L'Hôpital's Rule - Review

Ex: $\lim_{x\to\infty} \ln(3x) - \ln(x+7)$

$= \lim_{x\to\infty} \ln\left(\frac{3x}{x+7}\right)$

$= \ln\left(\lim_{x\to\infty} \frac{3x}{x+7}\right) = \ln(3)$

$\infty - \infty$

$\lim_{x\to\infty} \frac{x+\sin(2x)}{x} \to \frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{1+2\cos(2x)}{1}$ DNE

$\lim_{x\to\infty} 1 + \frac{\sin(2x)}{x} = 1 + \lim_{x\to\infty} \frac{\sin(2x)}{x}$

$= 1 + 0 = 1$

Ex: $\lim_{x\to\frac{\pi}{2}^+} \left(\frac{\pi}{2} - x\right)\tan x \to 0\cdot\infty$

$\tan x = \frac{\sin x}{\cos x} \to \frac{1}{0} \to \infty$

$\lim_{x\to a} f(x)g(x) \to 0\cdot\infty$

$= \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}$ or $\lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}}$

$\frac{0}{0}$ $\qquad$ $\frac{\infty}{\infty}$

$\lim_{x\to\frac{\pi}{2}^+} \frac{\left(\frac{\pi}{2} - x\right)}{\frac{1}{\tan x}} \to \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to\frac{\pi}{2}^+} \frac{-1}{-\csc^2 x} = \lim_{x\to\frac{\pi}{2}^+} \frac{1}{\frac{1}{\sin^2 x}} = 1$

Ex: $\lim_{x\to 1^+} x^{1/x-1} \to 1^\infty$

$\ln\left(x^{\frac{1}{x-1}}\right) = \frac{1}{x-1}\ln x = \frac{\ln x}{x-1}$

$\lim_{x\to 1^+} \frac{\ln x}{x-1} \to \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 1^+} \frac{\frac{1}{x}}{1} = \lim_{x\to 1^+} \frac{1}{x} = 1$

So, $\lim_{x\to 1^+} x^{\frac{1}{x-1}} = e$

Math 141: Section 4.8 Antiderivatives - Notes

Definition: A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example 1 Find an antiderivative for each of the following functions:

(a) $f(x) = 2x$

$F(x) = x^2$ $\qquad$ $F_1(x) = 1 + x^2$

Check: $F'(x) = 2x$ ✓ $\qquad$ $F_1'(x) = 2x = f(x)$

(b) $g(x) = \cos x$

$G(x) = \sin x$

Check: $G'(x) = \cos x$ ✓

(c) $h(x) = \frac{1}{x} + 2e^{2x}$

$H(x) = \ln|x| + e^{2x}$ $\qquad$ Check: $H'(x) = \frac{1}{x} + e^{2x}(2)$

$= \frac{1}{x} + 2e^{2x}$

Theorem If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Example 2 Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

$F(x) = x^3 + C \qquad F(1) = -1$

$F(1) = 1^3 + c = -1$

$c = -2$

$F(x) = x^3 - 2$

Example 3 Find the general antiderivative of each of the following functions:

(a) $f(x) = x^5$

(b) $g(x) = \frac{1}{\sqrt{x}}$

(c) $h(x) = \sin x$

(d) $k(x) = e^{-3x}$

(c) $H(x) = -\cos x + C$

check:

$H'(x) = \sin x$

sinx
-cosx
cos x
-sinx

(a) $f(x) = x^5 \qquad F(x) = \frac{1}{6}x^6 + C$

(b) $g(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$G(x) = 2x^{1/2} + c$

$G(x) = 2x^{1/2} + c$

$= 2\sqrt{x} + c$

(d) $k(x) = e^{-3x}$

$K(x) = -\frac{1}{3}e^{-3x} + c$

$(x^n)' = nx^{n-1}$

$f(x) = x^n \qquad F(x) = \frac{x^{n+1}}{n+1} + c, \; n \neq -1$

$g(x) = x^{-1} = \frac{1}{x} \qquad G(x) = \ln|x| + c$

General Formulas The following table includes a list of general formulas:

TABLE 4.2 Antiderivative formulas, k a nonzero constant

Function	General antiderivative	Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	**8.** e^{kx}	$\frac{1}{k}e^{kx} + C$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$	**9.** $\frac{1}{x}$	$\ln \lvert x \rvert + C, \quad x \neq 0$
3. $\cos kx$	$\frac{1}{k}\sin kx + C$	**10.** $\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$	**11.** $\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	**12.** $\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1} kx + C, kx > 1$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	**13.** a^{kx}	$\left(\frac{1}{k \ln a}\right)a^{kx} + C, a > 0, a \neq 1$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$		

Antiderivative Linearity Rules If a function is being multiplied by a constant or combined with another function, the following rules apply:

Constant Multiple Rule

$$k f(x) \longrightarrow kF(x) + C$$

Negative Rule

$$-f(x) \longrightarrow -F(x) + C$$

Sum/Difference Rules

$$f(x) \pm g(x) \longrightarrow F(x) \pm G(x) + C$$

Example 4 Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x$$

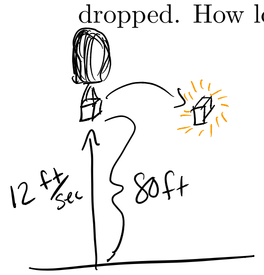
$$f(x) = 3x^{-1/2} + \sin 2x$$

$$F(x) = 3 \cdot \frac{x^{-1/2+1}}{-1/2+1} + -\frac{1}{2}\cos(2x) + C$$

$$= 3 \cdot \frac{x^{1/2}}{1/2} - \frac{1}{2}\cos(2x) + C$$

$$= 6x^{1/2} - \frac{1}{2}\cos(2x) + C$$

Example 5 - Differential Equations A hot-air balloon ascending at the rate of 12 ft/sec is at a height of 80 ft above the ground when a package is dropped. How long does it take the package to reach the ground?



Let $V(t)$ represent the velocity of the package at time t.

Let $s(t)$ represent the position (height) of the package at time t.

Acceleration due to gravity is 32 ft/sec²

$$V'(t) = -32$$

$$V(t) = -32t + C \qquad V(0) = 12$$

$$V(t) = -32t + 12$$

$$s(t) = -16t^2 + 12t + C_1 \qquad s(0) = 80$$

$$s(t) = -16t^2 + 12t + 80$$

$$-16t^2 + 12t + 80 = 0$$

$t \approx -1.89$ or 2.64

$\boxed{2.64 \text{ sec}}$

Indefinite Integrals - Definition: The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

We can restate each of the previous examples as finding the indefinite integral. Antiderivatives play a key role in computing limits of certain infinite sums, an unexpected and wonderfully useful role that is described in a central result of Chapter 5, called the *Fundamental Theorem of Calculus*.

Example 6 Evaluate

$$\int (x^2 - 2x + 5)dx.$$

$$= \frac{x^3}{3} - x^2 + 5x + C$$