Math 142: Section 11.2 - Notes

1 Calculus with Parametric Equations

Tangents and Areas A parametrized curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if f and g are differentiable at t.

Parametric Formula for dy/dx If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

If parametric equations define y as a twice-differentiable function of x, we can apply the above formula to the function $dy/dx = y'$ to calculate d^2y/dx^2 as a function of t:

Parametric Formula for d^2y/dx^2 If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}.$$

Example 1 Find the tangent to the curve

$$x = \sec t, \qquad y = \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$, where $t = \pi/4$.

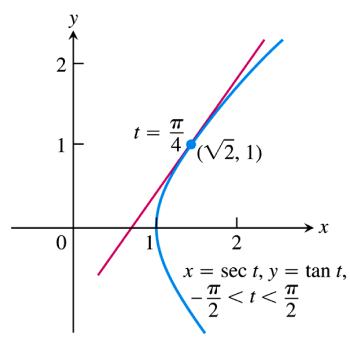

FIGURE 11.12 The curve in Example 1 is the right-hand branch of the hyperbola $x^2 - y^2 = 1$.

Example 1 (cont.)

Example 2 Find d^2y/dx^2 as a function of t if $x = t - t^2$ and $y = t - t^3$.

Example 3 Find the area enclosed by the astroid

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

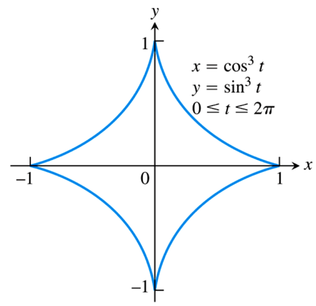

Figure 11.13 The astroid in Example 3.

Length of a Parametrically Defined Curve Let C be a curve given parametrically by the equations

$$x = f(t), \qquad y = g(t), \qquad a \le t \le b.$$

We assume the functions f and g are __________________________ (meaning they have continuous first derivatives) on the interval $[a, b]$. We also assume that the derivatives $f'(t)$ and $g'(t)$ are not simultaneously zero, which prevents the curve C from having any corners or cusps.

Such a curve is called a __________________________.

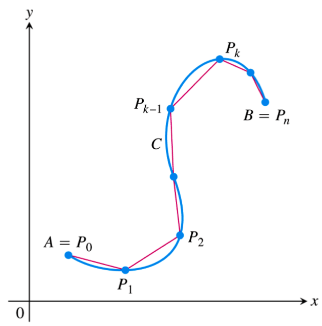

FIGURE 11.14 The smooth curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1, and so on, ending at $B = P_n$.

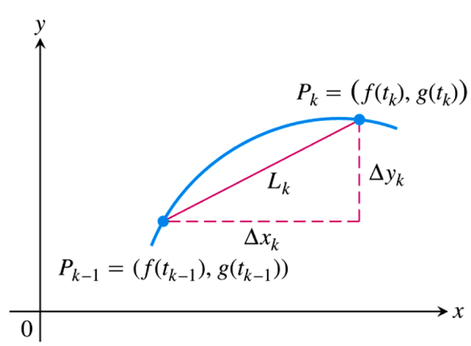

FIGURE 11.15 The arc $P_{k-1}P_k$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

Definition: If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$,

then the ______________________________ is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt.$$

Example 4 Using the definition, fund the length of the circle of radius r defined parametrically by

$$x = r\cos t, \qquad y = r\sin t, \qquad 0 \leq t \leq 2\pi.$$

Example 5 Find the length of the astroid

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \leq t \leq 2\pi.$$