The second-most beautiful mathematical argument

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This is a note on a lecture on the back-and-forth arguments, given by Pico Gilman.

Theorem 0.1

Let v_1, v_2, $\ldots$be a countable set of vertices with probability p, put and edge between v_i and v_j. Let such a graph be G_p. Then, $G_{p'} \cong G_p$ for all $0 < p$, $p' < 1$.

Do the back-and-forth construction, which works because we have countably many vertices. Make an isomorphism. Done.

Theorem 0.2 (Hilbert, 1885)

If $(X, <)$ is a total order, such that

- $\forall x$, $\exists a$, b such that $a < x < b$. (Unbounded)
- $\forall x$, y s.t. $x < y$, $\exists z$ such that $x < z < y$. (Dense)

and X is countable, then $X \cong \mathbb{Q}$.

Proof. Let $f : \mathbb{N} \to X$ and $g : \mathbb{N} \to \mathbb{Q}$ be bijections.

On step i, pick $q_i \in \mathbb{Q}$ such that $\forall j < i$, $q_i \neq q_j$ and $q_i \neq g(j)$, which also satisfies the following properties:

- $q_i < q_j$ iff $f(i) < f(j)$
- $q_i < g(j)$ iff $f(i) < h_j$

Doing this countably many times, we get an order-preserving bijection. $\square$

Corollary 0.3

Let $\overline{\mathbb{Q}}$ be the set of real algebraics. Then, $\overline{\mathbb{Q}}$ is countable (because $\mathbb{Q}[x]$ is countable), and moreover, there exists an order-preserving bijection between $\mathbb{Q}$ and $\overline{\mathbb{Q}}$.

Properties of $\mathbb{R}$:

(i) $\mathbb{Q} \subseteq \mathbb{R}$, and $\mathbb{Q}$ is dense in $\mathbb{R}$.

(ii) $\mathbb{R}$ has suprema.

(iii) If $\{\mathcal{U}_i\}_{i\in\mathcal{I}}$ is a family of disjoint open intervals, then $|\mathcal{I}| \leq |\mathbb{N}|$
(Because every interval has $q \in \mathbb{Q}$ due to $\mathbb{Q}$ being dense in $\mathbb{R}$.)

(iv) $\mathbb{R}$ is unbounded and dense.

> **Remark.** Just define $\mathbb{R}$ using Dedekind cuts!

In a total order X, dense and supremum implies infimum.

Proof. Let $A \subseteq X$ such that $y \leq A$. Let $L = \{x \mid x \leq A\}$. Then, $\sup(L) = z$, so $z \leq A$, hence A has an infimum. □

Theorem 0.4
ZFC cannot prove that you can or cannot get $\mathbb{R}$.

Definition 0.5 (Partially ordered set)**.** A poset $(T, \leq)$ is a tree of $\forall t \in T$ such that $s(t) := \{x \in T \mid x \leq t\}$. Then, $s(t)$ is well-ordered by $\leq$.

Definition 0.6 (Suslin line)**.** A *Suslin line* is an interval that satisfies the following three condition:

- unbounded and dense
- sup and inf exist
- no uncountable disjoint open intervals (CCC)
- not separable

Definition 0.7 (Suslin tree)**.** A *Suslin tree* is a tree of height ω_1 that satisfies the following three conditions:

- no uncountable antichains
- no uncountable branches
- height is uncountable

where $\omega_1 = \bigcup_{\alpha \text{ countable}} \alpha$ (the smallest uncountable ordinal).

Theorem 0.8
A Suslin line $\iff$ a Suslin tree.

Proof. Proof of ($\implies$). Let $(X, <)$ be a Suslin line. Our tree will be a subset of $\mathcal{O}_X = \{(a,b) \mid a, b \in X\}$ under $\subseteq$. Let $\mathcal{T}_0 = \emptyset$, and $\mathcal{T}_{i+1} = \mathcal{T}_i \cup \{(a,b) \mid (a,b) \cup$ endpoints of the leements of $\mathcal{T}_i\}$. Then, $\mathcal{T}_\alpha = \bigcup_{\beta<\alpha} \mathcal{T}_\beta$. Let $\mathcal{T} = \mathcal{T}_{\omega_1}$.

Claim 0.9 — $h(\mathcal{T}) \leq \omega_1$.

Proof. $h(\mathcal{T}) = \bigcup_{t \in I} h(t)$. Suppose $h(t) \geq \omega_1$. Take $t = (a, b)$. Consider $(a_0, b_0) \supsetneq (a_1, b_1) \supsetneq (a_2, b_2) \supsetneq \dots$ indexed by ω_1. At least one of $a_0, a_1, \dots$ or $b_0, b_1, \dots$ has to be uncountable (since otherwise the entire set is countable as well). WLOG, say $a_0, a_1, \dots$ was uncountable. But $(a_0, a_1), (a_1, a_2), \dots$ are disjoint, uncountable, and open, contradiction. ■

Claim 0.10 — $h(\mathcal{T}) > \alpha \quad \forall \alpha \in \omega_1$.

Proof. Suppose not, that is, $\exists \alpha \in \omega_1$. $\exists \beta \leq \alpha$ such that $|\{t \in \mathcal{T} : h(t) = \beta\}| \geq |\omega_1|$. But then at some point we get an uncountable number of elements for some height, which sucks. There exists a surjection from β to $\gamma < \beta$, but then there must exist at least one element that has uncountable pre-images, which is a contradiction to the condition of a Suslin tree. ■

Claim 0.11 — There are no uncountable branches.

Proof. $(a_0, b_0) \supsetneq (a_1, b_1) \supsetneq \dots$ (same idea). ■

Claim 0.12 — There are no uncountable antichains.

Proof. Dirty. ■

■

Proof of ($\Longleftarrow$). Prove $\mathcal{T}$.

Claim 0.13 — $\exists \mathcal{T}'$ that is Suslin such that $|\mathrm{succ}(t)| = |\omega_1|$ and $\forall t \; |\{x \mid x > t\}| = |\omega_1|$.

Let X be the set of maximal branches. For each branch B such that B cannot be *extended downwards*, i.e., $\nexists t$ such that both $\exists b \in B$ s.t. $t < b$ and $\{t\} \cup B$ is a branch. Let $\mathrm{succ}(B) = \{a \in \mathcal{T} \mid a > B, \nexists c : a > c > B\}$. Biject $\mathrm{succ}(B)$ with $\mathbb{N}$. Say $B_1 < B_2$ iff $B = B_1 \cap B_2$ and $\mathrm{succ}(B) \cap B_1 < \mathrm{succ}(B) \cap B_2$. Then, $<$ is total, since it has the entire set and everything satisfies totality (suffices to check transitivity).

Claim 0.14 — Unbounded.

Proof. Let B at level 0 be n. Then, construct B' such that B' is $n+1$ at level 0. Then, we naturally have $B < B'$. For the first time B is nonzero, takeB' to be 0 for $\alpha + 1$ levels, and nonzero at $\alpha + 2$ and anything from there. ■

Claim 0.15 — Dense.

Proof. For $B_1 < B_2$, and say they disagree at α. Then, say $B_1 = n$ at level α and $B_2 = m > n$ at level α. Then, take $B_3 = B_1 \cap B_2$ until levels $\alpha - 1$, then pick n at level α, then pick $> B_1$ above level α. Then, $B_1 < B_3 < B_2$, so we are done. ■

Claim 0.16 — CCC.

Proof. Let $I_1 = (A_1, B_1)$ and $I_\alpha = (A_\alpha, B_\alpha)$. Pick $x_\alpha \in (A_\alpha, B_\alpha)$. Let $t_\alpha \in X$ such that t_1 is above the last agreement of A_1 and B_1. Then, $\{t_\alpha\}$ is an uncountable antichain. Take $t_\alpha \leq t_\beta$. Then, $(A_\beta, B_\beta) \ni X_\beta \ni t_\beta \geq t_\alpha$, so $t_\alpha \in X_\beta$. Then, $X_\beta \in (A_\alpha, B_\alpha)$, so they have a common element, contradiction. ■

Claim 0.17 — Not dense.

Proof. It suffices to show that no uncountable set is dense. Let $Y \subseteq X$ be countable. Then, $\forall y \in Y$, $h(y) \in \omega_1$.We know that $\bigcup_{y \in Y} h(y) = \beta \in \omega_1$. WLOG $B_1 < B_2$. Take B_1, B_2 be maximal branches containing t, which implies $(B_1, B_2) \cap Y = \emptyset$, so Y is not dense. ■

■

□

Theorem 0.18 (Aronzsajin tree)

For any infinite cardinal, $\nexists$ Aronzsajin Tree with $h = |\mathbb{N}|$.
$\exists$ Aronzsajin Tree with $h - \aleph_1$.
The continuum hypothesis gives that $\exists A.T.$ with $h = \aleph_2$.
The $V = L$ gives that $\exists A.T >$ for every succ cardinal.

Theorem 0.19

A well-ordered set is isomorphic to a unique ordinal number.

Hence, we may define the height of t, denoted as $h(t)$, to be the ordinal that $s(t)$ is order isomorphic to. If T is a tree, then the height of T, denoted as $H(T)$, is

$$\bigcup_{t \in T} h(t)$$

which is uncountable.

Theorem 0.20 (Suslin, 1925)

A Suslin tree exists if and only if $\exists X$ satisfying (ii), (iii), and (iv) but not (i).

Proof. We prove ($\Longrightarrow$) first.

Proof of ($\Longrightarrow$). If a Suslin tree exists, then $\exists X$ satisfying (ii), (iii), and (iv) but not (i). Let $\mathcal{T}$ be a Suslin tree, and X be a countable, unbounded, dense set. Then, $X \cong \mathbb{Q}$. Hence, X is dense in Y, which is a "big" total order, contradiction. ■

Proof of ($\Longleftarrow$). If $\exists X$ satisfying (ii), (iii), and (iv) but not (i), then we may construct a Suslin tree. ■

□