Math 170: Section 2.3 Annuities and Loans - Notes

1 Sinking Funds

Future Value of a Sinking Fund A *sinking fund* is an account earning compound interest into which you make periodic deposits. Suppose that the account has an annual rate of r compounded m times per year, so that $i = r/m$ is the interest rate per compounding period. If you make a payment of PMT at the end of each period, then the future value after t years, or $n = mt$ periods, will be

$$FV = PMT\frac{(1+i)^n - 1}{i}.$$

Example 1 Your retirement account has \$5000 in it and earns 5% interest per year compounded monthly. Every month for the next 10 years you will deposit \$100 into the account. How much money will there be at the end of those 10 years?

Solution

This is a sinking fund with PMT =\$100, $r = 0.05$, $m = 12$, $i = 0.05/12$, and $n = 12 \times 10 = 120$. Let's ignore the \$5000 that is already in the account and calculate the future value of the payments:

$$FV = PMT\frac{(1+i)^n - 1}{i} = 100\frac{(1+0.05/12)^{120} - 1}{0.05/12} \approx \$15,528.23$$

The \$5000 that was already in the account is also going to earn the same compound interest of those 10 years so we use the formula from section 2.2 to calculate its future value:

$$FV = 5000(1+0.05/12)^{12\times 10} = \$8,235.05$$

Thus, the total amount of money in the account at the end of 10 years is \$15,528.23+\$8,235.05=\$23,763.28.

Example 2 Suppose you would like to start a college savings account for your new nephew/niece/child. You would like to have \$100,000 in the account at the end of 17 years. If the account pays 4% interest per year compounded quarterly, and you make deposits at the end of every quarter, how large must the payments be to reach your goal?

Solution

In this problem, instead of finding how much money will we have in the future, we want to know how much to deposit so we end up with a specified amount. So we're solving for PMT:

$$PMT = \frac{FV}{\frac{(1+i)^n - 1}{i}} = \frac{FVi}{(1+i)^n - 1}$$

In this problem, $FV = \$100,000$, $m = 4$, $n = 4 \times 17 = 68$, $r = 0.04$, and $i = 0.04/4 = 0.01$. Thus,

$$PMT = \frac{100,000 \times 0.01}{(1+0.01)^{68} - 1} \approx \$1,033.89$$

So you would need to deposit about \$1,033.89 at the end of each quarter to meet your goal.

2 Annuities

Suppose instead of making payments into an account at the end of each month, we put some money in an account and then periodically make withdrawals. For instance, suppose we deposit an amount PV now in an account earning 3.6% interest per year compounded monthly. Starting 1 month from now, the bank will send us monthly payments of \$100. What must PV be so that the account will be drawn down to \$0 in exactly 2 years?
So, $i = 0.36/12 = 0.003$ and $PMT = \$100$.
The first payment of \$100 will be made 1 month from now, so the present value of that payment is

$$\$100(1+0.003)^{-1} \approx \$99.70$$

Notice that the exponent here is negative. This is because we are thinking about this as solving for present value in compound interest:

$$PV = \frac{FV}{(1+i)^n} = FV(1+i)^{-n}.$$

The second payment of \$100 will occur two month from now, so its present value is

$$\$100(1+0.003)^{-2} \approx \$99.40$$

We can do the same calculation for all the future payments until the last one which occurs 24 months from now

$$\$100(1+0.003)^{-2} \approx \$93.06$$

Then PV is the sum of all these values. Just like for sinking funds, luckily there is a nice formula for this type of sum, as well:

$$x^{-1} + x^{-2} + x^{-3} + \cdots + x^{-n} = \frac{x^{n-1} + x^{n-2} + \cdots + 1}{x^n}.$$

Thus,

$$PV = 100\frac{1-(1+0.003)^{-24}}{(1+0.003)-1} \approx \$2,312.29$$

Present Value of an Annuity An *annuity* is an account earning compound interest from which periodic withdrawals are made. Suppose that the account has an annual rate of r compounded m times per year, so that $i = r/m$ is the interest rate per compounding period. Suppose also that the account starts with a balance of PV. If you receive a payment of PMT at the end of each compounding period, and the account is down to \$0 after t years, or $n = mt$ periods, then

$$PV = \frac{PMT(1-(1+i)^{-n})}{i}$$

If you want to know how much your payments will be given a starting balance of PV simply solve for PMT:

$$PMT = \frac{PVi}{1-(1+i)^{-n}}$$

Example 3 You have accumulated \$100,000 for your nephew/niece/child's college education! You would now like to make quarterly withdrawals over the next 4 years. How much money can you withdraw each quarter in order to draw down the account to zero at the end of 4 years? (Recall that the account pays 4% interest compounded quarterly.)

Solution

Now your account is acting like an annuity with a present value of \$100,000. So $PV = \$100,000$, $r = 0.04$, and $m = 4$, giving $i = 0.04/4 = 0.01$, and $n = 4 \times 4 = 16$. Thus,

$$PMT = \frac{100,000 \times 0.01}{1-(1+0.01)^{-16}} \approx \$6,794.46$$

So, you can withdraw \$6,794.46 each quarter for 4 years, at the end of which time the account balance is 0.

3 Installment Loans

In a typical installment loan, such as a car loan or a home mortgage, we borrow an amount of money and then pay it back with interest by making fixed payments (usually every month) over some number of years. From the point of view of the lender, this is an annuity. Thus, *loan calculations are* **identical** *to annuity calculations.*

Example 4 Alex and Blake are buying a house and have taken out a 30-year, \$90,000 mortgage at 8% interest per year. What will their monthly payment be?

Solution

From the bank's point of view, a mortgage is an annuity. In this case, the present value is $PV = \$90,000$, $r = 0.08$, $m = 12$, and $n = 12 \times 30 = 360$. Thus,

$$PMT = \frac{90,000 \times 0.08/12}{1-(1+0.08/12)^{-360}} \approx \$660.39$$

The word "mortgage" comes from the French word for "dead pledge" (just a little FYI from your textbook). The process of paying off a loan is called *amortizing* the loan, meaning to kill the debt owed.

Ever wondered why mortgages take so long to pay off? How much interest will Alex and Blake pay in the first year of their mortgage?

Solution

We should calculate how much of each month's payment is interest and how much goes to reducing the outstanding principal. At the end of the first month, Alex and Blake must pay 1 month's interest on \$90,000:

$$\$90,000 \times \frac{0.08}{12} = \$600$$

So the remainder of their first monthly payment, \$660.39-\$600=\$60.39, goes to reducing the principal. So, in the second month the outstanding principal is \$90,000-\$60.39=\$89,939.61, and part of their second monthly payment will be for the interest on this amount,

$$\$89,939.61 \times \frac{0.08}{12} = \$599.60$$

The remaining \$660.39-\$599.60=\$60.79 goes to reducing the principal. If we continue this calculation for the 12 months of the first year, we get the beginning of the mortgage's amortization schedule. (Excel would be a very practical tool to use for these calculations.) Adding up the interest paid for each of the 12 months we find that Alex and Blake pay \$7,172.81 in interest the first year. The amount paid on the principal is \$751.87.