
Name: ______________________________

Complete the following problems to the best of your ability. Clearly number each question and write your name on each sheet of paper you turn in. Algebraic support must be shown to receive full credit (i.e. show work!). Answers should be exact unless otherwise specified.

1. (15 pts.) Determine whether each of the following **sequences** is convergent or divergent. If the sequence converges, identify the value of its limit.

(a) $\left\{ \dfrac{n}{n^3+1} \right\}_{n=1}^{\infty}$

(b) $\left\{ \dfrac{n^3}{n^3+1} \right\}_{n=1}^{\infty}$

(c) $\left\{ \dfrac{n^{10}}{n^3+1} \right\}_{n=1}^{\infty}$

Solution.

(a) The sequence is convergent and $\displaystyle\lim_{n\to\infty} \frac{n}{n^3+1} = 0$.

(b) The sequence is convergent and $\displaystyle\lim_{n\to\infty} \frac{n^3}{n^3+1} = 1$.

(c) The sequence is divergent, since $\displaystyle\lim_{n\to\infty} \frac{n^{10}}{n^3+1} = \infty$.

2. (15 pts.) Determine whether each of the following **series** is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^3+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{n^3+1}$

Solution.

(a) The series is convergent, by the Comparison Test, since $\dfrac{n}{n^3+1} \le \dfrac{n}{n^3} = \dfrac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(b) The series is divergent, by the Divergence Test, since $\displaystyle\lim_{n\to\infty} \frac{n^3}{n^3+1} = 1 \neq 0$.

(c) The series is divergent, by the Divergence Test, since $\displaystyle\lim_{n\to\infty} \frac{n^{10}}{n^3+1} = \infty \neq 0$.

3. (15 pts.) Explain why the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\left(\sqrt{2}\right)^n}$ converges and find the value of its sum.

Solution. This is a Geometric Series with first term $a = 1$ and common ratio $r = \dfrac{1}{\sqrt{2}}$. Its sum is given by

$$s = \frac{a}{1-r} = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{1}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1} = 2+\sqrt{2}.$$

4. (10 pts.) Use the Integral Test to prove that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges.

Solution. Consider the function $f(x) = \dfrac{1}{x\ln x}$. By the method of substitution, with $u = \ln x$, we find that

$$\int f(x)\,dx = \int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln u + C = \ln(\ln x) + C.$$

Therefore,

$$\int_2^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_2^t f(x)\,dx = \lim_{t\to\infty} \left(\ln(\ln x)\Big|_2^{\infty}\right) = \lim_{t\to\infty} (\ln(\ln t) - \ln(\ln 2)) = \infty.$$

Since the integral diverges, the Integral Test implies that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges.

5. (15 pts.) Consider the alternating series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\ln(n)}$.

(a) Show that the series converges.

(b) What is the smallest positive integer, n, for which $|s - s_n| < 0.1$?

Solution.

(a) Let $b_n = \dfrac{1}{\ln(n)} \geq 0$. Since

$$n+1 \geq n \implies \ln(n+1) \geq \ln(n) \implies \frac{1}{\ln(n+1)} \leq \frac{1}{\ln(n)} \implies b_{n+1} \geq b_n$$

and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0,$$

the Alternating Series Test implies that the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges.

(b) Recall that for a convergent alternating series $\sum_{n=0}^{\infty}(-1)^n b_n$, we have the remainder estimate $|R_n| = |s - s_n| \leq b_{n+1}$. Solving the inequality $b_{n+1} \leq 0.1$ yields

$$b_{n+1} \leq 0.1 \implies \frac{1}{\ln(n+1)} \leq 0.1 \implies 10 \leq \ln(n+1) \implies n \geq e^{10} - 1 \approx 22025.46$$

so we need a minimum of $n = 22026$ terms in order to ensure an error of less than 0.1.

6. (15 pts.) Determine the radius of convergence and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n+1}$.

Solution. With $a_n = \dfrac{(x-2)^n}{n+1}$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{(n+1)+1} \cdot \frac{n+1}{(x-2)^n}\right| = \frac{n+1}{n+2} \cdot |x-2| \longrightarrow |x-2|.$$

By the Ratio Test, the series converges when $|x-2|<1$, so the radius of convergence is $R=1$. The interval of converge is obtain from

$$|x-2|<1 \implies -1<x-2<1 \implies 1<x<3,$$

but it remains to check what happens at the endpoints. At $x=1$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1},$$

which converges by the Alternating Series Test, while at $x=3$ we have

$$\sum_{n=0}^{\infty} \frac{1^n}{n+1},$$

which diverges by comparison against the Harmonic Series. Hence, the interval of convergence is $[1,3)$.

7. (15 pts.) Find the Maclaurin series for the function $f(x)=e^{-x^2}$. What is its radius of convergence?

Solution. Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. It follows that

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!},$$

and the radius of convergence of this Maclaurin series is also $R=\infty$.

E.C. (10 pts.) Find a power series representation for the function $f(x) = \frac{1}{(1+x)^2}$. Identify its radius of convergence and interval of convergence.

Solution. First, observe that

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

for $|x|<1$. Hence,

$$\begin{aligned}
\frac{1}{(1+x)^2} &= -\frac{d}{dx}\left(\frac{1}{1+x}\right) \\
&= -\frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n x^n\right) \\
&= -\sum_{n=1}^{\infty} (-1)^n n x^{n-1} \\
&= -\sum_{n=0}^{\infty} (-1)^{n+1} (n+1) x^n \\
&= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n.
\end{aligned}$$

The radius of convergence of this power series is also $R=1$. Checking the endpoints $x=-1$ and $x=1$ shows that the interval of convergence is $(-1,1)$.