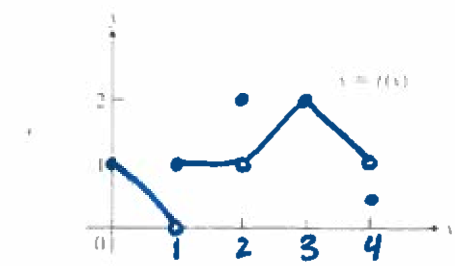
Math 141: Section 2.5 Continuity - Notes

Continuity at a Point Often when collecting data, we connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the points we did not measure. In doing so, we are assuming that we are working with a *continuous function*. Intuitively, any function $y = f(x)$ whose graph can be sketched over its domain in one unbroken motion is an example of a continuous function.

Example 1 Consider the graph of the function given below. At which numbers does the function appear to be not continuous?



$\lim_{x \to 1} f(x)$ DNE

$\lim_{x \to 2} f(x) = 1 \neq f(2)$

$\lim_{x \to 4^-} f(x) = 1 \neq f(4)$

$x = 2$ Jump!　$x = 1$ Jump!　$x = 4$ Jump!

Definition: Let c be a real number on the x-axis.

The function f is **continuous at** c if

$$\lim_{x \to c} f(x) = f(c).$$

The function f is **right-continuous at** c (or continuous from the right) if

$$\lim_{x \to c^+} f(x) = f(c).$$

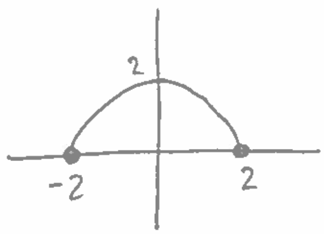
The function f is **left-continuous at** c (or continuous from the left) if

$$\lim_{x \to c^-} f(x) = f(c).$$

We say that a function is **continuous over a closed interval** $[a, b]$ if it is right-continuous at a, left-continuous at b, and continuous at all interior points of the interval.

Example 2 The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$.

$f(x) = \sqrt{4-x^2}$ is continuous on $[-2,2]$



$\lim_{x \to -2^+} f(x) = 0 = f(-2)$ Right-cts

$\lim_{x \to 2^-} f(x) = 0 = f(2)$ Left-cts

$\lim_{x \to c} f(x) = f(c)$, $-2 < c < 2$ Interior $(-2,2)$

Continuity Test A function $f(x)$ is continuous at a point $x = c$ <u>if and only if</u> it meets the following three conditions:

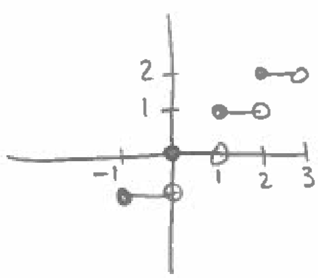
1) $f(c)$ exists $f(c)$ exists

2) $\lim_{x \to c} f(x)$ exists $\lim_{x \to c} f(x)$ exists, $\lim_{x \to c} f(x) = f(c)$

3) $\lim_{x \to c} f(x) = f(c)$

Example 3 Consider the greatest integer function. $f(x) = \lfloor x \rfloor$. Where is f discontinuous? Where is it right or left continuous?

$f(x) = \lfloor x \rfloor$



is discontinuous at every integer, n, since the left-hand and right-hand limits do not agree as $x \to n$.

Since $\lfloor n \rfloor = n$ for any integer,

$\lim_{x \to n^+} \lfloor x \rfloor = n = f(n)$, $f(x)$ is right continuous at any integer. Not left continuous.

Continuous Functions In general, we want to describe the continuity behavior of a function throughout its entire domain. We define a **continuous function** as one that is continuous at every point in its domain. Note: This is a property of the function. A function always has a specified domain - if we change the domain, we change the function. If a function is discontinuous at one or more points of its domain, we say it is a **discontinuous** function.

Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following are also continuous at $x = c$:

1. Sums: $f + g$
2. Differences: $f - g$
3. Constant multiples: kf, for any number k
4. Products: fg
5. Quotients: f/g, provided $g \neq 0$
6. Powers: f^n, n a positive integer
7. Roots: $f^{1/n}$, provided it is defined on an open interval containing c, where n is a positive integer

Example 4 Why is every polynomial continuous? If $P(x)$ and $Q(x)$ are polynomials, why is the rational function $P(x)/Q(x)$ continuous wherever it is defined ($Q(c) \neq 0$)?

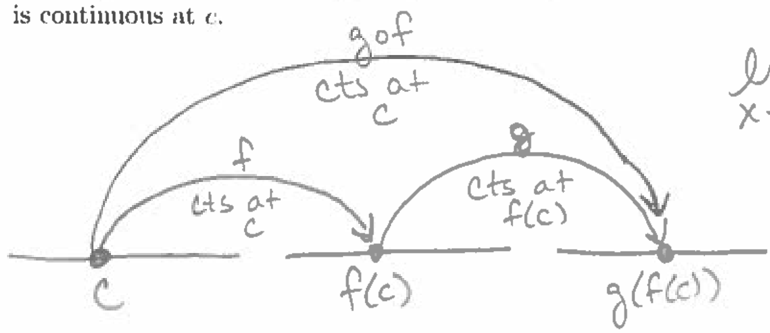
$\lim_{x \to c} P(x) = P(c)$, so every polynomial is continuous.

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)} \quad (Q(c) \neq 0)$$

Inverse Functions and Continuity The inverse function of any function continuous on an interval is continuous over its domain.

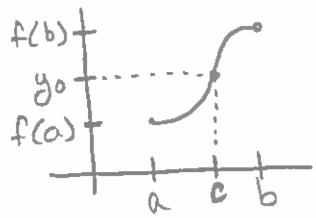
Intuitively, reflecting the graph of $f(x)$ over the line $y=x$ does not create any new holes or breaks, so $f^{-1}(x)$ is also continuous on its domain.

Composites All composites of continuous functions are continuous. That is, if f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.



$\lim_{x \to c} g(f(x)) = g(f(c))$

Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



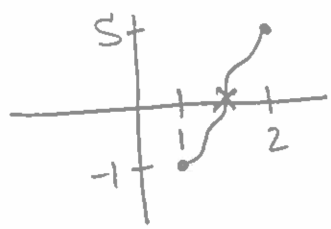
Example 5 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

$x^3 - x - 1 = 0 \quad [1, 2]$

Let $f(x) = x^3 - x - 1$, f is a polynomial so it is continuous on every closed interval

$f(1) = 1^3 - 1 - 1 = -1 < 0$

$f(2) = 2^3 - 2 - 1 = 5 > 0$



Since $y_0 = 0$ is a value between $f(1)$ and $f(2)$, the IVT says there must be a zero of f between 1 and 2.