Fundamental Theorem of Calculus (FTC)

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function.
Let $F\colon [a,b] \to \mathbb{R}$ be a function.

- If F is an antiderivative of f on $[a,b]$ (i.e. $F'(x) = f(x)$ for each $x \in [a,b]$), then

$$\int_a^b f(x)\,dx \;\equiv\; \int_a^b F'(x)\,dx \;=\; F(b) - F(a)\ .$$

- If $F(x) = \int_a^x f(t)\,dt$ for each $x \in [a,b]$, then F is an antiderivative of f on $[a,b]$, i.e.

$$F'(x) \equiv D_x\left[\int_a^x f(t)\,dt\right] \;=\; f(x)\ .$$

Basic Differentiation Rules

If the functions $y = f(x)$ and $y = g(x)$ are differentiable at x and a and b are constants, then:

(1) $D_x\left[af(x) \,+\, bg(x)\right] \;=\; af'(x) \,+\, bg'(x)$
(2) $D_x\left[f(x) \cdot g(x)\right] \;=\; f'(x) \cdot g(x) \,+\, f(x) \cdot g'(x)$
(3) $D_x\left[\frac{f(x)}{g(x)}\right] \;=\; \frac{f'(x)\,\cdot\, g(x) \,-\, f(x)\,\cdot\, g'(x)}{[g(x)]^2}$ provided $g(x) \neq 0$.

If f is differentiable at x and g is differentiable at $f(x)$, then:

(4) $D_x\left[g(f(x))\right] \;=\; g'(f(x))\; f'(x)$.

On this handout, a $(*)$ means that you do not have to memorize the formula but should be able to use the formula if you are given it.

Hyperbolic Trig Functions[1]

$$\cosh x \overset{(*)}{=} \frac{e^x + e^{-x}}{2} \qquad \sinh x \overset{(*)}{=} \frac{e^x - e^{-x}}{2} \qquad \cosh^2 x - \sinh^2 x \overset{(*)}{=} 1$$

$$\tanh x \overset{(*)}{=} \frac{\sinh x}{\cosh x} \qquad \coth x \overset{(*)}{=} \frac{\cosh x}{\sinh x} \qquad \operatorname{sech} x \overset{(*)}{=} \frac{1}{\cosh x} \qquad \operatorname{csch} x \overset{(*)}{=} \frac{1}{\sinh x}$$

<u>DERIVATIVES</u> $\overset{\text{FTC}}{\Longrightarrow}$ <u>INTEGRALS</u>

$$D_x \cosh u \overset{(*)}{=} \sinh u\,\frac{du}{dx} \qquad\qquad \int \sinh u\,du \overset{(*)}{=} \cosh u \,+\, C$$

$$D_x \sinh u \overset{(*)}{=} \cosh u\,\frac{du}{dx} \qquad\qquad \int \cosh u\,du \overset{(*)}{=} \sinh u \,+\, C$$

$$D_x \sinh^{-1} u \overset{(*)}{=} \frac{1}{\sqrt{1+u^2}}\,\frac{du}{dx} \qquad\qquad \int \frac{du}{\sqrt{a^2+u^2}} \underset{a>0}{\overset{(*)}{=}} \sinh^{-1}\left(\frac{u}{a}\right) + C$$

$$D_x \cosh^{-1} u \underset{u>1}{\overset{(*)}{=}} \frac{1}{\sqrt{u^2-1}}\,\frac{du}{dx} \qquad\qquad \int \frac{du}{\sqrt{u^2-a^2}} \underset{u>a>0}{\overset{(*)}{=}} \cosh^{-1}\left(\frac{u}{a}\right) + C$$

[1]Hyperbolic Trig Functions are covered in §7.3, pages 439–447.

Basic Integral Formulas

DERIVATIVES $\overset{\text{FTC}}{\Longrightarrow}$ INTEGRALS

DERIVATIVES		INTEGRALS
$D_x\, u^n = nu^{n-1}\dfrac{du}{dx}$		$\displaystyle\int u^n\,du \overset{n\neq -1}{=} \frac{u^{n+1}}{n+1} + C$
$D_x\, e^u = e^u\dfrac{du}{dx}$		$\displaystyle\int e^u\,du = e^u + C$
$D_x\, \ln\lvert u\rvert \overset{u\neq 0}{=} \dfrac{1}{u}\dfrac{du}{dx}$		$\displaystyle\int \frac{du}{u} \overset{u\neq 0}{=} \ln\lvert u\rvert + C$
$D_x a^u = a^u \ln a \dfrac{du}{dx}$	$0 < a \neq 1$	$\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C$
$D_x \log_a\lvert u\rvert \overset{u\neq 0}{=} \dfrac{1}{u}\dfrac{1}{\ln a}\dfrac{du}{dx}$	$0 < a \neq 1$	
$D_x \sin u = \cos u\dfrac{du}{dx}$		$\displaystyle\int \cos u\,du = \sin u + C$
$D_x \tan u = \sec^2 u\dfrac{du}{dx}$		$\displaystyle\int \sec^2 u\,du = \tan u + C$
$D_x \sec u = \sec u\tan u\dfrac{du}{dx}$		$\displaystyle\int \sec u\tan u\,du = \sec u + C$
$D_x \cos u = -\sin u\dfrac{du}{dx}$		$\displaystyle\int \sin u\,du = -\cos u + C$
$D_x \cot u = -\csc^2 u\dfrac{du}{dx}$		$\displaystyle\int \csc^2 u\,du = -\cot u + C$
$D_x \csc u = -\csc u\cot u\dfrac{du}{dx}$		$\displaystyle\int \csc u\cot u\,du = -\csc u + C$
$D_x \sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx} = -D_x\cos^{-1} u$		$\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} \overset{a>0}{=} \sin^{-1}\frac{u}{a} + C$
$D_x \tan^{-1} u = \dfrac{1}{1+u^2}\dfrac{du}{dx} = -D_x\cot^{-1} u$		$\displaystyle\int \frac{du}{a^2+u^2} \overset{a>0}{=} \frac{1}{a}\tan^{-1}\frac{u}{a} + C$
$D_x \sec^{-1} u = \dfrac{1}{\lvert u\rvert\sqrt{u^2-1}}\dfrac{du}{dx} = -D_x\csc^{-1} u$		$\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} \overset{a>0}{=} \frac{1}{a}\sec^{-1}\frac{\lvert u\rvert}{a} + C$

MORE INTEGRALS

$$\int \tan u\,du = -\ln\lvert\cos u\rvert + C = \ln\lvert\sec u\rvert + C$$

$$\int \cot u\,du = \ln\lvert\sin u\rvert + C = -\ln\lvert\csc u\rvert + C$$

$$\int \sec u\,du = \ln\lvert\sec u + \tan u\rvert + C = -\ln\lvert\sec u - \tan u\rvert + C$$

$$\int \csc u\,du = -\ln\lvert\csc u + \cot u\rvert + C = \ln\lvert\csc u - \cot u\rvert + C$$

Generalized Exponential $y = b^x$ and Logarithmic $y = \log_b x$ Functions with base b where $b > 0$ but $b \neq 1$. Also $0 < a \neq 1$.

$\ln \equiv \log_e$

$$f(x) = b^x \equiv e^{x \ln b} \qquad : (-\infty, \infty) \to (0, \infty)$$

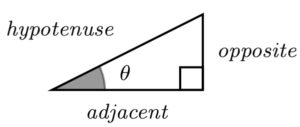
$$g(x) = \log_b x \equiv \text{the inverse of the fn. } f(x) = b^x \qquad : (0, \infty) \to (-\infty, \infty)$$

$$y = \log_b x \iff x = b^y$$

$$(\log_a b)(\log_b c) = \log_a c \implies \log_a x = \frac{\ln x}{\ln a}$$

$x, y > 0$ & $r \in \mathbb{R}$	$x, y \in \mathbb{R}$ & $r \in \mathbb{R}$
$b^{\log_b x} = x$	$\log_b(b^x) = x$
$\log_b 1 = 0$	$b^0 = 1$
$\log_b(xy) = \log_b x + \log_b y$	$b^x\, b^y = b^{x+y}$
$\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$	$\dfrac{b^x}{b^y} = b^{x-y}$
$\log_b(x^r) = r(\log_b x)$	$(b^x)^r = b^{xr}$
	$(ab)^x = a^x b^x$ and $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Basic Trig



$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

Basic Inverse Trig Functions

$y = \sin\theta$	$\Leftrightarrow$	$\theta = \sin^{-1} y$	where	$-1 \leq y \leq 1$	and	$\dfrac{-\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$
$y = \cos\theta$	$\Leftrightarrow$	$\theta = \cos^{-1} y$	where	$-1 \leq y \leq 1$	and	$0 \leq \theta \leq \pi$
$y = \tan\theta$	$\Leftrightarrow$	$\theta = \tan^{-1} y$	where	$y \in \mathbb{R}$	and	$\dfrac{-\pi}{2} < \theta < \dfrac{\pi}{2}$
$y = \cot\theta$	$\Leftrightarrow$	$\theta = \cot^{-1} y$	where	$y \in \mathbb{R}$	and	$0 < \theta < \pi$
$y = \sec\theta$	$\Leftrightarrow$	$\theta = \sec^{-1} y$	where	$\lvert y\rvert \geq 1$	and	$0 \leq \theta \leq \pi$, $\theta \neq \dfrac{\pi}{2}$
$y = \csc\theta$	$\Leftrightarrow$	$\theta = \csc^{-1} y$	where	$\lvert y\rvert \geq 1$	and	$\dfrac{-\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$, $\theta \neq 0$

Our Math 142 Course Homepage (CH) is http://people.math.sc.edu/girardi/w142.html

Integration by Parts

$$\int u\,dv \;=\; uv - \int v\,du$$

Key Ideas in Integration by Parts.

- For $\int x^n f(x)\,dx$ where $\int f(x)\,dx$ is easy, try $u = x^n$ and $dv = f(x)\,dx$. (Note that then $v = \int dv = \int f(x)\,dx$.) This often reduces x^n to x^{n-1}.
- For $\int f(x)\,dx$: if the integrand $f(x)$ is easy to differentiate but hard to integrate, then try letting $u = f(x)$ and so $dv = dx$.
- *Bring to the other side* (i.e., *loops*) method.
- Creatively look for a dv that is easy to integrate (since $v = \int dv$).
- None of the others. So what did you learn from this type of problem?

Trig Identities useful for Integration

Half-Angle Formulas:

$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad\qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

Double-Angle Formulas:

$$\cos(2x) = \cos^2 x - \sin^2 x \qquad\qquad \sin(2x) = 2\sin x\cos x$$

Addition/Subtraction Formulas:

$$\cos(s+t) \overset{(*)}{=} \cos s\cos t - \sin s\sin t \qquad\qquad \sin(s+t) \overset{(*)}{=} \sin s\cos t + \cos s\sin t$$

$$\cos(s-t) \overset{(*)}{=} \cos s\cos t + \sin s\sin t \qquad\qquad \sin(s-t) \overset{(*)}{=} \sin s\cos t - \cos s\sin t$$

Trig Substitution

IF INTEGRAND INVOLVES	THEN MAKE THE SUBSTITUTION	RESTRICTION ON θ
$a^2 - u^2$	$u \;=\; a\sin\theta \;\rightsquigarrow\; \theta \;=\; \sin^{-1}\frac{u}{a}$	$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$
$a^2 + u^2$	$u \;=\; a\tan\theta \;\rightsquigarrow\; \theta \;=\; \tan^{-1}\frac{u}{a}$	$\frac{-\pi}{2} < \theta < \frac{\pi}{2}$
$u^2 - a^2$	$u \;=\; a\sec\theta \;\rightsquigarrow\; \theta \;=\; \sec^{-1}\frac{u}{a}$	$0 \le \theta \le \pi\ ,\ \theta \ne \frac{\pi}{2}$

Space for your personal notes.