A GENTLE INTRODUCTION TO ARITHMETIC DYNAMICS

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Abstract. We provide an introduction to the field of arithmetic dynamics. We start by defining basic terminology and notation from classical dynamics, including periodic points, wandering points, rational maps, power maps, Chebyshev polynomials, Lattès maps, Julia sets, and Fatou sets. Together, we cover some basic machinery from algebraic geometry, number theory, and p-adic theory, including discriminants and resultants, $\mathbb{Q}_p$ and $\mathbb{Z}_p$, completions of $\mathbb{Q}$, Ostrowski's theorem, affine and projective space, ideals and varieties, Hilbert's Nullstellensatz, and basic elliptic curve theory. Continuing the discussion, we discuss dynamics over global fields, including height functions, local canonical heights, and Diophantine approximation, ultimately leading to the proof of Northcott's theorem. We then discuss several pathways emerging from it, including the uniform boundedness conjecture. After that, we move on to dynamics over local fields of good reduction, where we introduce some common machinery, including the nonarchimedean chordal metric and reduction of maps modulo p.

All the lonely objects, where do they all belong?

Joseph H. Silverman

1. Introduction

As does any interesting journey start, we start with some motivation. There are types of mathematics that people like to study. Of course, there are just so many mathematical objects. What do people do with interesting objects? They prove theorems about those objects, sometimes with additional restrictions. Lots of great mathematics is done in that form, where people restrict their attention to a very small subset of objects, then prove cool properties that hold within those objects. But there are also great mathematics that aim to prove *general* properties that hold within *all* objects of that type. In that case, it makes natural sense to look at the family $\mathcal{F}$ of objects. For example, as we will introduce later, one looks at the set of all morphisms $\mathbb{P}^n \to \mathbb{P}^n$, and try to prove properties that hold as generally as possible, within those set of objects. But there is a trade-off: too much zooming out often leads to general, but pretty boring statements; these sets can be too large and unwieldy. In order to avoid this problem, we study better behaved subsets by adding restrictions. For instance, there are "too many" maps $\mathbb{P}^n \to \mathbb{P}^n$, so in dynamics, we restrict our attention to *finite* maps $\mathbb{P}^n \to \mathbb{P}^n$ of *fixed degree*.

In dynamics, we look at the *composition* of functions and their behavior. For example, consider the function $\phi(z) = z^2 + 1$ in the complex plane. Traditional complex dynamics asks the following questions: "What are the fixed points of ϕ, that is, points $z \in \mathbb{C} \cup \{\infty\}$ such that $\phi(z) = z$? How can we describe the local behavior near those fixed points? Can we classify periodic points, that is, which points $z \in \mathbb{C}$ satisfy $\phi^n(z) = \phi^{n+m}(z)$ for some $n, m \in \mathbb{N}$? Which points $z \in \mathbb{C}$ *repel* the points near them, that is, they move farther away from z as we iteratively apply ϕ? Which points *attract* other

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points?" These properties of the map ϕ, are often colloquially called the *dynamics* of ϕ. In order to study the dynamics of a map, we consider the orbit of α, which is the set

$$\mathcal{O}_\phi(\alpha) = \{\phi^n(\alpha) : n \geq 0\}$$

The principal goal of dynamics is to classify the points α in the set S, according to the behavior of their orbits $\mathcal{O}_\phi(\alpha)$. Within $\mathbb{C}$, this is much of what complex dynamics is devoted to, and if one wants to study in this direction, an excellent reference is [Mil06]. The topic of this paper, arithmetic dynamics, instead looks at the dynamics of mappings under an *arithmetic* setting, such as $\mathbb{Q}$, $\mathbb{Z}_p$, or $\mathbb{Q}_p$, as any number theorist would do, asking the same questions as above. However, the nonarchimedean nature of $\mathbb{Q}_p$ leads to interesting theories, worth pursuing by itself. We will reach some of the deepest conjectures and results of interest, but just as how good theories are developed, we start with the basics.

2. The Basics

First, we define some commonly used notation, and give some basic definitions. Throughout this paper, we use the standard symbols

$$\mathbb{Z},\ \mathbb{Q},\ \mathbb{R},\ \mathbb{C},\ \mathbb{F}_q,\ \mathbb{Z}_p,\ \mathbb{A}^N,\ \mathbb{P}^N$$

to represent the integers, rational numbers, real numbers, complex numbers, field with q elements, ring of p-adic integers, N-dimensional affine space, and N-dimensional projective space, respectively. Much of the following definitions are borrowed from [Sil07].

Let us first define a dynamical system, which is the central object of study in dynamics.

Definition 2.1. A *dynamical system* consists of a set S and a function $\phi : S \to S$, where we consider the iteration of ϕ:

$$\phi^n = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}} = n^{\text{th}} \text{ iterate of } \phi$$

for $n \in \mathbb{N}$ with ϕ^0 being the identity map on S.

Now, as previously said, we look at the orbit of points.

Definition 2.2. For a given point $\alpha \in S$, the *orbit* of α is the set

$$\mathcal{O}_\phi(\alpha) = \{\phi^n(\alpha) : n \geq 0\}$$

We call a point α to be *periodic* if $\phi^n(\alpha) = \alpha$ for some $n \geq 1$. Moreover, the smallest such n is called the *exact period* of α. We also call a point α to be *preperiodic* if some iterate $\phi^m(\alpha)$ is periodic. Indeed, the set of periodic points of ϕ in S are denoted by

$$\mathrm{Per}(\phi, S) = \{\alpha \in S : \phi^n(\alpha) = \alpha \text{ for some } n \geq 1\}$$

and the set of preperiodic points of ϕ in S are denoted by

$$\mathrm{PrePer}(\phi, S) = \left\{\alpha \in S : \phi^{m+n}(\alpha) = \phi^m(\alpha) \text{ for some } n \geq 1, m \geq 0\right\}$$

which is the set of $\alpha \in S$ such that $\mathcal{O}_\phi(\alpha)$ is finite. For convenience, if the base set S is fixed, we just write $\mathrm{Per}(\phi)$ and $\mathrm{PrePer}(\phi)$.

Indeed, it makes sense to look at points of period n and exact period n:

Definition 2.3. The set of periodic points of ϕ with period n are denoted by

$$\mathrm{Per}_n(\phi) = \left\{\alpha \in \mathbb{P}^1(\mathbb{C}) : \phi^n(\alpha) = \alpha\right\}$$

Moreover, the set of periodic points of ϕ with exact period n are denoted by

$$\mathrm{Per}_n^{**}(\phi) = \{\alpha \in \mathrm{Per}_n(\phi) : \alpha \text{ has exact period } n\}$$

The reason for this weird $\mathrm{Per}_n^{**}(\phi)$ notation is that we also have $\mathrm{Per}_n^*(\phi)$, which is the set of points with *formal period* n. The consideration for formal periods comes from Galois theory, more specifically, the theory of cyclotomic polynomials, where the roots of a cyclotomic polynomial are said to be algebraically indistinguisable. Indeed, it makes sense to consider the following dynamic analogue of a cyclotomic polynomial:

$$\Phi_n^*(z) = \prod_{m|n} (\phi^m(z) - z)^{\mu(n/m)}$$

where μ is the Möbius function. The product makes sense because we are essentially performing PIE to quotient out all the smaller factors, and only leave the highest factor. Then, the roots of $\Phi_n^*(z)$ are said to have *formal period* n. The roots of $\Phi_n^*(z)$ behave in many ways as if they have exact period n, although their actual period is smaller than n.

Exercise 2.4. Prove that $\Phi_n^*(z)$ is well-defined, that is, $\phi^m(z) - z \mid \phi^n(z) - z$ whenever $m \mid n$ and $\Phi_n^*(z)$ is a polynomial.

Exercise 2.5. Prove that

$$\text{exact period } n \implies \text{formal period } n \implies \text{period } n$$

and that the reverse implication does not hold, that is, find a counterexample for each reverse implication.

In dynamics, we will mostly work with projective spaces (since they have various nice properties), so let's first define it.

Definition 2.6. Given a vector space V over a field K, the *projective space* $\mathbb{P}(V)$ is the set of equivalence classes of $V^* = V \setminus \{0\}$ with equivalence under scaling, that is, $\mathbb{P}(V) = V^*/\sim$ where $\sim$ is an equivalence relation such that $x \sim y$ iff $x = \lambda y$ for some nonzero λ.

If V is a topological vector space (TVS), the quotient space $\mathbb{P}(V)$ is also a TVS, equipped with the quotient topology of the subspace topology of V^*. For example, this is certainly the case when $K = \mathbb{R}$ or $K = \mathbb{C}$.

Exercise 2.7. Prove that if $\dim V < \infty$, then $\dim \mathbb{P}(V) = \dim(V) - 1$.

Indeed, with the above result, when $V = K^{n+1}$, we may also write $\mathbb{P}(V)$ as $\mathbb{P}^n(K) = K\mathbb{P}^n = \mathbb{P}^n K$. (For example, $\mathbb{C}\mathbb{P}^1$ denotes the complex projective line, and $\mathbb{R}\mathbb{P}^2$ denotes the real projective plane. The complex projective line $\mathbb{C}\mathbb{P}^1$ is also known as the Riemann sphere, since $\mathbb{P}^1(\mathbb{C}) \cong S^2$.)

We also deal with affine spaces, which are Euclidean spaces without a fixed origin (that has "forgotten" its origin). To be precise, they are defined as follows.

Definition 2.8. Given a vector space V over a field K, whose underlying set is A, the *affine space* $\mathbb{A}(V)$ is the pair (A, V), that is, a set A together with a vector space V, and a transitive and free action of the additive group of V on the set A.

The elements of the affine space $\mathbb{A}(V)$ are called *points*, and the vector space V is said to be *associated* to the affine space. As with projective spaces, if $V = K^n$, we equivalently write $\mathbb{A}(V)$ as $\mathbb{A}^n(K) = K\mathbb{A}^n = \mathbb{A}^n K$.

Remark 2.9. *Note that affine spaces are contained in projective spaces. For instance, one may obtain an affine place from a projective plane by removing a single line; conversely, any affine plane has an extension to a projective plane, namely by adding a line at infinity.*

Now, we move on to rational functions, which are the main objects of interest in both classical and arithmetic dynamics.

Definition 2.10. A *rational function* $\phi(z) \in \mathbb{C}(z)$ is a quotient of polynomials

$$\phi(z) = \frac{F(z)}{G(z)} = \frac{a_0 + a_1 z + \cdots + a_d z^d}{b_0 + b_1 z + \cdots + b_d z^d}$$

with no common factors; the *degree* of ϕ, denoted by $\deg \phi$, is $\max\{\deg F, \deg G\}$.

Definition 2.11. A *rational map of degree d* between projective spaces is a map $\phi : \mathbb{P}^N \to \mathbb{P}^M$ such that

$$\phi(P) = [f_0(P), \ldots, f_M(P)]$$

where $f_0, \ldots, f_M \in \overline{K}[X_0, \ldots, X_N]$ are fully reduced homogeneous polynomials of degree d.

Note that ϕ is defined at P if at least one of the values $f_0(P), \ldots, f_M(P)$ is nonzero, since 0 doesn't make sense in projective space.

Definition 2.12. A *rational morphism* ϕ is a rational map that is defined at every point of $\mathbb{P}^N(\overline{K})$, that is, a rational map with $\phi^{-1}(0) = \{0\}$. If the polynomials $f_0, \ldots, f_N$ have coefficients in K, we say that ϕ *is defined over* K.

Note that a rational function of degree d naturally induces a rational map of the complex projective line $\mathbb{P}^1(\mathbb{C})$, which is just the evaluation map.

Remark 2.13. *In dynamics, we generally consider maps with* $\deg \phi \geq 2$*, since degree one maps are just Möbius transformations* $\phi(z) = \frac{az+b}{cz+d} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{C})$*, i.e., automorphisms of* $\mathbb{P}^1$*, whose behaviors are very well-studied. They can be further classified into four types with respect to the trace* $\operatorname{tr} \phi = a + d$ *(given that we rescale the numerator and the denominator so that* $\det \phi = ad - bc = 1$*): parabolic, elliptic, hyperbolic, and loxodromic. Conversely, the behaviors of maps* ϕ *with* $\deg \phi \geq 2$ *are very rich, which is why we are mostly interested in the* $\deg \phi \geq 2$ *case.*

Two very important properties of rational maps are that they are continuous and open, that is, they preserve open sets.

Exercise 2.14. Prove that a rational map is continuous and open, with a rigorous $\varepsilon - \delta$ argument. (Recall that a function $f : X \to Y$ is an *open map* if it maps open sets to open sets, that is, for all U open, $f(U)$ is also open. Conversely, a function is *continuous* if the preimage of an open set is open, that is, for all V open, $f^{-1}(V)$ is also open.)

In order to study the arithmetic properties of points in projective space, we must have a good notion of "size" for the points, so that we can measure the arithmetical complexity of a point via its size. Similar to how we defined the degree of a rational function as the maximum of the degrees of the numerator and the denominator, provided that both the numerator and the denominator are fully reduced, we define the size (called the "height") of a point as follows:

Definition 2.15. The *height* of a point $P \in \mathbb{P}^N(K)$, denoted as $H(P)$, is

$$\max_{i \in [\![0,N]\!]} |x_i|$$

where $P = [x_0, x_1, \ldots, x_N]$ is homogenized with $\gcd_{i \in [\![0,N]\!]}(x_i) = 1$ and $x_i \in \mathbb{Z}$, that is, fully reduced with no common factors.

In algebraic geometry and dynamics, we typically work with commutative rings, so all rings R are assumed to be commutative unless specified otherwise.

Definition 2.16. For a ring R, a subset $I \subset R$ is called an *ideal* of R if it is an additive subgroup of R that "absorbs" multiplication by the elements of R, that is, for every $r \in R$ and every $x \in I$, we have $rx = xr \in I$.

Next, we need the definition of the *radical* of an ideal, to state the Nullstellensatz, which means, in German, "the zero-locus theorem."

Definition 2.17. In a ring R, the *radical* of an ideal I, denoted by $\mathrm{rad}(I)$ or $\sqrt{I}$, is defined as

$$\sqrt{I} = \left\{r \in R : r^n \in I \text{ for some } n \in \mathbb{Z}^+\right\}$$

In dynamics, there are several common maps that arise from the study of commuting rational functions. What are commuting rational functions? We say f commutes with g if $f = \Theta^{-1} \circ g \circ \Theta$ for some "nice" map Θ (for an actual definition, read [Mil04]); usually, we take the domain to be a quotient of the complex plane by a lattice Λ, that is, $\mathbb{C}/\Lambda$, with some points on the boundary, which are called the *exceptional points*. The goal of studying commuting rational functions is to, well, study which rational functions commute, and it is known that commuting rational functions can be classified into the following trichotomy: power maps, Chebyshev polynomials, and Lattès maps. The criterion for classification of these maps is by the rank of the lattice Λ; the first two maps have $\operatorname{rank}\Lambda = 1$, and the second one has $\operatorname{rank}\Lambda = 2$.

First, the d^{th} power map is just a function $\phi(z) = z^d$ with $d \in \mathbb{N}$. Note that $\phi^n(z) = z^{d^n}$. Moreover, ϕ^n and ϕ^m commute. In some sense, power maps show behavior that is relatively trivial to analyze, compared to generic rational functions.

Next, for Chebyshev polynomials, these are the first non-trivial examples of commuting rational functions. Actually, there are two kinds of Chebyshev polynomials, but in this paper, we only consider Chebyshev polynomials of the first kind, that is, the unique polynomial satisfying $T_n(\frac{z+z^{-1}}{2}) = \frac{z^n+z^{-n}}{2}$. Noting that $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, we see that $T_n(\cos\theta) = \cos n\theta$. Such a trigonometric interpretation directly implies that Chebyshev polynomials commute, that is, $T_n(T_m(z)) = T_m(T_n(z))$. Sometimes, in dynamics, one rather considers the closely related polynomial $C_n(z) = 2T_n(\frac{x}{2})$ for convenient reasons, since $C_n(z + z^{-1}) = z^n + z^{-n}$. The polynomials C_n are called the *Vieta-Lucas* polynomials [Hor02].

Lastly, for Lattès maps, they are defined as follows: the map $f = \Theta \circ L \circ \Theta^{-1}$ is said to be a *Lattès map*, if $\operatorname{rank}\Lambda = 2$ (whence the quotient $\mathcal{T} = \mathbb{C}/\Lambda$ is a torus), L is an affine automorphism of the torus, and $\Theta : \mathcal{T} \to \hat{\mathbb{C}}$ is holomorphic. These maps exhibit the most interesting behaviors, and there is extensive literature on the behavior of Lattès maps alone; they are crucial to the theory of commuting rational functions. For deeper reference on Lattès maps and their rather varied behavior, consult [Mil04], which provides an excellent exposition to Lattès maps and their dynamical behavior.

Later on, we will perform analysis on v-adic absolute values, which will be crucial to local theory of height functions.

Definition 2.18. An *absolute value* on a field K is a map $|\cdot| : K \to \mathbb{R}$ with the following properties:

- $|\alpha| \geq 0$, and $|\alpha| = 0$ if and only if $\alpha = 0$.
- $|\alpha\beta| = |\alpha| \cdot |\beta|$ for all $\alpha, \beta \in K$.
- $|\alpha + \beta| \leq |\alpha| + |\beta|$ for all $\alpha, \beta \in K$ (triangle inequality).

3. Dynamics over Global Fields

Naturally, we want to know if the set of preperiodic points has bounded height, that is, with respect to the morphism ϕ. A famous result of Northcott [Nor50] tells us that the set of preperiodic points indeed has bounded height.

Theorem 3.1 (Northcott's theorem)**.** *Let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$ defined over a number field K. Then, the set of preperiodic points $\mathrm{PrePer}(\phi) \subset \mathbb{P}^N(\overline{K})$ is a set of bounded height.*

Remark 3.2. *We look at the algebraic closure $\overline{K}$ instead of K for* $\mathrm{PrePer}(\phi)$*, since preperiodic points are solutions of $\phi^n(z) = \phi^m(z)$, whose solutions are all contained in $\overline{K}$ by definition of algebraic closure.*

Observe that Northcott's theorem immediately implies the following two nice-to-have results:

Corollary 3.3. *The set*

$$\mathrm{PrePer}(\phi, \mathbb{P}^N(K)) = \mathrm{PrePer}(\phi) \cap \mathbb{P}^N(K)$$

is finite.

Corollary 3.4. *For fixed $D \in \mathbb{N}$, the set*

$$\bigcup_{[L:K] \leq D} \mathrm{PrePer}(\phi, \mathbb{P}^N(L))$$

is finite.

In order to prove Northcott's theorem, we first need some preliminary results, some of which are cited without proof, since it suffices to only know the statement for our purposes.

Theorem 3.5 (Hilbert's Nullstellensatz)**.** *Let I and J be homogeneous ideals properly contained in $\overline{K}[X_0, \ldots, X_N]$, then*

$$V(I) = V(J) \iff \sqrt{I} = \sqrt{J}$$

Proof. One direction is relatively trivial: $\sqrt{I} = \sqrt{J}$ implies $V(I) = V(J)$, since for some point $P \in V(I)$ and $f \in J$, we have $f^n \in I$ for some $n \in \mathbb{N}$, so $f^n(P) = 0$, thus $f(P) = 0$, implying $P \in V(J)$. This implies that $V(I) \subset V(J)$, and vice versa. The other direction typically needs some techniques, and although there are some proofs that avoid it, this is beyond the scope of this paper, so we leave it to the interested reader to look up the proof. For reference, some often cited proofs are in [Art98] and [AM69]. □

For ease of notation, we define the following:

Definition 3.6. For $P = [x_0, x_1, \ldots, x_N] \in \mathbb{P}^N(K)$ and any absolute value $v \in M_K$, write the *absolute value of a point* to be

$$|P|_v = \max_i |x_i|_v$$

More generally, define *the absolute value of a polynomial*

$$f(X_0, \ldots, X_N) = \sum_{i_0, \ldots, i_N} a_{i_0 \ldots i_N} X_0^{i_0} \ldots X_N^{i_N}$$

to be

$$|f|_v = \max_{i_0, \ldots, i_N} |a_{i_0 \ldots i_N}|_v$$

Further, if $\phi = [f_0, \ldots, f_M]$ is a collection of polynomials, let $|\phi|_v = \max_j |f_j|_v$.

Note that the height of a point $P \in \mathbb{P}^N(K)$ can be written in the form

$$H(P) = \left(\prod_{v \in M_K} |P|_v^{n_v} \right)^{1/[K:\mathbb{Q}]}$$

We define the height of a polynomial f or a collection of polynomials ϕ similarly,

$$H(f) = \left(\prod_{v \in M_K} |f|_v^{n_v} \right)^{1/[K:\mathbb{Q}]}$$

and also

$$H(\phi) = \left(\prod_{v \in M_K} |\phi|_v^{n_v} \right)^{1/[K:\mathbb{Q}]}$$

Following the notation of [Sil07], we define $\delta_v(m)$ as follows, which will let us write a uniform version of the triangle inequality as

$$|x_1 + \cdots + x_m|_v \le \delta_v(m) \max\{|x_1|_v, \ldots, |x_m|_v\}$$

Definition 3.7. For any absolute value $v \in M_K$ and any number m, we set

$$\delta_v(m) = \begin{cases} m & \text{if } v \in M_K^\infty \text{ (i.e., if } v \text{ is archimedean),} \\ 1 & \text{if } v \in M_K^0 \text{ (i.e., if } v \text{ is nonarchimedean).} \end{cases}$$

Now, with these definitions, we state an important lemma that will almost immediately imply Northcott's theorem:

Lemma 3.8. *Let $\phi : \mathbb{P}^N(\overline{K}) \to \mathbb{P}^M(\overline{K})$ be a morphism of degree d. Then, there exist constants $C_1 = C_1(\phi)$ and $C_2 = C_2(\phi)$ with $C_1, C_2 > 0$, such that*

$$C_1 \le \frac{H(\phi(P))}{H(P)^d} \le C_2$$

for all $P \in \mathbb{P}^N(\overline{K})$, or equivalently, $H(\phi(P)) = \Theta(H(P)^d)$, using asymptotic notation.

Intuitively, this means that a morphism of degree d raises the height to the d^{th} power.

Proof. Let $P = [x_0, \ldots, x_N] \in \mathbb{P}^N(K)$ be a rational point, and $f \in K[X_0, \ldots, X_N]$ a homogeneous polynomial of degree d. Then, for any $v \in M_K$ we can estimate

$$\begin{aligned} |f(P)|_v &= \left| \sum_{\substack{i_0,\ldots,i_N \ge 0 \\ i_0+\cdots+i_N=d}} a_{i_0\ldots i_N} x_0^{i_0} \ldots x_N^{i_N} \right|_v \\ &\le \delta_v(\#\text{ of terms}) \max_{i_0,\ldots,i_N} \left| a_{i_0\ldots i_N} x_0^{i_0} \ldots x_N^{i_N} \right|_v \end{aligned}$$

The number of terms in the sum is equal to at most the number of monomials of degree d in $N+1$ variables, which is, by stars and bars, $\binom{N+d}{d}$. Continuing with the computation, we find that

$$\begin{aligned} |f(P)|_v &\le \delta_v\left(\binom{N+d}{d}\right) \max_{i_0,\ldots,i_N} |a_{i_0\ldots i_N}|_v \max_{i_0,\ldots,i_N} \max_{j \in [\![0,N]\!]} |x_j|_v^{i_0+\cdots+i_N} \\ &= \delta_v\left(\binom{N+d}{d}\right) |f|_v |P|_v^d \end{aligned}$$

Applying this for $f_0, f_1, \ldots, f_N$ and taking the maximum again, we get

$$|\phi(P)|_v \le \delta_v\left(\binom{N+d}{d}\right) |\phi|_v |P|_v^d$$

Now, raising this to the $n_v{}^{\text{th}}$ power, then multiplying over all valuations $v \in M_K$, we get

$$\prod_{v \in M_K} |\phi(P)|_v^{n_v} \le \prod_{v \in M_K} \delta_v^{n_v}\left(\binom{N+d}{d}\right) \prod_{v \in M_K} |\phi|_v^{n_v} |P|_v^{d \cdot n_v}$$

Taking the $[K:\mathbb{Q}]^{\text{th}}$ root, we obtain

$$H(\phi(P)) \leq \binom{N+d}{d} H(\phi)H(P)^d$$

recalling

$$\prod_{v\in M_K} \delta_v(a)^{n_v} = \prod_{v\in M_K^\infty} a^{n_v} = a^{\sum_{v\in M_K^\infty} n_v} = a^{[K:\mathbb{Q}]}$$

This gives the desired constant upper bound for $\frac{H(\phi(P))}{H(P)^d}$. For the lower bound, take ϕ be a morphism, then we may write $\phi = [f_0, \ldots, f_M]$ for homogeneous polynomials $f_0, \ldots, f_M$ which do not have any common zeros in $\mathbb{P}^N(\overline{K})$, as we can clear all common factors. This implies that the ideals $(f_0, \ldots, f_M)$ and $(X_0, \ldots, X_N)$ in $\overline{K}[X_0, \ldots, X_N]$ have the same algebraic set, that is, the empty set. Now, by the Nullstellensatz, those two ideals have the same radical, that is, $X_0, X_1, \ldots, X_N \in \sqrt{(f_0, f_1, \ldots, f_M)}$, thus $\exists e = \max_i e_i$ such that $X_i^{e_i} \in (f_0, \ldots, f_M)$, implying $X_i^e \in (f_0, \ldots, f_M)$. By this, we know that there are homogeneous polynomials $g_{ij} \in \overline{K}[X_0, \ldots, X_N]$ such that $X_i^e = \sum_{j\in[\![0,M]\!]} g_{ij} f_j$ for each $i \in [\![0, N]\!]$. Note that $\deg g_{ij} = e - d$ since $\deg f_j = d$ for all j. Evaluating this at $P = [x_0, \ldots, x_N] \in \mathbb{P}^N(K)$, we have

$$x_i = g_{i0}(P)f_0(P) + g_{i1}(P)f_1(P) + \cdots + g_{iM}(P)f_M(P)$$

for all $0 \leq i \leq N$. Now, estimating v-adic absolute values, we have

$$\begin{aligned}
|P|_v^e &= \max_{0\leq i\leq N} |x_i|_v^e \\
&= \max_{0\leq i\leq N} \left| \sum_{j\in[\![0,M]\!]} g_{ij}(P)f_j(P) \right|_v \\
&\leq \delta_v(M+1) \max_{(i,j)\in[\![0,N]\!]\times[\![0,M]\!]} |g_{ij}(P)f_j(P)|_v \\
&\leq \delta_v(M+1) \max_{(i,j)\in[\![0,N]\!]\times[\![0,M]\!]} \left\{ \delta_v\left(\binom{N+e-d}{e-d}\right) |g_{ij}|_v \, |P|_v^{e-d} \, |f_j(P)|_v \right\} \\
&\leq \delta_v\left((M+1)\binom{N+e-d}{e-d}\right) \left(\max_{(i,j)\in[\![0,N]\!]\times[\![0,M]\!]} |g_{ij}|_v \right) |P|_v^{e-d} |\phi(P)|_v
\end{aligned}$$

With this, we have some constant $C = C(M, N, d, e) = (M+1)\binom{N+e-d}{e-d}$, and letting $|g|_v = \max_{i,j} |g_{ij}|_v$, we get

$$|P|_v^d \leq \delta_v(C)|g|_v|\phi(P)|_v$$

and as done before, we take the ${n_v}^{\text{th}}$ power, then multiplying over all valuations $v \in M_K$, and finally take the $[K:\mathbb{Q}]^{\text{th}}$ root, which gives $H(P)^d \leq CH(g)H(\phi(P))$, where $H(g)$ is a constant with respect to P. Indeed, this gives the desired constant lower bound for $\frac{H(\phi(P))}{H(P)^d}$, and we are done. □

Lemma 3.8 tells us that a morphism ϕ of degree d basically sends the height $H(P)$ to approximately the d^{th} power, which tells us that H acts like a multiplicative function. Regarding notation, it is often easier to work with additive functions instead, since otherwise we have to write power towers all the time; nobody likes to do that. This prompts the following definition:

Definition 3.9. The *logarithmic height*, with respect to K, is the function $h_K : \mathbb{P}^N(K) \to \mathbb{R}$ such that $h_K(P) = \log H_K(P)$. Analogously, the *absolute logarithmic height* is the function $h : \mathbb{P}^N(\overline{Q}) \to \mathbb{R}$ where $h(P) = \log H(P)$.

Using this notation, Lemma 3.8 is just $h(\phi(P)) = dh(P) + O(1)$. Now, we are ready to prove Northcott's theorem.

Proof of Theorem 3.1. From Lemma 3.8, we know that $h(\phi(P)) \geq dh(P) - C$ for some constant $C = C(\phi)$ and all points $P \in \mathbb{P}^N(\overline{K})$. Inductively applying this to P, $\phi(P)$, $\phi^2(P)$, $\ldots$, $\phi^{n-1}(P)$ gives

$$h(\phi^n(P)) \geq d^n h(P) - C(1 + d + d^2 + \cdots + d^{n-1}) \geq d^n h(P) - d^n C$$

and thus $h(\phi^n(P)) \geq d^n(h(P) - C)$. Since $P \in \mathrm{PrePer}(\phi)$, we know that $\exists n, k$ with $n \geq 0$ and $k \geq 1$ such that $\phi^n(P) = \phi^{n+k}(P)$. Hence, from $h(\phi^n(P)) \geq d^n(h(P) - C)$, substituting $\phi^n(P)$ in place of P, we get

$$h(\phi^n(P)) = h(\phi^{n+k}(P)) \geq d^k(h(\phi^n(P)) - C)$$

thus

$$h(\phi^n(P)) \leq \frac{d^k}{d^k - 1} C \leq 2C$$

Now, from the original equation

$$h(P) \leq \frac{h(\phi^n(P)) + d^n C}{d^n}$$

combining this with the previous equation, we get

$$h(P) \leq \frac{2C + d^n C}{d^n} \leq 3C$$

Thus, for some $C = C(\phi)$, we know that $h(P)$ is bounded, and so is $H(P)$, as desired. □

Indeed, it is possible to give explicit bounds for $|\operatorname{PrePer}\left(\phi, \mathbb{P}^N(K)\right)$ in terms of ϕ, and there are several results on this, especially when $N = 1$. Our method of proving Northcott's theorem was in a sense global, whereas other local methods of using primes of good reduction also exist.

Let us take a brief moment to recall what we had until now. Theorem 3.1 tells us that the height of preperiodic points is bounded by a constant that is solely dependent on the morphism ϕ. Can we do any better? Is there a universal upper bound that works for any morphism of fixed degree, or even bounded degree? The following conjecture, still open at the time of writing this article, asks this question.

Conjecture 3.10 (Uniform boundedness conjecture)**.** *Fix a number field $K/\mathbb{Q}$ with $[K : \mathbb{Q}] \leq D$, and consider all finite morphisms $\phi : \mathbb{P}^N \to \mathbb{P}^N$ defined over K with $\deg \phi = d$ for some fixed d. Then, there is a universal constant $C = C(d, N, D)$ such that*

$$\left|\operatorname{PrePer}\left(\phi, \mathbb{P}^N(K)\right)\right| \leq C(d, N, D)$$

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