Math 141: Section 2.2 Limit of a Function and Limit Laws - Notes

Limits of Function Values Often when studying a function $y = f(x)$, we are interested in the function's behavior near a particular point c, but not precisely at c. For instance, if c is an irrational number, like π or $\sqrt{2}$, whose values can only be approximated by "close" rational numbers. Another instance would be when trying to evaluate a function at c leads to division by zero.

Example 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Generalizing, suppose $f(x)$ is defined on an open interval about c, except possibly at c itself. If $f(x)$ is arbitrarily close to the number L for all x sufficiently close to c, we say that f approaches the **limit** L as x approaches c, and write

$$\lim_{x \to c} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches c is L."

Example 2 The limit value of a function does not depend on how the function is defined at the point being approached.

Example 3 If f is the **identity function** $f(x) = x$, then for any value of c,

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = x.$$

If f is the **constant function** $f(x) = k$, then for any value of c,

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k.$$

A function may not have a limit at a particular point:

Theorem 1: Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \text{ and } \lim_{x \to c} g(x) = M, \text{ then}$$

1) Sum Rule:

2) Difference Rule:

3) Constant Multiple Rule:

4) Product Rule:

5) Quotient Rule:

6) Power Rule:

7) Root Rule:

Example 4 Use the observations from Example 3 and Limit Laws to find the following limits:

(a)

$$\lim_{x \to c} (x^3 + 4x^2 - 3)$$

(b)

$$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$$

(c)

$$\lim_{x \to c} \sqrt{4x^2 - 3}$$

Theorem 2: Limits of Polynomials If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

Theorem 3: Limits of Rational Function If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 5 Evaluate

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}.$$

Theorem 4: The Sandwich (Squeeze) Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Theorem 5 If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$