Math 141: Section 5.4 The Fundamental Theorem of Calculus - Notes

The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \underbrace{\frac{1}{b-a}\int_a^b f(x)dx}_{av(f)}.$$

Proof :

(Uses Max-Min Inequality from last section)

The Fundamental Theorem of Calculus, Part I If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^{x} \underline{f(t)}dt = f(x).$$

Example 1 Use the Fundamental Theorem, Part I to find dy/dx if

$$y = \int_a^x \underbrace{(t^2+1)}_{f(t)}dt$$

$$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^2+1)dt = x^2+1$$

Ex: $y = \int_1^{x^2} \cos t \, dt$, $f(t) = \cos t$

$$\frac{dy}{dx} = \frac{d}{dx}\int_1^{x^2} \cos(t)dt = \cos(x^2)(x^2)' = 2x\cos(x^2)$$

The Fundamental Theorem of Calculus, Part II If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example 2 Calculate the following definite integrals using the Fundamental Theorem instead of taking limits of Riemann Sums:

(a) $\int_0^\pi \cos x dx$

$$= \sin x \cancel{+C} \Big|_0^\pi = (\sin(\pi) \cancel{+C}) - (\sin(0) \cancel{+C})$$

$$= (0 \cancel{+C}) - (0 \cancel{+C}) = 0$$

* **(b)** $\int_0^1 (1 - x^2)dx$

$$= \left[x - \frac{x^3}{3}\right]_0^1 = \left(1 - \frac{1^3}{3}\right) - \left(0 - \frac{0^3}{3}\right)$$

$$= 1 - 1/3 = \boxed{2/3}$$ see 5.2

(c) $\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$

$$= \int_1^4 \left(\frac{3}{2}x^{1/2} - 4x^{-2}\right)dx = \left[x^{3/2} + 4x^{-1}\right]_1^4 = \left(4^{3/2} + 4(4)^{-1}\right) - \left(1^{3/2} + 4(1)^{-1}\right)$$

$$= (8+1) - (1+4)$$

$$= \boxed{4}$$

(d) $\int_0^1 \frac{dx}{x+1}$

$$= \int_0^1 \frac{1}{x+1} dx$$

$$= \ln|x+1| \Big|_0^1 = \ln(1+1) - \ln(0+1)$$

$$= \boxed{\ln(2)}$$

The Net Change Theorem The net change in a differentiable function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)dx.$$

Interpretations :

(a) If $c(x)$ is the cost of producing x units of a certain commodity, then $c'(x)$ is the marginal cost. Then,

$$\int_{x_1}^{x_2} c'xdx = c(x_2) - c(x_1),$$

which is the cost of increasing production from x_1 units to x_2 units.

(b) Displacement vs Total Distance Traveled

If an object with position function $s(t)$ moves along a coordinate line, its velocity is $v(t) = s'(t)$. The Net Change Theorem says

$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1),$$

so the integral of velocity is the displacement over the time interval $[t_1, t_2]$.

On the other hand, the integral of the speed, $|v(t)|$, is the total distance traveled over the time interval.

Total Area Area is always a nonnegative quantity. When working with Riemann sums, we were adding terms of the form $f(c_k)\Delta x_k$ that represented the area of a rectangle. When $f(c_k)$ is positive, the product is positive. What if $f(c_k)$ is negative? Then the product, $f(c_k)\Delta x_k$ is also negative and represents the negative of the rectangle's area. By taking the absolute value, we obtain the correct positive area.

Example 3 For each of the following functions, find the definite integral over the interval $[-2, 2]$ and the area between the graph and the x-axis over $[-2, 2]$.

$$y = 4 - x^2 \qquad\qquad y = x^2 - 4$$

(a) $\int_{-2}^{2}(4-x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$

$= \left(4(2) - 2^3/3\right) - \left(4(-2) - \frac{(-2)^3}{3}\right)$

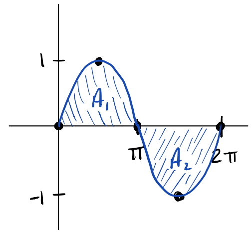
$= 32/3$

$\int_{-2}^{2}(x^2-4)\,dx = \left[x^3/3 - 4x\right]_{-2}^{2} = \left(\frac{2^3}{3} - 4(2)\right) - \left(\frac{(-2)^3}{3} - 4(-2)\right)$

$= -32/3$

(b) Total Area in both cases is $32/3$

Example 4 Find the total area between the graph of $y = \sin x$ and the x-axis over $[0, 2\pi]$.



$\int_{0}^{2\pi}\sin x\,dx = -\cos x\Big|_{0}^{2\pi} = -1+1 = 0$?

$\left|\int_{0}^{\pi}\sin x\,dx\right| + \left|\int_{\pi}^{2\pi}\sin x\,dx\right|$

$= \left|\left[-\cos x\right]_{0}^{\pi}\right| + \left|\left[-\cos x\right]_{\pi}^{2\pi}\right|$

$= |(1+1)| + |(-1-1)| = |2| + |-2| = 4$