Math 141: Section 4.4 Concavity and Curve Sketching - Notes

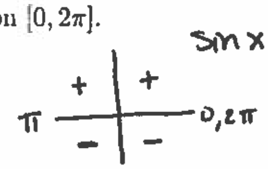
Definition: The graph of a differentiable function $y = f(x)$ is
(a) **concave up** on an open interval I if f' is increasing on I;
(b) **concave down** on an open interval I if f' is decreasing on I.

If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to the first derivative function...

Second Derivative Test for Concavity
Let $y = f(x)$ be twice-differentiable on an interval I,
1. If $f'' > 0$ on I, the graph of f over I is concave up;
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Example 1 Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

$y' = \cos x$



$y'' = -\sin x$

$y'' > 0$ on $(\pi, 2\pi)$, $y'' < 0$ on $(0, \pi)$

y is concave up on $(\pi, 2\pi)$, concave down on $(0, \pi)$

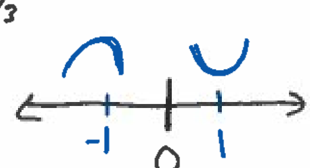
Definition: A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection (inflection point)**. At an inflection point $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

Example 2 Consider the function $f(x) = x^{5/3}$. Find the inflection point(s) if they exist.

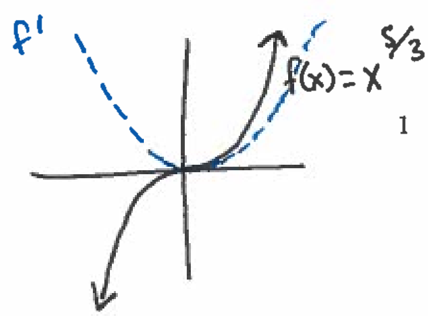
$f'(x) = \frac{5}{3}x^{2/3}$, when $x = 0$, $f'(x) = 0$ so we have a horizontal tangent at $(0,0)$

$f''(x) = \frac{10}{9}x^{-1/3} = \frac{10}{9x^{1/3}}$

Fails to exist when $x = 0$.

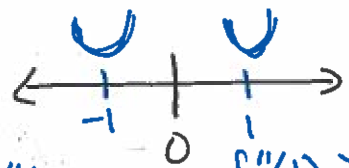


$f''(-1) < 0$ $\quad$ $f''(1) > 0$



Example 3 Consider the function $f(x) = x^4$. Find the inflection point(s) if they exist.

$f'(x) = 4x^3$ $\quad$ $f''(x) = 12x^2$, $f''(x) = 0$ when $x = 0$



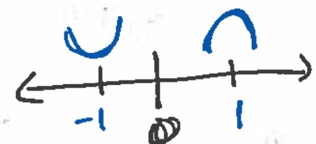
$f''(-1) > 0$ $\quad$ $f''(1) > 0$

$f(x)$ does <u>not</u> have an inflection point at $x=0$ even though $f''(x) = 0$ there.

Example 4 Consider the function $f(x) = x^{1/3}$. Find the inflection point(s) if they exist.

$f'(x) = \frac{1}{3}x^{-2/3}$ $\quad$ $f''(x) = -\frac{2}{9}x^{-5/3} = \frac{-2}{9x^{5/3}}$

$f''(x)$ is undefined when $x = 0$



$f''(-1) > 0$ $\quad$ $f''(1) < 0$

The point $(0, f(0)) = (0,0)$ is an inflection point.

Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, minimum, or neither.

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.

2. Find the derivatives y' and y''.

3. Find the critical points of f, if any, and identify the function's behavior at each one.

4. Find where the curve is increasing and where it is decreasing.

5. Find the points of inflection, if any occur, and determine the concavity of the curve.

6. Identify any asymptotes that may exist.

7. Plot key points, such as the intercepts and the points found in Steps 3-5, and sketch the curve together with any asymptotes that exist.

Example 5 Sketch the graph of

$$f(x) = \boxed{\frac{(x+1)^2}{1+x^2}} = \frac{(x+1)(x+1)}{1+x^2} = \frac{x^2+2x+1}{1+x^2}$$

Step 1: Domain: $\mathbb{R}$, $(-\infty, \infty)$; Symmetry: $f(-x) = \begin{cases} -f(x) & \text{odd} \\ f(x) & \text{even} \\ \text{neither} & \end{cases}$

No symmetry

Step 2: Find y', y''

$$f'(x) = \frac{(1+x^2)\left[(x+1)^2\right]' - (x+1)^2\left[1+x^2\right]'}{(1+x^2)^2}$$

$$= \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2(2x)}{(1+x^2)^2}$$

$$= \frac{2(x+1)\left(1+x^2-(x+1)x\right)}{(1+x^2)^2}$$

$$= \frac{2(x+1)(1+x^2-x^2-x)}{(1+x^2)^2}$$

$$= \frac{2(x+1)(1-x)}{(1+x^2)^2}$$

$$= \frac{-2(x+1)(x-1)}{(1+x^2)^2}$$

$$f'(x) = \frac{-2(x^2-1)}{(1+x^2)^2}$$

$$f''(x) = \frac{[-2(x^2-1)]'[(1+x^2)^2] - [-2(x^2-1)][(1+x^2)^2]'}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)^2 + 2(x^2-1)\cdot 2(1+x^2)(2x)}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)^2 + 8x(x^2-1)(1+x^2)}{(1+x^2)^4}$$

$$= \frac{4x(1+x^2)\left(\overset{-1-x^2}{-(1+x^2)} + \overset{+2x^2-2}{2(x^2-1)}\right)}{(1+x^2)^4}$$

$$= \frac{4x\cancel{(1+x^2)}(x^2-3)}{(1+x^2)^{\cancel{4}\,3}}$$

$$f''(x) = \frac{4x(x^2-3)}{(1+x^2)^3}$$

Step 2 $\quad f'(x) = \dfrac{-2(x^2-1)}{(1+x^2)^2}, \quad f''(x) = \dfrac{4x(x^2-3)}{(1+x^2)^3}$

Step 3 Find the critical points
first derivative

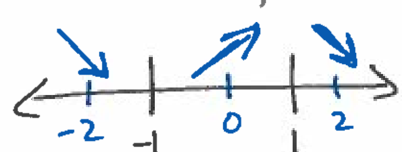
$x = 1, \ x = -1 \quad (f'(x) = 0)$

Second Derivative Test: $f''(1) < 0$, the function ($f(x)$) is concave down at $x=1$

so, local max at $x=1$

$f''(-1) > 0$, the function is concave up at $x=-1$, So, local min at $x=-1$.

Step 4: Find intervals of increase, decrease
(first derivative)



$f'(-2)<0 \quad f'(0)>0 \quad f'(2)<0$

Increase: $(-1, 1)$

Decrease: $(-\infty, -1) \cup (1, \infty)$

Step 5: Find inflection points & concavity.

$$f''(x) = \frac{4x(x^2-3)}{(1+x^2)^3} \qquad f''(x)=0 \text{ when } 4x(x^2-3)=0$$

$$4x = 0 \quad \text{or} \quad x^2-3=0$$

$$x=0 \qquad\qquad x=\sqrt{3}, -\sqrt{3}$$

$f''(-2)<0$

$f''(-1)>0$

$f''(1)<0$

$f''(2)>0$

Inflection Points at

$x=-\sqrt{3}, 0, \sqrt{3}$

(Plug in to original function to find coordinates)

Step 6: Find asymptotes

$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2}$$

$\lim_{x\to\pm\infty} f(x) = 1$ Horizontal Asymptote at $y=1$

Step 7: Plot all the information from steps 1-6.

$$f(x) = \frac{(x+1)^2}{1+x^2}$$