Math 141: Section 3.1 Tangents and the Derivative at a Point - Notes

Finding a Tangent to the Graph of a Function To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the ideas introduced in Section 2.1:

Definition: The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \text{ (provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Example 1 a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

$f(x) = 1/x$, The slope at $(a, 1/a)$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \to 0} \frac{1}{h} \cdot \frac{a - (a+h)}{a(a+h)}$$

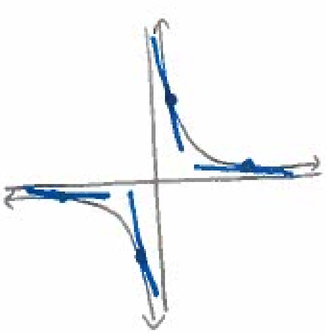
$$= \lim_{h \to 0} \frac{-h}{h \cdot a(a+h)} = \lim_{h \to 0} \frac{-1}{a(a+h)} = -1/a^2$$

when $a = -1$, the slope is $-1/(-1)^2 = -1$.

b) Where does the slope equal $-1/4$?

$$\frac{-1}{a^2} = -\frac{1}{4} \quad \text{or} \quad a^2 = 4$$

So $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$.



c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$. As $a \to 0^-$, again the slope approaches $-\infty$. As a moves away from the origin, the slope gets closer and closer to zero.

Definition The expression

$$\frac{f(x_0+h)-f(x)}{h)},\ h \neq 0,$$

is called the **difference quotient of f at x_0 with increment h.**

The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$\boxed{f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}}$$

provided this limit exists.

Example 2 We previously looked at the speed of a freely falling rock near the surface of the earth. We knew that the rock fell $y = 16t^2$ feet during the first t sec, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $x = 1$. What was the rock's *exact* speed at this time?

Let $f(t) = 16t^2$. The average speed of the rock over the interval between $t=1$ and $t=1+h$ seconds, for $h>0$ was

$$\frac{f(1+h)-f(h)}{h} = \frac{16(1+h)^2-16h^2}{h} = 16(h+2).$$

The rock's speed at the instant $t=1$ is then

$$f'(1) = \lim_{h \to 0} 16(h+2) = 16(0+2) = 32 \text{ ft/sec.}$$

Our estimate was correct.

Summary The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1) The slope of the graph of $y = f(x)$ at $x = x_0$.
2) The slope of the tangent to the curve $y = f(x)$ at $x = x_0$.
3) The rate of change of $f(x)$ with respect to x at $x = x_0$.
4) The derivative $f'(x_0)$ at a point.