Math 141:
Section 5.1 Area and Estimating with Finite Sums,
Section 5.2 Sigma Notation and Limits of Finite Sums
Notes

Estimating Area Under a Curve We saw that it is easy to find the area between a curve and the x-axis on an interval when that curve is a constant or a linear function. The curve and the x-axis make familiar shapes (rectangle or triangle) and we use known formulas.

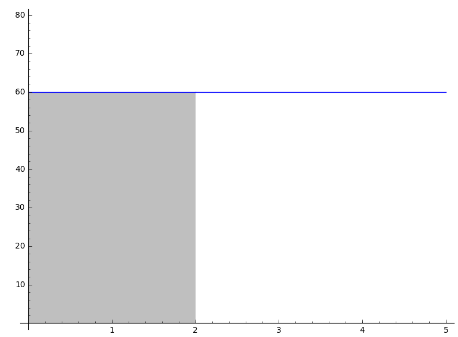


Figure 1: Area Under a Constant Function

Figure 2: Area Under a Linear Function

When the curve is not constant or linear, the best we can do is provide an approximation. We find this estimation by partitioning the interval into equally spaced subintervals and using rectangles.

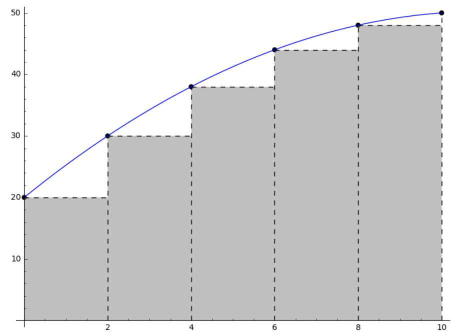


Figure 3: Left Hand Sum

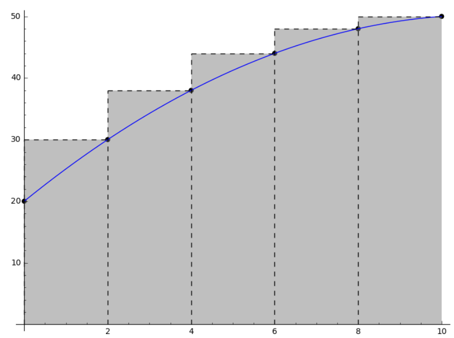


Figure 4: Right Hand Sum

To calculate the area estimated by our rectangles, we need to know the width of each rectangle and the height. The width is given by

$$\Delta t = \frac{b-a}{n},$$

where n is the number of rectangles or subintervals we use and $[a, b]$ is the interval. The height is determined by the function. In Figure 3, we find an underestimate of the area and in Figure 4, an overestimate. Figure 5 labels the necessary pieces of a Right Hand Sum.

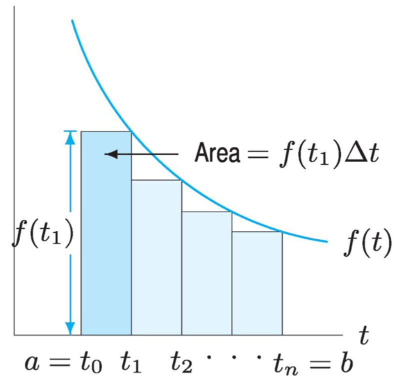


Figure 5: What we need to find the area

We could improve our estimate by taking the average of the two or, even better, using more rectangles and hence more subintervals. In fact, the more subintervals we use, the more accurate our estimation becomes.

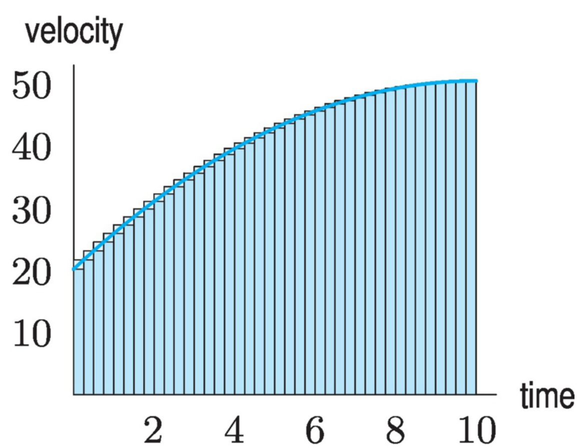


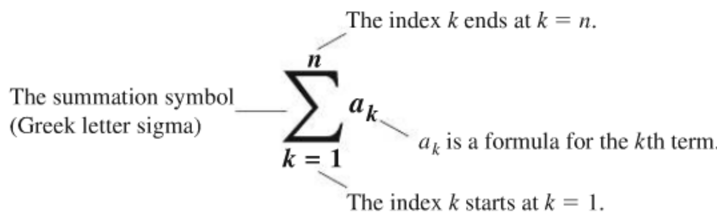
Figure 6: More subintervals - Better Estimate

Once we start using many subintervals, it is cumbersome to write out the sum of, say, 1000 terms. We introduce some notation to easily write a sum with many terms.

Finite Sums and Sigma Notation Sigma notation enables us to write a sum with many terms in the compact form:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

The Greek letter Σ stands for "sum." The **index of summation** k tells us where the sum begins (at the number below the Σ symbol) and where it ends (at the number above Σ). Any letter can be used to denote the index, but the letters i, j, and k are customary.



Thus, we can write

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2$$

The lower limit of summation does not have to be 1; it can be any integer.

Example 1

A sum in sigma notation	**The sum written out, one term for each value of k**	**The value of the sum**
$\sum_{k=1}^{5} k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

Algebra Rules for Finite Sums

1. *Sum Rule:* $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. *Difference Rule:* $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. *Constant Multiple Rule:* $\sum_{k=1}^{n} c a_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number c)
4. *Constant Value Rule:* $\sum_{k=1}^{n} c = n \cdot c$ (c is any constant value.)

Example 2 Show that the sum of the first n integers is

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\begin{array}{rl} & 1+2+3+4+\ldots+100 \\ + & 100+99+98+97+\ldots+1 \\ \hline & 101+101+101+101+\ldots+101 \end{array}$$

$= 100(101)$

$$\sum_{k=1}^{100} k = \frac{100(101)}{2}$$

$$\begin{array}{rl} & 1+2+3+\ldots+n \\ + & n+(n-1)+(n-2)+\ldots+1 \\ \hline & (n+1)+(n+1)+\ldots+(n+1) \end{array}$$

$= n(n+1)$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

The first n squares

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

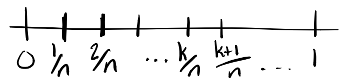
The first n cubes

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Example 3 Find the limiting value of lower sum approximations to the area of the region R below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$ on the x-axis using equal-width rectangles whose widths approach zero and whose number approaches infinity.

$[a,b]$
$[0,1]$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$



Each interval has width $1/n$.
Since we want an underestimate (lower sum) and $y = 1-x^2$ is decreasing on the interval, we use right endpoints.

$$\frac{1}{n}f(1/n) + \frac{1}{n}f(2/n) + \dots + \frac{1}{n}f(k/n) + \dots + \frac{1}{n}f(1)$$

In sigma notation:

$$\sum_{k=1}^{n} \frac{1}{n} f(k/n) = \sum_{k=1}^{n} \frac{1}{n}\left(1 - (k/n)^2\right)$$

$f(x) = 1 - x^2$

$$= \sum_{k=1}^{n}\left(\frac{1}{n} - \frac{k^2}{n^3}\right)$$

$$= \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{n} \frac{k^2}{n^3}$$

$\frac{1}{n}$ does not depend on k so we treat it like a constant

$$= n \cdot \frac{1}{n} - \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= 1 - \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= 1 - \frac{2n^3 + 3n^2 + n}{6n^3}$$