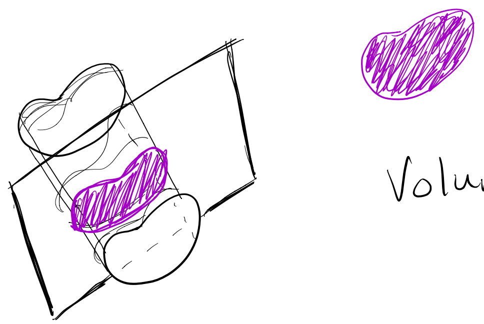
Ch. 6 Applications of the Definite Integral

6.1 Volumes Using Cross-sections

A cross-section of a solid S is the plane region formed by intersecting S with a plane



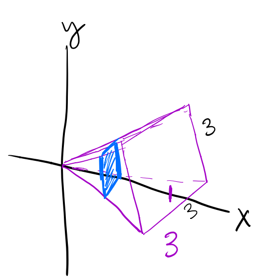
Cross-section $S(x)$ with area $A(x)$

Volume = Area · Height

- Slicing Method

The volume of a solid of integrable cross-sectional area $A(x)$ from $x=a$ to $x=b$ in the integral of $A(x)$ from a to b:

$$V = \int_a^b A(x)\,dx$$



x
x

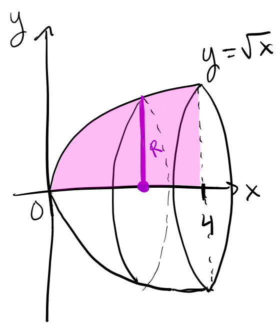
$A(x) = x^2$

$$V = \int_0^3 x^2\,dx$$

* Disk Method

The solid generated by rotating (or revolving) a plane region about an x-axis in its plane is called a <u>solid of revolution</u>.

<u>Ex</u>:



Cross-section is a circle!

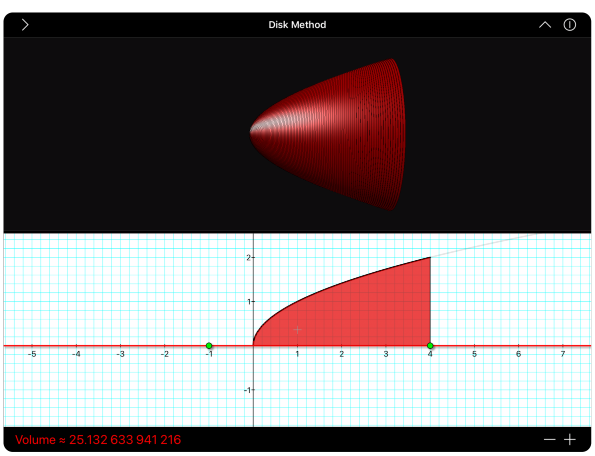
$$A(x) = \pi(\text{radius})^2 = \pi(R(x))^2$$

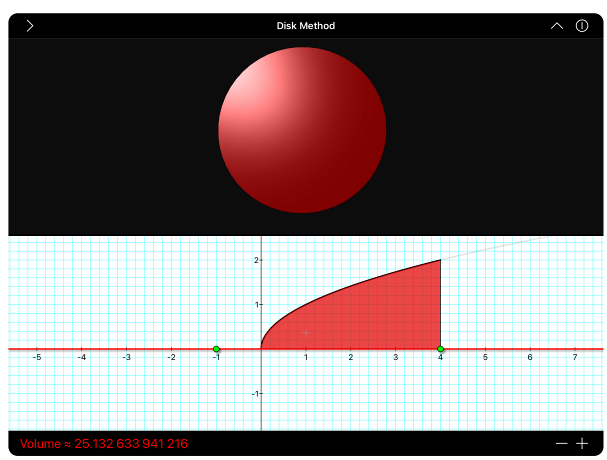
The cross-sectional area is the area of a circle with radius $\sqrt{x}$ ranging from $x=0$ to $x=4$.

$$V = \int_0^4 \overbrace{\pi(\sqrt{x})^2}^{A(x)}\,dx = \pi\int_0^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^4$$

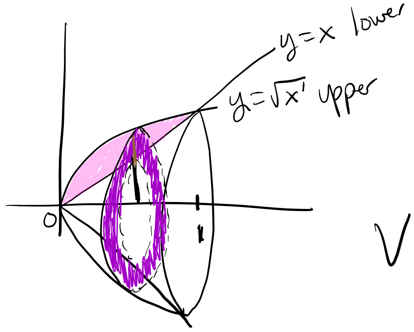
$$= \pi\left(\frac{4^2}{2} - 0\right)$$

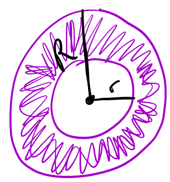
$$= 8\pi$$





* Washer Method





$$V=\int_a^b \pi\left(R^2-r^2\right)dx$$

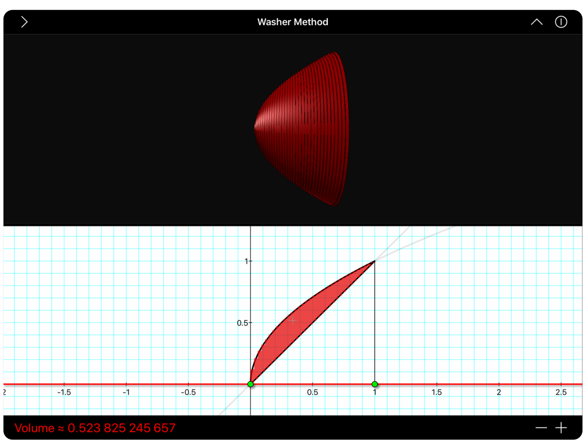
$$V=\int_0^1 \pi\left(\sqrt{x}\right)^2dx-\int_0^1\pi(x)^2dx$$

$$=\int_0^1\pi\left(\left(\sqrt{x}\right)^2-(x)^2\right)dx$$

$$=\pi\int_0^1\left(x-x^2\right)dx$$

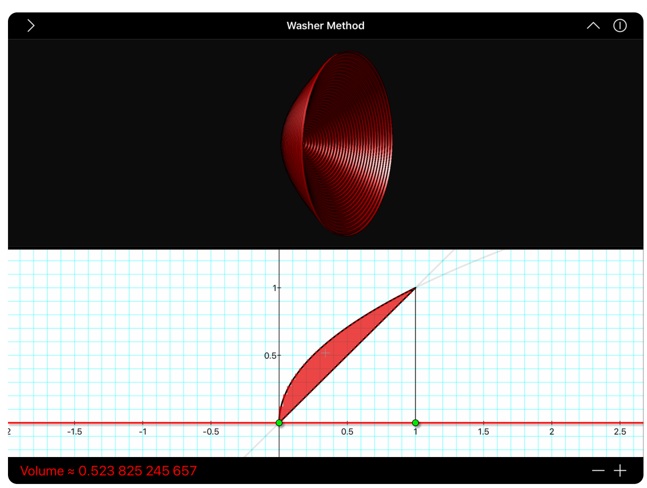
$$=\pi\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{3}-0\right)$$

$$=\boxed{\pi/6}$$



3D models of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and $y=x$ about the x-axis.

Note, $\frac{\pi}{6}\approx 0.52$



Ex: The region bounded by the curve $y = x^2+1$ and the line $y = -x+3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

1) Find the limits of integration by setting the two functions equal to each other and solving for x:

$$x^2+1 = -x+3$$

$$x^2+x-2 = 0$$

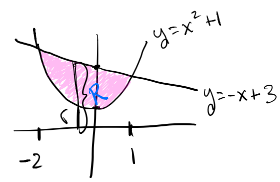
$$(x+2)(x-1) = 0$$

$$x=-2, \quad x=1$$

2) Determine the upper (outer) radius and lower (inner) radius.

$x=0$:

Outer R	Inner r
$y=-x+3$	$y=x^2+1$
$y=3$	$y=1$



3) $$V = \int_{-2}^{1} \pi \left((-x+3)^2 - (x^2+1)^2 \right) dx$$

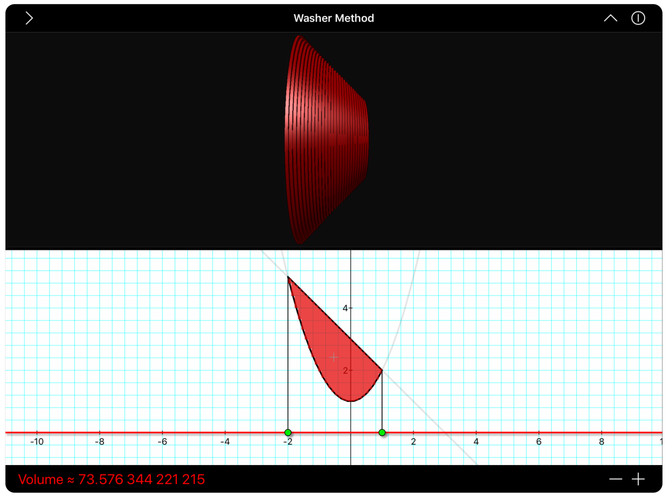
$$= \pi \int_{-2}^{1} (x^2-6x+9) - (x^4+2x^2+1)\, dx$$

$$= \pi \int_{-2}^{1} (-x^4-x^2-6x+8)\, dx$$

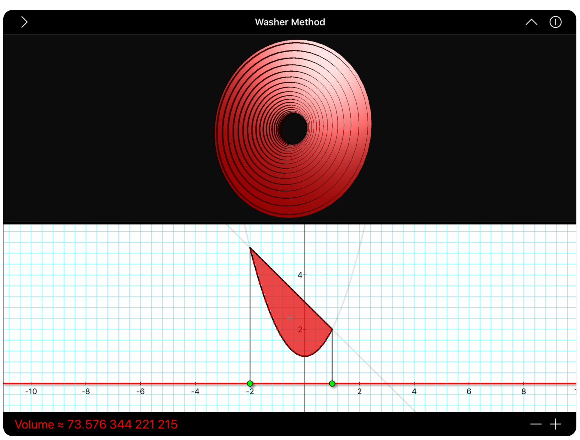
$$= \pi \left[-\frac{x^5}{5} - \frac{x^3}{3} - 3x^2 + 8x \right]_{-2}^{1}$$

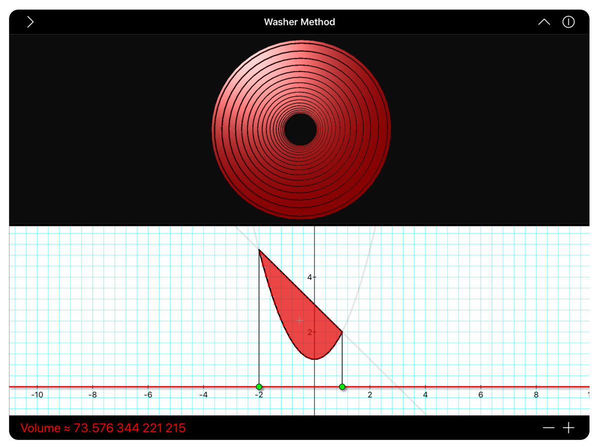
$$= \pi \left[\left(-\tfrac{1}{5} - \tfrac{1}{3} - 3 + 8\right) - \left(-\frac{(-2)^5}{5} - \frac{(-2)^3}{3} - 3(-2)^2 + 8(-2) \right) \right]$$

$$= \boxed{\frac{117\pi}{5}} \quad (\approx 73.5)$$

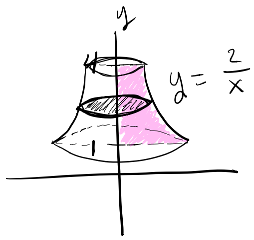


Different views of the solid obtained by revolving the region bounded by $y=x^2+1$ and $y=-x+3$ about the x-axis.





What if we revolve about the y-axis?

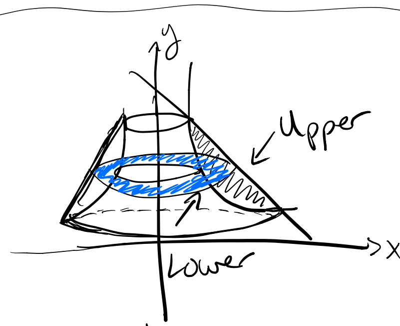


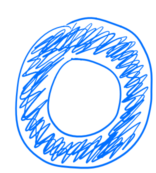
Instead of slicing vertically (perpendicular to the x-axis), we slice horizontally (perpendicular to the y-axis).

Cross-section is still a circle but the radius is now a function of y:

$$y = \frac{2}{x} \rightarrow x = \frac{2}{y}$$

$$V = \int_a^b \pi (R(y))^2 \, dy \qquad V = \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy$$



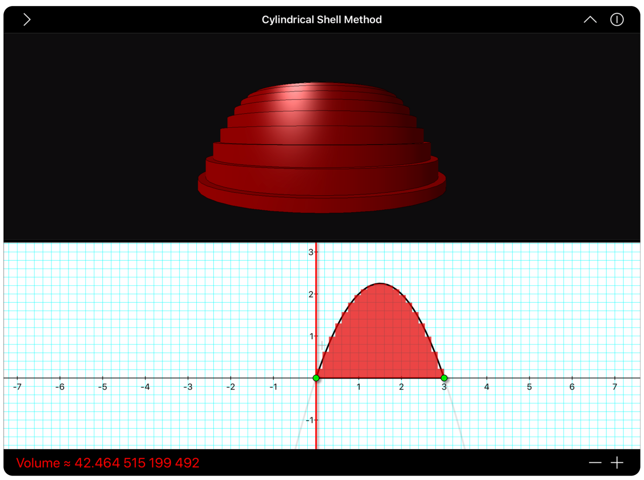


Still a washer!

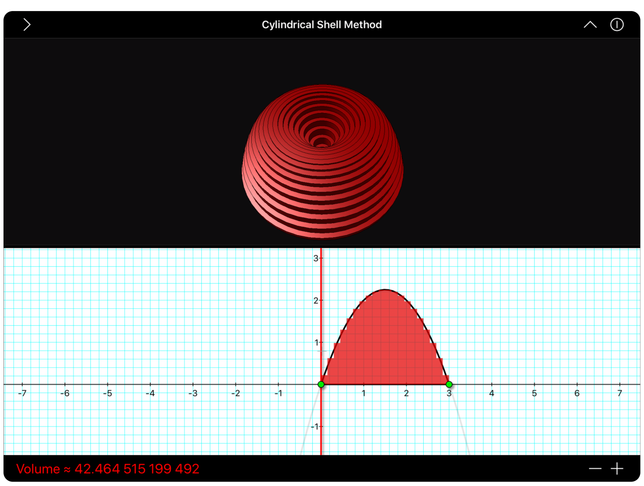
$$V = \int_a^b \pi \left(R(y)^2 - r(y)^2\right) dy$$

6.2 Volume Using Cylindrical Shells

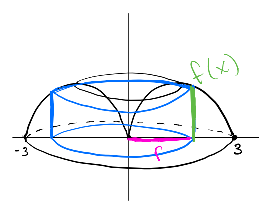
Consider the solid generated by revolving the region bounded by the curve $y = -x^2 + 3x$ and the x-axis about the y-axis. (See left)

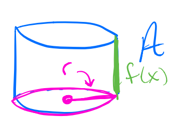


We could try using our washer method from 6.1 but this requires solving $y = -x^2 + 3x$ for x in terms of y... hard.



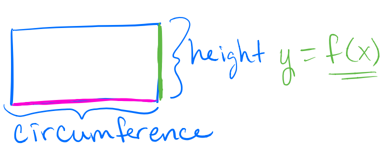
Let's instead still slice vertically (so we can keep y in terms of x) but with "cookie cutters" instead of a plane.





A cylinder!

Now we make use of what we know about another familiar shape



$$V = \int_a^b 2\pi \left(\text{shell radius}\right)\left(\text{shell height}\right) dx$$

For our example,

$$V = \int_0^3 2\pi(x)(-x^2+3x)\,dx$$

$$= 2\pi \int_0^3 (-x^3+3x^2)\,dx$$

$$= 2\pi \left[-\frac{x^4}{4} + x^3\right]_0^3$$

$$= 2\pi\left(-\frac{3^4}{4} + 3^3 - 0\right)$$

$$= 2\pi\left(-\frac{81}{4} + 27\right) = \frac{27\pi}{2} \quad (\approx 42.5)$$