Math 141: Section 5.5 Indefinite Integrals and the Substitution Method - Notes

The Indefinite Integral In Section 4.8, we defined the **indefinite integral** of the function f with respect to x as the set of *all* antiderivatives of f, symbolized by $\int f(x)dx$. Since any two antiderivatives of f differ by a constant, the indefinite integral $\int$ notation means that for any antiderivative F of f,

$$\int f(x)dx = F(x) + C,$$

where C is any constant.

Be careful with the difference between definite integrals and indefinite integrals:

A definite integral $\int_a^b f(x)dx$ is a number

An indefinite integral $\int f(x)dx$ is a function + C

Our First Technique of Integration We have seen a list of general rules for antiderivatives but a lot of functions are not easily recognizable. For instance, after applying the quotient rule and simplifying, it would be hard for someone to figure out which function you started with. Our first technique of integration is a way to run the Chain Rule backwards.

u-Substitution If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

Thus,

$$\int u^n \frac{du}{dx}\, dx = \frac{u^{n+1}}{n+1} + C$$

More simply,

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C$$

Example 1 Find the integral $\int (x^3+x)^5(3x^2+1)dx$.

Previously, to compute this integral we would need to multiply everything out to get a polynomial.

Now,

let $u = x^3 + x$

$du = (3x^2+1)dx$

Then,

$$\int (x^3+x)^5 (3x^2+1)\,dx$$
$$= \int u^5 du$$
$$= \frac{1}{6}u^6 + C$$
$$= \frac{1}{6}(x^3+x)^6 + C$$

Check:

$$\frac{1}{6}\cdot 6\,(x^3+x)^5(3x^2+1)$$
$$= (x^3+x)^5(3x^2+1) \checkmark$$

The Substitution Method

1) Identify a piece of the integrand that looks like it could be the derivative of another piece of the integrand, and let u be that function. In the previous example, $u = x^3 + x$.

2) Use the differential $du = \frac{du}{dx}dx$ to change variables from x to u. In Example 1, $du = (3x^2+1)dx$.

3) Finally, since these expressions are equal, we can evaluate the simpler integral. Previously, $\int (x^3+x)^5(3x^2+1)dx = \int u^5 du$.

Example 2 Find $\int \sqrt{2x+1}dx$.

$$= \int (2x+1)^{1/2} dx$$

Let $u = 2x+1$

$du = 2\,dx$

$\frac{1}{2} du = dx$

$$\int (2x+1)^{1/2} dx$$

$$= \int u^{1/2} \cdot \frac{1}{2} du$$

$$= \frac{1}{2} \int u^{1/2} du$$

$$= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3} u^{3/2} + C$$

$$= \frac{1}{3} (2x+1)^{3/2} + C$$

Example 3 Find $\int 5 \sec^2(5x+1)dx$..

Let $u = 5x+1$

$du = 5dx$

$$\int \sec^2(u)du = \tan(u) + C$$

$$= \tan(5x+1) + C$$

In general, we use Substitution for integrals of the form $\int f(g(x))\,g'(x)dx$.

1) Substitute $u = g(x)$ and $du = g'(x)dx$ to obtain $\int f(u)du$

2) Integrate wrt u

3) Replace u by $g(x)$

Example 4 Evaluate $\displaystyle\int x\sqrt{2x+1}dx$.

$u = 2x+1$

$du = 2\,dx$

$$\int x\sqrt{u}\,\tfrac{1}{2}du$$

$u-1 = 2x$

$\dfrac{u-1}{2} = x$

$$\int \frac{u-1}{2}\cdot u^{1/2}\cdot \frac{1}{2}du$$

$$\frac{1}{4}\int (u-1)u^{1/2}du$$

$$= \frac{1}{4}\int (u^{3/2}-u^{1/2})du$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{10}u^{5/2} - \frac{1}{6}u^{3/2} + C = \boxed{\frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C}$$

Substitution for Definite Integrals If g' is continuous on the interval $[a,b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Example 5 Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$.

Sol 1: $u = x^3+1 \quad du = 3x^2dx$

$$\int_{\square}^{\square} u^{1/2}du = \frac{2}{3}u^{3/2}\Big|_{\square}^{\square} = \frac{2}{3}(x^3+1)^{3/2}\Big|_{-1}^{1}$$

$$= \frac{2}{3}(1^3+1)^{3/2} - \frac{2}{3}((-1)^3+1)^{3/2}$$

$$= \frac{2}{3}(2)^{3/2} = \frac{4\sqrt{2}}{3}$$

Sol 2: $u = x^3+1 \quad du = 3x^2dx$

$x=-1 \quad u=0$

$x=1 \quad u=2$

$$\int_0^2 u^{1/2}du = \frac{2}{3}u^{3/2}\Big|_0^2$$

$$= \frac{2}{3}(2)^{3/2} - \frac{2}{3}(0)^{3/2}$$

$$= \frac{4\sqrt{2}}{3}$$