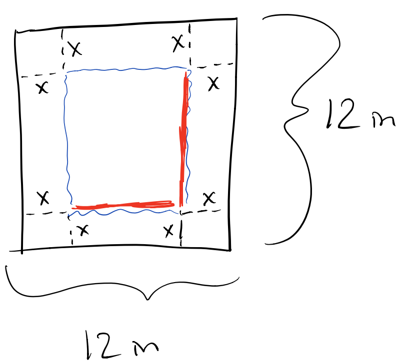
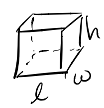
Math 141: Section 4.6 Applied Optimization - Notes

Solving Applied Optimization Problems:

1. Read the problem carefully. What is given? What is the unknown quantity to be optimized?
2. Draw a picture. Label any part that may be important to the problem.
3. Introduce variables. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. Write an equation for the unknown quantity. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown.* Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

Example 1 An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?





$V = l \cdot w \cdot h$

$l = w = (12-2x)$

$h = x$

$$V(x) = x(12-2x)^2, \quad 0 \le x \le 6$$

$$\begin{aligned} V'(x) &= (12-2x)^2 + x \cdot 2(12-2x)(-2) \\ &= (12-2x)^2 - 4x(12-2x) \\ &= (12-2x)(12-2x-4x) \\ &= (12-2x)(12-6x) \end{aligned}$$

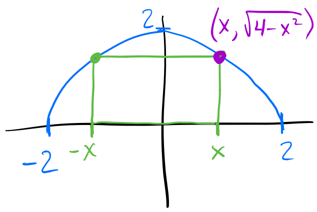
$V'(x) = 0$ when $(12-2x)(12-6x) = 0$

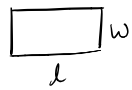
CPs $x = 2$, $x = 6$

To find absolute max, plug CPs and endpoints into $V(x)$: $V(0) = 0$, $V(2) = 2(12-2(2))^2 = 128$,
$V(6) = 0$

Max volume is 128 in^3, the squares should be $2\text{in} \times 2\text{in}$

Example 2 A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?





$A = l \cdot w$

$w = \sqrt{4-x^2}$

$l = 2x$

Equation of a circle centered at the origin with radius 2:

$$x^2 + y^2 = 4$$

$$y = \sqrt{4-x^2}$$

$$A(x) = 2x\sqrt{4-x^2}, \quad 0 \le x \le 2$$

$$A'(x) = 2\sqrt{4-x^2} + 2x \cdot \frac{1}{2}(4-x^2)^{-1/2}(-2x)$$

$$= 2\sqrt{4-x^2} - \frac{2x^2}{(4-x^2)^{1/2}}$$

$$= \frac{2(4-x^2) - 2x^2}{(4-x^2)^{1/2}}$$

$$= \frac{8-4x^2}{(4-x^2)^{1/2}}$$

CPs: $A'(x)$ is undefined when $x = 2$

and $A'(x) = 0$ when

$$8 - 4x^2 = 0$$

$$x = \pm\sqrt{2}$$

$x = \sqrt{2}$ $\quad A(0) = 0, \ A(2) = 0$

$$A(\sqrt{2}) = 2\sqrt{2}\sqrt{4-(\sqrt{2})^2} = 2\sqrt{2}\sqrt{2} = 4$$

Max area is 4 square units with length $2\sqrt{2}$ and width $\sqrt{2}$