Math 141: Section 3.10 Related Rates - Notes

Related Rates Equations Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

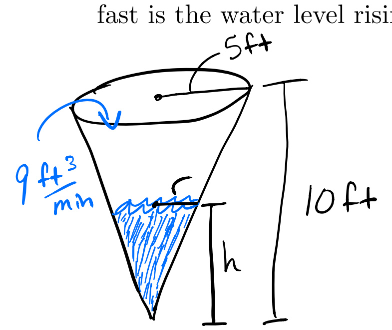
$$V = \frac{4}{3}\pi r^3.$$

Using the chain rule, we can differentiate both sides with respect to t to find an equation relating the rates of change of V and r,

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant.

Example 1 Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?



$$\frac{dV}{dt} = 9$$

V = volume

r = radius of the surface of the water

h = height of water

$$\frac{dh}{dt} = ? \text{ when } h = 6$$

Volume of a cone

$$V = \frac{1}{3}\pi r^2 h$$

Need to eliminate the variable r before differentiating.

$$\frac{r}{h} = \frac{5}{10} \Rightarrow r = \frac{h}{2}$$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{3}\pi \frac{h^2}{4} \cdot h$$

$$= \frac{\pi h^3}{12}$$

$$V = \frac{\pi h^3}{12}$$

Differentiate implicitly with respect to t.

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi h^2}{4} \frac{dh}{dt}$$

$\frac{dV}{dt} = 9$, $\frac{dh}{dt} = ?$ when $h=6$

Example 1 (cont.)

$$9 = \frac{\pi (6)^2}{4} \cdot \frac{dh}{dt}$$

$$36 = 36\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi} \text{ ft/min}$$

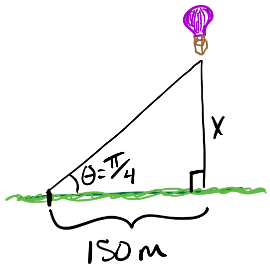
* Note: Only plug in given information AFTER differentiating

Ex: $h=6$ $V = \frac{\pi(6)^3}{12} \Rightarrow \frac{dV}{dt} = 0$?

Related Rates Problem Strategy:

1) Draw a picture and name the variables and constants. Use t for time and assume all variables are differentiable functions of time.
2) Write down the numerical information.
3) Write down what you are asked to find.
4) Write an equation that relates the variables. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rate you know.
5) Differentiate with respect to t.
6) Evaluate. Use known values to find the unknown rate.

Example 2 A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?



$$\frac{d\theta}{dt} = 0.14 \ \frac{\text{rad}}{\text{min}} \quad \text{when } \theta = \pi/4$$

$$\frac{dx}{dt} = ? \quad \text{when } \theta = \pi/4$$

$$\tan\theta = \frac{x}{150} \quad \text{or} \quad x = 150\tan\theta$$

Differentiate both sides wrt t

$$\frac{dx}{dt} = 150\sec^2(\theta)\cdot\frac{d\theta}{dt}$$

$$= 150\,(\sec(\pi/4))^2\,(0.14)$$

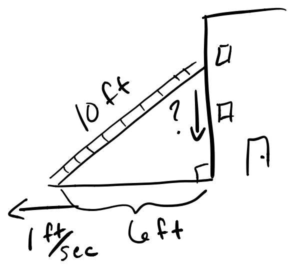
$$= 150\,(2)(0.14)$$

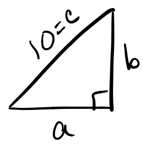
$$= 42 \ \text{m/min}$$

$$\sec(\pi/4) = \frac{1}{\cos(\pi/4)} = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}}$$

Math 141: Section 3.10 Related Rates Part II - Notes

Example 1 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the top of the ladder sliding down when the bottom is 6 ft from the wall?





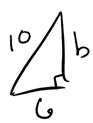
$\frac{da}{dt} = 1$ when $a = 6$

$\frac{db}{dt} = ?$

$$a^2 + b^2 = c^2$$

$$a^2 + b^2 = 100 \qquad 2c\underbrace{\frac{dc}{dt}}_{=0}$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$$



$36 + b^2 = 100$

$b^2 = 64$

$b = 8$

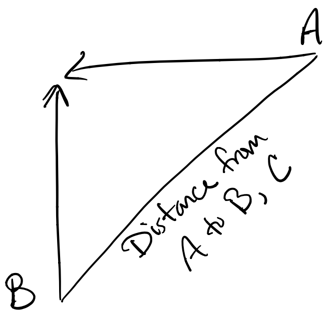
$$2(6)(1) + 2(8)\frac{db}{dt} = 0$$

$$16\frac{db}{dt} = -12$$

$$\frac{db}{dt} = -3/4 \text{ ft/sec}$$

Rates have direction associated; the negative indicates that the top of the ladder is sliding down

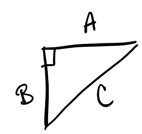
Example 2 Car A is traveling west at 50 mph and Car B is traveling north at 60 mph. Both are headed for the intersection of two roads. At what rate are the cars approaching each other when Car A is 0.3 mi and Car B is 0.4 mi from the intersection?

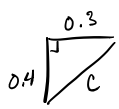


$$\frac{dA}{dt} = 50 \text{ mph}$$

$$\frac{dB}{dt} = 60 \text{ mph}$$

$$\frac{dC}{dt} = ? \quad \text{when } A = 0.3\text{mi}, B = 0.4\text{mi}$$





$$C = \sqrt{(0.3)^2 + (0.4)^2}$$
$$= \sqrt{0.9 + 1.6}$$
$$= \sqrt{2.5}$$
$$= 0.5$$

$$A^2 + B^2 = C^2$$

$$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2C\,dC/dt$$

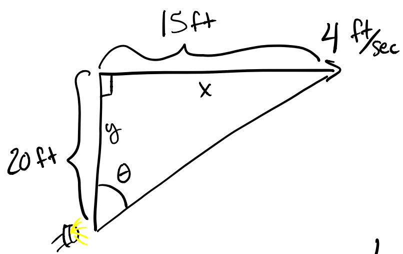
$$2(0.3)(50) + 2(0.4)(60) = 2(0.5)\frac{dC}{dt}$$

$$30 + 48 = \frac{dC}{dt}$$

$$\frac{dC}{dt} = 78 \text{ mph}$$

We should expect a negative answer since the distance between the cars is decreasing.
To fix this, we can either specify at the end, add the negative, or let $\frac{dA}{dt}$ and dB/dt be negative at the beginning.

Example 3 A searchlight is following a boat moving east at 4 ft/sec. If the searchlight is located 20ft due south of where the boat left dock, how fast is the searchlight rotating when the boat is 15 ft away from the dock?



$$\frac{dx}{dt} = 4 \text{ ft/sec}$$

$$\frac{d\theta}{dt} = ? \text{ when } x = 15\text{ft}$$

$$\tan\theta = \frac{x}{y} \quad , y \text{ is fixed at } 20\text{ft}$$

$$\tan\theta = \frac{x}{20}$$

$$\tan\theta = \frac{15}{20} = \frac{3}{4}$$

$$\theta = \tan^{-1}(3/4)$$

$$\tan^2 u + 1 = \sec^2 u$$

$$u = \arctan(3/4)$$

$$\sec^2(\arctan(3/4))$$

$$= (\tan(\arctan(3/4)))^2 + 1$$

$$= (3/4)^2 + 1$$

$$= 9/16 + 1 = \frac{25}{16}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{20}(4) \qquad \theta = ?$$

$$\frac{d\theta}{dt} = \frac{1}{5\sec^2(\arctan(3/4))}$$

$$= \frac{1}{5 \cdot 25/16} = \frac{16}{125} \text{ rad/sec}$$