

Measure Theory

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This is a note on a short talk about the basics of measure theory, given by Pico Gilman.

Definition 0.1. We define a Lebesgue measure $\mu : 2^{\mathbb{R}} \rightarrow \mathbb{R}^{\geq 0} \cup \infty$.

Example 0.2

$$\mu([a, b]) = b - a. \quad \mu([a, b)) = b - a.$$

Example 0.3

$$\mu(A \cap B) = \mu(A) + \mu(B) \text{ when } A \cap B = \emptyset.$$

Theorem 0.4

$$\sum \mu(A_i) = \mu(\cup A_i).$$

Theorem 0.5

For the Cantor set C , we have $\mu(C) = 0$.

Theorem 0.6

If $B \subseteq A$ and $\mu(A) = 0$ then $\mu(B) = 0$.

Definition 0.7. The Borel σ -algebra is the smallest subset $\subseteq 2^{\mathbb{R}}$ such that it is

- closed under \cup, \cap, \cdot^c
- contains all intervals.

Borel σ -algebra is basically everything you care to have a measure of.

Exercise 0.8. Is it possible to be “the same” as the Cantor set C topologically, but have positive measure?

Proof. Let $X \subseteq \mathbb{R}$ be countable. We want to show that $\mu(X) = 0$.

Let $X = \{x_1, x_2, \dots\}$, and $A_i = \{x_i\} = [x_i, x_i]$. Hence $\mu(A_i) = 0$. Now, $\mu(A) = \mu(\cup A_i) = \sum 0 = 0$. \square

Theorem 0.9

For Cantor set C , since C is in the Borel σ -algebra, $\mu(C) = 0$.

Proof. It suffices to show that $\forall \epsilon > 0, \mu(C) < \epsilon$.

Let $\epsilon > \frac{2^{n-1}}{3^n}$.

Since $C \subset C_n$ where C_n is the n^{th} Cantor set and $\mu(C_n) = (\frac{2}{3})^n$.

Hence, $\mu(C_n) = \mu(C) + \mu(C_n - C)$ and thus $\mu(C_n) \geq \mu(C)$, equivalently, $e > \frac{2^{n-1}}{3^n} \geq \mu(C)$, implying $\mu(C) = 0$. \square

Example 0.10

There are also weird measures like

$$\frac{1}{\ln 2} \int_A \frac{1}{1+x} \quad A \subseteq [0, 1]$$

where the points count less and less as you go towards 1.