Math 141: Section 3.10 Related Rates - Notes

Related Rates Equations Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3}\pi r^3.$$

Using the chain rule, we can differentiate both sides with respect to t to find an equation relating the rates of change of V and r,

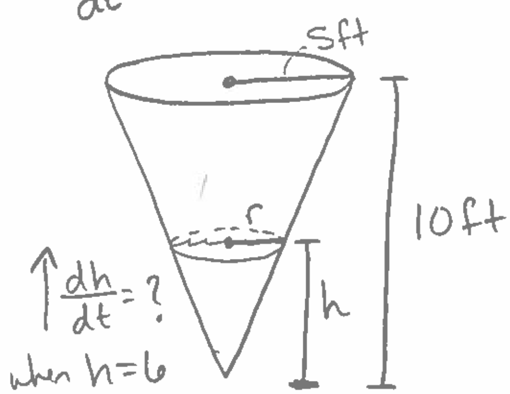
$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \cdot \frac{dr}{dt}$$

$$dV/dt = 4\pi r^2 \, dr/dt$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant.

Example 1 Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

$\frac{dV}{dt} = 9 ft^3/m$



V = volume at time t

r = radius of the surface of the water

h = depth of the water

$$V = \frac{1}{3}\pi r^2 h \qquad \frac{dV}{dt} = 9\ ft^3/m, \quad \frac{dh}{dt} = ? \text{ when } h = 6$$

Don't have enough information about r so we need to eliminate it.

$$\frac{r}{h} = \frac{5}{10} \quad \text{or} \quad r = h/2$$

$$V = \frac{1}{3}\pi (h/2)^2 h$$

$$V = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \cdot \frac{dh}{dt}$$

Example 1 (cont.)

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

Want dh/dt when $h = 6$ given $dV/dt = 9$ ft³/min

$$9 = \frac{1}{4}\pi (6)^2 \frac{dh}{dt}$$

$$9 = \frac{36\pi}{4} \frac{dh}{dt} \rightarrow 36 = 36\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi} \text{ ft/min}$$

Related Rates Problem Strategy:

1) Draw a picture and name the variables and constants. Use t for time and assume all variables are differentiable functions of time.
2) Write down the numerical information.
3) Write down what you are asked to find.
4) Write an equation that relates the variables. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rate you know.
5) Differentiate with respect to t.
6) Evaluate. Use known values to find the unknown rate.

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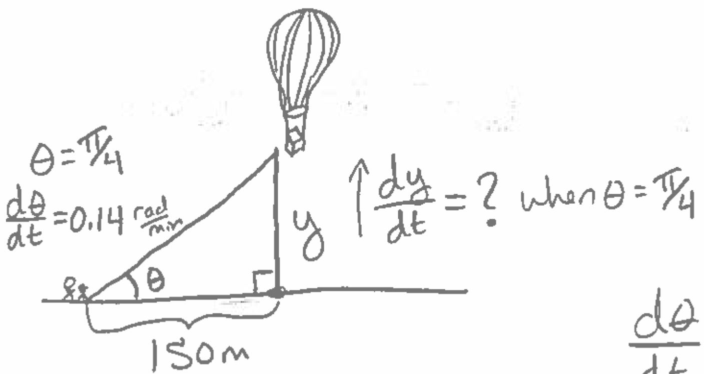
Example 2 A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?



θ = angle in radians

y = height of balloon

t = time in minutes

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min when } \theta = \pi/4, \quad \frac{dy}{dt} = ?$$

$$\tan\theta = \frac{y}{150} \quad \text{or} \quad y = 150\tan\theta$$

$$\frac{dy}{dt} = 150\sec^2\theta \cdot \frac{d\theta}{dt}$$

$$\frac{dy}{dt} = 150\sec^2(\pi/4) \cdot 0.14$$

$$= 150(2)(0.14)$$

$$= 42 \text{ m/min}$$