Math 142: Section 10.3 - Notes

1 The Integral Test

Tests for Convergence The most basic question we can ask about a series is whether or not it converges. In the next few sections, we will build the tools necessary to answer that question. If we establish that a series does converge, we generally do not have a formula for its sum (unlike the case for Geometric Series). So, for a convergent series, we need to investigate the error involved when using a partial sum to approximate its total sum.

Nondecreasing Partial Sums Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor since $s_{n+1} = s_n + a_n$, so

Since the partial sums form a nondecreasing sequence, the Monotone Convergence Theorem gives us the following result:

Corollary of the MCT A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

Example 1 Consider the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

The Integral Test We now introduce the Integral Test with a series that is related to the harmonic series, but whose nth term is $1/n^2$ instead of $1/n$.

Example 2 Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Theorem: The Integral Test Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.

Example 3 Show that the **p-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \ldots$$

(where p is a real constant) converges if $p > 1$ and diverges if $p \leq 1$.

Example 4 Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

Error Estimation For some convergent series, such as a geometric series or the telescoping series, we can actually find the total sum of the series. For most convergent series, however, we cannot easily find the total sum. Nevertheless, we can *estimate* the sum by adding the first n terms to get s_n, but we need to know how far off s_n is from the total sum S.

Bound for the Remainder in the Integral Test Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$, and that $\sum a_k$ converges to S. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx.$$

Example 5 Estimate the sum, S, of the series $\sum(1/n^2)$ with $n = 10$.