3.3 Differentiation Rules

Derivative of a Constant Function

If $f(x) = c$, then $\frac{df}{dx} = \frac{d}{dx}(c) = 0$.

Proof:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c-c}{h} = \lim_{h \to 0} 0 = 0.$$

Derivative of a Positive Integer Power (Power Rule)

Use the definition to find $(x^4)'$, $(x^3)'$, and $(x^2)'$.

Seems that if n is a positive integer, then

$$\frac{d}{dx} x^n = n x^{n-1}$$

Proof: $z^n - x^n = (z-x)(z^{n-1} + z^{n-2}x + \dots + z x^{n-2} + x^{n-1})$

Using the alternative definition of the derivative gives us

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{z^n - x^n}{z - x}$$

$$= \lim_{z \to x} (z^{n-1} + z^{n-2}x + \dots + z x^{n-2} + x^{n-1})$$

$$= n x^{n-1}$$

Power Rule (General)

If n is any real number, then

$$\frac{d}{dx} x^n = n x^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

<u>Ex</u>: Differentiate the following powers of x.

(a) x^3

$$\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$$

(b) $x^{2/3}$

$$\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-1/3}$$

(c) $x^{\sqrt{2}}$

$$\frac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}\,x^{\sqrt{2}-1}$$

(d) $\frac{1}{x^4}$

$$\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}x^{-4} = -4x^{-4-1} = -4x^{-5}$$

(e) $x^{-4/3}$

$$\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-\frac{4}{3}-1} = -\frac{4}{3}x^{-7/3}$$

(f) $\sqrt{x^{2+\pi}}$

$$\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(\left(x^{2+\pi}\right)^{1/2}\right) = \frac{d}{dx}\left(x^{1+\pi/2}\right) = \left(1+\frac{\pi}{2}\right)x^{\pi/2}$$

Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

Proof:

$$\frac{d}{dx}(cu) = \lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h}$$

$$= \lim_{h\to 0}\frac{c\left(u(x+h)-u(x)\right)}{h}$$

$$= c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

$$= c\frac{du}{dx}$$

Sum/Difference Rule

If u and v are differentiable functions of x, then their sum (or difference) $u+v$ ($u-v$) is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \frac{du}{dx}+\frac{dv}{dx} \quad \left(\frac{d}{dx}(u-v) = \frac{du}{dx}-\frac{dv}{dx}\right)$$

Ex: Find the derivative of the polynomial

$$y = x^3 + \tfrac{4}{3}x^2 - 5x + 1.$$

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\tfrac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}1$$

$$= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0$$

$$= 3x^2 + \tfrac{8}{3}x - 5$$

Ex: Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Horizontal tangents occur when the slope dy/dx is zero.

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x$$

Solve $\frac{dy}{dx} = 0$ for x: $4x^3 - 4x = 0$

$$4x(x^2 - 1) = 0$$

$$4x(x+1)(x-1) = 0$$

$$x = 0, 1, -1$$

The corresponding points are $(0,2)$, $(1,1)$, and $(-1,1)$.

Derivatives of Exponential Functions

Let $f(x) = a^x$. Then,

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= \lim_{h \to 0} a^x \cdot \frac{a^h - 1}{h}$$

$$= a^x \cdot \underbrace{\lim_{h \to 0} \frac{a^h - 1}{h}}_{\text{a fixed number } L.}$$

Note that $\lim_{h \to 0} \frac{a^h - 1}{h}$ is equal to the derivative of $f(x) = a^x$ at $x = 0$, $f'(0) = \lim_{h \to 0} \frac{a^h - a^0}{h} = \lim_{h \to 0} \frac{a^h - 1}{h} = L$.

It turns out (we'll see in Ch. 7) that this limit L exists and is equal to $\ln(a)$!

Recall that $f'(0)$ is the slope of the tangent line to the curve at $x = 0$.

Let $f(x) = e^x$, then $f'(0) = \lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

Thus, $\frac{d}{dx}(e^x) = \lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) \cdot e^x$

$$= 1 \cdot e^x$$

$$= e^x.$$

Products & Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product is NOT the product of the derivatives:

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx} x^2 = 2x \neq \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$$

Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Equivalently,

$$\frac{d}{dx}(f(x)\cdot g(x)) = f(x)g'(x) + g(x)f'(x)$$

Ex: Differentiate:

(a) $y = \frac{1}{x}(x^2+e^x)$

$$\left(\tfrac{1}{x}\right)' = (x^{-1})' = -1x^{-2} = -\tfrac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x}(x^2+e^x)' + (x^2+e^x)(1/x)'$$

$$= \frac{1}{x}(2x+e^x) + (x^2+e^x)(-1/x^2)$$

$$= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2}$$

$$= 1 + e^x/x - e^x/x^2 = 1 + \frac{xe^x - e^x}{x^2} = 1 + (x-1)e^x/x^2$$

(b) $y = e^{2x}$

$$\frac{dy}{dx} = \frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x \cdot e^x)$$

$$= e^x\frac{d}{dx}e^x + e^x\frac{d}{dx}e^x$$

$$= e^x \cdot e^x + e^x e^x$$

$$= e^{2x} + e^{2x}$$

$$= 2e^{2x}$$

Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Equivalently,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Ex: Find the derivative of

$$y = \frac{(x-1)(x^2-2x)}{x^4}$$

$$y = \frac{x^3-2x^2-x^2+2x}{x^4} = \frac{x^3-3x^2+2x}{x^4} = x^{-1}-3x^{-2}+2x^{-3}$$

$$\frac{dy}{dx} = -1x^{-2}-3(-2)x^{-3}+2(-3)x^{-4}$$

$$= -x^{-2}+6x^{-3}-6x^{-4}$$

$$= -\frac{1}{x^2}+\frac{6}{x^3}-\frac{6}{x^4}$$

Second- and Higher-Order Derivatives

If $f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So, $f''=(f')'$. The function f'' is called the <u>second derivative</u> of f. It is denoted by:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y''$$

$$= D^2(f)(x) = D_x^{\,2}f(x)$$

If y'' is differentiable, its derivative $y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3}$, is the <u>third derivative</u> of y with respect to x.

In general, the <u>n^{th} derivative</u> of y with respect to x (for any positive integer n) is denoted

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

Ex: Find the first four derivatives of $y = x^3-3x^2+2$

$y' = 3x^2-6x$ $\qquad$ $y''' = 6$

$y'' = 6x-6$ $\qquad$ $y^{(4)} = 0$