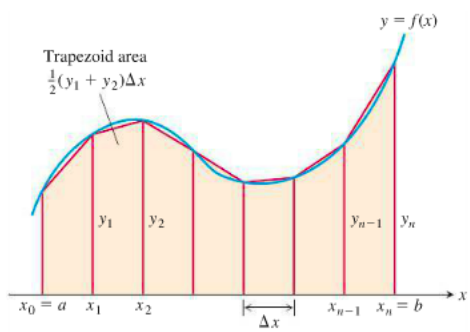
Math 142: Section 8.7 - Notes

1 Numerical Integration

What to do when there's no nice antiderivative? The antiderivatives of some functions, like $\sin(x^2)$, $1/\ln x$, and $\sqrt{1+x^4}$, have no elementary formulas. When we cannot find a workable antiderivative for a function f that we have to integrate, we can partition the interval of integration, replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to *approximate* the definite integral of f. This is an example of numerical integration. There are many methods of numerical integration but we will study only two: the *Trapezoidal Rule* and *Simpson's Rule*.

Trapezoidal Approximations As the name implies, the Trapezoidal Rule for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapezoids instead of rectangles (see the figure below).



The Trapezoidal Rule To approximate $\int_a^b f(x)dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

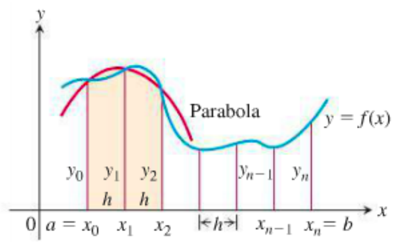
The y's are the values of f at the partition points

$$x_0 = a, x_1 = a+\Delta x, x_2 = a+2\Delta x, \ldots, x_{n-1} = a+(n-1)\Delta x, x_n = a+n\Delta x = b,$$

where $\Delta x = (b-a)/n$.

Example 1 Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2 dx$. Compare the estimate with the exact value.

Simpson's Rule: Approximating Using Parabolas Instead of using the straight-line segments that produced the trapezoids, we can use parabolas to approximate the definite integral of a continuous function. We partition the interval $[a, b]$ into n subintervals of equal length $h = \Delta x = (b-a)/n$ but this time we require that n be an even number. On each consecutive pair of intervals we approximate the curve $y = f(x) \geq 0$ by a parabola. A typical parabola passes through three consecutive points (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i+1}, y_{i+1}) on the curve.



Simpson's Rule To approximate $\int_a^b f(x)dx$, use

$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

The number n is even and $\Delta x = (b-a)/n$.

Example 2 Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4 dx$.

Theorem 1: Error Estimates in the Trapezoidal and Simpson's Rules If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

Example 3 Find an upper bound for the error in estimating $\int_0^2 5x^4 dx$ using Simpson's Rule with $n = 4$ (See previous example).