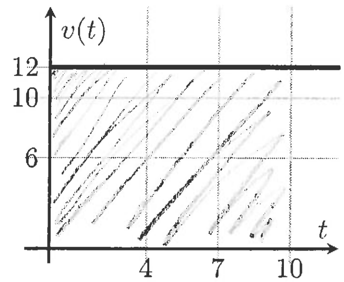
Sols

1. A girl is running at a velocity of 12 feet per second for 10 seconds, as shown in the velocity graph below.



How far does she travel during this time?

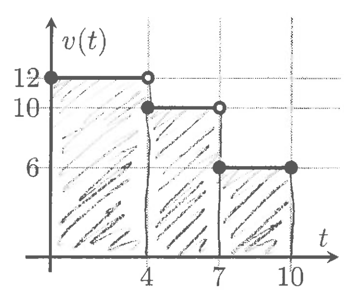
Distance = Rate · Time

= 12 (10) = 120 ft

This distance can be depicted graphically as a rectangle. Shade such a rectangle and explain why it gives the distance.

The area of the rectangle is found by multiplying v(t) by the time, t, which is exactly the distance formula.

2. Now the girl changes her velocity as she runs. Her velocity graph is approximately as shown:

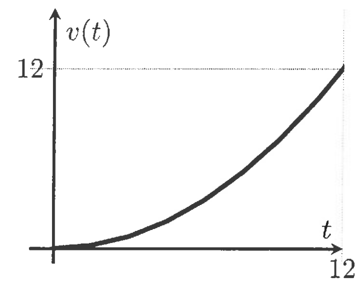


How far does she travel this time?

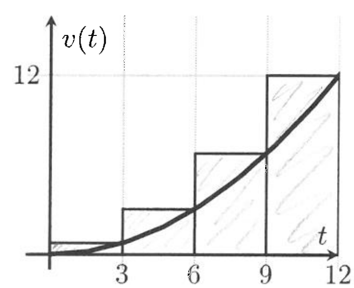
$12(4) + 10(3) + 6(3)$

$= 48 + 30 + 18 = 96$ ft

3. This time she starts off slowly and speeds up.



The velocity is given by $\boxed{v(t) = \frac{t^2}{12}}$ (time in seconds, velocity in ft/sec). We can no longer exactly find the distance travelled using areas of rectangles. But we can estimate it using areas of rectangles.



Find her velocity at time $t = 3, 6, 9, 12$ and use it to estimate her distance travelled in the first 12 seconds.

$$v(3) = \frac{3^2}{12} = \frac{9}{12} = \frac{3}{4}$$

$$v(6) = \frac{6^2}{12} = \frac{36}{12} = 3$$

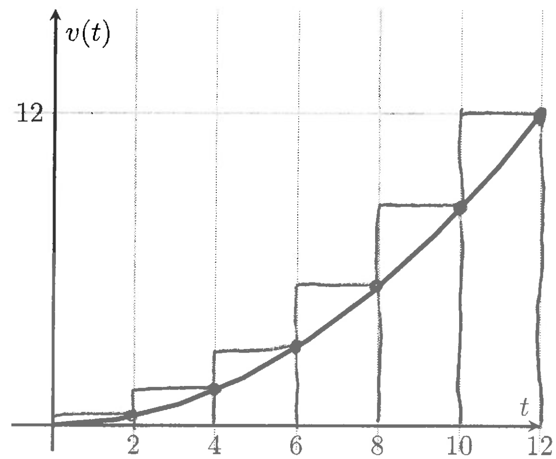
$$v(9) = \frac{9^2}{12} = \frac{27}{4}$$

$$v(12) = \frac{12^2}{12} = 12$$

$$\text{Distance} \approx 3\left(\frac{3}{4} + 3 + \frac{27}{4} + 12\right) = 67.5 \text{ ft}$$

4. Now, for the same velocity function $v(t) = \frac{t^2}{12}$, get a better estimate of how far she travelled using $n = 6$ rectangles. Draw a graph showing the areas, and use their areas to estimate her distance travelled in the first 12 seconds.

$$\Delta t = \frac{b-a}{n} = \frac{12-0}{6} = 2$$



$$\text{Distance} \approx \frac{2^2}{12}(2) + \frac{4^2}{12}(2) + \frac{6^2}{12}(2) + \frac{8^2}{12}(2) + \frac{10^2}{12}(2) + \frac{12^2}{12}(2)$$

$$\approx 60.67 \text{ ft}$$

5. Now we will estimate the area when there are $n = 37$ rectangles.

   (a) Width of each rectangle:

   $$\Delta t = \frac{12-0}{37} = 12/37$$

   (b) List of right-hand endpoint of each rectangle:

   $12/37$, $24/37$, $36/37$, ..., $\frac{432}{37}$, 12

   (c) List of heights of each rectangle:

   $\frac{(12/37)^2}{12}$, $\frac{(24/37)^2}{12}$, $\frac{(36/37)^2}{12}$, ..., $\frac{\left(\frac{432}{37}\right)^2}{12}$, $\frac{12^2}{12} = 12$

   (d) List of areas of rectangles:

   $\frac{(12/37)^2}{12}\left(\frac{12}{37}\right)$, $\frac{(24/37)^2}{12}(12/37)$, $\frac{(36/37)^2}{12}(12/37)$, ..., $\frac{\left(\frac{432}{37}\right)^2}{12}(12/37)$, $12(12/37)$

   (e) Sum of all areas:

   $\frac{12^2}{37^3} + \frac{24^2}{37^3} + \frac{36^2}{37^3} + \ldots + \frac{432^2}{37^3} + \frac{12^2}{37}$

6. Now we will figure out the estimate when there are an arbitrary number of rectangles, or n rectangles.

   (a) Width of each rectangle:

   $$\Delta t = \frac{12-0}{n} = \frac{12}{n}$$

   (b) List of right-hand endpoint of each rectangle:

   $$\frac{12}{n},\ 2\left(\frac{12}{n}\right),\ 3\left(\frac{12}{n}\right), \ldots, (n-1)\frac{12}{n},\ 12$$

   (c) List of heights of each rectangle:

   $$\frac{\left(\frac{12}{n}\right)^2}{12},\ \frac{4\left(\frac{12}{n}\right)^2}{12},\ \frac{9\left(\frac{12}{n}\right)^2}{12}, \ldots, \frac{(n-1)^2\left(\frac{12}{n}\right)^2}{12},\ \frac{12^2}{12}=12$$

   (d) List of areas of rectangles:

   $$\frac{12}{n^2}\left(\frac{12}{n}\right),\ 4\left(\frac{12}{n^2}\right)\left(\frac{12}{n}\right),\ 9\left(\frac{12}{n^2}\right)\left(\frac{12}{n}\right), \ldots, (n-1)^2\left(\frac{12}{n^2}\right)\left(\frac{12}{n}\right),\ 12\left(\frac{12}{n}\right)$$

   (e) Sum of all areas:

   $$\frac{12^2}{n^3} + 4\left(\frac{12^2}{n^3}\right) + 9\left(\frac{12^2}{n^3}\right) + \ldots + (n-1)^2\left(\frac{12^2}{n^3}\right) + \frac{12^2}{n}$$

(c) Simplified: $\frac{12}{n^2},\ 4\left(\frac{12}{n^2}\right),\ 9\left(\frac{12}{n^2}\right), \ldots, (n-1)^2\left(\frac{12}{n^2}\right),\ 12$

7. Manipulate the sum algebraically until it is of the form *stuff* $\cdot$ $(1 + 4 + 9 + ... + n^2)$.

$$\frac{12^2}{n^3}\left(1+4+9+\ldots+(n-1)^2+n^2\right)$$

$$\frac{12^2}{n} = \frac{12^2 n^2}{n^3}$$

8. Simplify further by substituting $1 + 4 + 9 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ into your answer above. Check that it gives the same answer for $n = 6$ that you got in problem 4.

$$\frac{12^2}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{24n(n+1)(2n+1)}{n^3}$$

$$= \frac{24(n+1)(2n+1)}{n^2}$$

9. As n approaches infinity we find her exact distance travelled (the exact area under the curve). Take the **limit** as n goes to infinity for your answer to the previous problem.

$$\lim_{n\to\infty} \frac{24(n+1)(2n+1)}{n^2} = \boxed{48 \text{ ft}}$$

Notice that you just found the area inside a region with a curved edge!