Degree Requirements *After an exhausting day at the office, John O'Hagan returns home and finds himself having to assist his son Billy-Sean, who continues to have a terrible time planning his first-year college course schedule. The latest Bulletin of Suburban State University reads as follows:*

> *All candidates for the degree of Bachelor of Arts at SSU must take, in their first year, at least 10 courses in the Sciences, Fine Arts, Liberal Arts, and Mathematics combined, of which at least 2 must be in each of the Sciences and Fine Arts, and exactly 3 must be in each of the Liberal Arts and Mathematics.*

Help him with the answers to Exercises 41–43.

41. If the Bulletin lists exactly five first-year-level science courses and six first-year-level courses in each of the other categories, how many course combinations are possible that meet the minimum requirements?

42. Reading through the course descriptions in the bulletin a second time, John O'Hagan notices that Calculus I (listed as one of the mathematics courses) is a required course for many of the other courses, and so decides that it would be best if Billy-Sean included Calculus I. Further, two of the Fine Arts courses cannot both be taken in the first year. How many course combinations are possible that meet the minimum requirements and include Calculus I?

43. To complicate things further, in addition to the requirement in Exercise 42, Physics II has Physics I as a prerequisite (both are listed as first-year science courses, but it is not necessary to take both). How many course combinations are possible that include Calculus I and meet the minimum requirements?

Case Study: Designing a Puzzle

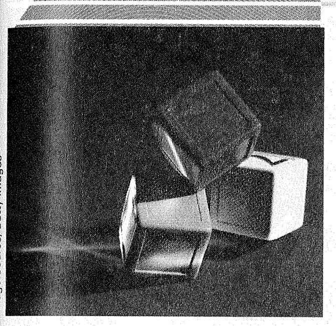


Image Source/Getty Images

As Product Design Manager for Cerebral Toys Inc., you are constantly on the lookout for ideas for intellectually stimulating yet inexpensive toys. You recently received the following memo from Felix Frost, the developmental psychologist on your design team.

> To: Felicia
> From: Felix
> Subject: Crazy Cubes
>
> We've hit on an excellent idea for a new educational puzzle (which we are calling "Crazy Cubes" until Marketing comes up with a better name). Basically, Crazy Cubes will consist of a set of plastic cubes. Two faces of each cube will be colored red, two will be colored blue, and two white, and there will be exactly two cubes with each possible configuration of colors. The goal of the puzzle is to seek out the matching pairs, thereby enhancing a child's geometric intuition and three-dimensional manipulation skills. The kit will include every possible configuration of colors. We are, however, a little stumped on the following question: How many cubes will the kit contain? In other words, how many possible ways can one color the faces of a cube so that two faces are red, two are blue, and two are white?

Looking at the problem, you reason that the following three-step decision algorithm ought to suffice:

Step 1 Choose a pair of faces to color red; $C(6, 2) = 15$ choices.

Step 2 Choose a pair of faces to color blue; $C(4, 2) = 6$ choices.

Step 3 Choose a pair of faces to color white; $C(2, 2) = 1$ choice.

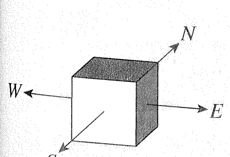


Figure 9

This algorithm appears to give a total of $15 \times 6 \times 1 = 90$ possible cubes. However, before sending your reply to Felix, you realize that something is wrong, because there are different choices that result in the same cube. To describe some of these choices, imagine a cube oriented so that four of its faces are facing the four compass directions (Figure 9). Consider choice 1, with the top and bottom faces blue, north and south faces white, and east and west faces red; and choice 2, with the top and bottom

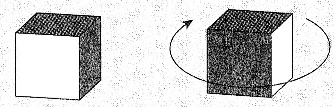


Figure 10

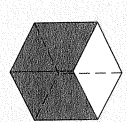


Figure 11

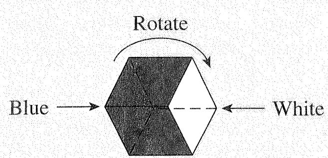


Figure 12

faces blue, north and south faces red, and east and west faces white. These cubes are actually the same, as you see by rotating the second cube 90 degrees (Figure 10).

You therefore decide that you need a more sophisticated decision algorithm. Here is one that works.

Alternative 1: Faces with the same color opposite each other. Place one of the blue faces down. Then the top face is also blue. The cube must look like the one drawn in Figure 10. Thus there is only one choice here.

Alternative 2: Red faces opposite each other and the other colors on adjacent pairs of faces. Again there is only one choice, as you can see by putting the red faces on the top and bottom and then rotating.

Alternative 3: White faces opposite each other and the other colors on adjacent pairs of faces; one possibility.

Alternative 4: Blue faces opposite each other and the other colors on adjacent pairs of faces; one possibility.

Alternative 5: Faces with the same color adjacent to each other. Look at the cube so that the edge common to the two red faces is facing you and horizontal (Figure 11). Then the faces on the left and right must be of different colors because they are opposite each other. Assume that the face on the right is white. (If it's blue, then rotate the die with the red edge still facing you to move it there, as in Figure 12.) This leaves two choices for the other white face, on the upper or the lower of the two back faces. This alternative gives two choices.

It follows that there are $1 + 1 + 1 + 1 + 2 = 6$ choices. Because the Crazy Cubes kit will feature two of each cube, the kit will require 12 different cubes.*

* There is a beautiful way of calculating this and similar numbers, called Pólya enumeration, but it requires a discussion of topics well outside the scope of this book. Take this as a hint that counting techniques can use some of the most sophisticated mathematics.

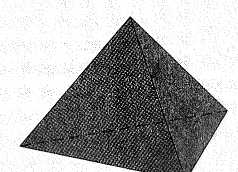
EXERCISES

In all of the following exercises, there are three colors to choose from: red, white, and blue.

1. In order to enlarge the kit, Felix suggests including two each of two-colored cubes (using two of the colors red, white, and blue) with three faces one color and three another. How many additional cubes will be required?

2. If Felix now suggests adding two each of cubes with two faces one color, one face another color, and three faces the third color, how many additional cubes will be required?

3. Felix changes his mind and suggests the kit use tetrahedral blocks with two colors instead (see the figure). How many of these would be required?



A tetrahedron

4. Once Felix finds the answer to the preceding exercise, he decides to go back to the cube idea, but this time insists that all possible combinations of up to three colors should be included. (For instance, some cubes will be all one color, others will be two colors.) How many cubes should the kit contain?