DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS

Key: Midterm Exam

MATH 122 Spring 2014

Your Name: ______________________

KUID Number: ____________________

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10	(20)	________
11	(20)	________
12	(20)	________
Total	(200)	________

Some Useful Formulas

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

$$< a_1, a_2, a_3 > \times < b_1, b_2, b_3 >= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n.$$

- The area of a region enclosed by a curve in polar coordinates $r = r(\theta),\ \alpha < \theta < \beta$ is
$$S = \frac{1}{2} \int_{\alpha}^{\beta} [r(\theta)]^2 d\theta.$$

- The length of a curve in polar coordinates $r = r(\theta),\ \alpha < \theta < \beta$ is
$$l = \int_{\alpha}^{\beta} \sqrt{r^2(\theta) + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Multiple Choice Questions (10 points each)

(1) Let $\vec{a} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, $\vec{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$. Find $\vec{a} \cdot \vec{b}$.

(A) $< -22, 14, -10 >$
(B) -1
(C) 0 **correct answer**
(D) $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

(2) Let $\vec{a} =< 1, 1, -1 >$, $\vec{b} =< 2, 4, 6 >$. Find $\vec{a} \times \vec{b}$.

(A) $< 10, -8, 2 >$ **correct answer**
(B) 0
(C) $< 10, -6, 4 >$
(D) $< -10, -4, 6 >$

(3) Find the sum of the series

$$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$$

(A) 6
(B) 2
(D) 4
(C) 3 **correct answer**

(4) Find the angle (in radians) between the planes $x+2y+2z = 1$, $2x-y+2z = 1$.

(A) 1.11 **correct answer**
(B) 1.07
(C) 1.15
(D) 1.04

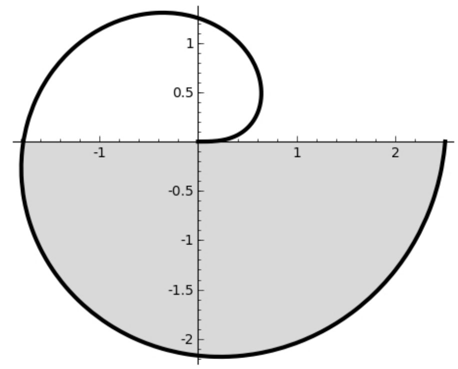
"Show Your Work" Questions (20 points each)

(5) Expand the function
$$f(x) = \frac{x}{(1-2x)^2}$$
as a Maclaurin series. Determine the interval of convergence.
Solution: Starting with $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$, we have after differentiation
$$\frac{1}{(1-y)^2} = \sum_{n=1}^{\infty} n y^{n-1}.$$
Using $y = 2x$ yields $\frac{1}{(1-2x)^2} = \sum_{n=1}^{\infty} n 2^{n-1} x^{n-1}$. Thus,
$$f(x) = \frac{x}{(1-2x)^2} = \sum_{n=1}^{\infty} n 2^{n-1} x^n.$$
Since the series in y converge for $|y| < 1$, we have convergence for $x : |2x| < 1$, which is $|x| < 1/2$.

For the endpoints, we have for $x = 1/2$, the series $\sum_{n=1}^{\infty} n$, which diverges $\lim_n n = \infty \neq 0$. Similarly, for $x = -1/2$, the series $\sum_{n=1}^{\infty} (-1)^n n$, which also diverges because $\lim_n (-1)^n n \neq 0$.

(6) Determine whether the series
$$s = \sum_{n=1}^{\infty} n e^{-n},$$
converges. If it does, give an approximation of s accurate to within 0.1. Justify your answer.
Solution: First, we check that the function $f(x) = xe^{-x}$ is decreasing for $x \geq 1$. Indeed,
$$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) \leq 0.$$
Thus, the integral test applies and since
$$\int_0^{\infty} xe^{-x} dx = -xe^{-x} \,|_0^{\infty} + \int_0^{\infty} e^{-x} dx = -e^{-x}|_0^{\infty} = 1,$$
the series converges. Moreover, $|s - s_N| \leq \int_N^{\infty} xe^{-x} dx$ and
$$\int_N^{\infty} xe^{-x} dx = -xe^{-x} \,|_N^{\infty} + \int_N^{\infty} e^{-x} dx = Ne^{-N} + e^{-N}$$
Solving $(N+1)e^{-N} \leq 0.1$ yields $N \geq 3.89$, thus $N = 4$ will suffice. Thus, the approximation is
$$s \sim e^{-1} + 2e^{-2} + 3e^{-3} + 4e^{-4} = 0.8611.$$

(7) Find the area of the *shaded region* shown in the diagram to the right. The curve is $r = \sqrt{\theta},\ 0 \le \theta \le 2\pi$.



Solution: The shaded region corresponds to $\pi \le \theta \le 2\pi$. Thus,

$$Area = \frac{1}{2}\int_{\pi}^{2\pi} \theta d\theta = \frac{\theta^2}{4}|_{\pi}^{2\pi} = \frac{3\pi^2}{4}.$$

(8) Find the length of the curve $r = e^{\theta},\ 0 \le \theta \le 2\pi$.
Solution: From the formula for length, we have

$$l = \int_0^{2\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2} d\theta = \sqrt{2}\int_0^{2\pi} e^{\theta} d\theta = \sqrt{2}(e^{2\pi} - 1).$$

(9) A sled is pulled by a rope along a level path through the snow. A 30 lb force, acting at an angle of 40° above the horizontal, moves the sled 80 ft. Find the work done by the force.
Solution:

$$W = \vec{F} \cdot \vec{r} = 30(80)\cos(40^\circ) = 1838.5$$

(10) Verify whether the points $A(4,0,0)$, $B(0,6,0)$, $C(0,0,12)$, $D(2,3,0)$ lie on the same plane. If yes, then write the equation of the plane.
Solution: We have that

$$\vec{AB} = (-4,6,0), \vec{AC} = (-4,0,12), \vec{AD} = (-2,3,0)$$

Taking their triple product yields the volume of the parallelepiped spanned on them

$$V = \begin{vmatrix} -4 & 6 & 0 \\ -4 & 0 & 12 \\ -2 & 3 & 0 \end{vmatrix} = 0.$$

Hence, the point are on the same plane. Take the cross product of any two vectors, say $\vec{n} = \vec{AB} \times \vec{AC}$ to produce a normal vector to this plane.

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -4 & 6 & 0 \\ -4 & 0 & 12 \end{vmatrix} =< 72, 48, 24 > .$$

One might as well take $\vec{n} =< 3,2,1 >$, which is parallel to $\vec{AB} \times \vec{AC}$. The plane now can be written by using $\vec{n}$ and the fact that A belongs to it:
$3(x-4) + 2y + z = 0$ or

$$3x + 2y + z - 12 = 0.$$

(11) Find the equation of the line of intersection of the planes $3x - 2y + z = 1$ and $2x + y - 3z = 3$.
Solution: A vecor along the line $\vec{v}$ can be found as a cross product of the normal vectors $\vec{n}_1 =< 3, -2, 1 >$ and $\vec{n}_2 =< 2, 1, -3 >$. We have

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 2 & 1 & -3 \end{vmatrix} =< 5, 11, 7 > .$$

Then, we need a point on the line, let us take $z = 0$. Then, $3x - 2y = 1, 2x + y = 3$, which has the solution $x = y = 1$. Thus, the point $P(1, 1, 0)$ is on the line. Thus, the equations of the line are

$$x = 1 + 5t, y = 1 + 11t, z = 7t.$$

(12) Find the equation of the plane through $P(1, 2, -2)$ that contains the line $x = 2t,\ y = 3 - t,\ z = 1 + 3t$.
Solution: A vector along the line is $\vec{v} =< 2, -1, 3 >$. a point on the line (corresponding to $t = 0$) is $Q = (0, 3, 1)$. Thus, another vector in the plane is $\vec{PQ} =< -1, 1, 3 >$. A normal vector to the plane may be constructed by

$$\vec{n} = \vec{v} \times \vec{PQ} = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ -1 & 1 & 3 \end{vmatrix} =< -6, -9, 1 >$$

The equation of the plane may be written as (using that $\vec{n}$ is orthogonal and P belongs to the plane)

$$-6(x - 1) - 9(y - 2) + (z + 2) = 0,$$

or

$$-6x - 9y + z + 26 = 0.$$