Math 142: Section 10.1 (cont.) - Notes

1 Sequences

The Continuous Function Theorem for Sequences Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Example 1

$$\lim_{n\to\infty} \sin(\pi/n)$$

Example 2

$$a_n = \frac{n!}{n^n}$$

Example 3 For what values of r is the sequence $\{r^n\}_{n=1}^{\infty}$ convergent?

Definitions Two concepts that play a key role in determining the convergence of a sequence are those of a *bounded* sequence and a *monotonic* sequence.

a) A sequence $\{a_n\}$ is ______________________________ if there exists a number M such that $a_n \leq M$ for all n.
The number M is an ______________________________ for $\{a_n\}$.
If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the ______________________________.

b) A sequence $\{a_n\}$ is ______________________________ if there exists a number m such that $a_n \geq m$ for all n.
The number m is a ______________________________ for $\{a_n\}$.
If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the ______________________________.

c) If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is ______________________________.
If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an ______________________________ sequence.

Example 4 Consider the following sequences:

i)

$$1, 2, 3, \ldots, n, \ldots$$

ii)

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

Completeness Axiom If S is any non-empty set of real numbers that has an upper bound M, then S has a least upper bound, b. That is, b is an upper bound and if M is any other upper bound, then $b \leq M$.

Definitions Although it is true that every convergent sequence is bounded, there are bounded sequences that fail to converge. We saw the example $\{(-1)^n\}$. The problem is that some bounded sequences bounce around in the band determined by any lower bound m and any upper bound M. We have terms to describe sequences that do not behave in this way:

A sequence $\{a_n\}$ is ______________________________ if $a_n \leq a_{n+1}$ for all n. That is, $a_1 \leq a_2 \leq a_3 \leq \ldots$

The sequence is ______________________________ if $a_n \geq a_{n+1}$ for all n.

The sequence $\{a_n\}$ is ______________________________ if it is either nondecreasing or nonincreasing.

Example 5 $\left\{\frac{3}{n+5}\right\}$ is decreasing.

Example 6 $a_n = \frac{n}{n^2+1}$ is decreasing.

The Monotone Convergence Theorem Every bounded, monotonic sequence converges.

Proof: Suppose $\{a_n\}_{n=1}^{\infty}$ is bounded and increasing. Let L be the least upper bound of a_n (which exists by the Completeness Axiom). Let $\epsilon > 0$ be given. Observe that $L - \epsilon < L$ implies that $L - \epsilon$ is not an upper bound for $\{a_n\}_{n=1}^{\infty}$ and thus there exists some $N \in \mathbb{N}$ such that

$$L - \epsilon < a_N < L.$$

Moreover, since $\{a_n\}_{n=1}^{\infty}$ is increasing, it follows that whenever $n \geq N$, $L - \epsilon < a_n < L$ must hold. Hence, $L - \epsilon < a_n$ implies $-a_n < \epsilon - L$, from which it follows that $L - a_n < \epsilon$. But also, $a_n < L$ implies

$$-\epsilon < 0 \leq L - a_n < \epsilon.$$

Therefore, $|L - a_n| < \epsilon$, as desired.
If $\{a_n\}_{n=1}^{\infty}$ is decreasing, then $\{-a_n\}_{n=1}^{\infty}$ is increasing. Then L is the least upper bound of $\{-a_n\}$ and is the limit of the sequence $\{-a_n\}$, from which it follows that $\lim_{n\to\infty} a_n = -L$.

Note! The Monotone Convergence Theorem ONLY tells us that the limit exists, NOT the value of the limit. It also tells us that a nondecreasing sequence converges when it is bounded from above, but diverges to infinity otherwise.

Example Consider the sequence defined recursively by

$$a_1 = 2,\ a_{n+1} = \frac{1}{2}\left(a_n + 6\right).$$