Math 141: Section 2.3 The Precise Definition of a Limit - Notes

Example 1 We need to replace the vague phrases such as "gets arbitrarily close to" with precise conditions that can be applied to any particular example. To show that the limit of $f(x)$ as $x \to c$ equals the number L, we need to show that the gap between $f(x)$ and L can be made "as small as we choose" if x is kept "close enough" to c.

Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it appears that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x - 1) = 7$. However, how close to $x = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Definition Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of f(x) as x approaches c is the number L**, and write

$$\lim_{x\to c} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

How to Find Algebraically a δ for a Given f, L, c, and $\epsilon > 0$ The process of finding a $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

can be accomplished in two steps:
1. Solve the inequality $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing c on which the inequality holds for all $x \neq c$.
2. Find a value of $\delta > 0$ that places the open interval $(c-\delta, c+\delta)$ centered at c inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq c$ in the δ-interval.

Example 2 Prove that $\lim_{x\to 2} f(x) = 4$ if $f(x) = x^2$.