Math 142: Section 10.2 - Notes

1 Infinite Series

Sum of an Infinite Sequence An **infinite series** is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it.
Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead, we look at the result of summing the first n terms of the sequence:

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

s_n is called the n**th partial sum**. As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense as the terms of a sequence approach a limit.

Example 1 To assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

we add the terms one at a time from the beginning to look for a pattern in how these partial sums grow:

Partial Sum		**Value**
First:	$s_1 = 1$	1
Second:	$s_2 = 1 + \frac{1}{2}$	$\frac{3}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4}$
.	.	.
.	.	.
.	.	.
nth:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}$	$\frac{2^n-1}{2^{n-1}}$

Definitions Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

is an ______________________________. The number a_n is the n**th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \textstyle\sum_{k=1}^{n} a + k \\ &\vdots \end{aligned}$$

is the ______________________________ of the series, the number s_n being the n**th partial sum**.

If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case we write

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Geometric Series A **geometric series** is of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The **ratio** r can be positive as in the previous example or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} \cdots + \left(-\frac{1}{3}\right)^{n-1} + \ldots$$

If $r = 1$, the nth partial sum of the geometric series is

If $r = -1$, the series diverges since the nth partial sums alternate between a and 0.

Convergence of Geometric Series If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \dots$ converges to $a/(1-r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \ |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Example 2 Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}$$

Example 3 Express the repeating decimal $5.232323\dots$ as the ratio of two integers.

Example 4 Find the sum of the "telescoping" series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Theorem If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

The nth Term Test for Divergence $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Combining Series If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then:

1) $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$

2) $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$

3) $\sum ka_n = k \sum a_n = kA$ (for any number k).

Some True Facts

1) Every nonzero constant multiple of a divergent series diverges.
2) If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n \pm b_n)$ diverges.

Caution! $\sum(a_n + b_n)$ can converge when both $\sum a_n$ and $\sum b_n$ diverge:

Adding or Deleting Terms and Reindexing