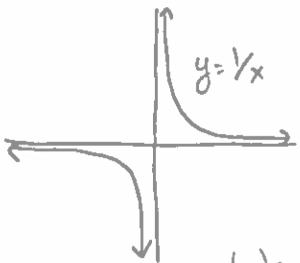
Math 141: Section 2.6 Limits Involving Infinity; Asymptotes of Graphs - Notes

Example 1 Consider the function $f(x) = \frac{1}{x}$.



As x gets large in the positive direction, $f(x)$ gets very small.
As x gets large in the negative direction, again, $f(x)$ gets very small.

We say $\lim_{x \to \infty} 1/x = 0$ and $\lim_{x \to -\infty} 1/x = 0$.

Theorem All the Limit Laws in Theorem 1 are true when we replace $\lim_{x \to c}$ by $\lim_{x \to \infty}$ or $\lim_{x \to -\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

*Note: "∞" is NOT a real number

Example 2 We can use limit laws to calculate limits in the same was as when x approaches a finite number c.

a)

$$\lim_{x \to \infty} \left(5 + \frac{1}{x}\right)$$

$$= \lim_{x \to \infty} 5 + \lim_{x \to \infty} 1/x = 5$$

b)

$$\lim_{x \to -\infty} \frac{\pi\sqrt{3}}{x^2}$$

$$= \lim_{x \to -\infty} \pi\sqrt{3} \cdot \frac{1}{x^2} = \lim_{x \to -\infty} \pi\sqrt{3} \cdot \lim_{x \to -\infty} 1/x \cdot \lim_{x \to -\infty} 1/x$$

$$= \pi\sqrt{3} \cdot 0 \cdot 0$$

$$= 0$$

Limits at Infinity of Rational Functions Various things can happen when we consider the limit of a rational function as $x \to \pm\infty$. The next example considers when the degree of the numerator is less than or equal to the degree of the denominator.

Example 3 Evaluate the following limits:

a)

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2}$$

Divide by the highest power of x in the denominator

$$= \lim_{x\to\infty} \frac{5+8/x-3/x^2}{3+2/x^2}$$

$$= \frac{5+0-0}{3+0} = 5/3$$

b)

$$\lim_{x\to-\infty} \frac{11x+2}{2x^3-1}$$

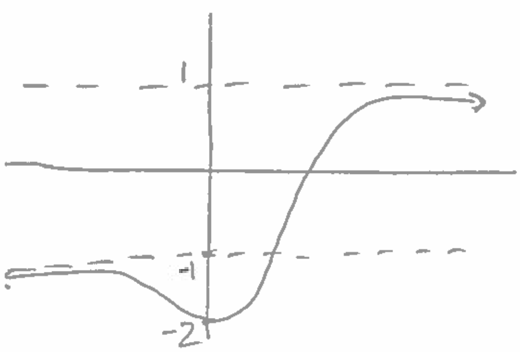
$$= \lim_{x\to-\infty} \frac{11/x^2+2/x^3}{2-1/x^3} = \frac{0+0}{2-0} = 0$$

Definition: A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b.$$

Example 4 Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$



$x \geq 0 \quad \lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} = \lim_{x\to\infty} \frac{x^3-2}{x^3+1} = 1$

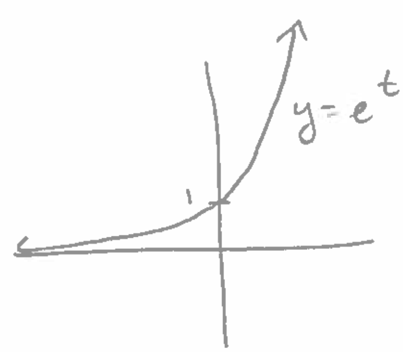
$x < 0 \quad \lim_{x\to-\infty} \frac{x^3-2}{|x|^3+1} = \lim_{x\to-\infty} \frac{x^3-2}{-x^3+1} = -1$

Example 5 Find

$$\lim_{x\to 0^-} e^{1/x}.$$

Substitute: Let $t = 1/x$. As $x \to 0^-$, $t \to -\infty$

So $\lim_{x\to 0^-} e^{1/x} = \lim_{t\to-\infty} e^t = 0$



Example 6 Find

$$\lim_{x\to\infty}(x-\sqrt{x^2+16}).$$

CanNOT just plug in "∞". Remember! "∞" is a symbol, not a real number.

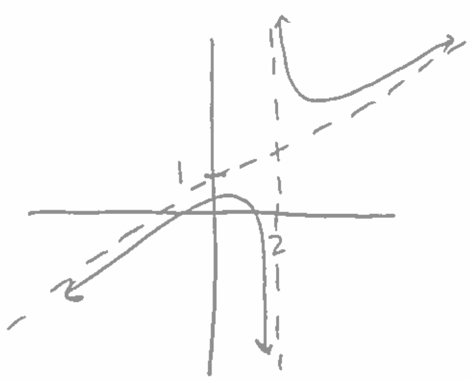
$$\lim_{x\to\infty}(x-\sqrt{x^2+16}) = \lim_{x\to\infty}(x-\sqrt{x^2+16})\cdot\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}$$

$$= \lim_{x\to\infty}\frac{x^2-(x^2+16)}{x+\sqrt{x^2+16}} = \lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}} = 0$$

Oblique Asymptotes What if the degree of the numerator is exactly one more than the degree of the denominator?

Example 7 Consider the function

$$f(x)=\frac{x^2-3}{2x-4}.$$



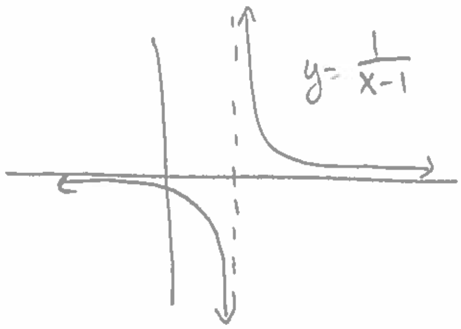
$$\begin{array}{r} \frac{x}{2}+1 \\ 2x-4\overline{\big)\,x^2-3} \\ \overline{} \\ 1 \end{array}$$

$$f(x)=\frac{x^2-3}{2x-4} = \underbrace{\left(\frac{x}{2}+1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x-4}\right)}_{\text{remainder}}$$

Oblique Asymptotes If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant** asymptote. We find an equation for the asymptote by dividing the numerator by the denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

Example 8: Infinite Limits Find

$$\lim_{x \to 1^+} \frac{1}{x-1}.$$



$$\lim_{x \to 1^+} \frac{1}{x-1} = \infty$$

"Does not exist because its values become arbitrarily large"

Vertical Asymptotes A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \text{ or } \lim_{x \to a^-} f(x) = \pm\infty.$$

Example 9 Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x+3}{x+2}.$$

Domain: $(-\infty, -2) \cup (-2, \infty)$

Consider $\lim_{x \to \pm\infty} \frac{x+3}{x+2}$ and $\lim_{x \to -2^{+,-}} \frac{x+3}{x+2}$

$\lim_{x \to \pm\infty} \frac{x+3}{x+2} = 1$ $\quad \lim_{x \to -2^+} \frac{x+3}{x+2} = \infty$ $\quad \lim_{x \to -2^-} \frac{x+3}{x+2} = -\infty$