Math 141: Section 5.3 The Definite Integral - Notes

Definition: Let $f(x)$ be a function defined on a closed interval $[a.b]$. We say that a number J is the **definite integral of f over** $[a,b]$ and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ if the following condition is satisfied:
Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

"the norm of P"
Just means the width of the rectangles, Δx, is getting smaller as the number of rectangles, n, increases

$$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \epsilon.$$

Notation :

Upper limit of integration
Integral sign
Integrand
Lower limit of integration
Variable of integration
Riemann Sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} f(c_k)\Delta x_k}_{\text{Riemann Sum}}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(a+k\Delta x)\Delta x, \quad \Delta x = \frac{b-a}{n}$$

Theorem 1 If a function f is continuous over the interval $[a,b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\displaystyle\int_a^b f(x)dx$ exists and f is integrable over $[a,b]$.

Example 1 The function

$$f(x) = \begin{cases} 1, & \text{if x is rational} \\ 0, & \text{if x is irrational} \end{cases}$$

has no Riemann integral over $[0,1]$.

f has infinitely many jump discontinuities
(Completeness Axiom)

Properties of Definite Integrals :

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A definition when $f(a)$ exists
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b - a) \leq \int_a^b f(x)\,dx \leq \max f \cdot (b - a).$$

7. *Domination:* $f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

   $f(x) \geq 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq 0$ (Special case)

Example 2 Suppose that

$$\int_{-1}^{1} f(x)dx = 5, \quad \int_{1}^{4} f(x)dx = -2, \text{ and } \int_{-1}^{1} h(x)dx = 7.$$

Find:

(a) $\int_1^1 f(x)dx = 0$

(c) $\int_4^1 f(x)dx = -\int_1^4 f(x)dx$
$= -(-2)$
$= 2$

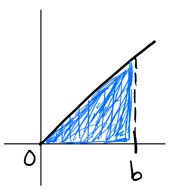
(b) $\int_{-1}^4 f(x)dx = \int_{-1}^1 f(x)dx + \int_1^4 f(x)dx$
$= 5 + (-2)$
$= 3$

(d) $\int_{-1}^1 (f(x) + h(x))dx$
$= \int_{-1}^1 f(x)dx + \int_{-1}^1 h(x)dx$
$= 5 + 7 = 12$

Definition: If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve** $y = f(x)$ **over [a,b]** is the integral of f from a to b,

$$A = \int_a^b f(x)dx.$$

Example 3 Compute $\int_0^b x dx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.



Sol 1 $\lim_{\substack{\|P\| \to 0 \\ (n \to \infty)}} \sum_{k=1}^{n} f(c_k)\Delta x$, $c_k = a + k\Delta x$

Partition the interval $[0, b]$ into n subintervals of equal width:

$$\Delta x = \frac{b-a}{n} = \frac{b-0}{n} = \frac{b}{n}$$

$$P = \left\{0, \frac{b}{n}, \frac{2b}{n}, \ldots, \frac{kb}{n}, \ldots, b\right\}$$

$$c_k = \frac{kb}{n}$$

$$\sum_{k=1}^{n} f(c_k)\Delta x$$

$$= \sum_{k=1}^{n} \frac{kb}{n} \cdot \frac{b}{n} = \sum_{k=1}^{n} \frac{b^2 k}{n^2} = \frac{b^2}{n^2} \sum_{k=1}^{n} k = \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2}$$

So, $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x = \lim_{n \to \infty} \frac{b^2 n(n+1)}{2n^2} = b^2/2 = A$

Sol 2: $A = \frac{1}{2}(\text{base})(\text{height})$ $\quad A = \frac{1}{2} \cdot b \cdot b = b^2/2$ ✓

Definition: If f is integrable on $[a, b]$, then its **average value on [a,b]** is

$$av(f) = \frac{1}{b-a} \int_a^b f(x)dx.$$