Math 141: Section 5.3 The Definite Integral - Notes

Definition: Let $f(x)$ be a function defined on a closed interval $[a.b]$. We say that a number J is the **definite integral of** f **over** $[a, b]$ and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ if the following condition is satisfied:
Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \epsilon.$$

Notation :

Theorem 1 If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\displaystyle\int_a^b f(x)dx$ exists and f is integrable over $[a, b]$.

Example 1 The function

$$f(x) = \begin{cases} 1, & \text{if x is rational} \\ 0, & \text{if x is irrational} \end{cases}$$

has no Riemann integral over $[0, 1]$.

Properties of Definite Integrals :

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A definition when $f(a)$ exists

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special case)

Example 2 Suppose that

$$\int_{-1}^{1} f(x)dx = 5, \quad \int_{1}^{4} f(x)dx = -2, \text{ and } \int_{-1}^{1} h(x)dx = 7.$$

Definition: If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve** $y = f(x)$ **over [a,b]** is the integral of f from a to b,

$$A = \int_a^b f(x)dx.$$

Example 3 Compute $\int_0^b xdx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.

Definition: If f is integrable on $[a, b]$, then its **average value on [a,b]** is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)dx.$$