Solutions

2.1: Instantaneous Rate of Change

Definition: The **instantaneous rate of change** of f at a, is defined to be the limit of the average rates of change of f over shorter and shorter time intervals around a. We can write this mathematically as

$$\lim_{\substack{b \to a \\ \text{"}b\text{ approaches }a\text{"}}} \frac{f(b) - f(a)}{b - a}.$$

Definition: The **derivative** of f at a, written $f'(a)$, is defined to be the instantaneous rate of change of f at the point a. It is common to write the derivative mathematically as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Question: Explain in your own words how the two limits above actually represent the same thing, i.e., explain how they are equal.

As $b \to a$, $f(b) \to f(a)$, $(b-a) \to 0$ and as $h \to 0$, $f(a+h) \to f(a)$, $h \to 0$

Remark: It is common that we will see

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

but this should not confuse you at this point since the variable x (and h) above is (are) just "dummy variables."

Exercise 1: In a time of t seconds, a particle moves a distance of s meters from its starting point, where $s = 4t^2 + 3$.

(a) Find the average velocity between $t = 1$ and $t = 1 + h$ if:

(i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$.

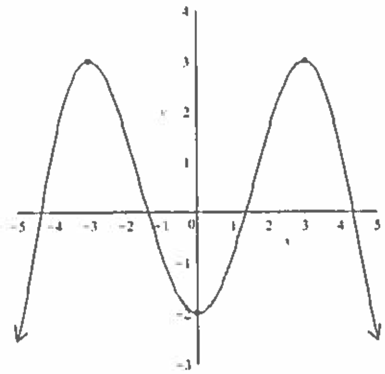
(b) Use your answers to part (a) to estimate the instantaneous velocity of the particle at time $t = 1$.

(a)

h	$\frac{s(1+h) - s(1)}{h}$
0.1	$\frac{4(1.1)^2 + 3 - (4(1)^2 + 3)}{0.1} = 8.4$
0.01	8.04
0.001	8.004

(b) It seems that $\lim_{h \to 0} \frac{s(1+h) - s(1)}{h} = 8.$

Exercise 2: Consider the function f given by the graph below. For each pair of numbers, determine which is larger.



(a) $f(3)$ or $f(4)$

(b) $f(4) - f(3)$ or $f(4) - f(2)$

(c) $\frac{f(4)-f(3)}{4-3}$ or $\frac{f(4)-f(2)}{4-2}$

(d) $f'(3)$ or $f'(4)$

Exercise 3: The graph of a function $y = f(x)$ is shown below. Indicate whether each of the following quantities is positive or negative and illustrate your answers graphically.

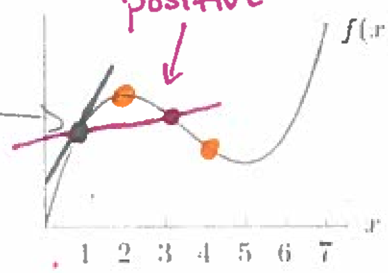
(a) $f'(1)$ (b) $\dfrac{f(3) - f(1)}{3 - 1}$ slope of secant line (c) $f(4) - f(2)$

positive

negative

$f'(1) > 0$

Slope of tangent line



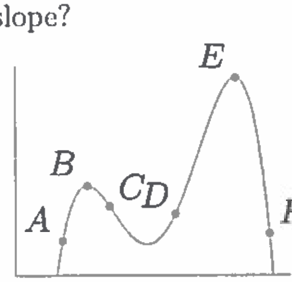
Exercise 4: For the function below, at what labeled points is the slope of the graph positive? Negative? At which labeled point does the graph have the greatest slope? The least slope?

Positive

A, D

Negative

C, F



Greatest (steepest, positive)

A

Least (steepest, negative)

F

Zero

B, E

Exercise 2

(a) $f(3) \approx 3$ $f(3)$ is larger

$f(4) \approx 2$

(b) $f(4) - f(3) \approx -1$ $f(4) - f(2)$ is larger

$f(4) - f(2) \approx 0$

(c) $\dfrac{f(4)-f(3)}{4-3} \approx -1$

$\dfrac{f(4)-f(2)}{4-2} \approx 0$ larger

(d) $f'(3) = 0$ $f'(3)$ is larger

$f'(4) < 0$

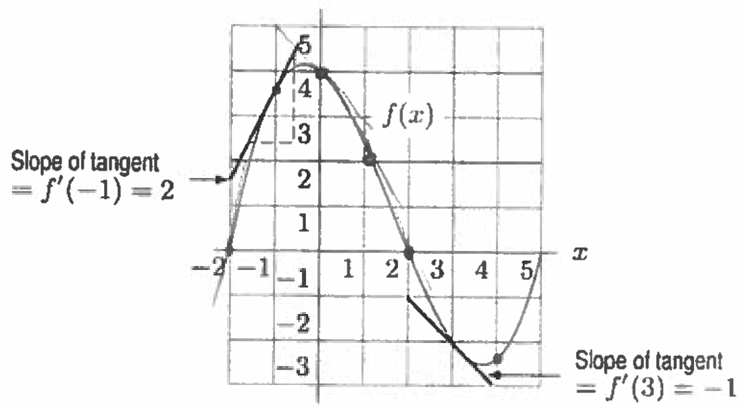
↳ Since f is decreasing at $x=4$, the derivative is negative

2.2: The Derivative Function

Definition: For a function f, we define the **derivative function**, f', by

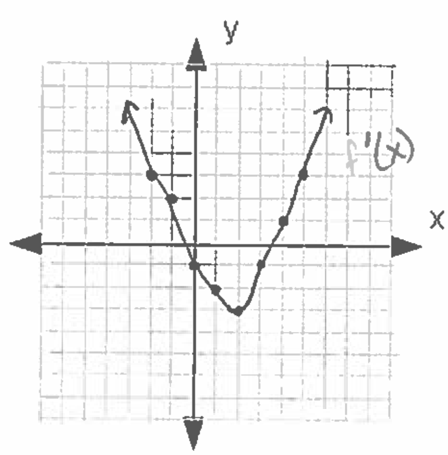
$$f'(x) = \text{Instantaneous rate of change of } f \text{ at } x = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Example 1: Estimate the derivative of the function $f(x)$ below at $x = -2, -1, 0, 1, 2, 3, 4, 5$.



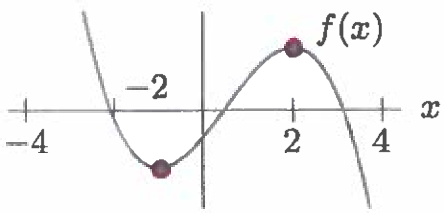
x	-2	-1	0	1	2	3	4	5
Derivative at x	3	2	-1	-2	-3	-1	1	3

Now we can draw the derivative of f.



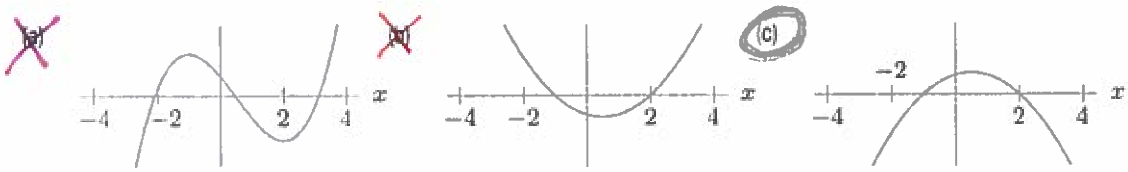
* **Note:** We will see that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$. So taking the derivative decreases the degree of a power function by 1. The graph of f above is the graph of a cubic so we should expect the graph of f' to be quadratic (a parabola)

Example 2: Consider the graph of f below. Which of the graphs (a)-(c) is a graph of the derivative, f'?



1) $f'(x) = 0$ at $x = -1$ and $x = 2$

2) f is decreasing from $(-\infty, -1)$ and $(2, \infty)$ so f' should be negative on those intervals



The derivative of a graph, f', can tell us a few things about the graph of f itself:

If $f' > 0$ on an interval, then f is *increasing* on that interval.
If $f' < 0$ on an interval, then f is *decreasing* on that interval.
If $f' = 0$ on an interval, then f is *constant* on that interval.

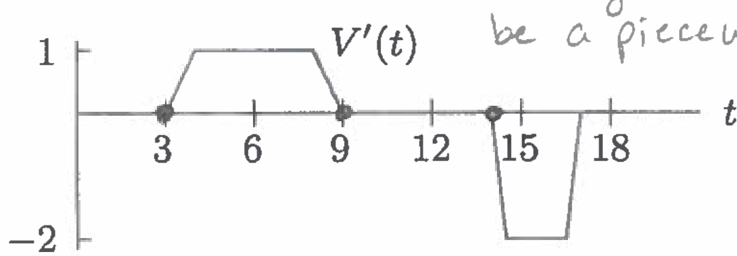
Example 3: A child inflates a balloon, admires it for a while and then lets the air out at a constant rate. If $V(t)$ gives the volume of the balloon at time t, then below is the graph of $V'(t)$ as a function of t. At what time does the child:

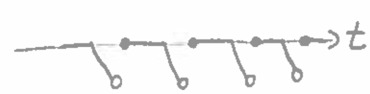
(a) Begin to inflate the balloon? $t = 3$

(b) Finish inflating the balloon? $t = 9$

(c) Begin to let the air out? $t = 14$

(d) What would the graph of $V'(t)$ look like if the child had alternated between pinching and releasing the open end of the balloon, instead of letting the air out at a constant rate? Starting at $t = 14$, the graph would be a piecewise function





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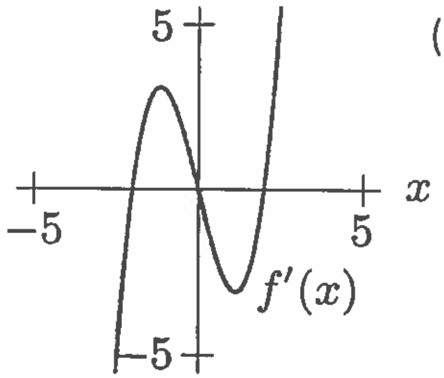
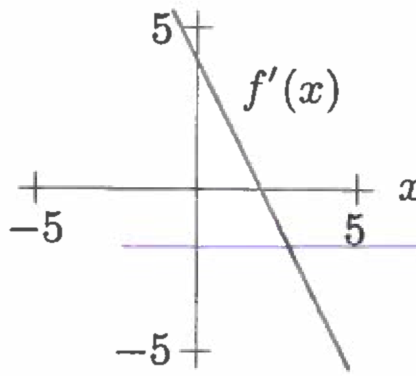
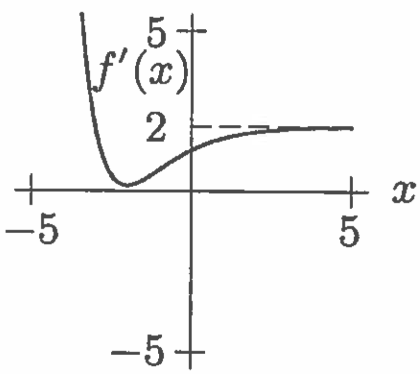
(I)



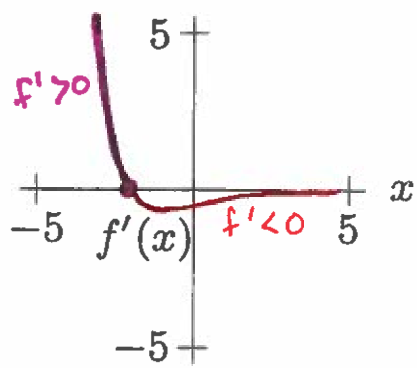
(II)



(V)



(VI)

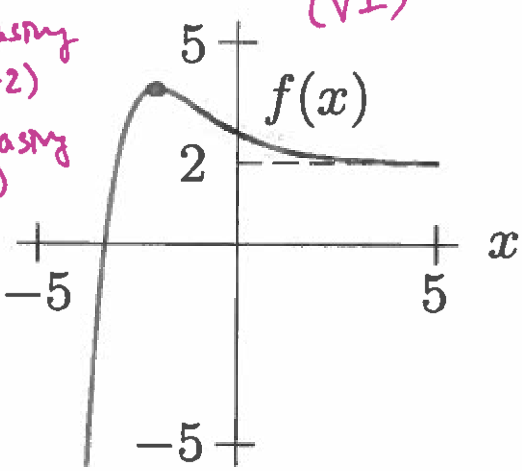


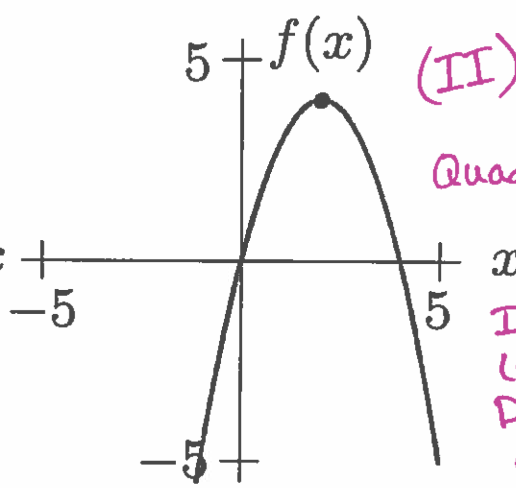
Exercise 1: Match each of the two graphs below to one of the derivative graphs above.

Increasing $(-\infty, -2)$

Decreasing $(-2, \infty)$

(VI)





(II)

Quadratic so f' should be linear

Increasing $(-\infty, 2)$

Decreasing $(2, \infty)$

Exercise 2: Draw a possible graph of $y = f(x)$ given the following information about its derivative.

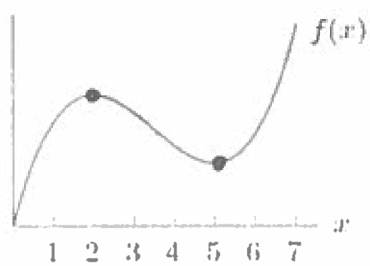
1. $f'(x) > 0$ for $x < -1$
2. $f'(x) < 0$ for $x > -1$
3. $f'(x) = 0$ for $x = -1$

Exercise 3: A vehicle moving along a straight road has distance $f(t)$ from its starting point at time t. Which of the graphs below could be $f'(t)$ for the following scenarios?

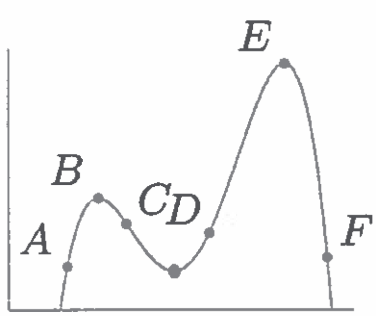
f'(t) represents velocity

1. A bus on a popular route, with no traffic (II)
2. A car with no traffic and all green lights (I)
3. A car in heavy traffic conditions (III)

Exercise 4: Sketch the derivative of the function $f(x)$ given below.

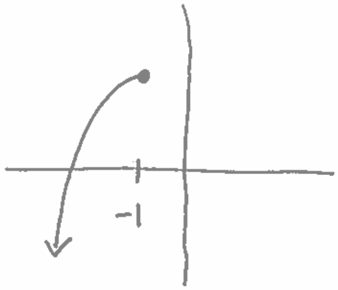


Exercise 5: Sketch the derivative of the function $f(x)$ given below.

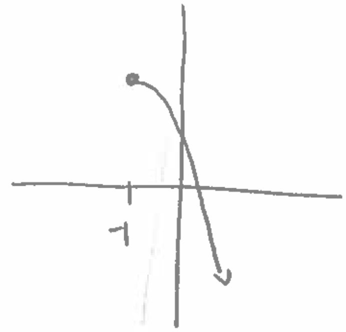


Exercise 2

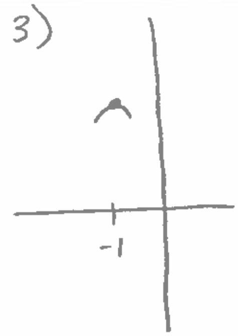
1)



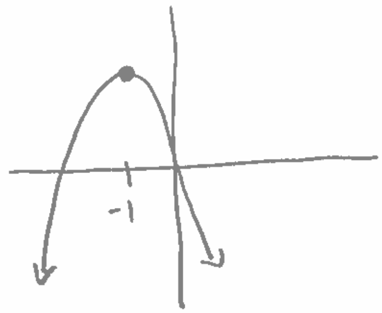
2)



3)



Possible graph (combine 1, 2, 3)



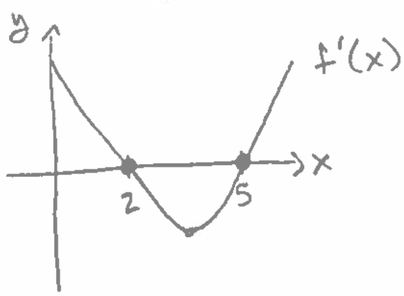
Exercise 4

$f'(x) = 0$ at $x = 2$ and $x = 5$

$f'(x) > 0$ on $(0, 2)$ and $(5, 7)$

$f'(x) < 0$ on $(2, 5)$

$f(x)$ is cubic so $f'(x)$ should be quadratic



Exercise 5

$f'(x) = 0$ at B and E and also between C and D