^2}{\sqrt{x^2+4}+x} \cdot \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{\frac{4}{x}}{\sqrt{1+\frac{4}{x^2}}+1} = 0.$$

Remark 2.5.2. Parts (a) - (c) of Example 2.5.8 illustrate special cases of a very general result. Namely, if $f(x) = P(x)/Q(x)$, where P and Q are polynomial functions of degree m and n with leading coefficients a and b respectively, then f exhibits the one of the three following types of limiting behavior:

1. $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$, if $m < n$.
2. $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$, if $m > n$.
3. $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = a/b$, if $m = n$.

2.6. **Derivatives and Rates of Change.**

Definition 2.6.1. The **derivative of a function f at a number a**, denoted by $f'(a)$, is defined by $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$, provided that the limit exists.

Remark 2.6.1. As alluded to in Section 2.1, the derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$. In particular, $f'(a)$ represents:

1. The **slope of the tangent line** to the graph of $y = f(x)$ at the point $(a, f(a))$ on the curve.
2. The **instantaneous velocity** at time a, when $f(x)$ is a position function.

Example 2.6.1. Find the equation of the tangent line to the curve $f(x) = \frac{2}{x+1}$ at the point $(0, 2)$.

Solution. The slope of the tangent line to $f(x)$ at $(0, 2)$ is given by the derivative

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{2}{x+1} - 2}{x} = \lim_{x\to 0} \frac{\frac{2-2(x+1)}{x+1}}{x} = \lim_{x\to 0} \frac{-2x}{x(x+1)} = \lim_{x\to 0} \frac{-2}{x+1} = -2.$$

Hence, the equation of the tangent line at the point $(0, 2)$ is given by $y - 2 = -2(x - 0)$, or $y = -2x + 2$.

Example 2.6.2. The displacement of a particle moving in a straight line is given by $s(t) = \sqrt{t+16} - 4$. Find the velocity of the particle at $t = 9$, $t = 20$, and $t = 33$.

Solution. First, we find a general formula for $s'(a)$.

$$\begin{aligned}
s'(a) &= \lim_{t\to a} \frac{s(t) - s(a)}{t - a} \\
&= \lim_{t\to a} \frac{\left(\sqrt{t+16} - 4\right) - \left(\sqrt{a+16} - 4\right)}{t - a} \\
&= \lim_{t\to a} \frac{\sqrt{t+16} - \sqrt{a+16}}{t - a} \cdot \frac{\sqrt{t+16} + \sqrt{a+16}}{\sqrt{t+16} + \sqrt{a+16}} \\
&= \lim_{t\to a} \frac{(t+16) - (a+16)}{(t-a)\left(\sqrt{t+16} + \sqrt{a+16}\right)} \\
&= \lim_{t\to a} \frac{(t-a)}{(t-a)\left(\sqrt{t+16} + \sqrt{a+16}\right)} \\
&= \lim_{t\to a} \frac{1}{\sqrt{t+16} + \sqrt{a+16}} \\
&= \frac{1}{2\sqrt{a+16}}.
\end{aligned}$$

Then it is easy to see that $s'(9) = 1/10$, $s'(20) = 1/12$, and $s'(33) = 1/14$.

2.7. **The Derivative as a Function.**

In Section 2.6, we considered the derivative of a function f at a fixed number a, defined by

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

If we let $x = a + h$ and substitute above for x, we obtain the alternative formula

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Since in this discussion a is arbitrary, it is reasonable to suppose that a is actually a variable quantity. In fact, this is a particularly useful generalization; it allows us to regard the derivative f' as a new function, derived from f by one of the equivalent limiting operations above. In general, with a replaced by the variable x, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Example 2.7.1. Let $f(x) = x^3 - x$. Calculate $f'(x)$ and determine the slope of the tangent line to the curve f when $x = 1$ and $x = 2$.

Solution. By the formula introduced above, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\big((x+h)^3 - (x+h)\big) - \big(x^3 - x\big)}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 1 \\ &= 3x^2 - 1. \end{aligned}$$

Then $f'(1) = 2$ and $f'(2) = 11$ represent the slope of the tangent line to f at $x = 1$ and $x = 2$ respectively.

Example 2.7.2. Let $f(x) = 1/x$. Find a formula for $f'(x)$.

Solution. As in the previous example, we carefully compute $f'(x)$ using the limit definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

If we use $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x).$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of differentiation. The symbol dy/dx, introduced by Leibniz, is especially useful and will be substituted frequently for the "prime" notation. If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a, we write $dy/dx|_{x=a}$, which is synonymous with $f'(a)$.

Definition 2.7.1. A function f is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Theorem 2.7.1. *If f is differentiable at a, then f is continuous at a.*

Example 2.7.3. Illustrate the common graphical features that cause a function not to be differentiable, e.g. corners, discontinuities, and vertical tangents.

Remark 2.7.1. Example 2.7.3 motivates the vital observation that the converse of Theorem 2.7.1 is false! That is, a function can be continuous but not differentiable at a point. In fact, there exist functions defined on $\mathbb{R}$ that everywhere continuous and yet nowhere differentiable.

Definition 2.7.2 (Higher Derivatives)**.** Suppose $y = f(x)$ is a differentiable function. Since f' is also a function, it may have a derivative of its own. This new function, denoted by f'', is called the **second**

derivative of f because it is the derivative of the derivative of f. In Leibniz notation, we write the second derivative of f as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

The **third derivative**, denoted by f''' or $\frac{d^3y}{dx^3}$, is the derivative of the second derivative, and so on. For $n \geq 4$, as the prime notation becomes impractical, we generally denote the nth derivative by $f^{(n)}$ or $\frac{d^ny}{dx^n}$.

Remark 2.7.2. We have already observed that if $s(t)$ is a position function, then $v(t) = s'(t)$ represents velocity. Furthermore, $a(t) = v'(t) = s''(t)$ represents acceleration, i.e. the rate of change of velocity.

2.8. What Does f' Say About f?

Proposition 2.8.1. *Suppose that f is a differentiable function. The sign of its derivative, f', provides the following information about the behavior of the graph of f:*

- *If $f'(x) > 0$ on an interval, then f is increasing on that interval.*
- *If $f'(x) < 0$ on an interval, then f is decreasing on that interval.*

An important issue that is not addressed by Proposition 2.8.1 is what happens when $f'(x) = 0$. In this case, the tangent line to the graph of f through the point $(x, f(x))$ has a slope of zero. In particular, the function f has a local minimum or a local maximum at that point, as we will discuss further in Section 4.2.

Proposition 2.8.2. *Suppose that f is a twice-differentiable function. The sign of its second derivative, f'', provides the following information about the behavior of the graph of f:*

- *If $f''(x) > 0$ on an interval, then f is* **concave upward** *on that interval.*
- *If $f''(x) < 0$ on an interval, then f is* **concave downward** *on that interval.*

Definition 2.8.1. An **inflection point** is a point where the curve changes its direction of concavity.

Example 2.8.1. Apply Propositions 2.8.1 and 2.8.2 to simple a polynomial function. Compare the shape of the curve with the tangential slope at various points. Identify intervals where the function is increasing and decreasing, local minima and maxima, intervals of concavity, and inflection points.

Example 2.8.2. Sketch a possible graph of a function f that satisfies the following conditions:

1. $f'(x) > 0$ on $(-\infty, 1)$; $f'(x) < 0$ on $(1, \infty)$.
2. $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$; $f''(x) < 0$ on $(-2, 2)$.
3. $\lim_{x\to-\infty} f(x) = -2$; $\lim_{x\to\infty} f(x) = 0$.

When given a function f, it may be useful to find a function F whose derivative is f; that is, $F'(x) = f(x)$. If such a function F exists, we call it an *antiderivative* of f. The concept of antiderivatives is of fundamental importance in connecting the ideas of differential and integral Calculus, as seen in Section 5.4.

Example 2.8.3. Consider a graph representing some derivative $y = f'(x)$.

(a) On what intervals is f increasing? On what intervals is f decreasing?
(b) At what values of x does f have a local maximum or minimum?
(c) On what intervals is f concave upward? On what intervals is f concave downward?
(d) What are the (x-coordinates of the) points of inflection of f?
(e) Assuming that $f(0) = 0$, sketch a possible graph of f.

3. Differentiation Rules.

3.1. Derivatives of Polynomials and Exponential Functions.

We have already informally discussed, or are aware of, the derivatives of the most basic polynomial functions. The constant function $f(x) = c$ is graphically represented by a horizontal line, so its slope is zero everywhere and hence $f'(x) = \frac{d}{dx}(c) = 0$ for all $x \in \mathbb{R}$. Of equal simplicity is the case of the linear function $g(x) = x$; its slope is one everywhere and hence $g'(x) = \frac{d}{dx}(x) = 1$ for all $x \in \mathbb{R}$. Now, if $f(x) = x^n$, for some integer $n \geq 2$, it is useful to apply a limit definition of the derivative in order to determine f'.

Example 3.1.1. Differentiate $f(x) = x^5$.

Solution. By the limit definition of the derivative from Section 2.7, we obtain

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^5 - x^5}{h}.$$

Now, to expand the term $(x+h)^5$, we can use the **binomial coefficients** given by **Pascal's Triangle**. In particular, the sixth row of the triangle contains the coefficients $1, 5, 10, 10, 5, 1$, which implies that

$$(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5.$$

Thus, the derivative of $f(x) = x^5$ is given by

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^5 - x^5}{h} \\ &= \lim_{h\to 0} \frac{(x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5) - x^5}{h} \\ &= \lim_{h\to 0} \frac{5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5}{h} \\ &= \lim_{h\to 0} (5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4) \\ &= 5x^4. \end{aligned}$$

Theorem 3.1.1 (Power Rule)**.** *If n is any real number, then* $\frac{d}{dx}(x^n) = nx^{n-1}$.

Proof. We proceed much as in Example 3.1.1, but it is necessary to apply the general form of the Binomial Theorem in order to expand the term $(x+h)^n$. □

Example 3.1.2. Differentiate each of the following: (a) $f(x) = x^{10}$, (b) $g(x) = \sqrt[5]{x^2}$, (c) $h(x) = \frac{1}{x^4}$.

Solution. By the Power Rule, we have:

(a) $f'(x) = \frac{d}{dx}\left(x^{10}\right) = 10x^9$.
(b) $g'(x) = \frac{d}{dx}\left(x^{2/5}\right) = \frac{2}{5}x^{-3/5}$.
(c) $h'(x) = \frac{d}{dx}\left(x^{-4}\right) = -4x^{-5}$.

Definition 3.1.1. The **normal line** to a curve at the point P is the line through P that is perpendicular to the tangent line at P.

Example 3.1.3. Find the equation of the normal line to the curve $y = x\sqrt{x}$ at the point $(4, 8)$.

Solution. Let $y = f(x) = x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$. Then by the Power Rule

$$f'(x) = \frac{d}{dx}\left(x^{3/2}\right) = \frac{3}{2}x^{1/2}.$$

In particular, the slope of the tangent line at $(4, 8)$ is given by $f'(4) = 3$. The slope of any line perpendicular to the tangent is the negative reciprocal of 3; that is, $-\frac{1}{3}$. Thus, the equation of the normal line to the curve $y = x\sqrt{x}$ at the point $(4, 8)$ is $y - 8 = -\frac{1}{3}(x - 4)$, or $y = -\frac{1}{3}x + \frac{28}{3}$.

The following theorems combine with the Power Rule to allow us to easily differentiate any polynomial.

Theorem 3.1.2 (Constant Multiple Rule)**.** *If c is a constant and f is a differentiable function, then*

$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right].$$

Proof. Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] \\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= cf'(x). \end{aligned}$$

□

Theorem 3.1.3 (Sum Rule)**.** *If f and g are differentiable functions, then*

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right].$$

Proof. Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{h(x+h)-h(x)}{h} \\ &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right] \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= f'(x)+g'(x). \end{aligned}$$

□

Theorem 3.1.4 (Difference Rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx}\left[f(x)-g(x)\right] = \frac{d}{dx}\left[f(x)\right] - \frac{d}{dx}\left[g(x)\right].$$

Proof. The argument is the same as in the case of the Sum Rule, except that we must use the Difference Law for limits in place of the Sum Law. □

Example 3.1.4. Find $\frac{d}{dx}\left(x^7 - 6x^6 + 3x^5 + 7x^3 - 8x + 11\right)$.

Solution. By the preceding theorems, we find that:

$$\begin{aligned} \frac{d}{dx}\left(x^7 - 6x^6 + 3x^5 + 7x^3 - 8x + 11\right) &= \frac{d}{dx}\left(x^7\right) - 6\frac{d}{dx}\left(x^6\right) + 3\frac{d}{dx}\left(x^5\right) + 7\frac{d}{dx}\left(x^3\right) - 8\frac{d}{dx}(x) + \frac{d}{dx}(11) \\ &= 7x^6 - 6\cdot 6x^5 + 3\cdot 5x^4 + 7\cdot 3x^2 + 8 \\ &= 7x^6 - 36x^5 + 15x^4 + 21x^2 - 8. \end{aligned}$$

Example 3.1.5. Find the points on the curve $y = x^4 - 2x^2$ where the tangent line is horizontal.

Solution. A horizontal tangent line occurs wherever the slope of the derivative is zero. Note that

$$y' = 4x^3 - 4x = 0 \implies 4x\left(x^2 - 1\right) = 0 \implies 4x(x-1)(x+1) = 0 \implies x = 0,\ x = 1,\ x = -1.$$

Thus, the curve $y = x^4 - 2x^2$ has horizontal tangent lines at the points $(-1,-1)$, $(0,0)$, and $(1,-1)$.

Recall from Ex. 1.5.5 that the function $f(x) = e^x$ has the property that the slope of its tangent line at the point $(0, 1)$ is exactly 1. By the limit definition of the derivative, it follows that e is the unique number satisfying

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1,$$

which yields the important differentiation rule:

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x\left(e^h - 1\right)}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x.$$

Example 3.1.6. If $f(x) = 3e^x - 4x^2$, find f' and f''.

Solution. Combining the rule above with the Difference, Constant Multiple, and Power Rules, we have:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(3e^x - 4x^2\right) = 3\frac{d}{dx}(e^x) - 4\frac{d}{dx}\left(x^2\right) = 3\left(e^x\right) - 4(2x) = 3e^x - 8x, \\ f''(x) &= \frac{d}{dx}(3e^x - 8x) = 3\frac{d}{dx}(e^x) - 8\frac{d}{dx}(x) = 3\left(e^x\right)) - 8(1) = 3e^x - 8. \end{aligned}$$

Example 3.1.7. At what point on the curve $y = 4e^x - 6$ is the tangent line parallel to $y = 4x$?

Solution. Let $x = a$ be the x-coordinate of the point in question. As in the previous example

$$f'(x) = \frac{d}{dx}(4e^x - 6) = 4e^x.$$

The slope of the tangent line to the curve $y = 4e^x - 6$ at the point $x = a$ is thus $4e^a$, while the slope of $y = 4x$ at any point is 4. Setting $4e^a = 4$ and solving for a, we have $e^a = 1$ or simply $a = 0$. Hence, the required point is $(0, 4)$.

3.2. **The Product and Quotient Rules.**

By analogy with the Sum and Difference Rules, one might suppose that similar rules exist for the products and quotients of functions. This is not the case, however, as illustrated by the following simple example.

Example 3.2.1. Let $f(x) = \frac{x}{3}$ and $g(x) = x^2$, and show that $(fg)'(x) \neq f'(x)g'(x)$.

Solution. Clearly $f'(x) = \frac{1}{3}$ and $g'(x) = 2x$, so that $f'(x)g'(x) = \frac{2}{3}x$. On the other hand, $(fg)(x) = \frac{x^3}{3}$, which has the derivative $(fg)'(x) = x^2$.

To develop a general rule for correctly differentiating products of functions, we return to the limit definition of the derivative.

Theorem 3.2.1 (Product Rule)**.** *If f and g are differentiable functions, then*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Proof. Let $F(x) = f(x)g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{h(x+h) - h(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)g(x+h) - f(x)g(x+h)] + [f(x)g(x+h) - f(x)g(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{g(x+h)\,[f(x+h) - f(x)]}{h} + \frac{f(x)\,[g(x+h) - g(x)]}{h}\right] \\
&= \lim_{h\to 0} g(x+h) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= g(x)f'(x) + f(x)g'(x).
\end{aligned}$$

$\square$

Example 3.2.2. If $f(x) = x^2e^x$, find $f'(x)$.

Solution. By the Product Rule, we have

$$f'(x) = \frac{d}{dx}\left(x^2e^x\right) = x^2\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(x^2\right) = x^2e^x + 2xe^x = xe^x(x+2).$$

Example 3.2.3. Differentiate $f(t) = \left(\sqrt{t}+1\right)\left(t^2+t+1\right)$.

Solution. Again by the Product Rule, we have

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\left(\sqrt{t}+1\right)\left(t^2+t+1\right)\right] \\
&= \left(\sqrt{t}+1\right)\frac{d}{dx}\left(t^2+t+1\right) + \left(t^2+t+1\right)\frac{d}{dx}\left(\sqrt{t}+1\right) \\
&= \left(\sqrt{t}+1\right)(2t+1) + \left(t^2+t+1\right)\left(\frac{1}{2}t^{-1/2}\right).
\end{aligned}$$

Example 3.2.4. If $f(x) = (\sqrt{x}+1)\,g(x)$, where $g(9) = -6$ and $g'(9) = \frac{1}{2}$, find $f'(9)$.

Solution. First, notice that

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\left(\sqrt{x}+1\right)g(x)\right] \\
&= \left(\sqrt{x}+1\right)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left(\sqrt{x}+1\right) \\
&= \left(\sqrt{x}+1\right)g'(x) + g(x)\left(\frac{1}{2}x^{-1/2}\right).
\end{aligned}$$

Thus,

$$f'(9) = \left(\sqrt{9}+1\right)g'(9) + g(9)\left(\frac{1}{2}(9)^{-1/2}\right) = (4)\left(\frac{1}{2}\right) + (-6)\left(\frac{1}{6}\right) = 1.$$

As with the Product Rule, the Quotient Rule involves a somewhat complicated expression.

Theorem 3.2.2 (Quotient Rule)**.** *If f and g are differentiable functions, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}\left[f(x)\right] - f(x)\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}.$$

Example 3.2.5. If $y = \frac{x^3+6x-6}{x^2+4}$, find y'.

Solution. By the Quotient Rule, we have that

$$\begin{aligned} y' &= \frac{d}{dx}\left[\frac{x^3+6x-6}{x^2+4}\right] \\ &= \frac{\left(x^2+4\right)\frac{d}{dx}\left(x^3+6x-6\right)-\left(x^3+6x-6\right)\frac{d}{dx}\left(x^2+4\right)}{\left(x^2+4\right)^2} \\ &= \frac{\left(x^2+4\right)\left(3x^2+6\right)-\left(x^3+6x-6\right)(2x)}{\left(x^2+4\right)^2}. \end{aligned}$$

Example 3.2.6. Find the equation of the tangent line to the curve $y = \frac{e^x}{e^x+1}$ at the point $(0, 1/2)$.

Solution. Again by the Quotient Rule, we have that

$$\begin{aligned} y' &= \frac{d}{dx}\left[\frac{e^x}{e^x+1}\right] \\ &= \frac{\left(e^x+1\right)\frac{d}{dx}\left(e^x\right)-\left(e^x\right)\frac{d}{dx}\left(e^x+1\right)}{\left(e^x+1\right)^2} \\ &= \frac{\left(e^x+1\right)\left(e^x\right)-\left(e^x\right)\left(e^x\right)}{\left(e^x+1\right)^2} \\ &= \frac{e^x}{\left(e^x+1\right)^2}. \end{aligned}$$

Thus, the slope of the tangent line to the curve when $x = 0$ is given by $y'|_{x=0} = \frac{e^0}{(e^0+1)^2} = \frac{1}{2^2} = \frac{1}{4}$, and the equation of the tangent line to the curve at the point $(0, 1/2)$ is $y = \frac{1}{4}x + \frac{1}{2}$.

3.3. **Derivatives of Trigonometric Functions.**

Example 3.3.1. Estimate $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$.

Solution. Consider the following table.

θ	$\frac{\sin\theta}{\theta}$
± 0.1	0.99833416
± 0.01	0.99998333
± 0.001	0.99999983
± 0.0001	0.99999999

From this, we conclude that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

Similarly, we can determine that $\lim_{\theta\to 0} \frac{\sin\theta-1}{\theta} = 0$. We are now equipped to establish a formula for the derivative of the sine function.

Theorem 3.3.1. $\frac{d}{dx}(\sin x) = \cos x$.

Proof. Let $f(x) = \sin x$. By the definition of the derivative and a well-known trigonometric identity, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x\cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \left[\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right] \\
&= \lim_{h\to 0} \left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right] \\
&= \lim_{h\to 0}\sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\left(\frac{\sin h}{h}\right) \\
&= \sin x\cdot 0 + \cos x \cdot 1 \\
&= \cos x.
\end{aligned}$$

□

By an analogous proof, we obtain a similar formula for the derivative of the cosine function.

Theorem 3.3.2. $\frac{d}{dx}(\cos x) = -\sin x.$

Remark 3.3.1. The derivatives of sine and cosine can be recalled with the aid of a simple circular diagram.

Since the tangent function is a quotient of the sine and cosine functions, its derivative can now be determined.

Theorem 3.3.3. $\frac{d}{dx}(\tan x) = \sec^2 x.$

Proof. By the Quotient Rule, we obtain

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x\cdot\frac{d}{dx}(\sin x) - \sin x\cdot\frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x.
\end{aligned}$$

□

We can further apply the Quotient Rule to establish formulas for the derivative of each of the remaining basic trigonometric functions.

Theorem 3.3.4. *1.* $\frac{d}{dx}(\csc x) = -\csc x\cot x$, *2.* $\frac{d}{dx}(\sec x) = \sec x\tan x$, *3.* $\frac{d}{dx}(\cot x) = -\csc^2 x.$

Example 3.3.2. For what values of x does the graph of $f(x) = \frac{\sec x}{\tan x + 1}$ have a horizontal tangent line?

Solution. Recalling the trigonometric identity $\tan^2 x + 1 = \sec^2 x$, the Quotient Rule gives

$$\begin{aligned} f'(x) &= \frac{(1+\tan x)\cdot \frac{d}{dx}(\sec x) - \sec x \cdot \frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \\ &= \frac{(1+\tan x)\cdot \sec x \tan x - \sec x \cdot \sec^2 x}{(1+\tan x)^2} \\ &= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} \\ &= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2} \end{aligned}$$

Since $\sec x$ is never equal to 0, we see that $f'(x) = 0$ when $\tan x = 1$, which is for $x = n\pi + \pi/4$, where n is any integer. At these points, the tangent line to the graph of f is horizontal.

Example 3.3.3. A 12-foot ladder rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/4$?

Solution. By the geometric definition of the sine function, we may write $\sin\theta = \frac{x}{12}$. That is, $x = 12\sin\theta$. The rate of change of x with respect to θ is then given by $\frac{dx}{d\theta} = \frac{d}{d\theta}(12\sin\theta) = 12\cos\theta$. Thus, when $\theta = \pi/4$, the rate of change of x with respect to θ is simply $12\cos(\pi/4) = 12\cdot\sqrt{2}2 = 6\sqrt{2} \approx 8.5$ ft/rad.

3.4. **The Chain Rule.**

Theorem 3.4.1 (Chain Rule)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product*

$$F'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example 3.4.1. Find the derivative of $f(x) = \sqrt[3]{x^5 + x}$.

Solution. Rewriting the function f, we have by the Chain Rule that

$$f'(x) = \frac{d}{dx}(x^5 + x)^{1/3} = (1/3)(x^5 + x)^{-2/3} \cdot (5x+1).$$

Example 3.4.2. Find the derivative of $f(x) = \sin\left(x^4\right)$ and $g(x) = \sin^4 x$.

Solution. By the Chain Rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\sin(x^4) = \cos(x^4)\cdot 4x. \\ g'(x) &= \frac{d}{dx}(\sin x)^4 = 4(\sin x)^3 \cdot \cos x. \end{aligned}$$

Corollary 3.4.2 (Power Rule combined with Chain Rule)**.** *If n is any real number and $u = g(x)$ is differentiable, then*

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x).$$

Example 3.4.3. Differentiate $h(x) = \frac{1}{\sqrt{x^2+5x+10}}$.

Solution. Rewriting the function h and applying the Chain Rule, we obtain

$$h'(x) = \frac{d}{dx}(x^2+5x+10)^{-1/2} = (-1/2)(x^2+5x+10)^{-3/2} \cdot (2x+5).$$

Example 3.4.4. Find y' if $y = (x^2+3x+2)^7(3x-8)^4$.

Solution. Combining the Chain Rule with the Product Rule, we find

$$y' = \frac{d}{dx}\left[(x^2+3x+2)^7(3x-8)^4\right] = (x^2+3x+2)^7 \cdot 4(3x-8)^3 \cdot 3 + (3x-8)^4 \cdot 7(x^2+3x+2)^6 \cdot (2x+3).$$

Example 3.4.5. Find the derivative of $y = e^{\tan x}$.

Solution. By the Chain Rule, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\tan x}\right) = e^{\tan x} \cdot \frac{d}{dx}(\tan x) = e^{\tan x} \cdot \sec^2 x.$$

Example 3.4.6. Let $a > 0$ be a constant. Find $\frac{d}{dx}(a^x)$.

Solution. Recall from Section 1.6 that $a = e^{\ln a}$ for any $a > 0$. In particular, $a^x = \left(e^{\ln a}\right)^x = e^{(\ln a)x}$. It follows by the Chain Rule that

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{(\ln a)x}\right) = e^{(\ln a)x} \cdot \frac{d}{dx}\left((\ln a)x\right) = e^{(\ln a)x} \cdot \ln a = a^x \ln a.$$

Example 3.4.7. If $f(x) = \cos(\sin(e^x))$, find $f'(x)$.

Solution. Applying the Chain Rule twice, we obtain

$$f'(x) = \frac{d}{dx}\left(\cos(\sin(e^x))\right) = -\sin(\sin(e^x)) \cdot \frac{d}{dx}(\sin(e^x)) = -\sin(\sin(e^x)) \cdot \cos(e^x) \cdot e^x.$$

Example 3.4.8. If $f(x) = e^{\sec(3x)}$, find $f'(x)$.

Solution. Again using the Chain Rule twice, we have

$$f'(x) = \frac{d}{dx}\left(e^{\sec(3x)}\right) = e^{\sec(3x)} \cdot \frac{d}{dx}(\sec(3x)) = e^{\sec(3x)} \cdot \sec(3x)\tan(3x) \cdot \frac{d}{dx}(3x) = 3\sec(3x)\tan(3x)e^{\sec(3x)}.$$

Remark 3.4.1. The Chain Rule gives us a way of finding the slope of tangent lines to parametric cubes. In particular, suppose that $x = f(t)$ and $y = g(t)$ are differentiable functions that define a parametric curve. We wish to find the tangent line at a point on the curve where y is also a differentiable function of x. By the Chain Rule, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$. Assuming that $\frac{dx}{dt} \neq 0$, it follows that $\frac{dy}{dx} = \left(\frac{dy}{dt}\right)/\left(\frac{dx}{dt}\right)$.

Example 3.4.9. Find an equation of the tangent line to the parametric curve $x = 2\sin(2t)$ and $y = 2\sin t$ at the point $(\sqrt{3}, 1)$. Where does this curve have horizontal or vertical tangents?

Solution. At the point with parameter value t, the slope of the tangent line to the curve is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(2\sin t)}{\frac{d}{dt}(2\sin(2t))} = \frac{2\cos t}{2\cos(2t)\cdot 2} = \frac{\cos t}{2\cos(2t)}.$$

The point $(\sqrt{3}, 1)$ corresponds to the parameter value $t = \pi/6$, so the slope of the tangent at that point is

$$\frac{dy}{dx}\Big|_{t=\pi/6} = \frac{\cos(\pi/6)}{2\cos(\pi/3)} = \frac{\sqrt{3}/2}{2(1/2)} = \frac{\sqrt{3}}{2}.$$

An equation of the tangent line is therefore: $y - 1 = \frac{\sqrt{3}}{2}\left(x - \sqrt{3}\right)$.

The tangent line is horizontal when $\frac{dy}{dx} = 0$. This occurs when $\cos t = 0$ (and $\cos(2t) \neq 0$); that is, when $t = \pi/2$ or $t = 3\pi/2$ (note that the entire curve is given by the parameter values $0 \leq t \leq 2\pi$). Thus, the curve has horizontal tangents at the points $(0, 2)$ and $(0, -2)$.

The tangent line is vertical when $\frac{dx}{dt} = 4\cos(2t) = 0$ (and $\cos t \neq 0$); that is, when $t = \pi/4$, $3\pi/4$, $5\pi/4$, or $7\pi/4$. The corresponding four points on the curve are $\left(\pm 2, \pm\sqrt{2}\right)$.

3.5. **Implicit Differentiation.**

The functions we have dealt with up until now have been described by expressing one variable explicitly in terms of another, e.g. $y = e^x$ or $y = x\cos x$. Some functions, however, are defined implicitly by a relationship between x and y such as $x^2 + y^2 = 25$ or $x^3 + y^3 = 3xy$.

In some cases, it is possible to solve such an equation for y as an explicit function (or functions) of x. For instance, the equation $x^2 + y^2 = 25$ can be written equivalently as the two equations $y = \pm\sqrt{25 - x^2}$. On the other hand, it's not easy to solve $x^3 + y^3 = 3xy$ for either variable explicitly. Fortunately, the method of **implicit differentiation** allows us to find the derivative of y without solving for y as a function of x.

The general approach to implicit differentation is as follows. Given an equation in x and y that defines y implicitly as a differentiable function of x, we can find y' by differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

Example 3.5.1. Find the equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(-3, 4)$.

Solution. First, we differentiate both sides of the equation with respect to x

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \implies \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0.$$

Since y is a function of x, the Chain Rule implies that $\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$. Thus, we have the equation

$$2x + 2y\frac{dy}{dx} = 0 \implies 2y\frac{dy}{dx} = -2x \implies \frac{dy}{dx} = -\frac{x}{y}.$$

At the point $(-3, 4)$, we have $x = -3$ and $y = 4$, so

$$\frac{dy}{dx}\Big|_{(x,y)=(-3,4)} = \frac{3}{4}.$$

An equation of the tangent line to the circle at the point $(-3, 4)$ is therefore $y - 4 = \frac{3}{4}(x + 3)$.

Example 3.5.2. Find y' if $x^3 + y^3 = 3xy$. At what point in the first quadrant does this curve have a horizontal tangent line?

Solution. Regarding y as a function of x, and differentiating both sides of the equation with respect to x via the Chain and Product Rules, we have

$$\begin{aligned}\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3xy) &\implies \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3\left(x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right)\\ &\implies 3x^2 + 3y^2y' = 3(xy' + y)\\ &\implies y^2y' - xy' = y - x^2\\ &\implies y'(y^2 - x) = y - x^2\\ &\implies y' = \frac{y - x^2}{y^2 - x}.\end{aligned}$$

Now, the tangent line is horizontal if and only if $y' = 0$. By the formula above, this happens whenever $y - x^2 = 0$ (provided that $y^2 - x \neq 0$). Substituting $y = x^2$ into the original equation, we obtain

$$x^3 + (x^2)^3 = 3x(x^2) \implies x^3 + x^6 = 3x^3 \implies x^6 - 2x^3 = 0 \implies x^3(x^3 - 2) = 0.$$

Since $x \neq 0$ in the first quadrant, this implies $x^3 - 2 = 0$ or $x = 2^{1/3}$. At this x-value, we have $y = (2^{1/3})^2 = 2^{2/3}$. Thus, the tangent line is horizontal at the point $(2^{1/3}, 2^{2/3})$ in the first quadrant.

Example 3.5.3. Find y' if $\sin(x + y) = y\cos x$.

Solution. Again using the Chain and Product Rules, we have

$$\begin{aligned}\frac{d}{dx}(\sin(x + y)) = \frac{d}{dx}(y\cos x) &\implies \cos(x + y)\frac{d}{dx}(x + y) = y\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(y)\\ &\implies \cos(x + y)\cdot(1 + y') = -y\sin x + \cos x\cdot y'\\ &\implies y'(\cos x - \cos(x + y)) = y\sin x + \cos(x + y)\\ &\implies y' = \frac{y\sin x + \cos(x + y)}{\cos x - \cos(x + y)}.\end{aligned}$$

3.6. **Inverse Trigonometric Functions and Their Derivatives.**

Definition 3.6.1. The **inverse trigonometric functions** are defined as follows:

- $\sin^{-1} x = y$ or $\arcsin x = y \iff \sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- $\cos^{-1} x = y$ or $\arccos x = y \iff \cos y = x$ and $0 \leq y \leq \pi$
- $\tan^{-1} x = y$ or $\arctan x = y \iff \tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Remark 3.6.1. The inverse sine or **arcsine** function, $\sin^{-1} x$, represents the angle y between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x. Similar intuition describes the other inverse trigonometric functions. Furthermore,

- The domain of both $\sin^{-1} x$ and $\cos^{-1} x$ is $[-1, 1]$.
- The domain of $\tan^{-1} x$ is $(-\infty, \infty)$, and $\lim_{x\to\pm\infty} \tan^{-1} x = \pm\frac{\pi}{2}$ as discussed in Ex 2.5.3.

Example 3.6.1. Explore the inverse trigonometric functions (arcsine, arccosine, arctangent) and their graphs in comparison to the ordinary trigonometric functions (sine, cosine, tangent).

Example 3.6.2. Note that the cancelation equations for inverse functions hold for arcsine, arccosine, and arctangent. For example,

$$\sin^{-1}(\sin x) = x \text{ for } -\tfrac{\pi}{2} \leq x \leq \tfrac{\pi}{2} \quad\text{and}\quad \sin(\sin^{-1} x) = x \text{ for } -1 \leq x \leq 1.$$

In particular, $\sin^{-1}(1/2) = \pi/6$, $\sin^{-1}(-\sqrt{2}/2) = -\pi/4$, $\tan^{-1}(1) = \pi/4$, and $\tan^{-1}(0) = 0$.

Example 3.6.3. Simplify the expression $\cos(\tan^{-1} x)$.

Solution (1)**.** Let $y = \tan^{-1} x$. Then $\tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$. We wish to determine $\cos y$, but it is easier to find $\sec y$ first using the Pythagorean identity $\sec^2 y = 1 + \tan^2 y = 1 + x^2$. Indeed, we have $\sec y = \sqrt{1+x^2}$ $\left(\text{since } \sec y > 0 \text{ for } -\frac{\pi}{2} < y < \frac{\pi}{2}\right)$. Thus, $\cos(\tan^{-1} x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1+x^2}}$.

Solution (2)**.** Alternatively, let $y = \tan^{-1} x$ and construct a right triangle such that y is one of its acute angles. Then the opposite and adjacent side-lengths are x and 1 respectively, and hence the hypotenuse has length $\sqrt{1+x^2}$. It follows that $\cos(\tan^{-1} x) = \cos y = \frac{1}{\sqrt{1+x^2}}$.

Example 3.6.4. Evaluate $\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right)$.

Solution. If we let $t = \frac{1}{x-2}$, we know that $t \to \infty$ as $x \to 2^+$. Therefore, by the H.A. properties of the arctangent function, we have

$$\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right) = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}.$$

Using the implicit differentiation method from Section 3.5 we are now prepared to establish the derivative of the arcsine function.

Theorem 3.6.1. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, *for* $-1 < x < 1$.

Proof. Let $y = \sin^{-1} x$. Then $\sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Differentating implicitly with respect to x yields

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x) \implies \cos y \cdot \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\cos y}.$$

Since $\cos y \geq 0$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we have by the Pythagorean identity that

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1-x^2}.$$

Hence, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$, as desired. □

By similar proofs, we can also obtain formulas for the derivatives of the arccosine and arctangent functions.

Theorem 3.6.2. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, *for* $-1 < x < 1$.

Theorem 3.6.3. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

Example 3.6.5. Differentiate $f(x) = \sin^{-1}(\sqrt{x})$.

Solution. Combining the Chain Rule with Theorem 3.6.1 above, we obtain

$$f'(x) = \frac{d}{dx}\sin^{-1}\left(\sqrt{x}\right) = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(x^{1/2}) = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x(1-x)}}.$$

3.7. **Derivatives of Logarithmic Functions.**

In this section, we use implicit differentiation to find the derivatives of logarithmic functions $y = \log_a x$, and in particular of the natural logarithmic function $y = \ln x$.

Theorem 3.7.1. $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$.

Proof. Let $y = \log_a x$. Then $a^y = x$. Differentating implicitly with respect to x yields

$$\frac{d}{dx}(a^y) = \frac{d}{dx}(x) \implies a^y \ln a \cdot \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{a^y \ln a} \implies \frac{dy}{dx} = \frac{1}{x \ln a}.$$

□

In the special case of the natural logarithm function, the substitution $a = e$ yields the following simple formula.

Corollary 3.7.2. $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

As a consequence of the Chain Rule, we also have a very useful generalized formula for the derivative of the natural logarithm function composed with some other differentiable function.

Corollary 3.7.3. *If $u = g(x)$ is a differentiable function of x, then $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ or $\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$* .

Example 3.7.1. Differentiate $y = \ln(x^4 + 3x^2 + 5x)$.

Solution. By Corollary 3.7.3 we easily obtain

$$y' = \frac{\frac{d}{dx}(x^4 + 3x^2 + 5x)}{x^4 + 3x^2 + 5x} = \frac{4x^3 + 6x + 5}{x^4 + 3x^2 + 5x}.$$

Example 3.7.2. Find $\frac{d}{dx}(\ln(\cos x))$.

Solution. Again it is very easy to compute that

$$\frac{d}{dx}(\ln(\cos x)) = \frac{\frac{d}{dx}(\cos x)}{\cos x} = -\frac{\sin x}{\cos x}.$$

Example 3.7.3. Differentiate $y = \sqrt{\ln x}$.

Solution. Using the Chain Rule, we have

$$y' = \frac{d}{dx}(\ln x)^{1/2} = \frac{1}{2}(\ln x)^{-1/2} \cdot \frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}.$$

Example 3.7.4. Differentiate $f(x) = \ln\left[\frac{\sqrt{x+5}}{(x-7)^2}\right]$.

Solution. By the Quotient and Chain Rules, we find that

$$\begin{aligned}
f'(x) &= \frac{1}{\left[\frac{\sqrt{x+5}}{(x-7)^2}\right]} \cdot \frac{d}{dx}\left(\frac{(x+5)^{1/2}}{(x-7)^2}\right) \\
&= \frac{(x-7)^2}{\sqrt{x+5}} \cdot \frac{(x-7)^2\frac{d}{dx}(x+5)^{1/2} - (x+5)^{1/2}\frac{d}{dx}(x-7)^2}{(x-7)^4} \\
&= \frac{(x-7)^2}{\sqrt{x+5}} \cdot \frac{(x-7)^2 \cdot (1/2)(x+5)^{-1/2} - (x+5)^{1/2} \cdot 2(x-7)}{(x-7)^4} \\
&= \frac{(1/2)(x-7)(x+5)^{-1/2} - 2(x+5)^{1/2}}{(x-7)\sqrt{x+5}} \\
&= \frac{1}{2(x+5)} - 2(x-7).
\end{aligned}$$

Example 3.7.5. Show that $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x \neq 0$.

Solution. Let $f(x) = \ln|x|$, and notice that we can write

$$f(x) = \begin{cases} \ln x, & \text{if } x > 0; \\ \ln(-x), & \text{if } x < 0. \end{cases}$$

By Corollary 3.7.3, it follows that

$$f'(x) = \begin{cases} \frac{1}{x}, & \text{if } x > 0; \\ \frac{1}{-x}(-1) = \frac{1}{x}, & \text{if } x < 0. \end{cases}$$

Hence, $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x \neq 0$.

The calculation of derivatives of complicated functions involving products, quotients, and/or powers can often be simplified by taking logarithms. This method is called **logarithmic differentation**, and it generally consists of the following three steps:

1. Take the natural logarithm of both sides of an equation $y = f(x)$ and simplify using the Laws of Logarithms,
2. Differentiate implicitly with respect to x,
3. Solve the resulting for y'.

Example 3.7.6. Logarithmic differentation allows us to easily prove the generalized Power Rule from Section 3.1. Indeed, let n be any real number and set $y = x^n$. Then $\ln|y| = \ln|x|^n = n\ln|x|$, for $x \neq 0$ (if $x = 0$, it is obvious that $\frac{d}{dx}(x^n) = \frac{d}{dx}(0) = 0$). Implicitly differentiating both sides, we obtain

$$\frac{y'}{y} = \frac{n}{x} \implies y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}.$$

Example 3.7.7. Differentiate $y = \frac{(2x+3)^{2/5}\sqrt{x^4+3x+1}}{(3x+4)^6}$.

Solution. First, we take the natural logarithm of both sides of the equation and simplify

$$\ln y = \ln\left[\frac{(2x+3)^{2/5}\sqrt{x^4+3x+1}}{(3x+4)^6}\right] \implies \ln y = \frac{2}{5}\ln(2x+3) + \frac{1}{2}\ln(x^4+3x+1) - 6\ln(3x+4).$$

Now, by implicit differentiation we have

$$\frac{y'}{y} = \frac{2}{5}\cdot\frac{2}{2x+3} + \frac{1}{2}\cdot\frac{4x^2+3}{x^4+3x+1} - 6\cdot\frac{3}{3x+4}.$$

Hence

$$y' = \frac{(2x+3)^{2/5}\sqrt{x^4+3x+1}}{(3x+4)^6}\left[\frac{4}{5(2x+3)} + \frac{4x^2+3}{2(x^4+3x+1)} - \frac{18}{3x+4}\right].$$

Example 3.7.8. Differentiate $y = x^{\sqrt{x}}$.

Solution. Taking the natural logarithm of both sides of the equation yields $\ln y = \ln\left(x^{\sqrt{x}}\right) = \sqrt{x}\ln x$. Now, by implicit differentiation and the Product Rule, we have

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + \ln x\cdot\frac{1}{2\sqrt{x}} \implies y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right).$$

Remark 3.7.1. It is important to distinguish carefully between the various exponential-type expressions when differentiating. In general, there are four cases for exponents and bases:

1. (constant base, constant exponent) use the fact that the derivative of a constant is zero: $\frac{d}{dx}(a^b) = 0$.
2. (variable base, constant exponent) use the Power Rule: $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$.
3. (constant base, variable exponent) use the diff. rule for exp. functions: $\frac{d}{dx}[a^{g(x)}] = a^{g(x)}\ln(a)g'(x)$.
4. (variable base, variable exponent) use logarithmic differentiation, as in Ex 3.7.8.

3.8. **Rates of Change in the Natural and Social Sciences.**

We know that if $y = f(x)$, then the derivative $\frac{dy}{dx}$ can be interpreted as the rate of change of y with respect to x. Now, with a myriad of differentiation rules at our disposal, we explore some of the applications of this concept to physics, economics, and other sciences.

Example 3.8.1 (Physics: Velocity and Acceleration)**.** The position of a particle is given by the equation $s(t) = t^3 - 9t^2 + 24t$, where s is measured in meters and t is in seconds.

(a) Find the velocity of the particle at time t.
(b) What is the velocity of the particle after 1 s? After 3 s?
(c) When is the particle at rest?
(d) When is the particle moving forward (that is, in the positive direction)?
(e) Draw a diagram to represent the motion of the particle.
(f) Find the total distance traveled by the particle during the first five seconds.
(g) Find the acceleration of the particle at time t after after 4 s.
(h) When is the particle speeding up? When is it slowing down?

Solution. We use the fact that velocity is the instantaneous rate of change of position ($v(t) = s'(t)$) and acceleration is the instantaneous rate of change of velocity ($a(t) = v'(t) = s''(t)$).

(a) $v(t) = s'(t) = 3t^2 - 18t + 24$.
(b) $v(1) = 9$ m/s, $v(3) = -3$ m/s.
(c) The particle is at rest when $v(t) = 0$. That is, when $3t^2 - 18t + 24 = 0 \implies 3(t-2)(t-4) = 0$, so at $t = 2$ and $t = 4$ s.
(d) The particle moves in the positive direction when $v(t) > 0$. That is, when $(t-2)(t-4) > 0$. This inequality is true when both factors are positive ($t > 4$) or when both factors are negative ($t < 2$). Consequently, the particle is moving forward for $t < 2$ and $t > 4$ and backward for $2 < t < 4$.
(e) Using the information from part (d), we can sketch the motion of the particle along the s-axis.
(f) From parts (d) and (e), the particle travels $|s(2) - s(0)| = 20$ meters from $t = 0$ to $t = 2$, $|s(4) - s(2)| = 4$ meters from $t = 2$ to $t = 4$, and $|s(5) - s(4)| = 4$ meters from $t = 4$ to $t = 5$. Thus, the total distance traveled by the particle during the first five seconds is $20 + 4 + 4 = 28$ m.
(g) $a(t) = v'(t) = 6t - 18$, so $a(4) = 6$ m/s^2.
(h) The particle speeds up when $a(t) > 0$ and slows down when $a(t) < 0$. Since $a(t)$ is a linear function, it is easy to see these inequalities hold when $t > 3$ and when $t < 3$ respectively.

Definition 3.8.1. If $C(x)$ represents the cost of producing x items, then $C'(x)$ is the instantaneous rate of change of the cost function and is referred to as the **marginal cost**.

Remark 3.8.1. The marginal cost of producing x items is approximately equal to the cost of producing one more item if x items have already been produced. Indeed, $C'(x) \approx \frac{C(x+1)-C(x)}{1} = C(x+1) - C(x)$.

Example 3.8.2 (Economics: Marginal Cost)**.** A company estimates that the cost (in dollars) of producing x items is given by $C(x) = 15000 + 6x + 0.02x^2$. Find the marginal cost of producing 500 items and interpret its meaning.

Solution. Clearly $C'(x) = 6 + 0.04x$, and it follows that $C'(500) = \$26$/item. This gives the rate at which the production cost is increasing when $x = 500$, and predicts the additional cost of producing the 501st item. Note that the actual cost of producing the 501st item is $C(501) - C(500) = \$26.02$.

In general, given an equation relating two or more variables, we can use a derivative to compute the instantaneous rate of change of one variable with respect to another.

Example 3.8.3. The volume of a spherical cell is $V(r) = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1\mu\text{m} = 10^{-6}$m). Find the instantaneous rate of change of V with respect to r when $r = 5\mu$m.

Solution. It is easy to compute that $V'(r) = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$, and hence $V'(5) = 100\pi\ \mu\text{m}^2$. Notice that the formula for V' is identical to the formula for the surface area of a sphere (even the units are correct!). In fact, this observation makes a lot of sense if you think about how an infinitesimal change in radius affects the volume of the sphere.

4. Applications of Differentiation.

4.1. **Related Rates.**

In a related rates problem, the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differeniate both sides with respect to time.

Example 4.1.1. If $z^2 = x^2 + y^2$, and we know that $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 3$, find $\frac{dz}{dt}$ when $x = 5$ and $y = 12$.

Solution. Differentiating both sides of the equation with respect to t using the Chain Rule, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \implies 2\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

When $x = 5$ and $y = 12$, the original equation implies that $z = 13$. Thus,

$$13\frac{dz}{dt} = 5 \cdot 2 + 12 \cdot 3 \implies \frac{dz}{dt} = \frac{46}{13}.$$

The general strategy for solving related rates application problems consists of the following steps.

1. Read the problem carefully.
2. Draw a diagram, if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use geometry to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Example 4.1.2. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

Solution. Let S be the snowball's surface area, let r be its radius, and let D be its diameter. Then $D = 2r$, so $S = 4\pi r^2 = 4\pi\left(\frac{D}{2}\right)^2 = \pi D^2$. We are given that $\frac{dS}{dt} = -1$, and we wish to find $\frac{dD}{dt}\big|_{D=10}$. Differentiating the equation $S = \pi D^2$ with respect to t using the Chain Rule gives

$$\frac{d}{dt}(S) = \frac{d}{dt}(\pi D^2) \implies \frac{dS}{dt} = 2\pi D\frac{dD}{dt} \implies \frac{dD}{dt} = -\frac{1}{2\pi D}.$$

Thus,

$$\frac{dD}{dt}\Big|_{D=10} = -\frac{1}{2\pi(10)} = -\frac{1}{20\pi} \text{ cm/min.}$$

Example 4.1.3. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 p.m.?

Solution. At time t hours after noon, let $x = 35t$ be the distance that ship A has traveled east, let $y = 25t$ be the distance that ship B has traveled north, and let D be the distance between the ships. We are given that $\frac{dx}{dt} = 35$ and $\frac{dy}{dt} = 25km$, and we wish to find $\frac{dD}{dt}$ when $t = 4$. From a diagram of the positions and movements of the two ships, we see that x, y, and D are related by $D^2 = (150 - x)^2 + y^2$. Differentiating this equation with respect to t, we have

$$\frac{d}{dt}(D^2) = \frac{d}{dt}\left((150 - x)^2 + y^2\right) \implies 2D\frac{dD}{dt} = -2(150 - x)\frac{dx}{dt} + 2y\frac{dy}{dt} \implies D\frac{dD}{dt} = (x - 150)\frac{dx}{dt} + y\frac{dy}{dt}.$$

When $t = 4$, we have $x = 140$ and $y = 100$, and hence $D = \sqrt{(150 - 140)^2 + 100^2} = 100.5$. In this case, it follows that

$$100.5\frac{dD}{dt} = (140 - 150) \cdot 35 + 100 \cdot 25 \implies \frac{dD}{dt} = 21.39 \text{ km/h.}$$

Example 4.1.4. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 in deep?

Solution. Let h be the height of the water in the trough, and let V be its volume. We are given $\frac{dV}{dt} = 12$ and we wish to find $\frac{dh}{dt}\big|_{h=1/2}$. Since the trough is 10 ft long, the volume V is equal to ten times the area of the iscosceles triangle of height h. From a simple diagram, we can use similar triangles to establish that this area must be $\frac{3}{2}h^2$. Thus, $V = 10 \cdot \frac{3}{2}h^2 = 15h^2$. Differentiating both sides of this equation with respect to t yields

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(15h^2\right) \implies \frac{dV}{dt} = 30h\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{1}{30h}\frac{dV}{dt}.$$

Hence,

$$\frac{dh}{dt}\Big|_{h=1/2} = \frac{1}{30(1/2)} \cdot 12 = \frac{4}{5} \text{ ft/min.}$$

Example 4.1.5. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

Solution. Let x be the distance from the man to the point on the path closest to the searchlight, and let θ be the angle between the beam of the searchlight and the perpendicular to the path. We are given that $\frac{dx}{dt} = 4$, and we wish to find $\frac{d\theta}{dt}$ when $x = 15$. The equation relating x and θ can be written as $\frac{x}{20} = \tan\theta$. Differentiating both sides of this equation with respect to t, we obtain

$$\frac{d}{dt}\left(\frac{x}{20}\right) = \frac{d}{dt}(\tan\theta) \implies \frac{1}{20}\frac{dx}{dt} = \sec^2\theta\frac{d\theta}{dt} \implies \frac{d\theta}{dt} = \frac{\cos^2\theta}{20}\frac{dx}{dt} = \frac{\cos^2\theta}{20}\cdot 4 = \frac{\cos^2\theta}{5}.$$

When $x = 15$, the length of the beam is 25, so $\cos\theta = \frac{20}{25} = \frac{4}{5}$. It follows that at this point the searchlight is rotating at a rate of

$$\frac{d\theta}{dt} = \frac{(4/5)^2}{5} = \frac{16}{125} = 0.128 \text{ rad/s.}$$

4.2. **Maximum and Minimum Values.**

Definition 4.2.1. Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all $x \in D$.
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all $x \in D$.

Definition 4.2.2. The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ for all x in an open interval containing the point c.
- **local minimum** value of f if $f(c) \leq f(x)$ for all x in an open interval containing the point c.

The maxima and minima of a function are commonly referred to as **extrema** or **extreme values**. It is thus common to discuss the absolute or global extrema and the local or relative extrema.

Example 4.2.1. Graphically illustrate the differences between absolute and relative extrema.

Example 4.2.2. Find and categorize all extrema of: (a) $f(x) = \sin x$, (b) $g(x) = x^2$, (c) $h(x) = x^3$.

Example 4.2.3. Use a calculator to graph the function $f(x) = 3x^4 - 16x^3 + 18x^2$, defined on $-1 \leq x \leq 4$, and conclude that absolute extrema can occur at the endpoints of an interval.

Theorem 4.2.1 (Extreme Value Theorem)**.** *If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c, d \in [a, b]$.*

Example 4.2.4. Note that if a function is not continuous on its domain, or if its domain is not a closed interval, then the Extreme Value Theorem does not guarantee the existence of any absolute extrema. Explore this concept by graphing different functions that fail to exhibit absolute extrema.

Theorem 4.2.2 (Fermat's Theorem)**.** *If f has a local maximum or local minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.*

Remark 4.2.1. When $f'(c) = 0$, it is not necessarily true that f has a local maximum or minimum at c (e.g. $f(x) = x^3$ at the origin). Furthermore, a function may have a local extreme value even at a point where its derivative does not exist (e.g. $f(x) = |x|$ at the origin). In other words, although Fermat's Theorem is useful, it is important to avoid reading too much into it.

Definition 4.2.3. A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

In terms of critical numbers, Fermat's Theorem can be rephrased as follows.

Theorem 4.2.3. *If f has a local maximum or minimum at c, then c is a ciritcal number of f.*

To find the absolute extrema of a continuous function on a closed interval, the following three-step procedure always works.

Proposition 4.2.1 (Closed Interval Method)**.** *To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:*

1. *Find the values of f at the critical numbers of f in (a, b).*
2. *Find the values of f at the endpoitns of the interval.*
3. *The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.*

Example 4.2.5. On the interval $[-1, 4]$, find the values of x where each of the following functions has an absolute maximum and absolute minimum: (a) $f(x) = x^{3/5}(4 - x)$, (b) $g(x) = 3x^4 - 16x^3 + 18x^2$.

Solution. For each function, we follow the steps of the Closed Interval Method.

(a) The derivative of f is given by $f'(x) = x^{3/5} \cdot (-1) + (4 - x) \cdot \frac{3}{5} x^{-2/5} = \frac{-5x+12-3x}{5x^{2/5}} = \frac{12-8x}{5x^{2/5}}$. Clearly $f'(x) = 0 \implies x = 3/2$, and $f'(x)$ is undefined for $x = 0$. Comparing $f(-1) = -5$, $f(0) = 0$, $f(3/2) = 3.1886$, and $f(4) = 0$, we conclude that f has an absolute minimum at $(-1, -5)$ and an absolute maximum at $(3/2, 3.1886)$.

(b) The derivative of g is given by $g'(x) = 12x^3 - 48x^2 + 36x = 12x(x - 1)(x - 3)$. Thus, $g'(x)$ is defined everywhere, and $g'(x) = 0 \implies x = 0, x = 1, x = 3$. Comparing $g(-1) = 37$, $g(0) = 0$, $g(1) = 5$, $g(3) = -27$, and $g(4) = 32$, we conclude that g has an absolute minimum at $(3, -27)$ and an absolute maximum at $(-1, 37)$.

4.3. **Derivatives and the Shapes of Curves.**

Consider an object moving in a straight line according to the differentiable position function $s = f(t)$. The following theorem implies that between any two times $t = a$ and $t = b$ there is a time $t = c$ at which the instantaneous velocity, $f'(c)$, is equal to the average velocity, $(f(b) - f(a))/(b - a)$, over the interval $[a, b]$.

Theorem 4.3.1 (Mean Value Theorem)**.** *If f is a differentiable function on the interval $[a, b]$, then there exists a number c between a and b such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

or, equivalently, $f(b) - f(a) = f'(c)(b - a)$.

Graphically, the Mean Value Theorem states that between $x = a$ and $x = b$ there is at least one point on the curve $y = f(x)$ where the tangent line is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Recall the Increasing/Decreasing Test given in Proposition 2.8.1, which can be proved using the Mean Value Theorem.

Example 4.3.1. Where is the function $f(x) = 3x^4 - 4x^3 - 36x^2 - 35$ is increasing? decreasing?.

Solution. First, we compute that $f'(x) = 12x^3 - 12x^2 - 72x = 12x(x-3)(x+2)$. To use the Increasing/Decreasing Test, we need to identify where $f'(x) > 0$ and where $f'(x) < 0$. Dividing the real line into four intervals whose endpoints are the critical numbers -2, 0, 3, we perform a sign test on the factors of $f'(x)$. For example, on the interval $(-\infty, -2)$, the factors $12x$, $x-3$, and $x+2$ are all negative, implying that $f'(x) < 0$. A similar analysis of the remaining intervals leads to the conclusion that the function f is decreasing on $(-\infty, -2)$ and $(0, 3)$, and increasing on $(-2, 0)$ and $(3, \infty)$. A graph confirms this result.

As a consequence of the Increasing/Decreasing Test, we have the following useful fact.

Proposition 4.3.1 (First Derivative Test)**.** *Suppose that c is a critical number of a continuous function f.*

(a) *If f' changes from positive to negative at c, then f has a local maximum at c.*
(b) *If f' changes from negative to positive at c, then f has a local minimum at c.*
(c) *If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.*

Example 4.3.2. Use the First Derivative Test to identify the local extrema of the function from Ex 4.3.1.

Solution. Since $f'(x)$ changes from negative to positive at $x = -2$, it follows that $f(-2) = -99$ is a local minimum of f. Similarly, $f(0) = -35$ is a local maximum and $f(3) = -224$ is a local minimum.

In view of the Concavity Test given in Proposition 2.8.2, we have an alternate derivative-based test for identifying the local extreme values of a function.

Proposition 4.3.2 (Second Derivative Test)**.** *Suppose f'' is continuous around c.*

(a) *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*
(b) *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*
(c) *If $f''(c) = 0$, the test is inconclusive, so $f(c)$ may be a maximum, a minimum, or neither.*

Example 4.3.3. Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Solution. If $f(x) = x^4 - 4x^3$, then $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ and $f''(x) = 12x^2 - 24x = 12x(x-2)$. Setting $f'(x) = 0$ gives the critical numbers $x = 0$ and $x = 3$, while setting $f''(x) = 0$ yields $x = 0$ and $x = 2$.

Since $f''(0) = 0$ and $f''(3) = 36 > 0$, the Second Derivative Test implies that $f(3) = -27$ is a local minimum but reveals nothing about $f(0)$. However, since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test implies that f does not have a local maximum or minimum at 0.

Dividing the real line into three intervals whose endpoints are 0 and 2, we use a sign test of the second derivative to determine that f is concave upward on $(-\infty, 0)$ and $(2, \infty)$, concave downward on $(0, 2)$, and hence has inflection points at $(0, 0)$ and $(2, -16)$. We are now equipped to roughly sketch the graph of f.

Example 4.3.4. Use the First and Second Derivative Tests to sketch a graph of the function $f(x) = x^{2/3}(6-x)^{1/3}$.

Solution. First, we compute $f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}}$ and $f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}$. Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4, 6.

A sign test of the factors of f', together with the First Derivative Test, reveals that $f(0) = 0$ is a local minimum, $f(4) = 2^{5/3}$ is a local maximum, and $f(6) = 0$ is neither.

Interval	$4-x$	$x^{1/3}$	$(6-x)^{2/3}$	$f'(x)$	f
$x < 0$	+	-	+	-	decreasing on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increasing on $(0, 4)$
$4 < x < 6$	-	+	+	-	decreasing on $(4, 6)$
$x > 6$	-	+	+	-	decreasing on $(6, \infty)$

Furthermore, since $|f'(x)| \to \infty$ as $x \to 0$ and $x \to 6$, it follows that f has vertical tangents at $(0, 0)$ and $(0, 6)$. Since $x^{4/3} \geq 0$ for all x, it is easy to see that $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$, and that $f''(x) > 0$ for $x > 6$. Thus, f is concave downward on $(-\infty, 0)$ and $(0, 6)$, concave upward on $(6, \infty)$, and hence has an inflection point at $(6, 0)$. Finally, we combine this information to sketch a graph of the function f.

Example 4.3.5. Discuss the curve $f(t) = t + \cos t$, defined on $-2\pi \leq t \leq 2\pi$, with respect to intervals of increase and intervals of decrease, local maxima and minima, concavity, and points of inflection. Use this information to sketch the curve.

Solution. We first obtain $f'(t) = 1 - \sin t$ and $f''(t) = -\cos t$, which are both defined on $-2\pi \leq t \leq 2\pi$. Since $f'(t) = 0$ if and only if $\sin t = 1$, it is clear that $t = -\frac{3\pi}{2}$ and $t = \frac{\pi}{2}$ are the only critical numbers of f on its domain.

Notice that $f'(t) \geq 0$ for all t. In particular, $f'(t) > 0$ on the intervals between the critical numbers. This implies that f is increasing on $\left(-2\pi, -\frac{3\pi}{2}\right)$, $\left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{\pi}{2}, 2\pi\right)$, and it follows that f has no local maxima nor local minima.

A sign test of the second derivative, together with the Second Derivative Test, shows that f is concave downward on $\left(-2\pi, -\frac{3\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$, concave upward on $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and hence has inflection points at $\left(-\frac{3\pi}{2}, -\frac{3\pi}{2}\right)$, $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$. Now we sketch a graph of f.

4.4. **Graphing with Calculus and Calculators.**

This section is omitted. It describes the intricacies of digital curve-plotting via graphing calculators, and in particular the necessity of selecting an appropriate window in order to graphically discover as much information as possible about a function and its derivatives.

4.5. **Indeterminate Forms and l'Hopital's Rule.**

The function $f(x) = \frac{\sqrt{x}-2}{x-4}$ is not defined at $x = 4$, and both the numerator and denominator are zero when the $x = 4$. Nevertheless, we can determine the exact value of $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ by factoring the denominator. Indeed

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}.$$

Likewise, the function $g(x) = \frac{\cos x}{x-\pi/2}$ is not defined at $x = \frac{\pi}{2}$, but in this case there is no obvious way to simplify the expression in order to evaluate $\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x-\pi/2}$. The next result provides a method for determining such limits.

Theorem 4.5.1 (L'Hôpital's Rule)**.** *Let f and g be differentiable functions, with $g'(x) \neq 0$ near a (except possibly at a). Suppose that*

$$\lim_{x\to a} f(x) = 0 \qquad \textit{and} \qquad \lim_{x\to a} g(x) = 0,$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x\to a} g(x) = \pm\infty$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right-hand side exists (or is $\pm\infty$).

Remark 4.5.1. L'Hôpital's Rule is also valid for one-sided limits and for limits at infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

Remark 4.5.2. Before using L'Hôpital's Rule, always be sure to check that the numerator and denominator either both converge to 0 or both converge to $\pm\infty$.

Example 4.5.1 (Indeterminate form of type 0/0)**.** Find $\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x-\pi/2}$.

Solution. As we have already observed, $\lim_{x\to\frac{\pi}{2}} \cos x = 0$ and $\lim_{x\to\frac{\pi}{2}}(x-\pi/2) = 0$. Applying L'Hôpital's Rule, we find that

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x-\pi/2} = \lim_{x\to\frac{\pi}{2}} \frac{-\sin x}{1} = \lim_{x\to\frac{\pi}{2}} (-\sin x) = -\sin\left(\frac{\pi}{2}\right) = -1.$$

Example 4.5.2. Find $\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$.

Solution. A naively application of L'Hôpital's Rule yields

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \lim_{x\to\pi^-} \frac{\cos x}{\sin x} = \lim_{x\to\pi^-} \cot x = -\infty.$$

This is **wrong**! Indeed, although the numerator approaches zero as $x \to \pi^-$, the denominator does not, so L'Hôpital's Rule cannot be applied. Instead, since the function $\frac{\sin x}{1-\cos x}$ is continuous at $x = \pi$, this limit can be easily determined by direct substitution as follows:

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \frac{\sin \pi}{1-\cos \pi} = \frac{0}{1-(-1)} = 0.$$

Example 4.5.3. Find $\lim_{x\to\left(\frac{\pi}{2}\right)^+} \frac{\cos x}{1-\sin x}$.

Solution. Since $\lim_{x\to\left(\frac{\pi}{2}\right)^+} \cos x = 0$ and $\lim_{x\to\left(\frac{\pi}{2}\right)^+}(1-\sin x) = 0$, we may apply L'Hôpital's Rule to obtain

$$\lim_{x\to\left(\frac{\pi}{2}\right)^+} \frac{\cos x}{1-\sin x} = \lim_{x\to\left(\frac{\pi}{2}\right)^+} \frac{-\sin x}{-\cos x} = \lim_{x\to\left(\frac{\pi}{2}\right)^+} \tan x = -\infty.$$

Example 4.5.4 (Indeterminate form of type ∞/∞)**.** Find $\lim_{x\to\infty} \frac{\ln(\ln x)}{x}$.

Solution. Notice that $\lim_{x\to\infty} \ln(\ln x) = \infty$ and $\lim_{x\to\infty} x = \infty$. By L'Hôpital's Rule, it follows that

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{x} = \lim_{x\to\infty} \frac{\left(\frac{1/x}{\ln x}\right)}{1} = \lim_{x\to\infty} \frac{1}{x\ln x} = 0.$$

Example 4.5.5 (Using L'Hôpital's Rule multiple times)**.** Find $\lim_{x\to 0} \frac{e^x-1-x}{x^2}$.

Solution. We see that $\lim_{x\to 0}(e^x - 1 - x) = 0$ and $\lim_{x\to 0} x^2 = 0$. Hence, L'Hôpital's Rule gives

$$\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \lim_{x\to 0} \frac{e^x-1}{2x}.$$

This is yet another indeterminate form, with $\lim_{x\to 0}(e^x-1) = 0$ and $\lim_{x\to 0} 2x = 0$, so a second application of L'Hôpital's Rule yields

$$\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \lim_{x\to 0} \frac{e^x-1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x\to a} f(x)g(x)$ will be, if it even exists. However, we can evalute this type of limit explicitly by writing the product fg as a quotient, either $fg = \frac{f}{1/g}$ or $fg = \frac{g}{1/f}$, which converts the limit into an indeterminate form of type $0/0$ or ∞/∞ so that we can use L'Hôpital's Rule.

Example 4.5.6 (Indeterminate form of type $0 \cdot \infty$)**.** Find $\lim_{x\to-\infty} x^2 e^x$.

Solution. Notice that $\lim_{x\to-\infty} x^2 = \infty$, whereas $\lim_{x\to-\infty} e^x = 0$. Rewriting the product x^2e^x as the quotient $\frac{x^2}{e^{-x}}$, we may apply L'Hôpital's Rule twice in order to obtain

$$\lim_{x\to-\infty} x^2e^x = \lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = \lim_{x\to-\infty} 2e^x = 0.$$

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit $\lim_{x\to a}[f(x) - g(x)]$ may be either $\pm\infty$ or some finite number. To find out which it is, we convert this indeterminate difference into an indeterminate quotient (for instance, by using a common denominator, rationalization, or factoring out a common factor) in order to apply L'Hôpital's Rule.

Example 4.5.7 (Indeterminate form of type $\infty - \infty$)**.** Find $\lim_{x\to1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

Solution. Since $\lim_{x\to1^+} \frac{1}{\ln x} = \infty$ and $\lim_{x\to\infty} \frac{1}{x-1} = \infty$, we are dealing with an indeterminate difference. By establishing a common demoninator between the terms, we can combine the difference into a single quotient: $\frac{1}{\ln x} - \frac{1}{x-1} = \frac{x-1-\ln x}{(x-1)(\ln x)}$. Then $\lim_{x\to1^+} \frac{x-1-\ln x}{(x-1)(\ln x)}$ is an indeterminate form of type $0/0$, so by two applications of L'Hôpital's Rule we have

$$\begin{aligned}
\lim_{x\to1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) &= \lim_{x\to1^+} \frac{x-1-\ln x}{(x-1)(\ln x)} \\
&= \lim_{x\to1^+} \frac{1-\frac{1}{x}}{(x-1)\cdot\frac{1}{x} + \ln x \cdot 1} \\
&= \lim_{x\to1^+} \frac{\frac{x-1}{x}}{\frac{x-1+x\ln x}{x}} \\
&= \lim_{x\to1^+} \frac{x-1}{x-1+x\ln x} \\
&= \lim_{x\to1^+} \frac{1}{1+x\cdot\frac{1}{x} + \ln x \cdot 1} \\
&= \frac{1}{2}.
\end{aligned}$$

Example 4.5.8. Find $\lim_{x\to\infty}(xe^{1/x} - x)$.

Solution. Notice that $\lim_{x\to\infty} xe^{1/x} = \infty$ and $\lim_{x\to\infty} x = \infty$, and that $xe^{1/x} - x = x(e^{1/x} - 1)$ can then be rewritten as the quotient $\frac{x}{(e^{1/x}-1)^{-1}}$. Then using L'Hôpital's Rule (three times!) we obtain

$$\begin{aligned}
\lim_{x\to\infty}(xe^{1/x} - x) &= \lim_{x\to\infty} \frac{x}{(e^{1/x}-1)^{-1}} \\
&= \lim_{x\to\infty} \frac{1}{-1(e^{1/x}-1)^{-2}\cdot e^{1/x}\cdot(-x^{-2})} \\
&= \lim_{x\to\infty} \frac{(e^{1/x}-1)^2}{x^{-2}e^{1/x}} \\
&= \lim_{x\to\infty} \frac{2(e^{1/x}-1)\cdot e^{1/x}\cdot(-x^{-2})}{x^{-2}\cdot e^{1/x}\cdot(-x^{-2}) + e^{1/x}\cdot(-2x^{-3})} \\
&= \lim_{x\to\infty} \frac{2(e^{1/x}-1)}{x^{-2}+2x^{-1}} \\
&= \lim_{x\to\infty} \frac{2e^{1/x}\cdot(-x^{2})}{-2x^{-3}-2x^{-2}} \\
&= \lim_{x\to\infty} \frac{e^{1/x}}{x^{-1}+1} \\
&= 1.
\end{aligned}$$

Several indeterminate forms arise from the limit $\lim_{x\to a}[f(x)]^{g(x)}$, namely

1. Type 0^0, if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.
2. Type ∞^0, if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = 0$.
3. Type 1^∞, if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$.

Each of these cases can be treated either by taking the natural logarithm

$$y = [f(x)]^{g(x)} \implies \ln y = g(x)\ln[f(x)]$$

or by writing the function as an exponential

$$[f(x)]^{g(x)} = e^{g(x)\ln[f(x)]}.$$

Example 4.5.9 (Indeterminate form of type 1^∞)**.** Find $\lim_{x\to 0}(1-2x)^{1/x}$.

Solution. Since $\lim_{x\to0}(1-2x) = 1$, and both $\lim_{x\to0^-}(1/x)$ and $\lim_{x\to0^+}(1/x)$ are infinite, we are dealing with an indeterminate power. Letting $y = (1-2x)^{1/x}$, and taking the natural logarithm of both sides, we obtain $\ln y = \frac{\ln(1-2x)}{x}$. Notice that $\lim_{x\to0}\frac{\ln(1-2x)}{x}$ is an indeterminate quotient of type $0/0$, so by L'Hôpital's Rule it follows that

$$\lim_{x\to0}\ln y = \lim_{x\to0}\frac{\ln(1-2x)}{x} = \lim_{x\to0}\frac{\left(\frac{-2}{1-2x}\right)}{1} = \lim_{x\to0}\frac{-2}{1-2x} = -2.$$

Now, using the fact that $y = e^{\ln y}$, we obtain the solution

$$\lim_{x\to0}(1-2x)^{1/x} = \lim_{x\to0} y = \lim_{x\to0} e^{\ln y} = e^{-2}.$$

4.6. **Optimization Problems.**

The methods we have learned in this chapter for finding extreme values can be practically applied to the class of so-called **optimization problems**, which consist of maximizing or minimizing some variable quantity. The general strategy for solving optimization problems consists of the following steps.

1. Read the problem carefully.
2. Draw a diagram.
3. Introduce notation. Assign a symbol, say Q, to the quantitiy that is to be optimized. Also select letters, such as $a, b, c, \ldots, x, y$, to represent other unknown quantities, and label the diagram with as much information as possible.
4. Express Q in terms of some of the other symbols from Step 3.

5. Use other relationships between the unknown quantities to eliminate all but one of the variables in the expression for Q. Thus, we are left with a function of the form $Q = f(x)$.
6. Use the methods of Section 4.2 and Section 4.3 to find the absolute maximum or minimum value of f. In particular, if the domain of f is a closed interval, then use the Closed Interval Method.

Notice the similarity between the steps above and those presented in Section 4.1 for solving related rate problems. In fact, many of these techniques are ubiquitous in the problem-solving strategies for all types of mathematical applications.

Example 4.6.1. A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four identical side-by-side pens. What is the largest possible total area of the pens?

Solution. Let A, l, and w be the area, length, and width respectively of the rectangular area. Then $A = lw$, and assuming that the area is divided into fourths along its length we also have $2l + 5w = 750$. Solving the latter equation for $l = \frac{750-5w}{2}$, and substituting into the former equation, we obtain $A = \left(\frac{750-5w}{2}\right) w = 375w - \frac{5}{2}w^2$. Having expressed $A = f(w)$ in terms of one variable, we now compute $f'(w) = 375 - 5w$. It follows that $w = 75$ is the only critical number of f. Since the domain of f is easily observed to be $[0, 150]$, and $f(0) = f(150) = 0$, the Closed Interval Method implies that $f(75) = 14062.5$ ft^2 is the largest possible total area of the pens.

Example 4.6.2. A box with a square base and open top must have a volume of 32000 cm^3. Find the dimensions of the box that minimize the amount of material used to construct it.

Solution. Let x and h be the side length and height respectively of the box, and let S be the amount of material used to construct it. Then $S = x^2 + 4xh$ and $x^2h = 32000$. Solving the latter equation for $h = \frac{32000}{x^2}$, and substituting into the former equation, we obtain $S = x^2 + 4x\left(\frac{32000}{x^2}\right) = x^2 + 128000x^{-1}$. Having expressed $S = f(x)$ in terms of one variable, we now compute $f'(x) = 2x - 128000x^{-2} = \frac{2x^3-128000}{x^2}$. It follows that $x = 0$ and $x = 40$ are the critical numbers of f. In this case, $x > 0$ is the only obvious domain restriction for f, so we apply the First Derivative Test. Since $f'(x) < 0$ for $x < 40$ and $f'(x) > 0$ for $x > 40$, we conclude that $x = 40$ is a local minimum, and is hence the absolute minimum, of f. Therefore, the dimensions $40 \times 40 \times 20$ cm minimize the amount of material needed to construct the box.

Expanding on the preceding example, wherein it was argued that a local minimum of the function f was actually the absolute minimum, we have the following general result based on the First Derivative Test.

Proposition 4.6.1 (First Derivative Test for Absolute Extreme Values)**.** *Suppose that c is a critical number of a continuous function f defined on an interval.*

(a) *If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.*
(b) *If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.*

Example 4.6.3. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Solution. A point (x, y) on the parabola satisfies $x = \frac{1}{2}y^2$. Thus, we wish to minimize the distance

$$d = \sqrt{(x-1)^2 + (y-4)^2} = \sqrt{\left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2}.$$

Equivalently, we can minimize $f(y) = d^2$, since it is easier to work with. Differentiating, we obtain

$$f'(y) = 2\left(\frac{1}{2}y^2 - 1\right)y + 2(y-4) = y^3 - 8,$$

so $f'(y) = 0$ when $y = 2$. Observe that $f'(y) < 0$ when $y < 2$ and $f'(y) > 0$ when $y > 2$, so by the First Derivative Test for Absolute Extreme Values, the absolute minimum occurs when $y = 2$. The corresponding x-value is $x = \frac{1}{2}(2)^2 = 2$. Thus, the point on the parabola that is closest to $(1, 4)$ is $(2, 2)$.

Example 4.6.4. Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution. Let (x, y) be the vertex of the rectangle that lies on the ellipse in the first quadrant. Then the area of the rectangle is $A = (2x)(2y) = 4xy$. Solving the equation of the ellipse for $y = \frac{b}{a}\sqrt{a^2 - x^2}$, and substituting into the expression for A, we obtain $A = 4x\frac{b}{a}\sqrt{a^2 - x^2}$. Having expressed $A = f(x)$ as a function of one variable, with domain $[0, a]$, we now compute

$$f'(x) = 4\frac{b}{a}x\left[\frac{1}{2}\left(a^2 - x^2\right)^{-1/2} \cdot (-2x)\right] + 4\frac{b}{a}\left(a^2 - x^2\right)^{1/2} = \frac{4\frac{b}{a}\left(a^2 - 2x^2\right)}{\sqrt{a^2 - x^2}}.$$

It follows that

$$f'(x) = 0 \implies 4\frac{b}{a}\left(a^2 - 2x^2\right) = 0 \implies x^2 = \frac{a^2}{2} \implies x = \frac{a}{\sqrt{2}}.$$

Since $f(0) = f(a) = 0$, the Closed Interval Method implies that $f\left(a/\sqrt{2}\right) = 2ab$ is the area of the largest rectangle that can be inscribed in the ellipse.

Example 4.6.5. An 8 ft tall fence runs parallel to a building at a distance of 4 ft from its base. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building.

Solution. Let L, x, and y be the length of the ladder, the distance from the base of the building to the base of the latter, and the distance from the base of the building to the top of the ladder respectively. Then $L = \sqrt{x^2 + y^2}$ and, by similar triangles, $\frac{y}{x} = \frac{8}{x-4}$. Solving the latter equation for $y = \frac{8x}{x-4}$, and substituting into the former equation, we obtain

$$L = \sqrt{x^2 + \left(\frac{8x}{x-4}\right)^2}.$$

Rather than working with L directly, we will minimize the simpler function $L^2 = f(x)$. Differentiating with respect to x yields

$$f'(x) = 2x + 2\left(\frac{8x}{x-4}\right) \cdot \left(\frac{8(x-4) - 8x}{(x-4)^2}\right) = \frac{2x((x-4)^3 - 256)}{(x-4)^3}.$$

It follows that $f'(x) = 0 \implies x = 4 + 256^{1/3} \approx 10.35$ (note that $x = 0$ and $x = 4$ are not critical numbers because the domain of f is $(4, \infty)$). Since $f'(x) > 0$ for $x < 10.35$ and $f'(x) < 0$ for $x > 10.35$, the First Derivative Test for Absolute Extreme Values implies that $f(10.35)$ is the absolute minimum value of f. The corresponding y-value is $y = \frac{8(10.35)}{10.35-4} = 13.04$, so the shortest the ladder can be is $L = 16.65$ ft.

Recall that if the $C(x)$ represents is the cost of producing x units of a certain product, then the **marginal cost** is given by the derivative $C'(x)$. The function $c(x) = C(x)/x$ represents the **average cost** of producing x units. Note that average cost is minimized when average cost equals marginal cost. Indeed

$$c'(x) = 0 \implies \frac{xC'(x) - C(x)}{x^2} = 0 \implies \frac{C(x)}{x} = C'(x).$$

If $p(x)$ is the price per unit that a company can charge if it sells x units, then p is called the **demand function** or **price function**. If x units are sold at price $p(x)$, then the total revenue is $R(x) = xp(x)$ and the total profit is $P(x) = R(x) - C(x)$. Then the derivatives R' and P' are respectively called the **marginal revenue function** and **marginal profit function**. Note that profit is maximized when marginal revenue equals marginal cost, since $P'(x) = 0 \implies R'(x) - C'(x) = 0 \implies R'(x) = C'(x)$.

Example 4.6.6. Assume that the cost function and demand function for a certain item are $C(x) = 680 + 4x + 0.01x^2$ and $p(x) = 12 - \frac{x}{500}$ respectively. Find the production level that will maximize profit.

Solution. As remarked above, maximum profit occurs when marginal revenue equals marginal cost. In this case, $C'(x) = 4 + 0.02x$, and

$$R(x) = xp(x) = 12x - \frac{x^2}{500} \implies R'(x) = 12 - \frac{x}{250}.$$

Setting $R'(x) = C'(x)$, we have

$$12 - \frac{x}{250} = 4 + 0.02x \implies 3000 - x = 1000 + 5x \implies 2000 = 6x \implies x \approx 333.33.$$

Thus, a production level of about $x = 333$ items will maximize profit.

Example 4.6.7. A company has been selling 1000 television sets a week at \$450 each. A survey indicates that for each \$10 rebate offered to the buyer, the number of sets sold will increase by 100 per week.

(a) Find the demand function.
(b) How large a rebate should the company offer the buyer in order to maximize revenue.
(c) If the weekly cost function is $C(x) = 68000 + 150x$, what rebate amount will maximize profit?

Solution.

(a) If x is the number of television sets sold per week, then the weekly increase in sales is $x - 1000$. For each increase of 100 units sold the price is decreased by \$10, so for each additional unit sold the decrease in price will be $\frac{1}{100} \times 10$. Thus, the demand function is given by
$$p(x) = 450 - \frac{10}{100}(x - 1000) = 550 - \frac{1}{10}x.$$
(b) The revenue function is $R(x) = xp(x) = 550x - \frac{1}{10}x^2$. It follows that
$$R'(x) = 0 \implies 550 - \frac{1}{5}x = 0 \implies 550 = \frac{1}{5}x \implies x = 2750.$$
Since $R'(x) < 0$ for $x < 2750$ and $R'(x) > 0$ for $x > 2750$, the First Derivative Test for Absolute Extreme Values implies that maximum revenue is achieved when $x = 2750$. The corresponding price is $p(2750) = 275$ and the rebate is 450 - 275 = 175. Therefore, to maximize revenue, the manufacturer should offer a rebate of \$175.
(c) Setting $R'(x) = C'(x)$, we have
$$550 - \frac{1}{5}x = 150 \implies 400 = \frac{1}{5}x \implies x = 2000.$$
The corresponding price is $p(2000) = 350$ and the rebate is 450 - 350 = 100. Therefore, to maximize profit, the manufacturer should offer a rebate of \$100.

4.7. **Newton's Method.**

This section is omitted. It introduces an iterative numerical scheme, called Newton's Method, that is particularly useful for approximating the roots of complicated equations.

4.8. **Antiderivatives.**

Definition 4.8.1. A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Theorem 4.8.1. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.*

Example 4.8.1. Find the most general antiderivative of: (a) $f(x) = \sec^2 x$, (b) $g(x) = e^x$, (c) $h(x) = x^6$.

Solution.

(a) Since $\frac{d}{dx}(\tan x) = \sec^2 x$, the most general antiderivative of f is $F(x) = \tan x + C$.
(b) Since $\frac{d}{dx}(e^x) = e^x$, the most general antiderivative of g is $G(x) = e^x + C$.
(c) Since $\frac{d}{dx}\left(\frac{1}{7}x^7\right) = x^6$, the most general antiderivative of h is $H(x) = \frac{1}{7}x^7 + C$.

The following table gives particular antidifferentiation formulas for many common functions.

Function	Particular antiderivative
$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
$x^n\ (n \neq -1)$	$(x^{n+1})/(n+1)$
$1/x$	$\ln \lvert x \rvert$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$1/\sqrt{1-x^2}$	$\sin^{-1} x$
$1/(1+x^2)$	$\tan^{-1} x$

In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function. The second formula says that the antiderivative of a sum is the sum of the antiderivatives. We assume the convention $F' = f$ and $G' = g$.

Example 4.8.2. Find all functions f such that $f'(x) = 5\sec^2 x + \frac{2\sqrt{x}+4}{x}$.

Solution. With $f'(x) = 5\sec^2 x + 2x^{-1/2} + 4x^{-1}$, we see that the general antiderivative is given by

$$f(x) = 5\tan x + 2\left(\frac{x^{1/2}}{1/2}\right) + 4\ln|x| + C = 5\tan x + 4\sqrt{x} + 4\ln|x| + C.$$

Example 4.8.3. Find f if $f'(x) = 7e^x + 5\sin x$ and $f(0) = 1$.

Solution. The general antiderivative of f' is $f(x) = 7e^x - 5\cos x + C$, and

$$f(0) = 1 \implies 7e^0 - 5\cos(0) + C = 1 \implies C = -1.$$

Hence, $f(x) = 7e^x - 5\cos x - 1$.

Example 4.8.4. Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 3$, and $f'(1) = 5$.

Solution. The general antiderivative of f'' is $f'(x) = 4x^3 + 3x^2 - 4x + C$, and

$$f'(1) = 5 \implies 4(1)^3 + 3(1)^2 - 4(1) + C = 5 \implies C = 2.$$

That is, $f'(x) = 4x^3 + 3x^2 - 4x + 2$. Next, the general antiderivative of f' is $f(x) = x^4 + x^3 - 2x^2 + 2x + D$, and

$$f(0) = 3 \implies (0)^4 + (0)^3 - 2(0)^2 + 2(0) + D = 3 \implies D = 3.$$

Hence, $f(x) = x^4 + x^3 - 2x^2 + 2x + 3$.

Example 4.8.5. A stone is dropped off a cliff and hits the ground with a speed of 120 ft/s. Determine the height of the cliff.

Solution. The stone accelerates due to gravity at a constant rate given by $a(t) = -32$ ft/s^2. With s and v representing position and velocity respectively, we know that $a(t) = v'(t)$ and $v(t) = s'(t)$. It follows that the general antiderivative of a is $v(t) = -32t + C$, and

$$v(0) = 0 \implies -32(0) + C = 0 \implies C = 0.$$

That is, $v(t) = -32t$. Next, the general antiderivative of v is $s(t) = -16t^2 + D$. Notice that $s(0) = D$ is the unknown height of the cliff, and we are given that $v(t) = -120$ when $s(t) = 0$. Solving $v(t) = -32t = -120$ for $t = 3.75$ s, the time at which the stone hits the ground, we then have that

$$s(3.75) = 0 \implies -16(3.75)^2 + D = 0 \implies D = 225 \text{ ft}.$$

5. Integrals.

5.1. Areas and Distances.

We begin by attempting to solve the **area problem**; that is, the problem of finding the area under a curve $y = f(x)$ between two numbers $x = a$ and $x = b$. The area of a rectangle is easily computed, and we apply this concept in a limiting sense to determine the area of more general regions with curved edges.

Example 5.1.1. Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

Solution. We first notice that the area, A, of the region in question must be less than 1 (and greater than 0) since it is contained within a square of side length one. To improve this estimate, we divide the unit interval $[0, 1]$ into four equal subintervals: $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$, $[3/4, 1]$. Now, using the values of $y = x^2$ at either the left or right endpoints of these subintervals, we can construct four rectangles with combined area approximately equal to A. In particular, the left endpoint approximation is

$$L_4 = \frac{1}{4}(0)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.2188,$$

while the right endpoint approximation is

$$R_4 = \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}(1)^2 = \frac{15}{32} = 0.4688.$$

Thus, we can conclude that $0.2188 < A < 0.4688$. We can obtain better estimates by increasing the number of subintervals, as shown in the following table.

n	L_n	R_n
10	0.2850000	0.3850000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

From this, it appears that a good estimate for the area under the curve is $A = 1/3$.

Example 5.1.2. Continuing the example above, show that $\lim_{n\to\infty} L_n = 1/3$ and $\lim_{n\to\infty} R_n = 1/3$.

Solution. Dividing the unit interval $[0, 1]$ into n equal subintervals $[0, 1/n]$, $[1/n, 2/n]$, …, $[(n-1)/n, 1]$, the right endpoint approximation of A is given by

$$\begin{aligned} R_n &= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\ &= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \\ &= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\ &= \frac{2n^2 + 3n + 1}{6n^2}. \end{aligned}$$

Since the numerator and denominator of this rational expression have the same degree, it follows that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2n^2 + 3n + 1}{6n^2} = \frac{2}{6} = \frac{1}{3}.$$

A similar calculation shows that $\lim_{n\to\infty} L_n = 1/3$. This confirms that $A = 1/3$ is indeed the area under the parabola $y = x^2$ from 0 to 1.

Definition 5.1.1. The **area** A of the region that lies under the graph of the continuous function f on some interval $[a, b]$ is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $x_0\,(= a)$, x_1, …, $x_n\,(= b)$ partition the interval $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$, and the **sigma notation**, $\sum_{i=1}^{n}$, is used to compactly denote a sum of many terms.

Remark 5.1.1. It can be proved that the limit in Definition 5.1.1 always exists, since we are assuming that f is continuous. Furthermore, the left endpoint approximation yields the same value in the limit:

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

In fact, instead of using left or right endpoints, we could take the height of the ith rectangle to be the value of f at any number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. We call the numbers x_1^*, x_2^*, $\ldots$, x_n^* the **sample points**, and a more general expression for the area A is:

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Example 5.1.3. Estimate the area, A, of the region under the curve $f(x) = e^{-x}$ between $x = 0$ and $x = 2$ by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

Solution. With $n = 4$, the subintervals of equal width $\Delta x = 0.5$ are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of these subintervals are $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$, and $x_4^* = 1.75$ respectively, and the sum of the areas of the four approximating rectangles is

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(x_i^*)\Delta x \\ &= f(0.25)\Delta x + f(0.75)\Delta x + f(1.25)\Delta x + f(1.75)\Delta x \\ &= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5) \\ &\approx 0.8557. \end{aligned}$$

Thus, an estimate for the area is $A \approx 0.8557$. With $n = 10$, the subintervals are $[0, 0.2]$, $[0.2, 0.4]$, $\ldots$, $[1.8, 2]$, and the midpoints are $x_1^* = 0.1$, $x_2^* = 0.3$, $\ldots$, $x_{10}^* = 1.9$. Thus, a better approximation for the area is

$$A \approx M_{10} = \sum_{i=1}^{10} f(x_i^*)\Delta x = 0.2\left(e^{-0.1} + e^{-0.3} + \cdots + e^{-1.9}\right) \approx 0.8632.$$

5.2. **The Definite Integral.**

Definition 5.2.1. If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0\ (= a)$, x_1, $\ldots$, $x_n\ (= b)$ be the endpoints of these subintervals, and we let x_1^*, x_2^*, $\ldots$, x_n^* be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided that this limit exists. If it does exist, we say that f is **integrable** on $[a, b]$.

Remark 5.2.1. Along with the introduction of the definite integral comes new notation and terminology.

- The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums.
- In the notation $\int_a^b f(x)\,dx$, the function $f(x)$ is called the **integrand**. The numbers a and b are called the **limits of integration**; in particular, a is the **lower limit** and b is the **upper limit**.
- The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a **Riemann sum**.

Remark 5.2.2. The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral, i.e. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(s)\,ds$.

We have defined the definite integral for an integrable function. It turns out that not all functions are integrable, but the following theorem shows that the most commonly occurring functions are in fact integrable.

Theorem 5.2.1. *If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.*

If f is integrable on $[a, b]$, then the limit in Definition 5.2.1 exists and gives the same value no matter how we choose the sample points x_i^*. To simplify the calculation of the integral we often take the sample points to be right endpoints. Then $x_i^* = x_i$, and the definition of the integral simplifies as follows.

Theorem 5.2.2. *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 1, 2, \ldots, n$.

Although right endpoint sample points are convenient for calculating limits, a better approximation of an integral using a finite number of rectangles is generally achieved by choosing "midpoint" sample points. Recall that this is precisely the approach that was used in Example 5.1.3.

Theorem 5.2.3 (Midpoint Rule)**.** *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i\Delta x = \Delta x\left[f(\bar{x}_1) + \cdots + f(\bar{x}_n)\right],$$

where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ for $i = 1, 2, \ldots, n$.

When we use a limit to evaluate a definite integral, we need to know how to work with sums. The following formulas are often useful.

Proposition 5.2.1. *For any positive integer n, the following properties hold.*

1. $\sum_{i=1}^n i = \frac{n(n+1)}{2}$.
2. $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.
3. $\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$.
4. $\sum_{i=1}^n c = nc$.
5. $\sum_{i=1}^n ca_i = c\sum_{i=1}^n a_i$.
6. $\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$.
7. $\sum_{i=1}^n (a_i - b_i) = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$.

When we know something about the definite integral of one or more functions, the following proposition provides formulas for useful generalizations.

Proposition 5.2.2. *For real numbers a, b, c, and integrable functions f and g, the following properties hold.*

1. $\int_a^a f(x)\,dx = 0$.
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.
3. $\int_a^b c\,dx = c(b - a)$.
4. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.
5. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.
6. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x) - \int_a^b g(x)\,dx$.
7. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$, *assuming that $a < b < c$.*

Additionally, we have the following comparison properties.

8. *If* $f(x) \geq 0$ *for* $a \leq x \leq b$*, then* $\int_a^b f(x)\,dx \geq 0$.
9. *If* $f(x) \geq g(x)$ *for* $a \leq x \leq b$*, then* $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
10. *If* $m \leq f(x) \leq M$ *for* $a \leq x \leq b$*, then* $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Example 5.2.1. Discuss positive, negative, and zero area "under" a curve in the context of Proposition 5.2.2 (specifically, properties 1, 2, and 7). Compute the exact area under a curve comprised of straight line segments and circular arcs.

5.3. **Evaluating Definite Integrals.**

In Section 5.2 we saw that computing definite integrals from the definition of a limit of Riemann sums is often a long and tedious process. The following important result, which forms one half of the Fundamental Theorem of Calculus (see Section 5.4), provides a useful shortcut for evaluating definite integrals when we happen to know an antiderivative of the **integrand**.

Theorem 5.3.1 (Evaluation Theorem)**.** *If* f *is continuous on the interval* $[a, b]$*, then*

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F *is any antiderivative of* f*, i.e.* $F' = f$.

Example 5.3.1. Use the Evaluation Theorem to exactly determine $\int_1^6 \frac{1}{x}\,dx$.

Solution. We have previously seen that $F(x) = \ln|x|$ is a particular antiderivative of $f(x) = \frac{1}{x}$. Therefore,

$$\int_1^6 \frac{1}{x}\,dx = \int_1^6 f(x)\,dx = F(6) - F(1) = \ln|6| - \ln|1| = \ln(6) \approx 1.7918.$$

Definition 5.3.1. If $F'(x) = f(x)$, then the **indefinite integral of $\boldsymbol{f(x)}$**, is given by

$$\int f(x)\,dx = F(x) + C,$$

where C is any constant. In other words, $\int f(x)\,dx$ is the general antiderivative of f.

Remark 5.3.1. It is important to distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a number, whereas an indefinite integral $\int f(x)\,dx$ is a function (or family of functions).

From Definition 5.3.1, we see that the table of antiderivative formulas given in Section 4.9 is equivalently a table of indefinite integrals.

Example 5.3.2. Find the indefinite integral $\int \left(5\sec x \tan x - 6\sqrt{x}\right)\,dx$.

Solution. Since $\frac{d}{dx}(\sec x)\,dx = \sec x \tan x$ and $\frac{d}{dx}\left(\frac{2}{3}x^{3/2}\right) = x^{1/2} = \sqrt{x}$, we find that

$$\int \left(5\sec x \tan x - 6\sqrt{x}\right)\,dx = 5\sec x - 9x^{3/2} + C.$$

Example 5.3.3. Evaluate $\int_0^{\pi/4} (\sin x + \cos x)\,dx$.

Solution. Since $\int (\sin x + \cos x)\,dx = -\cos x + \sin x + C$, we have

$$\int_0^{\pi/4} (\sin x + \cos x)\,dx = (-\cos x + \sin x)\Big|_0^{\pi/4} = (-\cos(\pi/4) + \sin(\pi/4)) - (-\cos(0) + \sin(0)) = 1.$$

Example 5.3.4. Evaluate $\int_1^9 \left(\frac{y^2 - y^2\sqrt{y}+1}{y^2}\right)\, dy$.

Solution. Rewriting the integrand as $1 - y^{1/2} + y^{-2}$, we compute

$$\begin{aligned}
\int_1^9 \left(\frac{y^2 - y^2\sqrt{y}+1}{y^2}\right) dy &= \int_1^n \left(1 - y^{1/2} + y^{-2}\right) dy \\
&= \left(y - \frac{2}{3}y^{3/2} - y^{-1}\right)\Big|_1^9 \\
&= \left(9 - \frac{2}{3}(9^{3/2}) - 9^{-1}\right) - \left(1 - \frac{2}{3}(1^{3/2}) - 1^{-1}\right) \\
&= -76/9 \approx -8.4444.
\end{aligned}$$

Corollary 5.3.2 (Net Change Theorem)**.** *The integral of a rate of change is the net change:*

$$\int_a^b F'(x)\, dx = F(b) - F(a).$$

Example 5.3.5. A particle moves along a straight line with velocity at time t given by $v(t) = t^2 - 8t + 12$.

(a) Find the displacement of the particle from $t = 0$ to $t = 9$.
(b) Find the total distance traveled during this time period.

Solution.

(a) If position is given by s, where $s' = v$, then the displacement from $t = 0$ to $t = 9$ is

$$s(9) - s(0) = \int_0^9 (t^2 - 8t + 12)\, dt = \left(\frac{1}{3}t^3 - 4t^2 + 12t\right)\Big|_0^9 = \left(\frac{1}{3}(9)^3 - 4(9)^2 + 12(9)\right) - (0) = 27.$$

(b) To calculate total distance, we need to pay attention to when the particle moves in the positive direction ($v(t) > 0$) and when it moves in the negative direction ($v(t) < 0$). In both cases, the distance traveled is $\int |v(t)|\, dt$. Since $v(t) = t^2 - 8t + 12 = (t-6)(t-2)$, we have $v(t) > 0$ when $0 < t < 2$ and $6 < t < 9$, and $v(t) < 0$ when $2 < t < 6$. Thus

$$\begin{aligned}
\int_0^9 |v(t)|\, dt &= \int_0^2 v(t)\, dt - \int_2^6 v(t)\, dt + \int_6^9 v(t)\, dt \\
&= \left(\frac{1}{3}t^3 - 4t^2 + 12t\right)\Big|_0^2 - \left(\frac{1}{3}t^3 - 4t^2 + 12t\right)\Big|_2^6 + \left(\frac{1}{3}t^3 - 4t^2 + 12t\right)\Big|_6^9 \\
&= 32/2 - (-32/3) + 27 \\
&= 145/3 \approx 48.3333.
\end{aligned}$$

5.4. **The Fundamental Theorem of Calculus.**

Theorem 5.4.1 (Fundamental Theorem of Calculus)**.** *Suppose f is continuous on an interval I.*

1. *If $g(x) = \int_a^x f(t)\, dt$ for any fixed $a \in I$, then $g'(x) = f(x)$ for all $x \in I$.*
2. *If $[a, b] \subset I$, then $\int_a^b f(x)\, dx = F(b) - F(a)$, where F is any antiderivative of f, i.e. $F' = f$.*

Proof. If f has an antiderivative, say F, in closed form, then the first part of the theorem is easily verified. Indeed,

$$g(x) = \int_a^x f(t)\, dt = F(t)\big|_a^x = F(x) - F(a),$$

which implies that

$$g'(x) = \frac{d}{dx}(F(x) - F(a)) = F'(x) = f(x).$$

Now suppose that f does not have an antiderivative in closed form (e.g. $f(t) = e^{t^2}$ or $f(t) = \frac{\sin t}{t}$). We will not give a formal proof for this more difficult case, but rather an intuitive justification. Let x and $x + h$ be in the open interval (a, b), and observe that $g(x + h) - g(x)$ is the area underneath the curve $y = f(t)$

between $t = x$ and $t = x + h$. This narrow strip of area is closely approximated by a rectangle of height $f(x)$ and width h. That is, $g(x+h) - g(x) \approx f(x) \cdot h$, or

$$\frac{g(x+h) - g(x)}{h} \approx \frac{f(x) \cdot h}{h} = f(x).$$

As h approaches 0, this approximation becomes better and better, and we conclude that

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} f(x) = f(x).$$

To prove the second part of the theorem, let $g(x) = \int_a^x f(t)\,dt$. Then $g'(x) = f(x)$ (by the first part of the Theorem) and $F'(x) = f(x)$ (because F is assumed to be an antiderivative of f), so $g(x) = F(x) + C$ for some constant C. To find C, we substitute $x = a$ into the above equation and obtain $g(a) = F(a) + C$. Since

$$g(a) = \int_a^a f(t)\,dt = 0,$$

it follows that $0 = F(a) + C$ or $C = -F(a)$. Therefore, $g(x) = F(x) - F(a)$. Substituting $x = b$ into this equation yields $g(b) = F(b) - F(a)$, but $g(b) = \int_a^b f(t)\,dt$ implies that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

□

The second part of the Fundamental Theorem of Calculus (FTC2) is the same as Theorem 5.3.1 (the Evaluation Theorem), which was illustrated in Examples 5.3.2, 5.3.3, and 5.3.4. We now turn our attention to the following examples of the first part of the Fundamental Theorem of Calculus (FTC1).

Example 5.4.1. If $g(x) = \int_0^x \sqrt{1+2t}\,dt$, find $g'(x)$.

Solution. Since $f(t) = \sqrt{1+2t}$ is continuous on $(-1/2, \infty)$, FTC1 gives $g'(x) = \sqrt{1+x^2}$ for $x > -1/2$.

Example 5.4.2. If $g(x) = \int_x^{10} e^{t^2}\,dt$, find $g'(x)$.

Solution. Notice that

$$g(x) = \int_x^{10} e^{t^2}\,dt = -\int_{10}^x e^{t^2}\,dt = \int_{10}^x \left(-e^{t^2}\right)\,dt.$$

Since $f(t) = -e^{t^2}$ is continuous on $\mathbb{R}$, FTC1 gives $g'(x) = -e^{x^2}$ for $x \in \mathbb{R}$.

Example 5.4.3. If $g(x) = \int_{e^x}^0 \sin^3 t\,dt$, find $g'(x)$.

Solution. First, we rewrite g as

$$g(x) = \int_{e^x}^0 \sin^3 t\,dt = -\int_0^{e^x} \sin^3 t\,dt = \int_0^{e^x} \left(-\sin^3 t\right)\,dt.$$

Now, let $u = e^x$. Given that $f(t) = -\sin^3 t$ is continuous on $\mathbb{R}$, the Chain Rule and FTC1 imply that

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\left[\int_0^{e^x} \left(-\sin^3 t\right)\,dt\right] \\
&= \frac{d}{dx}\left[\int_0^{u} \left(-\sin^3 t\right)\,dt\right] \\
&= \frac{d}{du}\left[\int_0^{u} \left(-\sin^3 t\right)\,dt\right] \cdot \frac{du}{dx} \\
&= -\sin^3 u \cdot \frac{d}{dx}(e^x) \\
&= -e^x \sin^3(e^x), \quad \text{for } x \in \mathbb{R}.
\end{aligned}$$

5.5. **The Substitution Rule.**

Definition 5.5.1. If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number. The differential dy is then defined in terms of x and dx by the equation $dy = f'(y)\,dy$.

Example 5.5.1. If $y = \sin x$, then $dy = \cos x\,dx$. If $y = 4x^3$, then $dy = 12x^2\,dx$.

Proposition 5.5.1 (Substitution Rule)**.** *If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example 5.5.2. Find $\int x^2\sqrt{x^3+1}\,dx$.

Solution. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$, or $dx = \frac{1}{3x^2}\,du$, so we have

$$\int x^2\sqrt{x^3+1}\,dx = \int x^2\sqrt{u}\left(\frac{1}{3x^2}\,du\right) = \frac{1}{3}\int u^{1/2}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C = \frac{2}{9}(x^3+1)^{3/2} + C.$$

Example 5.5.3. Find $\int e^{\cos t}\sin t\,dt$.

Solution. Let $u = \cos t$. Then $du = -\sin t\,dt$, or $dt = -\frac{1}{\sin t}\,du$, so we have

$$\int e^{\cos t}\sin t\,dt = \int e^u \sin t\left(-\frac{1}{\sin t}\,du\right) = -\int e^u\,du = -e^u + C = -e^{\cos t} + C.$$

Example 5.5.4. Find $\int \frac{dx}{5-3x}$.

Solution. Let $u = 5 - 3x$. Then $du = -3\,dx$, or $dx = -\frac{1}{3}\,du$, so we have

$$\int \frac{dx}{5-3x} = \int \frac{-\frac{1}{3}\,du}{u} = -\frac{1}{3}\int\frac{du}{u} = -\frac{1}{3}\cdot\ln|u| + C = -\frac{1}{3}\ln|5-3x| + C.$$

Example 5.5.5. Find $\int \sin w\cos^6 w\,dw$.

Solution. Let $u = \cos w$. Then $du = -\sin w\,dw$, or $dw = -\frac{1}{\sin w}\,du$, so we have

$$\int \sin w\cos^6 w\,dw = \int \sin w\cdot u^6\left(-\frac{1}{\sin w}\,dw\right) = -\int u^6\,du = -\frac{1}{7}u^7 + C = -\frac{1}{7}\cos^7 x + C.$$

Example 5.5.6. Find $\int \frac{x}{1+x^4}\,dx$.

Solution. Let $u = x^2$. Then $du = 2x\,dx$, or $dx = \frac{1}{2x}\,du$, so we have

$$\int \frac{x}{1+x^4}\,dx = \int\frac{x}{1+u^2}\left(\frac{1}{2x}\,du\right) = \frac{1}{2}\int\frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}(x^2) + C.$$

Proposition 5.5.2 (Substitution Rule for Definite Integrals)**.** *If g' is continuous on $[a,b]$, and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof. Let F be an antiderivative of f. Observe that $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$, since by the Chain Rule we have

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x) = f'(g(x))g'(x).$$

Thus, FTC2 gives

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\big|_a^b = F(g(b)) - F(g(a)).$$

On the other hand, FTC2 also gives

$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)),$$

and this completes the proof. □

Example 5.5.7. Find $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$.

Solution (1)**.** Let $u = x^2$. Then $du = 2x\,dx$, or $dx = \frac{1}{2x}\,du$. Moreover, the new limits of integration are given by $u = (0)^2 = 0$ and $u = (\sqrt{\pi})^2 = \pi$, so we have

$$\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \int_{u=0}^{u=\pi} x\cos u\left(\frac{1}{2x}\,du\right) = \frac{1}{2}\int_{u=0}^{u=\pi}\cos u\,du = \frac{1}{2}\cdot\sin u\big|_{u=0}^{u=\pi} = \frac{1}{2}(0-0) = 0.$$

Solution (2)**.** An alternate approach is to compute an indefinte integral and then use the original limits of integration. Let $u = x^2$. Then $du = 2x\,dx$, or $dx = \frac{1}{2x}\,du$, so we have

$$\int x\cos(x^2)\,dx = \int x\cos u\left(\frac{1}{2x}\,du\right) = \frac{1}{2}\int\cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2) + C.$$

Hence, $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\big|_0^{\sqrt{\pi}} = 0$.

Example 5.5.8. Find $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Solution. Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$, or $dx = 2\sqrt{x}\,du = 2u\,du$. Moreover, the new limits of integration are given by $u = \sqrt{1} = 1$ and $u = \sqrt{4} = 2$, so we have

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_{u=1}^{u=2}\frac{e^u}{u}\cdot 2u\,du = 2\int_{u=1}^{u=2} e^u\,du = 2\cdot e^u\big|_{u=1}^{u=2} = 2(e^2 - e) = 2e(e-1).$$

Example 5.5.9. Find $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$.

Solution. Let $u = \sqrt{1+2x}$. Then $du = \frac{2}{2\sqrt{1+2x}}\,dx$, or $dx = \sqrt{1+2x}\,du = u\,du$. Moreover, the new limits of integration are given by $u = \sqrt{1+2(0)} = 1$ and $u = \sqrt{1+2(4)} = 9$, so we have

$$\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx = \int_{u=1}^{u=3}\frac{x}{u}\cdot u\,du = \int_{u=1}^{u=3} x\,du.$$

Now, since $u = \sqrt{1+2x} \implies u^2 = 1+2x \implies u^2 - 1 = 2x \implies \frac{1}{2}(u^2-1) = x$, it follows that

$$\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx = \frac{1}{2}\int_{u=1}^{u=3}\left(u^2-1\right)\,du = \frac{1}{2}\cdot\left(\frac{u^3}{3} - u\right)\Bigg|_{u=1}^{u=3} = \frac{1}{2}\left(6 - \left(-\frac{2}{3}\right)\right) = \frac{10}{3}.$$

Proposition 5.5.3 (Integrals of Symmetric Functions)**.** *Suppose f is continuous on $[-a,a]$.*

(a) *If f is even (i.e. $f(-x) = f(x)$), then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$.*

(b) *If f is odd (i.e. $f(-x) = -f(x)$), then $\int_{-a}^a f(x)\,dx = 0$.*

Example 5.5.10. Find $\int_{-2}^2 (x^4 - 1)\,dx$.

Solution. Let $f(x) = x^4 - 1$. Then

$$f(-x) = (-x)^4 - 1 = x^4 - 1,$$

so f is an even function. By Proposition 5.5.3, it follows that

$$\int_{-2}^2 (x^4-1)\,dx = 2\int_0^2 (x^4-1)\,dx = 2\left(\frac{1}{5}x^5 - x\right)\Bigg|_0^2 = 2\left(\frac{32}{5} - 2\right) = \frac{44}{5} = 8.8.$$

Example 5.5.11. Find $\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\, dx$.

Solution. Let $f(x) = \frac{\tan x}{1+x^2+x^4}$. Then

$$f(-x) = \frac{\tan(-x)}{1+(-x)^2+(-x)^4} = \frac{-\tan x}{1+x^2+x^4} = -f(x),$$

so f is an odd function. Consequently, we have

$$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\, dx = 0.$$

5.6. **Integration by Parts.**

Proposition 5.6.1 (Integration by Parts)**.** *If f and g are differentiable functions, then we have the formula*

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx.$$

Equivalently, letting $u = f(x)$ and $v = g(x)$, we may write $\int u\, dv = uv - \int v\, du$.

Proof. The Product Rule states that

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

In the notation for indefinite integrals, this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\, dx = f(x)g(x),$$

or

$$\int f(x)g'(x)\, dx + \int g(x)f'(x)\, dx = f(x)g(x).$$

Rearranging the terms, we obtain the desired formula

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx.$$

Now, with $u = f(x)$ and $v = g(x)$, the differentials are $du = f'(x)\, dx$ and $dv = g'(x)\, dx$, so by the Substitution Rule we have $\int u\, dv = uv - \int v\, du$. $\square$

Example 5.6.1. Find $\int x \sin x\, dx$

Solution. Let $u = x$ and $dv = \sin x\, dx$. Then $du = dx$ and $v = -\cos x$, and we find that

$$\int x \sin x\, dx = -x\cos x - \int (-\cos x)\, dx = -x\cos x + \sin x + C.$$

Example 5.6.2. Find $\int \ln x\, dx$

Solution. Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\, dx$ and $v = x$, and we find that

$$\int \ln x\, dx = x\ln x - \int x \cdot \frac{1}{x}\, dx = x\ln x - \int 1\, dx = x\ln x - x + C.$$

Remark 5.6.1. LIATE is a mnemonic device for recalling which functions have priority to be selected as u when integrating by parts. It stands for: logarithmic functions, inverse trigonometric functions, algebraic functions, trigonometric functions, exponential functions.

Example 5.6.3. Find $\int x^2 e^x \, dx$

Solution (1). Let $u = x^2$ and $dv = e^x \, dx$. Then $du = 2x \, dx$ and $v = e^x$, and we find that

$$\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx.$$

It appears that should now use integration by parts a second time. Letting $u = x$ and $dv = e^x \, dx$, so $du = dx$ and $v = e^x$, we obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2\left(xe^x - \int e^x \, dx\right) = x^2 e^x - 2xe^x + 2e^x + C.$$

Solution (2). When integrating by parts multiple times, the following "tabular" method can be helpful.

	derivatives of u	antiderivatives of v
$+$	x^2	e^x
$-$	$2x$	e^x
$+$	2	e^x
$-$	0	e^x

From the table, we find the formula for the indefinite integral to be $\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C$.

Example 5.6.4. Find $\int \cos x \, e^x \, dx$

Solution. Letting $u = \cos x$ and $dv = e^x$, we construct the following table:

	derivatives of u	antiderivatives of v
$+$	$\cos x$	e^x
$-$	$-\sin x$	e^x
$+$	$-\cos x$	e^x
$-$	$\sin x$	e^x

Although it is clear that differentiating u will never yield zero, we can write

$$\int \cos x \, e^x \, dx = \cos x \, e^x + \sin x \, e^x - \int \cos x \, e^x \, dx.$$

It follows that $2 \int \cos x \, e^x \, dx = \cos x \, e^x + \sin x \, e^x$, and hence $\int \cos x \, e^x \, dx = \frac{1}{2} e^x(\cos x + \sin x) + C$.

Proposition 5.6.2 (Integration by Parts for Definite Integrals)**.** *If f and g are differentiable functions on the interval $[a, b]$, then we have the formula*

$$\int_a^b f(x)g'(x) \, dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x) \, dx$$

Equivalently, letting $u = f(x)$ and $v = g(x)$, we may write $\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du$.

Example 5.6.5. Find $\int_0^1 \tan^{-1} x \, dx$

Solution. Let $u = \tan^{-1} x$ and $dv = dx$. Then $du = \frac{1}{1+x^2} \, dx$ and $v = x$, and we find that

$$\int_0^1 \tan^{-1} x \, dx = x \tan^{-1} x\Big|_0^1 - \int \frac{x}{1 + x^2} \, dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1 + x^2} \, dx.$$

We now apply the method of substitution to the remaining integral term. Let $w = 1 + x^2$ (since u is already in use). Then $dw = 2x \, dx$, or $dx = \frac{1}{2x} \, dw$. Moreover, the new limits of integration are $w = 1 + (0)^2 = 1$ and $w = 1 + 1^2 = 2$, and so we have

$$\int_0^1 \frac{x}{1 + x^2} \, dx = \int_{w=1}^{w=2} \frac{x}{w} \cdot \frac{1}{2x} \, dw = \frac{1}{2} \int_{w=1}^{w=2} \frac{1}{w} \, dw = \frac{1}{2} \cdot \ln|w|\Big|_{w=1}^{w=2} = \frac{1}{2} \ln 2.$$

Hence, $\int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{1}{2} \ln 2$.

5.7. **Additional Techniques of Integration.**

Trigonometric Integrals: we can use trigonometric identities to integrate certain combinations of trigonometric functions.

Example 5.7.1. Find $\int \sin^5 x \cos^2 x\,dx$.

Solution. Using the Pythagorean identity and the substitution $u = \cos x$, we have

$$\begin{aligned}\int \sin^5 x \cos^2 x\,dx &= \int (1-\cos^2 x)\sin x \cos x\,dx \\ &= \int (\cos x - \cos^3 x)\sin x\,dx \\ &= \int (u^3 - u)\,du \\ &= \frac{1}{4}u^4 - \frac{1}{2}u^2 + C \\ &= \frac{1}{4}\cos^4 x - \frac{1}{2}\cos^2 x + C.\end{aligned}$$

In general, we try to write an integrand involving powers of sine and cosine in such a way that it only contains one sine factor (and the remainder of the expression in terms of cosine) or one cosine factor (and the remainder of the expression in terms of sine). If the integrand contains only even powers of both sine and cosine, however, this strategy fails, and it is useful to recall the half-angle identities.

Example 5.7.2. Find $\int \cos^2 x\,dx$.

Solution. If we write $\cos^2 x = 1 - \sin^2 x$, the integral is no simpler to evaluate. Using the half-angle formula $\cos^2 x = \frac{1+\cos(2x)}{2}$, however, we have

$$\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos(2x))\,dx = \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.$$

Trigonometric Substitutions: when dealing with integrals involving expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$, it is often effective to make a trigonometric substitution that eliminates the radical.

Example 5.7.3. Find $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Solution. Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$, so

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\,(3\cos\theta\,d\theta) = \int \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta}\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta.$$

Now, using the identity $1+\cot^2\theta = \csc^2\theta$, and the fact that $\int \csc^2\theta\,d\theta = -\cot\theta + C$, we have

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int (\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C.$$

The above example suggests that if an integrand contains a factor of the form $\sqrt{a^2-x^2}$, then a trigonometric substitution of $x = a\sin\theta$ or $x = a\cos\theta$ may be effective. However, this is not always the best method. To evaluate $\int x\sqrt{a^2-x^2}\,dx$, for instance, a simpler substitution is $u = a^2 - x^2$, because then $du = -2x\,dx$.

When an integrand contains an expression of the form $\sqrt{a^2+x^2}$, the substitution $x = a\tan\theta$ should be considered because the identity $1+\tan^2\theta = \sec^2\theta$ eliminates the root sign. Similarly, if the factor $\sqrt{x^2-a^2}$ occurs, the substitution $x = a\sec\theta$ is effective.

Example 5.7.4. Find $\int \frac{dx}{x^2\sqrt{x^2+4}}$.

Solution. Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$, and we find that

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} = \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\tan^2\theta+1}}\, d\theta = \frac{1}{4}\int \csc\theta\cot\theta\, d\theta.$$

Using the fact that $\int \csc\theta\cot\theta\, d\theta = -\csc\theta + C$, we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{1}{4}\csc\theta + C = -\frac{\sqrt{x^2+4}}{4x} + C.$$

Partial Fractions: when integrating rational functions (ratios of polynomials), it is sometimes convenient to express them as sums of simpler quotients that we already know how to integrate.

Example 5.7.5. Find $\int \frac{5x-4}{2x^2+x-1}\, dx$.

Solution. Notice that $\frac{5x-4}{2x^2+x-1} = \frac{5x-4}{(x+1)(2x-1)}$. In a case like this, where the numerator has a smaller degree than the denominator, we can write the given rational function as a sum of **partial fractions**:

$$\frac{5x-4}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1}.$$

To find the values of A and B, we multiply both sides of the above equation by $(x+1)(2x-1)$, obtaining

$$5x-4 = A(2x-1) + B(x+1) \implies 5x-4 = (2A+B)x + (-A+B).$$

The coefficients of x on either side of the equation must be equal, and likewise the constant terms must be equal. This yields a system of linear equations:

$$\begin{cases} 2A+B &= 5, \\ -A+B &= -4. \end{cases}$$

Solving for $A = 3$ and $B = -1$, it follows that

$$\frac{5x-4}{(x+1)(2x-1)} = \frac{3}{x+1} - \frac{1}{2x-1},$$

and each of the partial fractions on the right-hand side is easy to integrate. Indeed, using the substitutions $u = x+1$ and $u = 2x-1$, respectively, we have

$$\int \frac{5x-4}{2x^2+x-1}\, dx = \int \frac{3}{x+1}\, dx - \int \frac{1}{2x-1}\, dx = 3\ln|x+1| - \frac{1}{2}\ln|2x-1| + C.$$

Remark 5.7.1. The method of partial fractions can be extended to more complicated rational functions.

(a) If the degree of the numerator is greater than or equal to the degree of the denominator, we must first take the preliminary step of performing long division. For instance,

$$\frac{2x^3-11x^2-2x+2}{2x^2+x-1} = x - 6 + \frac{5x-4}{(x+1)(2x-1)}.$$

(b) If the denominator has more than two linear factors, we need to include a term corresponding to each factor. For example,

$$\frac{x+6}{x(x-3)(4x+5)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{4x+5}.$$

(c) If a linear factor is repeated, we need to include extra terms in the partial fraction expansion. Here's an example:

$$\frac{x}{(x+2)^2(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1}.$$

(d) When we factor a denominator as far as possible, it might happen that we obtain an irreducible quadratic factor ax^2+bx+c, where the discriminant b^2-4ac is negative. Then the corresponding partial fraction is of the form

$$\frac{Ax+B}{ax^2+bx+c}.$$

To integrate this type of expression, we consider the two terms $\frac{Ax}{ax^2+bx+c}$ and $\frac{B}{ax^2+bx+c}$ separately. The former can often be integrated using the Substitution Rule, while for the latter it may be helpful to complete the square in the denominator and then apply the formula $\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$.

Example 5.7.6. Find $\int \frac{x^2+2}{x^3-4x}\,dx$.

Solution. Since $x^3 - 4x = x(x-2)(x+2)$, we write

$$\frac{x^2+2}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}.$$

Multiplying by the common denominator yields

$$\begin{aligned} x^2+2 &= A(x-2)(x+2) + B(x)(x+2) + C(x)(x-2) \\ &= A(x^2-4) + B(x^2+2x) + C(x^2-2x) \\ &= (A+B+C)x^2 + (2B-2C)x - 4A. \end{aligned}$$

Equating the coefficients of like-powered terms, we obtain the linear system

$$\begin{cases} A+B+C &= 1, \\ 2B-2C &= 0, \\ -4A &= 2. \end{cases}$$

Solving for $A = -1/2$ and $B = C = 3/4$, it follows that

$$\frac{x^2+2}{x^3-4x} = -\frac{1}{2x} + \frac{3}{4(x-2)} + \frac{3}{4(x+2)},$$

and we are able to evaluate the integral

$$\int \frac{x^2+2}{x^3-4x}\,dx = -\frac{1}{2}\ln|x| + \frac{3}{4}\ln|x-2| + \frac{3}{4}\ln|x+2| + C.$$

5.8. **Integration Using Tables and Computer Algebra Systems.**

This section is omitted. It discusses computational strategies for evaluating complicated integrals.

5.9. **Approximate Integration.**

This section is omitted. It introduces two methods (the Trapezoidal Rule and Simpson's Rule) for accurately approximating the area under a curve when it is impossible to find the exact value of a definite integral.

5.10. **Improper Integrals.**

Definition 5.10.1 (Improper Integral of Type 1)**.**

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx,$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx,$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are said to be **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent for some $a \in \mathbb{R}$, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

Example 5.10.1. Calculate $\int_1^\infty \frac{1}{x^2}\,dx$.

Solution. Let $t \geq 1$. Then

$$\int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^t = -\frac{1}{t} - \left(-\frac{1}{1}\right) = 1 - \frac{1}{t}.$$

Since $\lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$, we conclude that

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{t^2}\,dx = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1.$$

Example 5.10.2. Show that $\int_1^\infty \frac{1}{x}\,dx$ is divergent.

Solution. Let $t \geq 1$. Then

$$\int_1^t \frac{1}{x}\,dx = \ln|x|\big|_1^t = \ln|t| - \ln|1| = \ln t.$$

Since $\lim_{t\to\infty}\ln t = \infty$, we conclude that $\int_1^\infty \frac{1}{x}\,dx$ is divergent.

The results of Examples 5.10.1 and 5.10.2 are generalized in the following proposition.

Proposition 5.10.1. *The improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.*

Example 5.10.3. Calculate $\int_{-\infty}^0 x^2 e^{x^3}\,dx$.

Solution. Let $t \leq 0$. Then, substituting $u = x^3$, we have

$$\int_t^0 x^2 e^{x^3}\,dx = \frac{1}{3}\int_{t^3}^0 e^u\,du = \frac{1}{3}e^u\bigg|_{t^3}^0 = \frac{1}{3} - \frac{1}{3}e^{t^3}.$$

Since $\lim_{t\to-\infty}\left(\frac{1}{3} - \frac{1}{3}e^{t^3}\right) = \frac{1}{3}$, we conclude that

$$\int_{-\infty}^0 x^2 e^{x^3}\,dx = \lim_{t\to-\infty}\int_t^0 x^2 e^{x^3}\,dx = \lim_{t\to-\infty}\left(\frac{1}{3} - \frac{1}{3}e^{t^3}\right) = \frac{1}{3}.$$

Example 5.10.4. Calculate $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$.

Solution. First, notice that

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx,$$

provided that the improper integrals on the right-hand side are convergent. Since

$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\tan^{-1}x\big|_t^0 = \lim_{t\to-\infty}(-\tan^{-1}t) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2},$$

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\tan^{-1}x\big|_0^t = \lim_{t\to\infty}\tan^{-1}t = \frac{\pi}{2},$$

we conclude that

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Definition 5.10.2 (Improper Integral of Type 2)**.**

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx,$$

provided this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx,$$

provided this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is said to be **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 5.10.5. Find $\int_1^5 \frac{1}{\sqrt{x-1}}\,dx$.

Solution. This is an improper integral of Type 2, since $f(x) = \frac{1}{\sqrt{x-1}}$ is discontinuous at $x = 1$. Thus,

$$\int_1^5 \frac{1}{\sqrt{x-1}}\,dx = \lim_{t\to 1^+} \int_t^5 \frac{1}{\sqrt{x-1}}\,dx = \lim_{t\to 1^+} \left(2\sqrt{x-1}\Big|_t^5\right) = \lim_{t\to 1^+} 2\left(2-\sqrt{t-1}\right) = 4.$$

Example 5.10.6. Find $\int_0^{\pi/2} \tan\theta\,d\theta$.

Solution. Again, this is an improper integral of Type 2, since $f(x) = \tan\theta$ is discontinuous at $\theta = \pi/2$. It follows that

$$\int_0^{\pi/2} \tan\theta\,d\theta = \lim_{t\to\pi/2^-} \int_0^t \tan\theta\,d\theta = \lim_{t\to\pi/2^-} \ln|\sec\theta|\Big|_0^t = \lim_{t\to\pi/2^-} (\ln|\sec t| - \ln|\sec(0)|) = \infty,$$

since $\lim_{t\to\pi/2^-} \sec t = \infty$, and hence $\int_0^{\pi/2} \tan\theta\,d\theta$ must be divergent.

Example 5.10.7. Find $\int_1^4 \frac{1}{x-2}\,dx$.

Solution. Since the line $x = 2$ is a vertical asymptote for the integrand, we begin by writing

$$\int_1^4 \frac{1}{x-2}\,dx = \int_1^2 \frac{1}{x-2}\,dx + \int_2^4 \frac{1}{x-2}\,dx.$$

Next, notice that

$$\int_1^2 \frac{1}{x-2}\,dx = \lim_{t\to 2^-} \int_1^t \frac{1}{x-2}\,dx = \lim_{t\to 2^-} \ln|x-2|\Big|_1^t = \lim_{t\to 2^-} (\ln|t-2| - \ln|-1|) = \lim_{t\to 2^-} \ln(1-t) = -\infty.$$

Thus, $\int_1^2 \frac{1}{x-2}\,dx$ diverges, so even without computing $\int_2^4 \frac{1}{x-2}\,dx$ we can conclude that $\int_1^4 \frac{1}{x-2}\,dx$ diverges.

Theorem 5.10.1 (Comparison Theorem for Improper Integrals)**.** *Let $a \in \mathbb{R}$, and suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for all $x \geq a$.*

(a) *If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.*

(b) *If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.*

Example 5.10.8. Show that $\int_0^\infty e^{-x^2}\,dx$ converges.

Solution. First, we write

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx.$$

Since e^{-x^2} is continuous everywhere, the first term on the right-hand side is simply an ordinary definite integral; that is, it is equal to some finite number, and does not impact the convergence of $\int_0^\infty e^{-x^2}\,dx$. On the other hand, $x \geq 1 \implies x^2 \geq x \implies -x \geq -x^2 \implies e^{-x} \geq e^{-x^2} \geq 0$, and since

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}\left(-e^{-x}\Big|_1^t\right) = \lim_{t\to\infty}\left(e^{-1} - e^{-t}\right) = e^{-1} < \infty,$$

we conclude by Theorem 5.10.1 that $\int_0^\infty e^{-x^2}\,dx$ converges.

6. Applications of Integration.

6.1. More About Areas.

Theorem 6.1.1. *The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all $x \in [a, b]$, is*

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Remark 6.1.1. When $f(x) \geq g(x) \geq 0$ for all $x \in [a, b]$ in the above theorem, a quick sketch easily justifies the result. If f and g are not necessarily nonnegative, we can apply a vertical shift to both of them in such a way that the transformed functions are nonnegative and the area between them remains unchanged, and then the first case applies.

Remark 6.1.2. A more fundamental approach to establishing Theorem 6.1.1 is to consider the limit as n approaches infinity of the Riemann sum $\sum_{i=1}^{n}[f(x_i^*) - g(x_i^*)]\Delta x$, which represents an n-rectangle approximation of the area between the curves f and g.

Example 6.1.1. Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = -\pi/4$, and $x = \pi/4$.

Solution. A quick sketch shows that $\cos x \geq \sin x$ for $-\pi/4 \leq x \leq \pi/4$. Hence, the area of the region in question is

$$A = \int_{-\pi/4}^{\pi/4} (\cos x - \sin x)\, dx = (\sin x + \cos x)\Big|_{-\pi/4}^{\pi/4} = (\sin(\pi/4) + \cos(\pi/4)) - (\sin(-\pi/4) + \cos(-\pi/4)) = \sqrt{2}.$$

Example 6.1.2. Find the area of the region enclosed by the curves $f(x) = 1 + 3x - x^2$ and $g(x) = x^2 + x + 1$.

Solution. We first find the points of intersection of these two parabolas by solving their equations simultaneously:

$$f(x) = g(x) \implies 1 + 3x - x^2 = x^2 + x + 1 \implies 2x^2 - 2x = 0 \implies 2x(x-1) = 0 \implies x = 0,\ x = 1.$$

Thus, the points of intersection are $(0, 1)$ and $(1, 3)$. Notice that $f(x) \geq g(x)$ on the interval $[0, 1]$. Hence, the area of the region enclosed by the curves f and g is given by

$$\begin{aligned} A &= \int_0^1 (f(x) - g(x))\, dx \\ &= \int_0^1 \left(1 + 3x - x^2 - (x^2 + x + 1)\right) dx \\ &= \int_0^1 \left(-2x^2 + 2x\right) dx \\ &= \left(-\frac{2}{3}x^3 + x^2\right)\Big|_0^1 \\ &= \frac{1}{3}. \end{aligned}$$

Example 6.1.3. Find the area of the region enclosed by the curves $4x + y^2 = 12$ and $x = y$.

Solution. First, we rewrite the first equation in terms of x as a function of y, i.e. $x = 3 - \frac{1}{4}y^2$. Then the y-values of the points of intersection of $x = 3 - \frac{1}{4}y^2$ and $x = y$ are given by

$$3 - \frac{1}{4}y^2 = y \implies y^2 + 4y - 12 = 0 \implies (y+6)(y-2) = 0 \implies y = -6,\ y = 2.$$

Since $3 - \frac{1}{4}y^2 \geq y$ on the interval $[0, 1]$, it follows that the area of the region enclosed by the curves $4x + y^2 = 12$ and $x = y$ is given by

$$A = \int_{-6}^{2} \left(\left(3 - \frac{1}{4}y^2\right) - y\right) dy = \left(3y - \frac{1}{12}y^3 - \frac{1}{2}y^2\right)\Big|_{-6}^{2} = \frac{64}{3} \approx 21.3333.$$

6.2. **Volumes.**

Definition 6.2.1. Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\, dx.$$

Example 6.2.1. Find the volume of a sphere of radius r.

Solution. If we place the sphere so that its center is at the origin, then its intersection with the plane P_x is a circle whose radius (by the Pythagorean Theorem) is $y = \sqrt{r^2 - x^2}$ for $-r \le x \le r$. Thus, the cross-sectional area of the sphere in P_x is given by

$$A = \pi y^2 = \pi(r^2 - x^2).$$

Using the definition of volume, with $a = -r$ and $b = r$, and the fact that $A(x)$ is even, we then have

$$V = \int_{-r}^{r} A(x)\, dx = \int_{-r}^{r} \pi(r^2 - x^2)\, dx = 2\pi \int_0^r (r^2 - x^2)\, dx = 2\pi \left(r^2 x - \frac{1}{3}x^3\right)\Big|_0^r = \frac{4}{3}\pi r^3.$$

Example 6.2.2. Find the volume of a pyramid whose base is a square with side-length L and whose height is h

Solution. If we place the vertex of the pyramid at the origin, and its central axis along the x-axis, then it intersects the plane P_x in a square of side length s for $0 \le x \le h$. We can express s in terms of x by observing from similar triangles that

$$\frac{x}{h} = \frac{s/2}{L/2} = \frac{s}{L} \implies s = \frac{L}{h}x.$$

Thus, the cross-sectional area of the pyramid in P_x is $A(x) = \frac{L^2}{h^2}x^2$, and is volume is

$$V = \frac{L^2}{h^2}\int_0^h x^2\, dx = \frac{L^2}{h^2}\left(\frac{1}{3}x^3\right)\Big|_0^h = \frac{1}{3}L^2 h.$$

Proposition 6.2.1. *If f is continuous on $[a, b]$, and $f(x) \ge 0$ on $[a, b]$, then the volume of the solid obtained by rotating the region under the curve $y = f(x)$ from a to b about the x-axis is given by $V = \int_a^b \pi(f(x))^2\, dx$.*

Proof. For each x between a and b, the intersection of the solid with the plane P_x is a circle of radius $f(x)$. Thus, the cross-sectional area is $A(x) = \pi(f(x))^2$, and the volume of the solid of revolution is given by

$$V = \int_a^b \pi(f(x))^2\, dx.$$

□

Example 6.2.3. Find the volume of the solid obtained by rotating the region under the curve $y = e^x$ from $x = 0$ to $x = \ln 4$ about the x-axis.

Solution. By Proposition 6.2.1, we have

$$V = \int_0^{\ln 4} \pi(e^x)^2\, dx = \pi \int_0^{\ln 4} e^{2x}\, dx = \frac{\pi}{2}\, e^{2x}\Big|_0^{\ln 4} = \frac{\pi}{2}\left(e^{2\ln 4} - e^0\right) = \frac{15\pi}{2} \approx 23.5619.$$

Example 6.2.4. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 4$, and $x = 0$ about the y-axis.

Solution. First, we rewrite $y = \sqrt{x}$ as $x = y^2$ for $y \ge 0$. Then, replacing x by y in Proposition 6.2.1, we obtain

$$V = \int_0^4 \pi(y^2)^2\, dy = \pi \int_0^4 y^4\, dy = \frac{\pi}{5}y^5\Big|_0^4 = \frac{1024\pi}{5} \approx 643.3982.$$

Example 6.2.5. Find the volume of the solid obtained by rotating the region under the curve $y = e^x$ from $x = 0$ to $x = \ln 4$ about the line $y = -1$.

Solution. In this case, we cannot conveniently apply Proposition 6.2.1. It is easy to see, however, that the cross-sectional areas of the solid are circles of radius $e^x + 1$. Thus, $A(x) = \pi(e^x + 1)^2 = \pi(e^{2x} + 2e^x + 1)$ for $0 \le x \le \ln 4$, and the volume of the solid is given by

$$V = \int_0^{\ln 4} \pi(e^{2x} + 2e^x + 1)\,dx = \pi\left(\frac{1}{2}e^{2x} + 2e^x + x\right)\Bigg|_0^{\ln 4} = \pi\left(\frac{27}{2} + \ln 4\right) \approx 14.8863.$$

Example 6.2.6. The region enclosed by the curves $y = \sqrt{x}$ and $y = x$ is rotated about the x-axis. Find the volume of the resulting solid.

Solution. Notice that, for $0 \le x \le 1$, the intersection of the specified solid with the plane P_x is a "washer" with inner radius x and outer radius $\sqrt{x}$. Thus, the area is $A(x) = \pi(\sqrt{x})^2 - \pi x^2 = \pi(x - x^2)$, and the volume of the solid is

$$V = \int_0^1 \pi(x - x^2)\,dx = \pi\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6} \approx 0.5236.$$

The preceding examples illustrate two important approaches for computing the volume of a solid of revolution – the **disk method** and the **washer method** – which are distinguished in the following remark.

Remark 6.2.1. In general, we calculate the volume of a solid of revolution using the basic defining formula $V = \int_a^b A(x)\,dx$ or $V = \int_c^d A(y)\,dy$, and we find the cross-sectional area $A(x)$ or $A(y)$ in one of two ways:

- If the cross-section is a circle or disk (as in Examples 6.2.3, 6.2.4, and 6.2.5), we find its radius, r, and use $A = \pi r^2$.
- If the cross-section is a washer (as in Example 6.3.2), we find its inner radius, r_{in}, and its outer radius, r_{out}, and use $A = \pi(r_{\text{out}}^2 - r_{\text{in}}^2)$.

6.3. **Volumes by Cylindrical Shells.**

Some problems involving solids of revolution are very difficult, if not impossible, to solve by the disk and washer methods described in Section Section 6.2. For instance, consider the problem of finding the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis. Slicing this solid perpendicular to the y-axis, we obtain a washer, but to determine its inner and outer radius we would need to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y, a daunting prospect. Instead, we now consider the so-called **method of cylindrical shells**.

Proposition 6.3.1. *If f is continuous on $[a, b]$, and $f(x) \ge 0$ on $[a, b]$, then the volume of the solid obtained by rotating the region under the curve $y = f(x)$ from a to b about the y-axis is given by $V = \int_a^b (2\pi x)f(x)\,dx$.*

Example 6.3.1. Find the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis.

Solution. Notice that the specified region is the same as the region under the curve $y = 2x^2 - x^3$ between $x = 0$ and $x = 2$. Thus, by Proposition (6.3.1), we have

$$V = \int_0^2 (2\pi x)(2x^2 - x^3)\,dx = 2\pi\int_0^2 (2x^3 - x^4)\,dx = 2\pi\left(\frac{1}{2}x^4 - \frac{1}{5}x^5\right)\Bigg|_0^2 = 2\pi\left(8 - \frac{32}{5}\right) = \frac{16\pi}{5} \approx 10.0531.$$

Example 6.3.2. Find the volume of the solid obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ about the y-axis.

Solution. Clearly, the enclosed region is contained between $x = 0$ and $x = 1$, and a typical shell has radius x, circumference $2\pi x$, and height $x - x^2$. Thus, following the logic of Proposition (6.3.1), we obtain

$$V = \int_0^1 (2\pi x)(x - x^2)\,dx = 2\pi\int_0^1 (x^2 - x^3)\,dx = 2\pi\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Bigg|_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6} \approx 0.5236.$$

6.4. **Arc Length.**

Suppose that a curve C is given by the parametric equations $x = f(t)$ and $y = g(t)$, where $a \le t \le b$, and assume that C is **smooth** in the sense that the derivatives $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero for $a < t < b$ (this condition ensures that C has no sudden change in direction). To calculate the length of C, we first divide the interval $[a, b]$ into n subintervals of equal length $\Delta t = \frac{b-a}{n}$. If $t_0, t_1, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of the points $P_i(x_i, y_i)$ that lie on C. The length, L, of C is approximated by the sum of the lengths of the distances between successive points P_i, and the approximation improves as n increases. Thus, we set

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|.$$

For computational purposes, we need a more convenient expression for L. If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, then the length of the ith approximating line segment is

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}.$$

Then, by the definition of the derivative, we have $f'(t_i) \approx \frac{\Delta x_i}{\Delta t}$ and $g'(t_i) \approx \frac{\Delta y_i}{\Delta t}$ for Δt small. Equivalently, $\Delta x_i \approx f'(t_i)\Delta t$ and $\Delta y_i \approx g'(t_i)\Delta t$, so

$$|P_{i-1}P_i| \approx \sqrt{[f'(t_i)\Delta t]^2 + [g'(t_i)\Delta t]^2} \approx \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2} \cdot \Delta t.$$

Thus,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2} \cdot \Delta t = \int_a^b \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2}\, dt.$$

We now formalize the concept of arc length in the following definition.

Definition 6.4.1 (Arc Length Formula)**.** If a smooth curve defined by the parametric equations $x = f(t)$ and $y = g(t)$, where $a \le t \le b$, is traversed exactly once as t increases from a to b, then its **arc length** is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Remark 6.4.1. If we are given a curve with equation $y = f(x)$ for $a \le x \le b$, then regarding x as a parameter yields the parametric equations $x = x$ and $y = f(x)$, and the arc length formula becomes

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Similarly, if $x = f(y)$ for $a \le y \le b$, then we can regard y as a parameter and obtain the arc length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dx.$$

Example 6.4.1. Find the exact length of the curve defined by $x = \cos t$ and $y = \sin t$, for $-\pi/2 \le t \le \pi$.

Solution. It is easy to see that $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t$. Thus, by the definition, we have

$$L = \int_{-\pi/2}^{\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_{-\pi/2}^{\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt = \int_{-\pi/2}^{\pi} 1\, dt = \pi - (-\pi/2) = \frac{3\pi}{2}.$$

This conclusion can be verified by plotting the parametric curve. Indeed, its graph traces out three-fourths of the unit circle in the Cartesian plane, which has known circumference (i.e. arc length) 2π.

Example 6.4.2. Find the exact length of the curve defined by $x = t^3$ and $y = t^2$, for $0 \le t \le 1$.

Solution. Since $\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 2t$, we can use the Substitution Rule to obtain

$$L = \int_0^1 \sqrt{(3t^2)^2 + (2t)^2}\, dt = \int_0^1 \sqrt{9t^4 + 4t^2}\, dt = \int_0^1 t\sqrt{9t^2 + 4}\, dt = \frac{1}{27}\left(13^{3/2} - 8\right) \approx 1.4397.$$

6.5. Average Value of a Function.

Suppose that we would like to calculate the average value of a function, f, on an interval $[a,b]$. Dividing $[a,b]$ into n equal subintervals, each with length $\Delta x = \frac{b-a}{n}$, we can choose arbitrary points $x_1^*, \ldots, x_n^*$ in successive subintervals, and compute the average of the numbers $f(x_1^*), \ldots, f(x_n^*)$ to be

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{n} = \frac{f(x_1^*) + \cdots + f(x_n^*)}{\left(\frac{b-a}{\Delta x}\right)} = (f(x_1^*) + \cdots + f(x_n^*)) \cdot \frac{\Delta x}{b-a} = \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

For large n, the above expression represents the average of a large number of closely-spaced values of the function f on the interval $[a,b]$. Thus, the fact that

$$\lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a} \int_a^b f(x)\,dx$$

motivates the following definition.

Definition 6.5.1. The **average value** of a function f on an interval $[a,b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Example 6.5.1. Find the average value of $f(x) = \cos x$ on the interval $[0, \pi/2]$.

Solution. By the definition, we have

$$f_{\text{ave}} = \frac{1}{\pi/2 - 0} \int_0^{\pi/2} \cos x\,dx = \frac{2}{\pi}\left(\sin x\Big|_0^{\pi/2}\right) = \frac{2}{\pi}(\sin(\pi/2) - \sin(0)) = \frac{2}{\pi}.$$

6.6. Applications to Physics and Engineering.

Definition 6.6.1. If a constant force of magnitude F is acting on a rigid body that is moving translationally in the direction of the force, then the **work** W done by this force along a path of length d is given by $W = Fd$.

A natural question is: how do we calculate work if an object is being acted on by a variable force?

Suppose that the object moves along the x-axis from $x = a$ to $x = b$, and at each point x between a and b a continuous force $f(x)$ acts on the object. We divide the interval $[a,b]$ into n equal subintervals of equal length $\Delta x = \frac{b-a}{n}$, and select a sample point x_i^* from each subinterval.

If n is large, then Δx is small, and since f is continuous it stays nearly constant on each subinterval. Thus, the work done in moving the object across the ith subinterval is approximately $f(x_i^*)\Delta x$, and the total work done in moving the object from a to b is appoximately given by $\sum_{i=1}^{n} f(x_i^*)\Delta x$.

As n approaches infinity, Δx approaches 0, and the approximation $W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$ becomes more and more accurate. Therefore, we define the work done in moving the object from a to b as

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx.$$

Example 6.6.1. When a particle is located at a distance of x ft from the origin, a force of $x^2 + 2x$ lbs acts on it. How much work is done in moving the particle from $x = 1$ to $x = 3$.

Solution. Following the discussion above, we conclude that the work done is given by

$$W = \int_1^3 (x^2 + 2x)\,dx = \left(\frac{1}{3}x^3 + x^2\right)\Bigg|_1^3 = \left(\frac{1}{3}(3)^3 + 3^2\right) - \left(\frac{1}{3}(1)^3 - 1^2\right) = \frac{50}{3} \approx 16.6667 \text{ ft-lbs}.$$