Math 141: Section 3.6 The Chain Rule - Notes

Example 1 How do we differentiate the composite $f \circ g$ of two function $y = f(u)$ and $u = g(x)$? The function

$$y = (3x^2 + 1)^2$$

is the composite of

$$y = f(u) = u^2 \quad \text{and} \quad u = g(x) = 3x^2 + 1$$

Calculating derivatives, we have

$$\frac{dy}{du} = 2u \qquad \frac{du}{dx} = 6x$$

$$\begin{aligned} \frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 12u \cdot x \\ &= 12(3x^2+1)x \\ &= 12x(3x^2+1) \end{aligned}$$

The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx},$$

where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

Example 2 Differentiate $\sin(x^2 + e^x)$ with respect to x:

$y = \sin(x^2 + e^x)$ $\quad$ $f(u) = \sin(u)$

$u = x^2 + e^x = g(x)$

$$y' = f'(u) \cdot u'$$
$$= \cos u \cdot (2x + e^x)$$
$$= \cos(x^2 + e^x)(2x + e^x)$$

Example 3 Differentiate

$$y = e^{\cos x}.$$

$y = e^{\cos x}$ $\quad$ $f(u) = e^u$, $u = \underline{\cos x}$

$f'(u) = e^u$

$f'(3)$

$$y' = f'(u) \cdot u'$$
$$= e^u \cdot (-\sin x)$$
$$= -e^{\cos x} \cdot \sin x$$

Example 4 An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

Position: $x(t) = \cos(t^2+1)$

Velocity: $x'(t)$ $\quad f(u) = \cos u, \; u = t^2+1$

$= f'(u) \cdot u'$

$= -\sin u \cdot 2t$

$= -2t \sin(t^2+1)$

Example 5 Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

$g(t) = \tan(\underline{5 - \sin(2t)})$

$f(u) = \tan u \qquad u = 5 - \sin(2t)$

$u' = (-\sin(2t))'$

Apply second chain rule

$h(v) = -\sin v$

$v = 2t$

$u' = -\cos v \cdot 2$

$= -2\cos(2t)$

$g'(t) = f'(u) \cdot u'$

$= \sec^2 u \, (-2\cos(2t))$

$= \sec^2(5 - \sin(2t))(-2\cos(2t))$

$= \boxed{-2\cos(2t)\sec^2(5-\sin(2t))}$

Example 6 Compute the following

$$\frac{d}{dx}\left(5x^3 - x^4\right)^7$$

$$= 7\left(5x^3 - x^4\right)^6\left(15x^2 - 4x^3\right)$$

Example 7 Compute the following

$$\frac{d}{dx}\left(\frac{1}{3x-2}\right)$$

$$\frac{d}{dx}(3x-2)^{-1}$$

$$= -1\,(3x-2)^{-2}(3)$$

$$= -3\,(3x-2)^{-2}$$