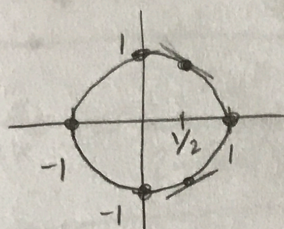
1. Consider the equation of a circle of radius one (the unit circle): $x^2 + y^2 = 1$
   a. Draw a picture of this circle.



   b. What would you guess the slope of the tangent line is to the circle at $x = \frac{1}{2}$? Why do you say this?

$m \approx 1$ or $m \approx -1$?

There are two points for $x = 1/2$!

   c. Check your work by first finding a formula for $\frac{dy}{dx} =$ $-x/y$

   and then finding the slope of the tangent line at $x = \frac{1}{2}$. Does your answer make sense with your picture? Why or why not? (Discuss this one with a partner or a group, as it isn't trivial.)

$2x + 2y \, dy/dx = 0$

$2y \frac{dy}{dx} = -2x$

$m = \frac{dy}{dx} = -\frac{x}{y}$

$x = 1/2: \ \frac{1}{4} + y^2 = 1$

$y^2 = 3/4$

$y = \pm \frac{\sqrt{3}}{2}$

$m = -\frac{1/2}{\sqrt{3}/2}$

$= -\frac{1}{\sqrt{3}}$

or

$m = -\frac{1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

2. In your own words, what is implicit differentiation and when is it useful?

Your own words! But some ideas:

Differentiating an unknown or unspecified function of x.

When the function is messy and you just need a placeholder.

3. Use implicit differentiation to find $\frac{dy}{dx}$.

a. $xy = 1$

$$1\cdot y + x\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} = -y$$

$$\boxed{\frac{dy}{dx} = -y/x}$$

b. $x^2 - y^2 = 1$

$$2x - 2y\,dy/dx = 0$$

$$-2y\frac{dy}{dx} = -2x$$

$$\boxed{dy/dx = x/y}$$

c. $\sqrt{x} - \sqrt{y} = 1$

$$x^{1/2} - y^{1/2} = 1$$

$$\frac{1}{2}x^{-1/2} - \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$$

$$-\frac{1}{2}y^{-1/2}\frac{dy}{dx} = -\frac{1}{2}x^{-1/2}$$

$$\boxed{\frac{dy}{dx} = \frac{\sqrt{y}}{\sqrt{x}} = \sqrt{y/x}}$$

d. $2x^2y + 3xy^3 = 1$

$$4xy + 2x^2\frac{dy}{dx} + 3y^3 + 9xy^2\frac{dy}{dx} = 0$$

$$2x^2\frac{dy}{dx} + 9xy^2\frac{dy}{dx} = -4xy - 3y^3$$

Factor out $\frac{dy}{dx}$ then divide to get

$$\boxed{\frac{dy}{dx} = \frac{-4xy - 3y^3}{2x^2 + 9xy^2}}$$

e. $(x-1)y^2 = x+1$

$$y^2 + 2(x-1)y\frac{dy}{dx} = 1$$

$$2(x-1)y\frac{dy}{dx} = 1-y^2$$

$$\boxed{\frac{dy}{dx} = \frac{1-y^2}{2y(x-1)}}$$

4. Pick two of the problems from #3 and find the second derivative with respect to x.
   a. What was different (or notable) about this process?

(a) $\frac{dy}{dx} = -\frac{y}{x}$

$$\frac{d^2y}{dx^2} = \frac{-\frac{dy}{dx}\cdot x - -y\cdot 1}{x^2}$$

$$= \frac{-x\frac{dy}{dx} + y}{x^2}$$

$$= \frac{-x(-\frac{y}{x}) + y}{x^2}$$

$$= \frac{2y}{x^2}$$

The second derivative is in terms of x, y, AND the first. Since we know $\frac{dy}{dx}$, we substitute

5. Find the slope of the tangent line at the given point.

"Use implicit differentiation then plug in these values for x and y"

a. $xy^5 + yx^5 = 1$ at (-1,-1)

$$y^5 + x \cdot 5y^4 y' + y'x^5 + y \cdot 5x^4 = 0$$

$$5xy^4y' + x^5y' = -y^5 - 5yx^4$$

$$y' = \frac{-y^5 - 5yx^4}{5xy^4 + x^5} \qquad m_{(-1,-1)} = \frac{-(-1)^5 - 5(-1)(-1)^4}{5(-1)(-1)^4 + (-1)^5} = \frac{6}{-6} = \boxed{-1}$$

b. $\frac{1}{x^3} + \frac{1}{y^3} = 2$ at (1,1)

$$x^{-3} + y^{-3} = 2 \qquad m_{(1,1)} = -\frac{1^4}{1^4} = \boxed{-1}$$

$$-3x^{-4} + -3y^{-4}y' = 0$$

$$-3y^{-4}y' = 3x^{-4}$$

$$y' = \frac{3x^{-4}}{-3y^{-4}} = -\frac{y^4}{x^4}$$

6. For those of you who need/want a challenge, try this problem. Consider the function $f(x) = x^2$ and let (a,b) be a point in the plane. Find conditions for a and b such that
   a. there are no tangent lines to *f* that pass through (a,b).
   b. there is exactly one tangent line to *f* that passes through (a,b).
   c. there are two tangent lines to *f* that pass through (a,b).

If you tried this problem and want to talk about your solution, come by during office hours!

Math 141: Section 3.8 Derivatives of Inverse Functions and Logarithms

The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

Example 1 Let $f(x) = x^2 - 4x - 5$, $x > 2$. Find the value of df^{-1}/dx at $x = 0 = f(5)$ without finding a formula for $f^{-1}(x)$.

$f'(x) = 2x - 4$

"$f(5)$" tells us the x-value to plug into f is $x=5$

"$x=0$" is the x-value for the inverse function, f^{-1}.

$f'(5) = 2(5) - 4 = 6$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(5)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=5}} = \boxed{1/6}$$

Derivative of the Natural Logarithm Function We can derive a formula for the derivative of the natural logarithm in the following way. Since we know the exponential function $f(x) = e^x$ is differentiable everywhere, we can apply the Derivative Rule for Inverses to find the derivative of its inverse, $f^{-1}(x) = \ln x$.

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\
&= \frac{1}{e^{f^{-1}(x)}} \quad \text{(How did we get this?)} \\
&= \frac{1}{e^{\ln x}} \\
&= \frac{1}{x}
\end{aligned}
$$

If $f(x) = e^x$, then $f'(x) = e^x$. Composing with $f^{-1}(x)$ we get $e^{f^{-1}(x)}$.

By properties of inverse/logs. $e^{\ln(A)} = \ln(e^A) = A$

Challenge Instead of using the rule for inverses, we can find the derivative of $y = \ln x$ using implicit differentiation. To get started, solve $y = \ln x$ for x and then apply implicit differentiation.

$$y = \ln x$$

$$e^y = e^{\ln x}$$

$$e^y = x$$

$$e^y \cdot y' = 1 \quad \text{Differentiate Implicitly}$$

$$y' = \frac{1}{e^y} \quad \text{Substitute for } y.$$

$$y' = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Derivative of the Natural Logarithm Let $y = \ln x$ and $x > 0$. Then, the derivative, y', is given by the following:

$$y' = \frac{1}{x}$$

Derivative of a^x and $\log_a x$ Now that we know how to find the derivative of the natural logarithm, we can derive the formula for the derivative of $y = a^x$ using the following fact:

$$a^x = e^{\ln(a^x)} = e^{x \ln a}, a > 0$$

Use the chain rule to find the derivative:

$$(a^x)' = (e^{\ln a^x})' = (e^{x \ln a})' = e^{x \ln a} \cdot (x \cdot \ln a)'$$

$$= e^{\ln(a^x)} \cdot \ln a$$

$$= a^x \cdot \ln a$$

Derivative of a^u If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \cdot \ln a \cdot \frac{du}{dx}$$

Chain Rule

Logarithmic Differentiation The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to simplify using laws of logarithms (Section 1.6) before differentiating.

Example 2 Find dy/dx if

$$y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$$

Solution Start by taking the natural logarithm of both sides:

$$\ln y = \ln\left(\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\right)$$

$$= \ln x + \ln(x^2+1)^{1/2} - \ln(x+1)^{2/3} \quad \text{(Rules 1 and 2)}$$

$$= \ln x + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1) \quad \text{(Rule 4)}$$

Now we take the derivative of both sides with respect to x. Remember! We need to use implicit differentiation for the left side.

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - \frac{2}{3}\cdot\frac{1}{x+1}$$

Chain Rule

Solve for dy/dx:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}$$

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right)$$

Finally, replace y with the function of x:

$$\frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right)$$