1.5 Exponential Functions

Suppose a population of 12 million doubles each year (starting in 2003)

2003	12 mil
2004	24 mil $= 12 \cdot 2^1$
2005	$48 = 24 \cdot 2 = 12 \cdot 2 \cdot 2 = 12 \cdot 2^2$
2006	$96 = 12 \cdot 2^3$

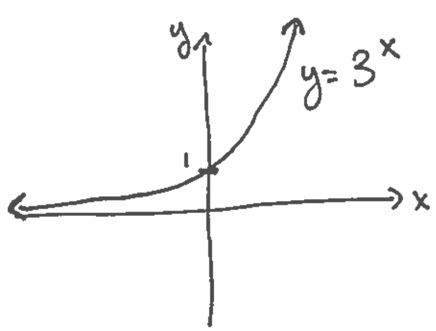
$P(t) = 12 \cdot 2^t$, where t represents number of years since 2003

$f(x) = a^x$, $a > 0$,

is the <u>exponential function</u> with base a.

For $a > 1$, the exponential function is increasing

For $0 < a < 1$, the exponential function is decreasing.



For integer and rational values of x, we know how to evaluate $f(x) = a^x$.

$x = 0$ $\quad f(0) = a^0 = 1$

$x = n$ $\quad$ for some positive integer n,

$$f(n) = a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n\text{-times}}$$

$x = -n$ $\quad$ for some positive integer n,

$$f(-n) = a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$$

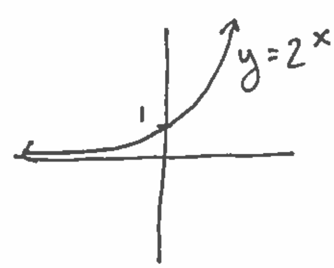
$x = \frac{1}{n}$ $\quad$ for some positive integer n,

$$f(1/n) = a^{1/n} = \sqrt[n]{a}$$

$x = p/q$ $\quad$ where p/q is any rational number,

$$f(p/q) = a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

If x is irrational (yikes!), things get fuzzy...
In Chapter 7, we'll have a precise definition
(yay calculus!)



Let $f(x) = 2^x$ and $x = \sqrt{3}$.

$$\sqrt{3} = 1.732050808\ldots$$

Approximate $2^{\sqrt{3}}$ using the decimal expansion of $\sqrt{3}$.

r	2^r
1.0	2
1.7	3.249009585
1.73	3.317278183
1.732	3.321880096
1.73205	3.321995226
$\vdots$	$\vdots$

$$2^{\sqrt{3}} \approx 3.3219970$$

This type of approximation keeps the graph of $y = 2^x$ "nice" (smooth, no holes or jumps)

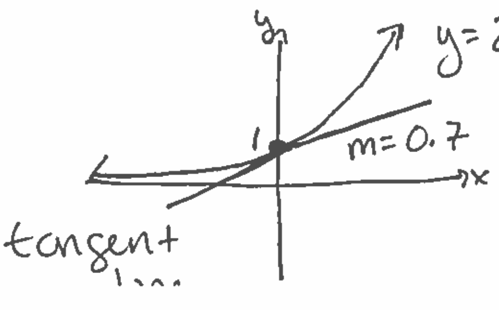
Rules for Exponents

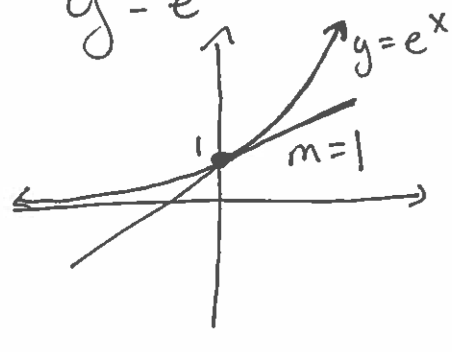
If $a > 0$, $b > 0$ then the following hold for all real numbers x, y.

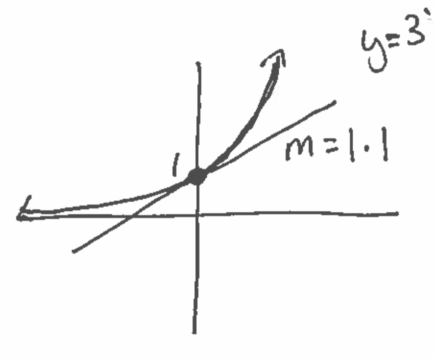
1. $a^x \cdot a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{x \cdot y}$
4. $a^x \cdot b^x = (a \cdot b)^x$
5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

The natural exponential function has base $e \approx 2.7182\ldots$

$y = e^x$







Ex: The initial population in 2003 was 12 mil and it increases by 3.4% each year.

$P(t) = P_0 a^t$ $\quad$ P_0 = initial population

a = growth factor

2003	12
2004	$12 + .034(12) = 12(1.034)$
2005	$[12(1.034)](1.034) = 12(1.034)^2$

$$P(t) = 12(1.034)^t$$

$$\frac{\text{Pop in 2004}}{\text{Pop in 2003}} = 1.034 \qquad \frac{\text{Pop in 2005}}{\text{Pop in 2004}} = 1.034$$

Exponential Growth and Decay

$y = e^{kx}$, where k is a nonzero constant

$y = y_0 e^{kx}$

If $k > 0$, then this models exponential growth

If $k < 0$, this models exponential decay

$y = Pe^{rt}$, P = initial investment

r = interest rate (in decimal form)

t = time (units consistent with r)

Ex: Track the growth of a \$100 investment, which was invested in 2014 with an annual interest rate of 5.5%.

Let $t=0$ represent 2014, $t=1$ represent 2015, etc.

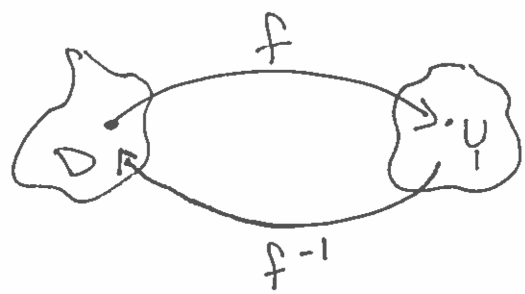
$$y = Pe^{rt} \qquad P = 100$$
$$r = .055$$

$$y = 100e^{.055t}$$

In 2018,

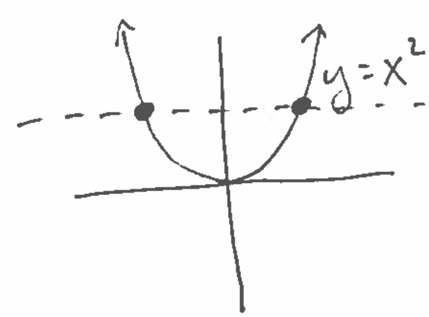
$$y(t) = y(4) = 100e^{.055(4)} = \$124.61$$

1.6 Inverse Functions and Logarithms

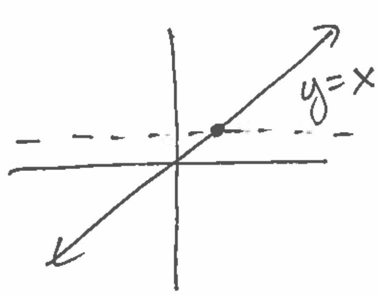


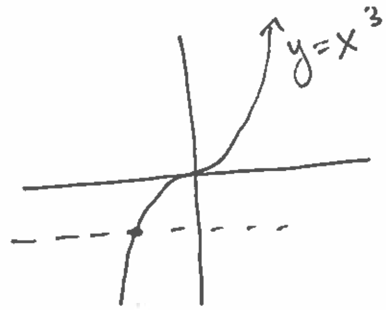
$f(x) = x^2 \qquad f(-1) = 1$

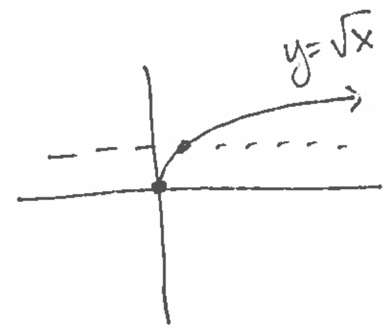
$f(1) = 1$



A function $f(x)$ is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.







Horizontal Line Test

A function, $y = f(x)$, is one-to-one if and only if its graph intersects each horizontal line at most once.

Def: Suppose that a function f is one-to-one on a domain D with range R. The inverse function, f^{-1}, is defined by $f^{-1}(b) = a$ if $f(a) = b$.

The domain of f^{-1} is R and the range of f^{-1} is D.

Ex: Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Step 1: interchange x and y

Step 2: Solve for y

$$y = \tfrac{1}{2}x + 1$$

$$x = \tfrac{1}{2}y + 1 \quad \text{Step 1}$$

$$x - 1 = \tfrac{1}{2}y$$

$$2(x-1) = y$$

$$2x - 2 = y = f^{-1}(x)$$

Domain of f^{-1} (Range of f) $= (-\infty, \infty)$

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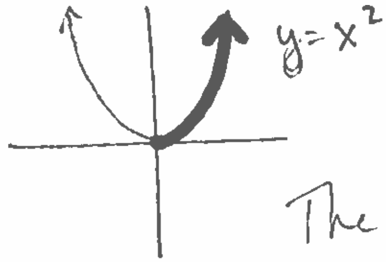

The function $y = x^2$ on the domain $\underline{[0, \infty)}$ is one-to-one, and we can find the inverse

$$f(x) = y = x^2$$
$$x = y^2$$
$$\sqrt{x} = y = f^{-1}(x)$$

Domain of $f = [0, \infty)$
Range of $f^{-1} = [0, \infty)$

Range of $f = [0, \infty)$
Domain of $f^{-1} = [0, \infty)$