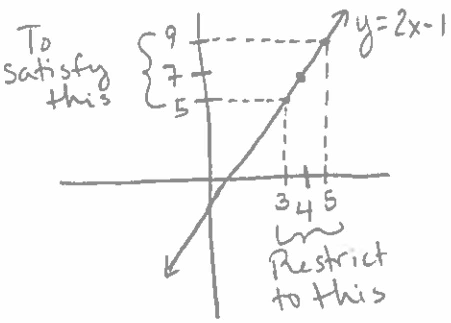
Math 141: Section 2.3 The Precise Definition of a Limit - Notes

Example 1 We need to replace the vague phrases such as "gets arbitrarily close to" with precise conditions that can be applied to any particular example. To show that the limit of $f(x)$ as $x \to c$ equals the number L, we need to show that the gap between $f(x)$ and L can be made "as small as we choose" if x is kept "close enough" to c.

Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it appears that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x - 1) = 7$. However, how close to $x = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?



Question: For what values of x is $|y-7|<2$?

Express $|y-7|$ in terms of x:

$|y-7| = |(2x-1)-7| = |2x-8|$.

Now, what values of x satisfy $|2x-8|<2$?

$$|2x-8|<2$$
$$-2<2x-8<2$$
$$6<2x<10$$
$$3<x<5$$
or $-1<x-4<1$

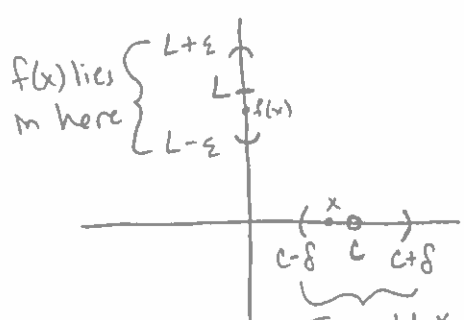
So, keeping x within 1 unit of $x=4$ will keep y within 2 units of 7.

Definition Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of f(x) as x approaches c is the number L**, and write

$$\lim_{x\to c} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$



How to Find Algebraically a δ for a Given f, L, c, and $\epsilon > 0$ The process of finding a $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

can be accomplished in two steps:

1. Solve the inequality $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing c on which the inequality holds for all $x \neq c$.

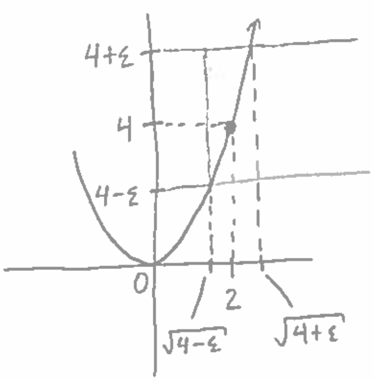
2. Find a value of $\delta > 0$ that places the open interval $(c-\delta, c+\delta)$ centered at c inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq c$ in the δ-interval.

Example 2 Prove that $\lim_{x\to 2} f(x) = 4$ if $f(x) = x^2$. $c=2, L=4$

We need to show that given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x

$0 < |x-2| < \delta$ implies $|f(x)-4| < \varepsilon$

1) Solve $|f(x)-4| < \varepsilon$ to find an open interval containing $x=2$ on which the inequality holds for all x (not necessarily $x=2$). $f(x)=x^2$ so,



$$|x^2-4| < \varepsilon$$
$$-\varepsilon < x^2-4 < \varepsilon$$
$$4-\varepsilon < x^2 < 4+\varepsilon$$
$$\sqrt{4-\varepsilon} < |x| < \sqrt{4+\varepsilon} \quad \text{(Assumes } \varepsilon < 4\text{)}$$
$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$

So, the inequality $|f(x)-4| < \varepsilon$ holds for all x (not necessarily $x=2$) in the open interval $(\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon})$

2) Find a value $\delta > 0$ that places the centered interval $(2-\delta, 2+\delta)$ inside $(\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon})$. We just take δ to be the distance from $x=2$ to the nearer endpoint of $(\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon})$. $\delta = \min\{2-\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}-2\}$

Then, for $\varepsilon < 4$, for all x $0 < |x-2| < \delta \Rightarrow |f(x)-4| < \varepsilon$.

* If $\varepsilon \geq 4$. take $\delta = \min\{2, \sqrt{4+\varepsilon}-2\}$. (Distance from $x=2$ to nearer endpoint