Math 141: Section 2.1 Rates of Change and Tangents to Curves - Notes

Average and Instantaneous Speed If y denotes the distance fallen in feet after t seconds, then Galileo's law is

$$y = 16t^2$$

, where 16 is the (approximate) constant of proportionality. (If y is measured in meters, the constant is 4.9).
A moving object's **average speed** during an interval of time is found by dividing the distance covered by the time elapsed.

Example 1 A rock breaks loose from the top of a tall cliff. What is its average speed
(a) during the first 2 sec of fall?
(b) during the 1-second interval between second 1 and second 2?

(a) $\frac{\Delta y}{\Delta t}$ $t=0$ to $t=2$ $\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2-0} = 32$ ft/sec

(b) $t=1$ to $t=2$ $\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2-1} = 48$ ft/sec

Example 2 Find the speed of the falling rock in Example 1 at t=1 sec.

Look at the interval $[t_0, t_0+h]$ where $\Delta t = h$

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0+h)^2 - 16t_0^2}{h}$$

Want $h=0$ but we can't divide by 0! :(

Length of time interval h	Avg speed over interval of length h starting at $t_0=1$
1	48
0.1	33.6
0.01	32.16
0.001	32.016
0.0001	32.0016

The average speed on the interval starting at $t_0 = 1$ seems to approach a limiting value of 32 as the length of the interval decreases. This suggests that the rock is falling at a speed of 32 ft/sec at $t_0 = 1$ sec. We can confirm this algebraically:

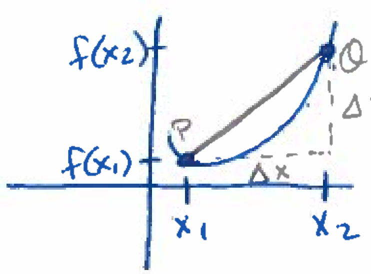
$$\frac{\Delta y}{\Delta t} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(1+2h+h^2) - 16}{h}$$

$$= \frac{32h + 16h^2}{h} = 32 + 16h$$

As h approaches 0, $\frac{\Delta y}{\Delta t}$ approaches 32 ft/sec

Average Rate of Change The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

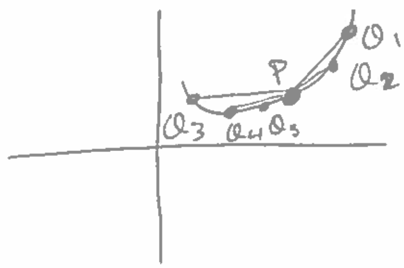
$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1+h) - f(x_1)}{h}, \quad h \neq 0$$



Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$. This line is called a <u>secant</u> to the curve.

The instantaneous rate of change (speed) at a point $P(x, f(x))$ would seem to be the slope of the line that touches the curve precisely at P.

Slope of a Curve To define tangency for general curves, we need an approach that takes into account the behavior of the secants through P and nearby points Q as Q moves toward P along the curve.



1) Start with what we can calculate, namely, the slope of the secant PQ.
2) Investigate the limiting value of the secant slope as Q approaches P along the curve.
3) If the *limit* exists, take it to be the slope of the curve at P and *define* the tangent to the curve P to be the line through P with this slope.

Example 3 Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

Begin with the secant line through $P(2,4)$ and $Q(2+h, (2+h)^2)$

$$\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{h} = \frac{4+4h+h^2-4}{h}$$

$$= \frac{4h+h^2}{h} = 4+h$$

If $h > 0$, then Q lies above and to the right of P.
If $h < 0$, then Q lies to the left of P.
Either way, as Q approaches P along the curve, h is approaching 0, the secant slope is approaching 4. Take $m = 4$ to be the slope of the tangent to $P(2,4)$

$$y - 4 = 4(x-2)$$

Instantaneous Rate of Change and Tangent Lines The rates at which the rock in Example 2 was falling at the instant $t = 1$ is called the **instantaneous rate of change**. Instantaneous rates and slopes of tangent lines are closely connected.

The instantaneous rate is the value the average rate approaches as the length h of the interval over which the change occurs approaches zero.

The average rate of change corresponds to the slope of a secant line.

The instantaneous rate of change corresponds to the slope of the tangent line as the independent variable approaches a fixed value.