Analysis & Topology

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This is a note on a series of lectures on real analysis and topology, given by Xinkai Wu, Mustafa Nawaz, and Pico Gilman, at the 2023 Ross Mathematics Program at Otterbein College.

§1 Terminology

Definition 1.1. $\mathbb{R}$ is the *completion* of $\mathbb{Q}$.

Claim — There is a set $\mathbb{R} \supset \mathbb{Q}$ such that $\mathbb{R}$ is *totally ordered* and *complete*.

Definition 1.2. A *set* is a collection of elements. (Naïve set theoretic definition of a set)

Definition 1.3. If S is a set, we write $x \in S$ to indicate an element x is in the set S.

Definition 1.4. If A, B are sets, then we write $A \subset B$ if $\forall x \in A \implies x \in B$.
For $A \supset B$, we take the dual definition.

Definition 1.5. We say $A = B$ if $A \supset B$ and $A \subset B$.

Definition 1.6. Let R be a ring. We define $R \times R = \{(r_1, r_2) \mid r_1, r_2 \in R\}$ and $(r_1, r_2) + (r_3, r_4) = \{(r_1 + r_3, r_2 + r_4) \mid (r_1, r_2), (r_3, r_4) \in R \times R\}$.

Definition 1.7. We have an equivalence relation $\sim$ on $\mathbb{Q} : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, where $(a, b) \sim (c, d) \iff ad - bc = 0$.

Example 1.8

For example, $(1, 2) \sim (3, 6)$.

Exercise 1.9. Check that $\sim$ is indeed an equivalence relation.

Theorem 1.10

$\mathbb{Q}$ is totally ordered with $<$, $>$, and $=$.

Definition 1.11. If S is an ordered set, $E \subset S$, and $\exists \alpha \in S$ s.t. $\forall x \in E$, $x \leq \alpha$, then we say α is an *upper bound* of E. If $\nexists \gamma \in S$ s.t. $\gamma < \alpha$, then we say α is a *least upper bound*.
We define the *greatest lower bound* dually.

Definition 1.12. We define a set $\mathbb{F}$ to be a *field* if it is a nontrivial commutative ring such that every nonzero element has an inverse.

Example 1.13

$\mathbb{Q}$ is a field.

Definition 1.14. We define the field of complex numbers $\mathbb{C} := \mathbb{R}[x]/(x^2+1)$ where x^2+1 is an ideal in $\mathbb{R}[x]$, equipped with the operations $+ : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ and $\cdot : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ where $(a,b)+(c,d) = (a+c, b+d)$ and $(a,b)\cdot(c,d) = (ac-bd, ad+bc)$.

§2 Point-Set Topology

In $\mathbb{R}$, a set $\mathcal{O}$ is open if $\forall x \in \mathcal{O}$, $\exists \delta > 0$ s.t. $(x-\delta, x+\delta) \in \mathcal{O}$.

A set $\mathcal{F}$ is closed if $\mathcal{F}^{\complement}$ is open.

Example 2.1

$\mathbb{N}$ is closed since it does not have any limit points.

Theorem 2.2

For $\{\mathcal{O}_i\}$, an arbitrary union $\bigcup_i \mathcal{O}_i$ is also open.

Corollary 2.3

For $\{\mathcal{F}_i\}$, an arbitrary (possibly uncountable) intersection $\bigcap_i \mathcal{F}_i$ is also closed.

Theorem 2.4

For $\{\mathcal{O}_i\}_{i=1}^n$, a finite intersection $\bigcap_i \mathcal{O}_i$ is also open.

Example 2.5

For an infinite intersection, $\{(-\frac{1}{n}, \frac{1}{n})\}_{n\in\mathbb{N}}$ consists solely of the point 0, which is closed in $\mathbb{R}$.

Definition 2.6. We say that P is a limit point in a set $\mathcal{S} \subseteq \mathbb{R}$ if for any $\varepsilon > 0$, in an open neighborhood of radius ε, you can find a distinct point other than p.

Example 2.7

0 is the limit point of $\{\frac{1}{n}\}_{n\in\mathbb{N}}$.

Not every interval has a limit point; think of $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ in $\mathbb{Q}$.

Definition 2.8. A set is *closed* iff it contains all of its limit points.

Example 2.9

There are *clopen* sets (think of $\mathbb{R}$, also $\emptyset$), also sets that are neither closed nor open (think of $(0, 1]$, for example).

Example 2.10

Is $\mathbb{Q}$ closed? Think of the sequence that goes to $\sqrt{2}$, whose elements are all rationals, yet its limit point is irrational. Hence, $\mathbb{Q}$ is not closed.

Definition 2.11. We call a set A to be *disconnected* in $\mathbb{R}$ if one can find two disjoint $\exists U, V$ in $\mathbb{R}$ such that $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$ and $A = (A \cap U) \cup (A \cap V)$.

Example 2.12

$\mathbb{Q}$ is disconnected; for example, consider $U = (-\infty, \sqrt{2})$ and $V = (\sqrt{2}, \infty)$.

Example 2.13

The cantor set C is disconnected; for example, think of $\frac{1}{2}$.

Example 2.14

In $\mathbb{R}$, only an interval is connected. The empty set is not.

Definition 2.15. The closure $\overline{S}$ of a set S in $\mathbb{R}$ is the smallest closed set that contains S.

Example 2.16

The closure of $\mathbb{Q}$ is $\mathbb{R}$.

Example 2.17

The closure of $[0, 1)$ is $[0, 1]$.

Definition 2.18. A set S is *dense* in $\mathbb{R}$ if its closure equals $\mathbb{R}$.

Example 2.19

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 2.20. A set S is *dense* in $\mathbb{R}$ if $\forall x \in \mathbb{R}$, $\forall \delta > 0$, $(x - \delta, x + \delta) \cap S \neq \emptyset$.

Example 2.21

The cantor set C is not dense in $[0, 1]$. Actually, it is *nowhere dense* in $[0, 1]$. Every single time you decrease the maximum length of any interval by $\frac{1}{3}$, hence for any open subset it is not dense.

Definition 2.22. We call a set $S \subseteq X$ *nowhere dense* in X if S is not dense in any open subset of X.

Definition 2.23. A set S is *sequentially compact* if for any sequence $\{s_i\}$ where $s_i \in S$, $\{s_i\}$ contains a convergent subsequence.

Theorem 2.24 (Bolzano-Weierstrass theorem)
In $\mathbb{R}^n$, a set S is sequentially compact if and only if it is closed and bounded.

Exercise 2.25. Show that the order topology on $\mathbb{Q}$ is disconnected.

Proof. Consider $\sqrt{2}$, and the two open intervals adjacent to them. They are disconnected. Hence we are done. $\square$

Exercise 2.26. Let $f : X \to Y$ be continuous, and $X' \subseteq X$. Show that if $f' = f|_X$, then f is continuous.

Definition 2.27. Let (X, τ) be a topology. We say X is separable if $\exists Y \subseteq X$ and $|Y| \leq |\mathbb{N}|$ such that $\overline{Y} = X$.

Exercise 2.28. $\mathbb{R}^n$ is separable.

Exercise 2.29. If X, Y are separable, then $X \times Y$ is separable.

Definition 2.30. Let (X, τ_X) and (Y, τ_Y) be topological spaces. Then, the *product topology*, denoted as $X \times Y$, is the smallest topology such that $U \times V$ is open for $U \leftarrow \tau_X$ and $V \leftarrow \tau_Y$.

Exercise 2.31. Let U, V be closed in X, Y respectively, then $U \times V$ is closed in the product topology.

Exercise 2.32. Let $f : X \to Y$ and $g : X \to Z$ be continuous. Then, show that $f \times g$ is continuous, where $f \times g : X \to Y \times Z$ and $x \mapsto (f(x), g(x))$.

Exercise 2.33. Show that $f : X \to Y$ is continuous iff for V closed in Y, $f^{-1}(V)$ is closed.

Exercise 2.34. Let $X' \subseteq X$, $Y' \subseteq Y$, then show that $X' \times Y'$ is same as the subspace topology on $X \times Y$ of $X' \times Y'$.

Exercise 2.35. Show that $[a, b)$ is in the Borel σ-algebra.

Exercise 2.36. Find the cardinality of the Borel σ-algebra.

§3 Topological spaces

Definition 3.1. A topological space (X, τ) is a set X equipped with a topology τ.

Definition 3.2. A topology τ is a set of subsets of X that satisfies the following properties:

1. $\{\emptyset, X\} \subseteq \tau$.
2. Arbitrary union: $\bigcup_{i \in I} X_i \in \tau$
3. Finite intersection: $\bigcap_{i=1}^{n} X_i \in \tau$.

Example 3.3

For elements A, B, C, we define $2^{\{A,B,C\}} = \{\emptyset, \{A\}, \{B\}, \{C\}, \dots, \{A, B, C\}\}$.

Then, $\{\{A, B, C\}, \emptyset\}$ is called the *trivial topology*.

Moreover, $2^{\{A,B,C\}}$ is called the *discrete topology*.

Example 3.4

The set $S = \{(-\infty, b), (\infty, \infty), (a, \infty), (a, b) \mid a, b \in \mathbb{R}\}$ has a topology τ. (We define $(a, b) = \emptyset$ if $b < a$.) We call this the *standard topology* on $\mathbb{R}$.

Definition 3.5. A topological space (X, τ) is called a *metric space* if it has a metric $d : X \times X \to \mathbb{R}^{\geq 0}$.

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x) \quad \forall x, y \in X$
3. $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$ (triangle inequality)

We denote such a metric space as (X, d).

Corollary 3.6

A metric induces a topology τ, where τ is the smallest set which contains all open balls $B_{\mathbf{x}}(r)$ under finite intersection and arbitrary union, where $B_{\mathbf{x}}(r) := \{(\mathbf{x}, \mathbf{y}) \mid d(\mathbf{x}, \mathbf{y}) < r\}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Definition 3.7. $C^k[0, 1]$ is defined as the set of continuous functions on the closed interval $[0, 1]$ with a continuous k^{th} derivative. We denote $C^0[0, 1] = C[0, 1]$, i.e., the set of continuous functions on the closed interval $[0, 1]$.

Example 3.8

Here are some examples of metric spaces:

- $d(x, y) = |x - y|$ is a metric.
- $\dfrac{d(x, y)}{1 + d(x, y)}$ is a metric if $d(x, y)$ is a metric.
- For $f, g \in C[0, 1]$, then
$$d(f, g) = \int_0^1 |f - g| dx$$
is a metric.
- The discrete metric
$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Definition 3.9. A metric space (X, d) is *complete* if all Cauchy sequences necessarily converge.

Theorem 3.10

Each metric space has a completion. That is, for all metric spaces (X, d), there exists a metric space (Y, d) where Y is complete.

Example 3.11

$\mathbb{Q} \subseteq \mathbb{R}$. ($\mathbb{R}$ is the completion of $\mathbb{Q}$.)

Theorem 3.12 (Baire category theorem (BCT))

For any metric space (X, d), a countable set $\{X_i\}_{i=1}^{\infty}$, $X_i \subseteq X$, and X_i being an open dense set in X, we have

$$\bigcap_{i=1}^{\infty} X_i \text{ is dense in } X$$

A metric space with this property is called a Baire space.

Proof. For an open subset $W \subseteq X$, we may construct a closed ball $\overline{B}(x_1, r_1) \subseteq X_1 \cap W$, since we may take its radius to be slightly smaller than its open counterpart, which always exists. Then, for all $n \geq 2$, $\overline{B}(x_n, y_n) \subseteq X_n \cap \overline{B}(x_{n-1}, r_{n-1})$ for $0 < r_n < \frac{1}{n}$, that is, we construct a decreasing sequence of closed balls. Then, consider $\{\overline{B}(x_i, y_i)\}_{i=1}^{\infty}$, then because X is a complete metric space, the sequence is Cauchy thus convergent, and the sequence $\{y_i\} \to y$ has its limit point $y \in X$ residing in each of the closed balls. Hence, $y \in (\bigcap X_i) \cap W$, so $\{X_i\}_{i=1}^{\infty}$ is dense in X, and we are done. □

Corollary 3.13

Liouville's approximation theorem can be proved via Baire category theorem.

Definition 3.14. A σ-algebra (X, Σ) on a set X is a nonempty collection Σ of subsets of X with the following properties:

- $\{\emptyset, X\} \subset \Sigma$
- closed under countable unions
- closed under countable intersections
- closed under complement

The ordered pair (X, Σ) is called a measurable space.

§4 Cardinalities and Equinumerosity

Definition 4.1. A set X is *countably infinite* if there is a bijection between X and $\mathbb{N}$.

Theorem 4.2 (Cantor-Schröder-Bernstein theorem)
$\forall X \nexists f : X \to 2^X$ such that f is surjective.

Proof. Let $f : X \to 2^X$ be a surjective function.
Let $A \subset X$ s.t. $A = \{x \in X \mid x \notin f(x)\}$.
Hence, $\exists y \in X$ such that $f(y) = A$, which implies $f(y) = \{x \in X \mid x \notin f(x)\}$.
We divide cases into whether y is in the RHS.
If $y \in f(y)$, then $y \notin f(y) = A$ by the given equation, contradiction.
If not, then $y \in f(y) = A$, contradiction.
Thus, we are done. □

Remark. Axiom of choice.

Example 4.3
$\nexists f : \mathbb{N} \twoheadrightarrow \mathbb{R}$, i.e., that f is surjective.

Definition 4.4. Let X, Y be sets. $|X| \le |Y|$ means there is an injection from $X \to Y$. Here, the binary operation $\le$ is a total order.

Example 4.5 (Beth numbers)
We may create the sequence of sets $|\mathbb{N}| < |\mathbb{R} = 2^{\aleph_0}| < |2^{\mathbb{R}}| < |2^{2^{\mathbb{R}}}| < \dots$.
The sequence of beth numbers is defined by setting $\beth_0 = \aleph_0$ and $\beth_{k+1} = 2^{\beth_k}$.

Theorem 4.6
If $|X| \le |Y|$ and $|Y| \le |X|$, then $|X| = |Y|$.

Definition 4.7. We define an orbit $\text{orb}_f(x)$ of an element $x \in X$ to be the set of all the elements we get when we apply a function $f : X \to X$ n times or undo a function n times. Formally, $\text{orb}_f(x) = \{x\} \cup \{f^n(x)\} \cup \{f^{-n}(x)\}$.

Proof. Note that $f : X \to Y$ and $g : Y \to X$ are both injective. Consider $f \circ g : Y \to Y$ which is injective. Let $y \in Y$. Then, we have

$$\text{orb}_{f\circ g}(y) = \{y\} \cup \{(f \circ g)^n(y)\} \cup \{(f \circ g)^{-n}(y)\}$$

and

$$\text{orb}_{g\circ f}(x) = \{x\} \cup \{(g \circ f)^n(x)\} \cup \{(g \circ f)^{-n}(x)\}$$

How does $\text{orb}(y)$ and $\text{orb}(g(y))$ relate? Take Axiom of Choice for each orbit, then inject. □

Exercise 4.8. Prove $|C[0,1]| \le |\mathbb{R}|$.

Proof. Let $f \in C[0,1]$. For some $x \in [0,1]$ s.t. $g(x) \neq f(x)$ and $g \in C[0,1]$, consider $f' := f|_{\mathbb{Q}\cap[0,1]}$ and $g' := g|_{\mathbb{Q}\cap[0,1]}$ and $f' = g'$. But if there exists a sequence $a_n \to x$ s.t. $f(a_n) \to f(x)$ and $g(a_n) \to g(x)$, then at some point they must differ, otherwise $f(x) = g(x)$, which is a contradiction. Thus, $|C[0,1]| \le |\mathbb{R}^{\mathbb{Q}}|$. Hence, ISTS $|\mathbb{R}^{\mathbb{Q}}| = |\mathbb{R}|$. But we have $|\mathbb{R}^{\mathbb{Q}}| = |(2^{\aleph_0})^{\aleph_0}| = |2^{\aleph_0 \times \aleph_0}| = |2^{\aleph_0}| = |\mathbb{R}|$. □

Exercise 4.9. Let x be countably infinite. Prove $|x^x| = |2^x|$.

Proof. We have $|x| = \aleph_0$. Now, $|x^x| = |\aleph_0{}^{\aleph_0}| = |2^{\aleph_0}|$, so we are done. □

§5 Sequences and series

Definition 5.1. We say $\{a_n\}_{n=1}^{\infty}$ *converges* to a limit a if $\forall \varepsilon > 0$, $\exists N$ such that $d(a_n, a) < \varepsilon \quad \forall n > N$.

Theorem 5.2

If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.

Proof. Let $\varepsilon > 0$.

Then, because a_n converges to a, $\exists N_1$ such that $d(a_n, a) < \frac{\varepsilon}{2} \quad \forall n > N_1$.

Similarly, because b_n converges to b, $\exists N_2$ such that $d(b_n, b) < \frac{\varepsilon}{2} \quad \forall n > N_2$.

Observe that $d(a_n, a) + d(b_n, b) \geq d(a_n + b_n, a + b)$ by the triangle inequality, so $d(a_n + b_n, a + b) < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall n > \max(N_1, N_2)$.

Hence, $a_n + b_n$ converges to $a + b$. □

Theorem 5.3

If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.

Proof. Let $\varepsilon > 0$.

Then, because a_n converges to a, $\exists N_1$ such that $d(a_n, a) < \frac{\varepsilon}{2} \quad \forall n > N_1$.

Similarly, because b_n converges to b, $\exists N_2$ such that $d(b_n, b) < \frac{\varepsilon}{2} \quad \forall n > N_2$.

Observe that $d(a_n b_n, ab) = d((a - a_n)(b - b_n), a(b_n - b) + a(a_n - a)) \leq d((a - a_n)(b - b_n), 0) + d(a(b_n - b), 0) + d(b(a_n - a), 0)$ by the triangle inequality, so $d(a_n b_n, ab) < \varepsilon \quad \forall n > \max(N_1, N_2)$.

Hence, $a_n b_n$ converges to ab. □

Theorem 5.4

If $a_n \to a$ and $b_n \to b$ (with $b_n \neq 0$ and $b \neq 0$), then $a_n / b_n \to a/b$.

Proof. Similar to above. □

Definition 5.5. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be *Cauchy* if $\forall \varepsilon > 0\ \exists N$ s.t. $\forall n, m > N$, $d(a_n, a_m) < \varepsilon$.

Theorem 5.6 (Cauchy implies convergence)

Let $\{a_n\}_{n=1}^{\infty}$ be Cauchy, then a_n converges.

Proof. Next class. □

Theorem 5.7 (Convergent sequences are Cauchy)

Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence, then a_n is Cauchy.

Proof. Obvious. □

§6 Fourier series

Definition 6.1 (Length of a curve)**.** Let $f : [0,1] \to \mathbb{R}^n$ be continuous.

Then, we define

$$\text{len}(f) = \lim_{(d_{i+1}-d_i)\to 0} \sum \|f(d_{i+1}) - f(d_i)\|$$

Definition 6.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period τ, i.e., $f(x+\tau) = f(x) \quad \forall x$.

Consider the complex integral

$$c_n = \frac{2\pi}{\tau} \int_0^\tau e^{\frac{-in\phi 2\pi}{\tau}} f(\phi) d\phi$$

then we have the following identity

$$f(\theta) = \sum_{n\in\mathbb{Z}} e^{in\theta} c_n$$

If $\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$, then equality actually holds.

Theorem 6.3

Let $f : [0, 2\pi] \to \mathbb{R}$ be a continuous function. Then,

$$\forall \varepsilon > 0 \; \exists g \in \mathbb{R}[x] \text{ s.t. } d(g,f) < \varepsilon$$

Theorem 6.4

If $f(0) = f(2\pi)$, then $\forall \varepsilon > 0 \; \exists g \in \text{span}(e^{in\theta})$ s.t. $d(g,f) < \varepsilon$.

Definition 6.5. A perfect set is a closed set without isolated points.

Definition 6.6. We say $x \in X$ is *isolated* if for $X \subseteq S$, $\exists \mathcal{O}$ open in S such that $\mathcal{O} \cap X = \{x\}$.

Example 6.7

There does not exist an isolated point in $\mathbb{Q}$.

Example 6.8

The Cantor set is a perfect set.

Definition 6.9. We say a set is *totally disconnected* if the "largest" connected set is a single point.

Definition 6.10. For a subset X with respect to the parent set Y, we say that X has a subspace topology τ in Y where $\tau = \{\mathcal{O} \cap X \mid \mathcal{O} \text{ open in } Y\}$ for $\mathcal{O} \subseteq Y$ and $\mathcal{O} \subseteq X$.

Exercise 6.11. Verify that τ is a topology in X.

Lemma 6.12

Let Y have topology τ. Let $X \subseteq Y$ and $\tau_x = \{\mathcal{O} \cap X \mid \mathcal{O} \in \tau\}$. Let $\{u_i\} \subseteq \tau_x$. We have $u_i = \mathcal{O}_i \cap X$ for $\mathcal{O}_i \in \tau$. Then, $\bigcup_{i \in I} u_i = [\bigcup_{i \in I} \mathcal{O}_i] \cap X$. Let $u, v \in \tau_x$.

It suffices to show that $U \cap V \in \tau_x$. Then, $U = \mathcal{O}_1 \cap X$ and $V = \mathcal{O}_2 \cap X$, so $U \cap V = (\mathcal{O}_1 \cap \mathcal{O}_2) \cap X$, hence we are done.

Theorem 6.13

Every nonempty perfect set $X \subseteq \mathbb{R}$ (or some other complete metric space) is uncountable.

Proof. We proceed with proof by contradiction. Assume there existed a countable perfect set, denoted as $\{x_i\}_{i=1}^{\infty}$.

Then, take an open set in $\mathbb{R}$ by taking a cut on some point x_j, i.e., $S = \{x_i\} \setminus x_j$, then for any x_j, S is open with respect to the subspace topology.

Lemma 6.14

For $\{x_i\} \subset \mathbb{R}$ and any $x_j \in \{x_i\}$, the set $\{x_i\} \setminus x_j$ is dense in $\{x_i\}$ with respect to the subspace topology of $\{x_i\}$.

Proof. Take any open, nonempty, dense set in $\{x_i\}$. Then, we have $\mathcal{O} \cap \{x_i\} \setminus x_j \neq \emptyset$. But then $\{x_i\}$ has no isolated point as a subset in $\mathbb{R}$. ☐

Then, $\bigcap (\{x_i\} \setminus x_j)$ is dense, but it is $\emptyset$, contradiction. ☐

§7 Uniform continuity

Example 7.1

A pathological example: let $f_n(x) = x^n$, then $f_n(x)$ is continuous for all $n \in \mathbb{N}$, yet

$$\lim_{n \to \infty} f_n(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

is not continuous on $[0, 1]$.

Definition 7.2 (Uniform continuity for a series of functions)**.** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions, then f_n is uniformly continuous if and only if $f := \lim_{n \to \infty} f_n$ where f is continuous.

Definition 7.3 (Uniform continuity)**.** A function f is called *uniformly continuous* if for every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that for every $x, y \in X$ with $d_1(x, y) < \delta$, we have $d_2(f(x), f(y)) < \varepsilon$.

For each x, the set

$$\{y \in X : d_1(x, y) < \delta\}$$

is a δ-*neighborhood* of x.

Definition 7.4 (Bounded in a metric space)**.** A set X is bounded iff $\forall x, y \in X$, $\exists M > 0$ such that $d(x, y) \leq M$.

Theorem 7.5 (Heine-Cantor theorem)

A continuous function on a compact set is uniformly continuous.

Definition 7.6 (Supremum norm)**.** Define the supremum norm of two functions $f, g \in C[0, 1]$ as

$$d(f, g) := \sup_{x \in [0,1]} |f(x) - g(x)|$$

Definition 7.7. Let

$$A_{n,m} := \{f \in C[0, 1] \mid \exists x \in [0, 1] \text{ such that } \frac{f(t) - f(x)}{t - x} \leq n \text{ if } t - x < \frac{1}{m}\}$$

Denote D_f to be the set of differentiable functions. Then, $D_f \subset \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m}$, that is, every differentiable function f is in $A_{n,m}$.

Definition 7.8. A set E is *meager* if it can be written as a countable union of nowhere dense sets. A set E is *nonmeager* if it is not meager. A set E is *comeager* if $E^{\complement}$ is meager.

§8 Pathological functions

Example 8.1 (Weierstrass function)

Let $f(x) = \sum_{k=1}^{\infty} a^k \cos(b^k \pi x)$, where $0 < a < 1$ and $ab > 1 + \frac{3}{2}\pi$. This function is continuous everywhere, yet differentiable nowhere.

Example 8.2 (Devil's staircase)

Discussed in detail in Cantor functions.

Example 8.3

Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R}/\mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$. Then, f is not Riemann integrable but is Lebesgue integrable.

Example 8.4

Let $f_n(x) = \sum_{n=0}^{\infty} \frac{1}{1+n^2x}$. Then, f is uniformly continuous on $\mathbb{R} - \{\frac{1}{n^2}\}_{n=1}^{\infty}$.

Example 8.5

$\exists f \in L^1(\mathbb{R})$ whose Fourier series does not converge at any point.

Example 8.6

$\exists f \in C[\mathbb{R}]$ that is continuous yet nowhere monotone. Take f be the Weierstrass function.