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1. Functions and Models.

1.1. Four Ways to Represent a Function.

Definition 1.1.1. A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of f, and the **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

A function can be thought of as a machine. For example, a machine that makes pies (e.g. if you put in apples, you get an apple pie; you don't also get a peach pie or a cherry pie).

The four possible ways to represent a function are: verbally (by a description in words), numerically (by a table of values), visually (by a graph), and algebraically (by an explicit formula).

Example 1.1.1 (Using function notation)**.** If $f(x) = x^2 + 5x$, evaluate the quotient $\frac{f(a+h)-f(a)}{h}$.

Solution. $\frac{f(a+h)-f(a)}{h} = \frac{[(a+h)^2+5(a+h)]-[a^2+5a]}{h} = \frac{[a^2+2ah+h^2+5a+5h]-[a^2+5a]}{h} = \frac{2ah+h^2+5h}{h} = 2a + h + 5.$

Example 1.1.2 (Domain)**.** Identify the domain of the following functions:

(a) $f(x) = 1 - x^7 + 12x^2$.
(b) $g(x) = \sqrt{x+2}$.
(c) $h(x) = \frac{1}{x^2-x}$.

Solution.

(a) The domain of a polynomial is all real numbers, so the domain of f is $(-\infty, \infty)$ or $\mathbb{R}$.
(b) The radicand can't be negative, so $x + 2 \geq 0$ implies that the domain of g is $[-2, \infty)$.
(c) The denominator can't be zero, so $x^2 - x \neq 0$ implies that the domain of h is $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$.

Proposition 1.1.1 (Vertical Line Test)**.** *A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects it more than once.*

Example 1.1.3. Illustrate the use of the Vertical Line Test by comparing the graphs of several functions and non-functions.

A **piecewise defined function** is one that is defined by different formulas over different parts of its domain.

Example 1.1.4. Consider the piecewise function $f(x) = \begin{cases} 4-x, & x < 1; \\ x^2, & 1 \leq x < 2. \end{cases}$ Find $f(0)$, $f(1)$, $f(3)$.

Solution. $f(0) = 4 - 0 = 4$, $f(1) = 1^2 = 1$, $f(3)$ is not defined because 3 is not in the domain of f.

It is straightforward to graph the function in the example above. Conversely, given a graph of a simple piecewise function, we should be able to recover its explicit formula.

Example 1.1.5. Sketch a simple piecewise function and use its graph to recover its formula.

Definition 1.1.2 (Function symmetry)**.**

- A function f is **even** if $f(-x) = f(x)$ for every x in its domain. If a function is even, its graph is symmetric about the x-axis.
- A function f is **odd** if $f(-x) = -f(x)$ for every x in its domain. If a function is odd, its graph is symmetric about the y-axis.

Example 1.1.6. Determine whether each of the following functions is even, odd, or neither:
(a) $f(x) = x^5 - 7x$.
(b) $g(x) = \frac{6+x^4}{x^2}$.
(c) $h(x) = 1 + x^3 + x^2$.

Solution. (a) odd, (b) even, (c) neither. The symmetry of polynomials is easy to determine!

Definition 1.1.3 (Increasing and decreasing functions)**.**
- A function f is **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$. Graphically, an increasing function "rises" from left to right.
- A function f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$. Graphically, a decreasing function "falls" from left to right.

1.2. **Mathematical Models: A Catalog of Essential Functions.**

Definition 1.2.1. A function P is called a **polynomial** if it has the form $P(x) = a_n x^n + \cdots + a_1 x + a_0$, where n is a nonnegative integer and the numbers $a_0, a_1, \ldots, a_n$ are constants called the **coefficients** of the polynomial. If the leading coefficient a_n is nonzero, then the **degree** of the polynomial is n. The domain of any polynomial is $\mathbb{R}$.

Example 1.2.1. Sketch various simple polynomials, e.g. linear, quadratic, cubic, etc.

A **linear function** is one whose graph is a straight line. Linear functions are commonly written in **slope-intercept form**. For example, if y is a linear function of x, we may write the relationship between the variables as $y = mx + b$, where m is the slope of the line and b is the y-intercept.

The slope of a linear function represents a rate of change of the dependent variable with respect to the independent variable. For example, in a real-life linear model, slope could represent the rate of change of temperature with respect to elevation.

The y-intercept generally represents some sort of initial state of the model. This is especially true when the independent variable represents time, such as in the case of population models or depreciation models.

A polynomial of degree 2 is called a **quadratic function**. Its graph is a parabola, so it can be used to model physical scenarios such as the height of an object with respect to time after it is thrown up in the air.

Definition 1.2.2. A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**.

Example 1.2.2. The general power function $f(x) = x^a$ can take on many forms:
- If $a = n$, where n is a positive integer, the function f is a simple polynomial.
- If $a = 1/n$, where n is a positive integer, then $f(x) = x^{1/n} = \sqrt[n]{x}$, is a **root function**.
- If $a = -1$, then $f(x) = x^{-1} = 1/x$ is the **reciprocal function**.

Definition 1.2.3. A **rational function** f is a ratio of two polynomial functions, e.g. $f(x) = P(x)/Q(x)$.

Example 1.2.3. A simple rational function is the reciprocal function $f(x) = 1/x$. Rational functions arise in physics and chemistry, such as in the case of Boyle's Law: when temperature is constant, the volume V of a gas is inversely proportional to the pressure P; in symbols, $V = C/P$.

Definition 1.2.4. A function f is called an **algebraic function** if it can be constructed using **algebraic operations**, i.e. the four arithmetic operations $+$, $-$, $\times$, $\div$, along with the taking of roots.

Example 1.2.4. An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity v is given by $m = f(v) = \frac{m_0}{1-v^2/c^2}$, where m_0 is the rest mass of the particle and $c = 3.0 \times 10^5$ km/sec is the speed of light in a vacuum.

Trigonometric functions such as sine, cosine, and tangent can also be used in mathematical modeling. The periodic nature of these functions makes them ideal representatives of repetitive phenomena such as vibrating springs and sound waves.

Exponential and logarithmic functions will be reviewed in Sections 1.5 and 1.6 respectively.

Remark 1.2.1. Choosing an appropriate mathematical model is the most difficult part of many practical problems, especially those in which the goal is to make accurate predictions about the future. Graphical tools such as scatter plots can be helpful for visualizing trends in the available data, and in turn these trends can hint at choosing one particular type of model over another. Many calculators offer the capability to analyze scatter plots and produce linear or quadratic regressions (i.e. "best-fit" lines).

1.3. **New Functions from Old Functions.**

Definition 1.3.1. There are ten basic transformations that can be used to modify functions. We can often interpret a seemingly complex function as some variant of a simple "parent function," arrived at by applying one or more of these transformations.

- Vertical and Horizontal Shifts: suppose $c > 0$.
  1. $y = f(x) + c$ is a vertical shift of $y = f(x)$ a distance of c units upward.
  2. $y = f(x) - c$ is a vertical shift of $y = f(x)$ a distance of c units downward.
  3. $y = f(x + c)$ is a horizontal shift of $y = f(x)$ a distance of c units to the left.
  4. $y = f(x - c)$ is a horizontal shift of $y = f(x)$ a distance of c units to the right.
- Vertical and Horizontal Stretching: suppose $c > 1$.
  5. $y = cf(x)$ is a vertical stretch of $y = f(x)$ by a factor of c.
  6. $y = (1/c)f(x)$ is a vertical shrink of $y = f(x)$ by a factor of c.
  7. $y = f(cx)$ is a horizontal shrink of $y = f(x)$ by a factor of c.
  8. $y = f(x/c)$ is a horizontal stretch of $y = f(x)$ by a factor of c.
- Reflecting:
  9. $y = -f(x)$ is a reflection of $y = f(x)$ across the x-axis.
  10. $y = f(-x)$ is a reflection of $y = f(x)$ across the y-axis.

Example 1.3.1. Illustrate the 10 transformations on a simple function such as $y = x^2$ or $y = \sin x$

Definition 1.3.2. To form an **arithmetic combination** of two functions, f and g, we simply apply one of the four arithmetic operations. The rules are exactly as expected:

- Sums: $(f + g)(x) = f(x) + g(x)$.
- Differences: $(f - g)(x) = f(x) - g(x)$.
- Products: $(fg)(x) = f(x)g(x)$.
- Quotients: $(f/g)(x) = f(x)/g(x)$, provided $g(x) \neq 0$.

Example 1.3.2. Let $f(x) = x^2 - 1$ and $g(x) = 2x + 1$. Find (a) $f + g$, (b) $f - g$, (c) fg, (d) f/g, and their domains.

Solution. The functions $f + g$, $f - g$, and fg each have domain $\mathbb{R}$. The domain of f/g is $2x + 1 \neq 0$; that is, $(-\infty, -1/2) \cup (-1/2, \infty)$. The rules for each function are:

(a) $(f + g)(x) = f(x) + g(x) = (x^2 - 1) + (2x + 1) = x^2 + 2x$.
(b) $(f - g)(x) = f(x) - g(x) = (x^2 - 1) - (2x + 1) = x^2 - 2x - 2$.
(c) $(f * g)(x) = f(x)g(x) = (x^2 - 1)(2x + 1) = 2x^3 + x^2 - 2x - 1$.
(d) $(f/g)(x) = f(x)/g(x) = (x^2 - 1)/(2x + 1)$.

Definition 1.3.3 (Function composition)**.** Given two functions, f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by $(f \circ g)(x) = f(g(x))$ for all x in the domain of g such that $g(x)$ is in the domain of f. That is, $(f \circ g)(x)$ is defined if and only if $g(x)$ and $f(g(x))$ are defined.

Example 1.3.3. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. Find the domain and formulas for $f \circ g$ and of $g \circ f$.

Solution. First note that the domain of f is $(-\infty, \infty)$ and the domain of g is $[0, \infty)$. It follows that the domain of $f \circ g$ is $[0, \infty)$ and the domain of $g \circ f$ is $(-\infty, \infty)$. It is easy to see that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, but the are not the same function because each has a different domain.

Example 1.3.4. Use the table below to find: (a) $f(g(0))$, (b) $f(f(2))$, (c) $(g \circ f)(1)$.

x	-2	-1	0	1	2
$f(x)$	6	5	0	-2	-1
$g(x)$	-3	-1	1	3	5

Solution. (a) $f(g(0)) = f(1) = -2$, (b) $f(f(2)) = f(-1) = 5$, (c) $(g \circ f)(1) = g(f(1)) = g(-2) = -3$.

1.4. **Graphing Calculators and Computers.**

This section is omitted. It essentially discusses some ways in which graphing calculators (and computers) can be useful in the study of calculus. Students in Math 121 are expected to have prior experience with graphing calculators (esp. the TI-83 or TI-84 series), and those that do not may arrange a meeting with the instructor in order to be brought up to speed.

1.5. **Exponential Functions.**

Definition 1.5.1. An **exponential function** has the form $f(x) = a^x$, where a is a positive constant.

Example 1.5.1. Graph and compare the function $f(x) = a^x$ for: (a) $0 < a < 1$, (b) $a = 1$, (c) $a > 1$.

Exponential functions are ideal for modeling populations that grow very rapidly. For example, if a bacterial culture initially contains P_0 bacteria and is known to double in size every half-hour, its growth can be modeled by the exponential function $P(t) = P_0\, 2^t$, where $t \geq 0$ is the elapsed time in half-hours.

Remark 1.5.1.

- The domain of $f(x) = a^x$ is $\mathbb{R}$ for any $a > 0$.
- If $x = n$ is a positive integer, then: $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}}$.
- If $x = 0$, then $a^0 = 1$.
- If $x = -n$, where n is a positive integer, then $a^{-n} = \dfrac{1}{a^n}$.
- If p and q are integers, with $q > 0$, then $a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.

Example 1.5.2. Simplify: (a) $64^{-2/3}$, (b) $(-16)^{3/4}$.

Solution.

(a) $64^{-2/3} = \frac{1}{64^{2/3}} = \frac{1}{\left(\sqrt[3]{64}\right)^2} = \frac{1}{(4)^2} = \frac{1}{16}$.

(b) $(-16)^{3/4} = \left(\sqrt[4]{16}\right)^3 \cdot (-1)^{3/4} = (2)^3 \cdot (-1)^{3/4} = 8 \cdot (-1)^{3/4}$.

Example 1.5.3. What is the meaning of a^x when x is an irrational number? For example, what is $3^{\sqrt{2}}$?

Proposition 1.5.1 (Laws of Exponents)**.** *If a and b are positive numbers and $x, y \in \mathbb{R}$, then:*

$$\textbf{1. } a^{x+y} = a^x a^y, \ \ \textbf{2. } a^{x-y} = \tfrac{a^x}{a^y}, \ \ \textbf{3. } (a^x)^y = a^x y, \ \ \textbf{4. } (ab)^x = a^x b^x.$$

Example 1.5.4. Use the laws of exponents to rewrite and simply the expression $\frac{\left(6y^3\right)^4}{2y^5}$.

Solution. It is straightforward to determine that: $\frac{\left(6y^3\right)^4}{2y^5} = \frac{6^4\left(y^3\right)^4}{2y^5} = \frac{1296y^{12}}{2y^5} = 648y^7$.

Definition 1.5.2. The **natural exponential function**, $f(x) = e^x$, is ubiquitous in mathematics. It is characterized by the **natural base** $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828$, which is a transcendental number (not a root of a nonzero polynomial with integer coefficients).

Example 1.5.5. The function $f(x) = e^x$ has the property that the slope of its tangent line at the point $(0, 1)$ is exactly 1. On the other hand, the functions $g(x) = 2^x$ and $h(x) = 3^x$ have tangential slopes of about 0.7 and 1.1 respectively at the point $(0, 1)$. Graph all three curves and note how the former lies between the latter two.

1.6. **Inverse Functions and Logarithms.**

Definition 1.6.1. A function f is called a **one-to-one function** if it never takes on the same value twice; that is: $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Proposition 1.6.1 (Horizontal Line Test)**.** *A function is one-to-one if and only if no horizontal line intersects its graph more than once.*

Example 1.6.1. Illustrate the use of the HLT by comparing the graphs of functions that are/aren't one-to-one.

Definition 1.6.2. Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by $f^{-1}(y) = x \iff f(x) = y$ for any $y \in B$.

We use the notation $f : A \to B$ to denote a function with domain A and range B. Thus, if $f : A \to B$ is a one-to-one function, its inverse is written $f^{-1} : B \to A$.

Remark 1.6.1.

- The function $f : A \to B$ is one-to-one if $f(x) = f(y)$ implies $x = y$ for any $x, y \in A$.
- The graph of f^{-1} is obtained by reflecting the graph of f across the line $y = x$.

Example 1.6.2. It follows from the definition that if the functions $f : A \to B$ and $f^{-1} : B \to A$ are inverses, then $f^{-1}(f(x)) = x$ for every $x \in A$ and $f(f^{-1}(x)) = x$ for every $x \in B$. For example, this property is easily verified in the case of the functions $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$.

In view of the preceding example, and the comment made in Section 1.1 about a function acting like a machine, we can think of an inverse function as a machine that works in reverse, i.e. it disassembles or undoes the result of the original function.

Definition 1.6.3. Let a be a positive constant. It is easy to verify that the exponential function $f(x) = a^x$ is one-to-one, i.e. its inverse exists. The **logarithmic function with base *a***, denoted $\log_a$, is uniquely defined by the rule $\log_a x = y \iff a^y = x$.

Remark 1.6.2.

- The domain of $f(x) = \log_a x$ is $(0, \infty)$ for any $a > 0$. The range is $\mathbb{R}$.

- $\log_a(a^x) = x$ for every $x \in \mathbb{R}$.
- $a^{\log_a x} = x$ for every $x > 0$.

Example 1.6.3. Graphically illustrate the inverse relationship between exponential and logarithmic functions.

Proposition 1.6.2 (Laws of Logarithms). *If a, x, and y are positive numbers and $r \in \mathbb{R}$, then:*

$$\textbf{1. } \log_a(xy) = \log_a x + \log_a y, \quad \textbf{2. } \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y, \quad \textbf{3. } \log_a(x^r) = r\log_a x.$$

Example 1.6.4. Simplify $\log_3 36 - \log_3 4$.

Solution. $\log_3 36 - \log_3 4 = \log_3\left(\frac{36}{4}\right) = \log_3 9 = \log_3 3^2 = 2$.

Definition 1.6.4. There are two "special" logarithms that each have a unique name and notation:

1. The logarithm with base 10 is called the **common logarithm**, and is denoted by $\log x$.
2. The logarithm with base e is called the **natural logarithm**, and is denoted by $\ln x$.

Remark 1.6.3.

- By definition, $\ln x = y \iff e^y = x$.
- $\ln(e^x) = x$ for every $x \in \mathbb{R}$.
- $e^{\ln x} = x$ for every $x > 0$.
- From the inverse relationship between the exponential and logarithmic functions it follows that $a^b = e^{\ln(a^b)} = e^{b\ln a}$ for any numbers a and b, which is a useful tool in some differentiation problems.

Example 1.6.5. Find x if $\ln(x-5) = 3$.

Solution. $\ln(x-5) = 3 \iff x - 5 = e^3 \iff x = 5 + e^3$.

Example 1.6.6. Solve the equation $e^{7+2x} - 6 = 5$.

Solution. $e^{7+2x} - 6 = 5 \iff e^{7+2x} = 11 \iff 7 + 2x = \ln 11 \iff x = (-7 + \ln 11)/2$.

Example 1.6.7. Express $2\ln x - 3\ln y$ as a single logarithm.

Solution. $2\ln x - 3\ln y = \ln x^2 - \ln y^3 = \ln\left(x^2/y^3\right)$.

Proposition 1.6.3 (Change of Base Formula). *For any positive a ($a \neq 1$), we have:* $\log_a x = \ln x / \ln a = \log x / \log a$.

1.7. **Parametric Curves.**

Definition 1.7.1. Supppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations $x = f(t)$ and $y = g(t)$ (called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve**.

Example 1.7.1. Consider the parametric equations $x = t^2 - 2t$ and $y = t + 1$.

(a) Plot points to sketch a graph of the curve.
(b) Eliminate the parameter to find a Cartesian equation of the curve.

Solution.

(a) Plotting the points (x, y) corresponding to $t = -2, -1, 0, 1, 2, 3, 4$, we sketch a horizontal parabola.
(b) First, we solve $y = t + 1$ for t. We then substitute $t = y - 1$ for t in $x = t^2 - 2t$; that is, $x = (y-1)^2 - 2(y-1) = y^2 - 2y + 1 - 2y + 2 = y^2 - 4y + 3$, which supports the result of part (a).

Example 1.7.2. Identify the curve represented by $x = \cos t$ and $y = \sin t$, where $0 \leq t \leq 2\pi$.

Solution. The ordered pairs $(x, y) = (\cos t, \sin t)$ correspond to points on the graph of a circle. This is easily verified by eliminating the parameter t; in particular, observe that $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. As t increases from 0 to 2π, the point (x, y) moves once around the unit circle in the counterclockwise direction starting from the point $(1, 0)$.

Example 1.7.3. What curve is represented by $x = \cos 2t$ and $y = \sin 2t$, where $0 \leq t \leq 2\pi$?

Solution. As in the previous example, $x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1$. In this case, however, as t increases from 0 to 2π the point (x, y) starts at $(1, 0)$ and moves twice around the unit circle in the counterclockwise direction.

Example 1.7.4. Find parametric equations for the circle with center (h, k) and radius r?

Solution. We begin with parametric equations for the unit circle, i.e. $x = \cos t$ and $y = \sin t$ for $0 \leq t \leq 2\pi$. To scale the radius of the circle to r, we multiply both x and y by r. To center the circle at the point (h, k), we add h to x and k to y. Hence, the circle centered at (h, k) with radius r can be represented by the parametric equations $x = h + r \cos t$ and $y = k + r \sin t$, where $0 \leq t \leq 2\pi$.

Most graphing calculators and computer graphing programs can be used to graph curves defined by parametric equations. In fact, it's instructive to watch a parametric curve being drawn by such a digital graphing tool because the points are plotted in order as the corresponding parameter values increase.

Example 1.7.5. Use a calculator to graph the curve $x = y^4 - 3y^2$.

Solution. If $t = y$ is a parameter, then we have the parametric equations $x = t^4 - 3t^2$ and $y = t$.

The example above illustrates the importance of choosing an appropriate range and increment-value for t. If the range is too small, then the true shape of the curve may not be fully revealed. If the increment-value is too large, the curve may have sharp or square edges. If the range is too large, or the increment-value is too small, the curve will be plotted very slowly as the calculator attempts to display a high level of detail.

Example 1.7.6. Use a calculator to plot complicated curves, such as those illustrated in Exercise 1.7.33.

2. Limits and Derivatives.

2.1. **The Tangent and Velocity Problems.**

Definition 2.1.1. A **tangent line** at a point P on a curve is a line that passes through P and approximates the curve as perfectly as possible near P.

Consider a nonlinear curve such as a parabola. The more we zoom in on a particular point P on the curve, the more the curve looks like a straight line passing through P.

To approximate the slope of the tangent line at P, we choose a point Q on the curve and calculate the slope of the **secant line** passing through P and Q. The closer Q is to P, the better the slope of the secant line approximates the slope of the tangent line at P. For this reason, we say that the slope of the tangent line to the curve at P is the limit as Q approaches P of the slope of the secant line through P and Q.

Example 2.1.1. Use secant lines to estimate the slope of the tangent line to $f(x) = x^2$ at $P = (1, 1)$.

Solution. We choose x-values closer and closer to $x = 1$, and determine the slope, m_{PQ}, of the secant line through P and $Q = (x, f(x))$ as follows:

$(x, f(x))$	m_{PQ}
$(2, 4)$	$\frac{4-1}{2-1} = 3$
$(1.5, 2.25)$	$\frac{2.25-1}{1.5-1} = 2.5$
$(1.1, 1.21)$	$\frac{1.21-1}{1.1-1} = 2.1$
$(1.01, 1.0201)$	$\frac{1.0201-1}{1.01-1} = 2.01$

It appears that the slope of the tangent line to the graph of $f(x) = x^2$ at the point $(1, 1)$ is $m = 2$. Hence, the equation of the tangent line is $y - 1 = 2(x - 1)$ or $y = 2x - 1$. This is easily verified graphically.

The Velocity Problem. We commonly understand velocity as the rate of change of position with respect to time, but what is the meaning of the **instantaneous velocity** at any particular point in time?

Example 2.1.2. A ball is dropped from 500 feet. Find its (instantaneous) velocity after 5 seconds.

Solution. After t seconds, the ball has dropped a distance of $s(t) = 32t^2$ feet. To approximate the velocity of the ball at $t = 5$ seconds, we can use this formula to calculate the average velocity of the ball over smaller and smaller increments of time starting at $t = 5$, as shown below:

Time Interval	Average Velocity (ft/sec)
$5 \le t \le 6$	$\frac{s(6)-s(5)}{6-5} = 352$
$5 \le t \le 5.1$	$\frac{s(5.1)-s(5)}{5.1-5} = 323.2$
$5 \le t \le 5.01$	$\frac{s(5.01)-s(5)}{5.01-5} = 320.32$
$5 \le t \le 5.001$	$\frac{s(5.001)-s(5)}{5.001-5} = 320.032$
$5 \le t \le 5.0001$	$\frac{s(5.0001)-s(5)}{5.0001-5} = 320.0032$

Note that as the time interval shrinks, the average velocity approaches 320 ft/sec. Thus, we estimate the instantaneous velocity of the falling ball after 5 seconds to be 320 ft/sec.

A Look Ahead... Notice that both of the preceding examples involve finding the limiting value of the quotient $\frac{f(x)-f(a)}{x-a}$ as x gets closer and closer to a. When this **limit** exists, it is called the **derivative of $f(x)$ at $x = a$** and we denote it by $f'(a)$. This concept will be studied in detail, beginning in Section 2.6.

2.2. **The Limit of a Function.**

Definition 2.2.1. We write $\lim_{x\to a} f(x) = L$ and say, "the limit of $f(x)$, as x approaches a, equals L," if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Example 2.2.1. Let $f(x) = \frac{x^2-9}{x-3}$. What is $f(3)$? What is $\lim_{x\to 3} f(x)$?

Solution. It is easy to see that $f(3)$ is undefined. Next, consider the following tables of values of $f(x)$ for values of x close, but not equal, to 3.

x (from the left)	f(x)
2	5
2.9	5.9
2.99	5.99
2.999	5.999

x (from the right)	$f(x)$
4	7
3.1	6.1
3.01	6.01
3.001	6.001

It appears that $\lim_{x\to 3} f(x) = 6$. Indeed, this can be verified graphically, since the graph of $f(x)$ is identical to the graph of the linear function $g(x) = x + 3$ except at the point $x = 3$.

Example 2.2.2. Explore the limiting behavior of a function that exhibits the various common types of discontinuity, e.g. jumps, holes, one- or two-sided asymptotes.

Example 2.2.3. Estimate $\lim_{x\to 0} \frac{1-\cos x}{x}$.

Solution. From the following table of values, it appears that $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

x	$\frac{1-\cos x}{x}$
± 0.1	± 0.04995
± 0.01	± 0.00410
± 0.001	± 0.00050
± 0.0001	± 0.00005

Indeed, this can be verified graphically, or by using L'Hôpital's Rule as seen in Section 4.5.

Example 2.2.4. Estimate $\lim_{x\to 0} \sin \frac{\pi}{x}$.

Solution. Consider the following two tables of values of $f(x)$ for values of x close to 0.

x	f(x)
± 0.1	0
± 0.01	0
± 0.001	0
± 0.0001	0

x	$f(x)$
± 0.4	1
± 0.016	1
± 0.00064	1
± 0.000256	1

From the table on the left we conclude that $\lim_{x\to 0} \sin \frac{\pi}{x} = 0$, but the table on the right suggests that $\lim_{x\to 0} \sin \frac{\pi}{x} = 1$. Which answer is correct? In fact, neither. Since the value of $\lim_{x\to 0} \sin \frac{\pi}{x}$ is dependent on how x is chosen to approach 0, this limit does not exist.

Definition 2.2.2. We write $\lim_{x\to a^-} f(x) = L$, and say the **left-hand limit of $f(x)$ as x approaches a** is equal to L, if we can make the values of $f(x)$ arbitrarily close to L by choosing x to be less than, and sufficiently close to, a. The **right-hand limit of $f(x)$ as x approaches a** is defined similarly, and is denoted by $\lim_{x\to a^+} f(x) = L$.

Example 2.2.5. Explore the one-sided limits of the functions from Example 2.2.2.

Proposition 2.2.1. *For any function f:* $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L$ *and* $\lim_{x\to a^+} f(x) = L$.

2.3. Calculating Limits Using the Limit Laws.

Proposition 2.3.1 (Limit Laws)**.** *If c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then:*

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
3. $\lim_{x\to a}[cf(x)] = c \lim_{x\to a} f(x)$.
4. $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)] \cdot [\lim_{x\to a} g(x)]$.
5. $\lim_{x\to a}[f(x)/g(x)] = [\lim_{x\to a} f(x)] \,/\, [\lim_{x\to a} g(x)]$, *provided that* $\lim_{x\to a} g(x) \neq 0$.
6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, *where n is a positive integer.*
7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, *where n is a positive integer.*

Some special additional laws are:

8. $\lim_{x\to a} c = c$.
9. $\lim_{x\to a} x = a$.
10. $\lim_{x\to a} x^n = a^n$, *where n is a positive integer (special case of Law 6).*
11. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, *where n is a positive integer (special case of Law 7).*

Example 2.3.1. Calculate $\lim_{x\to 5} 4x^3 - 10x^2 - 225$.

Solution. By Limit Laws 2, 3, 8, 9, 10, we find that $\lim_{x\to 5} 4x^3 - 10x^2 - 225 = 25$.

Example 2.3.2. Calculate $\lim_{x\to -2} \frac{x^3+5x+3}{x-7}$.

Solution. By Limit Laws 1, 2, 3, 5, 8, 9, 10, we find that $\lim_{x\to -2} \frac{x^3+5x+3}{x-7} = \frac{5}{3}$.

The following proposition shows that Examples 2.3.1 and 2.3.2 can actually be solved trivially.

Proposition 2.3.2 (Direct Substitution Property)**.** *If f is a polynomial or a rational function, and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$. This is a consequence of* **continuity**, *as seen in Section 2.4.*

Example 2.3.3. Calculate $\lim_{x\to 2} \frac{2-x}{4-x^2}$.

Solution. In this case, Proposition 2.3.2 does not apply because 2 is not in the domain of $\frac{2-x}{4-x^2}$. Instead, we apply the following algebraic simplification: $\lim_{x\to 2} \frac{2-x}{4-x^2} = \lim_{x\to 2} \frac{2-x}{(2-x)(2+x)} = \lim_{x\to 2} \frac{1}{2+x} = \frac{1}{4}$.

The following proposition justifies the above solution to Example 2.3.3.

Proposition 2.3.3. *If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limits exist.*

Example 2.3.4. Calculate $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} 2x+1, & \text{if } x \neq 3 \\ 4, & \text{if } x = 3 \end{cases}$.

Solution. By Proposition 2.3.3, it suffices to consider the function $g(x) = 2x + 1$. Then by Proposition 2.3.2, since g is a polynomial, we conclude that $\lim_{x\to 3} f(x) = \lim_{x\to 3} g(x) = g(3) = 2(3) + 1 = 7$.

Example 2.3.5. Calculate $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} -x+4, & \text{if } x < 3 \\ x, & \text{if } x \geq 3 \end{cases}$.

Solution. The one-sided limits of f are easily seen to be $\lim_{x\to 3^-} f(x) = 1$ and $\lim_{x\to 3^+} f(x) = 3$. Thus, invoking Proposition 2.2.1, we conclude that $\lim_{x\to 3} f(x)$ does not exist.

Example 2.3.6. Calculate $\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2}$.

Solution. Limit Law 5 does not apply, since 0 is not in the domain of the function. Instead, we will employ the method of **rationalizing the numerator** to determine this limit; that is,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2} &= \lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2} \cdot \frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2} \\
&= \lim_{x\to 0} \frac{(x^2+4)-4}{x^2\left(\sqrt{x^2+4}+2\right)} \\
&= \lim_{x\to 0} \frac{x^2}{x^2\left(\sqrt{x^2+4}+2\right)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x^2+4}+2} \\
&= \frac{1}{\sqrt{\lim_{x\to 0}(x^2+4)}+2} \\
&= \frac{1}{\sqrt{4}+2} \\
&= \frac{1}{2+2} \\
&= \frac{1}{4}.
\end{aligned}$$

Example 2.3.7. Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Solution. As in the case of Example 2.3.5, we compute:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{|x|}{x} &= \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1, \\
\lim_{x\to 0^-} \frac{|x|}{x} &= \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1.
\end{aligned}$$

Thus, again by Proposition 2.2.1, we conclude that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Theorem 2.3.1. *If $f(x) \leq g(x)$ when x is near a (except possible at a) and the limits of f and g both exist as x approaches a, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.*

Theorem 2.3.2 (Squeeze Theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possible at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

Example 2.3.8. Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

Solution. First, recall that $-1 \leq \sin x \leq 1$ for any x. Likewise, $-1 \leq \sin\frac{1}{x} \leq 1$ for any x. Since any non-strict inequality remains true when all sides are multiplied by a non-negative number, and we know that $x^2 \geq 0$ for all x, it follows that $-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$. Now, notice that $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$. Thus, by the Squeeze Theorem, we conclude that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

2.4. **Continuity.**

Definition 2.4.1. A function f is **continuous at a number a** if $\lim_{x\to a} f(x) = f(a)$. If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at a** (or f has a **discontinuity** at a) if f is not continuous at a.

Remark 2.4.1. Notice that Definition 2.4.1 implicitly requires three things if f is continuous at a:

$$\textbf{1. } f(a) \text{ exists}, \quad \textbf{2. } \lim_{x\to a} f(x) \text{ exists}, \quad \textbf{3. } \lim_{x\to a} f(x) = f(a).$$

Geometrically, we can think of a function that is continuous at every number in an interval as a function whose graph has no breaks in it, i.e. the graph can be drawn without removing pen from paper.

Example 2.4.1. Explore the various types of discontinuity illustrated by the graph in Example 2.2.2. In each case, discuss which of the three properties from Remark 2.4.1 are violated.

Example 2.4.2. Where are each of the following functions discontinuous?

(a) $f(x) = \frac{x^2-x-2}{x-2}$.

(b) $f(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$

(c) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2}; & \text{if } x \neq 2 \\ 1, & \text{if } x = 2. \end{cases}$

(d) $f(x) = [[x]]$ (the "greatest integer" function).

Solution.

(a) Since $f(2)$ is not defined, f is discontinuous at 2.

(b) Notice that $f(0) = 1$ is defined, but $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2}$ does not exist (because the function has an asymptote at $x = 0$), so f is discontinuous at 0.

(c) Notice that $f(2) = 1$ is defined and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3$$

exists, but $\lim_{x\to 2} f(x) \neq f(2)$, so f is discontinuous at 2.

(d) The function $f(x) = [[x]]$ is defined by: $[[x]]$ *is the largest integer that is less than or equal to x.* For example, $[[4]] = 4$, $[[4.8]] = 4$, $[[\pi]] = 3$, $\left[\left[\sqrt{2}\right]\right] = 1$, and $[[-1/2]] = -1$. It has discontinuuities at all of the integers, because $\lim_{x\to n}[[x]]$ does not exist if n is an integer.

Definition 2.4.2. A function f is **continuous from the right at a number a** if $\lim_{x\to a^+} f(x) = f(a)$, and f is **continuous from the left at a** if $\lim_{x\to a^-} f(x) = f(a)$.

Example 2.4.3. At each integer n, the function $f(x) = [[x]]$ is continuous from the right but discontinuous from the left, because

$$\lim_{x\to n^+} f(x) = \lim_{x\to n^+} [[x]] = n = f(n)$$

but

$$\lim_{x\to n^-} f(x) = \lim_{x\to n^-} [[x]] = n - 1 \neq f(n).$$

Definition 2.4.3. A function f is **continuous on an interval** if it is continuous at every number in the interval. If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continous from the left.*

Theorem 2.4.1. *If f and g are continuous at a, and c is a constant, then the following functions are also continuous at a:*

$$\textbf{1. } f+g, \quad \textbf{2. } f-g, \quad \textbf{3. } cf, \quad \textbf{4. } fg, \quad \textbf{5. } f/g \quad (\text{if } g(a) \neq 0).$$

Theorem 2.4.2. *The following types of functions are continuous at every number in their domains: polynomial functions, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions.*

Example 2.4.4. Where is the function $f(x) = \frac{\ln x + 7}{x^2 - 9}$ continuous?

Solution. By Theorem 2.4.2, we know that the logarithmic function $\ln x$ is continuous on $(0, \infty)$, while the constant function 7 and the polynomial function $x^2 - 9$ are both continuous on $\mathbb{R}$. Thus, by parts 1 and 5 of Theorem 2.4.1, it follows that f is continuous at all positive numbers x such that $x^2 - 9 = 0$. That is, f is continuous on $(0, 3) \cup (3, \infty)$.

Theorem 2.4.3. *If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words, $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$.*

Example 2.4.5. Find $\lim_{x\to 1} \ln\left(\frac{1-x}{1-\sqrt{x}}\right)$.

Solution. First, we calculate

$$\begin{aligned} \lim_{x\to 1} \frac{1-x}{1-\sqrt{x}} &= \lim_{x\to 1} \frac{1-x}{1-\sqrt{x}} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} \\ &= \lim_{x\to 1} \frac{(1-x)\left(1+\sqrt{x}\right)}{1-x} \\ &= \lim_{x\to 1} \left(1+\sqrt{x}\right) \\ &= 1 + \sqrt{\lim_{x\to 1} x} \\ &= 2. \end{aligned}$$

Since 2 is in the domain of the function $\ln x$, it follows from Theorem 2.4.3 that $\lim_{x\to 1} \ln\left(\frac{1-x}{1-\sqrt{x}}\right) = \ln(2)$.

Theorem 2.4.4. *If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$, given by $(f \circ g)(x) = f(g(x))$ is continuous at a.*

Example 2.4.6. Where are the following functions continuous? (a) $f(x) = \cos x^2$, (b) $g(x) = \ln\left(\sin^2 x\right)$.

Solution.

(a) Since $\cos x$ and x^2 are both continuous on $\mathbb{R}$, their composition $f(x) = \cos x^2$ is also continuous on $\mathbb{R}$ by Theorem 2.4.4.

(b) Note that $\sin^2 x$ is continuous on $\mathbb{R}$, and $\ln(x)$ is continuous on its domain, so by Theorem 2.4.4 the function $\ln\left(\sin^2 x\right)$ is continuous wherever it is defined; that is, for all values of x such that $\sin^2 x > 0$. In fact, $\sin^2 x = (\sin x)^2 \geq 0$ for all $x \in \mathbb{R}$, while $\sin x = 0$ if and only if $x = n\pi$ where n is an integer. Thus, $\ln\left(\sin^2 x\right)$ is discontinuous at every integer multiple of π and is continuous on the intervals between these values.

Theorem 2.4.5 (Intermediate Value Theorem)**.** *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.*

Example 2.4.7. Show that the polynomial $f(x) = 3x^4 - 4x^3 + 2x^2 - x - 10$ has a root between 1 and 2.

Solution. Since f is a polynomial, it is continuous at all real numbers; in particular, it is continuous on the interval $(1, 2)$. Since 0 is a number between $f(1) = -10$ and $f(2) = 12$, the Intermediate Value Theorem guarantees the existence of a number $c \in (1, 2)$ such that $f(c) = 0$. That is, the polynomial f has a root between 1 and 2.

2.5. **Limits Involving Infinity.**

Definition 2.5.1 (Infinite Limits)**.** The notation $\lim_{x\to a} f(x) = \infty$ means that the values of $f(x)$ an be made arbitrarily large (as large as we please) by taking x sufficiently close to a (on either side of a) but not equal to a. An analogous statement defines the expression $\lim_{x\to a} f(x) = -\infty$, and likewise the following one-sided infinite limits: $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^-} f(x) = -\infty$, $\lim_{x\to a^+} f(x) = -\infty$.

Remark 2.5.1. It is important to note that in Definition 2.5.1, we do not regard ∞ as a number, nor does it mean that the limit exists. It simply denotes a particular way in which a limit can fail to exist. The expression $\lim_{x\to a} f(x) = \infty$ is often read as "the limit of $f(x)$, as x approaches a, is infinity," or "$f(x)$ becomes infinite as x approaches a," or "$f(x)$ increases without bound as x approaches a."

Example 2.5.1. Find $\lim_{x\to 0} \frac{1}{x^2}$.

Solution. The values of $\frac{1}{x^2}$ are inversely proportional to (the square of) the values of x. In other words, very small values of x correspond to very large values of $\frac{1}{x^2}$. This is seen explicitly from a graphical analysis, or from the following table.

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.01	$10,000$
± 0.001	$1,000,000$

Thus, we conclude that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Example 2.5.2. Find $\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$.

Solution. If x is close to 3 but larger than 3, then the denominator $x-3$ is a small positive number, and $2x$ is close to 6. Thus, the quotient $\frac{2x}{x-3}$ is a large positive number, and we intuitively conclude that $\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$. Similarly, if x is close to 3 but smaller than 3, then $x-3$ is a small negative number, while $2x$ is still a positive number (close to 6), and hence $\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$.

Definition 2.5.2. The line $x = a$ is called a **vertical asymptote** (V.A.) of the curve $y = f(x)$ if any limit (or one-sided limit) of $f(x)$ as x approaches a is either ∞ or $-\infty$.

Example 2.5.3. Explore the vertical asymptotes of familiar functions such as $f(x) = \ln x$ (V.A. at $x = 0$) and $g(x) = \tan x$ (infinite number of V.A.'s, one at $x = (2n+1)\pi/2$ for every integer n).

Definition 2.5.3 (Limits at Infinity)**.** Let f be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large. An analogous statement defines the expression $\lim_{x\to -\infty} f(x) = L$.

Example 2.5.4. Note how $\lim_{x\to\infty} \frac{1}{x^2} = 0$, while $\lim_{x\to\infty} \cos x$ does not exist.

Definition 2.5.4. The line $y = L$ is called a **horizontal asymptote** (H.A.) of the curve $y = f(x)$ if the limit of $f(x)$ as x approaches either ∞ or $-\infty$ is equal to L.

Example 2.5.5. Explore the horizontal asymptotes of familiar functions such as $f(x) = e^x$ (H.A. at $y = 0$) and $g(x) = \arctan x$ (H.A.'s at $y = -\pi/2$ and $y = \pi/2$).

Definition 2.5.5 (Infinite Limits at Infinity)**.** The notation $\lim_{x\to\infty} f(x) = \infty$ is used to indicate that the values of $f(x)$ become arbitrarily large as x becomes arbitrarily large. Similar meanings are attached to each of the following expressions: $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = -\infty$.

Example 2.5.6. Many familiar functions exhibit infinite limiting behavior at infinity, e.g.

$$\lim_{x\to\infty} e^x = \infty,\ \lim_{x\to-\infty} x^2 = \infty,\ \lim_{x\to-\infty} x^3 = -\infty.$$

Example 2.5.7. Find $\lim_{x\to 0^-} e^{1/x}$ and $\lim_{x\to 0^+} e^{1/x}$.

Solution. Following Example 2.5.1, it is easy to verify that $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = \infty$. Letting $t = 1/x$ and using the results of Examples 2.5.5 and 2.5.6 respectively, it follows that:

$$\lim_{x\to 0^-} e^{1/x} = \lim_{t\to-\infty} e^t = 0 \ \text{ and } \ \lim_{x\to 0^+} e^{1/x} = \lim_{t\to\infty} e^t = \infty.$$

Proposition 2.5.1. *If n is a positive integer, then* $\lim_{x\to-\infty} \frac{1}{x^n} = 0$ *and* $\lim_{x\to\infty} \frac{1}{x^n} = 0$.

Example 2.5.8. Evaluate each of the following limits.

(a) $\lim_{x\to\infty} \frac{3x^2+9x+27}{5x^3-7x+6}$.

(b) $\lim_{x\to-\infty} \frac{3x^4+9x+27}{5x^3-7x+6}$.

(c) $\lim_{x\to-\infty} \frac{3x^3+9x+27}{5x^3-7x+6}$.

(d) $\lim_{x\to\infty} \frac{3x^2+9x+27}{\sqrt{16x^4+11x^3-12x+100}}$.

(e) $\lim_{x\to\infty} \left(\sqrt{x^2+4} - x\right)$.

Solution. We will use Proposition 2.5.1 together with Definitions 2.5.3 and 2.5.5.

(a) Both the numerator and denominator appear to approach infinity as x does. The largest power of x that appears in the expression is x^3. Multiplying both the numerator and denominator by $1/x^3$, we obtain

$$\lim_{x\to\infty} \frac{3x^2+9x+27}{5x^3-7x+6} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{3}{x} + \frac{9}{x^2} + \frac{27}{x^3}}{5 - \frac{7}{x^2} + \frac{6}{x^3}} = 0.$$

(b) This time we multiply both the numerator and denominator by $1/x^4$, simplify, and conclude that

$$\lim_{x\to-\infty} \frac{3x^4+9x+27}{5x^3-7x+6} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim_{x\to-\infty} \frac{3 + \frac{9}{x^3} + \frac{27}{x^4}}{\frac{5}{x} - \frac{7}{x^3} + \frac{6}{x^4}} = \infty.$$

(c) Again multiplying both the numerator and denominator by $1/x^3$, we have

$$\lim_{x\to\infty} \frac{3x^3+9x+27}{5x^3-7x+6} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{3 + \frac{9}{x^2} + \frac{27}{x^3}}{5 - \frac{7}{x^2} + \frac{6}{x^3}} = \frac{3}{5}.$$

(d) Notice that $x^2 = \sqrt{x^4}$ is the largest power of x that appears in the expression. Multiplying both the numerator and denominator by $1/x^2 = 1/\sqrt{x^4}$ yields

$$\lim_{x\to\infty} \frac{3x^2+9x+27}{\sqrt{16x^4+11x^3-12x+100}} \cdot \frac{\frac{1}{x^2}}{\frac{1}{\sqrt{x^4}}} = \lim_{x\to\infty} \frac{3 + \frac{9}{x} + \frac{27}{x^2}}{\sqrt{16 + \frac{11}{x} - \frac{12}{x^3} + \frac{100}{x^4}}} = \frac{3}{\sqrt{16}} = \frac{3}{4}.$$

(e) We first multiply by the conjugate of $\left(\sqrt{x^2+4} - x\right)$ over itself, and then multiply both the numerator and denominator of the resulting expression by $1/x = 1/\sqrt{x^2}$, in order to obtain

$$\lim_{x\to\infty} \left(\sqrt{x^2+4} - x\right) \cdot \frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}+x} = \lim_{x\to\infty} \frac{\left(x^2+4\right) - x^2}{\sqrt{x^2+4}+x} \cdot \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{\frac{4}{x}}{\sqrt{1+\frac{4}{x^2}}+1} = 0.$$

Remark 2.5.2. Parts (a) - (c) of Example 2.5.8 illustrate special cases of a very general result. Namely, if $f(x) = P(x)/Q(x)$, where P and Q are polynomial functions of degree m and n with leading coefficients a and b respectively, then f exhibits the one of the three following types of limiting behavior:

1. $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$, if $m < n$.
2. $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$, if $m > n$.
3. $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = a/b$, if $m = n$.

2.6. **Derivatives and Rates of Change.**

Definition 2.6.1. The **derivative of a function f at a number a**, denoted by $f'(a)$, is defined by $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$, provided that the limit exists.

Remark 2.6.1. As alluded to in Section 2.1, the derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$. In particular, $f'(a)$ represents:

1. The **slope of the tangent line** to the graph of $y = f(x)$ at the point $(a, f(a))$ on the curve.
2. The **instantaneous velocity** at time a, when $f(x)$ is a position function.

Example 2.6.1. Find the equation of the tangent line to the curve $f(x) = \frac{2}{x+1}$ at the point $(0, 2)$.

Solution. The slope of the tangent line to $f(x)$ at $(0, 2)$ is given by the derivative

$$f'(0) = \lim_{x\to0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to0} \frac{\frac{2}{x+1} - 2}{x} = \lim_{x\to0} \frac{\frac{2-2(x+1)}{x+1}}{x} = \lim_{x\to0} \frac{-2x}{x(x+1)} = \lim_{x\to0} \frac{-2}{x+1} = -2.$$

Hence, the equation of the tangent line at the point $(0, 2)$ is given by $y - 2 = -2(x - 0)$, or $y = -2x + 2$.

Example 2.6.2. The displacement of a particle moving in a straight line is given by $s(t) = \sqrt{t+16} - 4$. Find the velocity of the particle at $t = 9$, $t = 20$, and $t = 33$.

Solution. First, we find a general formula for $s'(a)$.

$$\begin{aligned} s'(a) &= \lim_{t\to a} \frac{s(t) - s(a)}{t - a} \\ &= \lim_{t\to a} \frac{\left(\sqrt{t+16} - 4\right) - \left(\sqrt{a+16} - 4\right)}{t - a} \\ &= \lim_{t\to a} \frac{\sqrt{t+16} - \sqrt{a+16}}{t - a} \cdot \frac{\sqrt{t+16} + \sqrt{a+16}}{\sqrt{t+16} + \sqrt{a+16}} \\ &= \lim_{t\to a} \frac{(t+16) - (a+16)}{(t-a)\left(\sqrt{t+16} + \sqrt{a+16}\right)} \\ &= \lim_{t\to a} \frac{(t-a)}{(t-a)\left(\sqrt{t+16} + \sqrt{a+16}\right)} \\ &= \lim_{t\to a} \frac{1}{\sqrt{t+16} + \sqrt{a+16}} \\ &= \frac{1}{2\sqrt{a+16}}. \end{aligned}$$

Then it is easy to see that $s'(9) = 1/10$, $s'(20) = 1/12$, and $s'(33) = 1/14$.

2.7. **The Derivative as a Function.**

In Section 2.6, we considered the derivative of a function f at a fixed number a, defined by

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

If we let $x = a + h$ and substitute above for x, we obtain the alternative formula

$$f'(a) = \lim_{h\to0} \frac{f(a+h) - f(a)}{h}.$$

Since in this discussion a is arbitrary, it is reasonable to suppose that a is actually a variable quantity. In fact, this is a particularly useful generalization; it allows us to regard the derivative f' as a new function, derived from f by one of the equivalent limiting operations above. In general, with a replaced by the variable x, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Example 2.7.1. Let $f(x) = x^3 - x$. Calculate $f'(x)$ and determine the slope of the tangent line to the curve f when $x = 1$ and $x = 2$.

Solution. By the formula introduced above, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\big((x+h)^3 - (x+h)\big) - \big(x^3 - x\big)}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 1 \\ &= 3x^2 - 1. \end{aligned}$$

Then $f'(1) = 2$ and $f'(2) = 11$ represent the slope of the tangent line to f at $x = 1$ and $x = 2$ respectively.

Example 2.7.2. Let $f(x) = 1/x$. Find a formula for $f'(x)$.

Solution. As in the previous example, we carefully compute $f'(x)$ using the limit definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

If we use $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x).$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of differentiation. The symbol dy/dx, introduced by Leibniz, is especially useful and will be substituted frequently for the "prime" notation. If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a, we write $dy/dx|_{x=a}$, which is synonymous with $f'(a)$.

Definition 2.7.1. A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Theorem 2.7.1. *If f is differentiable at a, then f is continuous at a.*

Example 2.7.3. Illustrate the common graphical features that cause a function not to be differentiable, e.g. corners, discontinuities, and vertical tangents.

Remark 2.7.1. Example 2.7.3 motivates the vital observation that the converse of Theorem 2.7.1 is false! That is, a function can be continuous but not differentiable at a point. In fact, there exist functions defined on $\mathbb{R}$ that everywhere continuous and yet nowhere differentiable.

Definition 2.7.2 (Higher Derivatives)**.** Suppose $y = f(x)$ is a differentiable function. Since f' is also a function, it may have a derivative of its own. This new function, denoted by f'', is called the **second**

derivative of f because it is the derivative of the derivative of f. In Leibniz notation, we write the second derivative of f as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

The **third derivative**, denoted by f''' or $\frac{d^3y}{dx^3}$, is the derivative of the second derivative, and so on. For $n \geq 4$, as the prime notation becomes impractical, we generally denote the nth derivative by $f^{(n)}$ or $\frac{d^ny}{dx^n}$.

Remark 2.7.2. We have already observed that if $s(t)$ is a position function, then $v(t) = s'(t)$ represents velocity. Furthermore, $a(t) = v'(t) = s''(t)$ represents acceleration, i.e. the rate of change of velocity.

2.8. **What Does f' Say About f?**

Proposition 2.8.1. *Suppose that f is a differentiable function. The sign of its derivative, f', provides the following information about the behavior of the graph of f:*

- *If $f'(x) > 0$ on an interval, then f is increasing on that interval.*
- *If $f'(x) < 0$ on an interval, then f is decreasing on that interval.*

An important issue that is not addressed by Proposition 2.8.1 is what happens when $f'(x) = 0$. In this case, the tangent line to the graph of f through the point $(x, f(x))$ has a slope of zero. In particular, the function f has a local minimum or a local maximum at that point, as we will discuss further in Section 4.2.

Proposition 2.8.2. *Suppose that f is a twice-differentiable function. The sign of its second derivative, f'', provides the following information about the behavior of the graph of f:*

- *If $f''(x) > 0$ on an interval, then f is* **concave upward** *on that interval.*
- *If $f''(x) < 0$ on an interval, then f is* **concave downward** *on that interval.*

Definition 2.8.1. An **inflection point** is a point where the curve changes its direction of concavity.

Example 2.8.1. Apply Propositions 2.8.1 and 2.8.2 to simple a polynomial function. Compare the shape of the curve with the tangential slope at various points. Identify intervals where the function is increasing and decreasing, local minima and maxima, intervals of concavity, and inflection points.

Example 2.8.2. Sketch a possible graph of a function f that satisfies the following conditions:

1. $f'(x) > 0$ on $(-\infty, 1)$; $f'(x) < 0$ on $(1, \infty)$.
2. $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$; $f''(x) < 0$ on $(-2, 2)$.
3. $\lim_{x\to-\infty} f(x) = -2$; $\lim_{x\to\infty} f(x) = 0$.

When given a function f, it may be useful to find a function F whose derivative is f; that is, $F'(x) = f(x)$. If such a function F exists, we call it an *antiderivative* of f. The concept of antiderivatives is of fundamental importance in connecting the ideas of differential and integral Calculus, as seen in Section 5.4.

Example 2.8.3. Consider a graph representing some derivative $y = f'(x)$.

(a) On what intervals is f increasing? On what intervals is f decreasing?
(b) At what values of x does f have a local maximum or minimum?
(c) On what intervals is f concave upward? On what intervals is f concave downward?
(d) What are the (x-coordinates of the) points of inflection of f?
(e) Assuming that $f(0) = 0$, sketch a possible graph of f.

3. Differentiation Rules.

3.1. Derivatives of Polynomials and Exponential Functions.

We have already informally discussed, or are aware of, the derivatives of the most basic polynomial functions. The constant function $f(x) = c$ is graphically represented by a horizontal line, so its slope is zero everywhere and hence $f'(x) = \frac{d}{dx}(c) = 0$ for all $x \in \mathbb{R}$. Of equal simplicity is the case of the linear function $g(x) = x$; its slope is one everywhere and hence $g'(x) = \frac{d}{dx}(x) = 1$ for all $x \in \mathbb{R}$. Now, if $f(x) = x^n$, for some integer $n \geq 2$, it is useful to apply a limit definition of the derivative in order to determine f'.

Example 3.1.1. Differentiate $f(x) = x^5$.

Solution. By the limit definition of the derivative from Section 2.7, we obtain

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^5 - x^5}{h}.$$

Now, to expand the term $(x+h)^5$, we can use the **binomial coefficients** given by **Pascal's Triangle**. In particular, the sixth row of the triangle contains the coefficients $1, 5, 10, 10, 5, 1$, which implies that

$$(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5.$$

Thus, the derivative of $f(x) = x^5$ is given by

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^5 - x^5}{h} \\ &= \lim_{h\to 0} \frac{(x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5) - x^5}{h} \\ &= \lim_{h\to 0} \frac{5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5}{h} \\ &= \lim_{h\to 0} (5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4) \\ &= 5x^4. \end{aligned}$$

Theorem 3.1.1 (Power Rule)**.** *If n is any real number, then* $\frac{d}{dx}(x^n) = nx^{n-1}$.

Proof. We proceed much as in Example 3.1.1, but it is necessary to apply the general form of the Binomial Theorem in order to expand the term $(x+h)^n$. □

Example 3.1.2. Differentiate each of the following: (a) $f(x) = x^{10}$, (b) $g(x) = \sqrt[5]{x^2}$, (c) $h(x) = \frac{1}{x^4}$.

Solution. By the Power Rule, we have:

(a) $f'(x) = \frac{d}{dx}\left(x^{10}\right) = 10x^9$.
(b) $g'(x) = \frac{d}{dx}\left(x^{2/5}\right) = \frac{2}{5}x^{-3/5}$.
(c) $h'(x) = \frac{d}{dx}\left(x^{-4}\right) = -4x^{-5}$.

Definition 3.1.1. The **normal line** to a curve at the point P is the line through P that is perpendicular to the tangent line at P.

Example 3.1.3. Find the equation of the normal line to the curve $y = x\sqrt{x}$ at the point $(4, 8)$.

Solution. Let $y = f(x) = x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$. Then by the Power Rule

$$f'(x) = \frac{d}{dx}\left(x^{3/2}\right) = \frac{3}{2}x^{1/2}.$$

In particular, the slope of the tangent line at $(4, 8)$ is given by $f'(4) = 3$. The slope of any line perpendicular to the tangent is the negative reciprocal of 3; that is, $-\frac{1}{3}$. Thus, the equation of the normal line to the curve $y = x\sqrt{x}$ at the point $(4, 8)$ is $y - 8 = -\frac{1}{3}(x - 4)$, or $y = -\frac{1}{3}x + \frac{28}{3}$.

The following theorems combine with the Power Rule to allow us to easily differentiate any polynomial.

Theorem 3.1.2 (Constant Multiple Rule). *If c is a constant and f is a differentiable function, then*

$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right].$$

Proof. Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] \\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= cf'(x). \end{aligned}$$

□

Theorem 3.1.3 (Sum Rule). *If f and g are differentiable functions, then*

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right].$$

Proof. Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{h(x+h)-h(x)}{h} \\ &= \lim_{h\to 0}\frac{\left[f(x+h)+g(x+h)\right]-\left[f(x)+g(x)\right]}{h} \\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right] \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= f'(x)+g'(x). \end{aligned}$$

□

Theorem 3.1.4 (Difference Rule). *If f and g are differentiable, then*

$$\frac{d}{dx}\left[f(x)-g(x)\right] = \frac{d}{dx}\left[f(x)\right] - \frac{d}{dx}\left[g(x)\right].$$

Proof. The argument is the same as in the case of the Sum Rule, except that we must use the Difference Law for limits in place of the Sum Law. □

Example 3.1.4. Find $\frac{d}{dx}\left(x^7 - 6x^6 + 3x^5 + 7x^3 - 8x + 11\right)$.

Solution. By the preceding theorems, we find that:

$$\begin{aligned} \frac{d}{dx}\left(x^7 - 6x^6 + 3x^5 + 7x^3 - 8x + 11\right) &= \frac{d}{dx}\left(x^7\right) - 6\frac{d}{dx}\left(x^6\right) + 3\frac{d}{dx}\left(x^5\right) + 7\frac{d}{dx}\left(x^3\right) - 8\frac{d}{dx}(x) + \frac{d}{dx}(11) \\ &= 7x^6 - 6\cdot 6x^5 + 3\cdot 5x^4 + 7\cdot 3x^2 + 8 \\ &= 7x^6 - 36x^5 + 15x^4 + 21x^2 - 8. \end{aligned}$$

Example 3.1.5. Find the points on the curve $y = x^4 - 2x^2$ where the tangent line is horizontal.

Solution. A horizontal tangent line occurs wherever the slope of the derivative is zero. Note that

$$y' = 4x^3 - 4x = 0 \implies 4x\left(x^2-1\right) = 0 \implies 4x(x-1)(x+1) = 0 \implies x = 0,\ x = 1,\ x = -1.$$

Thus, the curve $y = x^4 - 2x^2$ has horizontal tangent lines at the points $(-1,-1)$, $(0,0)$, and $(1,-1)$.

Recall from Ex. 1.5.5 that the function $f(x) = e^x$ has the property that the slope of its tangent line at the point $(0,1)$ is exactly 1. By the limit definition of the derivative, it follows that e is the unique number satisfying

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1,$$

which yields the important differentiation rule:

$$\frac{d}{dx}\left(e^x\right) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x\left(e^h - 1\right)}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x.$$

Example 3.1.6. If $f(x) = 3e^x - 4x^2$, find f' and f''.

Solution. Combining the rule above with the Difference, Constant Multiple, and Power Rules, we have:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(3e^x - 4x^2\right) = 3\frac{d}{dx}\left(e^x\right) - 4\frac{d}{dx}\left(x^2\right) = 3\left(e^x\right) - 4(2x) = 3e^x - 8x, \\ f''(x) &= \frac{d}{dx}\left(3e^x - 8x\right) = 3\frac{d}{dx}\left(e^x\right) - 8\frac{d}{dx}(x) = 3\left(e^x\right)) - 8(1) = 3e^x - 8. \end{aligned}$$

Example 3.1.7. At what point on the curve $y = 4e^x - 6$ is the tangent line parallel to $y = 4x$?

Solution. Let $x = a$ be the x-coordinate of the point in question. As in the previous example

$$f'(x) = \frac{d}{dx}\left(4e^x - 6\right) = 4e^x.$$

The slope of the tangent line to the curve $y = 4e^x - 6$ at the point $x = a$ is thus $4e^a$, while the slope of $y = 4x$ at any point is 4. Setting $4e^a = 4$ and solving for a, we have $e^a = 1$ or simply $a = 0$. Hence, the required point is $(0,4)$.

3.2. **The Product and Quotient Rules.**

By analogy with the Sum and Difference Rules, one might suppose that similar rules exist for the products and quotients of functions. This is not the case, however, as illustrated by the following simple example.

Example 3.2.1. Let $f(x) = \frac{x}{3}$ and $g(x) = x^2$, and show that $(fg)'(x) \neq f'(x)g'(x)$.

Solution. Clearly $f'(x) = \frac{1}{3}$ and $g'(x) = 2x$, so that $f'(x)g'(x) = \frac{2}{3}x$. On the other hand, $(fg)(x) = \frac{x^3}{3}$, which has the derivative $(fg)'(x) = x^2$.

To develop a general rule for correctly differentiating products of functions, we return to the limit definition of the derivative.

Theorem 3.2.1 (Product Rule)**.** *If f and g are differentiable functions, then*

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right].$$

Proof. Let $F(x) = f(x)g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{h(x+h) - h(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)g(x+h) - f(x)g(x+h)] + [f(x)g(x+h) - f(x)g(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{g(x+h)\,[f(x+h) - f(x)]}{h} + \frac{f(x)\,[g(x+h) - g(x)]}{h}\right] \\
&= \lim_{h\to 0} g(x+h) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= g(x)f'(x) + f(x)g'(x).
\end{aligned}$$

□

Example 3.2.2. If $f(x) = x^2e^x$, find $f'(x)$.

Solution. By the Product Rule, we have

$$f'(x) = \frac{d}{dx}\left(x^2e^x\right) = x^2\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(x^2\right) = x^2e^x + 2xe^x = xe^x(x+2).$$

Example 3.2.3. Differentiate $f(t) = \left(\sqrt{t}+1\right)\left(t^2+t+1\right)$.

Solution. Again by the Product Rule, we have

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\left(\sqrt{t}+1\right)\left(t^2+t+1\right)\right] \\
&= \left(\sqrt{t}+1\right)\frac{d}{dx}\left(t^2+t+1\right) + \left(t^2+t+1\right)\frac{d}{dx}\left(\sqrt{t}+1\right) \\
&= \left(\sqrt{t}+1\right)(2t+1) + \left(t^2+t+1\right)\left(\frac{1}{2}t^{-1/2}\right).
\end{aligned}$$

Example 3.2.4. If $f(x) = \left(\sqrt{x}+1\right)g(x)$, where $g(9) = -6$ and $g'(9) = \frac{1}{2}$, find $f'(9)$.

Solution. First, notice that

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\left(\sqrt{x}+1\right)g(x)\right] \\
&= \left(\sqrt{x}+1\right)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left(\sqrt{x}+1\right) \\
&= \left(\sqrt{x}+1\right)g'(x) + g(x)\left(\frac{1}{2}x^{-1/2}\right).
\end{aligned}$$

Thus,

$$f'(9) = \left(\sqrt{9}+1\right)g'(9) + g(9)\left(\frac{1}{2}(9)^{-1/2}\right) = (4)\left(\frac{1}{2}\right) + (-6)\left(\frac{1}{6}\right) = 1.$$

As with the Product Rule, the Quotient Rule involves a somewhat complicated expression.

Theorem 3.2.2 (Quotient Rule)**.** *If f and g are differentiable functions, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}\left[f(x)\right] - f(x)\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}.$$

Example 3.2.5. If $y = \frac{x^3+6x-6}{x^2+4}$, find y'.

Solution. By the Quotient Rule, we have that

$$\begin{aligned} y' &= \frac{d}{dx}\left[\frac{x^3+6x-6}{x^2+4}\right] \\ &= \frac{\left(x^2+4\right)\frac{d}{dx}\left(x^3+6x-6\right) - \left(x^3+6x-6\right)\frac{d}{dx}\left(x^2+4\right)}{\left(x^2+4\right)^2} \\ &= \frac{\left(x^2+4\right)\left(3x^2+6\right) - \left(x^3+6x-6\right)(2x)}{\left(x^2+4\right)^2}. \end{aligned}$$

Example 3.2.6. Find the equation of the tangent line to the curve $y = \frac{e^x}{e^x+1}$ at the point $(0, 1/2)$.

Solution. Again by the Quotient Rule, we have that

$$\begin{aligned} y' &= \frac{d}{dx}\left[\frac{e^x}{e^x+1}\right] \\ &= \frac{\left(e^x+1\right)\frac{d}{dx}\left(e^x\right) - \left(e^x\right)\frac{d}{dx}\left(e^x+1\right)}{\left(e^x+1\right)^2} \\ &= \frac{\left(e^x+1\right)\left(e^x\right) - \left(e^x\right)\left(e^x\right)}{\left(e^x+1\right)^2} \\ &= \frac{e^x}{\left(e^x+1\right)^2}. \end{aligned}$$

Thus, the slope of the tangent line to the curve when $x = 0$ is given by $y'|_{x=0} = \frac{e^0}{(e^0+1)^2} = \frac{1}{2^2} = \frac{1}{4}$, and the equation of the tangent line to the curve at the point $(0, 1/2)$ is $y = \frac{1}{4}x + \frac{1}{2}$.

3.3. **Derivatives of Trigonometric Functions.**

Example 3.3.1. Estimate $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$.

Solution. Consider the following table.

θ	$\frac{\sin\theta}{\theta}$
± 0.1	0.99833416
± 0.01	0.99998333
± 0.001	0.99999983
± 0.0001	0.99999999

From this, we conclude that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Similarly, we can determine that $\lim_{\theta \to 0} \frac{\sin\theta - 1}{\theta} = 0$. We are now equipped to establish a formula for the derivative of the sine function.

Theorem 3.3.1. $\frac{d}{dx}(\sin x) = \cos x$.

Proof. Let $f(x) = \sin x$. By the definition of the derivative and a well-known trigonometric identity, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right] \\
&= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] \\
&= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \left(\frac{\sin h}{h} \right) \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x.
\end{aligned}$$

□

By an analogous proof, we obtain a similar formula for the derivative of the cosine function.

Theorem 3.3.2. $\frac{d}{dx}(\cos x) = -\sin x$.

Remark 3.3.1. The derivatives of sine and cosine can be recalled with the aid of a simple circular diagram.

Since the tangent function is a quotient of the sine and cosine functions, its derivative can now be determined.

Theorem 3.3.3. $\frac{d}{dx}(\tan x) = \sec^2 x$.

Proof. By the Quotient Rule, we obtain

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x} \right) \\
&= \frac{\cos x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x.
\end{aligned}$$

□

We can further apply the Quotient Rule to establish formulas for the derivative of each of the remaining basic trigonometric functions.

Theorem 3.3.4. *1.* $\frac{d}{dx}(\csc x) = -\csc x \cot x$, *2.* $\frac{d}{dx}(\sec x) = \sec x \tan x$, *3.* $\frac{d}{dx}(\cot x) = -\csc^2 x$.

Example 3.3.2. For what values of x does the graph of $f(x) = \frac{\sec x}{\tan x + 1}$ have a horizontal tangent line?

Solution. Recalling the trigonometric identity $\tan^2 x + 1 = \sec^2 x$, the Quotient Rule gives

$$\begin{aligned} f'(x) &= \frac{(1+\tan x)\cdot \frac{d}{dx}(\sec x) - \sec x \cdot \frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \\ &= \frac{(1+\tan x)\cdot \sec x \tan x - \sec x \cdot \sec^2 x}{(1+\tan x)^2} \\ &= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} \\ &= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2} \end{aligned}$$

Since $\sec x$ is never equal to 0, we see that $f'(x) = 0$ when $\tan x = 1$, which is for $x = n\pi + \pi/4$, where n is any integer. At these points, the tangent line to the graph of f is horizontal.

Example 3.3.3. A 12-foot ladder rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/4$?

Solution. By the geometric definition of the sine function, we may write $\sin\theta = \frac{x}{12}$. That is, $x = 12\sin\theta$. The rate of change of x with respect to θ is then given by $\frac{dx}{d\theta} = \frac{d}{d\theta}(12\sin\theta) = 12\cos\theta$. Thus, when $\theta = \pi/4$, the rate of change of x with respect to θ is simply $12\cos(\pi/4) = 12\cdot\sqrt{2}2 = 6\sqrt{2} \approx 8.5$ ft/rad.

3.4. **The Chain Rule.**

Theorem 3.4.1 (Chain Rule)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product*

$$F'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example 3.4.1. Find the derivative of $f(x) = \sqrt[3]{x^5 + x}$.

Solution. Rewriting the function f, we have by the Chain Rule that

$$f'(x) = \frac{d}{dx}(x^5 + x)^{1/3} = (1/3)(x^5 + x)^{-2/3} \cdot (5x + 1).$$

Example 3.4.2. Find the derivative of $f(x) = \sin\left(x^4\right)$ and $g(x) = \sin^4 x$.

Solution. By the Chain Rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\sin(x^4) = \cos(x^4)\cdot 4x. \\ g'(x) &= \frac{d}{dx}(\sin x)^4 = 4(\sin x)^3 \cdot \cos x. \end{aligned}$$

Corollary 3.4.2 (Power Rule combined with Chain Rule)**.** *If n is any real number and $u = g(x)$ is differentiable, then*

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}\cdot g'(x).$$

Example 3.4.3. Differentiate $h(x) = \frac{1}{\sqrt{x^2+5x+10}}$.

Solution. Rewriting the function h and applying the Chain Rule, we obtain

$$h'(x) = \frac{d}{dx}(x^2+5x+10)^{-1/2} = (-1/2)(x^2+5x+10)^{-3/2} \cdot (2x+5).$$

Example 3.4.4. Find y' if $y = (x^2+3x+2)^7(3x-8)^4$.

Solution. Combining the Chain Rule with the Product Rule, we find

$$y' = \frac{d}{dx}\left[(x^2+3x+2)^7(3x-8)^4\right] = (x^2+3x+2)^7 \cdot 4(3x-8)^3 \cdot 3 + (3x-8)^4 \cdot 7(x^2+3x+2)^6 \cdot (2x+3).$$

Example 3.4.5. Find the derivative of $y = e^{\tan x}$.

Solution. By the Chain Rule, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\tan x}\right) = e^{\tan x} \cdot \frac{d}{dx}(\tan x) = e^{\tan x} \cdot \sec^2 x.$$

Example 3.4.6. Let $a > 0$ be a constant. Find $\frac{d}{dx}(a^x)$.

Solution. Recall from Section 1.6 that $a = e^{\ln a}$ for any $a > 0$. In particular, $a^x = \left(e^{\ln a}\right)^x = e^{(\ln a)x}$. It follows by the Chain Rule that

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{(\ln a)x}\right) = e^{(\ln a)x} \cdot \frac{d}{dx}((\ln a)x) = e^{(\ln a)x} \cdot \ln a = a^x \ln a.$$

Example 3.4.7. If $f(x) = \cos(\sin(e^x))$, find $f'(x)$.

Solution. Applying the Chain Rule twice, we obtain

$$f'(x) = \frac{d}{dx}(\cos(\sin(e^x))) = -\sin(\sin(e^x)) \cdot \frac{d}{dx}(\sin(e^x)) = -\sin(\sin(e^x)) \cdot \cos(e^x) \cdot e^x.$$

Example 3.4.8. If $f(x) = e^{\sec(3x)}$, find $f'(x)$.

Solution. Again using the Chain Rule twice, we have

$$f'(x) = \frac{d}{dx}\left(e^{\sec(3x)}\right) = e^{\sec(3x)} \cdot \frac{d}{dx}(\sec(3x)) = e^{\sec(3x)} \cdot \sec(3x)\tan(3x) \cdot \frac{d}{dx}(3x) = 3\sec(3x)\tan(3x)e^{\sec(3x)}.$$

Remark 3.4.1. The Chain Rule gives us a way of finding the slope of tangent lines to parametric cubes. In particular, suppose that $x = f(t)$ and $y = g(t)$ are differentiable functions that define a parametric curve. We wish to find the tangent line at a point on the curve where y is also a differentiable function of x. By the Chain Rule, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$. Assuming that $\frac{dx}{dt} \neq 0$, it follows that $\frac{dy}{dx} = \left(\frac{dy}{dt}\right)/\left(\frac{dx}{dt}\right)$.

Example 3.4.9. Find an equation of the tangent line to the parametric curve $x = 2\sin(2t)$ and $y = 2\sin t$ at the point $(\sqrt{3}, 1)$. Where does this curve have horizontal or vertical tangents?

Solution. At the point with parameter value t, the slope of the tangent line to the curve is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(2\sin t)}{\frac{d}{dt}(2\sin(2t))} = \frac{2\cos t}{2\cos(2t)\cdot 2} = \frac{\cos t}{2\cos(2t)}.$$

The point $(\sqrt{3}, 1)$ corresponds to the parameter value $t = \pi/6$, so the slope of the tangent at that point is

$$\frac{dy}{dx}\Big|_{t=\pi/6} = \frac{\cos(\pi/6)}{2\cos(\pi/3)} = \frac{\sqrt{3}/2}{2(1/2)} = \frac{\sqrt{3}}{2}.$$

An equation of the tangent line is therefore: $y - 1 = \frac{\sqrt{3}}{2}\left(x - \sqrt{3}\right)$.

The tangent line is horizontal when $\frac{dy}{dx} = 0$. This occurs when $\cos t = 0$ (and $\cos(2t) \neq 0$); that is, when $t = \pi/2$ or $t = 3\pi/2$ (note that the entire curve is given by the parameter values $0 \leq t \leq 2\pi$). Thus, the curve has horizontal tangents at the points $(0, 2)$ and $(0, -2)$.

The tangent line is vertical when $\frac{dx}{dt} = 4\cos(2t) = 0$ (and $\cos t \neq 0$); that is, when $t = \pi/4$, $3\pi/4$, $5\pi/4$, or $7\pi/4$. The corresponding four points on the curve are $\left(\pm 2, \pm\sqrt{2}\right)$.

3.5. **Implicit Differentiation.**

The functions we have dealt with up until now have been described by expressing one variable explicitly in terms of another, e.g. $y = e^x$ or $y = x\cos x$. Some functions, however, are defined implicitly by a relationship between x and y such as $x^2 + y^2 = 25$ or $x^3 + y^3 = 3xy$.

In some cases, it is possible to solve such an equation for y as an explicit function (or functions) of x. For instance, the equation $x^2 + y^2 = 25$ can be written equivalently as the two equations $y = \pm\sqrt{25 - x^2}$. On the other hand, it's not easy to solve $x^3 + y^3 = 3xy$ for either variable explicitly. Fortunately, the method of **implicit differentiation** allows us to find the derivative of y without solving for y as a function of x.

The general approach to implicit differentation is as follows. Given an equation in x and y that defines y implicitly as a differentiable function of x, we can find y' by differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

Example 3.5.1. Find the equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(-3, 4)$.

Solution. First, we differentiate both sides of the equation with respect to x

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \implies \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0.$$

Since y is a function of x, the Chain Rule implies that $\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$. Thus, we have the equation

$$2x + 2y\frac{dy}{dx} = 0 \implies 2y\frac{dy}{dx} = -2x \implies \frac{dy}{dx} = -\frac{x}{y}.$$

At the point $(-3, 4)$, we have $x = -3$ and $y = 4$, so

$$\frac{dy}{dx}\Big|_{(x,y)=(-3,4)} = \frac{3}{4}.$$

An equation of the tangent line to the circle at the point $(-3, 4)$ is therefore $y - 4 = \frac{3}{4}(x + 3)$.

Example 3.5.2. Find y' if $x^3 + y^3 = 3xy$. At what point in the first quadrant does this curve have a horizontal tangent line?

Solution. Regarding y as a function of x, and differentiating both sides of the equation with respect to x via the Chain and Product Rules, we have

$$\begin{aligned}\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3xy) &\implies \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3\left(x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right)\\ &\implies 3x^2 + 3y^2y' = 3(xy' + y)\\ &\implies y^2y' - xy' = y - x^2\\ &\implies y'(y^2 - x) = y - x^2\\ &\implies y' = \frac{y - x^2}{y^2 - x}.\end{aligned}$$

Now, the tangent line is horizontal if and only if $y' = 0$. By the formula above, this happens whenever $y - x^2 = 0$ (provided that $y^2 - x \neq 0$). Substituting $y = x^2$ into the original equation, we obtain

$$x^3 + (x^2)^3 = 3x(x^2) \implies x^3 + x^6 = 3x^3 \implies x^6 - 2x^3 = 0 \implies x^3(x^3 - 2) = 0.$$

Since $x \neq 0$ in the first quadrant, this implies $x^3 - 2 = 0$ or $x = 2^{1/3}$. At this x-value, we have $y = (2^{1/3})^2 = 2^{2/3}$. Thus, the tangent line is horizontal at the point $(2^{1/3}, 2^{2/3})$ in the first quadrant.

Example 3.5.3. Find y' if $\sin(x + y) = y\cos x$.

Solution. Again using the Chain and Product Rules, we have

$$\begin{aligned}\frac{d}{dx}(\sin(x + y)) = \frac{d}{dx}(y\cos x) &\implies \cos(x + y)\frac{d}{dx}(x + y) = y\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(y)\\ &\implies \cos(x + y)\cdot(1 + y') = -y\sin x + \cos x\cdot y'\\ &\implies y'(\cos x - \cos(x + y)) = y\sin x + \cos(x + y)\\ &\implies y' = \frac{y\sin x + \cos(x + y)}{\cos x - \cos(x + y)}.\end{aligned}$$

3.6. **Inverse Trigonometric Functions and Their Derivatives.**

Definition 3.6.1. The **inverse trigonometric functions** are defined as follows:

- $\sin^{-1} x = y$ or $\arcsin x = y \iff \sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- $\cos^{-1} x = y$ or $\arccos x = y \iff \cos y = x$ and $0 \leq y \leq \pi$
- $\tan^{-1} x = y$ or $\arctan x = y \iff \tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Remark 3.6.1. The inverse sine or **arcsine** function, $\sin^{-1} x$, represents the angle y between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x. Similar intuition describes the other inverse trigonometric functions. Furthermore,

- The domain of both $\sin^{-1} x$ and $\cos^{-1} x$ is $[-1, 1]$.
- The domain of $\tan^{-1} x$ is $(-\infty, \infty)$, and $\lim_{x\to\pm\infty} \tan^{-1} x = \pm\frac{\pi}{2}$ as discussed in Ex 2.5.3.

Example 3.6.1. Explore the inverse trigonometric functions (arcsine, arccosine, arctangent) and their graphs in comparison to the ordinary trigonometric functions (sine, cosine, tangent).

Example 3.6.2. Note that the cancelation equations for inverse functions hold for arcsine, arccosine, and arctangent. For example,

$$\sin^{-1}(\sin x) = x \text{ for } -\tfrac{\pi}{2} \leq x \leq \tfrac{\pi}{2} \quad\text{and}\quad \sin(\sin^{-1} x) = x \text{ for } -1 \leq x \leq 1.$$

In particular, $\sin^{-1}(1/2) = \pi/6$, $\sin^{-1}(-\sqrt{2}/2) = -\pi/4$, $\tan^{-1}(1) = \pi/4$, and $\tan^{-1}(0) = 0$.

Example 3.6.3. Simplify the expression $\cos(\tan^{-1} x)$.

Solution (1)**.** Let $y = \tan^{-1} x$. Then $\tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$. We wish to determine $\cos y$, but it is easier to find $\sec y$ first using the Pythagorean identity $\sec^2 y = 1 + \tan^2 y = 1 + x^2$. Indeed, we have $\sec y = \sqrt{1+x^2}$ $\left(\text{since } \sec y > 0 \text{ for } -\frac{\pi}{2} < y < \frac{\pi}{2}\right)$. Thus, $\cos(\tan^{-1} x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1+x^2}}$.

Solution (2)**.** Alternatively, let $y = \tan^{-1} x$ and construct a right triangle such that y is one of its acute angles. Then the opposite and adjacent side-lengths are x and 1 respectively, and hence the hypotenuse has length $\sqrt{1+x^2}$. It follows that $\cos(\tan^{-1} x) = \cos y = \frac{1}{\sqrt{1+x^2}}$.

Example 3.6.4. Evaluate $\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right)$.

Solution. If we let $t = \frac{1}{x-2}$, we know that $t \to \infty$ as $x \to 2^+$. Therefore, by the H.A. properties of the arctangent function, we have

$$\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right) = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}.$$

Using the implicit differentiation method from Section 3.5 we are now prepared to establish the derivative of the arcsine function.

Theorem 3.6.1. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, *for* $-1 < x < 1$.

Proof. Let $y = \sin^{-1} x$. Then $\sin y = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Differentating implicitly with respect to x yields

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x) \implies \cos y \cdot \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\cos y}.$$

Since $\cos y \ge 0$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we have by the Pythagorean identity that

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1-x^2}.$$

Hence, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$, as desired. □

By similar proofs, we can also obtain formulas for the derivatives of the arccosine and arctangent functions.

Theorem 3.6.2. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, *for* $-1 < x < 1$.

Theorem 3.6.3. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

Example 3.6.5. Differentiate $f(x) = \sin^{-1}(\sqrt{x})$.

Solution. Combining the Chain Rule with Theorem 3.6.1 above, we obtain

$$f'(x) = \frac{d}{dx} \sin^{-1}\left(\sqrt{x}\right) = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(x^{1/2}) = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x(1-x)}}.$$

3.7. **Derivatives of Logarithmic Functions.**

In this section, we use implicit differentiation to find the derivatives of logarithmic functions $y = \log_a x$, and in particular of the natural logarithmic function $y = \ln x$.

Theorem 3.7.1. $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$.

Proof. Let $y = \log_a x$. Then $a^y = x$. Differentating implicitly with respect to x yields

$$\frac{d}{dx}(a^y) = \frac{d}{dx}(x) \implies a^y \ln a \cdot \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{a^y \ln a} \implies \frac{dy}{dx} = \frac{1}{x \ln a}.$$

□

In the special case of the natural logarithm function, the substitution $a = e$ yields the following simple formula.

Corollary 3.7.2. $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

As a consequence of the Chain Rule, we also have a very useful generalized formula for the derivative of the natural logarithm function composed with some other differentiable function.

Corollary 3.7.3. *If $u = g(x)$ is a differentiable function of x, then* $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ *or* $\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$.

Example 3.7.1. Differentiate $y = \ln(x^4 + 3x^2 + 5x)$.

Solution. By Corollary 3.7.3 we easily obtain

$$y' = \frac{\frac{d}{dx}(x^4 + 3x^2 + 5x)}{x^4 + 3x^2 + 5x} = \frac{4x^3 + 6x + 5}{x^4 + 3x^2 + 5x}.$$

Example 3.7.2. Find $\frac{d}{dx}(\ln(\cos x))$.

Solution. Again it is very easy to compute that

$$\frac{d}{dx}(\ln(\cos x)) = \frac{\frac{d}{dx}(\cos x)}{\cos x} = -\frac{\sin x}{\cos x}.$$

Example 3.7.3. Differentiate $y = \sqrt{\ln x}$.

Solution. Using the Chain Rule, we have

$$y' = \frac{d}{dx}(\ln x)^{1/2} = \frac{1}{2}(\ln x)^{-1/2} \cdot \frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}.$$

Example 3.7.4. Differentiate $f(x) = \ln\left[\frac{\sqrt{x+5}}{(x-7)^2}\right]$.

Solution. By the Quotient and Chain Rules, we find that

$$\begin{aligned}
f'(x) &= \frac{1}{\left[\frac{\sqrt{x+5}}{(x-7)^2}\right]} \cdot \frac{d}{dx}\left(\frac{(x+5)^{1/2}}{(x-7)^2}\right) \\
&= \frac{(x-7)^2}{\sqrt{x+5}} \cdot \frac{(x-7)^2\frac{d}{dx}(x+5)^{1/2} - (x+5)^{1/2}\frac{d}{dx}(x-7)^2}{(x-7)^4} \\
&= \frac{(x-7)^2}{\sqrt{x+5}} \cdot \frac{(x-7)^2 \cdot (1/2)(x+5)^{-1/2} - (x+5)^{1/2} \cdot 2(x-7)}{(x-7)^4} \\
&= \frac{(1/2)(x-7)(x+5)^{-1/2} - 2(x+5)^{1/2}}{(x-7)\sqrt{x+5}} \\
&= \frac{1}{2(x+5)} - 2(x-7).
\end{aligned}$$

Example 3.7.5. Show that $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x \neq 0$.

Solution. Let $f(x) = \ln|x|$, and notice that we can write

$$f(x) = \begin{cases} \ln x, & \text{if } x > 0; \\ \ln(-x), & \text{if } x < 0. \end{cases}$$

By Corollary 3.7.3, it follows that

$$f'(x) = \begin{cases} \frac{1}{x}, & \text{if } x > 0; \\ \frac{1}{-x}(-1) = \frac{1}{x}, & \text{if } x < 0. \end{cases}$$

Hence, $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x \neq 0$.

The calculation of derivatives of complicated functions involving products, quotients, and/or powers can often be simplified by taking logarithms. This method is called **logarithmic differentation**, and it generally consists of the following three steps:

1. Take the natural logarithm of both sides of an equation $y = f(x)$ and simplify using the Laws of Logarithms,
2. Differentiate implicitly with respect to x,
3. Solve the resulting for y'.

Example 3.7.6. Logarithmic differentation allows us to easily prove the generalized Power Rule from Section 3.1. Indeed, let n be any real number and set $y = x^n$. Then $\ln|y| = \ln|x|^n = n\ln|x|$, for $x \neq 0$ (if $x = 0$, it is obvious that $\frac{d}{dx}(x^n) = \frac{d}{dx}(0) = 0$). Implicitly differentiating both sides, we obtain

$$\frac{y'}{y} = \frac{n}{x} \implies y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}.$$

Example 3.7.7. Differentiate $y = \frac{(2x+3)^{2/5}\sqrt{x^4+3x+1}}{(3x+4)^6}$.

Solution. First, we take the natural logarithm of both sides of the equation and simplify

$$\ln y = \ln\left[\frac{(2x+3)^{2/5}\sqrt{x^4+3x+1}}{(3x+4)^6}\right] \implies \ln y = \frac{2}{5}\ln(2x+3) + \frac{1}{2}\ln(x^4+3x+1) - 6\ln(3x+4).$$

Now, by implicit differentiation we have

$$\frac{y'}{y} = \frac{2}{5}\cdot\frac{2}{2x+3} + \frac{1}{2}\cdot\frac{4x^2+3}{x^4+3x+1} - 6\cdot\frac{3}{3x+4}.$$

Hence

$$y' = \frac{(2x+3)^{2/5}\sqrt{x^4+3x+1}}{(3x+4)^6}\left[\frac{4}{5(2x+3)} + \frac{4x^2+3}{2(x^4+3x+1)} - \frac{18}{3x+4}\right].$$

Example 3.7.8. Differentiate $y = x^{\sqrt{x}}$.

Solution. Taking the natural logarithm of both sides of the equation yields $\ln y = \ln\left(x^{\sqrt{x}}\right) = \sqrt{x}\ln x$. Now, by implicit differentiation and the Product Rule, we have

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + \ln x\cdot\frac{1}{2\sqrt{x}} \implies y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right).$$

Remark 3.7.1. It is important to distinguish carefully between the various exponential-type expressions when differentiating. In general, there are four cases for exponents and bases:

1. (constant base, constant exponent) use the fact that the derivative of a constant is zero: $\frac{d}{dx}(a^b) = 0$.
2. (variable base, constant exponent) use the Power Rule: $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$.
3. (constant base, variable exponent) use the diff. rule for exp. functions: $\frac{d}{dx}[a^{g(x)}] = a^{g(x)}\ln(a)g'(x)$.
4. (variable base, variable exponent) use logarithmic differentiation, as in Ex 3.7.8.

3.8. **Rates of Change in the Natural and Social Sciences.**

We know that if $y = f(x)$, then the derivative $\frac{dy}{dx}$ can be interpreted as the rate of change of y with respect to x. Now, with a myriad of differentiation rules at our disposal, we explore some of the applications of this concept to physics, economics, and other sciences.

Example 3.8.1 (Physics: Velocity and Acceleration)**.** The position of a particle is given by the equation $s(t) = t^3 - 9t^2 + 24t$, where s is measured in meters and t is in seconds.

(a) Find the velocity of the particle at time t.
(b) What is the velocity of the particle after 1 s? After 3 s?
(c) When is the particle at rest?
(d) When is the particle moving forward (that is, in the positive direction)?
(e) Draw a diagram to represent the motion of the particle.
(f) Find the total distance traveled by the particle during the first five seconds.
(g) Find the acceleration of the particle at time t after after 4 s.
(h) When is the particle speeding up? When is it slowing down?

Solution. We use the fact that velocity is the instantaneous rate of change of position ($v(t) = s'(t)$) and acceleration is the instantaneous rate of change of velocity ($a(t) = v'(t) = s''(t)$).

(a) $v(t) = s'(t) = 3t^2 - 18t + 24$.
(b) $v(1) = 9$ m/s, $v(3) = -3$ m/s.
(c) The particle is at rest when $v(t) = 0$. That is, when $3t^2 - 18t + 24 = 0 \implies 3(t-2)(t-4) = 0$, so at $t = 2$ and $t = 4$ s.
(d) The particle moves in the positive direction when $v(t) > 0$. That is, when $(t-2)(t-4) > 0$. This inequality is true when both factors are positive ($t > 4$) or when both factors are negative ($t < 2$). Consequently, the particle is moving forward for $t < 2$ and $t > 4$ and backward for $2 < t < 4$.
(e) Using the information from part (d), we can sketch the motion of the particle along the s-axis.
(f) From parts (d) and (e), the particle travels $|s(2) - s(0)| = 20$ meters from $t = 0$ to $t = 2$, $|s(4) - s(2)| = 4$ meters from $t = 2$ to $t = 4$, and $|s(5) - s(4)| = 4$ meters from $t = 4$ to $t = 5$. Thus, the total distance traveled by the particle during the first five seconds is $20 + 4 + 4 = 28$ m.
(g) $a(t) = v'(t) = 6t - 18$, so $a(4) = 6$ m/s^2.
(h) The particle speeds up when $a(t) > 0$ and slows down when $a(t) < 0$. Since $a(t)$ is a linear function, it is easy to see these inequalities hold when $t > 3$ and when $t < 3$ respectively.

Definition 3.8.1. If $C(x)$ represents the cost of producing x items, then $C'(x)$ is the instantaneous rate of change of the cost function and is referred to as the **marginal cost**.

Remark 3.8.1. The marginal cost of producing x items is approximately equal to the cost of producing one more item if x items have already been produced. Indeed, $C'(x) \approx \frac{C(x+1)-C(x)}{1} = C(x+1) - C(x)$.

Example 3.8.2 (Economics: Marginal Cost)**.** A company estimates that the cost (in dollars) of producing x items is given by $C(x) = 15000 + 6x + 0.02x^2$. Find the marginal cost of producing 500 items and interpret its meaning.

Solution. Clearly $C'(x) = 6 + 0.04x$, and it follows that $C'(500) = \$26$/item. This gives the rate at which the production cost is increasing when $x = 500$, and predicts the additional cost of producing the 501st item. Note that the actual cost of producing the 501st item is $C(501) - C(500) = \$26.02$.

In general, given an equation relating two or more variables, we can use a derivative to compute the instantaneous rate of change of one variable with respect to another.

Example 3.8.3. The volume of a spherical cell is $V(r) = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1\mu\mathrm{m} = 10^{-6}\mathrm{m}$). Find the instantaneous rate of change of V with respect to r when $r = 5\mu\mathrm{m}$.

Solution. It is easy to compute that $V'(r) = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$, and hence $V'(5) = 100\pi\ \mu\mathrm{m}^2$. Notice that the formula for V' is identical to the formula for the surface area of a sphere (even the units are correct!). In fact, this observation makes a lot of sense if you think about how an infinitesimal change in radius affects the volume of the sphere.

4. Applications of Differentiation.

4.1. Related Rates.

In a related rates problem, the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differeniate both sides with respect to time.

Example 4.1.1. If $z^2 = x^2 + y^2$, and we know that $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 3$, find $\frac{dz}{dt}$ when $x = 5$ and $y = 12$.

Solution. Differentiating both sides of the equation with respect to t using the Chain Rule, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \implies 2\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

When $x = 5$ and $y = 12$, the original equation implies that $z = 13$. Thus,

$$13\frac{dz}{dt} = 5 \cdot 2 + 12 \cdot 3 \implies \frac{dz}{dt} = \frac{46}{13}.$$

The general strategy for solving related rates application problems consists of the following steps.

1. Read the problem carefully.
2. Draw a diagram, if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use geometry to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Example 4.1.2. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

Solution. Let S be the snowball's surface area, let r be its radius, and let D be its diameter. Then $D = 2r$, so $S = 4\pi r^2 = 4\pi\left(\frac{D}{2}\right)^2 = \pi D^2$. We are given that $\frac{dS}{dt} = -1$, and we wish to find $\frac{dD}{dt}\big|_{D=10}$. Differentiating the equation $S = \pi D^2$ with respect to t using the Chain Rule gives

$$\frac{d}{dt}(S) = \frac{d}{dt}(\pi D^2) \implies \frac{dS}{dt} = 2\pi D\frac{dD}{dt} \implies \frac{dD}{dt} = -\frac{1}{2\pi D}.$$

Thus,

$$\frac{dD}{dt}\Big|_{D=10} = -\frac{1}{2\pi(10)} = -\frac{1}{20\pi} \text{ cm/min.}$$

Example 4.1.3. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 p.m.?

Solution. At time t hours after noon, let $x = 35t$ be the distance that ship A has traveled east, let $y = 25t$ be the distance that ship B has traveled north, and let D be the distance between the ships. We are given that $\frac{dx}{dt} = 35$ and $\frac{dy}{dt} = 25km$, and we wish to find $\frac{dD}{dt}$ when $t = 4$. From a diagram of the positions and movements of the two ships, we see that x, y, and D are related by $D^2 = (150 - x)^2 + y^2$. Differentiating this equation with respect to t, we have

$$\frac{d}{dt}(D^2) = \frac{d}{dt}\left((150 - x)^2 + y^2\right) \implies 2D\frac{dD}{dt} = -2(150 - x)\frac{dx}{dt} + 2y\frac{dy}{dt} \implies D\frac{dD}{dt} = (x - 150)\frac{dx}{dt} + y\frac{dy}{dt}.$$

When $t = 4$, we have $x = 140$ and $y = 100$, and hence $D = \sqrt{(150 - 140)^2 + 100^2} = 100.5$. In this case, it follows that

$$100.5\frac{dD}{dt} = (140 - 150) \cdot 35 + 100 \cdot 25 \implies \frac{dD}{dt} = 21.39 \text{ km/h.}$$

Example 4.1.4. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 in deep?

Solution. Let h be the height of the water in the trough, and let V be its volume. We are given $\frac{dV}{dt} = 12$ and we wish to find $\frac{dh}{dt}\big|_{h=1/2}$. Since the trough is 10 ft long, the volume V is equal to ten times the area of the iscosceles triangle of height h. From a simple diagram, we can use similar triangles to establish that this area must be $\frac{3}{2}h^2$. Thus, $V = 10 \cdot \frac{3}{2}h^2 = 15h^2$. Differentiating both sides of this equation with respect to t yields

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(15h^2\right) \implies \frac{dV}{dt} = 30h\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{1}{30h}\frac{dV}{dt}.$$

Hence,

$$\frac{dh}{dt}\Big|_{h=1/2} = \frac{1}{30(1/2)} \cdot 12 = \frac{4}{5} \text{ ft/min}.$$

Example 4.1.5. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

Solution. Let x be the distance from the man to the point on the path closest to the searchlight, and let θ be the angle between the beam of the searchlight and the perpendicular to the path. We are given that $\frac{dx}{dt} = 4$, and we wish to find $\frac{d\theta}{dt}$ when $x = 15$. The equation relating x and θ can be written as $\frac{x}{20} = \tan\theta$. Differentiating both sides of this equation with respect to t, we obtain

$$\frac{d}{dt}\left(\frac{x}{20}\right) = \frac{d}{dt}(\tan\theta) \implies \frac{1}{20}\frac{dx}{dt} = \sec^2\theta\frac{d\theta}{dt} \implies \frac{d\theta}{dt} = \frac{\cos^2\theta}{20}\frac{dx}{dt} = \frac{\cos^2\theta}{20} \cdot 4 = \frac{\cos^2\theta}{5}.$$

When $x = 15$, the length of the beam is 25, so $\cos\theta = \frac{20}{25} = \frac{4}{5}$. It follows that at this point the searchlight is rotating at a rate of

$$\frac{d\theta}{dt} = \frac{(4/5)^2}{5} = \frac{16}{125} = 0.128 \text{ rad/s}.$$