Math 141: Section 5.6 Area Between Curves - Notes

Definition: If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves** $y = f(x)$ **and** $y = g(x)$ **from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]dx$$

Example 1 Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

$$y = 2e^{-x} + x \quad , \quad y = \frac{e^x}{2} \quad , \quad x = 0 \ , \ x = 1$$

$$A = \int_0^1 (2e^{-x} + x) - \left(\frac{e^x}{2}\right) dx$$

$$= \int_0^1 \left(2e^{-x} + x - \frac{1}{2}e^x\right) dx$$

$$= -2e^{-x} + \frac{x^2}{2} - \frac{1}{2}e^x \Big|_0^1$$

$$= \left(-2e^{-1} + \frac{1^2}{2} - \frac{1}{2}e^1\right) - \left(-2e^0 + \frac{0^2}{2} - \frac{1}{2}e^0\right)$$

$$= -\frac{2}{e} + \frac{1}{2} - \frac{e}{2} - \left(-2 - \frac{1}{2}\right)$$

$$= -\frac{2}{e} - \frac{e}{2} + 3 \ (\approx 0.9051)$$

Example 2 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Find where the curves intersect:

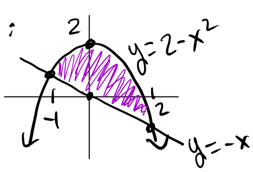
$$2 - x^2 = -x$$
$$0 = x^2 - x - 2$$
$$0 = (x-2)(x+1)$$
$$x = -1, \quad x = 2$$

$[-1, 2]$ Pick a x-value in the interval, evaluate both functions there, the larger y-value is the upper, the smaller is the lower.

$x = 0$ $\quad y = 2 - 0^2 = 2$ $\quad$ So $y = 2 - x^2$ is above

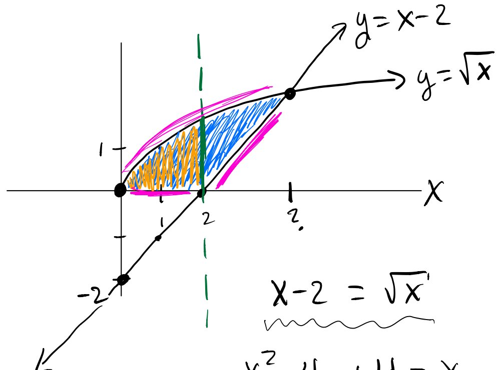
$\quad\quad\quad y = 0 \quad\quad\quad\quad y = -x$

Graph:



$$A = \int_{-1}^{2} (2 - x^2) - (-x)\,dx$$
$$= \int_{-1}^{2} (2 - x^2 + x)\,dx$$
$$= 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^{2}$$
$$= \left(2(2) - \frac{2^3}{3} + \frac{2^2}{2}\right) - \left(2(-1) - \frac{(-1)^3}{3} + \frac{(-1)^2}{2}\right)$$
$$= \boxed{9/2}$$

Example 3 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

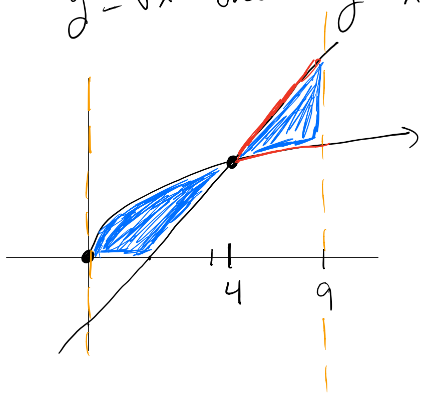


$$X - 2 = \sqrt{x}$$
$$X^2 - 4x + 4 = X$$
$$x^2 - 5x + 4 = 0$$
$$(x-4)(x-1) = 0$$
$$\cancel{x=1}, \quad x = 4 = ?$$

$$A = \int_0^2 \sqrt{x}\,dx + \int_2^{?} (\sqrt{x} - (x-2))\,dx$$
$$= \int_0^2 \sqrt{x}\,dx + \int_2^4 (\sqrt{x} - x + 2)\,dx$$
$$= \frac{2}{3}x^{3/2}\Big|_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$
$$= \frac{2}{3}(2)^{3/2} - 0 + \left[\left(\frac{2}{3}(4)^{3/2} - \frac{4^2}{2} + 2(4)\right) - \left(\frac{2}{3}(2)^{3/2} - \frac{2^2}{2} + 2(2)\right)\right]$$
$$= \cancel{\frac{4\sqrt{2}}{3}} + \left(\frac{16}{3} - \cancel{8} + \cancel{8} - \cancel{\frac{4\sqrt{2}}{3}} + 2 - 4\right)$$
$$= \frac{16}{3} - 2 = \boxed{10/3}$$

Find the area in Quadrant I bounded by

$y=\sqrt{x}$ and $y=x-2$ on $[0,9]$.



$$A = \frac{10}{3} + \int_4^9 (x-2)-\sqrt{x}\,dx$$

(from above)

Quiz 10

If you could have any super power, what would it be and why?