Hypergeometric Functions

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This is a note on a series of lectures on hypergeometric functions, instructed by Brian Grove, at the 2023 Ross Mathematics Program at Otterbein College.

§1 References

- Poonen - Arithmetic Geometry notes
- Silverman-Tate - Rational Points on Elliptic Curves
- Silverman - Arithmetic of EC's, Advanced topics in the Arithmetic of EC's

§2 Intro

> **Remark.** As typical, we assume $x \in \mathbb{R}$.

$$
\begin{aligned}
\sin(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \\
\cos(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \\
\tan^{-1}(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \text{ where } x \in [-1,1] \\
-\ln(1-x) &= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \\
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!}
\end{aligned}
$$

> **Remark.** Goal: find a "master power series".

Definition 2.1. Let $y \in \mathbb{Q}$ and $k \in \mathbb{N}$. Then define the rising factorial as

$$(y)_k = y(y+1)\dots(y+k-1)$$

and $(y)_0 = 1$.

Definition 2.2. Let $a, b, c \in \mathbb{Q}$ with $c \notin \mathbb{Z}^{\leq 0}$. Then define the ${}_2F_1\left[\ ;\ \right]$ hypergeometric (HG) function as

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(1)_k (c)_k} z^k $$

with $z \in \mathbb{C}$ with $\|z\| < 1$. (There's always a $(1)_k$ by convention.)

If $1 + c > a + b$, then the hypergeometric function is defined when $\|z\| = 1$.

> **Remark.** $c \notin \mathbb{Z}^{\leq 0}$ because we don't want to divide by zero :P

Example 2.3

Let $a = b = c = 1$, then we get ${}_2F_1\left[\begin{matrix} 1 & 1 \\ & 1 \end{matrix}; z\right] = \sum_{k=0}^{\infty} z^k$, the geometric series.

Claim — $\tan^{-1}(x) = x \cdot {}_2F_1\left[\begin{matrix} 1 & \frac{1}{2} \\ & \frac{3}{2} \end{matrix}; -x^2\right]$.

Proof. We have $\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$ where $x \in [-1, 1]$.

Moreover, we have

$$ \begin{aligned} & x \cdot {}_2F_1\left[\begin{matrix} 1, & \frac{1}{2} \\ & \frac{3}{2} \end{matrix}; -x^2\right] \\ &= \sum_{k=0}^{\infty} \frac{(1)_k (\frac{1}{2})_k}{(1)_k (\frac{3}{2})_k} (-1)^k x^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \end{aligned} $$

by cancellation, hence we are done. □

Example 2.4

Let $x = 1$, then $\frac{\pi}{4} = {}_2F_1\left[\begin{matrix} 1 & \frac{1}{2} \\ & \frac{3}{2} \end{matrix}; -1\right]$.

Definition 2.5. In general, we define the generalized hypergeometric function (GHF) as

$$ {}_nF_{n-1}\left[\begin{matrix} a_1 & a_2 & a_3 & \ldots & a_n \\ & b_2 & b_3 & \ldots & b_n \end{matrix}; z\right] := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k (a_3)_k, \ldots, (a_n)_k}{(1)_k (b_2)_k (b_3)_k, \ldots, (b_n)_k} z^k $$

> **Remark.** This is called the *sum definition* of the hypergeometric function. (There's also an integral definition.)

Example 2.6

$$ {}_3F_2\left[\begin{matrix} a_1 & a_2 & a_3 \\ & b_2 & b_3 \end{matrix}; z\right] := \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k}{(1)_k(b_2)_k(b_3)_k} z^k $$

Remark. Application of hypergeometric functions on elliptic curves.

§3 Elliptic curves

Definition 3.1. An elliptic curve over $\mathbb{Q}$ is an equation of the form $y^2 = x^3 + ax + b$ with the discriminant $\Delta = -16(4a^3 + 27b^2) \neq 0$.

- nonsingular
- projective
- $\mathbb{Q}$-rational point

Definition 3.2. A singularity is a node (there's a point with "X-like" derivatives) or a cusp (not smooth).

Example 3.3

$y^2 = x^3 + x$ is nonsingular ($\Delta = -64 \neq 0$).

Definition 3.4. Let $\Bbbk$ be a field. We define an *affine n-space* as $\mathbb{A}^n(\Bbbk) = \Bbbk^n$.

We define a *projective n-space* as

$$ \mathbb{P}^n(\Bbbk) = {}^{\Bbbk^{n+1} - \{\mathbf{0}\}}\big/_{\sim} $$

where $\sim$ is some equivalence relation and $(x_0, \ldots, x_n) = \lambda(y_0, \ldots, y_n)$ and $\lambda \in \Bbbk - \{\mathbf{0}\}$ is the determinant of $\sim$.

We want to make it nice, that is, to make it respect the projective n-space.

Remark. Goal: write a homogeneous equation for the elliptic curve.

Definition 3.5 (Homogenization)**.** We send $x \mapsto \frac{x}{z}$ and $y \mapsto \frac{y}{z}$, where $z \neq 0$. This homogenizes the equation.

Example 3.6

For $y^2 = x^3 + Ax + B$, it becomes $y^2z = x^3 + Axz^2 + Bz^3$, so it's homogenized.

Example 3.7

Why $z \neq 0$? In projective space $\mathbb{P}^n(\Bbbk)$, we don't have $(0,0,0)$.

Let $z = 0$, in our previous example, then $x^3 = 0 \implies x = 0$, so we get $\mathcal{O} = (0,1,0)$, the point at infinity.

Definition 3.8. Let $E : y^2 = x^3 + Ax + B$. Then, define

$$E(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 \text{ satisfies } E\} \cup \{\mathcal{O}\}$$

Claim — $E(\mathbb{Q})$ is an abelian group.

Theorem 3.9 (Bézout's theorem)
For a line L, $L \cup E$ has exactly 3 intersection points (if we count multiple points and point at infinity).

Proof. By Bézout's theorem, we call $P \star Q$ the third point on the line with P, Q.
Then, we take the second intersection point of the tangent of $P \star Q$ as $P + Q$, that is,

$$P + Q = \mathcal{O} \star (P \star Q)$$

Then, since the line-point labeling is not order-dependent, it is obviously abelian. □

Claim 3.10 — The identity of E is the point at infinity $\mathcal{O}$.

Proof. Obviously $P + \mathcal{O} = \mathcal{O} \star (P \star \mathcal{O}) = P$. □

We want $P + (-P) = \mathcal{O}$.

Claim 3.11 — The inverse of P, denoted as $(-P)$, is constructed as follows:
We take the tangent line from $\mathcal{O}$, whose intersection is $P \star (-P)$.

Proof. We have

$$P + (-P) = \mathcal{O} \star (P \star (-P)) = \mathcal{O}$$

Thus, by construction, inverses are unique. □

Remark. No one actually cares about the underlying lines once we prove that they form a group.

Definition 3.12. The Legendre form of E is the following:

$$y^2 = x(1 - x)(1 - \lambda x)$$

where $\lambda \in \mathbb{Q} \setminus \{0, 1\}$.

Definition 3.13. An alternative form is to take $x \mapsto \frac{1}{\lambda}x$ and $y \mapsto \frac{1}{\lambda}y$, thus

$$y^2 = x(x - 1)(x - \lambda)$$

Definition 3.14. Let $s \in \mathbb{C}$. Define

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$$

for $\Re(s) > 0$. An alternative definition is

$$\Gamma(s) = \lim_{k\to\infty} \frac{k^{s-1}k!}{(s)_k}$$

for $s \in \mathbb{C} \setminus \{\mathbb{Z}_{\leq 0}\}$.

Example 3.15

We have the following facts about $\Gamma(s)$:

- $\Gamma(1) = 1$
- $\Gamma(s+1) = s\Gamma(s)$ for $s \in \mathbb{C} \setminus \{\mathbb{Z}_{\leq 0}\}$ (functional equation)
- $\Gamma(k+1) = k!$
- $\Gamma(a+k) = (a)_k\Gamma(a)$
- ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+k)} \cdot \frac{2^k}{k!}$
- $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$, where $s \in \mathbb{C} \setminus \mathbb{Z}$
- $(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!}$ for $|z| < 1$
- $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$
- $\pi = \Gamma\left(\frac{1}{2}\right)^2 = B\left(\frac{1}{2}, \frac{1}{2}\right)$

Exercise 3.16. Prove the above facts.

Definition 3.17. Define $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, for $\Re(x), \Re(y) > 0$.

Exercise 3.18. Prove that $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ for $x, y > 0$.

Theorem 3.19 (Differential forms of elliptic curves)

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}} = \pi \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix}; \lambda\right] \text{ for } \lambda \in \mathbb{Q} \setminus \{0,1\}.$$

Proof.

$$
\begin{aligned}
&\int_0^1 (x(1-x))^{-\frac{1}{2}}(1-\lambda x)^{-\frac{1}{2}}dx \\
&= \int_0^1 (x(1-x))^{-\frac{1}{2}} \left[\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{k!}(\lambda x)^k\right] dx \\
&= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{k!}\lambda^k \int_0^1 x^{k-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx \\
&= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{k!}\lambda^k \int_0^1 x^{k+\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx \\
&= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{k!}\lambda^k B\left(k+\frac{1}{2}, \frac{1}{2}\right) \\
&= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{k!} \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(k+1)}\lambda^k \\
&= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{k!k!}\lambda^k \Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \\
&= \Gamma\left(\frac{1}{2}\right)^2 \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}{k!k!}\lambda^k \\
&= \pi \cdot {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; \lambda\right]
\end{aligned}
$$

□

Example 3.20

We denote ${}_2P_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; -1\right] = B(\frac{1}{2}, \frac{1}{2}) \cdot {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; \lambda\right]$.

Definition 3.21. Define ${}_2F_1\left[\begin{matrix}a\ b \\ c\end{matrix}; z\right] := B(b, c-b) \cdot {}_2P_1\left[\begin{matrix}a\ b \\ c\end{matrix}; z\right]$.

Assume $c > b$, then

$$
\begin{aligned}
&{}_2P_1\left[\begin{matrix}a, b \\ c\end{matrix}; z\right] = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \cdot {}_2F_1\left[\begin{matrix}a, b \\ c\end{matrix}; z\right] \\
&\implies {}_2P_1\left[\begin{matrix}a, b \\ c\end{matrix}; z\right] = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-2t)^{-a}dt \text{ when } z \in \mathbb{C} \setminus [1, \infty) \\
&\implies {}_2F_1\left[\begin{matrix}a, b \\ c\end{matrix}; z\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-2t)^{-a}dt
\end{aligned}
$$

Theorem 3.22 (Gauss)

If $c > b$ and $c - a - b > 0$, then

$$
{}_2F_1\left[\begin{matrix}a\ b \\ c\end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}
$$

Proof. By Abel continuity theorem, letting $z \to 1^-$,

$$\begin{aligned}
{}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; 1\right] &= \frac{\Gamma c}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{(c-a-b)-1} \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} B(b, c-a-b) \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)} \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}
\end{aligned}$$

hence we are done. □

Example 3.23

Let $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = \frac{3}{2}$. Then, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(s+1) = s\Gamma(s)$, we have

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} \end{matrix}; 1\right] = \Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{2}$$

hence $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$.

Theorem 3.24 (Pfaff transformation)

$${}_2F_1\left[\begin{matrix} a & b \\ c \end{matrix}; x\right] = (1-x)^{-a}{}_2F_1\left[\begin{matrix} a & c-b \\ c \end{matrix}; \frac{x}{x-1}\right].$$

Proof. We have ${}_2F_1\left[\begin{matrix} a & b \\ c \end{matrix}; x\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-2t)^{-a}dt$.

Let $t \mapsto 1-s$, then

$$\begin{aligned}
{}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 s^{c-k-1}(1-s)^{b-1}(1-x)^{-a}(1+s(\frac{x}{1-x}))^{-a} \\
&= (1-x)^{-a}{}_2F_1\left[\begin{matrix} a, c-b \\ c \end{matrix}; \frac{x}{x-1}\right]
\end{aligned}$$

and we are done. □

Theorem 3.25 (Euler)

$${}_2F_1\left[\begin{matrix} a & b \\ c \end{matrix}; x\right] = (1-x)^{c-a-b}{}_2F_1\left[\begin{matrix} c-a & c-b \\ c \end{matrix}; x\right].$$

Theorem 3.26 (Binet's formula)

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Exercise 3.27. Prove $F_{-n} = (-1)^{n-1}F_n$. (Use Binet's formula or induction)

Theorem 3.28

Hypergeometric functions are recursive.

Theorem 3.29 (Dilcher)

Let $a = \frac{1-n}{2}$ and $z = \sqrt{5}$. Then,

$$\begin{aligned} {}_2F_1\left[\begin{matrix} \frac{1-n}{2}, 1-\frac{n}{2} \\ \frac{3}{2} \end{matrix}; 5\right] &= \frac{1}{2n\sqrt{5}}\left[(1+\sqrt{5})^n - (1-\sqrt{5})^n\right] \\ \implies F_n &= \frac{n}{2^{n-1}} \cdot {}_2F_1\left[\begin{matrix} \frac{1-n}{2}, 1-\frac{n}{2} \\ \frac{3}{2} \end{matrix}; 5\right] \end{aligned}$$

Theorem 3.30

$${}_2F_1\left[\begin{matrix} a \quad a+\frac{1}{2} \\ \frac{3}{2} \end{matrix}; z^2\right] = \frac{1}{2z(1-2a)}\left[(1+z)^{1-2a} - (1-z)^{1-2a}\right]$$

Theorem 3.31

$${}_2F_1\left[\begin{matrix} a \quad a+\frac{1}{2} \\ \frac{1}{2} \end{matrix}; z\right] = \frac{1}{2}\left[(1+\sqrt{z})^{-2a} + (1-\sqrt{z})^{-2a}\right]$$

Exercise 3.32. For $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$, show that $C_n = {}_2F_1\left[\begin{matrix} 1-n \quad -n \\ 2 \end{matrix}; 1\right]$.

Proof. Expand by definition, then represent the summation as

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{n}{n-k-1}}{n}$$

which is just $\frac{\binom{2n}{n}}{n+1}$ by Vandermonde's identity. □

§4 Relation with the Riemann zeta function

Definition 4.1 (Riemann, 1859)**.**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-\frac{1}{p^s}}$$

for $\Re(s) > 1$.

Example 4.2 (Basel problem)

For example, $\zeta(2) = \frac{\pi^2}{6}$.

Note that $\pi = {}_2F_1\left[\begin{matrix}1 & \frac{1}{2} \\ \frac{3}{2}\end{matrix}; -1\right]$, so

$$\zeta(2) = \frac{1}{6}\left({}_2F_1\left[\begin{matrix}1 & \frac{1}{2} \\ \frac{3}{2}\end{matrix}; -1\right]\right)^2$$

Definition 4.3. Let $B_0 = 1$ and $\sum_{k=0}^{n-1}\binom{n}{k}B_k = 0$.

$B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $\ldots$.

Exercise 4.4. Prove that $B_{2k+1} = 0$ for $k \geq 1$.

We may write $\zeta(2k) = \dfrac{(-1)^{k+1}B_{2k}(2\pi)^{2k}}{2(2k)!}$ for $k \in \mathbb{N}$.

> **Remark.** Special ζ values $\leftrightarrow$ Bernoulli numbers $\overset{\text{Byrd}}{\leftrightarrow}$ Fibonacci numbers $\overset{\text{Dilcher}}{\leftrightarrow}$ Truncated ${}_pF_q$'s.

Theorem 4.5 (Byrd)
If $N \geq 0$, then

$$F_{2N+2} = 2\sum_{k=0}^{N} A_{2k,N}B_{2k}$$

where

$$A_{2k,N} = \sum_{n=0}^{N-k}\binom{2N+1-n}{n}\binom{2N+1-2n}{2k}\frac{1}{2N-2n-2k+2}$$

We also have $B_2 = \frac{F_4}{2} - \frac{4}{3}$ and

$$\begin{aligned}
F_4 &= \frac{1}{2}{}_2F_1\left[\begin{matrix}-\frac{3}{2}, -1 \\ \frac{3}{2}\end{matrix}; 5\right] \\
\implies B_2 &= \frac{1}{4}{}_2F_1\left[\begin{matrix}-\frac{3}{2}, -1 \\ \frac{3}{2}\end{matrix}; 5\right] - \frac{4}{3} \implies \zeta(2) \\
&= \left(\frac{1}{4}{}_2F_1\left[\begin{matrix}-\frac{3}{2}, -1 \\ \frac{3}{2}\end{matrix}; 5\right] - \frac{4}{3}\right)\cdot\left({}_2F_1\left[\begin{matrix}-1, \frac{1}{2} \\ \frac{3}{2}\end{matrix}; -1\right]\right)^2
\end{aligned}$$

thus

$$\zeta(4) = \left(\frac{64}{3}{}_2F_1\left[\begin{matrix}-\frac{3}{2} & -1 \\ \frac{3}{2}\end{matrix}; 5\right] - \frac{11392}{45}\cdot\left({}_2F_1\left[\begin{matrix}1 & \frac{1}{2} \\ \frac{3}{2}\end{matrix}; -1\right]\right)^4\right)$$

and by using $\zeta(s) = \zeta(1-s)$ and $\zeta(-k) = \dfrac{(-1)^{k+1}B_{k+1}}{k+1}$, we have

$$\zeta(-1) = \frac{2}{3} - \frac{1}{8}\cdot{}_2F_1\left[\begin{matrix}-\frac{3}{2}, -1 \\ \frac{3}{2}\end{matrix}; 5\right] = -\frac{1}{12}$$

$$\zeta(-3) = \frac{89}{120} - \frac{1}{8}\cdot{}_2F_1\left[\begin{matrix}-\frac{3}{2}, -1 \\ \frac{3}{2}\end{matrix}; 5\right] = \frac{1}{120}$$

Example 4.6

We have $L_p \equiv 1 \pmod{p}$ and $F_p \equiv \binom{p}{s} \pmod{p}$ (we can relate it to B_k, then to $\zeta(s)$ as well.). The relation chain is basically ${}_2F_1 \to F_n \to B_k \to \zeta$.

Example 4.7 (${}_pF_q$ in the p-adics)

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix}; x\right]_{p-1} = \sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k(\frac{1}{2})_k}{k!k!} x^k. $$

Lemma 4.8

The multiplicative group of a field is cyclic.

Definition 4.9. Let $\varphi : G \to H$ and $\chi : \mathbb{F}_p^\times \to \mathbb{C}^\times$ be a character.

Example 4.10

Let $p = 5$, that is, in $\mathbb{F}_5^\times$. Then, $\chi : \mathbb{F}_5^\times \to \mathbb{C}^\times$. $\chi(1) = 1$, $\chi(2) = i$, $\chi(3) = -i$, $\chi(4) = \chi(2)\chi(2) = -1$.

Example 4.11

The trivial character $\varepsilon : \mathbb{F}_p^\times \to \mathbb{C}^\times$, where $\varepsilon \equiv 1$.

Example 4.12

The Legendre symbol is also a character.

Example 4.13

$\widehat{\mathbb{F}_p^\times}$ is the group of characters on $\mathbb{F}_p^\times$.

Lemma 4.14

We have two character sums:

- Fix χ. Then,
$$ \sum_{q \in \mathbb{F}_p^\times} \chi(q) = \begin{cases} p-1 & \chi = \varepsilon \\ 0 & \text{otherwise} \end{cases} $$
- Fix $q \in \mathbb{F}_p^\times$. Then,
$$ \sum_{\chi \in \widehat{\mathbb{F}_p^\times}} \chi(q) = \begin{cases} p-1 & q = e \\ 0 & \text{otherwise} \end{cases} $$

Classical	Finite fields
$a \in \mathbb{Q}$	$\chi = \omega^{(p-1)a}$
$-a$	$\overline{\chi}$
$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$	$g(A) = \sum_{x\in\mathbb{F}_p^\times} A(x)\zeta_p^\times$ where $A(a) = \omega^{(p-1)a}$
$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$	$g(A)g(\overline{A}) = A(-1)p$ if $A \neq \varepsilon$
$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$	$J(A,B) = \sum_{x\in\mathbb{F}_p^\times} A(x)B(1-x)$
$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$	$J(A,B) = \frac{g(A)g(B)}{g(AB)}$ if $AB \neq \varepsilon$
x^a	$A(x)$
$a+b$	AB

Table 1: Finite field analogs of classical hypergeometric functions.

Example 4.15

For $a_1 = \frac{1}{2}$, we have $\chi = \omega^{\frac{p-1}{2}} = \phi$, which is the Legendre symbol.

Example 4.16

For $a_1 = \frac{3}{4}$, we have $\chi = \omega^{\frac{3(p-1)}{4}} = \eta$.

§5 Finite fields

Definition 5.1. Let ω be a generator of $\widehat{\mathbb{F}_p^\times}$, that is, $\widehat{\mathbb{F}_p^\times} = \langle\omega\rangle$.
Define $A := \omega^{(p-1)a}$ and $B := \omega^{(p-1)b}$.

Definition 5.2 (Beukers, Coher, Mellit, 2015)**.** A hypergeometric function over $\mathbb{F}_p$ looks like:

$$\begin{aligned} H_p\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right] &:= \sum_{k=0}^{p-2} \frac{g(A\omega^k)g(B\omega^k)g(\overline{C\omega^k})}{g(\varepsilon)g(A)g(B)g(\overline{C})}\chi(\lambda) \\ &= \frac{1}{1-p}\sum_{x\in\widehat{\mathbb{F}_p^\times}} \frac{g(Ax)g(Bx)g(\overline{Cx})}{g(\varepsilon)g(A)g(B)g(\overline{C})}\chi(\lambda) \quad \text{where } x = \omega^k \\ &= \frac{1]J(B,C\overline{B})}{\sum}_{xin\mathbb{F}_p^\times} B(x)C\overline{B}(1-x)\overline{A}(1-\lambda x) \end{aligned}$$

Theorem 5.3

We have $g(A)g(\overline{A}) = A(-1)p - (p-1)\delta(A)$, where

$$\delta(A) = \begin{cases} 1 & \text{if } A = \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Definition 5.4. Over $\mathbb{F}_{p^r}$, we define $\Phi(x) = \zeta_p^{\mathrm{Tr}(x)}$, where $\mathrm{Tr}(x) = x + x^p + \cdots + x^{p^{r-1}}$.

Proof. Let $\Phi(x) = \zeta_p^{\times}$. Then,

$$\begin{aligned}
g(A)g(\overline{A}) &= \sum_{x \in \mathbb{F}_p^\times} A(x)\Phi(x) \cdot \sum_{y \in \mathbb{F}_p^\times} A\left(\frac{1}{y}\right)\Phi(y) \\
&= \sum_{x,\, y \in \mathbb{F}_p^\times} A\left(\frac{x}{y}\right)\Phi(x+y) \\
&= \sum_{x,\, t \in \mathbb{F}_p^\times} A(t)\ \Phi\left(x\left(1+\frac{1}{t}\right)\right) \quad \text{where } t = \frac{x}{y} \\
&= \sum_{t \in \mathbb{F}_p^\times, t \neq -1} A(t) \sum_{x \in \mathbb{F}_p^\times} \Phi\left(x\left(1+\frac{1}{t}\right)\right) + A(-1)\sum_{x \in \mathbb{F}_p^\times} \Phi(0) \\
&= A(-1) + A(-1)(p-1) \\
&= A(-1) \cdot p
\end{aligned}$$

since

$$\sum_{t \in \mathbb{F}_p^\times} A(t) = 0$$

$$\sum_{t \in \mathbb{F}_p^\times} A(t) = -A(-1)$$

$$\sum_{x \in \mathbb{F}_p} \Phi\left(x\left(1+\frac{1}{t}\right)\right) = 0$$

and

$$\sum_{x \in \mathbb{F}_p^\times} \Phi\left(x\left(1+\frac{1}{t}\right)\right) = -1$$

□

Theorem 5.5

$$H_p\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; \lambda\right] = \frac{1}{1-p} \sum_{\chi \in \widehat{\mathbb{F}_p^\times}} \frac{g(\phi\chi)g(\phi\chi)g(\overline{\chi})}{g(\varepsilon)g(\phi)g(\phi)g(\varepsilon)}\chi(\lambda)$$

§6 Algebraic hypergeometric functions

Let $\boldsymbol{\alpha} = \{a_1, a_2, \ldots, a_n\}$ and $\boldsymbol{\beta} = \{b_1, b_2, \ldots, b_n\}$, where $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$.

Definition 6.1. We say $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are *interlacing* if $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ or $b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n$.

Theorem 6.2 (Beukers-Heckman, 1975)
The data $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ is algebraic iff $\boldsymbol{\alpha}, \boldsymbol{\beta}$ interlace.

Example 6.3
$H_p\left[\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & \frac{1}{2}\end{matrix}; \lambda\right]$ is algebraic, since $\boldsymbol{\alpha} = \{\frac{1}{3}, \frac{2}{3}\}$ and $\boldsymbol{\beta} = \{1, 1\}$ interlace.

Theorem 6.4 (Multiplication formula)
Let $m \in \mathbb{N}$. Then

$$\prod_{\chi \in \mathbb{F}_p^\times \ \chi^m = \varepsilon} \frac{g(A\chi)}{g(\chi)} = -g(A^m)A(m^{-m})$$

Theorem 6.5 (Special case)
If $m = 2$, then $g(A)g(\phi A) = g(A)g(\phi)\overline{A}(4)$.

Theorem 6.6

$$H_p\left[\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ & \frac{1}{2}\end{matrix}; \lambda\right] = \left(\frac{1+\phi(\lambda)}{2}\right)\left[\phi(1+\sqrt{\lambda}) + \phi(1-\sqrt{\lambda})\right]$$

for the quadratic character $\phi(x) = x^{\frac{p-1}{2}}$, where we assume $p \equiv 1 \pmod 4$.
H_p collapses to 0 if λ is not a square mod p.

Proof. Assume $\lambda \neq 0$ is a square. Let η_4 be a character of order 4.

Lemma 6.7 (Double-angle formula)
$g(A)g(\phi A) = g(A^2)g(\phi)\overline{A}(4)$.

Then, we have

$$
\begin{aligned}
H_p\left[\begin{matrix}\frac{1}{4}, \frac{3}{4} \\ \frac{1}{2}\end{matrix}; \lambda\right] &= \frac{1}{p-1}\sum_{\chi} \frac{g(\eta_4\chi)g(\overline{\eta}_4)g(\overline{\chi})g(\phi\overline{\chi})}{g(\eta_4)g(\overline{\eta}_4)g(\phi)}\chi(\lambda) \\
&= \frac{1}{p-1}\sum_{\chi} \left(\frac{g(\chi^4)}{g(\chi)}\right)\left(\frac{g(\phi)}{g(\phi\chi)}\right)\left(\frac{g(\overline{\chi}g(\phi\overline{\chi}))}{g(\phi)}\right)\chi(\frac{\lambda}{256}) \quad \text{iterating over } \chi \in \{\phi, \varepsilon, \eta_4, \overline{\eta}_4\} \\
&= \frac{1}{p-1}\sum_{\chi} \frac{g(\chi^4)}{g(\chi^2)g(\phi)} g(\overline{\chi})g(\phi\overline{\chi})\chi(\frac{\lambda}{64}) \quad \text{by the double-angle formula with } A = \chi \\
&= \frac{1}{p-1}\sum_{\chi} \frac{g(\chi^4)}{g(\chi^2)g(\phi)} g(\overline{\chi}^2)g(\phi)\chi(\frac{\lambda}{16}) \quad \text{by the double-angle formula with } A = \overline{\chi} \\
&= \frac{1}{p-1}\sum_{\chi} \frac{g(\chi^4)g(\overline{\chi}^2)}{g(\chi^2)}\chi(\frac{\lambda}{16}) \\
&= \frac{1}{p-1}\sum_{\chi} \frac{g(\phi\chi^2)g(\overline{\chi}^2)}{g(\phi)}\chi(\lambda) \quad \text{by the double-angle formula with } A = \chi^2 \\
&= \frac{1}{p-1}\sum_{\chi}\sum_{a\in\mathbb{F}_p\setminus\{0,1\}} \phi\chi^2(a)\overline{\chi}^2(1-a)\chi(\lambda) \quad \text{write as a Jacobi sum} \\
&= \frac{1}{p-1}\sum_{a\in\mathbb{F}_p\setminus\{0,1\}} \phi(a)\sum_{\chi}\chi\left(\frac{a^2\lambda}{(1-a)^2}\right) \quad \text{swap the order of summation} \\
&= \phi\left((1+\sqrt{\lambda})^{-1} + (1-\sqrt{\lambda})^{-1}\right) \quad \text{by evaluating cases where } \frac{a^2\lambda}{(1-a)^2} = 1 \\
&= \phi(1+\sqrt{\lambda}) + \phi(1-\sqrt{\lambda})
\end{aligned}
$$

□

Example 6.8 (BCM, G.)

$H_p\left[\begin{matrix}\frac{1}{3}\ \frac{2}{3} \\ \frac{1}{2}\end{matrix}; \lambda\right] = N_f(\lambda) - 1$, where $N_f(\lambda)$ is the number of zeros of $f(x) = x^3 + 3x^2 - 4\lambda$ over $\mathbb{F}_p$.

Example 6.9 (G.)

$H_p\left[\begin{matrix}\frac{1}{6}\ \frac{5}{6} \\ \frac{1}{2}\end{matrix}; \lambda\right] = \phi(\frac{\lambda}{27})(N_f(\lambda) - 1)$ where ϕ is the quadratic character. This is basically immediate from the previous example, since if we add $\frac{1}{2}$ (which is the equivalent of sending χ to $\phi(\chi)$, since ϕ is basically "$\frac{1}{2}$" in $\mathbb{F}_p^\times$) and quotient $\mathbb{Z}$, we get this HG.

Remark. We implicitly define it in $/\ \mathbb{Q}$, where $\boldsymbol{\alpha} = \{a_1, \ldots, a_n\}$ is $/\ \mathbb{Q}$ if $\prod_{i=1}^n (x - e^{2\pi i a_i}) \in \mathbb{Z}[x]$.

§7 Hypergeometric moments

Example 7.1

The intuition comes from $\int_0^1 \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}} = \pi \cdot {}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; \lambda\right]$.

> **Remark.** Certain H_p values have a relation with the points continuous on cubic curves over $\mathbb{F}_p$. Goal: count $\mathbb{F}_p$ solutions on $\tilde{E}$ = mod p reduction of E, where p is a good prime (i.e., doesn't make E singular).

$$\begin{aligned} |\tilde{E}(\mathbb{F}_p) &= 1 + \sum_{x\in\mathbb{F}_p}\left(1 + \left(\frac{x(1-x)(1-\lambda x)}{p}\right)\right) \quad \text{including } \mathcal{O}\text{, i.e., point at infty} \\ &= p + 1 + \sum_{x\in\mathbb{F}_p} \phi(x(1-x)(1-\lambda x)) \end{aligned}$$

Definition 7.2. Define $a_p = -\sum_{x\in\mathbb{F}_p} \phi\left(x(1-x)(1-\lambda x)\right)$.

Definition 7.3. Denote $H_p(\lambda) = H_p\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; \lambda\right]$.

Definition 7.4. Good reduction refers to the reduced variety having the same properties as the original, for example, an algebraic curve having the same genus, or a smooth variety remaining smooth.

Claim 7.5 — $a_p = H_p\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; \lambda\right]$ for primes of *good reduction*.

Proof. Let $a = b = \frac{1}{2}$ and $c = 1$, so

$$\begin{aligned} H_p\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; \lambda\right] &= \frac{1}{J(\phi,\phi)} \sum_{x\in\mathbb{F}_p} \phi(x(1-x)(1-\lambda x)) \\ &= -\phi(-1) \sum_{x\in\mathbb{F}_p} \phi(x(1-x)(1-\lambda x)) = -\phi(-1) a_p \end{aligned}$$

hence $a_p = H_p\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; \lambda\right]$. $\square$

Theorem 7.6 (Hasse bound)

$|H_p(\lambda)| \le 2\sqrt{p}$, or equivalently, $\frac{H_p(\lambda)}{\sqrt{p}} \in [-2, 2]$.

What is $\mathrm{End}(E)$? (For "nice" elliptic curves, since it forms an abelian group, $\mathrm{End}(E) \cong \mathbb{Z}$.)

Most of the time, $\mathrm{End}(E) \cong \mathbb{Z}$.

But sometimes, $\mathrm{End}(E) \supsetneq \mathbb{Z}$.

Example 7.7

$y^2 = x^3 - x$, then the map $(x, y) \mapsto (-x, iy)$ gives us back the original curve.

Remark. E has complex multiplication (CM) if $\mathrm{End}(E) \supsetneq \mathbb{Z}$.

Theorem 7.8 (Sato-Tate, 2011)

Fix E_λ that is not CM. Then, $\frac{H_p(\lambda)}{\sqrt{p}} \in [-2, 2]$ gives a semicircular distribution as $p \to \infty$.

Conjecture 7.9 (Sato-Tate for families, 2021)

Fix p. Let $\lambda \in \mathbb{F}_p \setminus \{0, 1\}$ vary in $\{E_\lambda\}$. Then, what is the distribution of $\frac{H_p(\lambda)}{\sqrt{p}} \in [-2, 2]$ as λ varies, for sufficiently large p? (Answer: semicircular.)

Take an "average" of the normalized H_p values. Let m be a fixed positive integer greater than 1. Consider the hypergeometric moment

$$\frac{1}{p} \sum_{\lambda \in \mathbb{F}_p} \left(\frac{H_p(\lambda)}{\sqrt{p}} \right)^m = \frac{1}{p^{\frac{m}{2}+1}} \sum_{\lambda \in \mathbb{F}_p} H_p(\lambda)^m$$

The expression is interesting (i.e., nontrivial) if $m > 1$, since for $m = 1$, it's basically orthogonality characters, so it sums to 0 or $p - 1$.

Theorem 7.10 (Ono-Saad-Saikia, 2021)

Let m be a fixed positive integer. Then,

$$\lim_{p \to \infty} \frac{1}{p^{\frac{m}{2}+1}} \sum_{\lambda \in \mathbb{F}_p} H_p(\lambda)^m = \begin{cases} 0 & \text{if } m \text{ is odd} \\ C(n) & \text{if } m = 2n \text{ for } n \in \mathbb{N} \end{cases}$$

where $C(n) = \frac{1}{n+1}\binom{2n}{n}$.

Proof. We have $H_p(\lambda) = -H_p(\frac{1}{\lambda})$ where $\lambda \in \mathbb{F}_p^\times$, so for $2 \nmid m$, everything cancels out nicely. $\square$

Theorem 7.11 (Grove)

Let m be a fixed positive integer. Then,

$$\lim_{p \to \infty} \frac{1}{p^{\frac{m}{2}+1}} \sum_{\lambda \in \mathbb{F}_p} H_p(\lambda^2)^m = \begin{cases} 0 & \text{if } m \text{ is odd} \\ C(n) & \text{if } m = 2n \text{ for } n \in \mathbb{N} \end{cases}$$

where $C(n) = \frac{1}{n+1}\binom{2n}{n}$.

Proof. We have $H_p(\lambda) = \phi(\lambda)H_p(1-\lambda)$ for $\lambda \in \mathbb{F}_p^\times$, so for $2 \nmid m$, everything cancels out nicely. ☐

Theorem 7.12 (Grove)
Let m be a fixed positive integer. Then,

$$\lim_{p\to\infty} \frac{1}{p^{\frac{m}{2}+1}} \sum_{\lambda\in\mathbb{F}_p} H_p\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ & 1 \end{bmatrix}; \lambda\Bigg]^m = \begin{cases} 0 & \text{if } m \text{ is odd} \\ C(n) & \text{if } m = 2n \text{ for } n \in \mathbb{N} \end{cases}$$

where $C(n) = \frac{1}{n+1}\binom{2n}{n}$.

The high level intuition comes from $\displaystyle\int_{SU(2)} (\mathrm{Tr}(X))^{2n} = C(n)$.

Theorem 7.13 (Ono-Saad-Saikia, 2021)
Let m be a fixed positive integer. Then,

$$\lim_{p\to\infty} \frac{1}{p^{m+1}} \sum_{\lambda\in\mathbb{F}_p} H_p\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{matrix}; \lambda\right]^m = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \sum_{i=0}^m (-1)^i \binom{m}{i} C(i) & \text{if } m \text{ is even} \end{cases}$$

The high level intuition comes from $\displaystyle\int_{O(3)} (\mathrm{Tr}(X))^m = \sum_{i=0}^m (-1)^i \binom{m}{i} C(i)$.

§8 Finale

Finally, here is an open problem to think about:

Conjecture 8.1
Let m be a fixed positive integer. Then,

$$\lim_{p\to\infty} \frac{1}{p^{\frac{m}{2}+1}} \sum_{\lambda\in\mathbb{F}_p} H_p\left[\begin{matrix} \frac{1}{6} & \frac{5}{6} \\ & 1 \end{matrix}; \lambda\right]^m = \begin{cases} 0 & \text{if } m \text{ is odd} \\ C(n) & \text{if } m = 2n \end{cases}$$