# Math 141: Section 3.2 The Derivative as a Function - Notes

**Definition** The **derivative** of the function $f(x)$ with respect to the variable $x$ is the function $f'$ whose value at $x$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. So we can consider the derivative as a *function*

derived from $f$ by considering the limit at each point $x$ in the domain of $f$.

The domain of $f'$ is the set of points in the domain of $f$ for which the limit exists, which means that the domain may be **the same or smaller than** the domain of $f$. If $f'$ exists at a particular $x$, we say that $f$ is **differentiable (has a derivative) at x**. If $f'$ exists at every point in the domain of $f$, we call $f$ **differentiable**.

**Alternative Definition**

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

If $h = z - x$ then $h \to 0$ if and only if $z \to x$

**Calculating Derivatives from the Definition** The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

as another way to denote the derivative $f'(x)$.

In Example 1 of section 3.1, we saw that for $x$ representing any point in the domain of $f(x) = 1/x$, we get

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{x^2}.$$

**Example 1** Differentiate

$$f(x) = \frac{x}{x-1}.$$

$$f(x+h) = \frac{x+h}{(x+h)-1} \qquad \text{So,}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{x+h}{(x+h)-1} - \frac{x}{x-1}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x((x+h)-1)}{((x+h)-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{\cancel{x^2} - \cancel{x} + \cancel{xh} - h - \cancel{x^2} - \cancel{xh} + \cancel{x}}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} = \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \boxed{\frac{-1}{(x-1)^2}}$$

**Example 2** Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$. Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

$f(x) = \sqrt{x}$ Use Alternative Definition (just for practice)

$$f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x} = \lim_{z\to x} \frac{\sqrt{z} - \sqrt{x}}{z - x}$$

$$= \lim_{z\to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \cdot \frac{\sqrt{z} + \sqrt{x}}{\sqrt{z} + \sqrt{x}} = \lim_{z\to x} \frac{\cancel{z-x}}{(\cancel{z-x})(\sqrt{z}+\sqrt{x})}$$

$$= \lim_{z\to x} \frac{1}{\sqrt{z}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

when $x=4$, slope $f'(4) = \frac{1}{2\sqrt{4}} = 1/4$.
The tangent is the line through $(4,2)$ with slope $1/4$,

$$\boxed{y - 2 = 1/4(x-4)}$$

**Notations** There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is $x$ and the dependent variable is $y$.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x)$$

At a point $x=a$,

$$f'(a) = \frac{dy}{dx}\Big|_{x=a} = \frac{df}{dx}\Big|_{x=a} = \frac{d}{dx} f(x)\Big|_{x=a}$$

**One-Sided Derivatives** A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior $(a, b)$ and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

and

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

exist at the endpoints.

**Example 3** Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Recall $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

The derivative of the line $y = mx + b$ is just the slope $m$. So, to the right of the origin

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \quad (0, \infty)$$

To the left of the origin

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1. \quad (-\infty, 0)$$

**Example 3, cont.** Right-hand and Left-hand limits do not agree at the origin:

Right-hand: $\lim\limits_{h\to 0^+} \dfrac{|0+h|-|0|}{h} = \lim\limits_{h\to 0^+} \dfrac{|h|}{h} = \lim\limits_{h\to 0^+} \dfrac{h}{h} = 1$

Left-hand: $\lim\limits_{h\to 0^-} \dfrac{|0+h|-|0|}{h} = \lim\limits_{h\to 0^-} \dfrac{|h|}{h} = \lim\limits_{h\to 0^-} \dfrac{-h}{h} = -1$

So there is no derivative at the origin since the one-sided derivatives differ there.

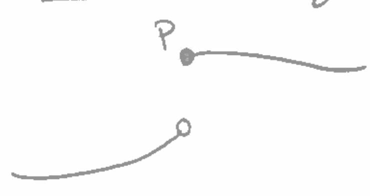
**When Does a Function NOT Have a Derivative at a Point?** A function *has* a derivative at a point $x_0$ if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point $Q$ on the graph approach a finite limit as $Q$ approaches $P$.

A corner


A cusp


Vertical Tangent


Discontinuity

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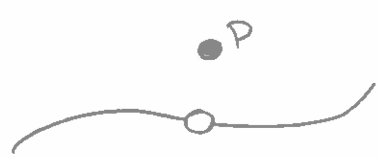


Oscillatory

$\sin(1/x)$, discontinuous at the origin

**Differentiable Functions Are Continuous** A function is continuous at every point where it has a derivative.

**Theorem** If $f$ has a derivative at $x = c$, then $f$ is continuous at $x = c$.

**Careful!** The converse of this theorem is *not* true. A function need not have a derivative at a point where it is continuous, as we saw with the absolute value function in the previous example.