Math 141: Section 3.2 The Derivative as a Function - Notes

Definition The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. So we can consider the derivative as a *function*

derived from f by considering the limit at each point x in the domain of f.
The domain of f' is the set of points in the domain of f for which the limit exists, which means that the domain may be **the same or smaller than** the domain of f. If f' exists at a particular x, we say that f is **differentiable (has a derivative) at x**. If f' exists at every point in the domain of f, we call f **differentiable**.

Alternative Definition

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

Calculating Derivatives from the Definition The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

as another way to denote the derivative $f'(x)$.

In Example 1 of section 3.1, we saw that for x representing any point in the domain of $f(x) = 1/x$, we get

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{x^2}.$$

Example 1 Differentiate
$$f(x) = \frac{x}{x-1}.$$

Example 2 Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$. Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Notations There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y.

One-Sided Derivatives A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

and

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

exist at the endpoints.

Example 3 Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Example 3, cont. Right-hand and Left-hand limits do not agree at the origin:

When Does a Function NOT Have a Derivative at a Point? A function *has* a derivative at a point x_0 if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a finite limit as Q approaches P.

Differentiable Functions Are Continuous A function is continuous at every point where it has a derivative.

Theorem If f has a derivative at $x = c$, then f is continuous at $x = c$.

Careful! The converse of this theorem is *not* true. A function need not have a derivative at a point where it is continuous, as we saw with the absolute value function in the previous example.