Solutions

Math 141: Practice Problems for Exam 2

Sections 3.3, 3.5, 3.6, and 3.9 Differentiation Rules/Formulas

See the 48 Derivatives Worksheet on the Course Webpage!

Find the derivatives of the following functions:

1) $y = 2\sqrt{x}\sin(\sqrt{x})$

2) $r = \left(\dfrac{1+\sin\theta}{1-\cos\theta}\right)^2$

3) $y = x^2 e^{-2/x}$

1) $y' = (2\sqrt{x})'\sin(\sqrt{x}) + 2\sqrt{x}(\sin(\sqrt{x}))'$

$= x^{-1/2}\sin(\sqrt{x}) + x^{1/2}x^{-1/2}\cos(\sqrt{x})$

$= \boxed{x^{-1/2}\sin(\sqrt{x}) + \cos(\sqrt{x})}$

2) $r' = 2\left(\dfrac{1+\sin\theta}{1-\cos\theta}\right)\left(\dfrac{1+\sin\theta}{1-\cos\theta}\right)'$

$= 2\left(\dfrac{1+\sin\theta}{1-\cos\theta}\right)\left(\dfrac{\cos\theta(1-\cos\theta)-(1+\sin\theta)\sin\theta}{(1-\cos\theta)^2}\right)$

$= 2\left(\dfrac{1+\sin\theta}{1-\cos\theta}\right)\left(\dfrac{\cos\theta-\cos^2\theta-\sin\theta-\sin^2\theta}{(1-\cos\theta)^2}\right)$

$= \boxed{2\left(\dfrac{(1+\sin\theta)(\cos\theta-\sin\theta-1)}{(1-\cos\theta)^3}\right)}$

3) $y' = 2xe^{-2/x} + x^2e^{-2/x}(2x^{-2})$

$= 2xe^{-2/x} + 2e^{-2/x} = \boxed{2e^{-2/x}(x+1)}$

Section 3.7 Implicit Differentiation

See the 48 Derivatives Worksheet and Implicit Differentiation Practice on the Course Webpage!

Find dy/dx by implicit differentiation:

4) $\ln(x/y) = 1$

5) $ye^{\tan^{-1} x} = 2$

4) $\frac{1}{\frac{x}{y}}\left(\frac{x}{y}\right)' = 0$

$\frac{\cancel{y}}{x}\left(\frac{y - x\frac{dy}{dx}}{y^{\cancel{2}}}\right) = 0$

$\frac{y - x\frac{dy}{dx}}{xy} = 0$

$\frac{1}{x} - \frac{1}{y}\frac{dy}{dx} = 0$

$-\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x}$

$\boxed{\frac{dy}{dx} = \frac{y}{x}}$

5) $\frac{dy}{dx}e^{\tan^{-1}x} + ye^{\tan^{-1}x}\cdot\frac{1}{1+x^2} = 0$

$\frac{dy}{dx}e^{\tan^{-1}x} = -\frac{ye^{\tan^{-1}x}}{1+x^2}$

$\boxed{\frac{dy}{dx} = -\frac{y}{1+x^2}}$

Section 3.8 Derivatives of Inverse Functions and Logarithms

Use logarithmic differentiation to find the derivative of y with respect to x.

6) $y = \dfrac{2(x^2+1)}{\sqrt{\cos 2x}}$

7) $y = \left(\dfrac{(x+1)(x-1)}{(x-2)(x+3)}\right)^5, \ x > 2$

6) $\ln y = \ln(2) + \ln(x^2+1) - \frac{1}{2}\ln(\cos 2x)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{\sin(2x)}{\cos(2x)}$$

$$\boxed{\frac{dy}{dx} = \frac{2(x^2+1)}{\sqrt{\cos 2x}}\left(\frac{2x}{x^2+1} + \tan(2x)\right)}$$

7) $\ln y = 5\left(\ln(x+1) + \ln(x-1) - \ln(x-2) - \ln(x+3)\right)$

$$\frac{1}{y}\frac{dy}{dx} = 5\left(\frac{1}{x+1} + \frac{1}{x-1} - \frac{1}{x-2} - \frac{1}{x+3}\right)$$

$$\boxed{\frac{dy}{dx} = 5\left(\frac{(x+1)(x-1)}{(x-2)(x+3)}\right)^5\left(\frac{1}{x+1} + \frac{1}{x-1} - \frac{1}{x-2} - \frac{1}{x+3}\right)}$$

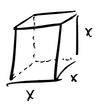
Section 3.10 Related Rates

See Related Rates Practice on the Course Webpage!

8) The volume of a cube is increasing at the rate of 1200 cm^3/min at the instant its edges are 20 cm long. At what rate are the lengths of the edges changing at that instant?

9) The top of a ladder slides down a vertical wall at a rate of 0.15m/s. At the moment when the bottom of the ladder is 3m from the wall, it slides away from the wall at a rate of 0.2m/s. How long is the ladder?

8)



$V = lwh$

$V = x^3$

$\frac{dV}{dt} = 1200 \text{ cm}^3/\text{min}$ when $x = 20\text{cm}$

$\frac{dx}{dt} = ?$

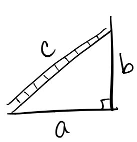
$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$1200 = 3(20)^2 \frac{dx}{dt}$$

$$1200 = 3(400) \frac{dx}{dt}$$

$$\boxed{\frac{dx}{dt} = 1 \text{ cm/min}}$$

9)



$\downarrow \frac{db}{dt} = -0.15 \text{ m/s}$

$c = ?$

$\leftarrow \frac{da}{dt} = 0.2 \text{ m/s}$ when $a = 3\text{m}$

$$a^2 + b^2 = c^2$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$\frac{dc}{dt} = 0$ since the ladder's length is constant

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$$

$$2(3)(0.2) + 2b(-0.15) = 0$$

$$1.2 - 0.3b = 0$$

$$b = 4\text{m}$$

$$3^2 + 4^2 = c^2 \rightarrow \boxed{c = 5\text{m}}$$

Section 4.1 Extreme Values of Functions

For problems 10-11 find the absolute maximum and minimum values of each function on the given interval.

10) $f(x) = -3x^{2/3}, \quad -1 \le x \le 1$

11) $f(x) = \csc x, \quad -\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$

Find all the critical points for the following functions

12) $f(x) = x^3(x-5)^2$

13) $g(x) = (x-1)^2(x-3)^2$

14) $h(x) = e^x - e^{-x}$

10) $f'(x) = -2x^{-1/3}$

$= -\frac{2}{x^{1/3}}$

$f(-1) = -3$

$f(0) = 0$

$f(1) = -3$

CP $x=0$

Abs max value is 0 at $x=0$

Abs min value is -3 at $x=-1$ and $x=1$

11) $f'(\theta) = -\csc\theta\cot\theta$

$= -\frac{\cos\theta}{\sin^2\theta}$

CP $\theta = \pi/2, 0$

$f(-\pi/3) = \csc(-\pi/3)$

$= -\frac{2}{\sqrt{3}}$

$f(0)$ undefined

$f(\pi/2) = 1$

$f(2\pi/3) = \frac{2}{\sqrt{3}}$

Abs max value of $2/\sqrt{3}$ at $\theta = 2\pi/3$

Abs min value of $-2/\sqrt{3}$ at $\theta = -\pi/3$

12) $f'(x) = 3x^2(x-5)^2 + 2x^3(x-5)$

$= x^2(x-5)(3(x-5)+2x)$

$= x^2(x-5)(5x-15)$

CPs $x=0$, $x=5$, $x=3$

13) $g'(x) = 2(x-1)(x-3)^2 + 2(x-1)^2(x-3)$

$= 2(x-1)(x-3)(x-3+x-1)$

$= 2(x-1)(x-3)(2x-4)$

CPs $x=1$, $x=3$, $x=2$

14) $h'(x) = e^x + e^{-x}$

$= e^x + \frac{1}{e^x}$

$= \frac{e^{2x}+1}{e^x}$

$e^{2x}+1 = 0$

$e^{2x} = -1$

No sol

$e^x = 0$

No sol

No CPs

Section 4.2 The Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in problems 35-36.

15) $f(x) = x^{2/3}, \quad [0,1]$

16) $f(x) = \ln(x-1), \quad [2,4]$

15) $\frac{f(1)-f(0)}{1-0} = 1 \qquad f'(c) = \frac{2}{3}c^{-1/3}$

$$\frac{2}{3}c^{-1/3} = 1$$

$$c^{-1/3} = \frac{3}{2}$$

$$\boxed{c = \sqrt[3]{2/3}}$$

16) $\frac{f(4)-f(2)}{4-2} = \frac{\ln(3)-\ln(1)}{2} = \frac{\ln(3)}{2}$

$f'(c) = \frac{1}{c-1}, \quad \frac{1}{c-1} = \frac{\ln(3)}{2}$

$$2 = c\ln(3) - \ln(3)$$

$$\boxed{c = \frac{2+\ln(3)}{\ln(3)}}$$

Section 4.3 Monotonic Functions and the First Derivative Test

Answer the following questions about functions whose derivatives are given in problems 17-18:

a) What are the critical points of f?

b) On what open intervals is f increasing or decreasing?

c) At what points, if any, does f assume local maximum and minimum values?

17) $f'(x) = \frac{x^2+2x-8}{x^2-2x-3}$

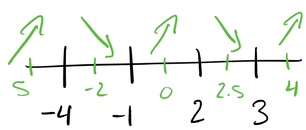
18) $f'(x) = (\sin x + \cos x)(\sin x - \cos x), \quad 0 \le x \le 2\pi$

17) $f'(x) = \dfrac{(x+4)(x-2)}{(x-3)(x+1)}$

$\sin^2 x - \sin x\cos x + \sin x \cos x - \cos^2 x = \sin^2 x - \cos^2 x < 0$ when $\sin^2 x < \cos^2 x$

a) CPs $x = -4, 2, 3, -1$

b)



Increasing $(-\infty,-4)\cup(-1,2)\cup(3,\infty)$

Decreasing $(-4,-1)\cup(2,3)$

c) Local max at $x=-4$ and $x=2$

Local min at $x=-1$ and $x=3$

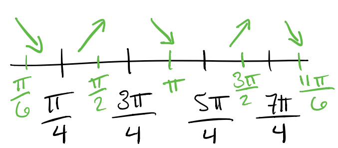
18) a) $f'(x)=0$ when

$\sin x + \cos x = 0$ or $\sin x - \cos x = 0$

$\sin x = -\cos x \qquad \sin x = \cos x$

CPs $x = \frac{3\pi}{4}, \frac{7\pi}{4} \qquad x = \frac{\pi}{4}, \frac{5\pi}{4}$

b)



Increasing $(\frac{\pi}{4},\frac{3\pi}{4})\cup(\frac{5\pi}{4},\frac{7\pi}{4})$

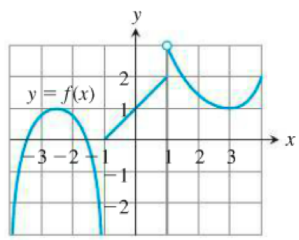
Decreasing $(0,\frac{\pi}{4})\cup(\frac{3\pi}{4},\frac{5\pi}{4})\cup(\frac{7\pi}{4},2\pi)$

c) Local max at $x=\frac{3\pi}{4},\frac{7\pi}{4}$ Local min at $x=\frac{\pi}{4},\frac{5\pi}{4}$

Section 4.3 Monotonic Functions and the First Derivative Test

Given the graph of the function $f(x)$, find the open intervals on which the function is increasing and decreasing and identify the function's local maximum and minimum values, saying where they occur. Also state where the derivative is zero or undefined.

19)



Increasing $(-\infty,-2.5)\cup(-1,1)\cup(3,\infty)$

Decreasing $(-2.5,-1)\cup(1,3)$

Local max at $x=-2.5$

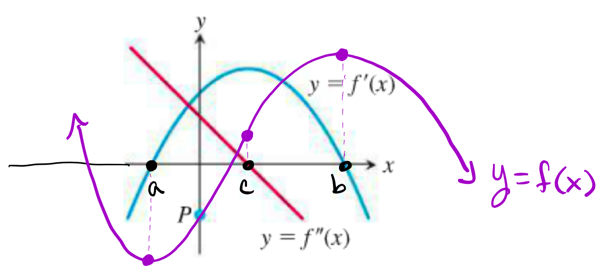
Local min at $x=3$

$f'(x)$ is undefined at $x=-1$ and $x=1$

Section 4.4 Concavity and Curve Sketching

Given the graphs of the first and second derivative of $f(x)$, sketch the approximate graph of f given that $f(x)$ passes through the point P.

20)



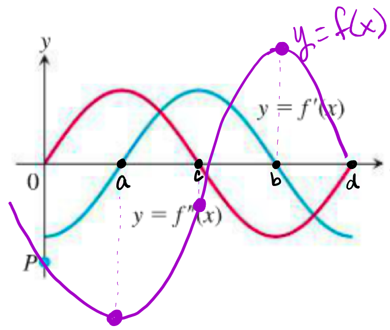
Decreasing $(-\infty,a)\cup(b,\infty)$ Increasing (a,b)

Concave up $(-\infty,c)$ Concave down (c,∞)

Section 4.4 Concavity and Curve Sketching

Given the graphs of the first and second derivative of $f(x)$, sketch the approximate graph of f given that $f(x)$ passes through the point P.

21)



Decreasing $(0,a) \cup (b,d)$

Increasing (a,b)

Concave up $(0,c)$

Concave down (c,d)

Graph the rational functions using all the steps in the graphing procedure from Section 4.4.

22) $y = \frac{x^2-4}{2x}$

23) $y = \frac{x^2}{x^2-1}$

22) Domain $(-\infty,0)\cup(0,\infty)$ Symmetry: $f(-x)=-f(x)$ Odd, symmetry about the origin

X-intercepts $x=2,\ x=-2$

y-intercept None

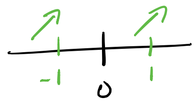
Asymptotes: Oblique $2x\,\overline{)\,x^2-4}$ with quotient $\frac{1}{2}x$, x^2, -4 $\quad y=\frac{1}{2}x$

Vertical $x=0$ $\quad \lim_{x\to0^-}\frac{x^2-4}{2x}=\infty \quad \lim_{x\to0^+}\frac{x^2-4}{2x}=-\infty$

Intervals of Increase/Decrease:

$$y' = \frac{2x(2x)-2(x^2-4)}{(2x)^2} = \frac{2x^2+8}{4x^2} = \frac{x^2+4}{2x^2}$$

CPs $y'=0$ No sol $\quad y'$ undefined when $x=0$



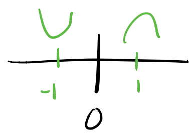
y is Increasing $(-\infty,0)\cup(0,\infty)$

Decreasing None

No local max or min

Concavity: $y'' = \dfrac{2x(2x^2)-4x(x^2+4)}{(2x^2)^2} = \dfrac{-16x}{4x^4} = \dfrac{-4}{x^3}$

y'' undefined when $x=0$

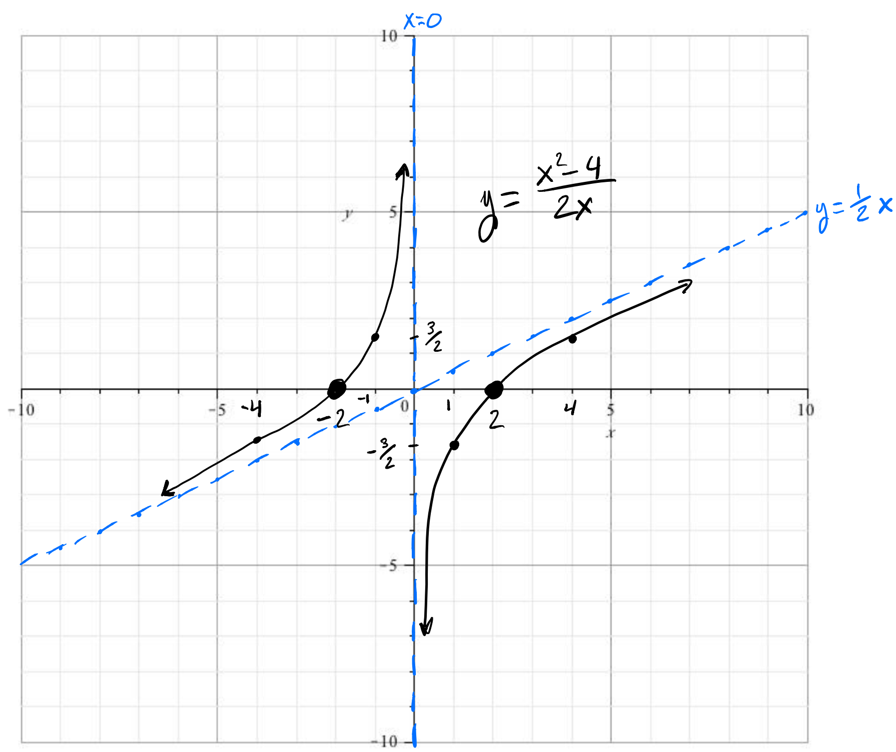


y is Concave up $(-\infty,0)$

Concave down $(0,\infty)$

$x=0$ would correspond to an inflection point if it were in the domain

Sketch:



23) $y = \dfrac{x^2}{x^2-1} = \dfrac{x^2}{(x+1)(x-1)}$

Domain: $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

Symmetry: $f(-x) = \dfrac{(-x)^2}{(-x)^2-1} = \dfrac{x^2}{x^2-1} = f(x)$ Even, y-axis symmetry

Intercepts: x-int: $x=0$ $(0,0)$

y-int: $y=0$

Asymptotes: Horizontal: $\lim\limits_{x\to\pm\infty} \dfrac{x^2}{x^2-1} = 1$, $y=1$

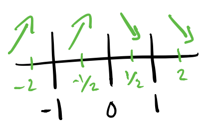
Vertical:

$x=-1$ $\lim\limits_{x\to-1^-} \dfrac{x^2}{x^2-1} = \infty$ $\lim\limits_{x\to-1^+} \dfrac{x^2}{x^2-1} = -\infty$

$x=1$ $\lim\limits_{x\to1^-} \dfrac{x^2}{x^2-1} = -\infty$ $\lim\limits_{x\to1^+} \dfrac{x^2}{x^2-1} = \infty$

Intervals of Increase / Decrease:

$$y' = \frac{2x(x^2-1)-2x(x^2)}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$ CPs $x=0,-1,1$



Increasing on $(-\infty,-1)\cup(-1,0)$

Decreasing on $(0,1)\cup(1,\infty)$

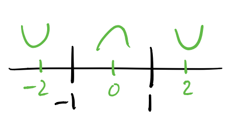
Local max at $x=0$

Concavity:

$$y'' = \frac{-2(x^2-1)^2+2x\cdot2(x^2-1)2x}{((x^2-1)^2)^2} = \frac{-2(x^2-1)^2+8x^2(x^2-1)}{(x^2-1)^4}$$

$$= \frac{-2\cancel{(x^2-1)}(x^2-1-4x^2)}{(x^2-1)^{\cancel{4}\,3}} = \frac{-2(-3x^2-1)}{(x^2-1)^3} = \frac{6x^2+2}{(x^2-1)^3}$$

y'' undefined at $x=-1,1$



y is Concave up $(-\infty,-1)\cup(1,\infty)$

Concave down $(-1,1)$

No inflection points

Sketch: