# Measure Theory

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June 14, 2023

This is a note on a short talk about the basics of measure theory, given by Pico Gilman.

**Definition 0.1.** We define a Lebesgue measure $\mu : 2^{\mathbb{R}} \to \mathbb{R}^{\geq 0} \cup \infty$.

**Example 0.2**

$\mu([a, b]) = b - a$. $\mu([a, b)) = b - a$.

**Example 0.3**

$\mu(A \cap B) = \mu(A) + \mu(B)$ when $A \cap B = \emptyset$.

**Theorem 0.4**

$\sum \mu(A_i) = \mu(\cup A_i)$.

**Theorem 0.5**

For the Cantor set $C$, we have $\mu(C) = 0$.

**Theorem 0.6**

If $B \subseteq A$ and $\mu(A) = 0$ then $\mu(B) = 0$.

**Definition 0.7.** The Borel $\sigma$-algebra is the smallest subset $\subseteq 2^{\mathbb{R}}$ such that it is

- closed under $\cup$, $\cap$, $\cdot^{\complement}$
- contains all intervals.

Borel $\sigma$-algebra is basically everything you care to have a measure of.

**Exercise 0.8.** Is it possible to be "the same" as the Cantor set $C$ topologically, but have positive measure?

*Proof.* Let $X \subseteq \mathbb{R}$ be countable. We want to show that $\mu(X) = 0$.

Let $X = \{x_1, x_2, \dots\}$, and $A_i = \{x_i\} = [x_i, x_i]$. Hence $\mu(A_i) = 0$. Now, $\mu(A) = \mu(\cup A_i) = \sum 0 = 0$. $\square$

**Theorem 0.9**

For Cantor set $C$, since $C$ is in the Borel $\sigma$-algebra, $\mu(C) = 0$.

*Proof.* It suffices to show that $\forall \epsilon > 0$, $\mu(C) < \epsilon$.

Let $\epsilon > \frac{2^{n-1}}{3^n}$.

Since $C \subset C_n$ where $C_n$ is the $n^{\text{th}}$ Cantor set and $\mu(C_n) = (\frac{2}{3})^n$.

Hence, $\mu(C_n) = \mu(C) + \mu(C_n - C)$ and thus $\mu(C_n) \geq \mu(C)$, equivalently, $e > \frac{2^{n-1}}{3^n} \geq \mu(C)$, implying $\mu(C) = 0$. $\square$

**Example 0.10**

There are also weird measures like

$$\frac{1}{\ln 2} \int_A \frac{1}{1+x} \; A \subseteq [0,1]$$

where the points count less and less as you go towards 1.