Math 142: Section 8.5 - Notes

1 Integration by Partial Fractions

Our next technique: We can integrate some rational functions using u-substitution or trigonometric substitution, but these methods do not always work. Our next method of integration allows us to express any rational function as a sum of functions that can be integrated using methods with which we are already familiar. That is, we cannot integrate $\frac{1}{x^2-x}$ as-is, but it is equivalent to $\frac{1}{x} - \frac{1}{x-1}$, each term of which we can integrate.

Example 1 Our goal is to compute

$$\int \frac{x-7}{(x+1)(x-3)} dx.$$

(a) $\int \frac{1}{x+1} dx =$

(b) $\frac{2}{x+1} - \frac{1}{x-3} =$

(c) $\int \frac{x-7}{(x+1)(x-3)} dx =$

Example 2 Compute $\int \frac{10x-31}{(x-1)(x-4)}dx$

(a) $\frac{10x-31}{(x-1)(x-4)} = \frac{3}{\qquad\qquad\qquad} + \frac{7}{\qquad\qquad\qquad}.$

(b) $\int \frac{10x-31}{(x-1)(x-4)}dx = \int \qquad\qquad\qquad dx + \int \qquad\qquad\qquad dx =$

Example 3 Goal: Compute $\int \frac{x+14}{(x+5)(x+2)}dx.$

(a) $\frac{x+14}{(x+5)(x+2)} = \frac{?}{x+5} + \frac{?}{x+2}$

There is no indicator of what the numerators should be, so there is work to be done to find them. If we let the numerators be variables, we can use algebra to solve. That is, we want to find constants A and B that make the equation below true for all $x \neq -5, -2$.

$$\frac{x+14}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2}$$

How can we solve for A and B?

Example 3 (cont.)

Example 4 Find

$$\int \frac{x+15}{(3x-4)(x+1)}dx$$

Example 5 Goal: Find $\int \frac{5x-2}{(x+3)^2} dx$

Here, there are not two different linear factors in the denominator. This canNOT be expressed in the form

$$\frac{5x-2}{(x+3)^2} = \frac{5x-2}{(x+3)(x+3)} \neq \frac{A}{x+3} + \frac{B}{x+3} =$$

However, it can be expressed in the form:

$$\frac{5x-2}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$$

(a) Find constants A and B that make the above equation true for all $x \neq -3$.

(b) Compute $\int \frac{5x-2}{(x+3)^2} dx$.

Example 6 What if the denominator is an irreducible quadratic of the form x^2+px+q? That is, it can not be factored (does not have any real roots). In this case, suppose that $(x^2+px+q)^n$ is the highest power of this factor that divides the denominator. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x+C_1}{(x^2+px+q)}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\frac{B_3x+C_3}{(x^2+px+q)^3}+\cdots+\frac{B_nx+C_n}{(x^2+px+q)^n}.$$

Compute $\displaystyle\int\frac{-2x+4}{(x^2+1)(x-1)^2}dx$.

Summary: Method of Partial Fractions when $f(x)/g(x)$ is Proper

1) Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3)} + \cdots + \frac{A_m}{(x-r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2) Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2+px+q)} + \frac{B_2x + C_2}{(x^2+px+q)^2} + \frac{B_3x + C_3}{(x^2+px+q)^3} + \cdots + \frac{B_nx + C_n}{(x^2+px+q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$.

3) Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4) Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Examples adapted from `http://math.colorado.edu/activecalc2/PartialFractionsDiscovery.pdf`