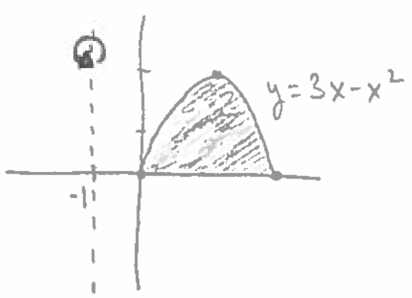
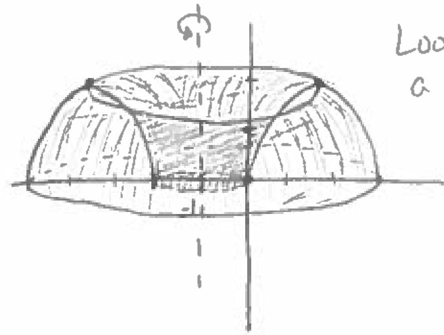
6.2 Volumes Using Cylindrical Shells

In section 6.1, we sliced through solids with a plane perpendicular to the axis of revolution. We then found the area of the cross-section and defined the volume as $V = \int_a^b A(x)\,dx$ (or $\int_c^d A(y)\,dy$).

Depending on the shape of the solid, this method of slicing can sometimes be tricky...

Ex: The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid. Find the volume of the solid.



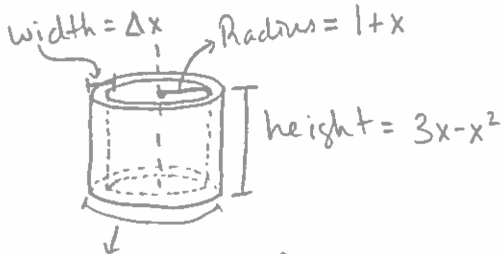


Looks sort of like a bundt cake.

Slicing with a plane would not always generate the same shape like we saw in the previous section.

Instead, we can slice using cylinders of increasing radii! Think about a cookie cutter: If we "cut" the solid from the top with larger and larger cookie cutters, we'll end up with a bunch of cylinders.

Let's examine one of these cylinders more closely:

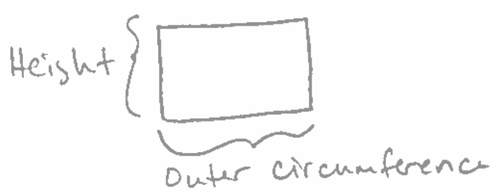


Outer circumference $= 2\pi \cdot$ radius $= 2\pi(1+x)$

Since we're revolving about the line $x=-1$, the radius is $x-(-1) = x+1 = 1+x$. The height of the cylinder is determined by the function (similar to last section).

Why might we need the outer circumference?

If we unroll the cylinder, we get a nearly rectangular solid which we know how to find the area of:



Using the same formula as in section 6.1, but using our newly defined area, we now have

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx.$$

For our example:

$$V = \int_0^3 2\pi(x+1)(3x-x^2)\,dx$$

$$= 2\pi \int_0^3 (2x^2+3x-x^3)\,dx$$

$$\vdots$$

$$= \frac{45\pi}{2}$$

The Shell Method

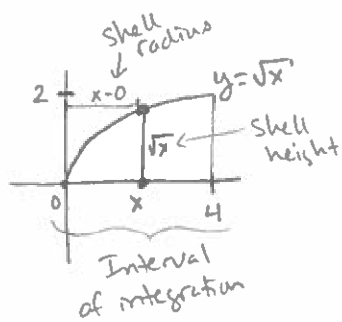
The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y=f(x)\geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x=L$ is

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\,dx$$

$$= \int_a^b 2\pi (x-L)\, f(x)\,dx$$

The variable of integration, here x, is called the <u>thickness variable</u>.

<u>Ex</u>: The region bounded by the curve $y=\sqrt{x}$, the x-axis, and the line $x=4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.



$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\,dx$$

$$= \int_0^4 2\pi (x)(\sqrt{x})\,dx$$

$$= 2\pi \int_0^4 x^{3/2}\,dx$$

$$\vdots$$

$$= \frac{128\pi}{5}$$