How Commutative Are Direct Products of Dihedral Groups?

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In his popular text on abstract algebra, Gallian describes a way to measure the commutativity of a finite group G [**3**, pp. 397–398]. An ordered pair $(a, b) \in G \times G$ is said to be *commuting* if $ab = ba$. If $\mathrm{Comm}(G)$ is the number of commuting pairs, then let

$$\Pr(G) = \mathrm{Comm}(G)/|G|^2$$

(where $|S|$ is the cardinality of the set S). In other words, $\Pr(G)$ is the probability that two randomly selected elements of the group actually commute.

A great deal is known about the set of fractions that can occur as $\Pr(G)$ for some group G [**2, 4, 6, 7**]. For example, if $1/2 < x \leq 1$, then there is a group G with $x = \Pr(G)$ if and only if $x = (1 + 4^k)/(2 \cdot 4^k)$ for some non-negative integer k (see the chart on p. 246 of [**7**]). For example, if $k = 0$, then $\Pr(G) = 1$ and G is abelian. And if $k = 1$, then $x = 5/8$ is the largest value of $P(G)$ for a non-abelian group. In addition, the only other possible values of $P(G)$ greater than $11/32$ are $3/8$, $25/64$, $2/5$, $11/27$, $7/16$, and $1/2$. (This upper bound for Pr(G) is generalized in [**5**].)

This note, which is based on [**1**], addresses the following question: Given a positive integer m, is there an easily constructed group G such that $\Pr(G) = 1/m$? For example, if $m = 100$, then we are asking if there is a straightforward way to find a group such that two randomly selected elements of the group commute precisely one percent of the time. Our main result (Theorem 2) produces such a group G that is a direct product of dihedral groups.

We also show (Theorem 3) that for any positive integer m there is a direct product of dihedral groups G such that $\Pr(G) = m/m'$, where m, m' are relatively prime; in fact, such a G can be found that is itself a dihedral group. We close by showing that there is a finite group H such that $\Pr(H)$ is not a member of the set $\{\Pr(G) : G$ is a direct product of dihedral groups$\}$.

Recall that if n is a positive integer, then the dihedral group D_n is generated by two elements, ρ (for "rotation") and ϕ (for "flip"), subject to the relations

$$\rho^n = \phi^2 = e \quad \text{and} \quad \phi\rho = \rho^{-1}\phi. \tag{1}$$

It follows that the elements of D_n can be written as

$$e, \rho, \ldots, \rho^{n-1}, \phi, \rho\phi, \ldots, \rho^{n-1}\phi,$$

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so that $|D_n| = 2n$ is even. If $n \geq 3$, then D_n is usually interpreted as the symmetries of a regular n-gon in the plane.

Our main computational tool will be the following result.

THEOREM 1. *If n is a positive integer, then*

$$\Pr(D_n) = \begin{cases} \dfrac{n+3}{4n} & \textit{if } n \textit{ is odd;} \\ \dfrac{n+6}{4n} & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. An easy computation using the relations (1) shows that, whether n is odd or even, we have commuting pairs (ρ^i, ρ^j) for all $0 \leq i, j < n$, as well as $(\rho^i\phi, e)$, $(e, \rho^i\phi)$ and $(\rho^i\phi, \rho^i\phi)$ for all $0 \leq i < n$.

If n is odd, this is actually a complete list, so that there are $n^2 + 3n$ commuting pairs. On the other hand, if n is even, then we have the additional commuting pairs $(\rho^i\phi, \rho^{i+(n/2)})$, $(\rho^{i+(n/2)}, \rho^i\phi)$, and $(\rho^i\phi, \rho^{i+(n/2)}\phi)$ for all $0 \leq i < n$. Therefore, when n is even there are $n^2 + 6n$ commuting pairs. Since $|D_n|^2 = 4n^2$, the result follows. ∎

If n is a positive integer, we let $d_n = \Pr(D_n)$. If n is odd, it follows that

$$d_n = \frac{n+3}{4n} = \frac{2n+6}{4(2n)} = d_{2n},$$

so that $\{d_n : n \text{ is a positive integer}\} = \{d_n : n \text{ is an even positive integer}\} = \{1, 5/8, 1/2, 7/16, 2/5, 3/8, 5/14, \dots\}$.

We denote the direct product of the groups G and H by $G \oplus H$, which is the cartesian product $G \times H$ with the usual coordinate-wise operation. It is easy to verify that $\mathrm{Comm}(G \oplus H) = \mathrm{Comm}(G) \cdot \mathrm{Comm}(H)$, so

$$\Pr(G \oplus H) = \frac{\mathrm{Comm}(G \oplus H)}{|G \oplus H|^2} = \frac{\mathrm{Comm}(G)}{|G|^2} \cdot \frac{\mathrm{Comm}(H)}{|H|^2} = \Pr(G) \cdot \Pr(H).$$

This gives the following well-known result (see, for example, p. 1033 of [**4**]).

LEMMA 1. *If G and H are finite groups, then* $\Pr(G \oplus H) = \Pr(G) \cdot \Pr(H)$.

Let $\mathcal{D}$ be the set of all possible fractions that can appear as $\Pr(G)$, where G is isomorphic to a direct product of dihedral groups. By the lemma, $\mathcal{D}$ is the set of all possible products of the form $d_{n_1} \cdots d_{n_k}$, where $n_1, \dots, n_k$ are positive integers. Clearly, $\mathcal{D}$ is closed under multiplication.

Building denominators

This brings us to our main result.

THEOREM 2. *For every positive integer m, there is a collection of dihedral groups, $D_{n_1}, \dots, D_{n_k}$, such that*

$$\Pr(D_{n_1} \oplus \cdots \oplus D_{n_k}) = \frac{1}{m}.$$

Proof. We want to show for all m, that $1/m \in \mathcal{D}$. Note that

$$\frac{1}{1} = d_1 \in \mathcal{D},\ \frac{1}{2} = d_3 \in \mathcal{D},\ \frac{1}{3} = d_9 \in \mathcal{D},$$

so assume $m \geq 4$ and the result holds for all positive integers $m' < m$.

If m is even, then $m = 2m'$ for some positive integer $m' < m$. It follows that $1/m' \in \mathcal{D}$, so that

$$\frac{1}{m} = \frac{1}{2} \cdot \frac{1}{m'} = d_3 \cdot \frac{1}{m'} \in \mathcal{D}.$$

If m is odd, then it is of the form either $4j + 1$ or $4j + 3$ for some positive integer j. If $m = 4j + 1$, then let $n = 8j + 2 = 2m$ and $m' = j + 1 < m$. We then have

$$d_n = \frac{n+6}{4n} = \frac{8j+8}{32j+8} = \frac{j+1}{4j+1} = \frac{m'}{m}.$$

On the other hand, if $m = 4j + 3$, let $n = 24j + 18 = 6m$ and $m' = j + 1 < m$. Now,

$$d_n = \frac{n+6}{4n} = \frac{24j+24}{96j+72} = \frac{j+1}{4j+3} = \frac{m'}{m}.$$

In either case, by induction, $1/m' \in \mathcal{D}$, so that

$$\frac{1}{m} = \frac{m'}{m} \cdot \frac{1}{m'} = d_n \cdot \frac{1}{m'} \in \mathcal{D},$$

which completes the proof. ■

The above argument is actually an algorithm for expressing $1/m$ as $\Pr(G)$, where G is a direct product of dihedral groups. For example, if we consider the question mentioned at the beginning of constructing a group such that the probability of two elements commuting is exactly one percent, it yields

$$\begin{aligned}\frac{1}{100} = d_3 \cdot \frac{1}{50} &= d_3 \cdot d_3 \cdot \frac{1}{25} = d_3 \cdot d_3 \cdot d_{50} \cdot \frac{1}{7} \\ &= d_3 \cdot d_3 \cdot d_{50} \cdot d_{42} \cdot \frac{1}{2} = d_3 \cdot d_3 \cdot d_{50} \cdot d_{42} \cdot d_3.\end{aligned}$$

So $1/100 = \Pr(G)$ where G is a group of order $6^3 \cdot 100 \cdot 84 = 1{,}814{,}400$. Clearly, though the method is easy to apply, it can produce groups that are exceptionally large.

Every fraction $1/m$ is a product of fractions of the form d_n, but this expression is not unique. For example,

$$d_4 \cdot d_5 = \frac{5}{8} \cdot \frac{2}{5} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = d_3 \cdot d_3.$$

Building numerators

We now show that every positive integer also appears as the numerator of an element of $\mathcal{D}$ written in lowest terms.

THEOREM 3. *If m is a positive integer, then there is a dihedral group D_n such that*

$$\Pr(D_n) = \frac{m}{m'},$$

where m' is an integer relatively prime to m.

Proof. Let $n = 24m - 6$, which is an even positive integer, and $m' = 4m - 1$. It follows that

$$\Pr(D_n) = \frac{24m - 6 + 6}{96m - 24} = \frac{m}{m'},$$

and since $1 = 4m - m'$, we can conclude that m and m' are relatively prime. ■

For example, if we want $m = 10$ as a numerator, we need only set $n = 234$, so that $d_{234} = 240/(4 \cdot 234) = 10/39$. Again, in Theorem 2 we might have to take the product of many dihedral groups to show that $1/m \in \mathcal{D}$, but in Theorem 3 it was only necessary to use a single dihedral group to show $m/m' \in \mathcal{D}$.

It is natural to ask if there are groups H for which $\Pr(H)$ is not in $\mathcal{D}$. To construct such an example, by [**7**] there is a group H such that $\Pr(H) = (1 + 16)/2 \cdot 16 = 17/32$. If n is an even positive integer with

$$\frac{17}{32} = d_n = \frac{n+6}{4n},$$

then we could conclude that $68n = 32n + 192$, i.e., $n = 16/3$, which is not an integer. On the other hand, any element of $\mathcal{D}$ which is the product of at least two $d_n < 1$ can be no larger than

$$\left(\frac{5}{8}\right)^2 = \frac{25}{64} < \frac{17}{32}.$$

Therefore, $\Pr(H)$ is not in $\mathcal{D}$.

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Summary If G is a finite group, then $\Pr(G)$ is the probability that two randomly selected elements of G commute. So G is abelian iff $\Pr(G) = 1$. For any positive integer m, we show that there is a group G which is a direct product of dihedral groups such that $\Pr(G) = 1/m$. We also show that there is a dihedral group G such that $\Pr(G) = m/m'$, where m' is relatively prime to m.