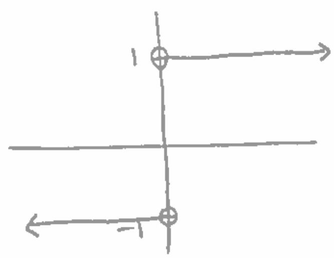
Math 141: Section 2.4 One-Sided Limits - Notes

Approaching a Limit from One Side To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. Ordinary limits are two-sided!

If f fails to have a two-sided limit at c, it may still have a one-sided limit. If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.

Example 1 Consider the function $f(x) = \dfrac{x}{|x|}$.



If we approach $x=0$ from the right, the limit is 1.
If we approach $x=0$ from the left, the limit is -1.
Since these values aren't the same, $f(x)$ does <u>not</u> have a two-sided limit.

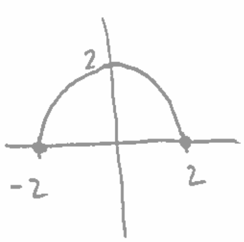
Right-hand Limit

$$\lim_{x \to c^+} f(x) = L$$

Left-hand Limit

$$\lim_{x \to c^-} f(x) = L$$

Example 2 The domain of $f(x) = \sqrt{4-x^2}$ is [-2,2]; its graph is the semicircle centered at the origin with radius 2.



$\lim_{x \to -2^+} \sqrt{4-x^2} = 0$ and $\lim_{x \to 2^-} \sqrt{4-x^2} = 0$

What about $\lim_{x \to -2^-} \sqrt{4-x^2}$? Does Not Exist (DNE)

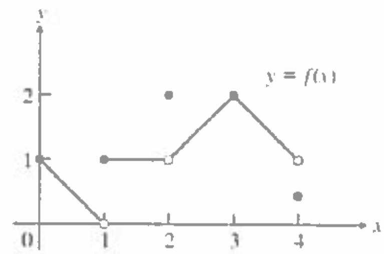
$\lim_{x \to 2^+} \sqrt{4-x^2}$ DNE

Also, $\lim_{x \to -2} f(x)$, $\lim_{x \to 2} f(x)$ DNE since the left-hand and right-hand limits do not agree

Theorem 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = L.$$

Example 3 Consider the graph of the function:



$\lim_{x \to 0^+} f(x) = 1$ $\lim_{x \to 0^-} f(x)$ DNE $\lim_{x \to 0} f(x)$ DNE

$\lim_{x \to 1^-} f(x) = 0$ $\lim_{x \to 1^+} f(x) = 1$ $\lim_{x \to 1} f(x)$ DNE

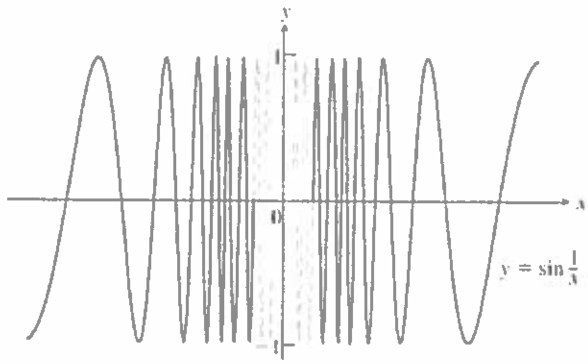
$\lim_{x \to 2^-} f(x) = 1$ $\lim_{x \to 2^+} f(x) = 1$ $\lim_{x \to 2} f(x) = 1$ $f(2) = 2$

$\lim_{x \to 3^-} f(x) = 2$ $\lim_{x \to 3^+} f(x) = 2$ $\lim_{x \to 3} f(x) = 2$ $f(3) = 2$

$\lim_{x \to 4^-} f(x) = 1$ $\lim_{x \to 4^+} f(x)$ DNE $\lim_{x \to 4} f(x)$ DNE

At every other point c in $[0,4]$ $f(x)$ has limit $f(c)$.

Example 4 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side.



As x approaches 0, the reciprocal $1/x$ grows without bound and the values of $\sin(1/x)$ cycle repeatedly between -1 and 1.
So, there is no single number L that the function's values stay increasingly close to as $x \to 0$.
Even if we restrict x to positive or negative values, the function still doesn't approach a single value.
Thus, the function has neither a right-hand limit nor a left-hand limit at $x=0$.