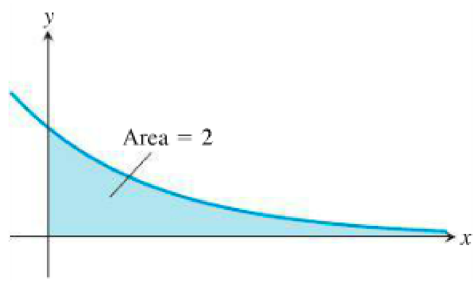
Math 142: Section 8.8 - Notes

1 Improper Integrals

Switching up the Limits of Integration Up until now, we have required two properties of *definite* integrals: 1) the domain of integration, $[a, b]$, is finite and 2) the range of the integrand is finite on this domain. We will now see what happens if we allow the domain or range to be infinite!

Infinite Limits of Integration Let's consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant.



Definition: Integrals with infinite limits of integration are **improper integrals of Type 1**.

1) If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx.$$

2) If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx.$$

3) If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dc + \int_c^\infty f(x)dx,$$

where c is any real number.
In each case, if the limit is finite we say that the improper integral ______________________________ and that the limit is the ______________________________ of the improper integral.
If the limit fails to exist, the improper integral ______________________________.

Any of the integrals in the above definition can be interpreted as an area if $f \geq 0$ on the interval of integration. If $f \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

Example 1 Evaluate

$$\int_1^\infty \frac{\ln x}{x^2} dx.$$

A Special Example For what values of p does the integral

$$\int_1^\infty \frac{dx}{x^p}$$

converge? When the integral does converge, what is its value?

Improper Integrals of Type II Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}dx.$$

Definition: Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1) If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx.$$

2) If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx.$$

3) If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

In each case, if the limit is finite we say the improper integral ______________________________ and that the limit is the ______________________________ of the improper integral.
If the limit does not exist, the integral ______________________________.