Math 170: Section 7.2, 7.3 Lecture

Section 7.2

Definition: When an experiment is performed a number of times, the relative frequency or estimated probability of an event E is the fraction of times that the event E occurs. If the experiment is performed N times and the event E occurs $fr(E)$ times, then the relative frequency is given by

$$P(E) = \frac{fr(E)}{N}.$$

The number $fr(E)$ is called the frequency of E. N, the number of times that the experiment is performed, is called the number of **trials** or the **sample size**. If E consists of a single outcomes s, then we refer to $P(E)$ as the relative frequency or estimated probability of the outcomes s, and we write $P(s)$.

The collection of the estimated probabilities of *all* the outcomes is the **relative frequency distribution** or **estimated probability distribution.**

Example 1 In a survey of 250 hybrid vehicles sold in the United States, 125 were Toyota Prii, 30 were Honda Civics, 20 were Toyota Camrys, 15 were Ford Escapes, and the rest were other makes. What is the relative frequency that a hybrid vehicle sold in the United States is not a Toyota Camry?

S = {Toyota Prius, Honda Civic, Toyota Camry, Ford Escape, Other}

E = {Toyota Prius, Honda Civic, Ford Escape, Other}

||

complement of the set {Toyota Camry} = F

$E = F'$

$N = 250 \quad fr(E) = ?$

$= 250 - \underset{=n(F)}{20}$

$= \underset{n(F')}{230}$

$$P(E) = \frac{fr(E)}{N} = \frac{230}{250} = .92$$

Bid Price	\$0-\$9.99	\$10-\$49.99	\$50-\$99.99	$\geq$\$100
~~Relative~~ Frequency	6	23	15	6

*   *

Example 2 The above chart shows the results of a survey of the bid prices for 50 paintings on eBay with the highest number of bids.

Consider the experiment in which a painting is chosen and the bid price is observed.

(a) Find the relative frequency distribution.

(b) Find the relative frequency that a painting in the survey had a bid price of less than \$50.

(a)

Bid Price	0-9.99	10-49.99	50-99.99	≥ 100
Rel. Frequency	$\frac{6}{50} = .12$	$\frac{23}{50} = .46$	$\frac{15}{50} = .30$	$\frac{6}{50} = .12$

(b) Method I Compute Directly

$E = \{\$0\text{-}9.99, \$10\text{-}49.99\}$

$$P(E) = \frac{fr(E)}{N} = \frac{6+23}{50} = \frac{29}{50} = .58$$

Method 2 Use Rel. Freq.

$$P(E) = .12 + .46 = .58$$

$F = \{0\text{-}9.99\}$

$G = \{10\text{-}49.99\}$

$$P(E) = P(F) + P(G)$$
$$= \frac{6}{50} + \frac{23}{50}$$
$$= \frac{6+23}{50}$$

Some Properties of Relative Frequency Distribution Let $S = \{s_1, s_2, \ldots, s_n\}$ be a sample space and let $P(s_i)$ be the relative frequency of the event $\{s_i\}$. Then

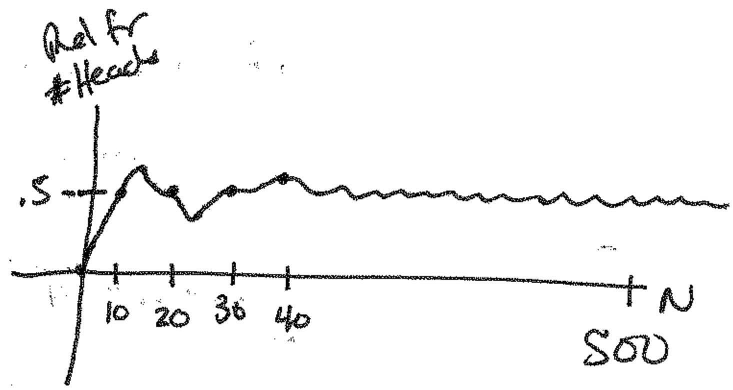
1. $0 \le P(s_i) \le 1$
2. $P(s_1) + P(s_2) + \cdots + P(s_n) = 1$
3. If $E = \{e_1, e_2, \ldots, e_r\}$ then $P(E) = P(e_1) + P(e_2) + \cdots + P(e_r)$.

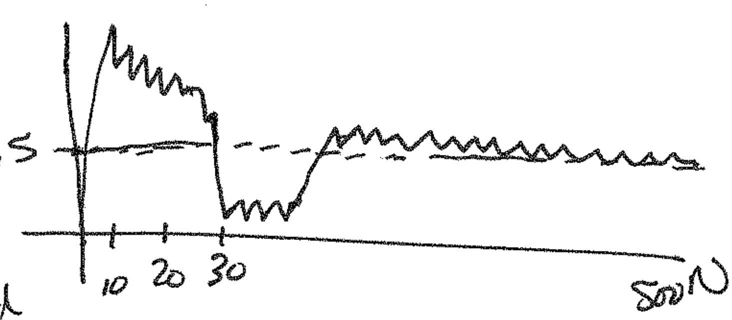
In words:

1. The relative frequency of each outcome is a number between 0 and 1 (inclusive).
2. The relative frequencies of all the outcomes add up to 1.
3. The relative frequency of an event E is the sum of the relative frequencies of the individual outcomes of E.

Relative Frequency and Increasing Sample Size A "fair" coin is one that is as likely to come up heads as it is to come up tails. In other words, we expect heads to come up 50% of the time if we toss such a coin many times. Put more precisely, we expect the relative frequency to approach .5 as the number of trials gets larger. Let's graph the behavior of the relative frequency for a sequence of coin tosses. For each N we will plot what fraction of times the coin comes up heads in the first N tosses.

N	# Heads
10	5
+10 20	5
+10 30	# Heads 5
+10 40	6





As N gets larger, the graph should approach .5. This idea is called the limit of a graph (a function)

Outcome	1	2	3	4	5	6
Probability	.3	.3		.1	.2	

Roll a die

.05 .05

.1 0

Table 1: Example 3

$P(1)+P(2)+P(3)+P(4)+P(5)+P(6) = 1$

Section 7.3: Probability and Probability Models

Probability Distribution; Probability

(Compare with the properties of relative frequency.)

A (finite) probability distribution is an assignment of a number $P(s_i)$, the probability of s_i, to each outcome of a finite sample space $S = \{s_1, s_2, \ldots, s_n\}$. The probabilities must satisfy

1. $0 \le P(s_i) \le 1$

2. $P(s_1) + P(s_2) + \cdots + P(s_n) = 1$.

We find the probability of an event E, written $P(E)$, by adding up the probabilities of the outcomes in E.

If $P(E) = 0$, we call E an impossible event. The empty event $\emptyset$ is always impossible, since *something* must happen.

Examples

1. Let us take $S = \{H, T\}$ and make the assignments $P(H) = .5$ and $P(T) =$ .5 Fair coin

2. We can instead make the assignments $P(H) = .2$ and $P(T) =$ .8

Unfair coin

3. The table at the top of this page gives a probability distribution for the sample space $S = \{1, 2, 3, 4, 5, 6\}$.

o	1	2	3	4	5	6
P	.3	.3	.1	.1	.2	0

It follows that $P(\{1, 6\}) =$ .3 + 0 = .3

$P(\{2, 3\}) =$ .3 + .1 = .4

$P(3) =$ .1

Probability Models

A probability model for a particular experiment is a probability distribution that predicts the relative frequency of each outcome if the experiment is performed a large number of times. Just as we think of relative frequency as *estimated probability,* we can think of modeled probability as theoretical probability

Examples

1. Fair Coin Model Flip a fair coin and observe the side that faces up. Because we expect that heads is as likely to come up as tails, we model this experiment with the probability distribution specified by:

$S = \{H, T\}$ $P(H) = .5$ $P(T) = .5$

2. Unfair Coin Model

$S = \{H, T\}$ $P(H) = .6$ $P(T) = .4$

3. Fair Die Model Roll a fair die and observe the number that faces up. Because we expect to roll each specific number one sixth of the time, we model the experiment with the probability distribution specified by:

$S = \{1, 2, 3, 4, 5, 6\}$ $P(1) = 1/6$ $P(2) = 1/6 = P(3) = P(4)$

$P(5) = 1/6$ $P(6) = 1/6$

4. Roll a pair of fair distinguishable dice

$n(S) = 36$ Probability of each outcome is $1/36$

5. In the experiment in Example 4, take E to be the event that the sum of the numbers that face up is 5, so $E = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$. By properties of probability distributions,

$$P(E) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}$$

$$= 4\left(\frac{1}{36}\right)$$

$$= \frac{4}{36} = \boxed{\frac{1}{9}}$$

For each $s \in S$,

$$P(s) = \frac{1}{n(S)}$$, if each outcome is equally likely.

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#6) Roll two <u>fair</u>, distinguishable dice; event: both numbers are prime

$n(S) = 36 \quad P(s_i) = 1/36$

$1, \boxed{2}, \boxed{3}, 4, \boxed{5}, 6$

$E = \{(2,3), (2,5), (3,5), (3,2), (5,2), (5,3), (2,2), (3,3), (5,5)\}$

$n(E) = 9$

$P(E) = 9/36 = \frac{1}{4}$

#7) 3 distinguishable & fair coins are tossed; event: the result is <u>at most</u> 1 head

$S = \{(H,H,H)$
(H,H,T)
(H,T,H)
(H,T,T)
(T,H,H)
(T,H,T)
(T,T,H)
$(T,T,T)\}$

$E = \{(H,T,T), (T,T,H), (T,H,T)$
$(T,T,T)\}$

$P(s) = 1/8$
$s \in S$

$P(E) = 4(1/8) = 1/2$

Example 1 (Compare to Example 1 in Section 7.2) A total of 1.9 million hybrid vehicles had been sold in the United States through October of 2011. Of these, 955,000 were Toyota Prii, 205,000 were Honda Civics, 170,000 were Toyota Camrys, 105,000 were Ford Escapes, and the rest were other makes.

(a) What is the probability that a randomly selected hybrid vehicle sold in the United States was either a Toyota Prius or a Honda Civic?

(b) What is the probability that a randomly selected hybrid vehicle sold in the United States was not a Toyota Camry?

(a) S = the set of all hybrid vehicles sold through Oct. 2011

$n(S) = 1{,}900{,}000$

E = the set of Toyota Prii and Honda Civics

$n(E) = 955{,}000 + 205{,}000 = 1{,}160{,}000$

$$P(E) = \frac{n(E)}{n(S)} = \frac{1{,}160{,}000}{1{,}900{,}000} = .61$$

(b) F = the set of Toyota Camrys

Want $P(F')$. $n(F')$

$n(F') = n(S) - n(F) = 1{,}900{,}000 - 170{,}000$

$= 1{,}730{,}000$

$$P(F') = \frac{n(F')}{n(S)} = \frac{1{,}730{,}000}{1{,}900{,}000} = .91$$

Example 2 Recall that the sample space when rolling a pair of fair indistinguishable dice is

$$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6),$$
$$(3,3), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,5), (5,6), (6,6)\}$$

Construct a probability model for this experiment.

$$n(S) = 21$$

$$\cancel{P(s_i) = 1/21}$$

	Outcome Indistinguishable	$(1,1)$	$(1,2)$	$(2,3)$	$(5,5)$	$(5,6)$
$n(S)=36$	Distinguishable	$\{(1,1)\}$	$\{(1,2),(2,1)\}$	$\{(2,3),(3,2)\}$	$\{(5,5)\}$	$\{(5,6),(6,5)\}$
	Probability	$1/36$	$2/36 = 1/18$	$2/36 = 1/18$	$1/36$	$2/36 = 1/18$

$$6 \cdot \frac{1}{36} + 15 \cdot \frac{2}{36}$$

$$= \frac{6}{36} + \frac{30}{36} = \frac{36}{36} = 1$$

Example 3 In order to impress your friends with you die-rolling skills, you have surreptitiously weighted your die in such a way that 6 is three times as likely to come up as any one of the other numbers. (All the other outcomes are equally likely.) Obtain a probability distribution for a roll of the die and use it to calculate the probability of an even number coming up. (a) (b)

(a) Let x = probability of rolling a 6

y = probability of rolling something other than a 6

$$x = 3y$$

$$x + y + y + y + y + y = x + 5y = 1$$

$$\begin{cases} x = 3y \\ x + 5y = 1 \end{cases}$$

Two equations in two unknowns

$$3y + 5y = 1$$

$$8y = 1$$

$$y = 1/8 \qquad x = 3 \cdot 1/8 = 3/8$$

Outcome	1	2	3	4	5	6
Probability	1/8	1/8	1/8	1/8	1/8	3/8

(b)

$$P(E) = P(\{2,4,6\})$$

$$= 1/8 + 1/8 + 3/8$$

$$= 5/8$$

Addition Principle Consider Example 3 on page 4.

Outcome	1	2	3	4	5	6
Prob	.3	.3	.1	.1	.2	0

$A = \{1, 2, 3\}$

$B = \{2, 3, 4\}$

$P(A) = .7$

$P(B) = .5$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 1.2 - .4$$

$$= .8$$

If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$