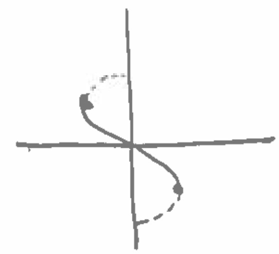
Math 141: Section 3.9 Inverse Trig Functions - Notes

Inverses of the six basic trigonometric functions

arcsin x $= y = \sin^{-1} x$	is the number in	$[-\pi/2, \pi/2]$	for which $\sin y = x$
$y = \cos^{-1} x$	is the number in	$[0, \pi]$	for which $\cos y = x$
$y = \tan^{-1} x$	is the number in	$(-\pi/2, \pi/2)$	for which $\tan y = x$
$y = \cot^{-1} x$	is the number in	$(0, \pi)$	for which $\cot y = x$
$y = \sec^{-1} x$	is the number in	$[0, \pi/2) \cup (\pi/2, \pi]$	for which $\sec y = x$
$y = \csc^{-1} x$	is the number in	$[-\pi/2, 0) \cup (0, \pi/2]$	for which $\csc y = x$

We use open or half-open intervals to avoid values for which the tangent, cotangent, secant, and cosecant functions are undefined.

The derivative of $\sin^{-1} u$: We know that the function $x = \sin y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. The theorem in section 3.8 therefore assures us that the inverse function, $y = \sin^{-1} x$ is differentiable throughout the interval $-1 < x < 1$. Let's find the derivative of $y = \sin^{-1} x$ by applying the theorem with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$.



$$(\sin^{-1} x)' = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \quad \text{(Theorem)}$$

$$= \frac{1}{\cos(\sin^{-1}(x))} \quad (f'(x) = \cos x)$$

$$\left(\begin{array}{l} \cos^2 u + \sin^2 u = 1 \\ \cos u = \sqrt{1 - \sin^2 u} \end{array} \right)$$

$$= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1}(x))}}$$

$$= \frac{1}{\sqrt{1 - (\sin(\sin^{-1} x))^2}}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

If u is a differentiable function of x with $|u| < 1$, we can use the chain to get a general formula.

Derivatives of Inverse Trig Functions

$$*\ \frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1$$

$$*\ \frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx},$$

$$\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1+u^2}\frac{du}{dx},$$

$$*\ \frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad |u| > 1$$

Ex: Find $\frac{d}{dx}(\sec^{-1}(5x^4))$

$$= \frac{1}{|5x^4|\sqrt{(5x^4)^2-1}} \cdot \frac{d}{dx}(5x^4)$$

$|5x^4| > 1$

$5x^4 > 1 > 0$

$$= \frac{1}{\cancel{5}x^4\sqrt{25x^8-1}} \cdot \cancel{20x^3}$$

$$= \boxed{\frac{4}{x\sqrt{25x^8-1}}}$$