Math 141: Section 5.4 The Fundamental Theorem of Calculus - Notes

The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Proof :

The Fundamental Theorem of Calculus, Part I If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x).$$

Example 1 Use the Fundamental Theorem, Part I to find dy/dx if

$$y = \int_a^x (t^2 + 1)dt$$

The Fundamental Theorem of Calculus, Part II If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example 2 Calculate the following definite integrals using the Fundamental Theorem instead of taking limits of Riemann Sums:

(a) $\displaystyle\int_0^\pi \cos x dx$

(b) $\displaystyle\int_0^1 (1 - x^2)dx$

(c) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$

(d) $\displaystyle\int_0^1 \frac{dx}{x+1}$

The Net Change Theorem The net change in a differentiable function $F(x)$ over an interval $a \leq x \leq b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)dx.$$

Interpretations :

(a) If $c(x)$ is the cost of producing x units of a certain commodity, then $c'(x)$ is the marginal cost. Then,

$$\int_{x_1}^{x_2} c'xdx = c(x_2) - c(x_1),$$

which is the cost of increasing production from x_1 units to x_2 units.

(b) Displacement vs Total Distance Traveled

Total Area Area is always a nonnegative quantity. When working with Riemann sums, we were adding terms of the form $f(c_k)\Delta x_k$ that represented the area of a rectangle. When $f(c_k)$ is positive, the product is positive. What if $f(c_k)$ is negative? Then the product, $f(c_k)\Delta x_k$ is also negative and represents the negative of the rectangle's area. By taking the absolute value, we obtain the correct positive area.

Example 3 For each of the following functions, find the definite integral over the interval $[-2, 2]$ and the area between the graph and the x-axis over $[-2, 2]$.

$$y = 4 - x^2 \qquad\qquad y = x^2 - 4$$

Example 4 Find the total area between the graph of $y = \sin x$ and the x-axis over $[0, 2\pi]$.