Case Study The Monty Hall Problem

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Here is a famous "paradox" that even mathematicians find counterintuitive. On the game show *Let's Make a Deal*, you are shown three doors, A, B, and C, and behind one of them is the Big Prize. After you select one of them–say, door A–to make things more interesting the host (Monty Hall), who knows what is behind each door, opens one of the other doors–say, door B–to reveal that the Big Prize is not there. He then offers you the opportunity to change your selection to the remaining door, door C. Should you switch or stick with your original guess? Does it make any difference?

Most people would say that the Big Prize is equally likely to be behind door A or door C, so there is no reason to switch.* In fact, this is wrong: The prize is more likely to be behind door C! There are several ways of seeing why this is so. Here is how you might work it out using Bayes' theorem.

Let A be the event that the Big Prize is behind door A, B the event that it is behind door B, and C the event that it is behind door C. Let F be the event that Monty has opened door B and revealed that the prize is not there. You wish to find $P(C \mid F)$ using Bayes' theorem. To use that formula you need to find $P(F \mid A)$ and $P(A)$ and similarly for B and C. Now, $P(A) = P(B) = P(C) = 1/3$ because at the outset, the prize is equally likely to be behind any of the doors. $P(F \mid A)$ is the probability that Monty will open door B if the prize is actually behind door A, and this is $1/2$ because we assume that he will choose either B or C randomly in this case. On the other hand, $P(F \mid B) = 0$, because he will never open the door that hides the prize. Also, $P(F \mid C) = 1$ because if the prize is behind door C, he must open door B to keep from revealing that the prize is behind door C. Therefore,

$$P(C \mid F) = \frac{P(F \mid C)P(C)}{P(F \mid A)P(A) + P(F \mid B)P(B) + P(F \mid C)P(C)}$$
$$= \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{2}{3}.$$

You conclude from this that you *should* switch to door C because it is more likely than door A to be hiding the Prize.

Here is a more elementary way you might work it out. Consider the tree diagram of possibilities shown in Figure 22. The top two branches of the tree give the cases in which the prize is behind door A, and there is a total probability of $1/3$ for that case. The remaining two branches with nonzero probabilities give the cases in which the prize is behind the door that you did not choose, and there is a total probability of $2/3$ for that case. Again, you conclude that you should switch your choice of doors because the one you did not choose is twice as likely as door A to be hiding the Big Prize.

* This problem caused quite a stir in late 1991 when this problem was discussed in Marilyn vos Savant's column in *Parade* magazine. Vos Savant gave the answer that you should switch. She received about 10,000 letters in response, most of them disagreeing with her, including several from mathematicians.

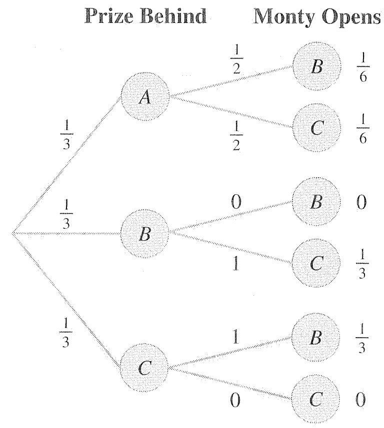

Figure 22

EXERCISES

1. The answer you came up with, to switch to the other door, depends on the strategy Monty Hall uses in picking the door to open. Suppose that he actually picks one of doors B and C at random, so that there is a chance that he will reveal the Big Prize. If he opens door B and it happens that the Prize is not there, should you switch or not?
2. What if you know that Monty's strategy is always to open door B if possible (i.e., it does not hide the Big Prize) after you choose A?

   a. If he opens door B, should you switch?
   b. If he opens door C, should you switch?
3. Repeat the analysis of the original game, but suppose that the game uses four doors instead of three (and still only one prize).
4. Repeat the analysis of the original game, but suppose that the game uses 1,000 doors instead of 3 (and still only one prize).