Math 141: Section 2.2 Limit of a Function and Limit Laws - Notes

Limits of Function Values Often when studying a function $y = f(x)$, we are interested in the function's behavior near a particular point c, but not precisely at c. For instance, if c is an irrational number, like π or $\sqrt{2}$, whose values can only be approximated by "close" rational numbers. Another instance would be when trying to evaluate a function at c leads to division by zero.

Example 1 How does the function

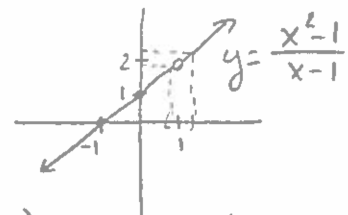
$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

$f(1)$ is undefined - can't divide by 0!

But for $x \neq 1$, we can simplify:

$$f(x) = \frac{(x+1)(x-1)}{x-1} = x+1, \ x \neq 1$$



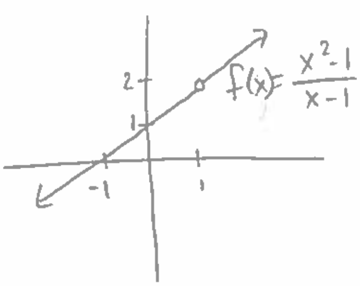
The graph is the line $y = x+1$ with the point $(1,2)$ removed.
We can make the value of $f(x)$ as close to 2 as we want by choosing x close to 1.

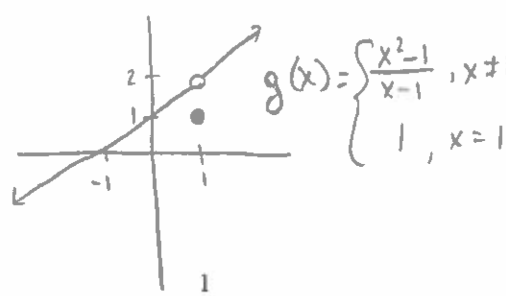
Generalizing, suppose $f(x)$ is defined on an open interval about c, except possibly at c itself. If $f(x)$ is arbitrarily close to the number L for all x sufficiently close to c, we say that f approaches the **limit** L as x approaches c, and write

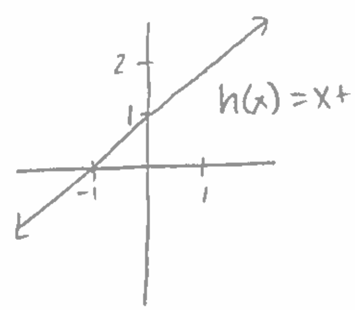
$$\lim_{x \to c} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches c is L."

Example 2 The limit value of a function does not depend on how the function is defined at the point being approached.







$$\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = 2$$

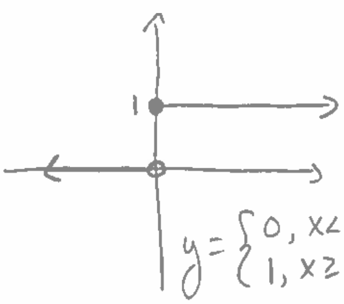
Example 3 If f is the **identity function** $f(x) = x$, then for any value of c,

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c.$$

If f is the **constant function** $f(x) = k$, then for any value of c,

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k.$$

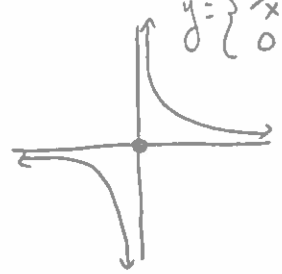
A function may not have a limit at a particular point:



$y = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

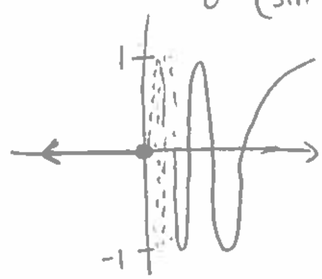
"Jump"

$y = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$



Grows too "large"
Not bounded

$y = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$



Oscillates too much:
The function's values oscillate between +1 and -1 in every open interval containing 0

Theorem 1: Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \text{ and } \lim_{x \to c} g(x) = M, \text{ then}$$

1) Sum Rule:

$$\lim_{x \to c} (f(x) + g(x)) = L + M$$

2) Difference Rule:

$$\lim_{x \to c} (f(x) - g(x)) = L - M$$

3) Constant Multiple Rule:

$$\lim_{x \to c} (k \cdot f(x)) = k \cdot L$$

4) Product Rule:

$$\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$$

5) Quotient Rule:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

6) Power Rule:

$$\lim_{x \to c} [f(x)]^n = L^n, \quad n \text{ a positive integer}$$

7) Root Rule:

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n \text{ a positive integer}$$

$$\left(\text{If } n \text{ is even, assume } \lim_{x \to c} f(x) = L > 0\right)$$

Example 4 Use the observations from Example 3 and Limit Laws to find the following limits:

(a)

$$\lim_{x \to c} (x^3 + 4x^2 - 3)$$

$$= \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 = c^3 + 4c^2 - 3$$

(Sum/Diff., Power/Mult. Rules)

(b)

$$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$$

$$= \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)} = \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5} = \frac{c^4 + c^2 - 1}{c^2 + 5}$$

(c)

$$\lim_{x \to c} \sqrt{4x^2 - 3}$$

$$= \sqrt{\lim_{x \to c} (4x^2 - 3)} = \sqrt{\lim_{x \to c} 4x^2 - \lim_{x \to c} 3} = \sqrt{4c^2 - 3}$$

Theorem 2: Limits of Polynomials If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then
$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

Theorem 3: Limits of Rational Function If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then
$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 5 Evaluate
$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}.$$

Limit Laws!

$$\frac{\lim_{x\to 0}(\sqrt{x^2+100}-10)}{\lim_{x\to 0} x^2} \leftarrow$$ Denominator is approaching 0! Ah!

Rationalize the numerator:
Multiply top and bottom by the conjugate
$$\sqrt{x^2+100}+10$$

$$\frac{(\sqrt{x^2+100}-10)}{(x^2)} \cdot \frac{(\sqrt{x^2+100}+10)}{(\sqrt{x^2+100}+10)} \overset{\text{FOIL the top}}{=} \frac{x^2+\cancel{100}+\cancel{10\sqrt{x^2+100}}-\cancel{10\sqrt{x^2+100}}-\cancel{100}}{x^2(\sqrt{x^2+100}+10)}$$

$$= \frac{\cancel{x^2}}{\cancel{x^2}(\sqrt{x^2+100}+10)} = \frac{1}{\sqrt{x^2+100}+10}$$

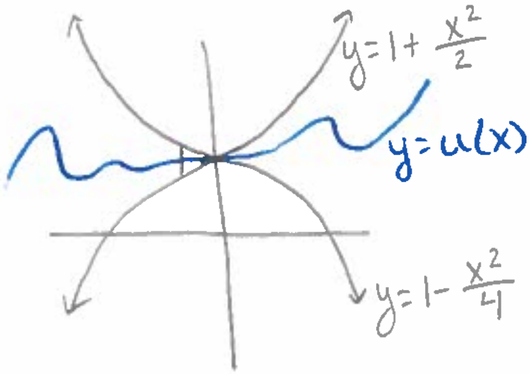
$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} = \frac{1}{\sqrt{0^2+100}+10} = \frac{1}{20}.$$

* This method may not always work. Sometimes we need geometry or methods of calculus

Theorem 4: The Sandwich (Squeeze) Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.



Given that

$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$ for all $x \neq 0$,

the limit as $x \to 0$ of $u(x)$ is 1 no matter how complicated u is

$$1 = \lim_{x \to 0}\left(1 - \frac{x^2}{4}\right) = \lim_{x \to 0}\left(1 + \frac{x^2}{2}\right)$$

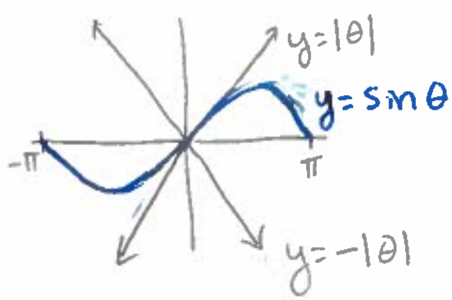
so $\lim_{x \to 0} u(x) = 1$.

Theorem 5 If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

Note the less than <u>or equal to</u> inequality

We can not replace this by a strict inequality ($<$). Example:



For $\theta \neq 0$, $-|\theta| < \sin\theta < |\theta|$

so $\lim_{\theta \to 0} \sin\theta = 0 = \lim_{\theta \to 0} |\theta|$

NOT $\lim_{\theta \to 0} \sin\theta < \lim_{\theta \to 0} |\theta|$