Ocean Dynamics

EAS 8803

week 1

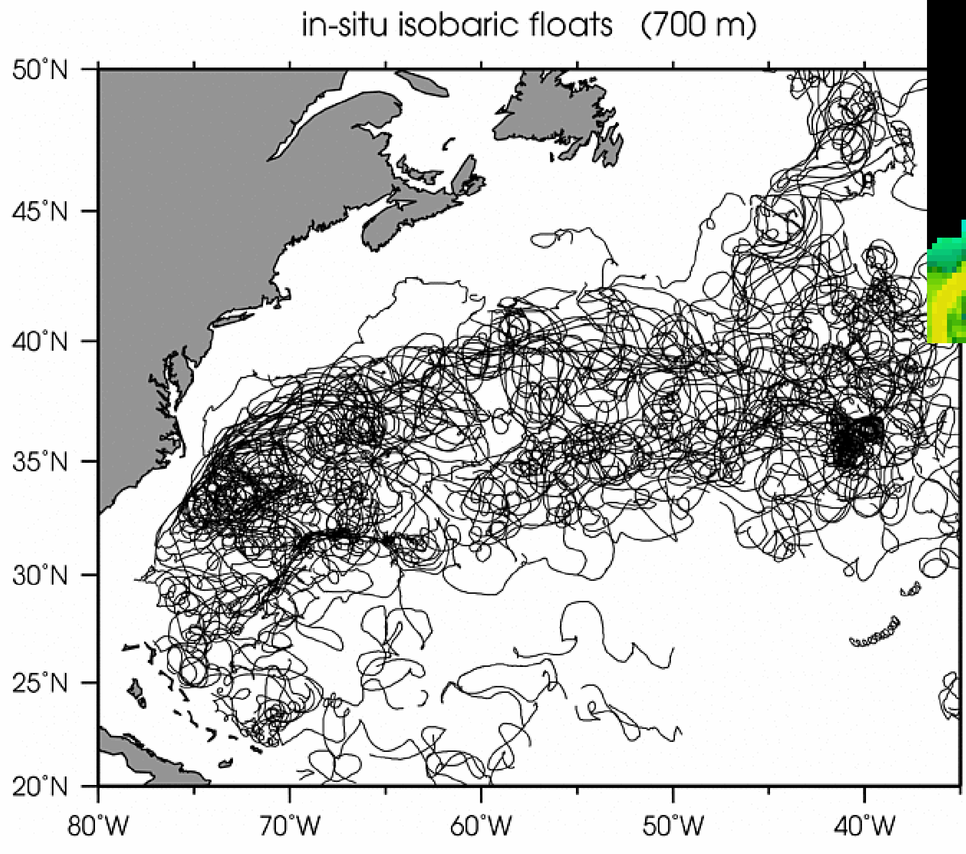
- Lagrangian vs Eulerian viewpoint: Time derivatives for fluids
- The mass continuity equation
- The momentum equation
- The equation of state (brief intro)
- (Compressible and) incompressible flows
- The energy budget

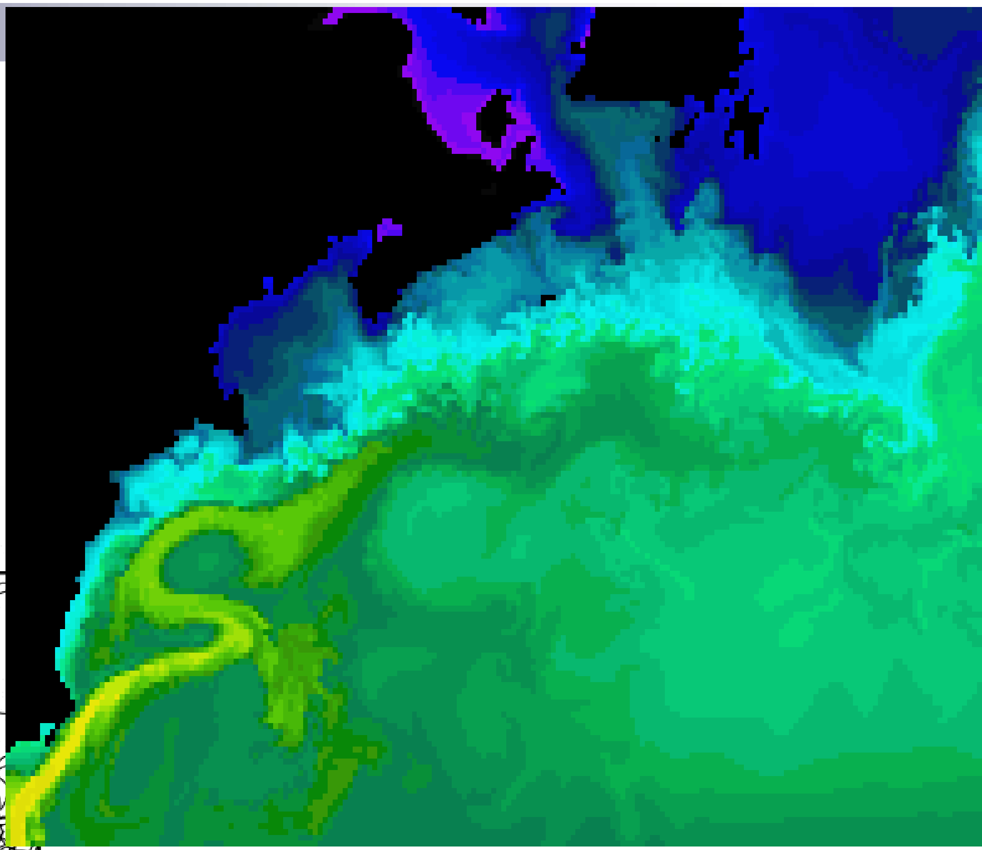
Lagrangian vs Eulerian

- Fluids form a continuum, flow and deform. Newton's law are still valid, but have to be expressed for fluids (same for thermodynamics)

- The description of fluid motions in terms of positions and momenta for each fluid particle (or fluid material volumes) is called ***Lagrangian view*** or material view. Very good in principle, but difficult to implement.

- The description of fluid motions at locations that are fixed is space is called ***Eulerian method***

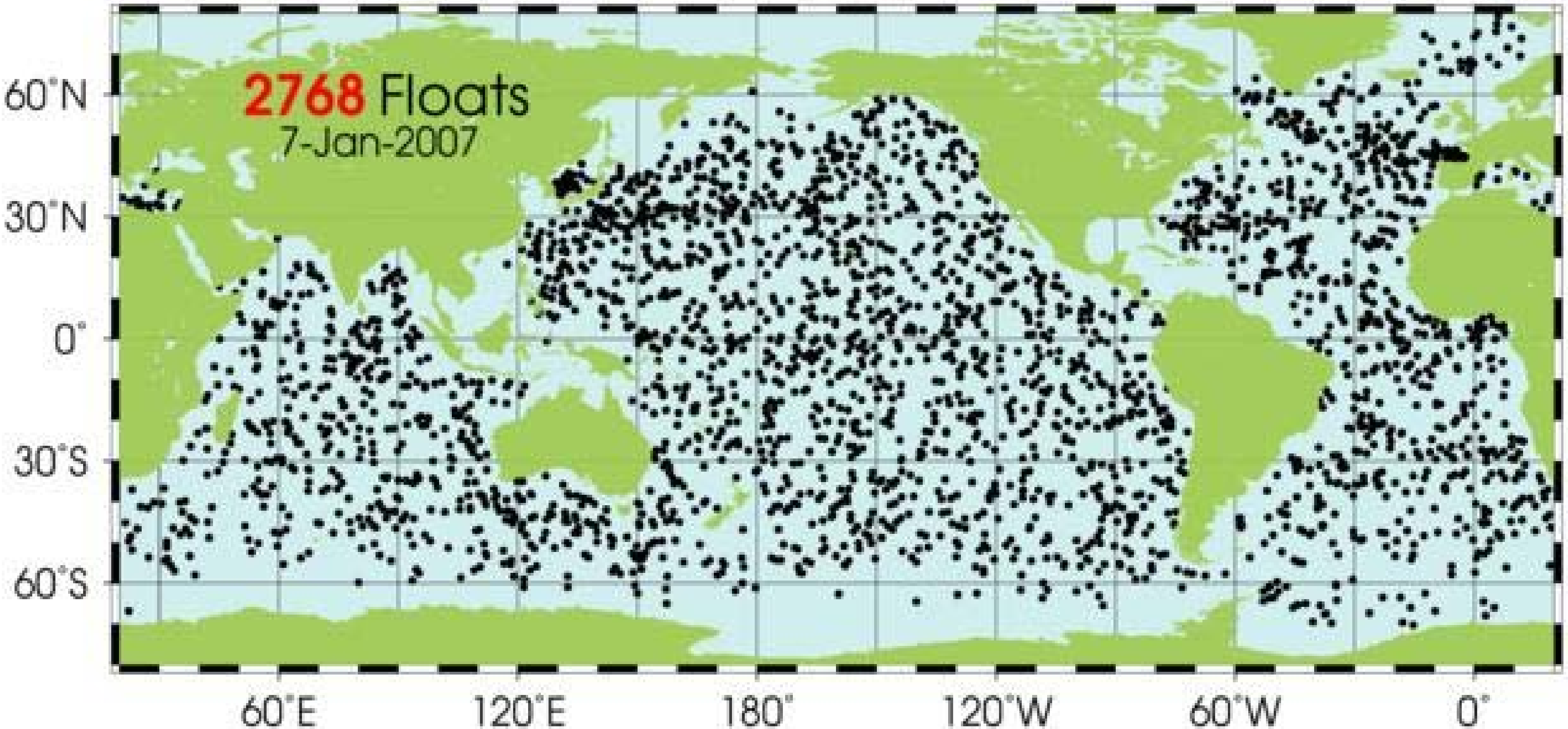

in-situ isobaric floats (700 m)
50°N
45°N
40°N
35°N
30°N
25°N
20°N
80°W
70°W
60°W
50°W
40°W

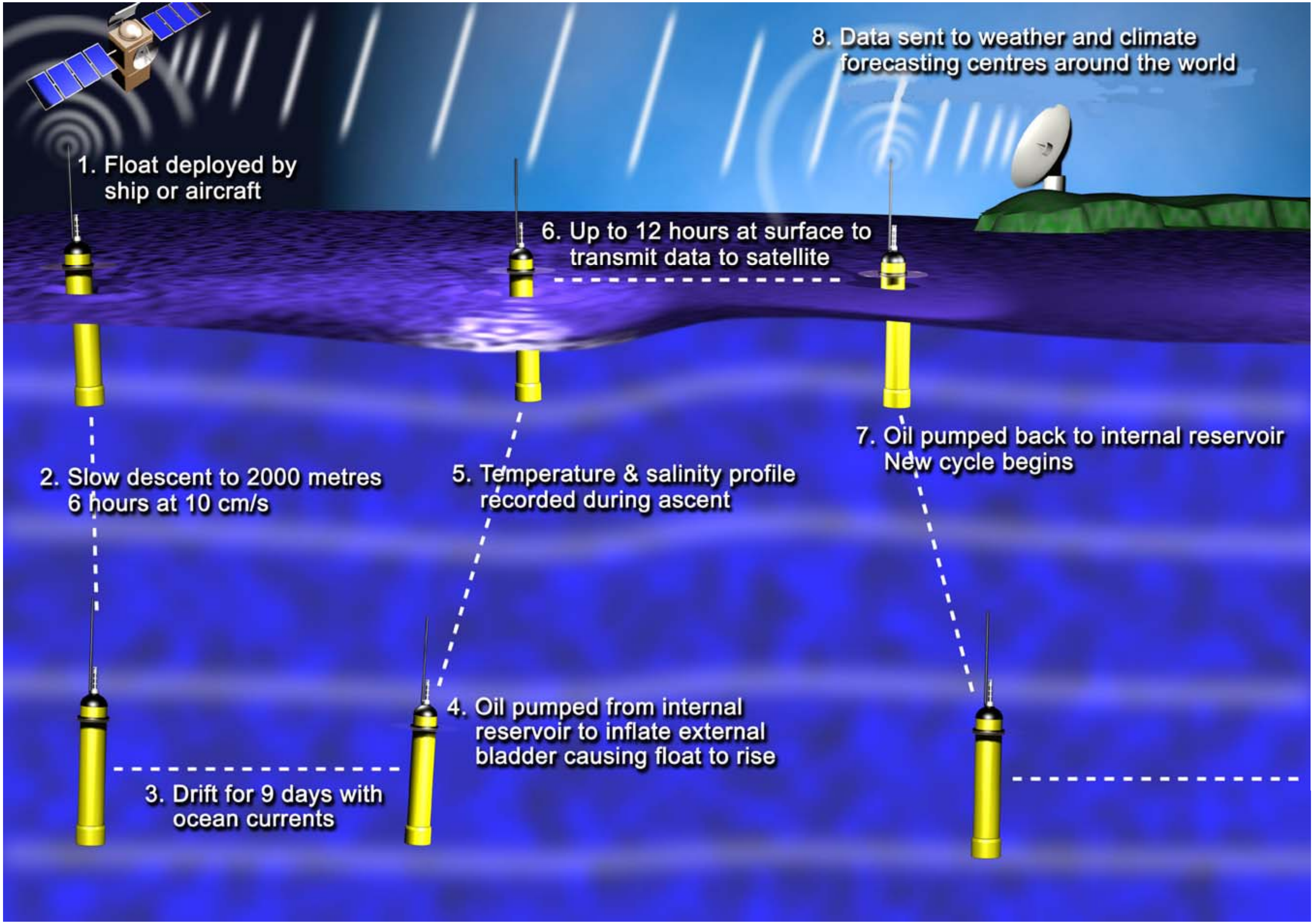


Argo floats

Argo is a global array of 3,000 free-drifting profiling floats that measures the temperature and salinity of the upper 2000 m of the ocean. This allows, for the first time, continuous monitoring of temperature, salinity, and velocity of the upper ocean, with all data made publicly available within hours after collection

http://www.argo.ucsd.edu/




8. Data sent to weather and climate forecasting centres around the world
1. Float deployed by ship or aircraft
6. Up to 12 hours at surface to transmit data to satellite
7. Oil pumped back to internal reservoir New cycle begins
2. Slow descent to 2000 metres 6 hours at 10 cm/s
5. Temperature & salinity profile recorded during ascent
4. Oil pumped from internal reservoir to inflate external bladder causing float to rise
3. Drift for 9 days with ocean currents

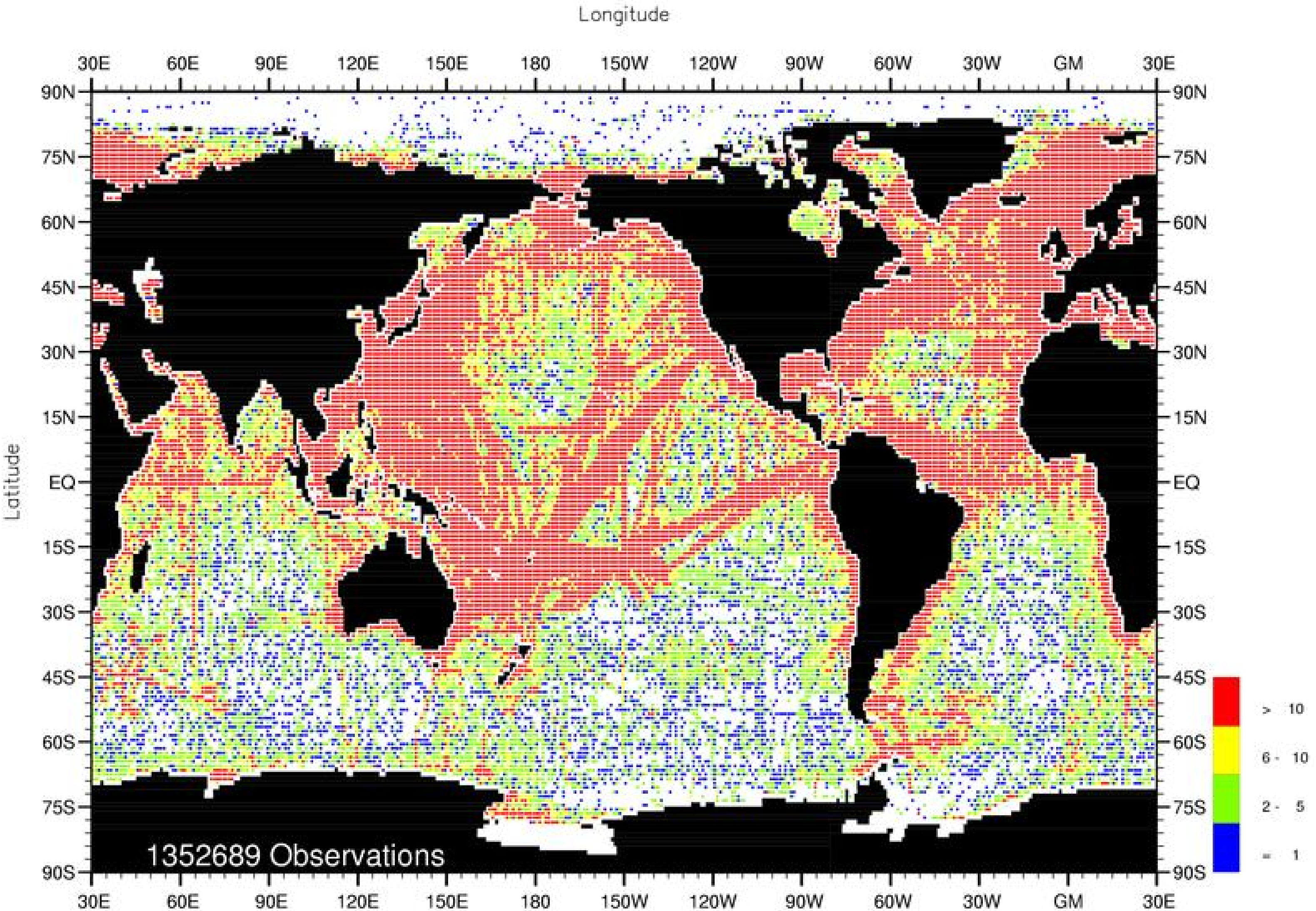


Fig. A1-1. Annual salinity observations at the surface.

World Ocean Atlas 2001
Ocean Climate Laboratory/NODC

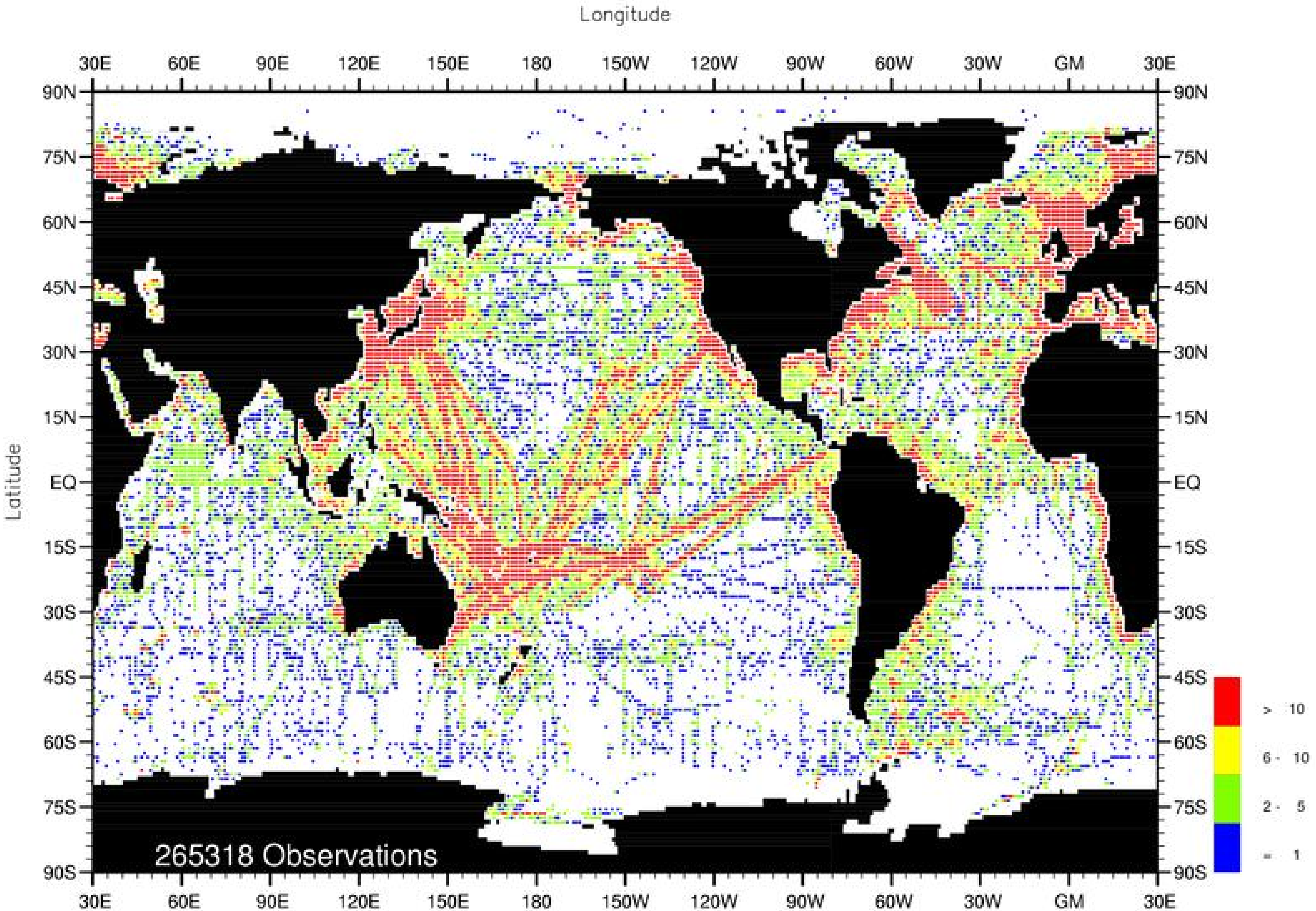


Fig. E1-1. Fall (Oct.-Dec.) salinity observations at the surface.

World Ocean Atlas 2001
Ocean Climate Laboratory/NODC

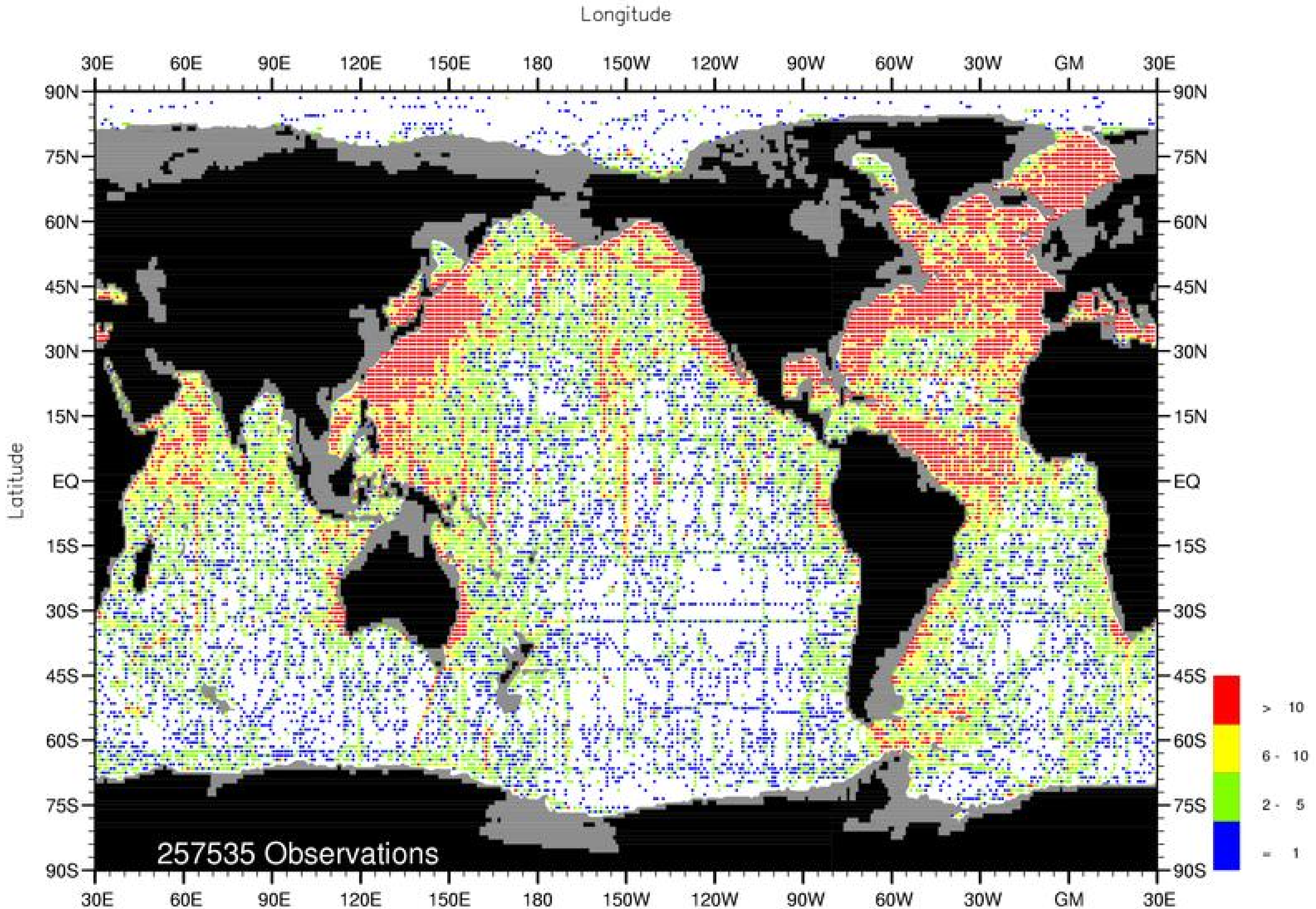


Fig. A1-19. Annual salinity observations at 1000 m. depth.

World Ocean Atlas 2001
Ocean Climate Laboratory/NODC

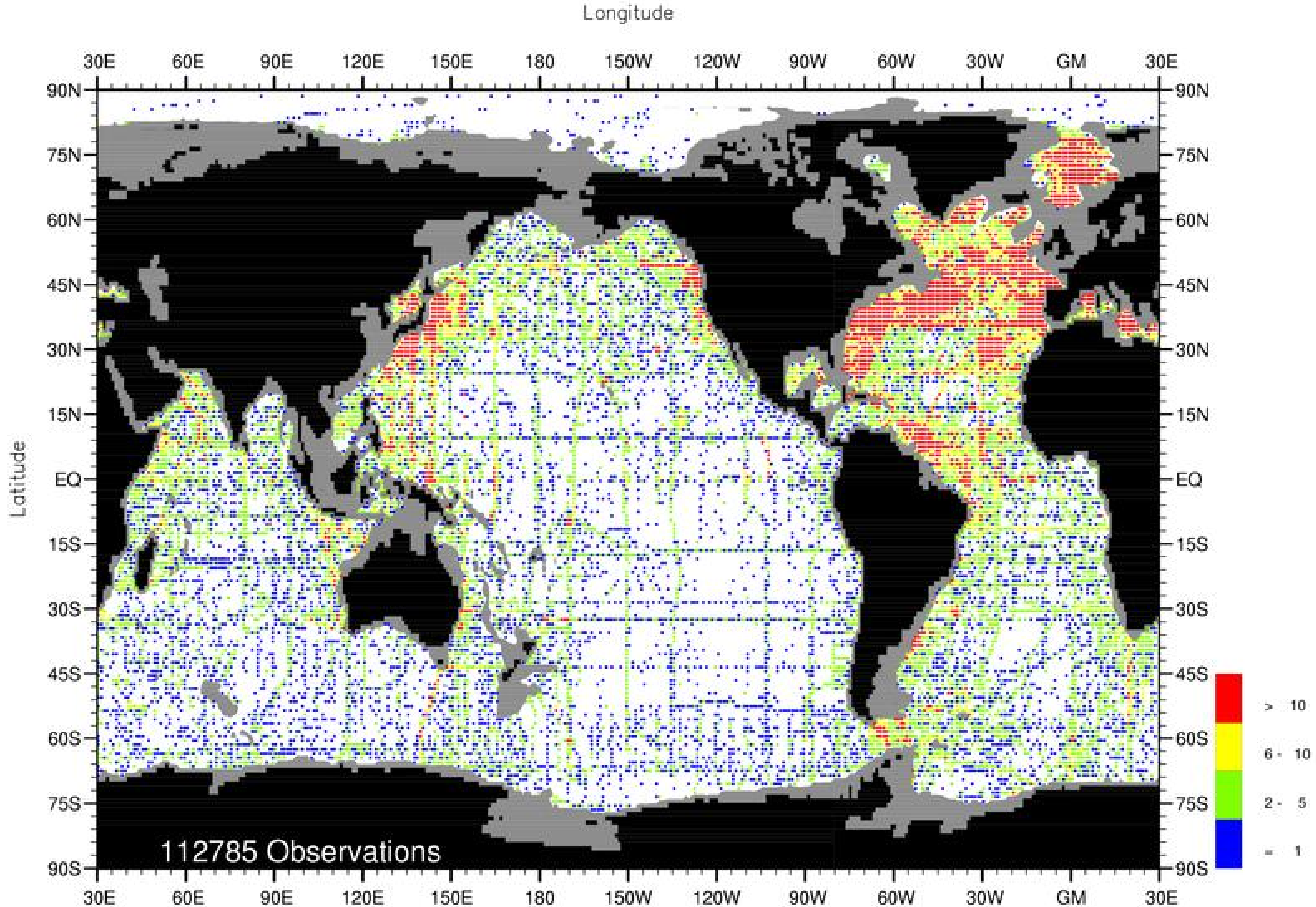


Fig. A1-26. Annual salinity observations at 2000 m. depth.

World Ocean Atlas 2001
Ocean Climate Laboratory/NODC

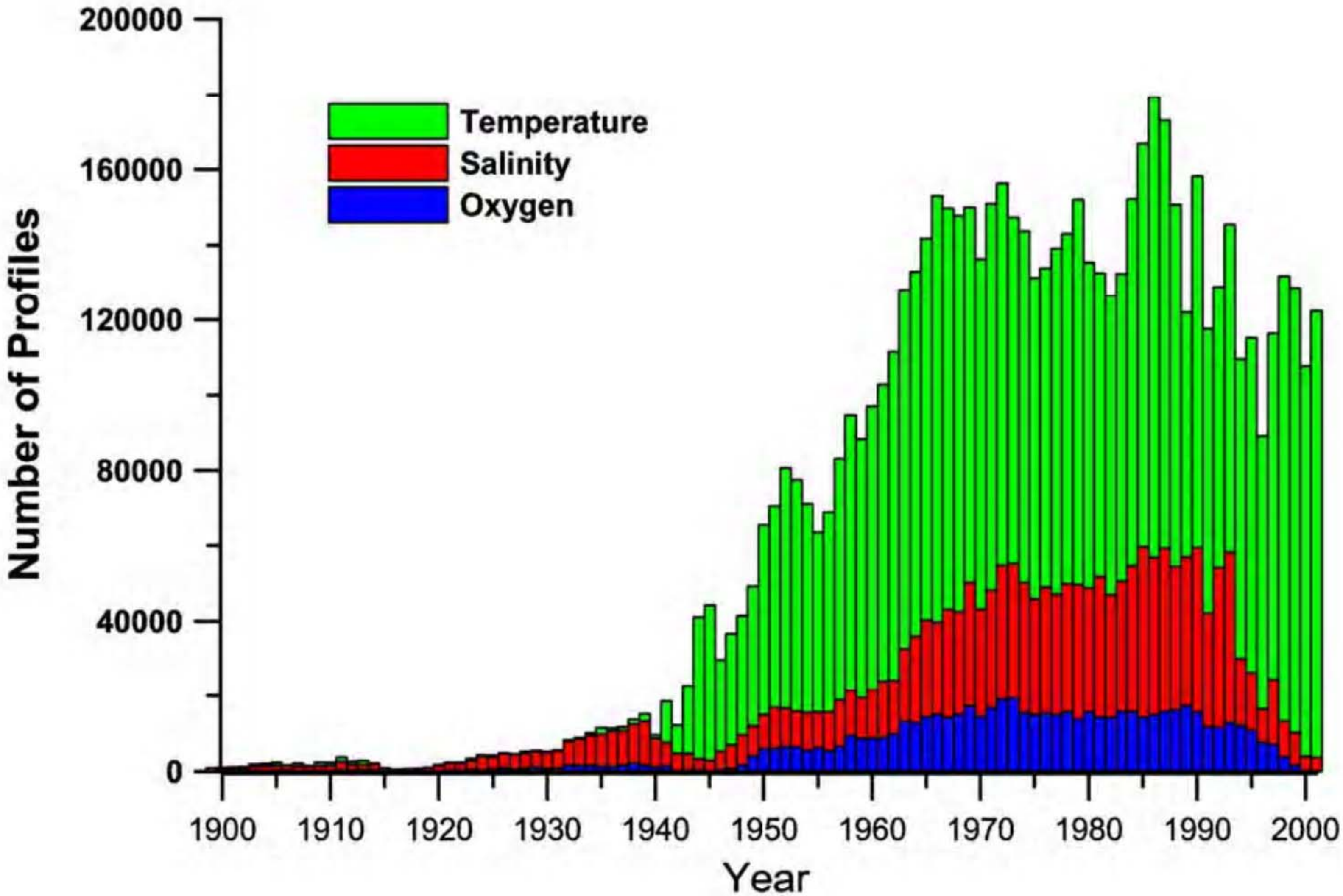


Figure 1. Number of Profiles per year in the World Ocean Database 2001

The Fundamental Principle of Kinematics

- In the Lagrangian framework let's use Greek letters to denote the position vector of a fluid parcel:

$$\vec{\xi} = (\xi, \psi, \omega)$$

- We need to identify different fluid parcels (we need to tag them) - overall we need a continuum description of our flow - let's use the initial position at t= 0 of the parcel,

$$\vec{A} = (\alpha, \beta, \gamma)$$

then the position of the parcel (or particle) at all later times to form a particle trajectory or ***pathline*** is given by

$$\vec{\xi} = \vec{\xi}(\vec{A}, t)$$

- In the Lagrangian description the trajectory **ξ** is a dependent variable (with **p** and **ρ**), while the initial position **A** and time are independent.
- The velocity is the rate of change of the parcel positioning holding **A** fixed

$$\frac{D}{Dt} = \frac{d}{dt} \qquad for \qquad \vec{A} = const$$

$$i.e.$$

$$\vec{V}_L(\vec{A},t) = \frac{D\vec{\xi}(\vec{A},t)}{Dt} = \frac{\partial\vec{\xi}(\vec{A},t)}{\partial t}$$

- If the velocity of a fluid is sampled at a fixed position **x** than velocity measured is called ***Eulerian velocity,*** $\vec{V}_E$

- In the Eulerian framework $\mathbf{V}_E$ is a dependent variable (along with **p** and **ρ**), while **x** and t are independent

- **The FPK states that**

$$\vec{V}_E(\vec{x},t)\,|_{\vec{x}=\vec{\xi}(\vec{A},t)} = \vec{V}_L(\vec{A},t)$$

- The FPK is valid instantaneously, and does not survive time-averaging.
- Note that **ξ** is the position of a moving particle, while **x** does not change

- The are **not** two different velocities in the flow: simply two different ways to sample them

- In principle the two representations are equivalent and can be inverted

Lagrangian repres

Eulerian repres

$$\vec{\xi} = \vec{\xi}(\vec{A}, t) \quad \Leftrightarrow \quad \vec{A} = \vec{A}(\vec{\xi}, t)$$

example (1D case)

- Given $$\xi(\alpha,t) = \alpha(1+2t)^{1/2}$$

find $V_L(\alpha,t)$ and the acceleration

and then calculate the Eulerian velocity

Solution

$$V_L(\alpha,t) = \frac{\partial \xi}{\partial t} = \alpha(1+2t)^{-1/2}$$

$$\frac{\partial^2 \xi}{\partial t^2} = -\alpha(1+2t)^{-3/2}$$

$$given \qquad that \quad V_E(x,t) = V_L(A(\xi,t),t)$$

$$and \qquad \alpha = \xi(1+2t)^{-1/2} \qquad then$$

$$V_E(x,t) = x(1+2t)^{-1}$$

The material derivative

- if the fluid parcel has another property φ that changes in time and space, an infinitesimal change of φ is given by

$$\delta\varphi = \frac{\partial\varphi}{\partial t}\delta t + \frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z = \frac{\partial\varphi}{\partial t}\delta t + \delta\vec{x}\cdot\nabla\varphi$$

The total derivative is then

$$\frac{d\varphi}{dt} \equiv \frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + \frac{d\vec{x}}{dt}\cdot\nabla\varphi = \frac{\partial\varphi}{\partial t} + \vec{v}\cdot\nabla\varphi = \frac{\partial\varphi}{\partial t} + (\vec{v}\cdot\nabla)\varphi$$

- for a vector field becomes

$$\frac{D\vec{b}}{Dt} = \frac{\partial \vec{b}}{\partial t} + u\frac{\partial \vec{b}}{\partial x} + v\frac{\partial \vec{b}}{\partial y} + w\frac{\partial \vec{b}}{\partial z}$$

- and for a volume

$$\frac{D}{Dt}\int_V dV = \int_S \vec{v}\cdot d\vec{S} = \int_V \nabla\cdot\vec{v}\,dV$$

leibnitz's formula   divergence theorem

Mass continuity equation

In classical mechanics **mass is conserved**. The mass conservation equation can be derived in different ways. We discuss two (two more on G. Vallis book)

1) Consider an infinitesimal control volume $\Delta V = \Delta x\, \Delta y\, \Delta z$ The change in the fluid content within the control volume happens through its surface

(from Vallis' book)

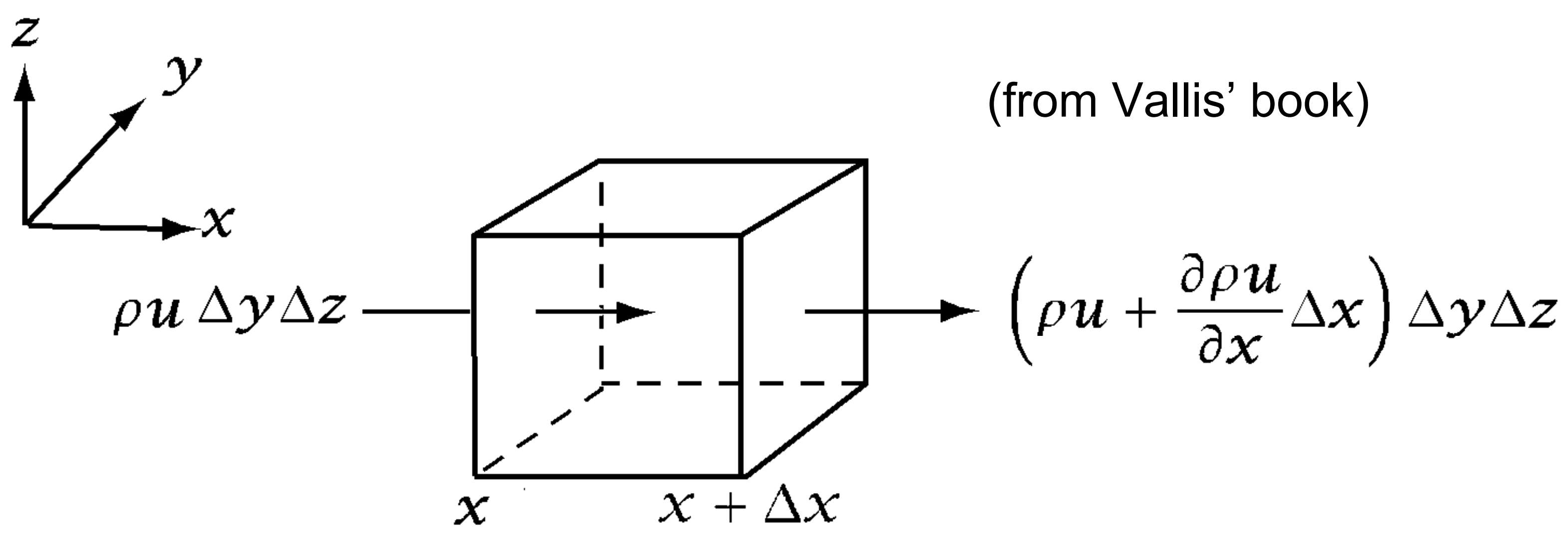


Fig. 1.1 Mass conservation in a cubic Eulerian control volume.

$$\Delta y \Delta z[(\rho u)(x,y,z)-(\rho u)(x+\Delta x,y,z)]+$$

$$\Delta x \Delta z[(\rho v)(x,y,z)-(\rho v)(x,y+\Delta y,z)]+$$

$$+\Delta x \Delta y[(\rho w)(x,y,z)-(\rho w)(x,y,z+\Delta z)]=$$

$$-\left|\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right|\Delta x \Delta y \Delta z$$

Which has to be balanced by an increase (or decrease) in fluid density within the volume

$$\frac{\partial}{\partial t}\left[density \times volume\right] = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

Therefore because mass is conserved we get

$$\Delta x \Delta y \Delta z \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] = 0 \quad \Rightarrow$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

2) The Lagrangian perspective

The mass conservation simply states that the mass of a fluid element is constant. Therefore

$$\frac{D}{Dt}\left(\rho\Delta V\right)=\frac{D}{Dt}\int_V \rho dV = 0$$

Both volume and density may change, so

$$\int_V \left(\frac{D\rho}{Dt}+\rho\nabla\cdot\vec{v}\right)dV = 0 \qquad \Rightarrow$$

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\vec{v}$$

The momentum equation

The momentum eq. is a partial differential equation that describes **how the momentum** (and therefore the velocity) **of a flow changes whenever internal and/or external forces are applied**

The momentum eq. expresses a balance of acceleration and forces (i.e., Newton's law, $F=ma$, where F is force, m is mass and $\mathbf{m}=\rho\mathbf{v}$ is momentum)

Let $\boldsymbol{m}(x,y,z,t)=\rho\boldsymbol{v}$ be the momentum density field (i.e. momentum per unit volume). The total momentum in a given volume is simply $\int_{v} \vec{m}\, dV$

Its rate of change for a fluid parcel is given by the material derivative, and is equal to the force acting on it (Newton' second law)

$$\frac{D}{Dt}\int_{V} \rho\vec{v}\, dV = \int_{V} \vec{F}\, dV$$

but ρdV is the mass of the fluid parcel which is constant. Therefore

$$\int_{V}\left(\rho\frac{D\vec{v}}{Dt}-\vec{F}\right)dV=0 \qquad \Rightarrow \qquad \frac{\partial\vec{v}}{\partial t}+\left(\vec{v}\cdot\nabla\right)\vec{v}=\frac{\vec{F}}{\rho}$$

nonlinear term

The pressure force

It is the most ‘obvious’ one. The pressure force is the normal force per unit area due to the collective action of molecular motion and directed inwards (whereas **S** is a vector normal to the surface and directed outwards)

$$\hat{F}_p = -\int_S p d\vec{S} = -\int_V \nabla p dV$$

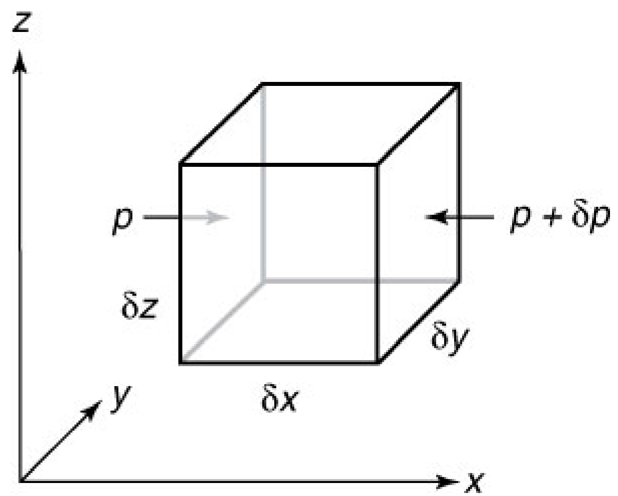
Therefore inserting it in the momentum eq. we obtain

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \nabla\right)\vec{v} = -\frac{\nabla p}{\rho} + \vec{F}\,'$$

where ***F’=(F-F**$_p$**)/ρ*** represents viscous and body forces per unit mass

Derivation of Pressure Term

Consider the forces acting on the sides of a small cube of fluid.



The net force δF_x in the x direction is

$\delta F_x = p\ \delta y \delta z - (p + \delta p)\ \delta y \delta z\ \delta F_x = -\delta p\ \delta y\ \delta z$

which can be re-written as

$$\delta F_x = -\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$$

$$\delta F_x = -\frac{\partial p}{\partial x}\,\delta V$$

and therefore **F'=** $-\frac{1}{\rho}\nabla p$

the *y* and *z* directions are derived in the same way.

The viscous forces: viscosity and diffusion

Viscosity is due to the internal motion of molecules.

For a constant density fluid viscosity is the only way energy can be removed from the fluid. It's very important if the fluid has to reach an equilibrium.

For most Newtonian fluids the viscous force per unit volume is ~ $\mu\nabla^2\vec{v}$, where μ is the viscosity coefficient. The momentum eq. becomes

$$\frac{\partial\vec{v}}{\partial t}+\left(\vec{v}\cdot\nabla\right)\vec{v}=-\frac{\nabla p}{\rho}+\nu\nabla^2\vec{v}+\vec{F}_b$$

where $\nu=\mu/\rho$ is the kinematic viscosity and F_b represents any body force

- A **Newtonian fluid** is a fluid that flows like water—its stress / strain curve is linear and passes through the origin. The constant of proportionality is known as the viscosity.
- A simple equation to describe Newtonian fluid behavior is $$\vec{\tau} = \mu \nabla \vec{v}$$
- where
  - τ is the shear stress exerted by the fluid ("drag") [Pa]
  - μ is the fluid viscosity [Pa·s]
  - $\nabla \vec{v}$ is the velocity gradient perpendicular to the direction of shear [s−1]

- This implies that the fluid continues to flow, regardless of the forces acting on it. For example, water is Newtonian, because it continues to exemplify fluid properties no matter how fast it is stirred or mixed. Contrast this with a non-Newtonian fluid, in which stirring can leave a "hole" behind (that gradually fills up over time - this behavior is seen in materials such as pudding, or, to a less rigorous extent, sand or toothpaste), or cause the fluid to become thinner, the drop in viscosity causing it to flow more (this is seen in non-drip paints).
- For a Newtonian fluid, the viscosity, by definition, depends only on temperature and pressure, not on the forces acting upon it.

Experimental values of viscosity for air and water:

	μ (kg $m^{-1}s^{-1}$)	ν (m^2s^{-1})
air	1.8×10^{-5}	1.5×10^{-5}
water	1.1×10^{-3}	1.1×10^{-6}

Viscosity is very small for both air and water. This brings us to three common statements about GFD:

1. **Advection usually dominates over molecular diffusion**
   Using a simple scale analysis if V is the characteristic velocity of the fluid and L its characteristic length for flow variations, this can be checked evaluating the **Reynolds number**, *Re*, given by $\mathrm{Re} = \frac{VL}{\nu}$
   In the ocean a modest velocity of 0.1 ms^{-1} over a distance of 100m gives *Re* ~ $10^8 >> 1$

2. Whenever *Re* >>1, the typical time scale for the evolution of the flow is the *advective time, T=L/V* (which is the passage time for some material pattern to be carried past a fixed point x). This is because the diffusion time scale ($T=L^2/\nu$) is much longer and hence relatively ineffective. btw: The ratio of those two time scale is equal to?
3. Almost all flows of geophysical interest are unstable and full of fluctuations (***turbulent***) within an advective time scale once *Re* is above a critical value of O(10-100), in contrast with laminar flows for which Re is smaller.

The mass continuity + momentum eqs., describing the motion of a fluid, are called *Euler equations* if the viscous term is omitted and *Navier-Stokes equations* if viscosity is included.

Hydrostatic balance

With a good approximation the component of the momentum equation parallel to the gravitational force is simply $\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$

if the fluid is static we have the so-called hydrostatic balance $\frac{\partial p}{\partial z} = -\rho g$

This is a good approximation if vertical accelerations are smalls compared to gravity, which is reasonable in the case of the ocean as a order 0 approximation.

When and how we can use the hydrostatic balance in the ocean?

Let's use a simple scaling argument: If we consider the vertical component of the inviscid momentum eq. we have

$$\frac{\partial w}{\partial t} + (\vec{v}\cdot\nabla)w + 2(\Omega_x v - \Omega_y u) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

the 'size' of those terms is $\frac{W}{T} + \frac{UW}{L} + \frac{W^2}{H} \sim \left|\frac{1}{\rho}\frac{\partial \rho}{\partial z}\right| + g$

The terms on the left side are all small in the ocean (typical values for mesoscale motions: W<1cm/s, L~100km, H~1km, U~0.1m/s, T=L/U) and the terms on the right-end side must compensate each other

We obtain $\frac{\partial p}{\partial z} = -\rho g$ which is the hydrostatic balance we introduced before

In this form is not always useful (we cannot put to 0 the right-end side of the momentum eq. or we will loose important infos! The motion is affected by both pressure and gravity indeed)

A better way to make use of the hydrostatic balance is to rewrite pressure as $p(x,y,z,t)=p_o(z)+p'(x,y,z,t)$. Under the assumption of constant density ρ_o then we can make use of $\frac{\partial p_o}{\partial z} = -\rho_o g$ and write the momentum eq. as

$$\frac{Dw}{Dt} = -\frac{1}{\rho_o}\frac{\partial p'}{\partial z}$$

gravity has no effect on the motion of constant density fluids (indeed there is no buoyancy!)

If a fluid is stratified, we will use a different argument and derive the Boussinesq equation (soon…)

The equation of state

The continuity and momentum equations provide 4 eqs for 5 unknown (velocity vector, density and pressure). The missing eq. is given by the eq. of state, which relates the thermodynamic variables to each other. Its general form is therefore simply $p=p(\rho,T,\mu_n)$.

For an ideal gas the eq. of state is very simple ($p=\rho RT$), but for water and particularly sea water is more complex and has been derived semi-empirically

$$\frac{1}{\rho}=\alpha=\alpha_0\left[1+\beta_T(1+\gamma^* p)(T-T_0)+\frac{\beta_T^*}{2}(T-T_0)^2-\beta_S(S-S_0)-\beta_p(P-P_0)\right]$$

with β_T being the thermal expansion coeff, β_S the saline or haline contraction and β_p compressibility coeff , β^*_T is the second thermal exp coef and γ^* is the thermobaric parameter

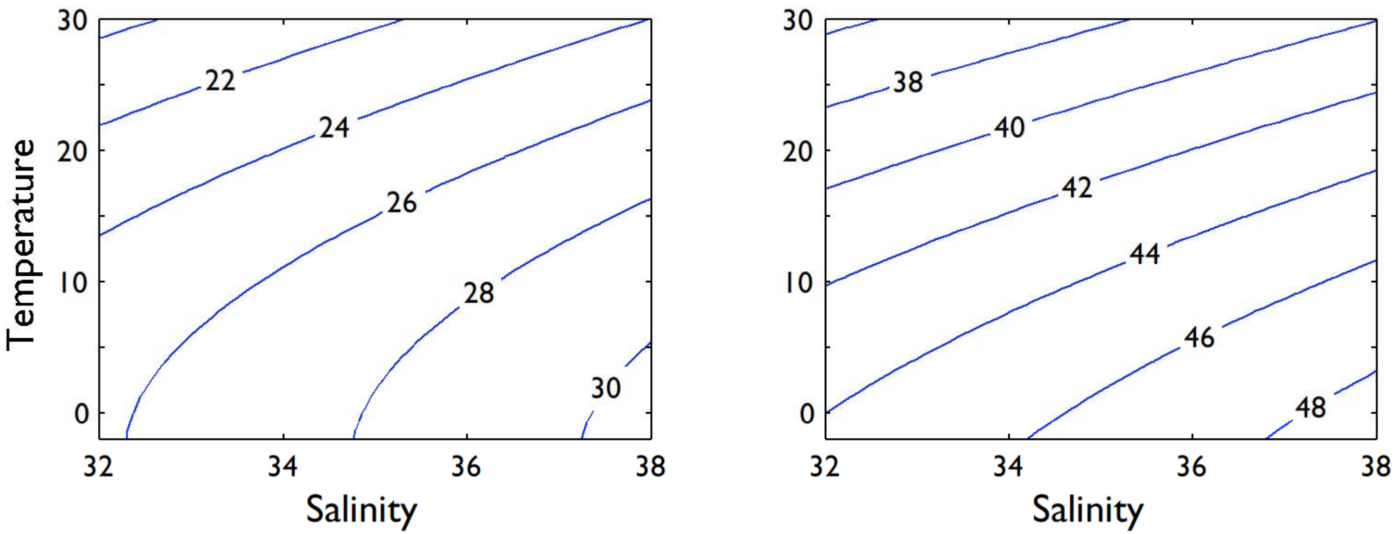


Fig. 1.3 A temperature–salinity diagram for seawater, calculated using an accurate empirical equation of state. Contours are (density $-$ 1000) kg m^{-3}, and the temperature is potential temperature, which in the deep ocean may be less than *in situ* temperature by a degree or so (see Fig. 1.4). Left panel: at sea-level ($p = 10^5$ Pa $= 1000$ mb). Right panel: at $p = 4 \times 10^7$ Pa, a depth of about 4 km. Note that in both cases the contours are slightly convex.

from Vallis' book

Definition: potential temperature and potential density

- Potential temperature: the temperature that a parcel would have if moved adiabatically to a given reference pressure (usually take as 1 bar = sea level)

- Potential density: the density that a fluid parcel would have if moved adiabatically and at constant composition to a reference pressure.

Compressible and incompressible flows

In most cases of geophysical interest the density can be considered constant <u>in the mass conservation eq</u>.: Fluids with constant ρ are called ***incompressible***

from $$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{v} = 0 \qquad we \qquad get \qquad \Rightarrow \qquad \nabla\cdot\vec{v} = 0$$

which is a diagnostic eq, because constrains the velocity field.

In reality *a fluid is incompressible if density changes are small enough to be negligible in the mass balance,* $\frac{\delta\rho}{\rho} << 1$

In the ocean $\frac{\delta\rho}{\rho} \approx 10^{-3}$

The energy budget

If the fluid is incompressible, the continuity and momentum equations are sufficient to determine the evolution of the flow. Let's start with the momentum eq. with Φ being the potential for any conservative force (e.g gravity):

$$\frac{D\vec{v}}{Dt} = -\nabla\left(\frac{p}{\rho} + \phi\right) + \nu\nabla^2\vec{v}$$

using the identity $(\vec{v}\cdot\nabla)\vec{v} = -\vec{v}\times\vec{\omega} + \nabla(\vec{v}^2/2)$

where $\vec{\omega} \equiv \nabla\times\vec{v}$ is the **vorticity** and omitting viscosity we obtain $\frac{\partial\vec{v}}{\partial t} + \vec{\omega}\times\vec{v} = -\nabla B$

where $B = \left(\frac{p}{\rho} + \phi + \vec{v}^2/2\right)$ is the Bernoulli function

Multiplying by **v**ρ we obtain

$$\frac{1}{2}\frac{\partial\rho\vec{v}^2}{\partial t} + \rho\vec{v}\cdot\left(\vec{\omega}\times\vec{v}\right) = -\rho\vec{v}\cdot\nabla B \quad \Rightarrow \quad \frac{\partial K}{\partial t} + \nabla\cdot\left(\rho\vec{v}B\right) = 0$$

where K is energy per unit volume

Φ is time-independent and therefore

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}\vec{v}^{\,2}+\phi\right)\right]+\nabla\cdot\left[\rho\vec{v}\left(\frac{1}{2}\vec{v}^{\,2}+\frac{p}{\rho}+\phi\right)\right]=0$$

where we made use of $\vec{v}\cdot\nabla B=\nabla\cdot(\vec{v}B)-B(\nabla\cdot\vec{v})$ and of the continuity eq. for incompressible fluids

or

$$\frac{\partial E}{\partial t}+\nabla\cdot\left[\vec{v}(E+p)\right]=0$$

where ***E=K+ρΦ*** is the total energy per unit volume. A local change of the energy is balanced by the divergence of its flux, which contains the additional term **v**p that represents the energy transfer when the fluid works against the pressure forces.

Things to remember from last week

- Ocean observations are taken in the Eulerian framework (mainly satellite data for surface variable over the last ~ 30 years plus CTDs (Conductivity Temperature Depth Profilers) and current meters) and in the Lagrangian one (floats, buoys)
- Most of obs for subsurface oceans from from Lagrangian measurements.
- Issues with sampling of subsurface water properties
- Differences between Eulerian and Lagrangian representations

- The equations of motion, for a fluid, are derived from conservation of mass (or continuity equation)

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0$$

+ momentum equation (Newton' law written for variable volume and density)

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \nabla \right) \vec{v} = \frac{\vec{F}}{\rho}$$

+ equation of state

$$\frac{1}{\rho} = \alpha \sim \alpha_0 \left[1 + \beta_T (1 + \gamma^* p)(T - T_0) - \beta_S (S - S_0) - \beta_p (P - P_0) \right]$$

- Incompressible flows: Fluids with constant ρ
  the continuity equation becomes $\Rightarrow \quad \nabla \cdot \vec{v} = 0$

- Hydrostatic balance under the assumption of constant density ($\rho = \rho_o$):

from $$\frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad \frac{\partial p_o}{\partial z} = -\rho_o g,$$

$$\text{being } \; p(x, y, z, t) = p_o(z) + p'(x, y, z, t)$$

we obtain $$\frac{Dw}{Dt} = -\frac{1}{\rho_o}\frac{\partial p'}{\partial z}$$

week 2

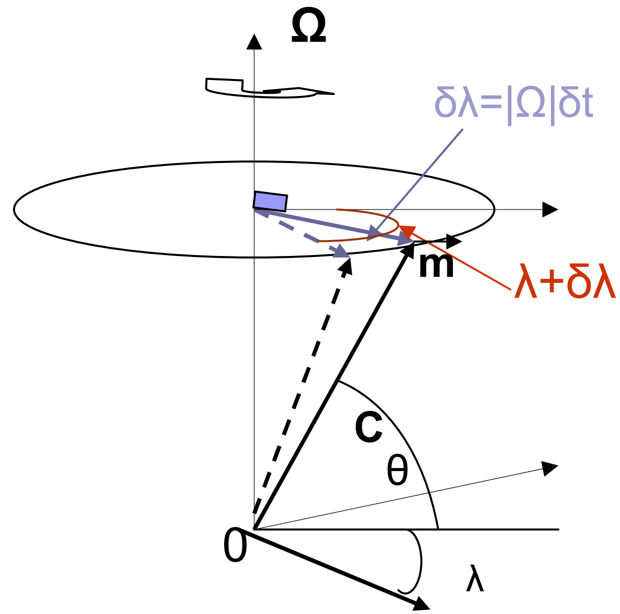
- Rotation and Stratification
- The f- and β-planes
- The Boussinesq approximation
- The geostrophic balance
- Static instability
- The Ekman layer

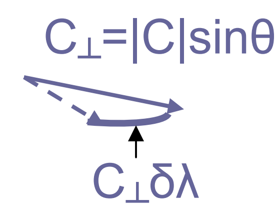
Rotation and Stratification

- The **Earth** rotates with a period of 23h 56m, therefore **is not an inertial frame**. It is convenient to make a transformation into the rotating frame of reference of an observed on the Earth, since the relative motions are much smaller that the absolute ones. Indeed if V is the typical scale for the relative motion it is true that $V \ll \Omega a$, where $\Omega = 2\pi \mathrm{rad}\ d^{-1} \sim 0.73 \cdot 10^{-4} s^{-1}$ and $a \sim 6400\mathrm{km}$
- The ocean (and the atmosphere) are shallow fluid layers on a sphere: their thickness is much less than their horizontal extension.
- The ocean is strongly stratified, i.e. there is a mean vertical gradient in the density field which is considerably larger than the horizontal one

Rate of change of a vector

let's consider a vector rotating relative to the inertial frame with a constant angular velocity Ω





$\delta\mathbf{C} = |\mathbf{C}|\sin\theta|\mathbf{\Omega}|\mathbf{m}\delta t = \mathbf{\Omega}\times\mathbf{C}\ \delta t \rightarrow$ $\boxed{\dfrac{d\vec{C}}{dt} = \vec{\Omega}\times\vec{C}}$

Let's now consider a vector B that changes in the inertial frame as well. If we measure that change in the rotating frame of reference, we will not 'see' the change associated with the rotation of the frame itself, i.e.

$$\delta\vec{B}_I = \delta\vec{B}_R + \delta\vec{B}_{rot}$$

but $\delta\vec{B}_{rot} = \vec{\Omega} \times \vec{B}\delta t$ from the previous slide

therefore:

$$\boxed{\left(\frac{d\vec{B}}{dt}\right)_I = \left(\frac{d\vec{B}}{dt}\right)_R + \vec{\Omega} \times \vec{B}}$$

Coriolis and centrifugal forces

we just saw that $$\left(\frac{d\vec{r}}{dt}\right)_I \equiv v_I = \left(\frac{d\vec{r}}{dt}\right)_R + \vec{\Omega}\times\vec{r} = v_R + \vec{\Omega}\times\vec{r}$$

we now want to derive an equivalent expression for the accelerations.

Because $\left(\frac{d\vec{v}_R}{dt}\right)_I = \left(\frac{d\vec{v}_R}{dt}\right)_R + \vec{\Omega}\times\vec{v}_R$ then $\left(\frac{d\vec{v}_R}{dt}\right)_I = \left(\frac{d}{dt}\left(\vec{v}_I - \vec{\Omega}\times\vec{r}\right)\right)_I = \left(\frac{d\vec{v}_R}{dt}\right)_R + \vec{\Omega}\times\vec{v}_R$

or $$\left(\frac{d\vec{v}_I}{dt}\right)_I = \left(\frac{d\vec{v}_R}{dt}\right)_R + \vec{\Omega}\times\vec{v}_R + \frac{d\vec{\Omega}}{dt}\times\vec{r} + \vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_I$$

but $\frac{d\vec{\Omega}}{dt} = 0$ *and* $\left(\frac{d\vec{r}}{dt}\right)_I = \vec{v}_R + \vec{\Omega}\times\vec{r}$ which implies

$$\boxed{\left(\frac{d\vec{v}_I}{dt}\right)_I = \left(\frac{d\vec{v}_R}{dt}\right)_R + 2\vec{\Omega}\times\vec{v}_R + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

Coriolis acceleration **Centrifugal acceleration**

- If **r** is the perpendicular distance from the axis of rotation, then using one of the vector identity we get

$$\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = \vec{\Omega}\left(\vec{\Omega}\cdot\vec{r}\right) - \vec{r}(\vec{\Omega}\cdot\vec{\Omega}) = -\Omega^2\vec{r} = \nabla\phi_{ce}$$

- The Coriolis force (= - Coriolis acceleration = -2$\mathbf{\Omega}\times\mathbf{v}_r$)

1. is not present for bodies stationary respect to the rotating frame
2. acts to deflect moving bodies in the horizontally perpendicular direction (i.e. to the right in the North. hemisp. where $\mathbf{\Omega}$>0)
3. does no work on the body

Momentum and continuity eqs. in the rotating frame

- the momentum eq. simply becomes:

$$\boxed{\frac{D\vec{v}_{(R)}}{Dt} + 2\vec{\Omega}\times\vec{v} = -\frac{\nabla p}{\rho} - \nabla\phi}$$

(where the centrifugal term has been incorporated in the potential)

- the continuity eq. does not change. Mass is conserved independently in the frame of reference!

The spherical coordinate system

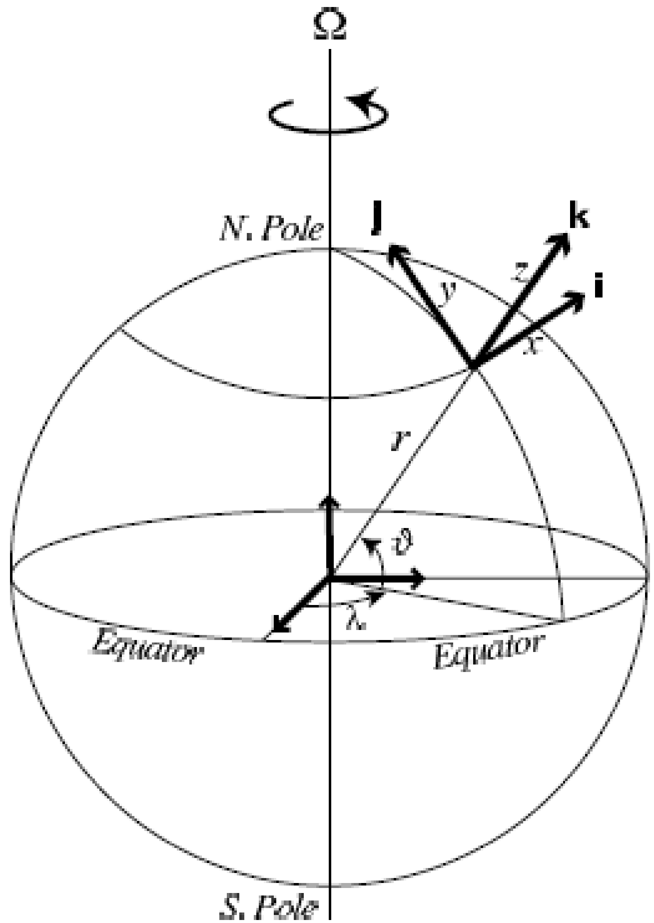


Figure 2.3 The spherical coordinate system. The orthogonal unit vectors **i**, **j** and **k** point in the direction of increasing longitude λ, latitude ϑ, and altitude z. Locally, one may apply a Cartesian system with variables x, y and z measuring distances along **i**, **j** and **k**.

a = r = radius of Earth

locally, for small excursion on a plane tangent to the surface at $\theta = \theta_o$

$$(x, y, z) \sim (a\lambda\cos\vartheta_o, a(\vartheta - \vartheta_o), z)$$

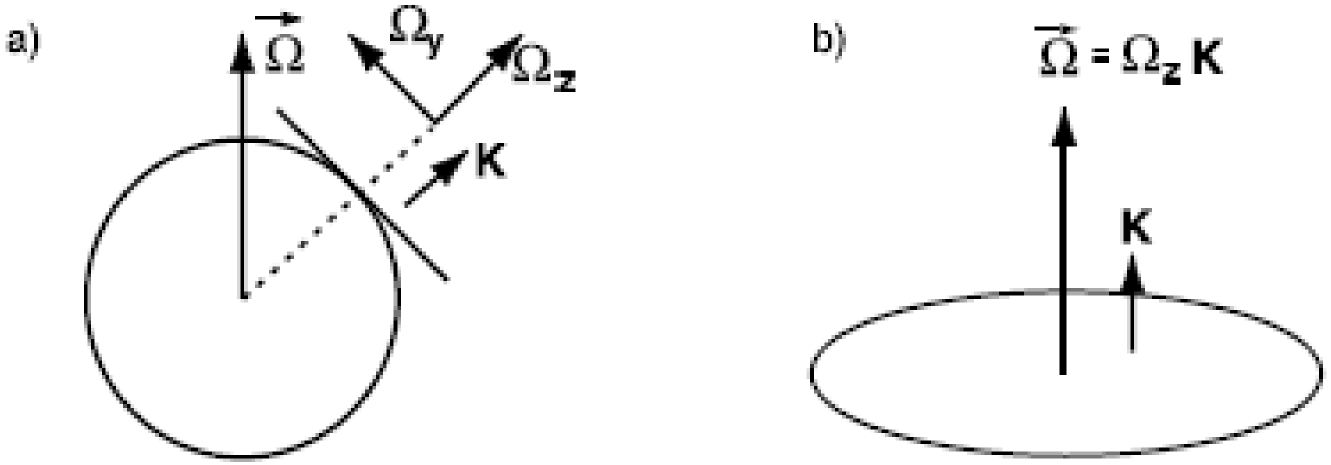
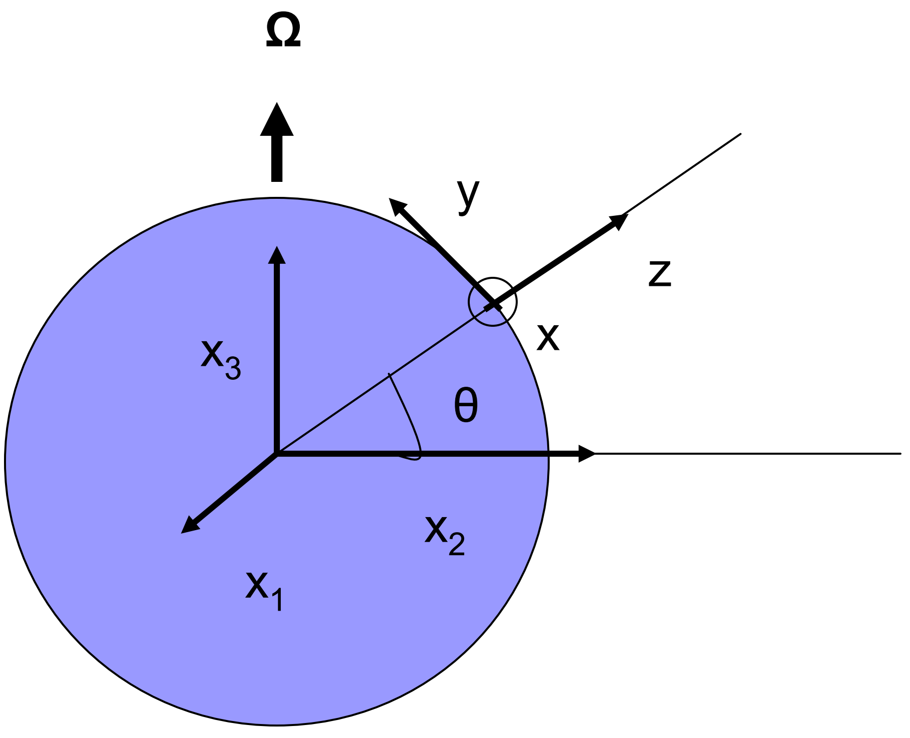


Fig. 2.4 a) On the sphere the rotation vector $\boldsymbol{\Omega}$ can be decomposed into two components, one in the local vertical and one in the local horizontal, pointing toward the pole. That is, $\boldsymbol{\Omega} = \Omega_y \mathbf{j} + \Omega_z \mathbf{k}$ where $\Omega_y = \Omega \cos\vartheta$ and $\Omega_z = \Omega \sin\vartheta$. In geophysical fluid dynamics, the rotation vector in the local vertical is often the more important component in the horizontal momentum equations. On a rotating disk, (b), the rotation vector $\boldsymbol{\Omega}$ is parallel to the local vertical $\mathbf{k}$.

The Cartesian approximation

The Earth is an approximately spherical body (see Vallis 2.2.1 for a discussion on the implications of the “approximately “) with a gravitationally bound fluid layer at the surface. Spherical shell coordinate are best suited, but Cartesian coordinates in rotation around an axis aligned with gravity are ok (more so in the ocean than in the atmosphere):

- Small-scale motions typically are not influenced very much by rotation because their time-scale is short compared to the rotation period;
- Large-scale motions are influenced but because of their small aspect ratio (the ocean is thin) only the vertical component is dynamically important (a rigorous discussion later on: see geostrophic balance)



By convection the xy-plane is tangent to the geoid with x,y,z pointing east, north, up

Using the expression for the rotation vector then

$$\vec{\Omega} = \vec{\Omega}^x \vec{i} + \vec{\Omega}^y \vec{j} + \vec{\Omega}^z \vec{k} = 0\vec{i} + |\Omega| \cos(\vartheta) \vec{j} + |\Omega| \sin(\vartheta) \vec{k} \approx$$

$$\approx (\vec{\Omega} \cdot \vec{k}) \vec{k} = |\Omega| \sin(\vartheta) \vec{k}$$

where θ is latitude and we made the approx of ignoring the component of Ω not in the direction of the local vertical

+ we can use a Taylor expansion to get a spatially local approximation in $\theta - \theta_o$ whenever $\theta - \theta_o \ll 1$ (i.e. we are considering small variations in latitude around latitude θ_o) →

$$\vec{\Omega} \approx |\Omega| (\sin(\vartheta_o) + \cos(\vartheta_o)(\vartheta - \vartheta_o) + \ldots.) = \frac{1}{2}(f_o + \beta y + \ldots)$$

where $f_o = 2\Omega \sin \vartheta_o$ and $\beta = \frac{2\Omega}{a} \cos(\vartheta_o)$

given that for small excursion on the plane

$$(x, y, z) \sim (a\lambda \cos \vartheta_o, a(\vartheta - \vartheta_o), z)$$

where a is the radius of the earth

$$\boxed{f = f_o + \beta y}$$

this is known as β-plane approximation.

The momentum equations are unaltered except that the term $2\mathbf{\Omega}\times\mathbf{v}_R$ is substituted by **f×u** when we retain only the vertical component of **Ω** and finally by $(f_o+\beta y)\mathbf{k}\times\mathbf{u}$

If we consider only the horizontal components of the momentum eq. →

$$\frac{D\vec{u}}{Dt} + \vec{f}\times\vec{u} = -\frac{1}{\rho}\nabla_h p, \qquad where \qquad \vec{f} = (f_o + \beta y)\vec{k}$$

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Oceanic approximations

In the ocean the density variations are small compared to the mean density. We can exploit this to derive the (widely used) **Boussinesq equations**.

Density variations are due to 3 effects: compression by pressure, thermal expansion and haline contraction (salinity changes) from the linearized eq. of state.

$$\rho = \rho_o\left[1-\beta_T(T-T_o)+\beta_S(S-S_o)+\beta_P(p-p_o)\right]$$

where we take $p_o = 0 \quad and \quad \beta_P = \dfrac{1}{\rho_o c_s^{\,2}}$

and c_s is the sound speed ~ $1500 ms^{-1}$ in the ocean

- pressure compressibility:

$$\Delta_p \rho \approx \Delta p / c_s^{\ 2} \approx \rho_o g H / c_s^{\ 2}$$

therefore

$$\frac{\left|\Delta_p \rho\right|}{\rho_o} << 1 \qquad if \qquad \frac{gH}{c_s^{\ 2}} << 1$$

which is verified given that H is ~ 5000m and max 10km

(alternatively one can consider the Mach number in the ocean -which is given by the ratio of the fluid velocity and the sound speed-. If this is small, pressure compressibility is negligible).

- Thermal expansion

we have $\Delta_T \rho / \rho_o \approx -\beta_T \Delta T$ which is <<1 in the ocean because β_T is ~ 10^{-4} K^{-1} and ΔT ~ 20K

- Haline contraction

we have $\Delta_S \rho / \rho_o \approx \beta_S \Delta S$ which is <<1 in the ocean because β_S is ~ 10^{-3} psu^{-1} and ΔS ~ 5psu

The Boussinesq approximation

Hyph: density variations are small (the flow in principle IS NOT incompressible)

$$\rho = \rho_o + \delta\rho(x, y, z, t), \qquad \delta\rho << \rho_o$$

and the pressure associated with the reference density is in hydrostatic balance

$$p = p_o(z) + \delta p(x, y, z, t), \qquad \delta p << p_o$$

$$\frac{dp_o}{dz} \equiv -g\rho_o$$

(incidentally: we're also assuming small variations in pressure doing so)

momentum equations

*if we multiply them by ρ, using the relation above we can write them as**

$$\left(\rho_o+\delta\rho\right)\left(\frac{D\vec{v}}{Dt}+2\vec{\Omega}\times\vec{v}\right)=-\nabla\delta p-\frac{\partial p_o}{\partial z}\vec{k}-g\left(\rho_o+\delta\rho\right)\vec{k}=$$

$$\simeq\left(\rho_o+\delta\rho\right)\left(\frac{D\vec{v}}{Dt}+2\vec{\Omega}\times\vec{v}\right)=-\nabla\delta p-g\delta\rho\vec{k}$$

** we are neglecting viscosity and body forces which expression does not change*

- if $\delta\rho$ is small then

$$\boxed{\frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \vec{v} = -\nabla\phi + b\vec{k}}$$

where $\Phi = \delta p/\rho_o$ and $b = -g\delta\rho/\rho_o$ is buoyancy

The Boussinesq approx ignores variations of density <u>EXCEPT</u> when associated with gravity

If δp and $\delta\rho$ are also in hydrostatic balance, then $\frac{\partial\phi}{\partial z} = b$

mass conservation

the mass conservation eq can be written as

$$\frac{D\delta\rho}{Dt} + (\rho_o + \delta\rho)\nabla \cdot \vec{v} = 0$$

that can be approximated by $\boxed{\nabla \cdot \vec{v} = 0}$

if D/Dt scales as $\vec{v} \cdot \nabla$

1. *This does not imply* Dδρ/Dt = 0!
2. *By removing the time-derivative of density we are excluding sound waves, which include shock waves (they have indeed relatively little energy in the ocean –not in stars, nuclear explosions, airplane wakes, volcanic eruptions! – compared to the large scale motion)*
3. *gives problems with the energy conservation*

Thermodynamic equation and equation of state

Adding a thermodynamic eq and an eq. for salinity + an eq. of state we can close the Boussinesq approximation

In the oceanic context the missing pieces are simply

$$\boxed{\frac{DS}{Dt} = \dot{S}\,, \qquad \frac{D\theta}{Dt} = \dot{\theta}}$$

where θ is potential temperature

With an eq. of state $b=b(\theta,S,p)$ we can close the system.

If we are using the hydrostatic balance, then p can be replaced by the hydrostatic pressure and taking $p_o=0$ we get $p=-g\rho_o z$ and **$b=b(\theta,S,z)$**

Summarizing

- momentum $$\boxed{\frac{D\vec{v}}{Dt} + \vec{f} \times \vec{v} = -\nabla\phi + b\vec{k}}$$
- mass conservation $$\boxed{\nabla \cdot \vec{v} = 0}$$
- thermodynamic eq. $$\boxed{\frac{D\theta}{Dt} = \dot{\theta}}$$
- salinity eq. $$\boxed{\frac{DS}{Dt} = \dot{S}}$$
- eq. of state $\boxed{b=b(\theta,S,\Phi)}$

The energy is conserved under this approx ONLY if b=b(θ,S,z)
(check section 2.4.3 on Vallis' book)

Geostrophic balance

Let's focus first on the horizontal components of the momentum eqs. (i.e. along geopotential surfaces): A scaling analysis to evaluate the importance of each term shows that for the left side

$$\frac{\partial \vec{u}}{\partial t} + (\vec{v}\cdot\nabla)\vec{u} + \vec{f}\times\vec{u}) = -\frac{1}{\rho}\nabla_z p$$

$$\frac{U}{T} + \qquad \frac{U^2}{L} + \qquad fU$$

under the assumption that W/H is ≤ U/L

The ratio of the sizes of the advective term and the Coriolis one is called Rossby number.

$$\boxed{R_o = \frac{U}{fL}}$$

In the ocean $R_o \sim 0.1\text{ms}^{-1} / 10^5\text{m}*7.3*10^{-5}\text{ s}^{-1} \sim$ **<u>0.01</u>:** the Coriolis term is much larger than the advective one

Another ‘informal’ definition of the Rossby number is in terms of time scales.

$$R_{oT} = \frac{1}{fT}$$

with T being the advective time scale and 1/f being the inertial time scale (about 2-3 hours at mid-latitudes)

For phenomena with a time scale longer than the inertial one rotation IS important (i.e.the Gulf Stream), otherwise rotation can be neglected.

if R_o is small enough, then the rotation term dominates and balances the pressure one, i.e.:

$$\vec{f} \times \vec{u} \approx -\frac{1}{\rho} \nabla_z p$$

and considering the two components separately

$$\boxed{fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad fv \approx \frac{1}{\rho}\frac{\partial p}{\partial x}}$$

whenever the relations above are exact, then is common to indicate the velocities with u_g and v_g

consequences:

- geostrophic balance implies that the horizontal velocities u_g and v_g are directed along isobars (u_g,v_g, and isolines of pressure are parallel) . If f>0 (northern hemisphere) the flow is anticlockwise around a low and clockwise around a high pressure region.

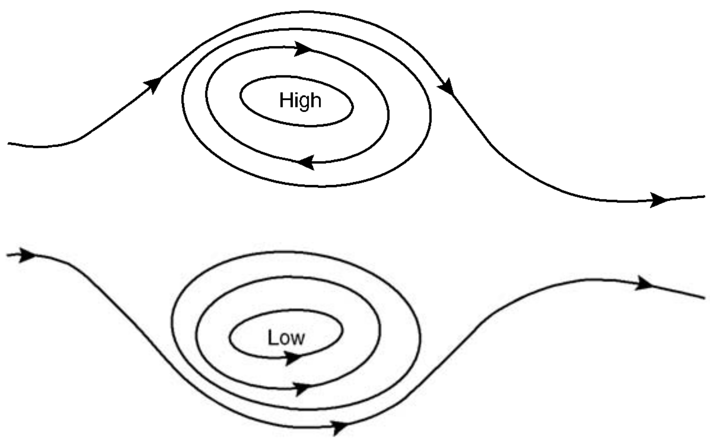


Fig. 2.5 Schematic of geostrophic flow with a positive value of the Coriolis parameter f. Flow is parallel to the lines of constant pressure (isobars). Cyclonic flow is anti-clockwise around a low pressure region and anticyclonic flow is clockwise around a high. If f were negative, as in the Southern Hemisphere, (anti)cyclonic flow would be (anti)clockwise.

from Vallis, 2006

- If f is constant and ρ does not vary on the horizontal plane then

$$\nabla_z \cdot \vec{u}_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = \frac{1}{f_o \rho_o}\left(\frac{\partial^2 p}{\partial x \partial y} - \frac{\partial^2 p}{\partial x \partial y} \right) = 0$$

and we can define a geostrophic streamfunction, Ψ

$$\psi \equiv \frac{p}{f_o \rho_o}$$

where $u_g = -\frac{\partial \psi}{\partial y}, \qquad v_g = \frac{\partial \psi}{\partial x}$

and defining vorticity **ζ** as $\nabla \times \vec{v}$,

then its vertical component ζ = $\vec{k} \cdot \nabla \times \vec{v} = \nabla^2{}_z \psi$

- If the Coriolis force is not constant (but density still is) we get (again cross-differentiating)

$$\frac{\partial f}{\partial x}u_g + f\frac{\partial u_g}{\partial x} = -\frac{\partial f}{\partial y}v_g - f\frac{\partial v_g}{\partial y} \Rightarrow \frac{\partial f}{\partial y}v_g + f\nabla_z \cdot u_g = 0$$

and using the mass continuity this implies

$$\frac{\partial f}{\partial y}v_g \equiv \beta v_g = f\frac{\partial w}{\partial z}$$

which is also known as Sverdrup balance (we'll encounter and re-derive this later on during this course)

Taylor-Proudman effect

if β=0 then the last eq. implies that for flows in geostrophic balance the vertical velocity does not depend on height. In fact none of the velocity components depends on height if a flow is in hydrostatic and geostrophic balance.

For such a flow and defining $\Phi = p/\rho_o$

$$v = \frac{1}{f_o}\frac{\partial \phi}{\partial x}, \qquad u = -\frac{1}{f_o}\frac{\partial \phi}{\partial y}, \qquad \frac{\partial \phi}{\partial z} = -g$$

if we differentiate with respect to z

$$\frac{\partial v}{\partial z} = -\frac{1}{f_o}\frac{\partial g}{\partial x} = 0, \qquad \frac{\partial u}{\partial z} = \frac{1}{f_o}\frac{\partial g}{\partial y} = 0$$

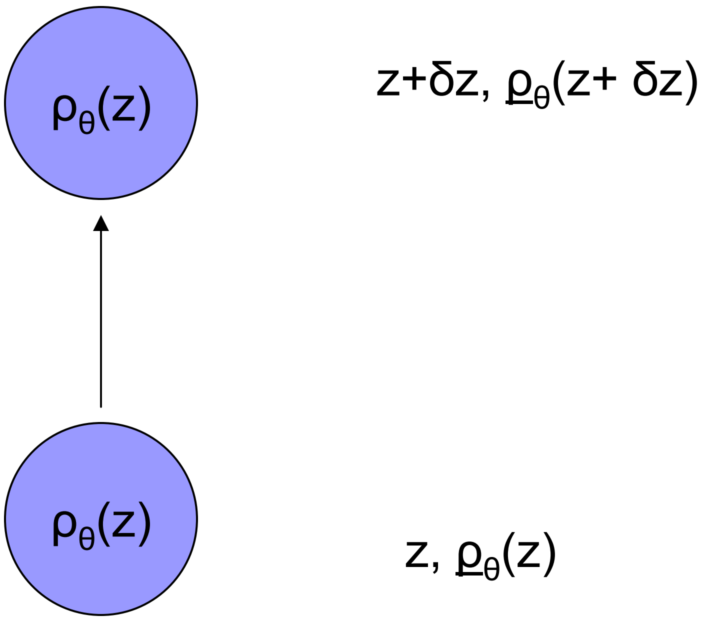
(we used the fact that horizontal velocities are non-divergent and that $\frac{\partial w}{\partial z} = 0 \quad for \quad \beta = 0$)

So for R_o=0 constant density flows in geostrophic and hydrostatic balance have vertical derivatives of ALL 3 components of the velocity vector = 0 → strong rotation induced a ‘stiffening’ of the fluid in the vertical.

Obviously this is not ‘completely’ true for geophysical flows, but the presence of rotation causes the vertical derivatives of velocity to be less important that the horizontal ones $W/H << U/L$

Static Instability

We now consider the vertical displacement of water parcels under the effect of gravity. We will neglect rotation and horizontal motion (overall we are considering small scale events)



the parcel moves adiabatically from a region of ambient potential density $\underline{\rho}_\theta(z) = \rho_\theta(z)$ (the parcel potential density) to a region of potential density $\underline{\rho}_\theta(z+\delta z)$

When the parcel is at $z+\delta z$ the difference between its density

and the in-situ one is simply $\delta\rho = \underline{\rho}_\theta(z) - \underline{\rho}_\theta(z+\delta z) = -\frac{\partial \underline{\rho}}{\partial z}\delta z$

The force acting on the parcel per unit volume is $F=-g\,\delta\rho=\rho a_z$

and we get

$$\frac{\partial^2 \delta z}{\partial t^2} = \frac{g}{\underline{\rho}}\frac{\partial \underline{\rho}}{\partial z}\delta z = -N^2\delta z$$

$$\boxed{N^2 = -\frac{g}{\underline{\rho}_\theta}\frac{\partial \underline{\rho}_\theta}{\partial z}}$$

N^2 is the Brunt-Vaisala frequency. If N^2 is >0 than the parcel is denser that the surrounding and will experience a restoring force. Viceversa if N^2 is < 0 the density profile is unstable and the parcel will continue ascending → convection takes place

In the ocean the expression for N^2 is more complicated due to the presence of salt and the different values of β_T and $\beta_{S.}$ Two parcels may have same potential density but different T and S. If we move them to the bottom of the ocean they may be compressed by a different amount. Cold water is more compressible than warm water: it is easier to deform a cold parcel than a warm parcel. Therefore cold water becomes denser than warm water when they are both submerged to the same pressure. If this were not true, or equivalently, if there were no salt so that density depended only on temperature and pressure, then potential density, using any single pressure for a reference, would be adequate for defining an isentropic surface.

Because of salinity, two water parcels at the same pressure can have the same density but differ in temperature and hence in compressibility.

The Brunt-Vaisala frequency can be found noticing that

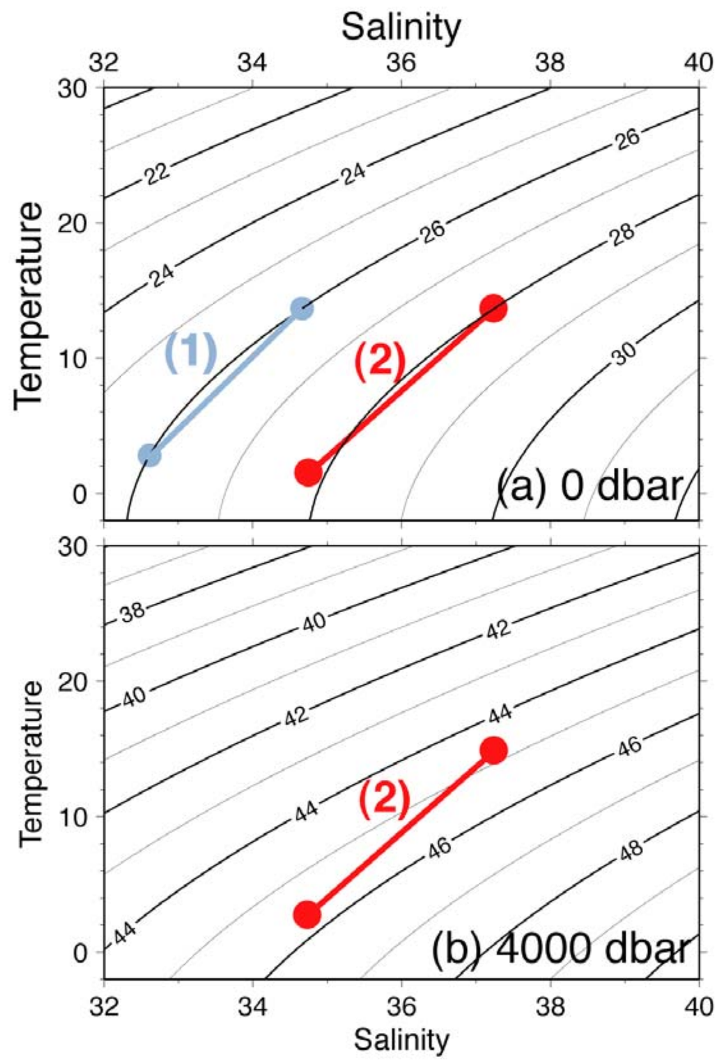
$$\delta\rho_\theta = \delta\rho - \frac{1}{{c_s}^2}\delta p$$

It follows that $$N^2 = -g\left[\frac{g}{{c_s}^2} + \frac{1}{\underline{\rho}}\frac{\partial\underline{\rho}}{\partial z}\right]$$

Typical values are ~ $0.01\,s^{-1}$ in the pycnocline and ten times smaller below

Cabbeling

it is an instability arising from the nonlinearity in the eq. of state for sea water.



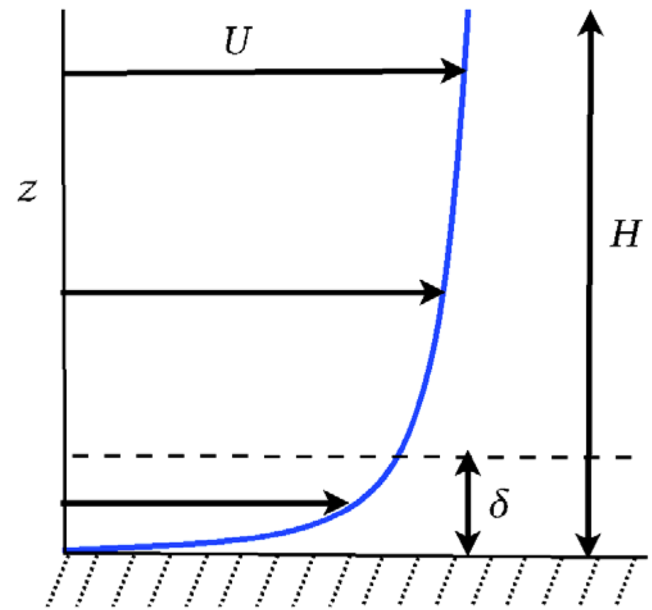
Potential density relative to (a) 0 dbar and (b) 4,000 dbar as a function of temperature and salinity. The blue pair (1) illustrates cabbeling, or density increase from mixing of two parcels of the same density. The red pair (2) in both panels indicates comparison of a warm, saline water parcel and a cold, fresh parcel. (from W. Emery Oceanography notes)

The Ekman layer

Now we return to the full 3d eq. to examine the boundary layers that exist close to the top and bottom of the ocean. In practice this can be done using the geostrophic eq. and considering the effects of friction, which dominates the dynamics close to the boundaries appearing in the terms of highest different order.

From Vallis, 2006

Idealized boundary layer. The value of U must vary rapidly close to the boundary to satisfy the BC



In boundary layers in lab experiments (ex. in turbulence exps) or in general in the absence of rotation (ex. turbines) the dominant balance is between advection and viscosity and/or diffusion. For geophysical flows the Coriolis term is the one to be balanced by the viscous one.

In the atmosphere the boundary layers occur near the ground. In the ocean the main Ekman or boundary layer is at the surface, and it's related to the presence of the wind. A weaker one is present at the bottom (analogous to the one in the atmosphere)

The eq. for the Ekman layer were derived by Ekman in 1905 (paper!). The assumptions are:

Hypothesis:

- The Boussinesq approx. is valid (ok for the ocean. More problematic in the atmosphere)
- The Ekman layer has a depth which is less that the total depth of the fluid. This implies that there is a region of the fluid below the Ekman layer where geostrophic balance holds
- The advective (non-linear) terms are negligible, the fluid is in hydrostatic balance in the vertical and buoyancy is constant
- Following the original derivation by Ekman we'll assume that the viscous term has the form $A\frac{\partial^2 \vec{u}}{\partial z^2} = \rho_o^{-1}\frac{\partial \tau}{\partial z}$ where τ is the stress (usually a tensor, but the vertical component dominates)

- The frictional-geostrophic balance is simply

$$\vec{f} \times \vec{u} = -\nabla_z \phi + \frac{1}{\rho_o} \frac{\partial \vec{\tau}}{\partial z} = -\nabla_z \phi + A \frac{\partial^2 \vec{u}}{\partial z^2}$$

to simplify our life we assume hydrostatic balance,

$$\frac{\partial \phi}{\partial z} = b = COST = 0$$

We can introduce the Ekman number as a measure of the importance of viscous effect vs the Coriolis ones

$$E_k = \frac{A}{f_o H^2} = \frac{1}{2}\left(\frac{d}{H}\right)^2 \qquad d = \sqrt{2A/f}$$

if E_k is << 1 then viscous effects can be neglected (as in the flow interior). At the boundaries however the viscous term is the one that allows for satisfying the boundary conditions. δ is the thickness of the Ekman layer

The Ekman layer and the pressure field

If the interior is in geostrophic balance then velocity and pressure fields can be written as $\mathbf{u}=\mathbf{u}_g+\mathbf{u}_E$ and $\Phi=\Phi_g+\Phi_{E.}$ Away from the boundaries the Ekman corrections are ~ 0 and by hydrostatic balance $\frac{\partial \phi_g}{\partial z}=0$

But the pressure field is continuous and must vanishes at the boundary to satisfy hydrostasy, so $\frac{\partial \phi_E}{\partial z}=0$.

There is no Ekman layer in the pressure field!

And the horizontal moment eq. at the Ekman boundary becomes

$$\vec{f}\times\vec{u}_E=+\frac{1}{\rho_o}\frac{\partial\vec{\tau}}{\partial z} \qquad or \qquad =\frac{\partial\vec{\tilde{\tau}}}{\partial z}$$

bottom boundary layer

we start with $\vec{f} \times (\vec{u} - \vec{u}_g) = A \dfrac{\partial^2 \vec{u}}{\partial z^2}$, where fu_g=- ∂Φ/∂y and fv_g= ∂Φ/∂x .

The boundary conditions are (for a reverse z coordinate):

$$\begin{cases} at \quad z = 0 \quad u = v = 0 & \text{(no slip)} \\ at \quad z \to \infty \quad u = u_g, \quad v = v_g & \end{cases}$$

We look for solution in the form $u = u_g + Be^{\beta z}, \qquad v = v_g + Ce^{\beta z}$

Substituting into our initial eq. we get two algebraic eqs.

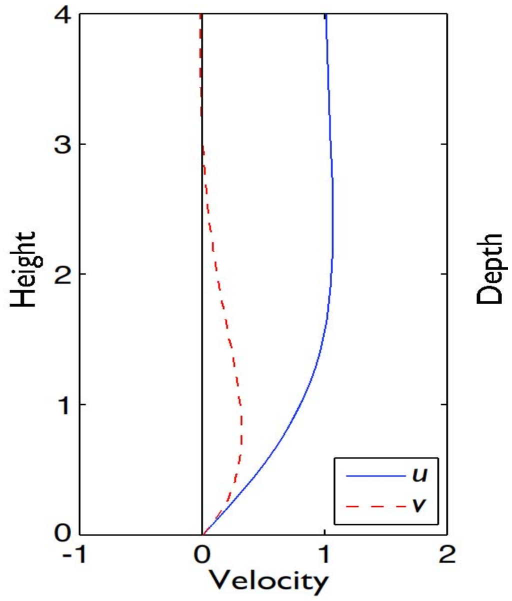
$$Bf - CA\beta^2 = 0; \qquad -BA\beta^2 - Cf = 0$$

which have non-trivial solutions only if $\beta^4 = -f^2 / A^2$ and therefore $\beta = \pm(1 \pm i)\sqrt{f / 2A}$

Using the BC we can obtain the solution

$$u = u_g - e^{-z/d}\left[u_g \cos(z/d) + v_g \sin(z/d)\right]$$

$$v = v_g + e^{-z/d}\left[u_g \sin(z/d) - v_g \cos(z/d)\right]$$



Bottom boundary layer with a given flow in the interior and with d=1 u_g=1,v_g=0 (from Vallis, 2006)

The upper ocean

At the ocean surface the Ekman layer is caused by the presence of the wind. It is extremely important because communicates the forcing by the wind to the ocean interior (in the vertical)

the BC now are

$$\begin{cases} at \quad z=0 \quad A\dfrac{\partial u}{\partial z}=\tau_x, \quad A\dfrac{\partial v}{\partial z}=\tau_y \\ at \quad z\to\infty \quad u=u_g, \quad v=v_g \end{cases}$$

and solutions (which now DO NOT depends on the interior flow in the correction terms, but only on the wind stress)

$$u=u_g+\frac{1}{fd}e^{-z/d}\left[(\tilde{\tau}^x+\tilde{\tau}^y)\cos(z/d)+(\tilde{\tau}^x-\tilde{\tau}^y)\sin(z/d)\right]$$

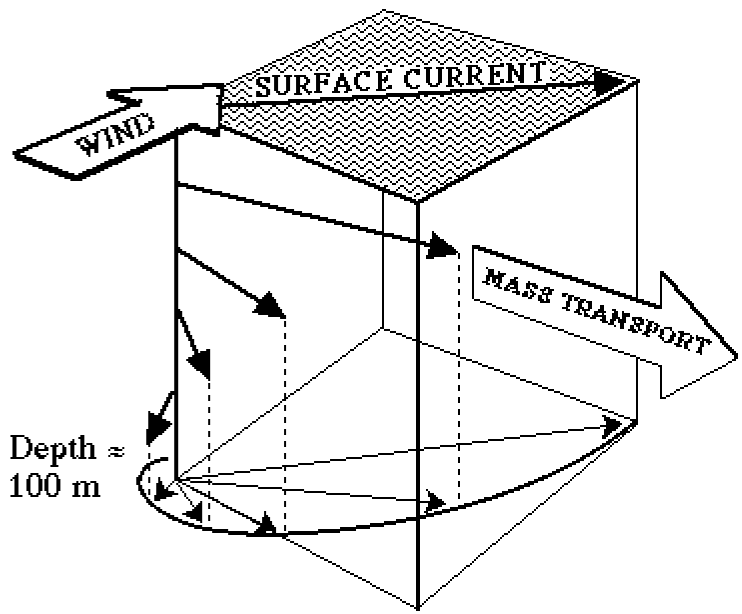
$$v=v_g+\frac{1}{fd}e^{-z/d}\left[(\tilde{\tau}^x+\tilde{\tau}^y)\sin(z/d)-(\tilde{\tau}^x-\tilde{\tau}^y)\cos(z/d)\right]$$

(see also equivalent formula on Vallis' book)

what is the impact on the transport?

Integrating the expression for u_E and v_E from 0 to $-\infty$ we can find the transport induced by the wind stress.

In the northern hemisphere, surface currents are deflected to the right of the wind direction and the ageostrophic flow is 45^o to the right. Results depends on the form of the friction, but not on the size of the viscosity! (There is not dependence on A). Directions are reversed in the southern hemisphere (see Vallis, 2006, fig 2.14).



Ekman Spiral in Northern Hemisphere

- The existence of the Ekman spiral was first proposed by Nansen, observing the ice drift in the Northern Sea.
- Ekman came up with the equations. The ‘Classical’ Ekman spiral, however, has not been found in either the atmosphere or the ocean. Reason is that the boundary layer is very turbulent, often dominated by eddies and jet, in addition to gravity waves. All those features are not included! (In the atmosphere the spiral is unstable to shear instability, in the ocean to wave-current interactions). Despite this, Ekman-spiral-like structures have been found in the ocean (Price et al., *Science*, 1987)

week 3

- Shallow water equations for a single layer
- Reduced gravity eqs.
- Multi-layer shallow water: the two-layer case
- Geostrophic balance and thermal wind
- Conservation properties
- Shallow water waves
- Geostrophic adjustment

Shallow water eqs: single layer case

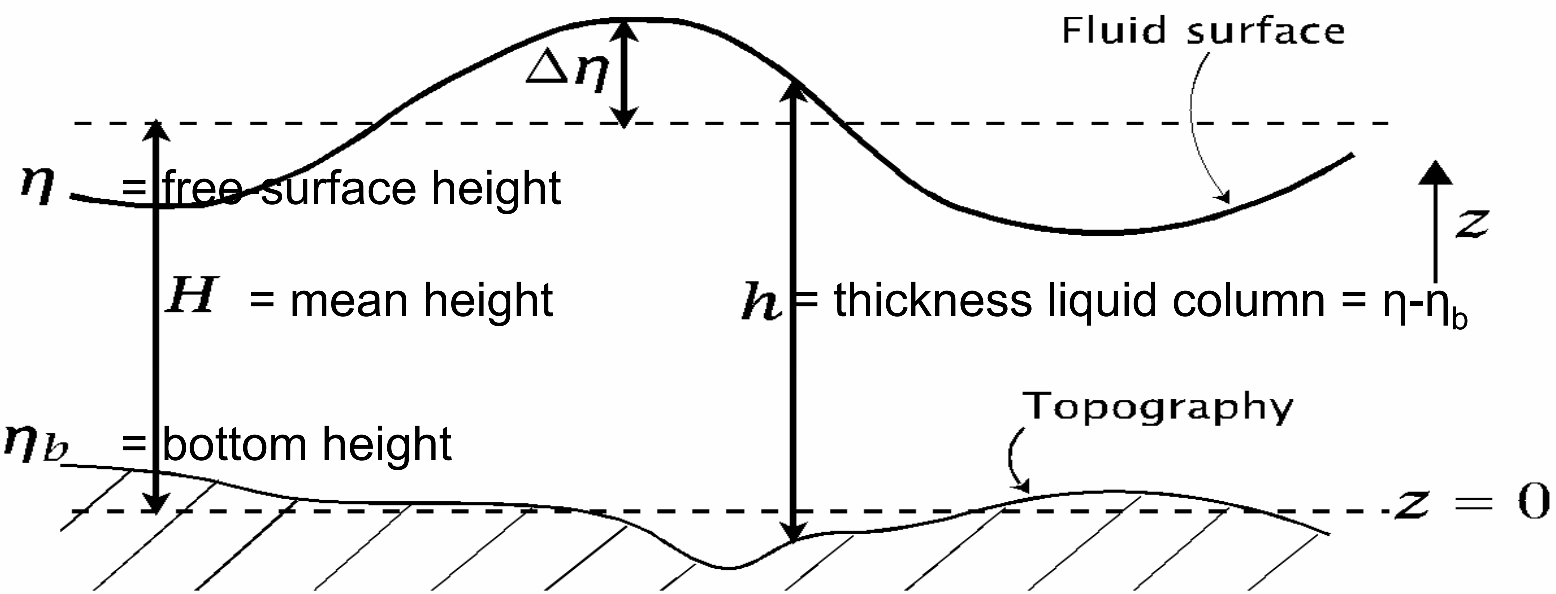


Fig. 3.1 A shallow water system. h is the thickness of a water column, H its mean thickness, η the height of the free surface and η_b is the height of the lower, rigid, surface, above some arbitrary origin, typically chosen such that the average of η_b is zero. $\Delta\eta$ is the deviation free surface height, so we have $\eta = \eta_b + h = H + \Delta\eta$.

From Vallis' book

hypothesis:

- a fluid layer of constant density
- the horizontal scale is >> than the vertical depth (small aspect ratio: the hydrostatic approx is satisfied)
- on top there is another fluid of negligible density (and inertia)

Momentum equations

The vertical component is simply the hydrostatic eq.

$$\frac{\partial p}{\partial z} = -g\rho$$

we have one layer of constant density and therefore we can integrate to get (assuming $p_0 = 0$)

$$p(x, y, z) = -\rho g z + p_0 = \rho g(\eta(x, y) - z)$$

because pressure at z=η is ~0 (the overlying fluid has ~0 inertia)

A very important consequence is that $\nabla_z p = g\rho\nabla_z \eta$ (gradient of pressure) is independent on z ! We can separate the horizontal and vertical eqs. completely

The horizontal eqs. then become

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla_z p = -\frac{1}{\rho}\nabla p = -g\nabla\eta$$

The right-end side is independent on z and so has to be the left. Therefore

$$\frac{D\vec{u}}{Dt} + \vec{f}\times\vec{u} = \frac{\partial u}{\partial t} + u\frac{\partial \vec{u}}{\partial x} + v\frac{\partial \vec{u}}{\partial y} + \vec{f}\times\vec{u} = -g\nabla\eta$$

We are left with only the horizontal components of the velocity as a consequence of the hydrostatic approx.

The *independence of the horizontal motion on height* is the characteristic of shallow water flows. In real flows does not hold exactly (imagine just what friction at the bottom does)

Mass continuity

Starting from $\nabla \cdot \vec{v} = 0$

we can re-write using the components

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -\nabla \cdot \vec{u}$$

Integrating from top to bottom of the water column (both sides!) $w(\eta) - w(\eta_b) = -h\nabla \cdot \vec{u}$

but $w(\eta^*) = \frac{D\eta^*}{Dt}$ and therefore

$$\boxed{\frac{D(\eta - \eta_b)}{Dt} + h\nabla \cdot \vec{u} = \frac{Dh}{Dt} + h\nabla \cdot \vec{u} = \frac{\partial h}{\partial t} + \nabla \cdot (h\vec{u}) = 0}$$

Rigid lid approx

If the top is kept at constant height (waves in the ocean are often neglected..) then $\frac{\partial h}{\partial t} = 0$ and $h_b = H - \eta_b$

The mass continuity reduces to $h_b \nabla \cdot \vec{u} = 0$

However we cannot assume anymore p_0=0 –a pressure is necessary to keep the top at constant height, and the horizontal momentum eq. goes back to

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho} \nabla p_{lid}$$

Reduced gravity equations: a VERY simple model for the upper ocean

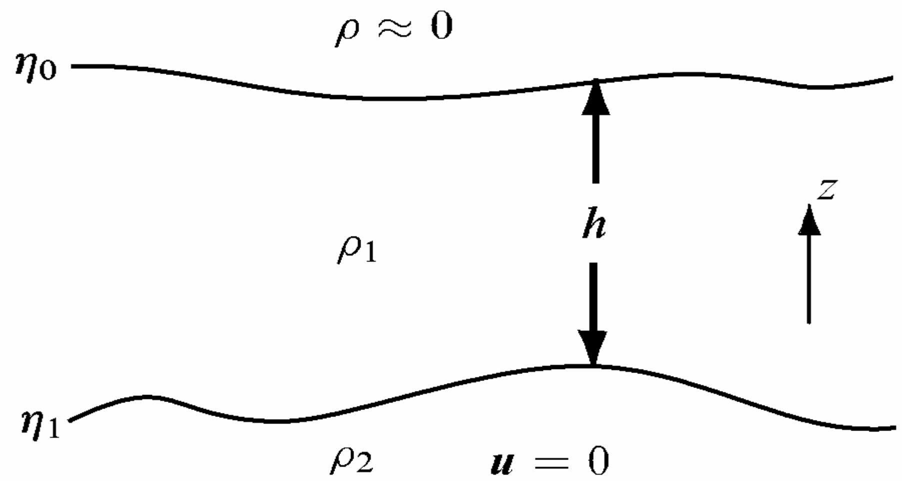


Fig. 3.3 The reduced gravity shallow water system. An active layer lies over a deep, more dense, quiescent layer. In a common variation the upper surface is held flat by a rigid lid, and $\eta_0 = 0$. from Vallis' book

a shallow layer where the flow has velocity **u** ≠ 0, on top of a quiescent denser layer where **u**=0 and beneath of a layer of negligible inertia

1) free upper surface

- for the upper layer : $p_1(z) = g\rho_1(\eta_o - z)$ hence $\frac{1}{\rho}\nabla p = g\nabla\eta_o$
  and the momentum eq. is

$$\boxed{\frac{D\vec{u}}{Dt} + \vec{f}\times\vec{u} = -g\nabla\eta_o}$$

- in the lower layer the pressure is due to the fluid in the layer + to the fluid above, so at some z in the bottom layers $p_2(z) = \rho_1 g(\eta_o - \eta_1) + \rho_2 g(\eta_1 - z)$
  but the layer is motionless and

$$\nabla p_2 = \nabla(\rho_1 g(\eta_o - \eta_1) + \rho_2 g\eta_1) = 0$$

which implies $\rho_1 g\eta_o = -\rho_1 g'\eta_1 + cons$ where $g' = (\rho_2 - \rho_1)/\rho_1$

the momentum eq. can be re-written as

$$\boxed{\frac{D\vec{u}}{Dt} + \vec{f}\times\vec{u} = g'\nabla\eta_1}$$

the mass conservation eq. does not change

$$\frac{Dh}{Dt} + h\nabla \cdot \vec{u} = 0, \quad with \qquad h = \eta_0 - \eta_1$$

if $\qquad \rho_1 g \eta_o = -\rho_1 g' \eta_1 + cons \qquad$ then the vertical displacement of the surface has to be MUCH smaller than the intra-layer one, because g'<<g. True in the ocean!

This is the justification for the rigid lid approx

2) Rigid lid approx

- the pressure at the lid is constant (P) and the horizontal pressure gradient on the upper layer is $\nabla p_1 = \nabla P$
- in the lower layer we have as before and using P

$$p_2(z) = -\rho_1 g \eta_1 + \rho_2 g(\eta_1 - z) + P = \rho_1 g h - \rho_2 g(h + z) + P$$

and therefore $\nabla p_2 = -g(\rho_2 - \rho_1)\nabla h + \nabla P = 0$

The momentum eq. is just $\dfrac{D\vec{u}}{Dt} + \vec{f} \times \vec{u} = -g'\nabla h$

Multi-layer shallow water eqs

from Vallis' book

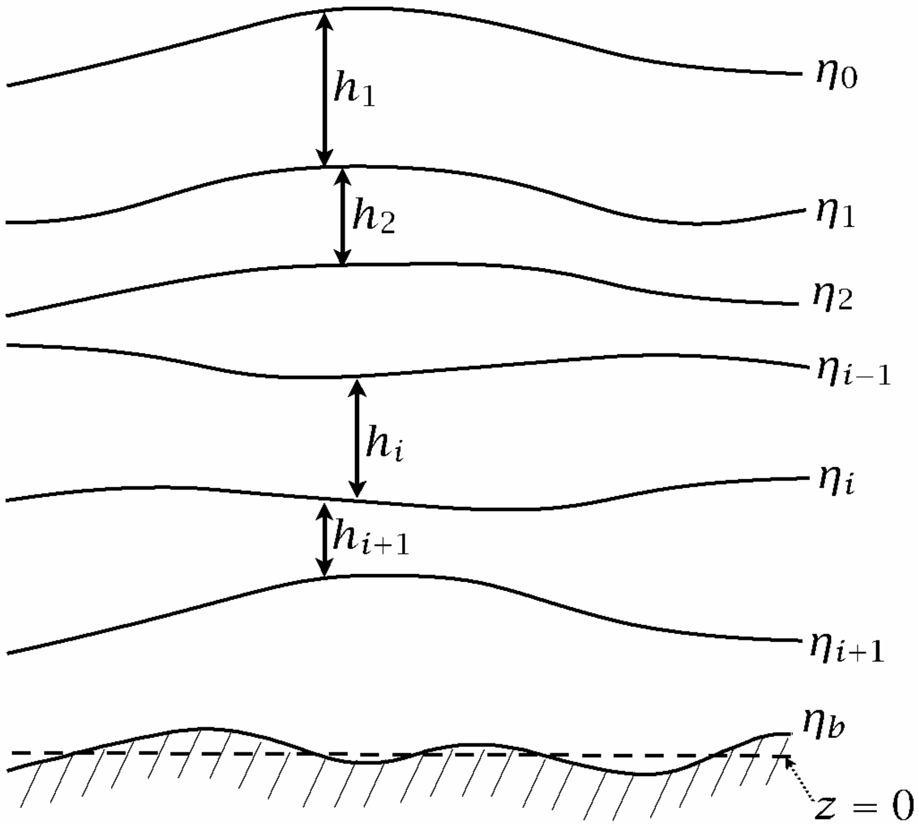


Fig. 3.4 The multi-layer shallow water system. The layers are numbered from the top down. The coordinates of the interfaces are denoted by η, and the layer thicknesses by h, so that $h_i = \eta_i - \eta_{i-1}$.

We consider multiple layer stack one on top of the other

Pressure is continuous between layer. Density ‘jumps’

In the first layer we have $p_1 = \rho_1 g(\eta_o - z)$

in the second $p_2 = \rho_1 g(\eta_0 - \eta_1) + \rho_2 g(\eta_1 - z) = \rho_1 g \eta_0 + \rho_1 g' \eta_1 - \rho_2 gz$

The terms in z are irrelevant for the dynamics. Only the horizontal derivatives enter in the eqs!

therefore we can write in a compressed form $p_n = \rho_1 \sum_{i=0}^{n-1} g'_i \eta_i$

where g_0=g and g_i=g(ρ_{i+1}-ρ_i)/ρ_1

and $\eta_n = \eta_b + \sum_{i=n+1}^{i=n} h_i$

we can generalize the momentum eq. for each layer as

$$\frac{D\vec{u}_n}{Dt} + \vec{f} \times \vec{u}_n = -\frac{1}{\rho_n}\nabla p_n$$

and the mass conservation is $$\frac{Dh_n}{Dt} + h_n \nabla \cdot \vec{u}_n = 0$$

for a two layer model therefore…

$$p_1 = \rho_1 g \eta_0 = \rho_1 g (h_1 + h_2 + \eta_b)$$

$$p_2 = \rho_1 \left[g\eta_0 + g'\eta_1 \right] = \rho_1 \left[g(h_1 + h_2 + \eta_b) + g'(h_2 + \eta_b) \right]$$

$$\frac{D\vec{u}_1}{Dt} + \vec{f} \times \vec{u}_1 = -g\nabla \eta_0 = -g\nabla (h_1 + h_2 + \eta_b)$$

$$\frac{D\vec{u}_2}{Dt} + \vec{f} \times \vec{u}_2 = -\frac{\rho_1}{\rho_2}(g\nabla \eta_0 + g'\nabla \eta_1) =$$

$$= -\frac{\rho_1}{\rho_2}[g\nabla(h_1 + h_2 + \eta_b) + g'\nabla(h_2 + \eta_b)]$$

Form Drag

- If the interface between layers or the bottom are not flat, the layers or the bottom layer and the topography exert a pressure on each other called form drag
- Given a layer comprised between the interfaces η_1 and η_2, the form drag has the form

$$F_p = -\frac{1}{L}\int_{x_1}^{x_2}\int_{\eta_1}^{\eta_2}\frac{\partial p}{\partial x}dxdz = -\frac{1}{L}\int_{x_1}^{x_2}\left[\frac{\partial p}{\partial x}z\right]_{\eta_2}^{\eta_1} = -\left\langle \eta_1\frac{\partial p_1}{\partial x}\right\rangle + \left\langle \eta_2\frac{\partial p_2}{\partial x}\right\rangle =$$

$$= +\left\langle p_1\frac{\partial \eta_1}{\partial x}\right\rangle - \left\langle p_2\frac{\partial \eta_2}{\partial x}\right\rangle = \tau_1 - \tau_2$$

Quantities conserved in SW

- Material invariants: $$\frac{D?}{Dt} = 0$$

  The most important is Potential Vorticity

- Integral invariants: a quantity conserved following an integration path (over a closed line, a volume, a surface..)

  Energy

Potential vorticity conservation

vorticity: $\omega = \nabla \times \vec{v}, \quad \omega' = \nabla \times \vec{u} = \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \vec{k}\varsigma$

Using $(\vec{A}\cdot\nabla)\vec{A} = \frac{1}{2}\nabla A^2 - \vec{A}\times(\nabla\times\vec{A})$ we can rewrite the SW momentum eq. As

$$\frac{\partial \vec{u}}{\partial t} + \vec{\omega}'\times\vec{u} = -\nabla\left(g\eta + \frac{1}{2}\vec{u}^2\right)$$

then using the vector identity

$$\nabla\times(\vec{\omega}'\times\vec{u}) = \vec{\omega}'(\nabla\cdot\vec{u}) + (\vec{u}\cdot\nabla)\vec{\omega}' - \vec{u}(\nabla\cdot\vec{\omega}') - (\vec{\omega}'\cdot\nabla)\vec{u}$$

and taking the curl of the mom. eq we get

$$\frac{\partial \varsigma}{\partial t} + (\vec{u}\cdot\nabla)\varsigma \equiv \frac{D\varsigma}{Dt} = -\varsigma\nabla\cdot\vec{u} \qquad (all \quad \cdot\vec{k})$$

multiplying the mass continuity eq. by ζ we get also

$$-\varsigma\nabla\cdot\vec{u} = \frac{\varsigma}{h}\frac{Dh}{Dt}$$

and therefore

$$\boxed{\frac{D\varsigma}{Dt} = \frac{\varsigma}{h}\frac{Dh}{Dt} \quad \Rightarrow \quad \frac{D}{Dt}\left(\frac{\varsigma}{h}\right) = 0}$$

ζ/h = Q is Potential Vorticity. Being conserved on a parcel, then any function of Q has to be a material invariant as well ($\frac{DF(Q)}{Dt} = F'(Q)\frac{DQ}{Dt} = 0$)

In presence of rotation is all the same but Q=(ζ+f)/h and

$$\boxed{\frac{D}{Dt}\left(\frac{\varsigma + f}{h}\right) = 0}$$

Circulation in SW

- Circulation: $$C = \int_A \varsigma dA = \int_A QhdA$$
- The material derivative of it is

$$\frac{DC}{Dt} = \int_A \frac{DQ}{Dt} hdA + \int_A Q \frac{D}{Dt}(hdA) = 0 = \int_A \frac{D\varsigma}{Dt} dA$$

and using Stoke's theorem

$$\frac{DC}{Dt} = \int_A \frac{D\varsigma}{Dt} dA = \frac{D}{Dt} \oint \vec{u} \cdot d\vec{l}$$

Conservation of energy in SW

- Energy is conserved BEFORE we make simplifications and assumptions. In SW is still an integral invariant
- The kinetic energy per unit area is $\frac{1}{2}\rho_o h\vec{u}^2$
- The potential (due to gravity) is $PE = \int_0^h \rho_o gzdz = \frac{1}{2}\rho_o gh^2$

In the assumption of a flat bottom, let's look for the evolution of PE:

$$\frac{D}{Dt}\frac{gh^2}{2} + gh^2\nabla\cdot\vec{u} = 0 = \frac{\partial}{\partial t}\frac{gh^2}{2} + \nabla\cdot\left(\vec{u}\frac{gh^2}{2}\right) + \frac{gh^2}{2}\nabla\cdot\vec{u}$$

for the kinetic part starting from the mom. eq and multiplying it by **u**h :

$$\frac{D}{Dt}\frac{h\vec{u}^2}{2}+\frac{\vec{u}^2 h}{2}\nabla\cdot\vec{u}=-g\vec{u}\cdot\nabla\frac{h^2}{2}$$

consider that $\frac{D}{Dt}\frac{h\vec{u}^2}{2}=\frac{\vec{u}^2}{2}\frac{Dh}{Dt}+h\vec{u}\frac{D\vec{u}}{Dt},\qquad g\,h\vec{u}\cdot\nabla h=g\vec{u}\cdot\nabla\frac{h^2}{2}$

which can be written as

$$\frac{d}{dt}\frac{h\vec{u}^2}{2}+\nabla\cdot\left(\vec{u}\,\frac{h\vec{u}^2}{2}\right)+g\vec{u}\cdot\nabla\frac{h^2}{2}=0$$

Adding the expressions for PE and KE we get

$$\boxed{\begin{gathered}\frac{\partial}{\partial t}\frac{1}{2}\left(h\vec{u}^2+gh^2\right)+\nabla\cdot\left[\frac{1}{2}\vec{u}\left(gh^2+h\vec{u}^2+gh^2\right)\right]=0, \\ \frac{\partial E}{\partial t}+\nabla\cdot\vec{F}=0\end{gathered}}$$

where E = KE+PE = Energy density = h**u**^{2}+gh^2 and

F = energy flux = **u**(h**u**^{2}+gh^2+gh^2)/2

If the fluid is such that at the (rigid) boundaries the normal component of the velocities is = 0, then F vanishes and integrating over the area given by the boundaries + using Gauss' we get the conservation of the total energy

SW waves in presence of rotation

Waves are solutions, in general, of the linearized eq. of motions: i.e. waves result from small disturbances. A parcel of water is disturbed, it oscillates and eventually goes back to its stable state. For simplicity will consider only gravity as restoring force (no diffusion). In SW in presence of rotation two kind of waves can emerge: Poincare waves and, in presence of boundaries, Kelvin waves.

Consider 1 layer and flat bottom. The fluid thickness is equal to the free surface displacement:

$$h(x,y,t) = H + \eta(x,y,t)$$

and the fluid velocity is $\mathbf{u}(x,y,t)$

The eq. of motion linearized about a state of rest are

$$\frac{\partial u}{\partial t} - f_0 v = -g\frac{\partial \eta}{\partial x}, \qquad \frac{\partial v}{\partial t} + f_0 u = -g\frac{\partial \eta}{\partial y}, \qquad \frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

It is useful non-dimensional the eqs. using L (the characteristic scale of the flow), U, its characteristic velocity, T=L/U, 1/T and H to scale horizontal length, velocity, time, Coriolis parameter and vertical scale respectively $\rightarrow$

$$\frac{\partial \hat{u}}{\partial \hat{t}} - \hat{f}_0 \hat{v} = -\hat{c}^2\frac{\partial \hat{\eta}}{\partial \hat{x}}, \qquad \frac{\partial \hat{v}}{\partial \hat{t}} + \hat{f}_0 \hat{u} = -\hat{c}^2\frac{\partial \hat{\eta}}{\partial \hat{y}}, \qquad \frac{\partial \hat{\eta}}{\partial \hat{t}} + \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}}\right) = 0$$

where $\hat{c} = \sqrt{gH}\,/U = 1/Fr$ Fr=Froude number (velocity of the fluid / wave speed)

We need a dispersion relation. To obtain one we set

$$(\hat{u},\hat{v},\hat{\eta}) = (\tilde{u},\tilde{v},\tilde{\eta})e^{i(\hat{\vec{k}}\cdot\hat{\vec{x}}-\hat{\omega}t)}$$

where $\hat{\vec{k}} = \hat{k}\vec{i} + \hat{l}\vec{j}$ and $\hat{\omega}$ is the non-dimensional frequency. Substituting we are left with the matrix

$$\begin{pmatrix} -i\hat{\omega} & -\hat{f}_o & i\hat{c}^2\hat{k} \\ \hat{f}_o & -i\hat{\omega} & i\hat{c}^2\ \hat{l} \\ i\hat{k} & i\hat{l} & -i\hat{\omega} \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{pmatrix} = 0$$

This homogeneous eq. has non-trivial solutions ONLY IF the determinant vanishes:

$$\hat{\omega}(\hat{\omega}^2 - \hat{f}_o^{\ 2} - \hat{c}^2\hat{K}^2) = 0$$

$$(\hat{K}^2 = \hat{k}^2 + \hat{l}^2)$$

Two possibilities:

1) $\hat{\omega} = 0$ a time-independent flow, which reduces the linearized eqs. to the ones for a geostrophically balanced flow

2) $\hat{\omega}^2 = \hat{f_o}^2 + \hat{c}^2 \hat{K}^2 \quad or \quad \omega^2 = f_o^{\,2} + gH(k^2 + l^2)$

known as Poincare' waves

Two limits:

Short wave: $K^2 >> \dfrac{f_o^{\,2}}{gH} \quad or \quad \dfrac{(2\pi)^2}{\left|\vec{k}\right|^2} << L_d^{\,2}(2\pi)^2$

Long wave: $K^2 << \dfrac{f_o^{\,2}}{gH} \quad or \quad \omega \approx f_0$

So called inertial oscillations because we can obtained them if $\dfrac{\partial u}{\partial t} - f_0 v = 0, \qquad \dfrac{\partial v}{\partial t} + f_0 u = 0$

$$L_d = \sqrt{gH} / f_o$$

L_d is known as Rossby radius of deformation or 'deformation radius'

It is the naturally occurring length-scale in problems involving rotation and gravity.

Whenever the flow is stratified, g is replaced by

$$g' = g \frac{\Delta\rho}{\rho}$$

L_d represents the scale at which rotation becomes as important as buoyancy

Coastal Kelvin waves

If we have a lateral boundary, for example at y=0 harmonic solutions in the y direction cannot exist: a boundary implies no normal flow! The existence of a wave solution also in this case was given by Kelvin in the second half of 1800.

The eqs. we used to derive the Poincare' waves are unchanged except that v=0 at y=0. For continuity a solution for the interior must converge to the solution of the eq. for v=0. We set v=0 everywhere and see where this takes us

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}, \qquad +f_0 u = -g\frac{\partial \eta}{\partial y}, \qquad \frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0$$

The left and right eqs together bring us to the standard wave eq. $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ where $c = \sqrt{gH}$

Solutions have to be in the form of two waves traveling in the positive and negative x-direction

$$u = F_1(x+ct, y) + F_2(x-ct, y)$$

and $$\eta = \sqrt{H/g}\left[-F_1(x+ct, y) + F_2(x-ct, y)\right]$$

Substituting to get the y-dependence we get

$$\frac{\partial F_1}{\partial y} = \frac{f_0}{\sqrt{gH}} F_1 = \frac{1}{L_d} F_1, \quad \frac{\partial F_2}{\partial y} = -\frac{f_0}{\sqrt{gH}} F_2 = -\frac{1}{L_d} F_2$$

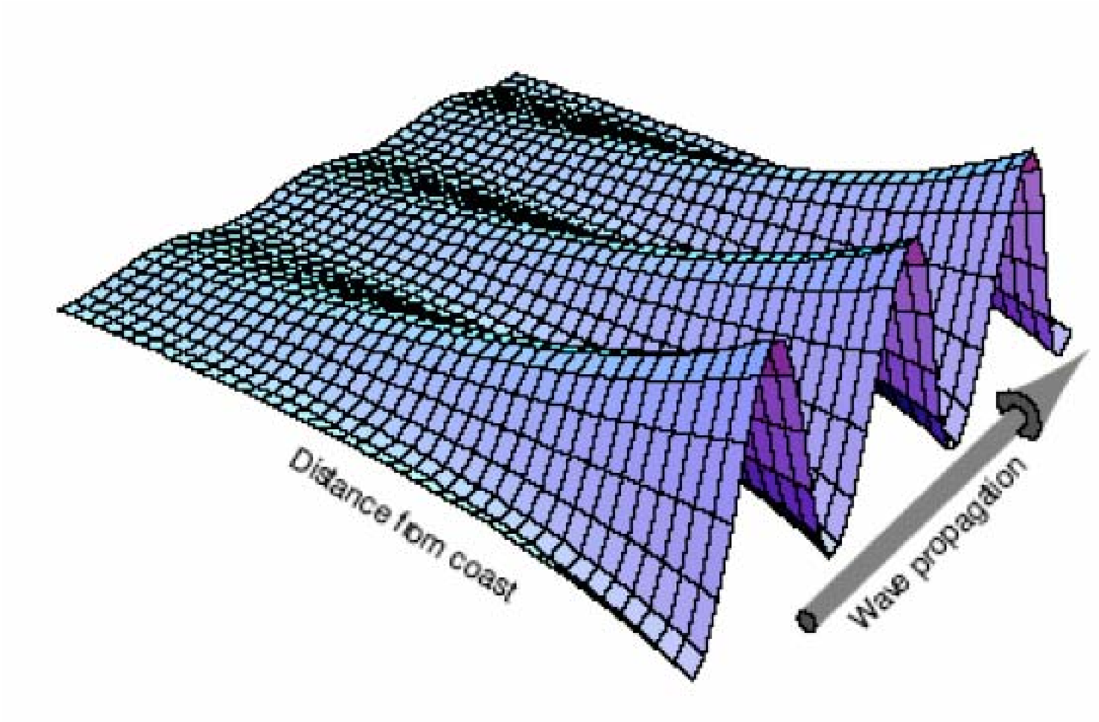
which solutions are

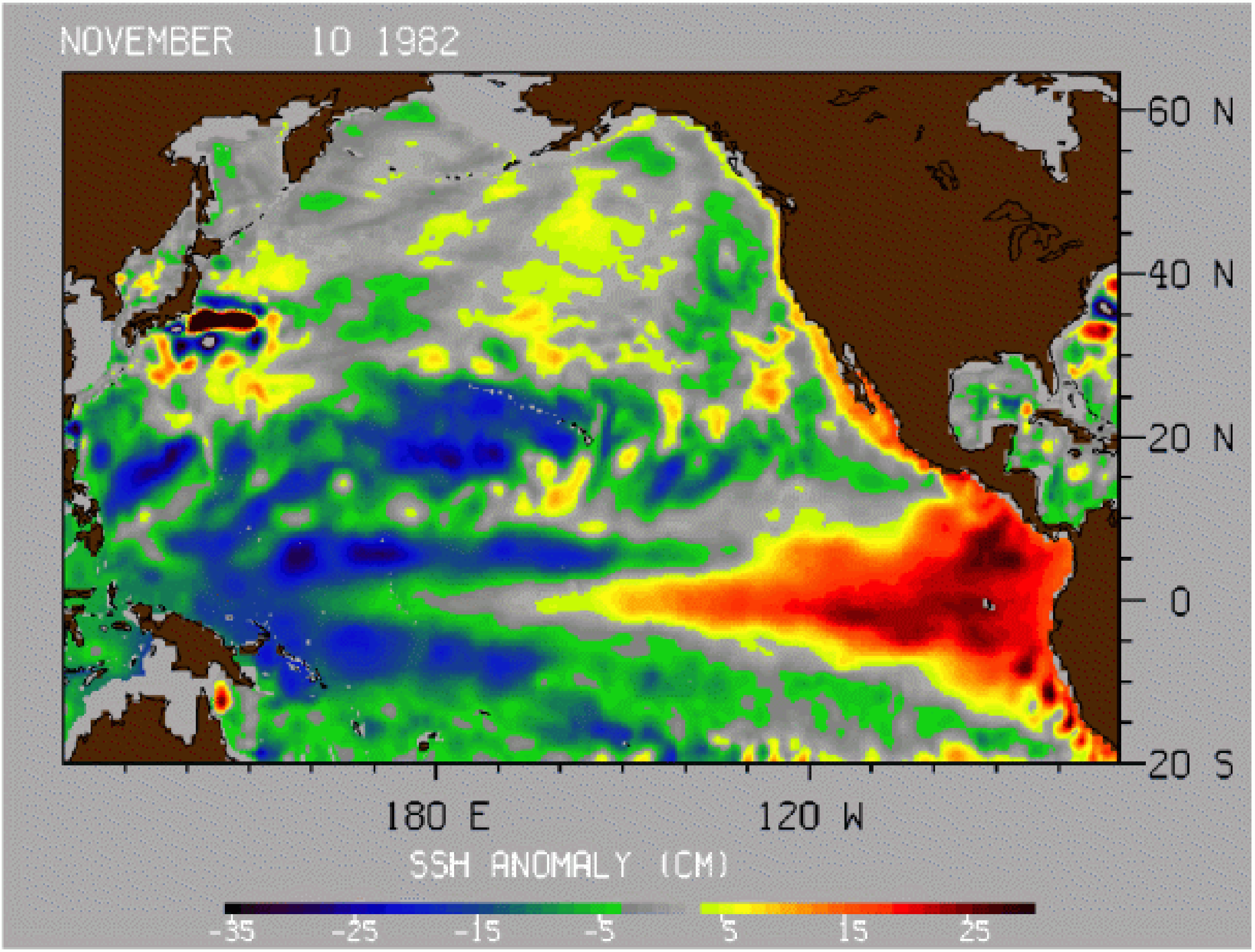
$$F_1 = F(x+ct)e^{y/L_d}, \qquad F_2 = G(x-ct)e^{-y/L_d}$$

and therefore $$\boxed{u = e^{-y/L_d} G(x-ct), \quad v = 0}$$

+ correspondent eq. for the surface elevation

As a result Kelvin waves in the ocean decade away from the boundary and always propagate with the shoreline on the right in the northern hemisphere, and with the shoreline on the left in the southern hemisphere.





The sea level anomaly from the NRL global ocean model (courtesy H. Hurlburt). The climatology was removed. The yellow band along the coast of N. America is the product of the coastal Kelvin wave. In this model, the speed of propagation is 2-3 m/s

Geostrophic adjustment

Geostrophic adjustment is an initial-value problem involving the evolution of an unbalanced initial state toward one in geostrophic balance.

In presence of rotation potential vorticity conservation plays an important role. The linearized eq. expressing PV conservation for f=cost=f_0 is (for $\eta \ll H$)

$$\frac{\partial Q}{\partial t}=0 \quad \Rightarrow \quad \frac{\partial q}{\partial t}=0 \quad with \quad q=\varsigma - f_0\frac{\eta}{H}$$

If we have the following initial condition

$$q(x,y) = \begin{cases} -f_o\eta_o / H & x < 0 \\ +f_o\eta_o / H & x > 0 \end{cases}$$

this cannot change in time and at the final steady state the system must satisfy the following eqs.

$$q = \varsigma - f_0 \frac{\eta}{H}, \qquad f_o u = -g\frac{\partial \eta}{\partial y}, \qquad f_o v = +g\frac{\partial \eta}{\partial x}$$

Using the fact that the Coriolis parameter is a constant and $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ the eqs. above reduce to

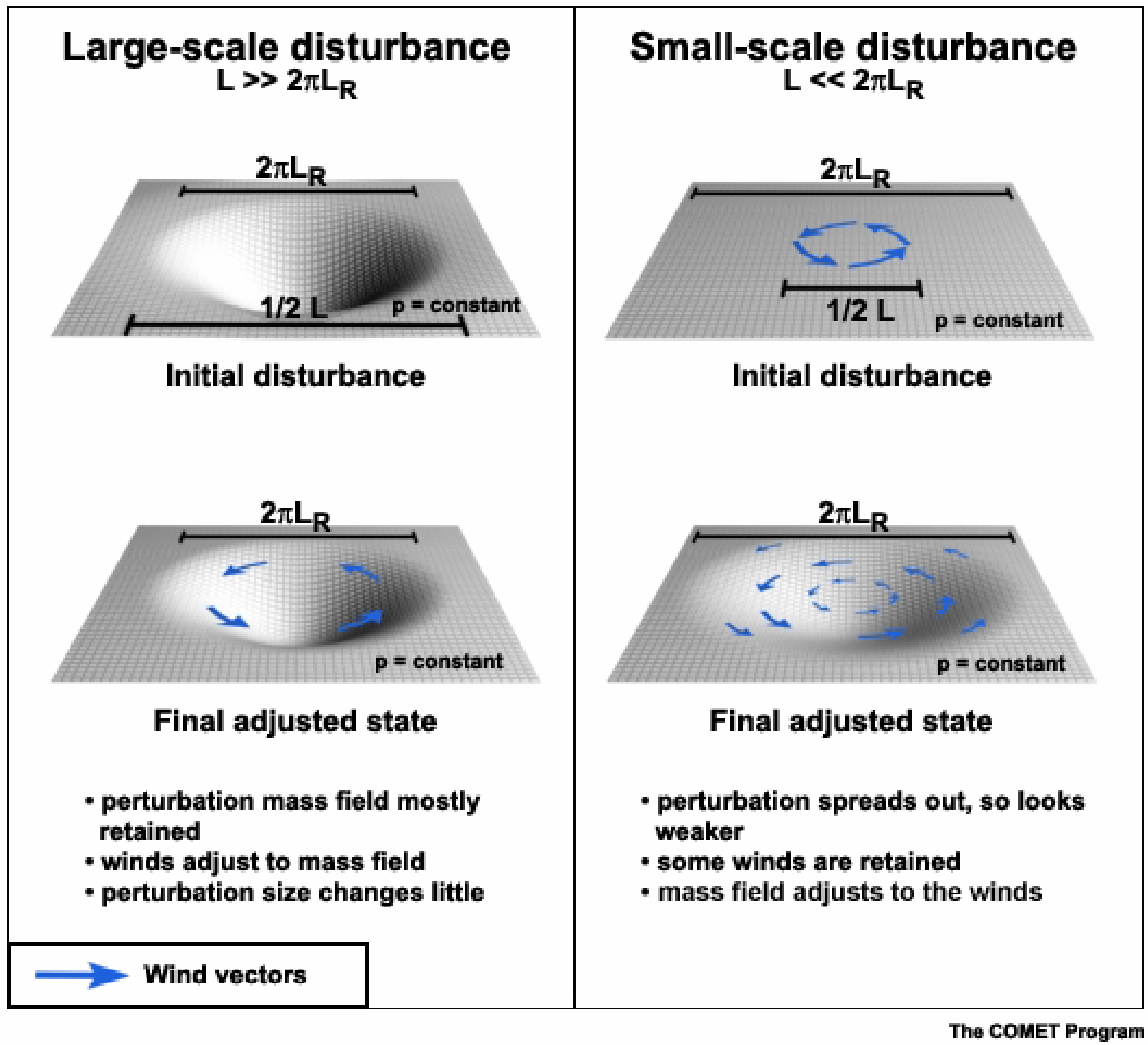
$$\left(\nabla^2 - \frac{1}{L_d{}^2}\right)\psi = q(x,y) \qquad with \qquad \psi = g\eta / f_o$$

the solution is (again) a decaying exp function. So the variations in Ψ (i.e. in height – η – and more generally in the streamfunction) are not radiated to infinity. The adjustment is constrained within a distance equal to a deformation radius from the source

$$\psi = \begin{cases} -(g\eta_o / f_0)(1 - e^{-x/L_d}) & x > 0 \\ +(g\eta_o / f_0)(1 - e^{x/L_d}) & x < 0 \end{cases}$$

Changes to the initial state occur only in a region of radius $O(L_d)$!

Using the variational approach (see Vallis' Book), it can be shown that In the linear approx geostrophic balance is the minimum energy state for a given PV field



Large-scale disturbance
L >> 2πLR
2πLR
1/2 L
p = constant
Initial disturbance
2πLR
p = constant
Final adjusted state
• perturbation mass field mostly retained
• winds adjust to mass field
• perturbation size changes little
Wind vectors
Small-scale disturbance
L << 2πLR
2πLR
1/2 L
p = constant
Initial disturbance
2πLR
p = constant
Final adjusted state
• perturbation spreads out, so looks weaker
• some winds are retained
• mass field adjusts to the winds
The COMET Program

- For a generic (and nice) derivation check http://snowball.millersville.edu/~adecaria/ESCI343/esci343_lesson17_geostrophic_adjustment.html

- Further reading: Gill, A. E., 1982: *Atmosphere-Ocean Dynamics*. Academic Press, New York, 662 pp.

- Check those web site: http://meted.ucar.edu/nwp/pcu1/d_adjust/1_0.htm
  and
  http://www.atmos.washington.edu/~lharris/adjust.html

APE

- The potential energy cannot be transferred all to the kinetic component
- Available potential energy: it is the amount of potential energy that can be in theory transferred and is equal to the difference between the total potential energy and the potential energy of the same fluid adiabatically re-arranged to a state in which the isoentropes are parallel to geopotential surfaces
- TPE>>APE>KE

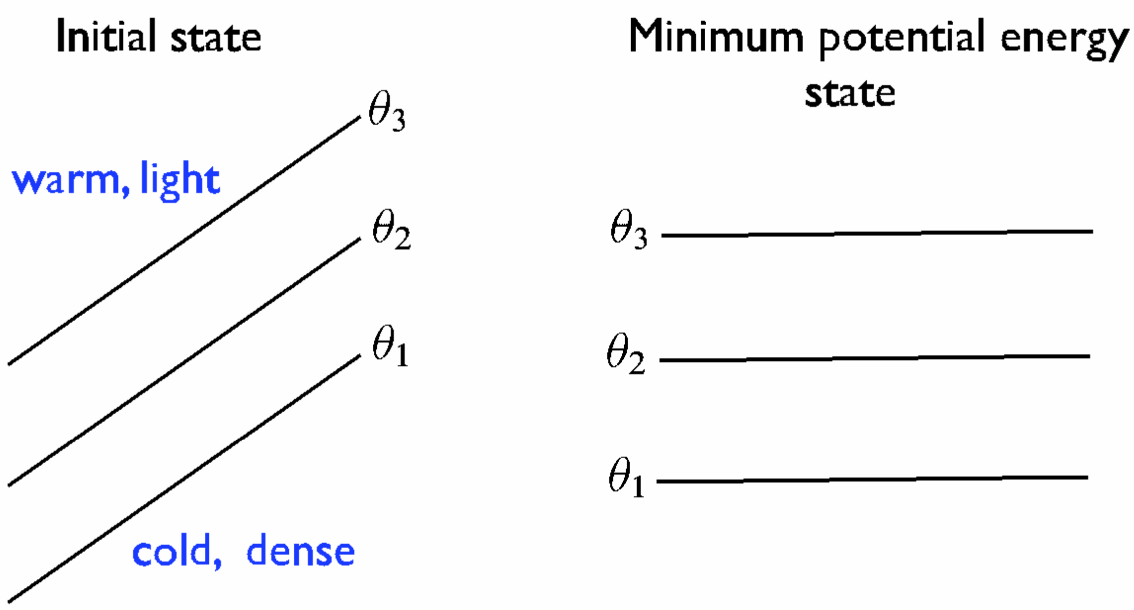


Fig. 3.11 If a stably stratified initial state with sloping isentropes (left) is adiabatically rearranged then the state of minimum potential energy has flat isentropes, as on the right, but the amount of fluid contained between each isentropic surface is unchanged. The difference between the potential energies of the two states is the *available potential energy*.

From Vallis' book

week 4

- Vorticity and circulation
- The vorticity equation
- The vorticity and circulation theorems
- Conservation of potential vorticity

Vorticity and circulation

$$\vec{\omega} = \nabla \times \vec{v}$$

$$C = \oint \vec{v} \cdot d\vec{r} = \int_S \vec{\omega} d\vec{S}$$

$$\vec{\hat{n}} \cdot (\nabla \times \vec{v}) = \frac{1}{\delta \vec{S}} \oint_{\delta r} \vec{v} \cdot d\vec{r}$$

If we consider an infinitesimally small δS pointing in the direction of the normal vector **n**, we get that the component of the vorticity in the direction of n is proportional to the circulation around the infinitesimal fluid element divided by the area bounded by the path δr

The vorticity is a measure of the local spin of a fluid element

(check the two examples on Vallis' book)

The vorticity equation

We use once more the following identity on the 3D momentum equation

$$(\vec{A}\cdot\nabla)\vec{A} = \frac{1}{2}\nabla A^2 - \vec{A}\times(\nabla\times\vec{A})$$

to obtain $$\frac{\partial\vec{v}}{\partial t} + \vec{\omega}\times\vec{v} = -\frac{1}{\rho}\nabla p - \frac{1}{2}\nabla\vec{v}^{\,2} + \vec{F}$$

and taking the curl

$$\frac{\partial\vec{\omega}}{\partial t} + \nabla\times(\vec{\omega}\times\vec{v}) = \frac{1}{\rho^2}\left(\nabla\rho\times\nabla p\right) + \nabla\times\vec{F}$$

Using $$\nabla\times(\vec{a}\times\vec{b})=(\vec{b}\cdot\nabla)\vec{a}-(\vec{a}\cdot\nabla)\vec{b}+\vec{a}\nabla\cdot\vec{b}-\vec{b}\nabla\cdot\vec{a}$$

we can rewrite the vorticity eq. as

$$\frac{\partial\vec{\omega}}{\partial t}+(\vec{v}\cdot\nabla)\vec{\omega}=(\vec{\omega}\cdot\nabla)\vec{v}+\frac{1}{\rho^2}\left(\nabla\rho\times\nabla p\right)+\nabla\times\vec{F}-\vec{\omega}\underbrace{\nabla\cdot\vec{v}}_{=0}-\vec{v}\underbrace{\nabla\cdot\vec{\omega}}_{=0}$$

or

$$\boxed{\frac{D\vec{\varpi}}{Dt}=(\vec{\varpi}\cdot\nabla)v+\frac{1}{\rho^3}\left(\nabla\rho\times\nabla p\right)+\frac{1}{\rho}\nabla\times\vec{F}\qquad with\qquad \vec{\varpi}=\vec{\omega}/\rho}$$

we’ll consider F=0 in the following….

The term $\frac{1}{\rho^2}(\nabla\rho\times\nabla p)=\vec{S}_o=-\nabla\alpha\times\nabla p$ is called

solenoidal or *baroclinic term*

(a solenoid is a tube perpendicular to the gradients of both pressure and density, with elements of length proportional to $\nabla p\times\nabla\alpha$)

in components the momentum eq. can be written as

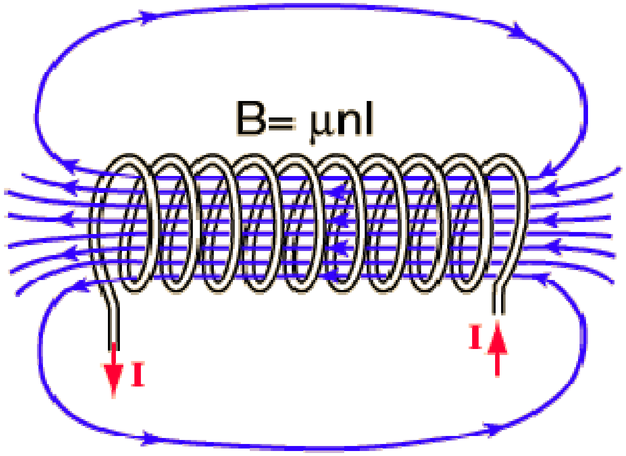
$$\frac{\partial\omega_i}{\partial t}+\frac{\partial}{\partial x_j}\left(v_j\omega_i-v_i\omega_j\right)=S_{oi}, \qquad i=x,y,z$$

IF DENSITY IS FUNCTION OF PRESSURE ONLY, I.E. THE ISOLINES OF PRESSURE AND DENSITY ARE PARALLEL THE BAROCLINIC TERM IS ZERO.

IF ρ=ρ(p) THE FLUID IS BAROTROPIC

For a barotropic fluid the momentum equation simplifies to

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{v}$$



2d flows

A 2d flow is barotropic by definition

The velocity field is **u**=(u,v) and vorticity is simply

$$\vec{\omega} = \varsigma \vec{k} = \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

If the flow is also incompressible the vorticity eq. is just the conservation of vorticity itself

$$\frac{D\varsigma}{Dt} = 0$$

Vorticity is conserved along the fluid elements. Each material parcel keeps its own value while being advected! We can write the velocity components in terms of a streamfunction ψ, as $u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}, \qquad \varsigma = \nabla^2 \psi$

Vorticity and circulation theorems

1. The Helmholtz's theorem

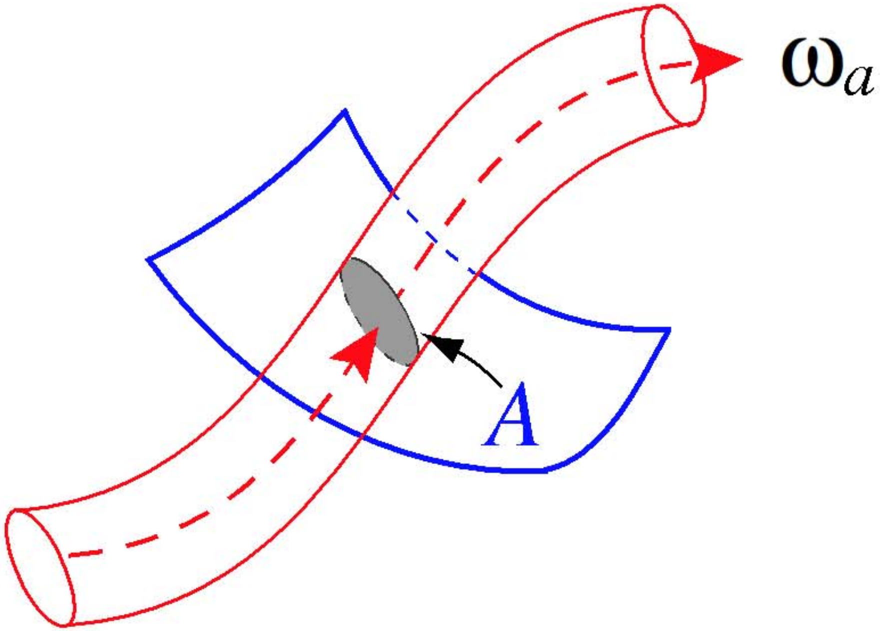


Figure 4.2 A vortex tube passing through a material sheet. The circulation is the integral of the velocity around the boundary of A, and is equal to the integral of the normal component of vorticity over A.

Vortex line: A line connecting fluid parcels which is everywhere in the direction of the local vorticity

Vortex tube: A collection of vortex lines passing through a close curve

Material line: a line connecting fluid parcels.

Helmholtz's theorem:

A vortex line defines also a material line. If a flow is inviscid, unforced and barotropic, a vortex line coincides with the same material line at all times. The vorticity is 'frozen' to the fluid parcel.

We know that for a barotropic flow if viscosity and F are zero $$\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \nabla \vec{v}$$

(not assuming incompressibility this time, nor 2d. Just **'barotropicity'**)

At time t = 0 we can define an infinitesimal material line such that $\delta \mathbf{l}(\mathbf{x}, t=0) = A\boldsymbol{\omega}(\mathbf{x}, t=0)$

The evolution of the infinitesimal material line $\delta \mathbf{l}$ is given by

$$\frac{D\delta\vec{l}}{Dt} = \frac{1}{\delta t}\left[\delta\vec{l}\,(t+\delta t) - \delta\vec{l}\,(t)\right] = \delta\vec{v}$$

but $\delta\vec{v} = \delta\vec{l} \cdot \nabla\vec{v}$

and therefore **ω** and **δl** evolve according to the same equation! The tendency **ω x δl** is zero

stretching and tilting

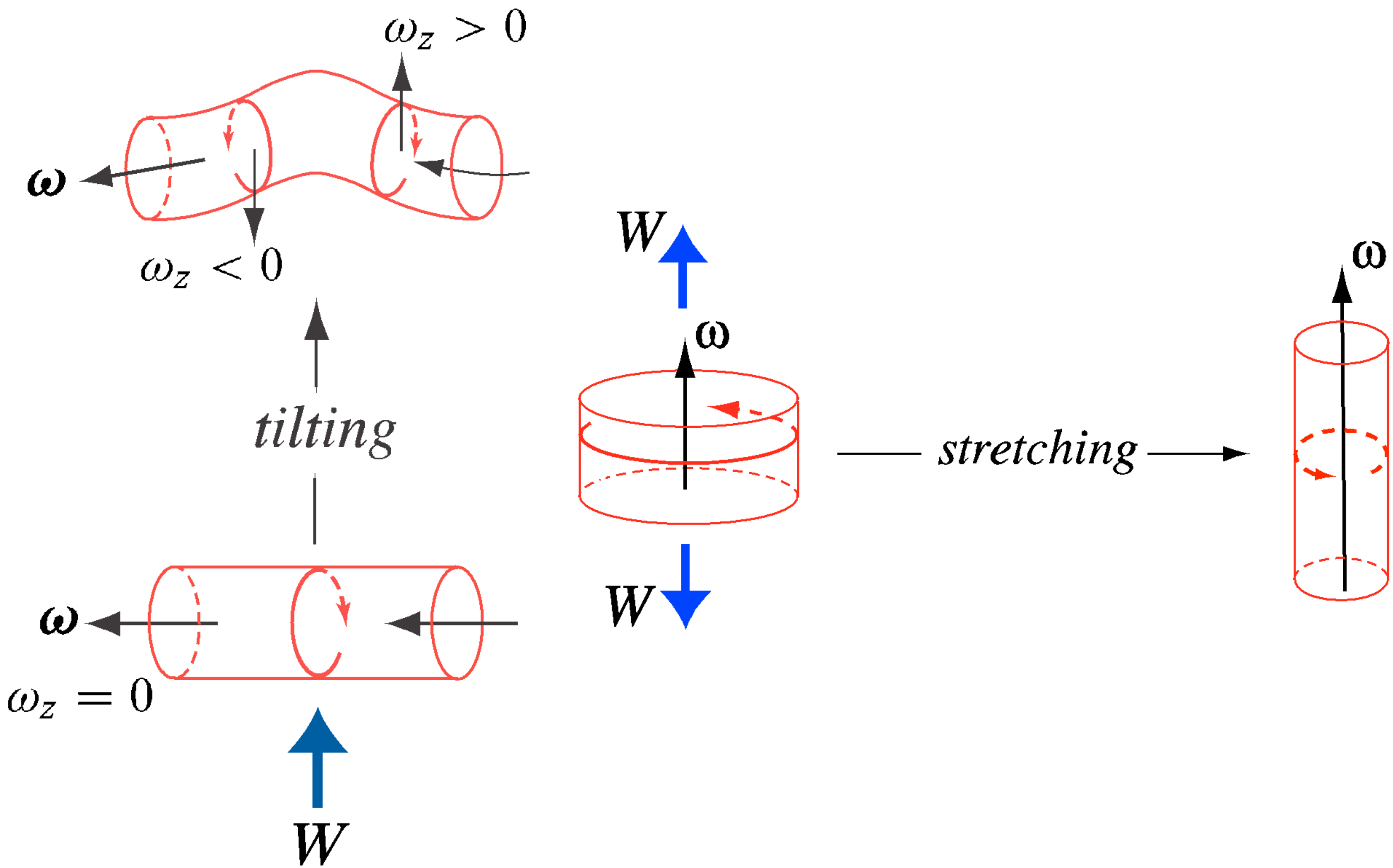
If we write the vorticity eq. for a barotropic flow in terms of the three components x,y,z we get

$$\frac{D\omega^x}{Dt} = \omega^x \frac{\partial u}{\partial x} + \omega^y \frac{\partial u}{\partial y} + \omega^z \frac{\partial u}{\partial z}$$

Tilting terms: describe changes in the orientation of the vorticity vector. x-component is generated by y- and z-components if advection tilts the material lines

Stretching term: x-component vorticity is amplified if the velocity increases in the x direction (tornadoes!)

If the flow is incompressible, then the sum of the three terms is zero. Stretching in one direction has to be compensated by convergence in another



$\omega_z > 0$
$\boldsymbol{\omega}$
$\omega_z < 0$
tilting
$\boldsymbol{\omega}$
$\omega_z = 0$
W
W
ω
W
stretching
ω

Kelvin circulation theorem

It states that under some conditions circulation is conserved (see shallow water case). Two conditions have to be satisfied:

1) the forces acting on the flow are conservative (no dissipation)
2) the flow is barotropic ($p=p(\rho)$)

The general form of the theorem states that for a barotropic flow circulation around a material close line is an invariant of motion if forces acting on the flow are conservative

Circulation is defined with respect to an inertial frame of reference

Momentum eq. : $\dfrac{D\vec{v}}{Dt} = -\dfrac{1}{\rho}\nabla p - \nabla\phi$

the material derivative of the circulation is therefore

$$\frac{DC}{Dt} = \frac{D}{Dt}\oint \vec{v}\cdot d\vec{r} = \oint \frac{D\vec{v}}{Dt}\cdot d\vec{r} + \vec{v}\cdot d\vec{v} =$$

$$\oint\left[\left(-\frac{1}{\rho}\nabla p - \nabla\phi\right)\cdot d\vec{r} + \vec{v}\cdot d\vec{v}\right] = \oint -\frac{1}{\rho}\nabla p\cdot d\vec{r}$$

(we used D(d**r**)/Dt=d**v** and the fact that the last two terms on the right side vanish when integrated over a closed loop because exact differentials. To see it use Stokes' theorem)

We are left with

$$\oint -\frac{1}{\rho}\nabla p \cdot d\vec{r} = \int_S -\nabla \times \left(\frac{\nabla p}{\rho}\right) \cdot d\vec{S} = \int_S \frac{\nabla \rho \times \nabla p}{\rho^2} \cdot d\vec{S}$$

if the flow is barotropic the integral is zero because the baroclinic term is zero, being density constant or function of p alone. As a result

$$\boxed{\frac{D}{Dt}\oint \vec{v} \cdot d\vec{r} = \frac{D}{Dt}\int_S \vec{\omega} \cdot d\vec{S} = 0}$$

- Baroclinic flows and Kelvin’ theorem

One exception to the ‘barotropic’ rule:

The solenoidal term may also be written as $S_o = -\nabla\Gamma \times \nabla T$

where Γ is entropy. So it vanishes if pressure and density are parallel (standard case), but also if isolines of temperature and entropy are parallel or if density, pressure, temperature or entropy are constant. If we can find a loop over which entropy remains constant, then Kelvin’ theorem is still satisfied. This is true for an ideal gas. It is relevant for the atmosphere.

- Circulation theory in a rotating frame

$$\frac{D}{Dt}\oint(\vec{v}_r+\vec{\Omega}\times\vec{r})\cdot d\vec{r}=0=\frac{D}{Dt}\int_S\left(\vec{\omega}_r+2\vec{\Omega}\right)\cdot d\vec{S}$$

- and for hydrostatic flows

$$\frac{D}{Dt}\oint(\vec{u}_r+\vec{\Omega}\times\vec{r})\cdot d\vec{r}=0=\frac{D}{Dt}\int_S\left(\vec{\omega}_{hy}+2\vec{\Omega}\right)\cdot d\vec{S}$$

$$with \qquad \vec{\omega}_{hy}=\nabla\times\vec{u}_r=-\vec{i}\,\frac{\partial v_r}{\partial z}+\vec{j}\,\frac{\partial u_r}{\partial z}+\vec{k}\left(\frac{\partial v_r}{\partial x}-\frac{\partial u_r}{\partial y}\right)$$

Circulation and beta effect

On a spherical planet the circulation theorem has important implications:

if we consider that $\vec{v}_r = \vec{v}_i - \vec{\Omega} \times \vec{r}$, then

$$C_r = C_i - \int_S 2\vec{\Omega} \cdot d\overleftarrow{S} = C_i - 2\vec{\Omega} A_\perp$$

and $\frac{D}{Dt}\left(C_r + 2\Omega A_\perp\right) = 0$

for a parcel at latitude θ $A_\perp$ = Asin θ: the vorticity must change with latitude. It is the so-called β-effect

$$\frac{D\varsigma_r}{Dt} = -\frac{2\Omega}{A}\frac{DA_\perp}{Dt} = -2\Omega\frac{D}{Dt}\sin\theta = -v_r\frac{2\Omega\cos\theta}{a} = -\beta v_r$$

$$\int_A (\omega + 2\Omega) \cdot \mathrm{d}A = 2\Omega A_\perp$$

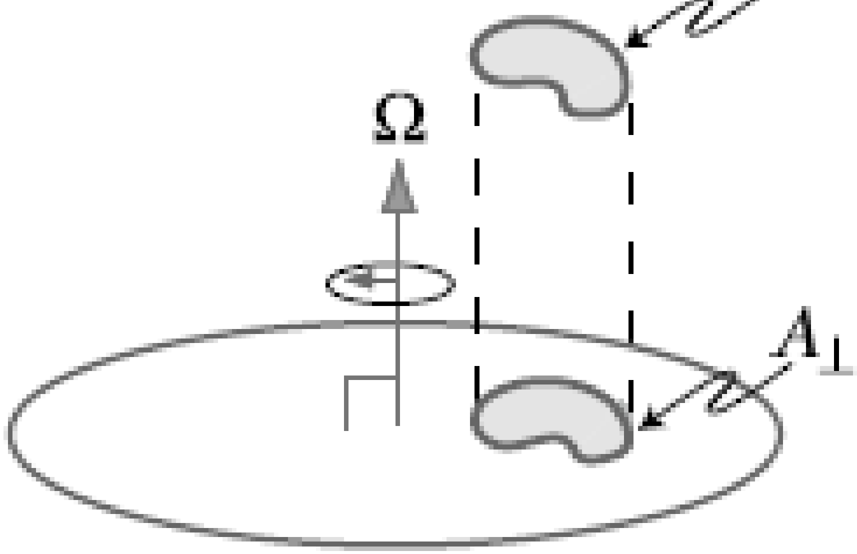


Fig. 4.7 The projection of a material circuit on to the equatorial plane. If a fluid element moves poleward, keeping its orientation to the local vertical fixed (e.g., it stays horizontal) then the area of its projection on to the equatorial plane increases. If its total (absolute) circulation is to be maintained, then the vertical component of the relative vorticity must diminish. That is, $\int_A (\omega + 2\Omega) \cdot \mathrm{d}A = \int_A (\zeta + f)\mathrm{d}A =$ constant. Thus, the β term in $\mathrm{D}(\zeta + f)/\mathrm{D}t = \mathrm{D}\zeta/\mathrm{D}t + \beta v = 0$ ultimately arises from the *tilting* of a parcel relative to the axis of rotation as it moves meridionally.

the vertical component of the vorticity equation

Can be obtained cross-differentiating and subtracting the horizontal component of the mom. eq.

$$\frac{D}{Dt}(\varsigma + f) = -(\varsigma + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial w}{\partial x}\right) +$$

$$+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) + \left(\frac{\partial F^y}{\partial x} - \frac{\partial F^x}{\partial y}\right)$$

$$D\varsigma / Dt =$$

the vorticity material derivative

$$Df / Dt = v\partial f / \partial y = v\beta$$

the β term: the system has differential rotation

$$-(\varsigma + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) =$$

divergence term. Responsible for vortex stretching can be written as $(\varsigma + f)\frac{\partial w}{\partial z}$ if the divergence of velocity is zero

$$\left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial w}{\partial x}\right) =$$

Tilting term: a vertical velocity acting on the horizontal vorticity

$$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) =$$

baroclinic term

$$\left(\frac{\partial F^y}{\partial x} - \frac{\partial F^x}{\partial y}\right) =$$

forcing (+ friction or viscosity)

Conservation of PV

It formally derives from the circulation theorem (just a different way to say the same thing..)

Introduced by Rossby in 1936 and in a more general form by Rossby and Ertel in 1940.

Implications: we can use a scalar to track the evolution of the fluid elements. In baroclinic flows the scalar has to be chosen opportunely as it needs to be function of pressure and density alone. Any scalar will work for barotropic flows.

The conservation of potential vorticity can be derived in various ways…

from the circulation theorem

- barotropic flow

we start from $\frac{D}{Dt}\left[\left(\vec{\omega}_i \cdot \vec{n}\right)\delta A\right]=0$ and we consider a volume bounded by two surfaces of any materially conserved tracer, so that $\vec{n}=\nabla\chi/\left|\nabla\chi\right|$ and $\delta V=\delta h\delta A$

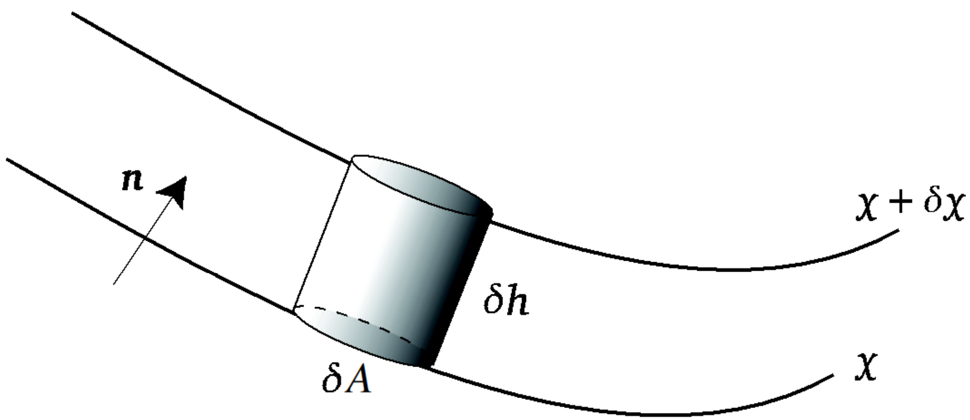


Fig. 4.8 An infinitesimal fluid element, bounded by two isosurfaces of the conserved tracer χ. As $D\chi/Dt = 0$, then $D\delta\chi/Dt = 0$.

then $$\left[(\vec{\omega}_i \cdot \vec{n})\delta A\right] = \vec{\omega}_i \cdot \frac{\nabla\chi}{|\nabla\chi|}\frac{\delta V}{\delta h}$$

but $$\delta\chi = \delta\vec{x}\cdot\nabla\chi = \delta h\left|\nabla\chi\right|$$

and therefore $$\frac{D}{Dt}\left[\left((\vec{\omega}_i \cdot \nabla\chi)\delta V\right)/\delta\chi\right] = 0$$

if χ is conserved on material lines also δχ is. But the mass infinitesimal volume ρδV is also conserved can be taken out of the integral. We get

$$\frac{\rho\delta V}{\delta\chi}\frac{D}{Dt}\left(\frac{\vec{\omega}_i}{\rho}\cdot\nabla\chi\right) = 0 \qquad \Rightarrow \qquad \frac{D}{Dt}\left(\tilde{\vec{\omega}}_i \cdot \nabla\chi\right) = 0$$

- For baroclinic flows the baroclinic term confines the choices of χ to be p, ρ, T or Γ because the solenoid has to be identically zero for the circulation theorem to be satisfied.

$$\frac{D}{Dt}\left[\left(\vec{\omega}_a \cdot \vec{n}\right)\delta A\right] = \vec{S}_o \cdot \vec{n}\delta A$$

$$\vec{S}_o = -\nabla\alpha \times \nabla p = -\nabla\eta \times \nabla T$$

Additionally χ has also to be materially conserved: the obvious choice is potential temperature (or density if the flow satisfies a thermodynamic equation in the form Dρ/Dt=0)

The conservation of PV for a baroclinic flow is

$$\boxed{\frac{D}{Dt}\left(\tilde{\vec{\omega}}_i \cdot \nabla \theta\right) = 0}$$

In sea water because of salinity the above eq. is never perfectly satisfied because potential temperature is function of S and is not a perfectly materially conserved quantity

Things to remember from last week

- The vorticity equation

$$\boxed{\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{v} + \frac{1}{\rho^2}\left(\nabla\rho\times\nabla p\right) + \nabla\times\vec{F}}$$

baroclinic term

- If the baroclinic term is zero (isolines of pressure and density are parallel) the flow is BAROTROPIC

For a barotropic flow the change in vorticity can happen by tilting and stretching

$$\frac{D\omega^x}{Dt} = \omega^x \frac{\partial u}{\partial x} + \omega^y \frac{\partial u}{\partial y} + \omega^z \frac{\partial u}{\partial z}$$

Tilting terms: describe changes in the orientation of the vorticity vector. x-component is generated by y- and z-components if advection tilts the material lines

Stretching term: x-component vorticity is amplified if the velocity increases in the x direction (tornadoes!)

If the flow is incompressible, then the sum of the three terms is zero. Stretching in one direction has to be compensated by convergence in another

Kelvin theorem

for a barotropic flow circulation around a material close line is an invariant of motion if forces acting on the flow are conservative

$$\boxed{\frac{D}{Dt}\oint \vec{v}\cdot d\vec{r} = \frac{D}{Dt}\int_S \vec{\omega}\cdot d\vec{S} = 0}$$

Conservation of PV

- for a barotropic flow

$$\frac{D}{Dt}\left(\tilde{\vec{\omega}}_i \cdot \nabla \chi\right) = 0$$

- for a baroclinic flow (ocean)

$$\frac{D}{Dt}\left(\tilde{\vec{\omega}}_i \cdot \nabla \theta\right) \cong 0$$

week 5

- Geostrophic balance (again) and the scaling for the atmosphere and ocean
- The planetary geostrophic equations
- The quasigeostrophic equations

- Balance in the vertical between pressure gradient and gravity: **hydrostatic balance**
- Balance in the horizontal between Coriolis force and pressure gradient: **geostrophic balance**

We can use those balances to simplify the Navier-Stokes equations whenever we are interested in horizontal motions at the synoptic scales -> ***quasigeostrophic equations***, or at the large (basin or planetary) scales -> ***planetary geostrophic equations***

Geostrophic scaling

Goal 1: rewrite the SW equations taking into account that large scale oceanic flows are in geostrophic balance

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \vec{f} \times \vec{u} = -g \nabla \eta$$

$$\frac{U}{T} \qquad \frac{U^2}{T} \qquad fU \qquad g\frac{\mathrm{K}}{L}$$

where K is the amplitude of the variations in the surface displacement

remembering that $(U^2/L)/(fU)=U/fL=R_o\sim0.01$ and that the local time derivative scales as the advective term, we obtain again the geostrophic balance relation

$$-fu \approx g\frac{\partial \eta}{\partial y}, \qquad -fv \approx -g\frac{\partial \eta}{\partial x}$$

This implies that variations if the free surface' height scale as

$$\Delta\eta \sim \mathrm{K} = \frac{fUL}{g} = R_o \frac{f^2 L^2}{g} = R_o H \frac{L^2}{{L_d}^2}$$

where L_d is as the (usual) = $\sqrt{gH}/f$

Finally, we obtain that the variations in free-surface height scale as $\frac{\Delta\eta}{H} \sim R_o \frac{L^2}{{L_d}^2}$

and can be non-dimensionalized by using $\hat{\eta}$ such that

$$\eta = H\left(1 + R_o \frac{L^2}{{L_d}^2}\hat{\eta}\right), \qquad \Delta\eta = R_o \frac{L^2}{{L_d}^2} H\hat{\eta}$$

In non-dimensional form, the SW eqs. for flows in geostrophic balance become

$$\boxed{R_o\left[\frac{\partial\hat{\vec{u}}}{\partial\hat{t}}+\hat{\vec{u}}\cdot\nabla\hat{\vec{u}}\right]+\hat{\vec{f}}\times\hat{\vec{u}}=-\nabla\hat{\eta}}$$

where $\hat{\vec{f}}=\vec{k}\hat{f}=\vec{k}f/f_o$

All the variables are O(1).

We can do the same to the continuity eq. and considering a flat bottom (i.e. $h = H + \Delta\eta$) we start from

$$\frac{1}{H}\frac{D\eta}{Dt} + \left(1 + \frac{\Delta\eta}{H}\right)\nabla \cdot \vec{u} = 0$$

that becomes

$$R_o \frac{L^2}{L_d{}^2}\frac{D\hat{\eta}}{D\hat{t}} + \left[1 + R_o \frac{L^2}{L_d{}^2}\hat{\eta}\right]\nabla \cdot \hat{\vec{u}} = 0$$

Using L_d to scale the ratio of the averaged fluid particle velocity to the typical wave speed in SW $\longrightarrow$

$$\boxed{Fr = \frac{U}{\sqrt{gH}} = \frac{U}{f_o L_d} = R_o \frac{L}{L_d}}$$

while the ratio of L_d and the scale of motion of the fluid is called Burger number and in SW is defined as

$$\boxed{Bu = \left(\frac{L_d}{L}\right)^2 = \left(\frac{R_o}{Fr}\right)^2}$$

Goal 2: rewrite the primitive equations taking into account that large scale oceanic flows are stratified and in hydrostatic balance

The primitive equations if δρ is small are

$$\boxed{\frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \vec{v} = -\nabla\phi + b\vec{k}}$$

where $\Phi = \delta p/\rho_o$ and $b = -g\delta\rho/\rho_o$ is buoyancy

The Boussinesq approx ignores variations of density EXCEPT when associated with gravity

If δp and δρ are in hydrostatic balance, then $\frac{\partial \phi}{\partial z} = b$

Remembering that the rotation coupled to stratification caused the vertical velocities to be ‘frozen’ (so much smaller than the horizontal ones) we focus on the horizontal components and we have the following system:

$$\frac{D\vec{u}}{Dt} + \vec{f} \times \vec{u} = -\nabla_z \phi$$

$$\frac{\partial \phi}{\partial z} = b$$

$$\frac{Db}{Dt} = 0$$

$$\nabla \cdot \left(\tilde{\rho} \vec{v} \right) = 0$$

b is buoyancy, $\tilde{\rho}$ the reference density profile

It is useful to separate in the stratification the reference one from the contribution due to advection, and in the potential the part which is hydrostatically balanced by $\tilde{b}$ as

$$b = \tilde{b}(z) + b'(x,y,z,t), \quad and \qquad \phi = \tilde{\phi}(z) + \phi'(x,y,z,t)$$

In this way we get $$\frac{Db}{Dt} = \frac{Db'}{Dt} + \frac{\partial \tilde{b}}{\partial z} w = \frac{Db'}{Dt} + N^2 w = 0$$

and $$\frac{\partial \phi'}{\partial z} = b'$$

now we are ready to obtain a non-dimensional eq. using

L, U, L/U, H and f_o as scaling factors, together with $Ro=U/(f_oL) << 1$

Considering that Φ’ and b’ (using the hydrostatic relation) scale as

$$\phi' \sim \phi \sim f_o UL$$

$$b' \sim B \sim f_o UL / H$$

$$\rightarrow \qquad \frac{\left(\partial b' / \partial z\right)}{N^2} \sim R_o \frac{L^2}{{L_d}^2}$$

where **L_d=NH/f_o** : this is the deformation radius for a continuously stratified fluid

If the scale of motion is ~ or < than the deformation radius and the Rossby number is small, than the variations in the stratification are small

The continuity eq. allows for scaling the vertical velocity as

$$w \sim W = UH / L$$

(using $\frac{1}{\tilde{\rho}} \frac{\partial \tilde{\rho} w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$)

and considering that the geostrophic balance insures also

$$\beta v \sim f \frac{\partial w}{\partial z} \qquad then \qquad w \sim W \sim \frac{\beta UH}{f_o}$$

finally we can write the primitive equations in non-dimensional form for a stratified geostrophic fluid in hydrostatic balance as

$$Hor.\quad mom.\quad eq \qquad R_o\left[\frac{\partial\hat{\vec{u}}}{\partial\hat{t}}+\hat{\vec{u}}\cdot\nabla\hat{\vec{u}}\right]+\hat{\vec{f}}\times\hat{\vec{u}}=-\nabla\hat{\phi}$$

$$Hydrostatic\quad eq \qquad \frac{\partial\hat{\phi}}{\partial\hat{z}}=\hat{b}$$

$$Mass\quad cont \qquad \left(\frac{\partial\hat{u}}{\partial\hat{x}}+\frac{\partial\hat{v}}{\partial\hat{y}}+\frac{1}{\tilde{\rho}}\frac{\partial\tilde{\rho}\hat{w}}{\partial\hat{z}}\right)=0$$

$$Thermo. \qquad R_o\frac{D\hat{b}}{D\hat{t}}+\left(\frac{L_d}{L}\right)^2\hat{w}=0$$

if $\tilde{\rho}=1$ the eqs. above reduce to the non-dimensional Boussinesq eqs

The planetary geostrophic eqs.

We simply use the fact that the Rossby number is small to derive a simpler set. This can be done using the SW and for stratified flows the stratified primitive eqs. written in non-dimensional form

<u>SW</u>:

momentum eq:

$$R_o\left[\frac{\partial\hat{\vec{u}}}{\partial\hat{t}}+\hat{\vec{u}}\cdot\nabla\hat{\vec{u}}\right]+\hat{\vec{f}}\times\hat{\vec{u}}=-\nabla\hat{\eta} \qquad \Rightarrow \qquad \hat{\vec{f}}\times\hat{\vec{u}}=-\nabla\hat{\eta}$$

Continuity eq:

$$R_o \frac{L^2}{L_d{}^2} \frac{D\hat{\eta}}{D\hat{t}} + \left[1 + R_o \frac{L^2}{L_d{}^2} \hat{\eta} \right] \nabla \cdot \hat{\vec{u}} = 0 \qquad \Rightarrow$$

if $(L/L_d)^2$ is such that $R_o\ (L/L_d)^2 = O(1)$ (i.e. $L \gg L_d$ or $Bu \ll 1$) we go back to the original SW continuity eq.

$$\frac{D\hat{\eta}}{D\hat{t}} + [1 + \hat{\eta}] \nabla \cdot \hat{\vec{u}} = 0 \qquad or \qquad \frac{Dh}{Dt} + h \nabla \cdot \vec{u} = 0$$

Those eqs. are valid ONLY for scales of motion >> than the Rossby deformation radius

Formally can be derived assuming

1) $R_o = U/f_oL << 1$
2) $Bu^{-1}=(L/L_d)^2 >> 1$
3) T=L/U (time scales as the advective term)

and expanding in Taylor' series using R_o both velocity and free surface displacement (non-dimensional)

$$\hat{u} = \hat{u}_o + R_o \hat{u}_1 + R_o{}^2 \hat{u}_2 + \ldots$$

$$\hat{\eta} = \hat{\eta}_o + R_o \hat{\eta}_1 + R_o{}^2 \hat{\eta}_2 + \ldots$$

at the lowest order we get, as expected,

$$\hat{\vec{f}} \times \hat{\vec{u}}_o = -\nabla \hat{\eta}_o$$

$$R_o \frac{L^2}{L_d{}^2}\left[\frac{\partial \hat{\eta}}{\partial \hat{t}} + \hat{\vec{u}} \cdot \nabla\right]\hat{\eta} + \left[1 + R_o \frac{L^2}{L_d{}^2}\hat{\eta}\right]\nabla \cdot \hat{\vec{u}} = 0$$

if the Coriolis parameter is constant, then from the momentum eq. follows that the divergence of $\mathbf{u}_o$ is zero. <u>The planetary geostrophic equations require L >> L_d and are useful only if the scale L is such that the Coriolis parameter varies (obvious in the atmosphere, not necessary in the ocean).</u>

In terms of potential vorticity, we can rewrite the SW equations as

$$\boxed{\frac{DQ}{Dt} = 0 \qquad where \qquad Q = f/h, \quad \vec{f} \times \vec{u} = -g\nabla\eta, \quad \eta = h + \eta_b}$$

Stratified Boussinesq eq. in hydrostatic approx:

momentum eq and mass conservations:

$$\hat{\vec{f}} \times \hat{\vec{u}}_o = -\nabla \hat{\phi}_o, \qquad \frac{\partial \hat{\phi}_o}{\partial \hat{z}} = \hat{b}_o, \qquad \nabla \cdot \hat{\vec{v}}_o = 0$$

assuming $L_d/L=O(1)$ the thermodynamic eq. becomes

$$\left(\frac{L_d}{L}\right)^2 \hat{w}_o = 0$$

not very useful.. all the time derivatives are gone. We must consider only scales such that $L^2 >> L_d^2$! and retain all the terms in the thermodynamic eq

$$R_o \frac{D\hat{b}}{D\hat{t}} + \left(\frac{L_d}{L}\right)^2 \hat{w} = 0$$

In dimensional form we have

$$\boxed{\begin{aligned}&\vec{f}\times\vec{u}=-\nabla\phi', \qquad \frac{\partial\phi'}{\partial z}=b', \qquad \nabla\cdot\vec{v}=0\\ &\frac{Db'}{Dt}+wN^2=0 \qquad (or \qquad S[b'] \quad if \quad there\\ &is \quad a \quad source)\end{aligned}}$$

and in terms of PV

$$\boxed{\begin{aligned}&\frac{DQ}{Dt}=\dot{Q}\\ &Q=f\frac{\partial b}{\partial z}\end{aligned}}$$

Applicability:

in the atmosphere not really: the deformation radius NH/f is ~ 1000km

Requiring that L >> 1000km does not leave too much space

in the ocean yes: NH/f ~ 100km: this approx is very useful. Most of the theory of the large scale circulation makes use of them

Shallow water quasigeostrophic equations

Assumptions:

1. R_o is small
2. The scale of motion is not significantly larger than L_d and
$$R_o \frac{L^2}{L_d^{\ 2}} = O(R_o)$$
3. Variations in the Coriolis parameter are small, i.e. $|\beta L| << |f_o|$
4. Time scales advectively as L/U

Assumption 2 implies that the variations in fluid depth are small compared to the total depth for a SW system and that variations in stratification are small compared to the background stratification for a continuously stratified fluid

Single layer sw qg

The formal way to derive the QG equations is again to expand the variables in an asymptotic series in R_o

$$\hat{u} = \hat{u}_o + R_o \hat{u}_1 + R_o{}^2 \hat{u}_2 + \ldots$$

$$\hat{v} = \hat{v}_o + R_o \hat{v}_1 + R_o{}^2 \hat{v}_2 + \ldots$$

$$\hat{\eta} = \hat{\eta}_o + R_o \hat{\eta}_1 + R_o{}^2 \hat{\eta}_2 + \ldots$$

$$\hat{f} = f / f_o = \hat{f}_o + R_o \hat{\beta} \hat{y} = 1 + R_o \hat{\beta} \hat{y}$$

($\hat{f}_o$=1 being the non-dimensional value of f) we insert this in the eq. of motion and equate the various powers of R_o obtaining from

$$R_o\left[\frac{\partial\hat{\vec{u}}}{\partial\hat{t}}+\hat{\vec{u}}\cdot\nabla\hat{\vec{u}}\right]+\hat{\vec{f}}\times\hat{\vec{u}}=-\nabla\hat{\eta}$$

at the lowest order (zero order in R_o)

$$\boxed{\hat{f}_o\hat{u}_o=-\frac{\partial\hat{\eta}_o}{\partial\hat{y}};\qquad \hat{f}_o\hat{v}_o=\frac{\partial\hat{\eta}_o}{\partial\hat{x}}}$$

by cross-differentiating we obtain, from the momentum equation, that the horizontal velocity field is divergence free

$$\boxed{\nabla\cdot\hat{\vec{u}}_o=0}$$

the continuity eq. at the lowest order gives us again $\boxed{\nabla \cdot \hat{\vec{u}}_o = 0}$

The next order (the one for R_o) leads to

$$Bu^{-1}\frac{\partial \hat{\eta}_o}{\partial \hat{t}} + Bu^{-1}\hat{\vec{u}}_o \cdot \nabla \hat{\eta}_o + \nabla \cdot \hat{\vec{u}}_1 = 0$$

which is not closed because we need the divergence of the first-order velocity term to describe the evolution of a zero-order term. We need the next order in the momentum equation as well

$$\frac{\partial \hat{\vec{u}}_o}{\partial \hat{t}} + \hat{\vec{u}}_o \cdot \nabla \hat{\vec{u}}_o + \hat{\beta}\hat{y}k \times \hat{\vec{u}}_o - \hat{f}_o k \times \hat{\vec{u}}_1 = -\nabla \hat{\eta}_1$$

and taking the curl we get

$$\frac{\partial \hat{\zeta}_o}{\partial \hat{t}} + \left(\hat{\vec{u}}_o \cdot \nabla\right)\left(\hat{\zeta}_o + \hat{\beta}\hat{y}\right) = -\hat{f}_o \nabla \cdot \hat{\vec{u}}_1$$

The right-hand side represents vortex stretching by the planetary vorticity. The eq. is not closed either. But if we eliminate the divergence of the velocity at the first order using the first order in R_o continuity eq. we are all set obtaining

$$\frac{\partial}{\partial \hat{t}}\left(\hat{\zeta}_o - \hat{f}_o Bu^{-1}\hat{\eta}_o\right) + \left(\hat{\vec{u}}_o \cdot \nabla\right)\left(\hat{\zeta}_o + \hat{\beta}\hat{y} - Bu^{-1}\hat{\eta}_o\right) = 0$$

Finally, from the lowest order for the momentum eq. we have that the velocity can be written as

$$\hat{u}_o = -\frac{\partial \hat{\psi}_o}{\partial \hat{y}}; \qquad \hat{v}_o = \frac{\partial \hat{\psi}_o}{\partial \hat{x}} \qquad with \qquad \hat{\zeta}_o = \nabla^2 \hat{\psi}_o$$

where $\hat{\psi}_o = \hat{\eta}_o / \hat{f}_o$

Then $$\frac{\partial}{\partial \hat{t}}\left(\hat{\zeta}_o - \hat{f}_o Bu^{-1}\hat{\eta}_o\right) + \left(\hat{\vec{u}}_o \cdot \nabla\right)\left(\hat{\zeta}_o + \hat{\beta}\hat{y} - Bu^{-1}\hat{\eta}_o\right) = 0$$

can be re-written as

$$\frac{\partial}{\partial \hat{t}}\left(\nabla^2\hat{\psi}_o - Bu^{-1}\hat{f}_o^{\,2}\hat{\psi}_o\right) + \left(\hat{\vec{u}}_o \cdot \nabla\right)\left(\nabla^2\hat{\psi}_o + \hat{\beta}\hat{y} - Bu^{-1}\hat{f}_o^{\,2}\hat{\psi}_o\right) = 0$$

or

$$\frac{D_o}{D\hat{t}}\left(\nabla^2\hat{\psi}_o + \hat{\beta}\hat{y} - Bu^{-1}\hat{f}_o^{\,2}\hat{\psi}_o\right) = 0$$

or restoring dimensions

$$\frac{D}{D\hat{t}}\left(\nabla^2\psi + \beta y - \frac{1}{L_d^{\,2}}\psi\right) = 0$$

here $\psi = \left(\frac{g}{f_o}\right)\eta,$

$$L_d^{\,2} = \frac{gH}{f_o^{\,2}}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = \frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial t} + J(\psi, \quad)$$

and we can define the shallow water QG PV as

$$\boxed{q \equiv \varsigma + \beta y - \frac{f_o}{H}\eta = \nabla^2\psi + \beta y - \frac{1}{L_d^{\,2}}\psi}$$

Obviously the SW QG PV is an approximation to the SW PV given by

$$Q = \frac{f + \varsigma}{h}$$

(see relation between the 2 on the book)

Two limits are dynamically interesting (let's take β=0 just to simplify the formulas, but it's not required):

1) L << L_d, i.e. Bu >> 1then:

$$\frac{\partial \varsigma}{\partial t} + J(\psi, \varsigma) = 0 \qquad where \qquad \varsigma = \nabla^2 \psi$$

we obtained the two-dimensional vorticity eq. <u>On small scales the deviations in height are negligible!</u>

2) L >> L_d (but in the limit of the QG approximation)

$$\frac{\partial \psi}{\partial t} + J(\psi, \psi) = 0 \qquad or \qquad \frac{\partial \eta}{\partial t} + J(\psi, \eta)$$

given that the Jacobian is identically zero, this means that there is no evolution of the stream function or of the height field. In this case the QG equations are useless, as we could expect given the hypothesis we started from.

Before we reach such a condition, the SQ QG eqs. loose validity if the variations in height are not small because we assumed $\eta / H \sim R_o (L / L_d)^2$

2-layer QG system

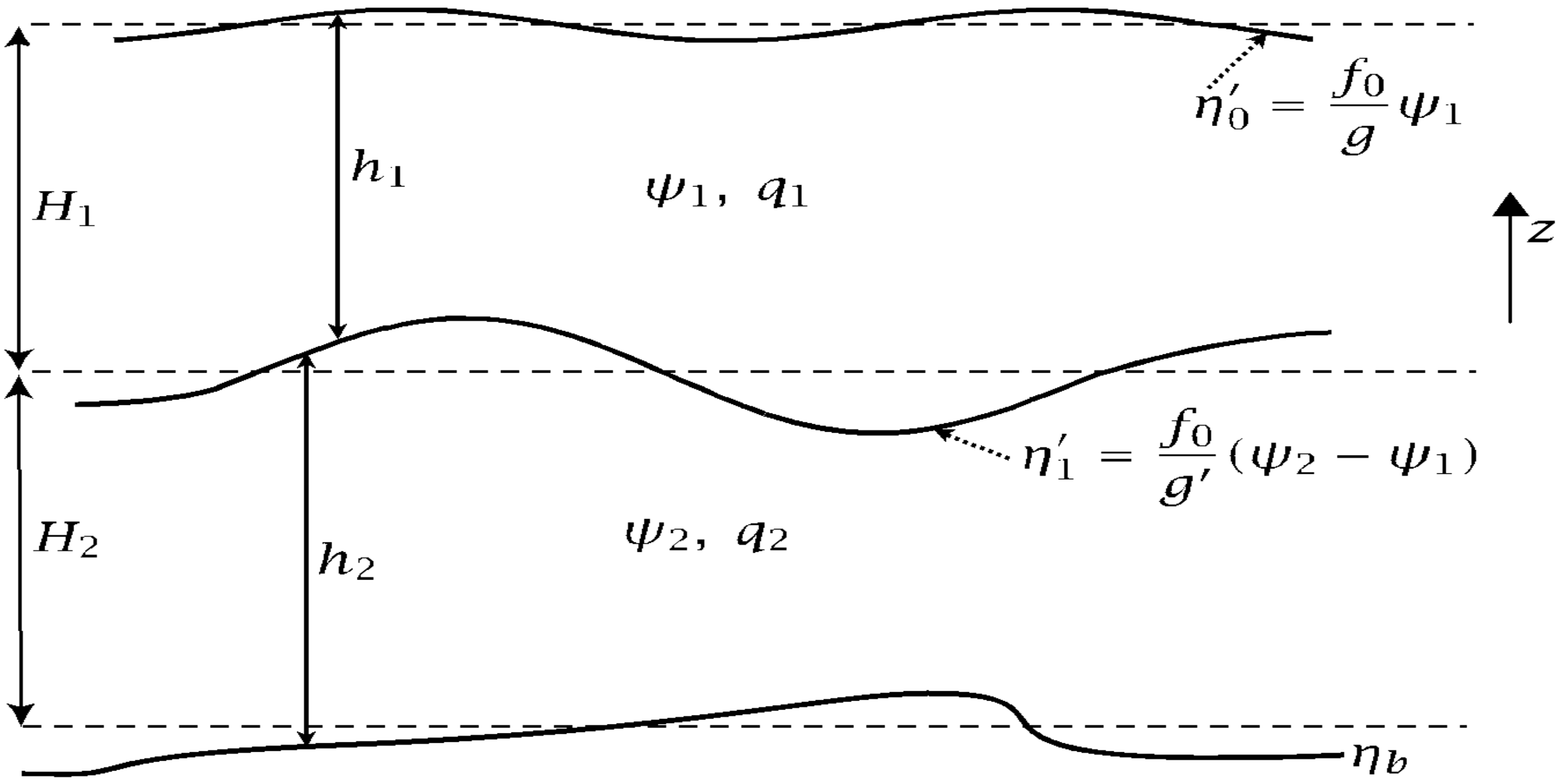


Fig. 5.1 A quasi-geostrophic fluid system consisting of two immiscible fluids of different density. The quantities η' are the interface displacemens from the resting basic state, denoted with dashed lines, with η_b being the bottom topography.

$$h_i' = h_i - H_i$$

3 conditions have to be satisfied:

1. the variations in the layer thickness are small
2. the Rossby number is small
3. the variations in the Coriolis parameter are small

This allows us to define a QG pv in each layer as

$$\boxed{q_i = \left(\varsigma_i + \beta y - f_o \frac{h_i}{H_i} \right)}$$

PV must be conserved and to obtain a close set of eq. we must use the geostrophic balance condition we obtain at the end a set of equations for the evolution of the layer thickness anomalies and an expression of the vorticity in terms stream function in each layer as

$$h'_1 = \frac{f_o}{g'}\left(\psi_1 - \psi_2\right) + \frac{f_o}{g}\psi_1, \qquad h'_2 = \frac{f_o}{g'}\left(\psi_2 - \psi_1\right) - \eta_b$$

$$q_1 = \beta y + \nabla^2\psi_1 + \frac{f_o^2}{g' H_1}\left(\psi_2 - \psi_1\right) + \frac{f_o^2}{g H_1}\psi_1$$

$$q_2 = \beta y + \nabla^2\psi_2 + \frac{f_o^2}{g' H_2}\left(\psi_1 - \psi_2\right) + f_o \frac{\eta_b}{H_2}$$

and $$\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = 0 \qquad i = 1, 2$$

multi layer

Same thing, just using for q_i the expression

$$q_i = \beta y + \nabla^2 \psi_i + \frac{f_o^2}{H_i}\left(\frac{\psi_{i-1} - \psi_i}{g'_{i-1}} - \frac{\psi_i - \psi_{i+i}}{g'_i}\right)$$

$$q_1 = \beta y + \nabla^2 \psi_1 + \frac{f_o^2}{H_1}\left(\frac{\psi_2 - \psi_1}{g'_1}\right) + \frac{f_o^2}{gH_1}\psi_1$$

$$q_N = \beta y + \nabla^2 \psi_N + \frac{f_o^2}{H_1}\left(\frac{\psi_{N-1} - \psi_N}{g'_{N-1}}\right) + \frac{f_o^2}{H_N}\eta_b$$

Transport and mixing in the Ocean (in lieu of continuously stratified QG system)

Annalisa Bracco

with

Jost von Hardenberg (CIMA)

Jim McWilliams (UCLA)

Jeff Weiss (CU)

Horizontal advection and spatially variable fluxes

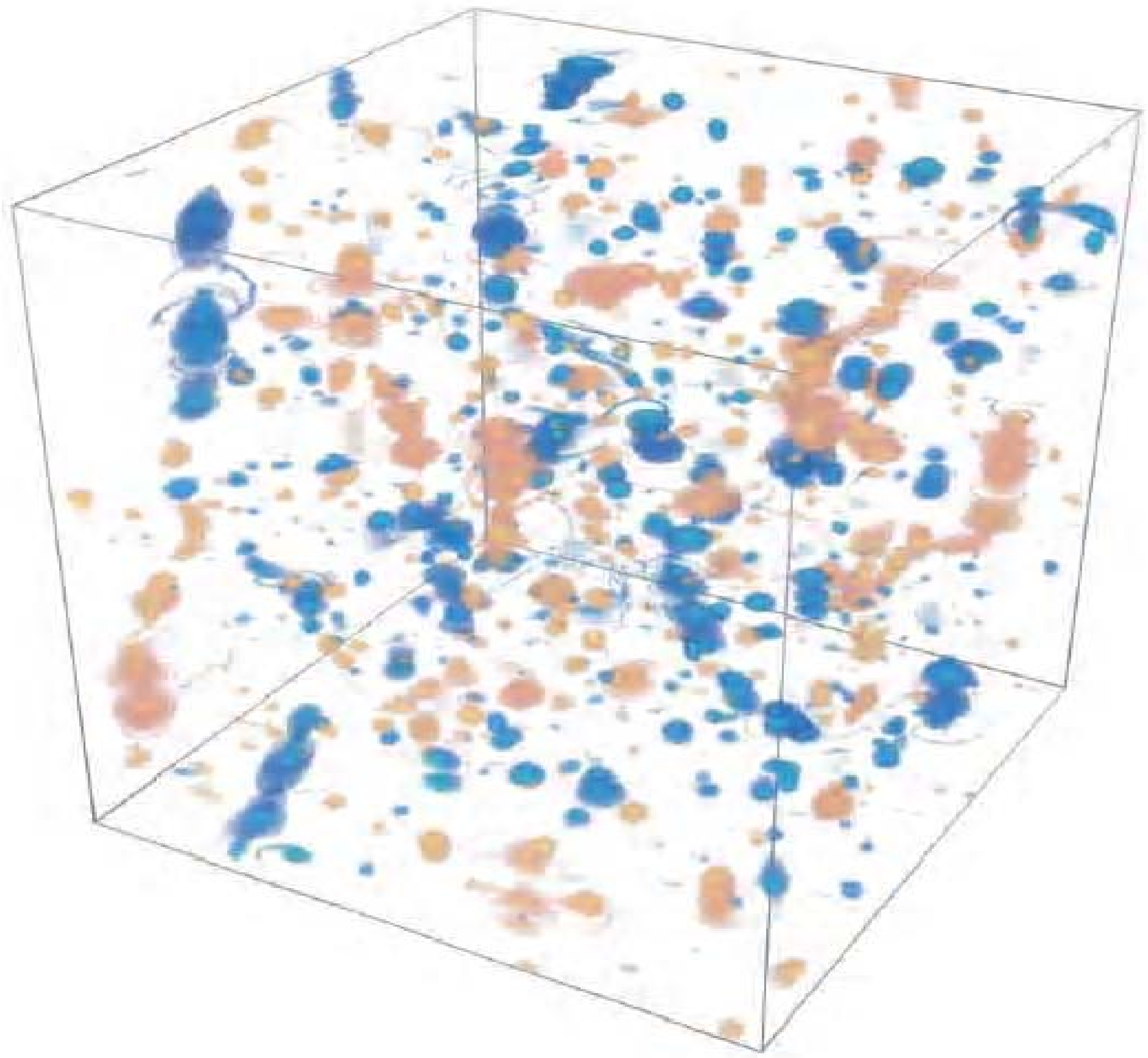
Looking for a simple, numerically fast, 'sounded' model for horizontal advection in the open ocean...

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + J[\psi, q]$$

Is QG turbulence a good candidate?

$$q = q_{3D} = \nabla^2_{2D}\psi + \frac{\partial}{\partial z} S(z) \frac{\partial \psi}{\partial z}$$

$$S(z) = \frac{\rho}{N^2}$$



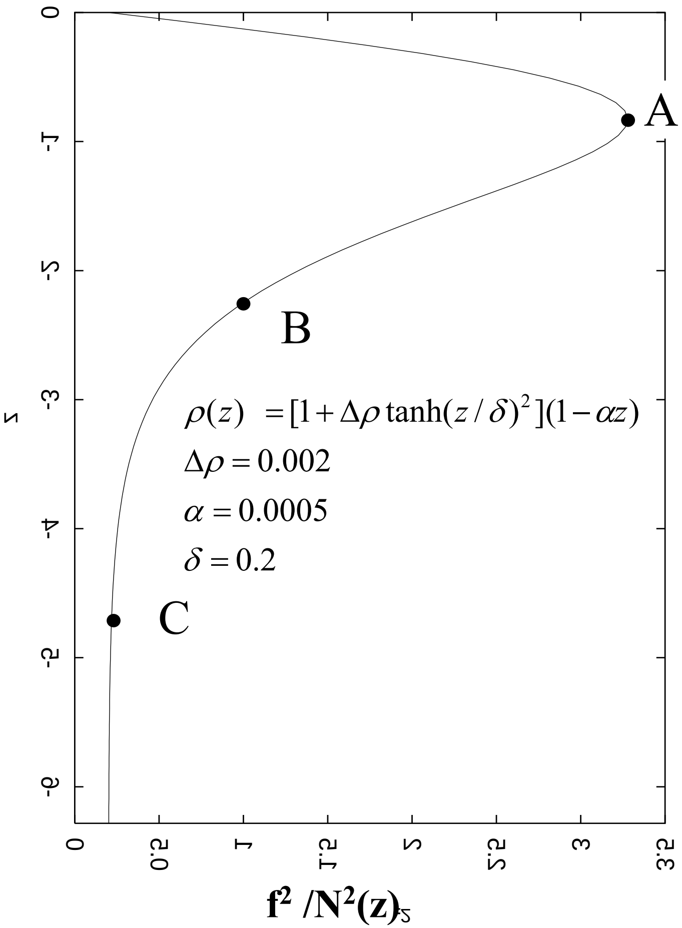
QG approx does not include internal gravity waves, tides, convection

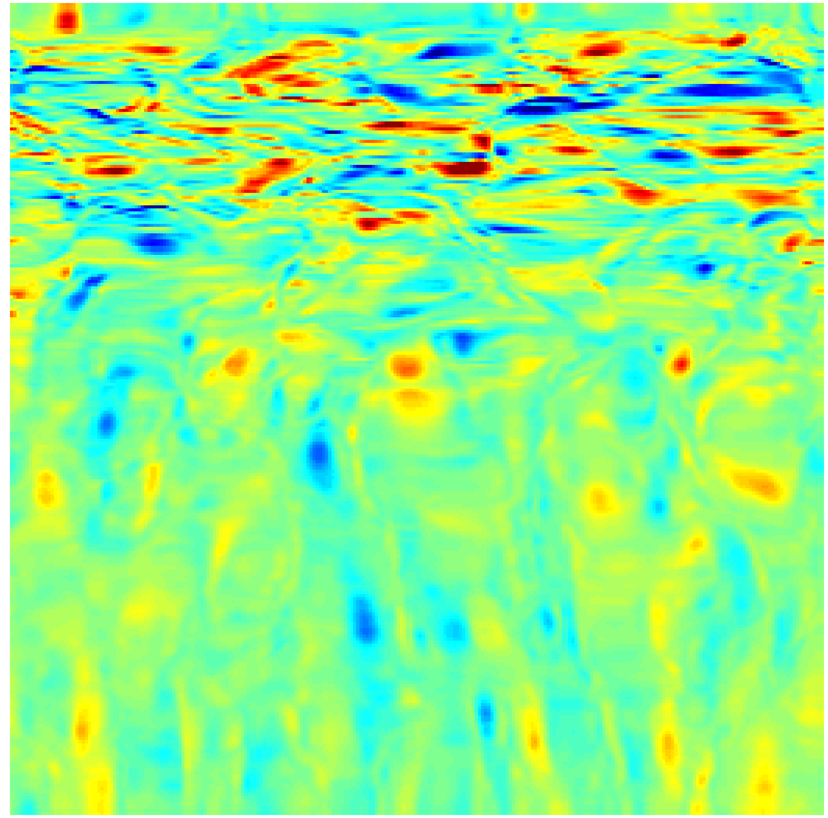
It does not work for currents over step topography

However...

profile as in Smith and Vallis, JPO, 2001

Relative vorticity yz





vertical res.: 256 grid points

horizontal res.: 256^2 ~ 4 Km 512^2 ~ 2 Km

From Holbrook et al., Science 2003:
A seismic reflection profiling section

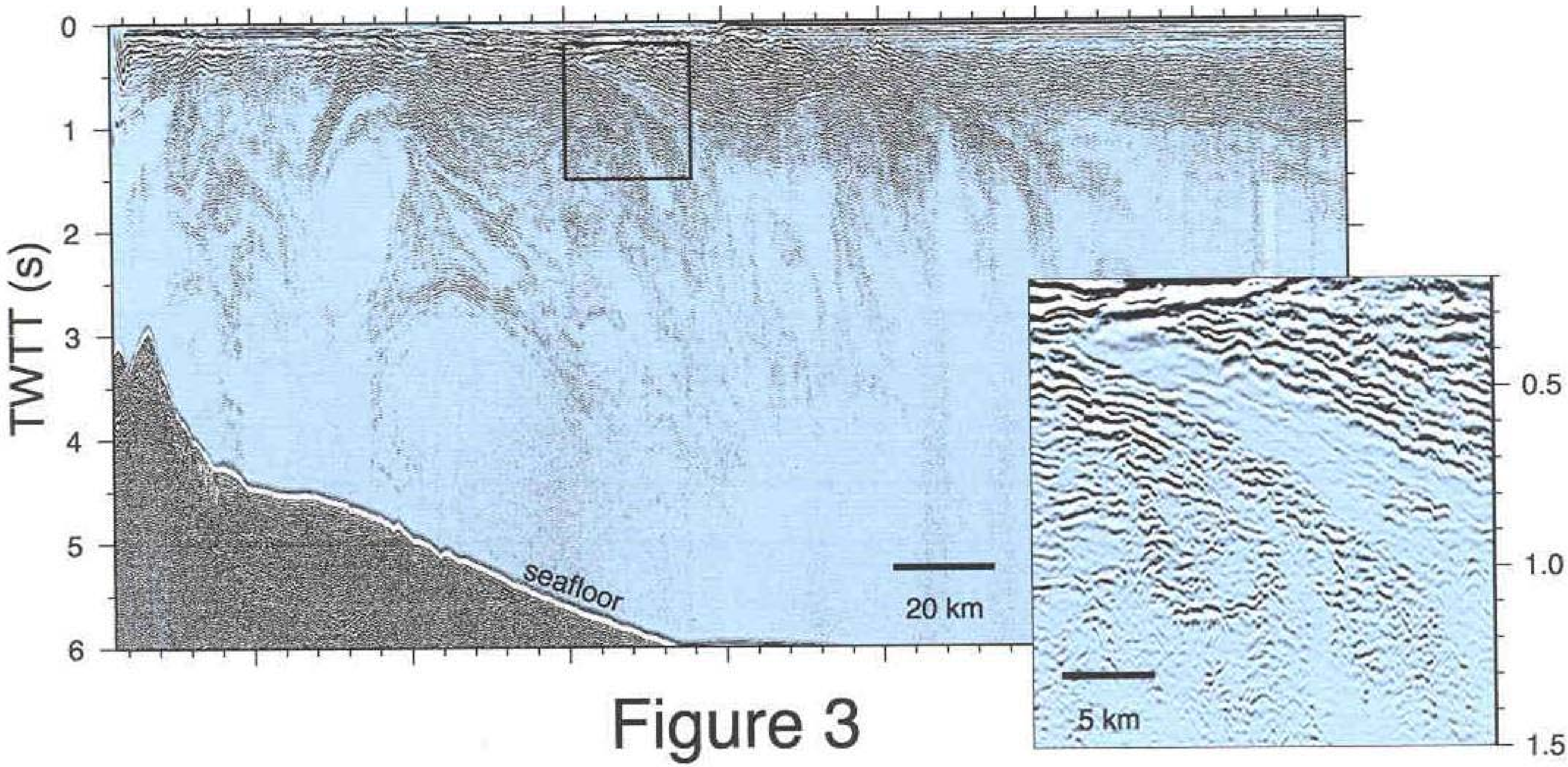
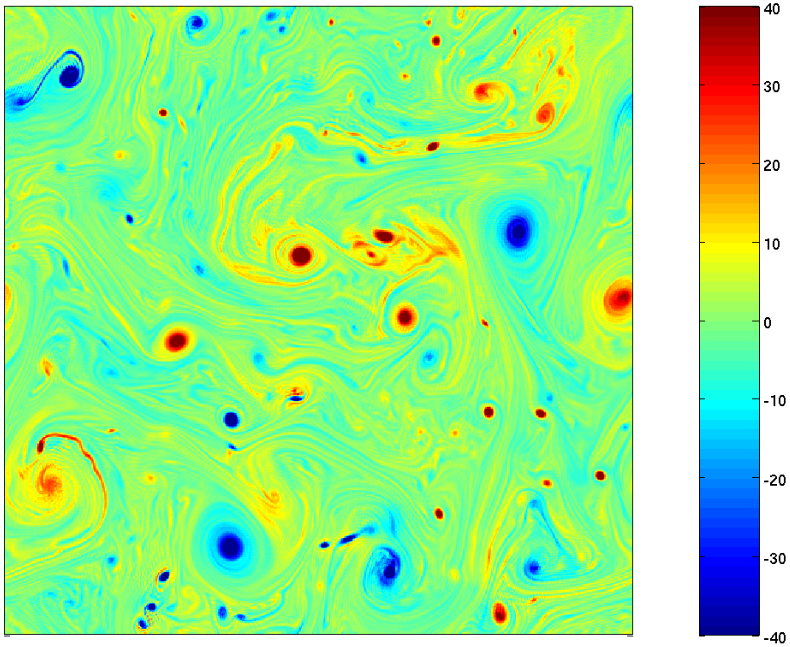
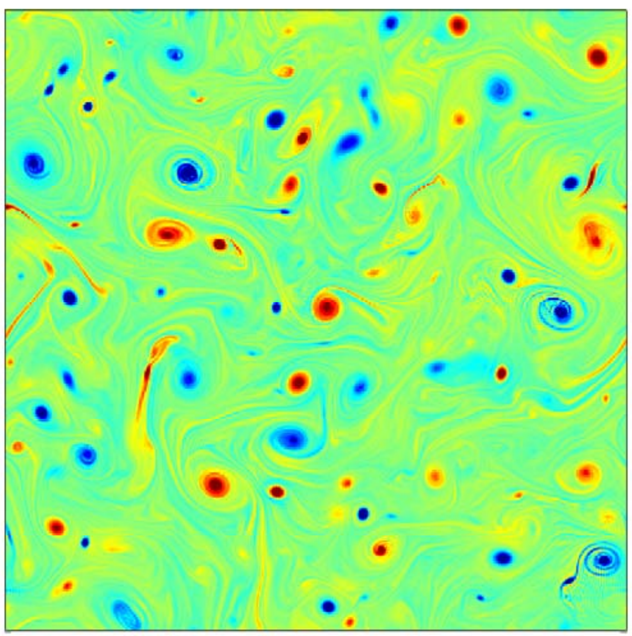


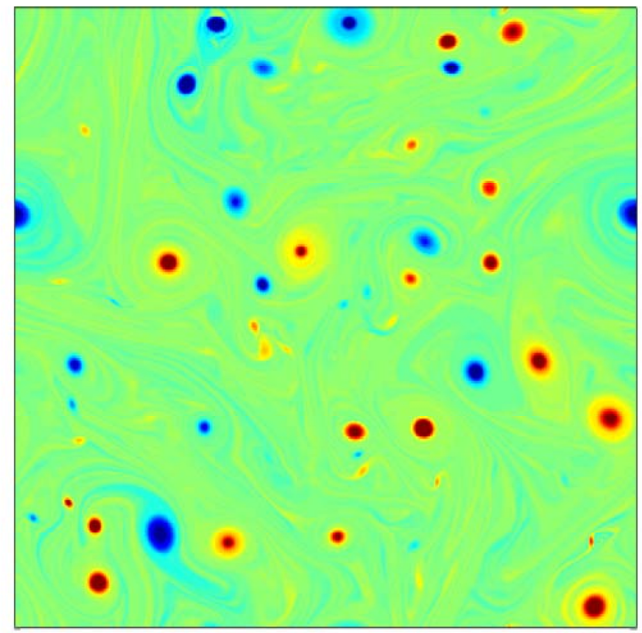
Figure 3



Level A



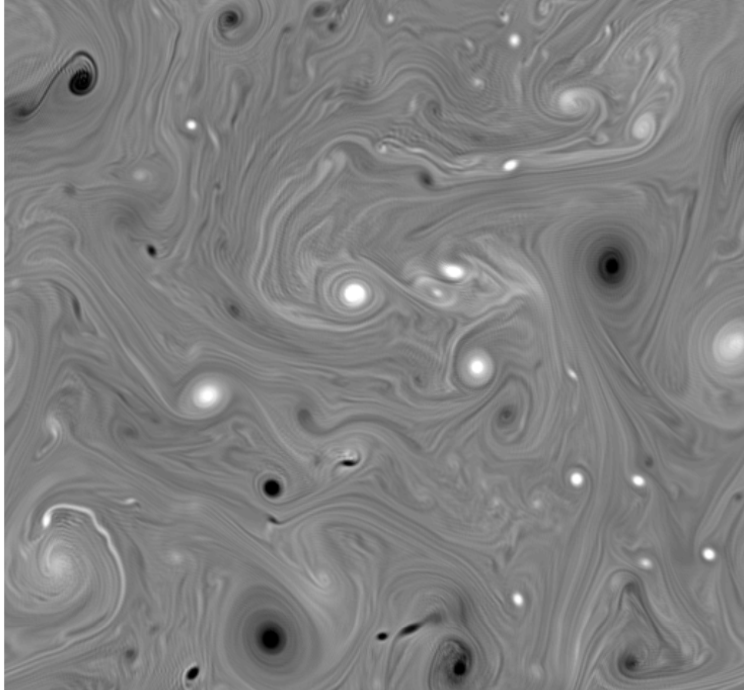
Level C

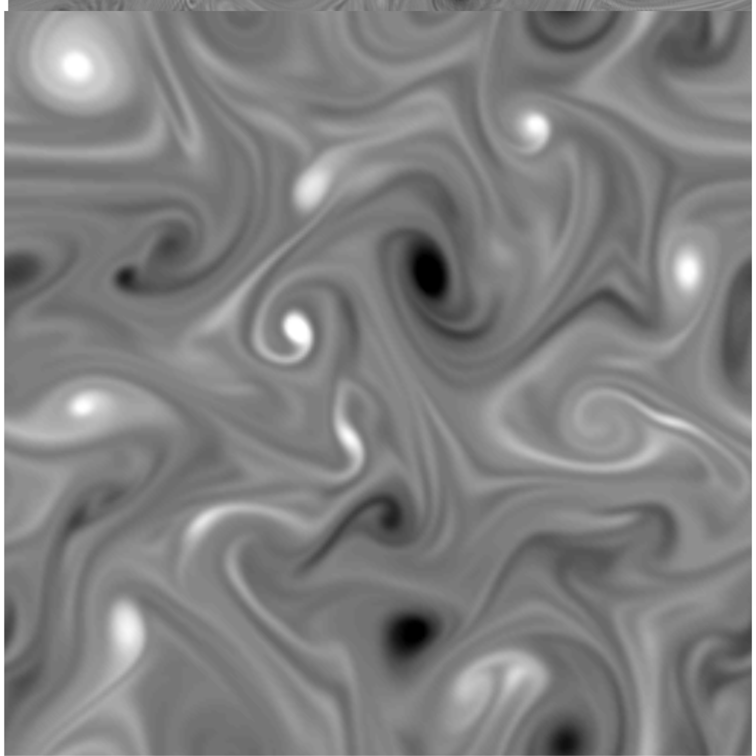


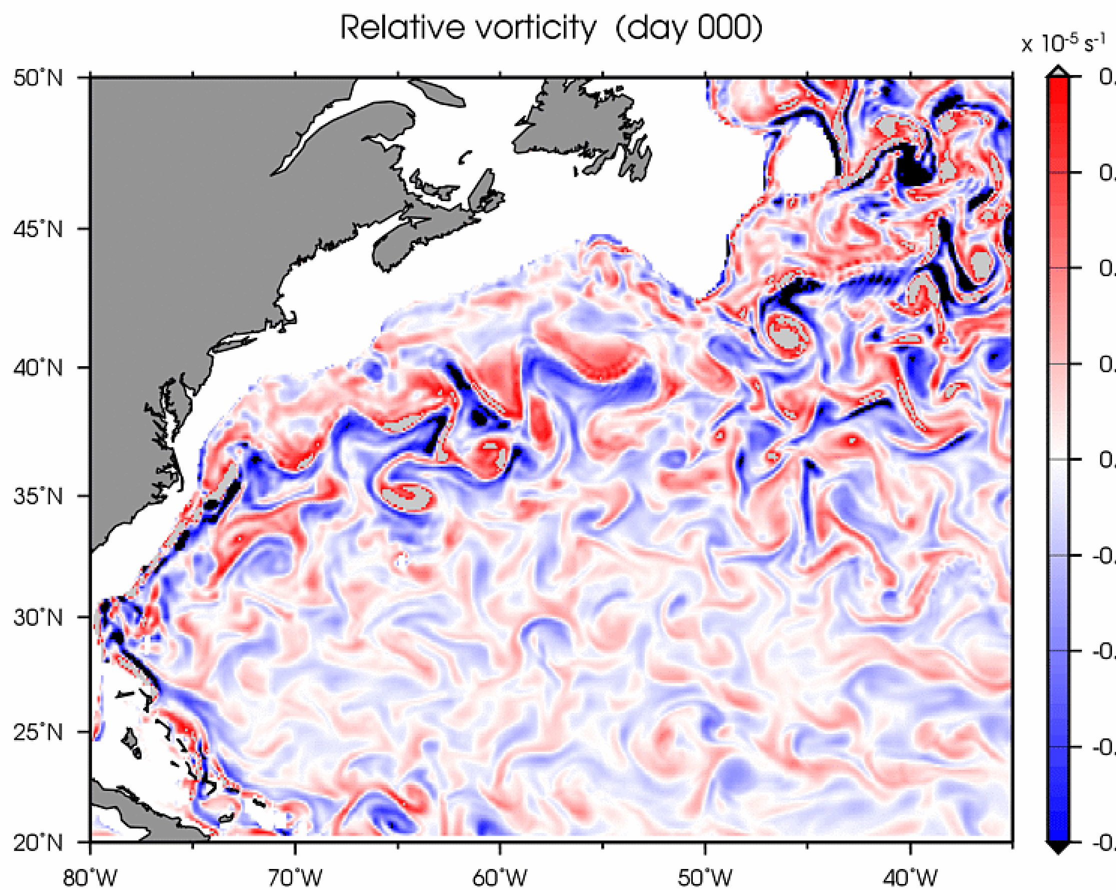
2D barotropic field

1 grid point ~ 2km

increasing in EKE 36%

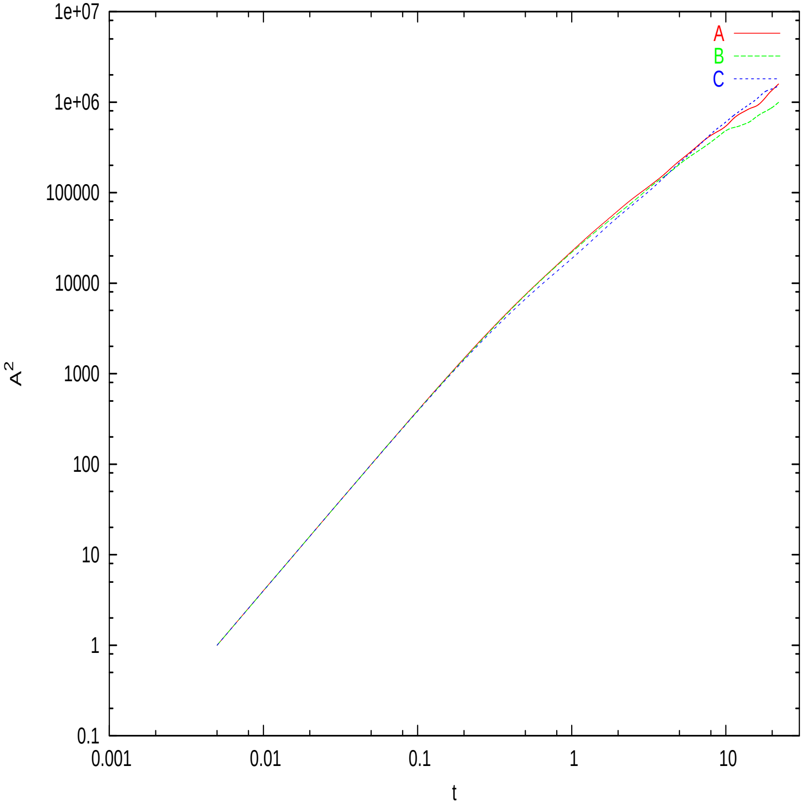


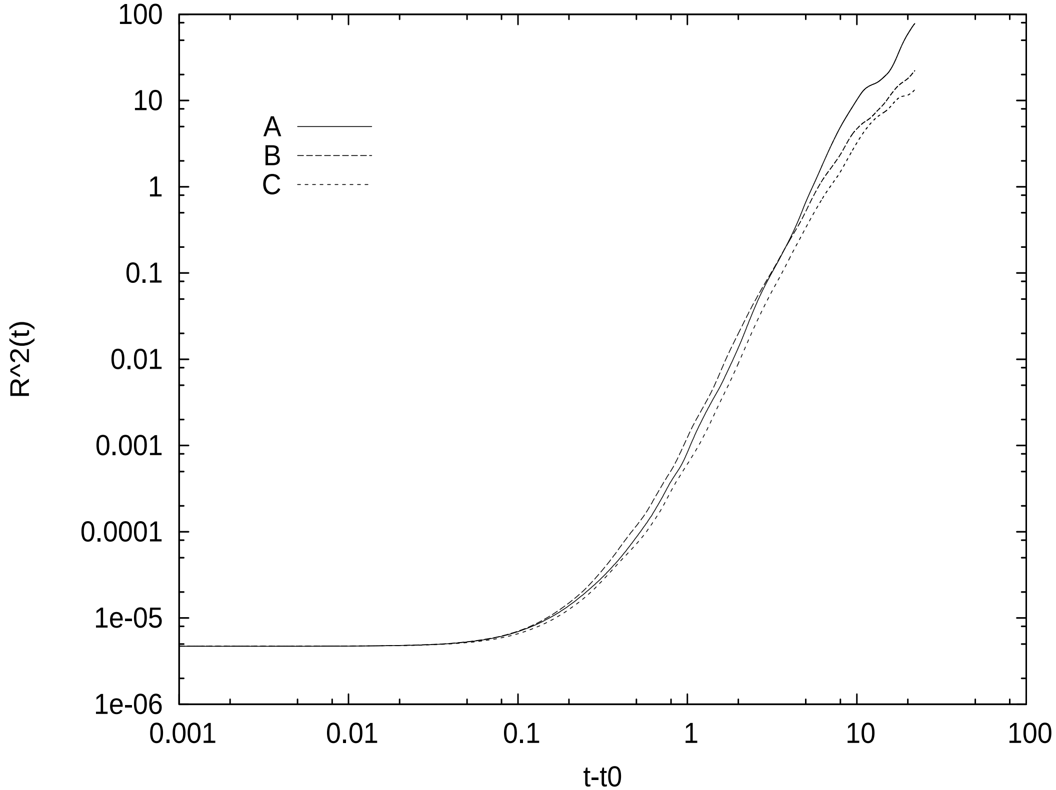




from Veneziani, Griffa, Garraffo, Chassignet, JMR, 2004 in press

Transport and mixing properties





$$A^2(t,t_0) = \left\langle \left| \vec{x}(t) - \vec{x}(t_0) \right|^2 \right\rangle, \qquad R^2(t) = \left\langle \left| (\vec{x}_1(t) - \vec{x}_2(t)) - (\vec{x}_1(t_0) - \vec{x}_2(t_0)) \right|^2 \right\rangle$$

- Lagrangian dynamics at the ocean mesoscale is governed by the **coherent vortices**, as for 2D barotropic turbulence

- Both **three-dimensionality** and **non-uniform stratification** do not play a significant role on the transport and mixing properties

- **Barotropic turbulent models** can be used as simple tools for representing mesoscale turbulent dynamics in the open ocean for scales **from few days to several weeks**

week 5-6 (repetita iuvant)

- Geostrophic balance (again) and the scaling for the atmosphere and ocean
- The planetary geostrophic equations
- The quasigeostrophic equations
- The energetic of the QG systems
- Rossby waves in one and two layers
- Phase speed and group velocity

Energetics in QG

A system of eqs. has to guarantee basic conservations. Energy in QG is conserved and this can be shown from for a continuously stratified flow of depth z =[0,1].

Using the PV equation for the interior we have

$$\frac{D}{Dt}\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right] + \beta\frac{\partial \psi}{\partial x} = 0 \qquad 0 < z < 1$$

and at the boundary $$\frac{D}{Dt}\left(\frac{\partial \psi}{\partial z}\right) = 0 \qquad z = 0,1$$

We also must assume that the stream function is periodic or constant at the lateral boundaries.

Multiplying by –Ψ and integrating over the domain, it is found that

$$E = \frac{1}{2}\int_V \left[\left(\nabla \psi\right)^2 + \frac{f_o{}^2}{N^2}\left(\frac{\partial \psi}{\partial z}\right)^2 \right] dV$$

$$\frac{dE}{dt} = 0$$

In the absence of viscous terms the energy in QG is conserved and is given by two terms, kinetic and available potential.

The APE can be re-written as

The β-term does not contribute to the energy budget

$$APE = \frac{1}{2}\int_V \frac{H}{L_d{}^2}\left(\frac{\partial \psi}{\partial z}\right)^2 dV, \qquad L_d = NH / f_o$$

Because the KE depends only on the gradient of the 2D stream-function squared, its scale is ~ $1/L^2$, where L is the horizontal scale of motion.

Therefore

$$\frac{KE}{APE} \sim \frac{{L_d}^2}{L^2}$$

If $L >> L_d$, APE dominates, and viceversa

For a two-layer flow, the energy conservation can be obtained from the eq. of motion (in the case of layer of equal thickness):

$$\frac{D}{Dt}\left[\nabla^2\psi_1 - \frac{1}{2}k_d{}^2\left(\psi_1 - \psi_2\right)\right] + \beta\frac{\partial\psi_1}{\partial x} = 0$$

$$\frac{D}{Dt}\left[\nabla^2\psi_2 - \frac{1}{2}k_d{}^2\left(\psi_2 - \psi_1\right)\right] + \beta\frac{\partial\psi_2}{\partial x} = 0$$

where $k_d^2/2=(2f_o/NH)^2$

Multiply the first by $-\Psi_1$ and the second one by $-\Psi_2$, integrate over the horizontal domain and (given that the advective and β terms vanish) obtain

$$\frac{d}{dt}\int_A \left[\frac{1}{2}(\nabla \psi_1)^2 + \frac{1}{2}k_d{}^2\psi_1(\psi_1 - \psi_2) \right] dA = 0$$

$$\frac{d}{dt}\int_A \left[\frac{1}{2}(\nabla \psi_2)^2 - \frac{1}{2}k_d{}^2\psi_2(\psi_1 - \psi_2) \right] dA = 0$$

Added together

$$\boxed{\frac{d}{dt}\int_A \left[\frac{1}{2}(\nabla \psi_1)^2 + \frac{1}{2}(\nabla \psi_2)^2 + \frac{1}{2}k_d{}^2(\psi_1 - \psi_2)^2 \right] dA = 0}$$

A useful way to write is in terms of barotropic and baroclinic streamfunction. The barotropic one is the vertically averaged streamfunction, and the baroclinic is the difference between the streamfunctions in the two layers. For 2 layers of equal thickness:

$$\boxed{\psi_{barot} = \bar{\psi} = \frac{1}{2}(\psi_1 + \psi_2), \qquad \psi_{baroc} = \tau = \frac{1}{2}(\psi_1 - \psi_2)}$$

and for the energy we obtain

$$\boxed{\frac{d}{dt}\int_A \left[(\nabla \bar{\psi})^2 + (\nabla \tau)^2 + k_d{}^2 \tau^2 \right] dA = 0}$$

the energy density in the barotropic mode is given by the first term, $(\nabla \bar{\psi})^2$,and in the baroclinic one by $(\nabla \tau)^2 + k_d{}^2 \tau^2$

ENSTROPHY:

in 2d PV is advected only horizontally, and thus is materially conserved on the horizontal surfaces at every height (or level or layer)

Therefore

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \vec{u} \cdot \nabla q = 0$$

Additionally the flow is divergence-free, which implies that $\vec{u} \cdot \nabla q = \nabla \cdot \left(\vec{u} q\right)$. If we multiply by q and integrate over a horizontal domain using periodic or no-normal flow boundary conditions we get

$$\frac{dZ}{dt} = 0, \qquad Z = \frac{1}{2} \int_A q^2 dA$$

Z is called enstrophy and it is conserved for each height (level or layer) as well. Obviously, being q materially conserved, any function of it is conserved as well, and enstrophy simply corresponds to $F(q)=q^2$.

Z is particularly important because is a quadratic invariant, like energy.

Rossby waves

Waves motion in qg → <u>Rossby waves</u>

the vorticity of a parcel has to be conserved. On the β-plane such a vorticity is simply $\zeta + \beta y$

If a parcel is displaced to the north (in BOTH hemispheres), the change in PV must be compensated by a negative change in relative vorticity and viceversa. But relative vorticity in a QG flow is directly linked to velocity, which further displaces the parcel, leading to the formation of a wave.

Schematic – from Vallis' book

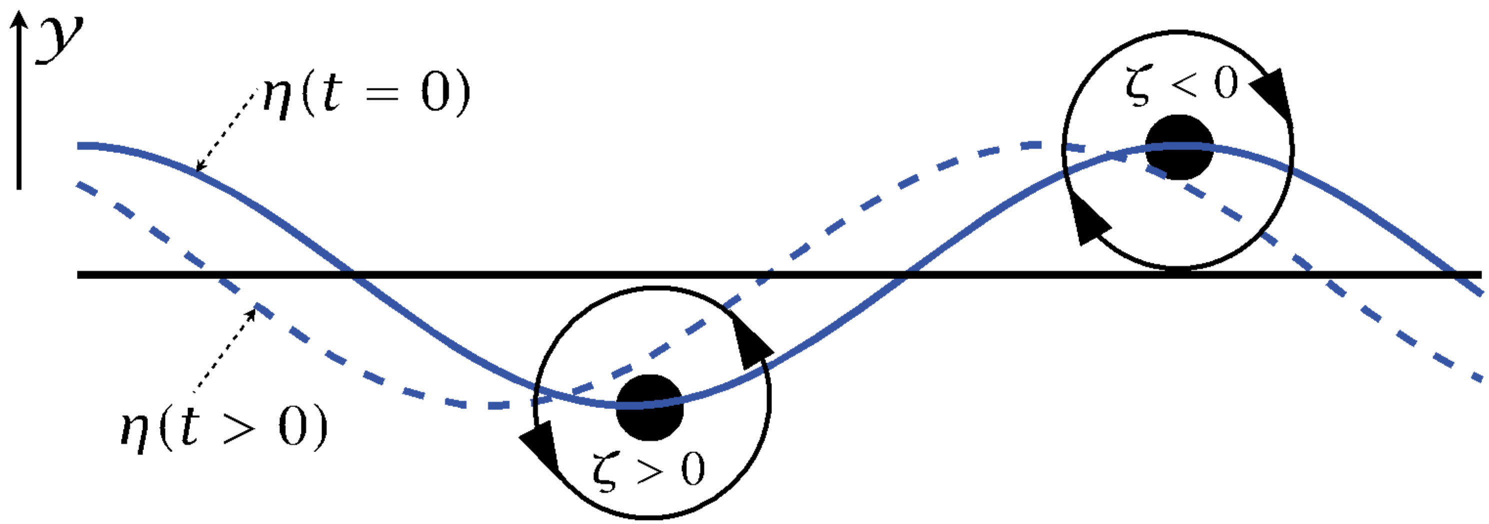


Fig. 5.4 The mechanism of a two-dimensional (x-y) Rossby wave. An initial disturbance displaces a material line at constant latitude (the straight horizontal line) to the solid line marked $\eta(t = 0)$. Conservation of potential vorticity, $\beta y + \zeta$, leads to the production of relative vorticity, as shown for two parcels. The associated velocity field (arrows on the circles) then advects the fluid parcels, and the material line evolves into the dashed line. The phase of the wave has propagated westwards.

For a single layer the vorticity eq without forcing term is simply (for vorticity)

$$\frac{D}{Dt}\left(\varsigma + f - \frac{\psi}{L_d{}^2}\right) = 0, \qquad \varsigma = \nabla^2\psi$$

Two limits to be considered: the scale of motion L is << than L_d or is comparable

- L_d is >> L:

The eq. of motion reduces to $\frac{D}{Dt}(\varsigma + \beta y) = 0, \qquad \varsigma = \nabla^2 \psi$

and expanding it $\frac{\partial \varsigma}{\partial t} + J(\psi, \varsigma) + \beta \frac{\partial \psi}{\partial x} = 0$

as usual we are interested in the linearized eq. Additionally we make the assumption that the flow contains a time independent component (mean state) + a perturbation,

$\psi = \Psi + \psi'(x, y, t)$ where $\Psi = -Uy$ and $|\psi'| << \Psi$

where U is the zonal flow of the mean state

Neglecting the nonlinear terms involving ψ', we can re-write the vorticity eq. as:

$$\frac{\partial \varsigma'}{\partial t} + J(\Psi, \varsigma') + \beta \frac{\partial \psi'}{\partial x} = 0 \qquad i.e.$$

$$\frac{\partial}{\partial t}\nabla^2 \psi' + U \frac{\partial \nabla^2 \psi'}{\partial x} + \beta \frac{\partial \psi'}{\partial x} = 0$$

solutions will be again in the form of a plane wave

$$\psi' = \mathrm{Re}\, \tilde{\psi} e^{i(kx+ly-\omega t)}$$

$$\tilde{\psi} = amplitude$$

$$k, l = x, y \qquad wavenumbers$$

$$\omega = frequency$$

Substituting in the linearized vorticity eq:

$$\left[\left(-\omega+Uk\right)K^2+\beta k\right]\tilde{\psi}=0, \qquad K^2=k^2+l^2$$

$$\Downarrow$$

$$\omega=Uk-\frac{\beta k}{K^2}=dispersion \quad relation$$

Phase speed (in -x direction) $$c_p{}^x\equiv\frac{\omega}{k}=U-\frac{\beta}{K^2}$$

Group velocity (in -x direction) $$c_g{}^x\equiv\frac{\partial\omega}{\partial k}=U+\frac{\beta\left(k^2-l^2\right)}{\left(k^2+l^2\right)^2}$$

Doppler shift. The phase for no mean flow is *westward*, with long wavelengths traveling faster

- L_d is ~ (or <) L:

All the same, but we must keep the $-\frac{\Psi}{{L_d}^2} = +\frac{Uy}{{L_d}^2}$ term and the derivative of the perturbation component.

The linearized vorticity equation becomes

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\nabla^2\psi' - \frac{\psi'}{{L_d}^2}\right) + \left(\beta + \frac{U}{{L_d}^2}\right)\frac{\partial \psi'}{\partial x} = 0$$

substituting the usual $\psi' = \mathrm{Re}\,\tilde{\psi} e^{i(kx+ly-\omega t)}$

the dispersion relation becomes

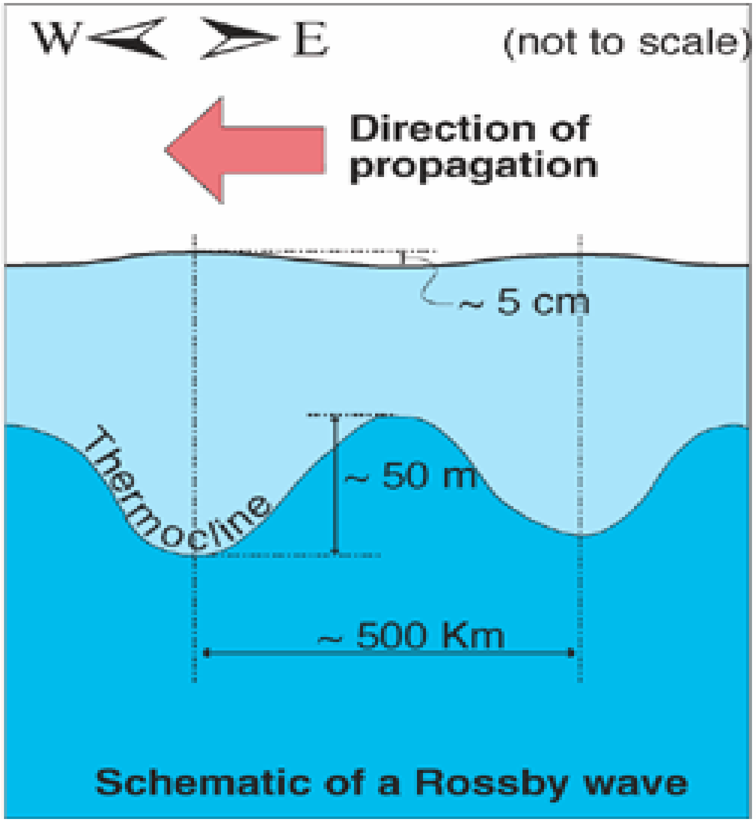
$$\omega = \frac{k\left(UK^2 - \beta\right)}{K^2 + 1/{L_d}^2} = Uk - k\frac{\beta + U/{L_d}^2}{K^2 + 1/{L_d}^2}$$

x-components of phase speed and group velocity:

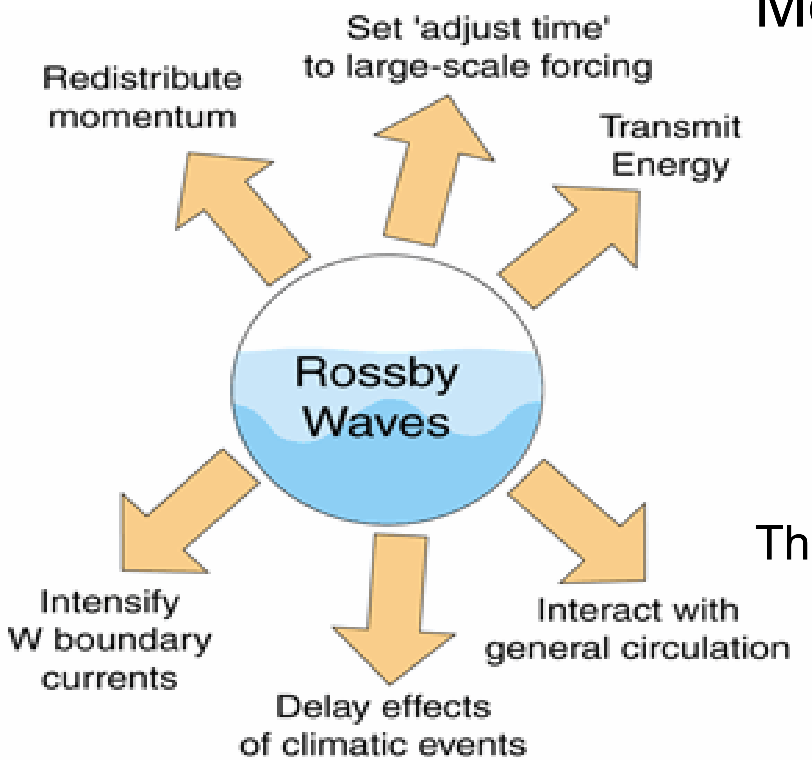
$$c_p^{\ x} \equiv \frac{\omega}{k} = U - \frac{\beta + U / L_d^{\ 2}}{K^2 + 1 / L_d^{\ 2}} = \frac{UK^2 - \beta}{K^2 + 1 / L_d^{\ 2}}$$

$$c_g^{\ x} \equiv \frac{\partial \omega}{\partial k} = U + \frac{\left(\beta + U / L_d^{\ 2}\right)\left(k^2 - l^2 - 1 / L_d^{\ 2}\right)}{\left(k^2 + l^2 + 1 / L_d^{\ 2}\right)^2}$$

no more simple Doppler shift nor anymore a uniform translation)! The presence of L_d modifies the basic potential vorticity gradient. The ambient potential vorticity increases with y.



Scale: order of hundred of km
Amplitude at the sea surface: is just a few centimeters (which makes them practically impossible to measure with in-situ techniques.)
Always travel from East to West, following the parallels.
Not very fast - the speed varies with latitude and increases equatorward, but is of the orded of just a few cm/s (or a few km/day) →
at mid-latitudes (say, 30 degrees N or S) one such wave may take several months - or even years - to cross the Pacific Ocean.
In some cases they may cross an entire oceanic basin, being originated close to the eastern boundaries and being (as a first approximation) non-dispersive.



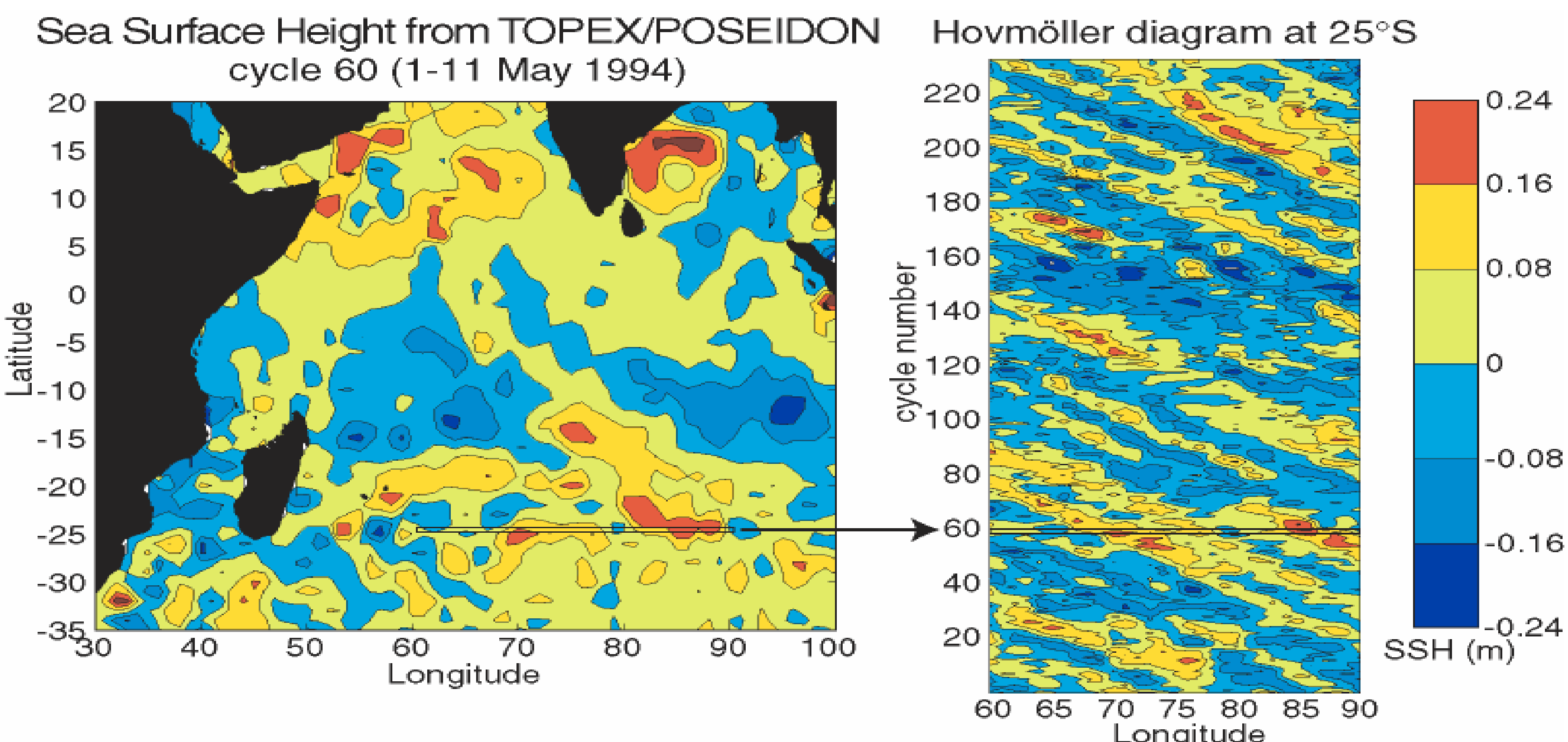
They have major effects on the large-scale ocean circulation.

Most important effect is on western boundary currents, such as the Gulf Stream or Kuroshio. Rossby waves can intensify the currents, as well as push them off their usual course (and they transport heat…).

This might have happened in the Kuroshio extension in 1993, the culprit Rossby wave being an effect of the 1982-83 El Niño - see the paper by Jacobs et al in the 4th august 1994 issue of *Nature*.

Measured by radar altimeter vi Sea Surface height Given that Rossby waves travel almost zonally, it is possible to observe them by taking zonal (west-east) sections of Sea Surface Height Anomalies from each orbital cycle and piling them up into a longitude-time plot (also known as a Hovmöller diagram).

In such a plot, the waves appear quite clearly as diagonal (i.e. going from bottom right to top left) alignments of crests and troughs. By measuring the slope of the alignments we can estimate the speed of propagation of the waves.



Rossby waves during ENSO

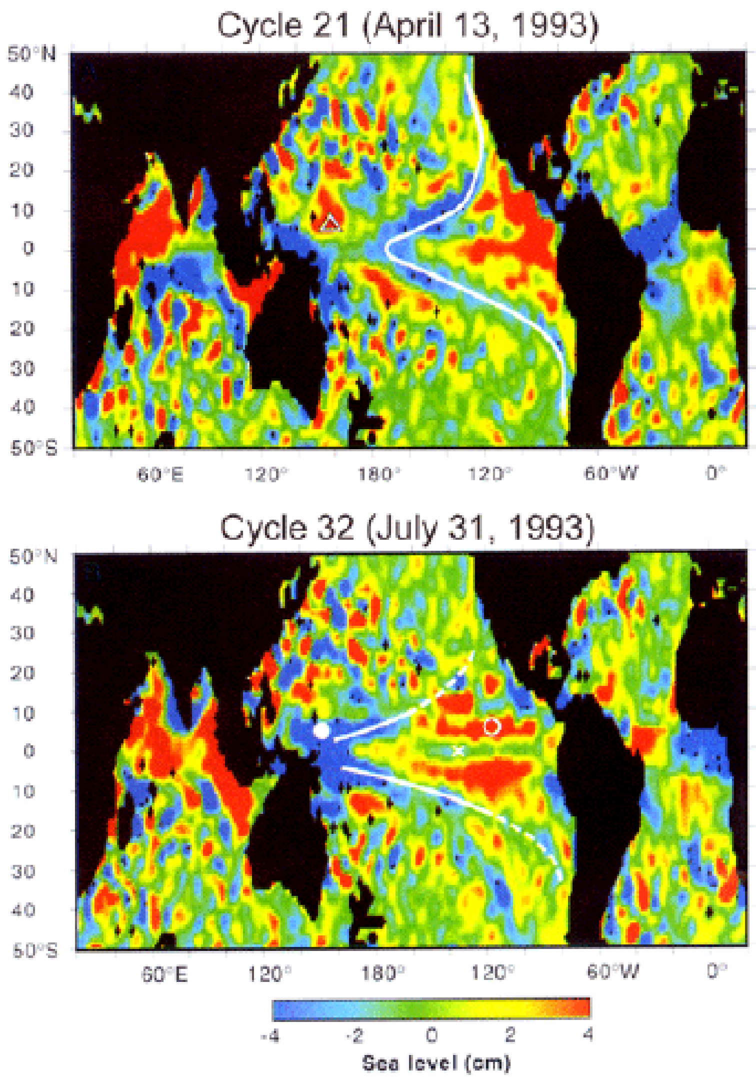
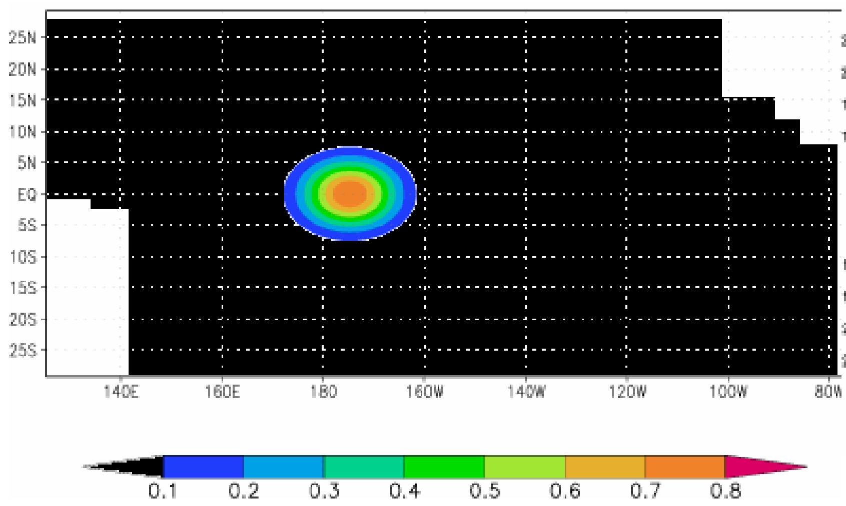
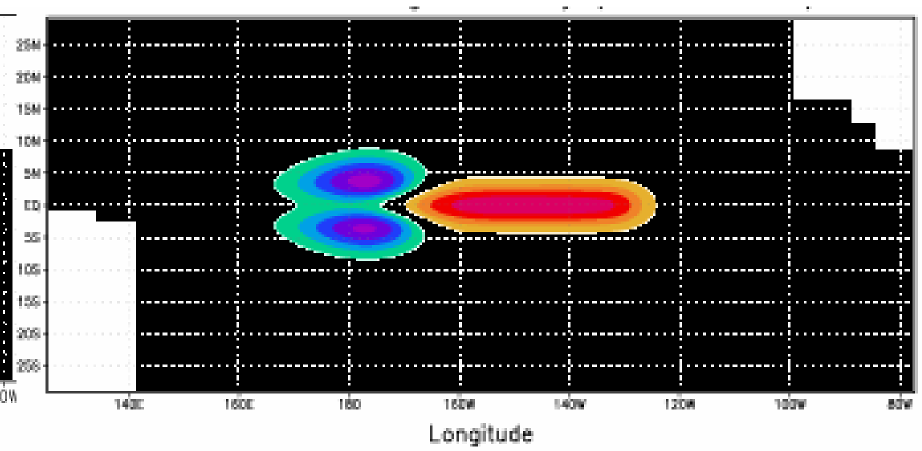


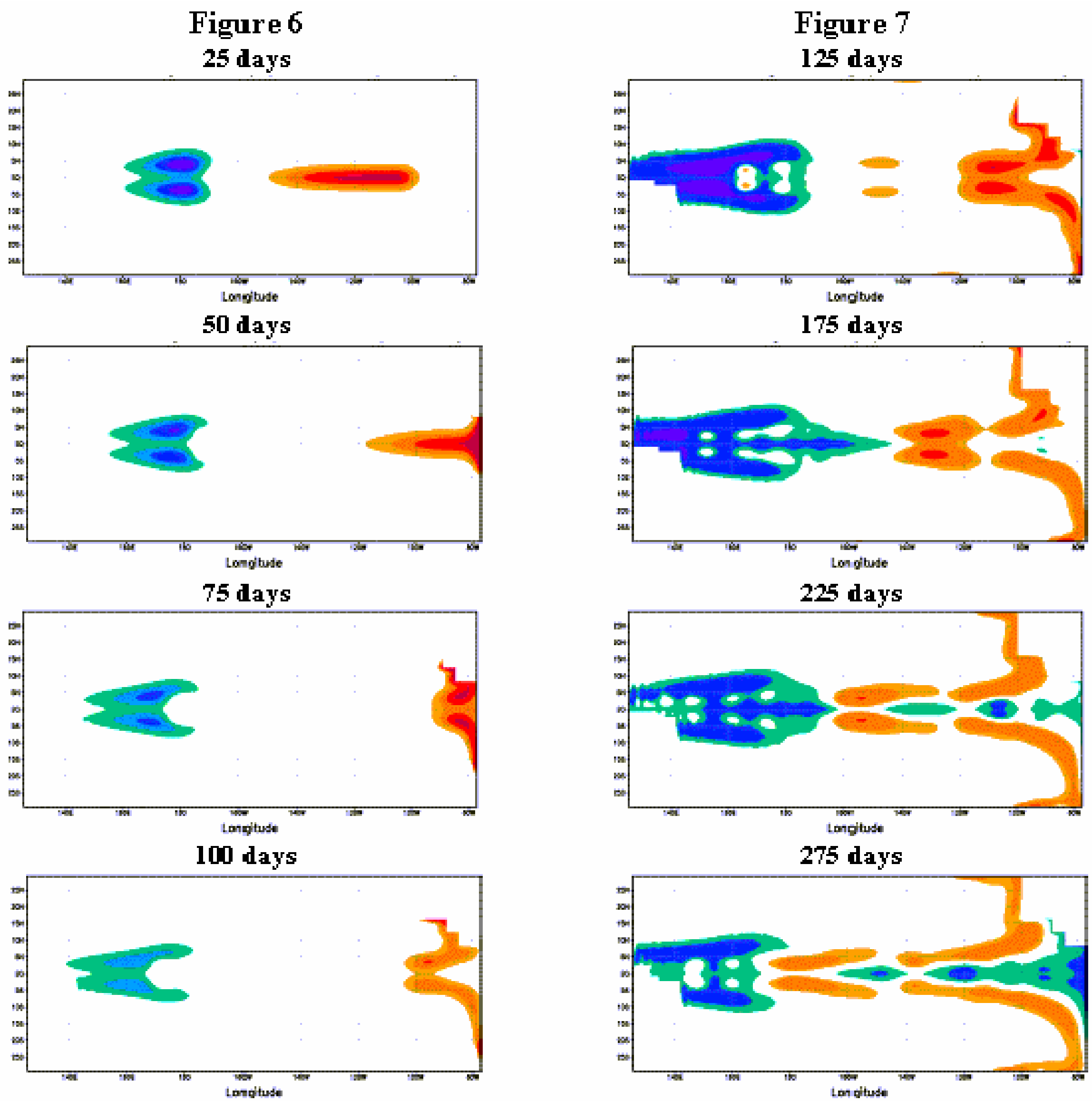
Figure 14.15 Snapshots of sea-surface height anomaly, spatially smoothed to emphasize scales longer than those of mesoscale eddies (Chelton and Schlax, 1996).

The delayed oscillator





A wind stress is applied in the middle of the Tropical Pacific Ocean.
The white areas represents the coasts.
The stronger wind stress is red and the weaker wind stress is blue.
The red signal is the Kelvin wave which propagates with a positive sea surface height anomaly and thus deepens the thermocline in the eastern Pacific. The blue-green signal is the Rossby wave which propagates with a negative sea surface height anomaly and thus rises the thermocline in the western Pacific.
Source : IRI



Propagation of the Kelvin and Rossby waves in the tropical Pacific (IRI)

The Kelvin and Rossby wave signals propagate at different speeds. The Kelvin wave travels eastward and in the idealized case has speed on the order of 2.9 meters per second (Kelvin wave will cross the Pacific Ocean ~ approximately 120° East to 80° West 17,760 Km~, in about 70 days. The Rossby mode travels westward at one third the speed of the Kelvin wave, ~about 0.93 meters per second. Thus a Rossby wave takes approximately 210 days to cross the Pacific.

(adapted from IRI)

The time evolution for the idealized delayed oscillator: After 25 days (Figure 6 upper left panel), the Kelvin wave (red and gold shading) has moved from the central Pacific forcing region to the east. At the same time, the Rossby wave (blue and green shading) has propagated to the west, but over a much shorter distance. Over days 50 through 100 the Kelvin wave reaches the eastern boundary and reflects as a Rossby wave with positive sea surface height anomalies. At the same time, the Rossby wave continues to propagate slowly to the west becoming visibly distorted by day 100 (associated with the interaction with the basin boundary).

(adapted from IRI)

- By day 125 (Figure 7), the Rossby wave has reached the western boundary and is starting to reflect as a same-signed Kelvin wave. We now see a time evolution similar to before, with a Kelvin wave propagating eastward along the equator (this time starting from the western boundary) and a Rossby wave propagating westward from the eastern boundary. However, now the Kelvin wave has negative sea surface height anomalies, and is an upwelling wave. Over the period from day 125 to day 275 the Kelvin wave propagates from the western to the eastern boundary resulting in negative sea surface height anomalies along the equator in the east. During this same period, the reflected Rossby wave has traveled from near 120° West to 170° West.

Rossby waves in two layers

For a two-layer system the equations of motion are

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_1' + F_1(\psi_2' - \psi_1')\right] + \beta\frac{\partial\psi_1'}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_2' - F_2(\psi_2' - \psi_1')\right] + \beta\frac{\partial\psi_2'}{\partial x} = 0$$

Multiply the first by F_2, the second by F_1 and take the sum and the difference of the equations above →

$$\frac{\partial}{\partial t}\nabla^2\overline{\psi} + \beta\frac{\partial\overline{\psi}}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left(\nabla^2 - k_d^{\,2}\right)\tau + \beta\frac{\partial\tau}{\partial x} = 0$$

where

$$\overline{\psi} = \frac{F_1\psi_2' + F_2\psi_1'}{F_1 + F_2}, \qquad \tau = \frac{1}{2}(\psi_1' - \psi_2')$$

$$L_d = k_d^{-1} = \frac{1}{f_o}\left(\frac{g' H_1 H_2}{H_1 + H_2}\right)^{1/2}$$

(to simplify the algebra –and for your problem set this week…- it is always possible to rewrite the eq. using L_d as a scale respect to which nondimensionalize the equations. If the two layers have equal depth, then $F_1=F_2=1$).

$\overline{\psi}$ and τ represent the *normal modes* of the system, and they oscillate independently of each other.

The equation for the barotropic mode is identical to the one for a 1-layer system with U=0. Therefore

$$\omega = -\frac{\beta k}{K^2}$$

It does not depend on the layer. The whole flow oscillates in a synchronous way.

The interface between the layers is given by $f_o(\psi_1 - \psi_2)/g'$ and is therefore proportional to τ.

For the baroclinic mode the dispersion relation is

$$\omega = -\frac{\beta k}{K^2 + k_d{}^2}$$

The deformation radius enters only in the baroclinic mode. If the scales considered are smaller that the depormation radius, then $K^2 >> k_d^2$. The dispersion relation for the two modes coincides. On the other extreme, for very large scales the baroclinic mode dominates

From the last two weeks

- SW equations for geostrophic flows (Ro~0.01):

$$\boxed{R_o\left[\frac{\partial\hat{\vec{u}}}{\partial\hat{t}}+\hat{\vec{u}}\cdot\nabla\hat{\vec{u}}\right]+\hat{\vec{f}}\times\hat{\vec{u}}=-\nabla\hat{\eta}}$$

$$R_o\frac{L^2}{L_d{}^2}\frac{D\hat{\eta}}{D\hat{t}}+\left[1+R_o\frac{L^2}{L_d{}^2}\hat{\eta}\right]\nabla\cdot\hat{\vec{u}}=0$$

- Planetary geostrophic equations:

$$R_o\left[\frac{\partial\hat{\vec{u}}}{\partial\hat{t}}+\hat{\vec{u}}\cdot\nabla\hat{\vec{u}}\right]+\hat{\vec{f}}\times\hat{\vec{u}}=-\nabla\hat{\eta} \qquad \Rightarrow \qquad \hat{\vec{f}}\times\hat{\vec{u}}=-\nabla\hat{\eta}$$

$$\frac{D\hat{\eta}}{D\hat{t}}+\left[1+\hat{\eta}\right]\nabla\cdot\hat{\vec{u}}=0 \qquad or \qquad \frac{Dh}{Dt}+h\nabla\cdot\vec{u}=0$$

$$\boxed{\frac{DQ}{Dt}=0 \qquad where \qquad Q=f/h,\quad \vec{f}\times\vec{u}=-g\nabla\eta,\quad \eta=h+\eta_b}$$

Valid ONLY for scales of motion >> than the Rossby deformation radius

- SW QG equations:

$$\frac{D}{D\hat{t}}\left(\nabla^2\hat{\psi}_o + \hat{\beta}\hat{y} - Bu^{-1}\hat{f}_o{}^2\hat{\psi}_o\right) = 0$$

or restoring dimensions

$$\frac{D}{D\hat{t}}\left(\nabla^2\psi + \beta y - \frac{1}{L_d{}^2}\psi\right) = 0$$

with

$$\boxed{q \equiv \varsigma + \beta y - \frac{f_o}{H}\eta = \nabla^2\psi + \beta y - \frac{1}{L_d{}^2}\psi}$$

- Rossby waves:

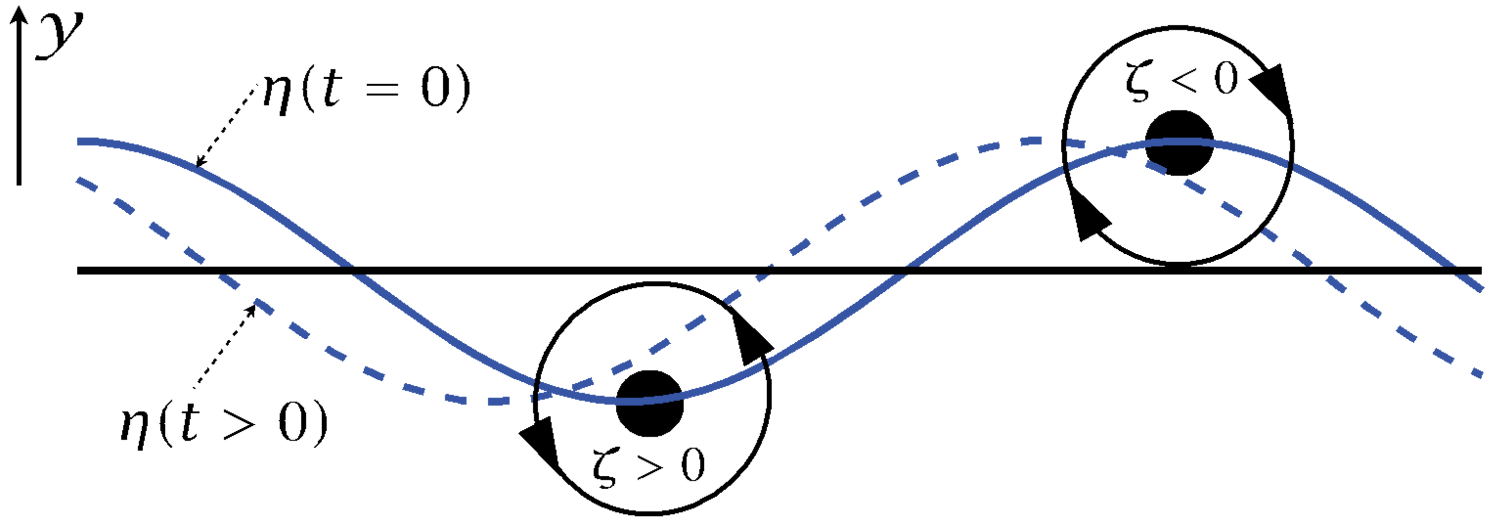


Fig. 5.4 The mechanism of a two-dimensional (x-y) Rossby wave. An initial disturbance displaces a material line at constant latitude (the straight horizontal line) to the solid line marked $\eta(t = 0)$. Conservation of potential vorticity, $\beta y + \zeta$, leads to the production of relative vorticity, as shown for two parcels. The associated velocity field (arrows on the circles) then advects the fluid parcels, and the material line evolves into the dashed line. The phase of the wave has propagated westwards.

Using in the linearized vorticity eq:

$$\left[\left(-\omega+Uk\right)K^2+\beta k\right]\tilde{\psi}=0, \qquad K^2=k^2+l^2$$

$$\Downarrow$$

$$\omega=Uk-\frac{\beta k}{K^2}=dispersion \quad relation$$

Phase speed (in -x direction) $$c_p^{\ x}\equiv\frac{\omega}{k}=U-\frac{\beta}{K^2}$$

Group velocity (in -x direction) $$c_g^{\ x}\equiv\frac{\partial\omega}{\partial k}=U+\frac{\beta\left(k^2-l^2\right)}{\left(k^2+l^2\right)^2}$$

Doppler shift. The phase for no mean flow is *westward*, with long wavelengths traveling faster

week 7

- Kelvin-Helmholtz instability
- Barotropic instability in parallel shear flows
- Conditions for instability
- Baroclinic instability: an example

Instability theory

The existence of fluctuations in the atmosphere and ocean can be attributed to the *instability* of the dynamical state without fluctuations to very small wavelike disturbances.

Two tasks:

- to demonstrate that the observed fluctuations are due to the instability one has to identify the fluctuation-free state. Such a state **IS NOT the observed mean state** of the flow (which indeed is influenced by the presence of the fluctuations and implicitly assumes the existence of the fluctuations). In general, if the time-averaged state is used as initial state, it is found more stable than the relevant initial state.

- Instead of calculating the mean state, we can arbitrarily prescribe an initial state, corresponding to a particular distribution of heat sources and frictional sources → we will end-up with *classes* of initial state for which we can identify the feature to which the instability can be attributed to. If the feature is sufficiently general and persistent, and the instability is geophysically relevant, then we can generalize a criterium for the specific instability to occur.

- Two major contributors: Charney (1947) and Eady (1949).

- Baroclinic instability: arises in rotating, stratified fluids that are subject to horizontal temperature gradient

- Barotropic instability: arises because of shear and may occur in constant density flows.

Kelvin-Helmholtz instability

- It is the simplest instability of relevance in geophysical flows. It is a barotropic instability (but obviously can involve fluids with different densities).

From Vallis' book

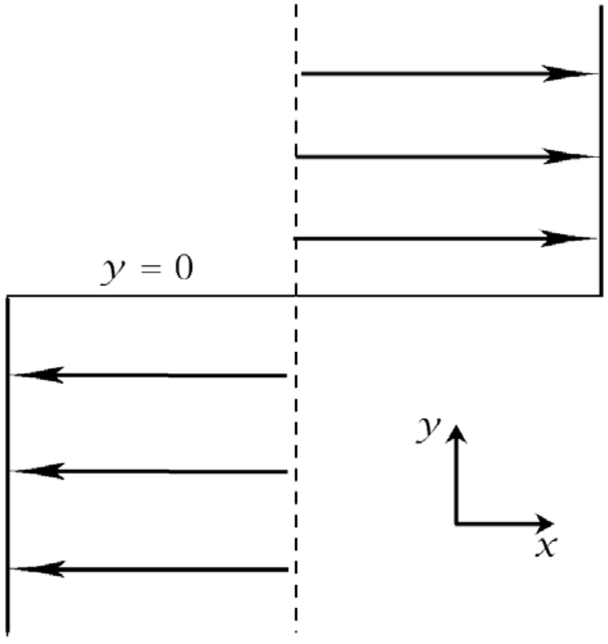


Fig. 6.1 A simple basic state giving rise to shear-flow instability. The velocity profile is discontinuous and the density is uniform.

- Two fluid masses, same density, 1D velocity field. u=U for y>0, u=-U for y<0, with common interface at y=0.
- Let's perturb the system (**u**', p' are the perturbations in the velocity and pressure field). The perturbation is small, and the flow is solution of the Euler equation. We have to consider the evolution of the perturbation using the linearized Euler eq. about the steady state.

for y > 0 the eq. for the perturbation is

$$\frac{\partial \vec{u}\,'}{\partial t} + U \frac{\partial \vec{u}\,'}{\partial x} = -\nabla p\,', \qquad \nabla \cdot \vec{u}\,' = 0$$

(same for y<0 with –U instead of U). Taking the divergence the left-hand side vanishes and we get $\nabla^2 p\,' = 0$

Assuming periodic boundary conditions in x, we look for solutions in the form

$$\phi'(x,y,t) = \mathrm{Re}\,\tilde{\phi}(y)e^{[ik(x-ct)]}$$

i.e. a solution in the form of a Fourier-like mode, where Φ can be velocity or pressure (any of the perturbed unknown of the system).

(if the system was nonlinear the solution should be found in the form of a superimposition of Fourier-like modes, not only one. If the system is linear the modes do not interact and we may consider them separately. it is enough to find one which grows in time to find an instability)

Using $\nabla^2 p' = 0$, the solution has the form

$$p' = \begin{cases} \tilde{p}_1 e^{ikx-ky} e^{\sigma t} & y > 0 \\ \tilde{p}_2 e^{ikx+ky} e^{\sigma t} & y < 0 \end{cases}$$

where σ = -ikc is a complex number. <u>If it has a positive real component, the amplitude of the perturbation will grow and there is an instability</u>. If it has a non-zero imaginary component the perturbation will display oscillatory motion.

Dispersion relation:

Consider one component of the momentum equation (for example y-component for y>0)

$$\frac{\partial v_1'}{\partial t} + U\frac{\partial v_1'}{\partial x} = -\frac{\partial p_1'}{\partial y}$$

substituting the expression for p' and using $v_1' = \tilde{v}_1 \, \mathrm{e}^{ikx+\sigma t}$ we get $\left(\sigma + ikU\right)\tilde{v}_1 = k\tilde{p}_1$

At the interface the velocity normal to the interface is simply the rate of change of the interface itself, i.e. at y=0

$$v_1 = \frac{\partial \eta'}{\partial t} + U\frac{\partial \eta'}{\partial x}$$

which implies that $\tilde{v}_1 = (\sigma + ikU)\tilde{\eta}$

and therefore $(\sigma + ikU)^2 \tilde{\eta} = k\tilde{p}_1$

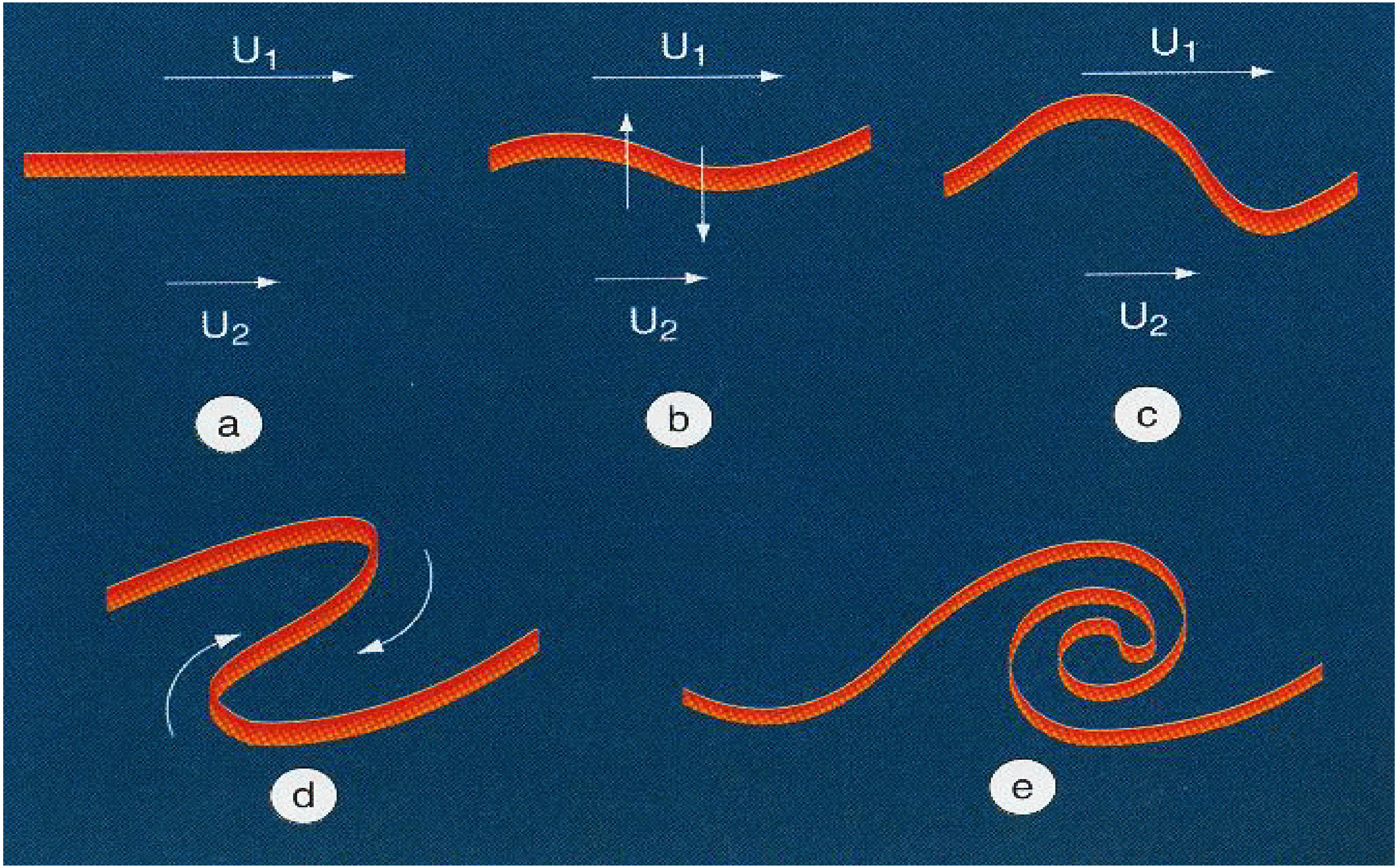
For y < 0 analogously $(\sigma - ikU)^2 \tilde{\eta} = -k\tilde{p}_2$

and at the interface the flow must be continuous and therefore the pressure must be the same (p_1=p_2)

The dispersion relation emerges requiring that

$$(\sigma - ikU)^2 = (\sigma + ikU)^2 \Rightarrow \sigma^2 = k^2 U^2$$

Two roots; one is positive: the flow is unstable!

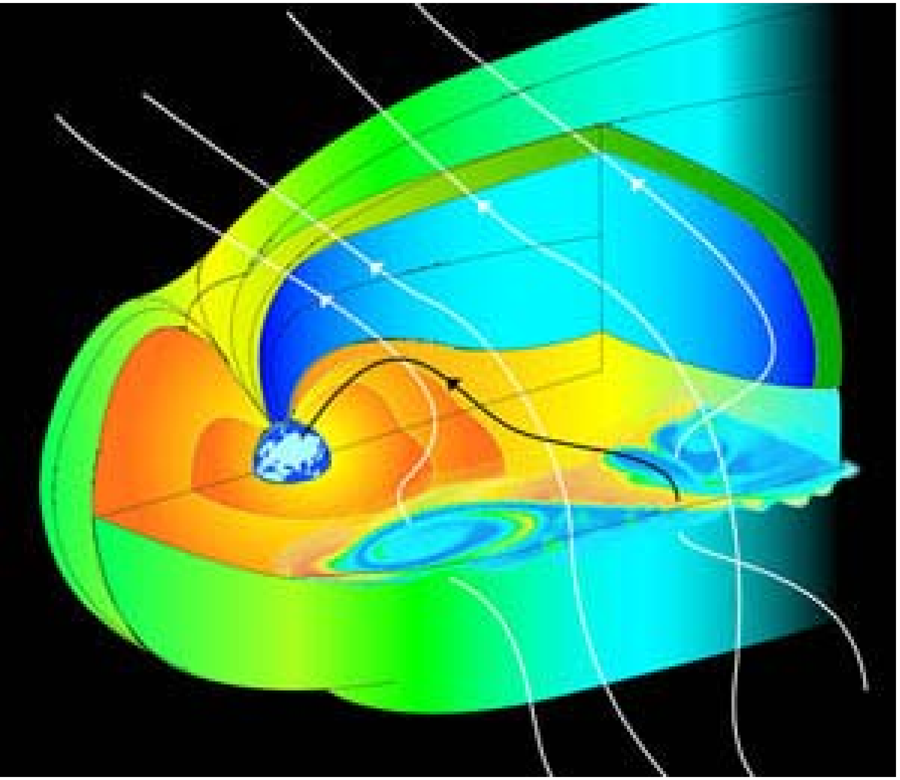


U1
U2
a
U1
U2
b
U1
U2
c
d
e

Wave clouds forming over Mount Duval, NSW. Photograph by Williamborg . From WP.
Excellent "wave cloud" image which is correctly described as an example of Kelvin-Helmholtz instability between two shear layers.

A detailed explanation of wind flow around Mount Shasta is described in "Airborne Doppler Lidar Wind Field Measurements of Waves in the Lee of Mount Shasta" in *Journal of the Atmospheric Sciences, 1988*. The Kelvin-Helmholtz wave cloud does not last very long because the upper layer of air is drier than the lower layer, which results in evaporation of the cloud.

- Check this out: **Transport of solar wind into Earth's magnetosphere through rolled-up Kelvin–Helmholtz vortices,** by Hasegawa et al., Nature 2004



A bevy of satellites buzzing around in the Earth's magnetosphere has found at least part of the answer to ***a long-standing puzzle about the source of the charged particles that feed the aurora.***
The charged particles come from explosions on the sun and smash into the Earth's magnetic field, which repels the bulk of them. But many slip through, often via a physical process called magnetic reconnection, where the magnetic field traveling with the particles breaks and reconnects with the Earth's field, opening a window for the particles to surge through. Once inside, these excited particles can spiral down toward the poles and create brilliant auroras when they hit the atmosphere. The vortices are huge structures, measuring more than 20,000 miles across. (Kentaro Tanaka of Tokyo Institute of Technology)

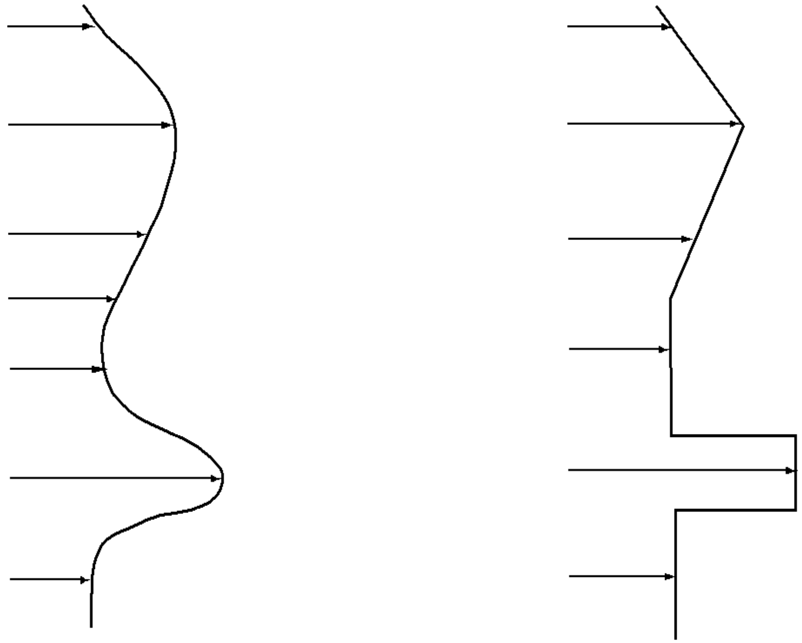
Instability of parallel shear flows

General case of flows characterized by same density but different velocities through the water column.

It is a classical problem in fluid dynamics and is important for geophysical flows for two reasons:

1) it is a very common example of barotropic instability
2) it is similar both conceptually and in the equations to baroclinic instability

We consider only 2d incompressible flows



generic velocity profiles that are barotropically unstable.

From Vallis' book

Fig. 6.2 Left: example of a smooth velocity profile — both the velocity and the vorticity are continuous. Right: example piecewise continuous profile — the velocity and vorticity may have finite discontinuities.

It is convenient to use the vorticity equation $\frac{D\zeta}{Dt}=0$

We assume that the mean state is given by a velocity field which is constant in the x-direction and vary in y:

$$\vec{u}=U(y)\vec{i}$$

The general form for the linearized vorticity equation is

$$\frac{\partial\zeta'}{\partial t}+U\frac{\partial\zeta'}{\partial x}+v'\frac{\partial Z}{\partial y}=0$$

where $Z=-\partial_y U$

Because it is a 2D incompressible flow, the mass continuity for the perturbation is just $\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0 \quad \Rightarrow \quad u'=-\frac{\partial\psi'}{\partial y}, \quad v'=\frac{\partial\psi'}{\partial x}$

The linear vorticity equation becomes

$$\frac{\partial \nabla^2 \psi'}{\partial t} + U \frac{\partial \nabla^2 \psi'}{\partial x} + \frac{\partial \psi'}{\partial x} \frac{\partial Z}{\partial y} = 0$$

again, we look for solution using $\psi' = \mathrm{Re}\, \tilde{\psi}(y) e^{ik(x-ct)}$

from which we can re-write the vorticity equation and obtain

$$\boxed{(U - c)\left(\tilde{\psi}_{yy} - k^2 \tilde{\psi}\right) - U_{yy} \tilde{\psi} = 0}$$

This is the Rayleigh's equation. Its general form in presence of a β term is

$$\boxed{(U - c)\left(\tilde{\psi}_{yy} - k^2 \tilde{\psi}\right) + \left(\beta - U_{yy}\right) \tilde{\psi} = 0}$$

The Rayleigh equation looks simple, but is not so easy to solve analytically for a continuous flow. It is possible however to move forward without computers assuming that U(y) is constant over a finite number of y intervals.

One can use the so-called matching conditions, in such a case and specifically:

1) for an inviscid flow the pressure must be continuous across the interfaces (because the normal stress is continuous)

From the linearized momentum equation

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} = -\frac{\partial p'}{\partial x}$$

we get $$ik\left(U-c\right)\tilde{\psi}_y - ik\tilde{\psi}U_y = -ik\tilde{p}$$

and the continuity at the interface implies

$$\Delta\left[\left(U-c\right)\tilde{\psi}_y - U_y\tilde{\psi}\right] = 0$$

where Δ indicates the ‘jump’ between two contiguous layers

2) the normal velocity at the interface is given by the variation of the interface itself

$$v' = \frac{\partial \psi'}{\partial x} = \frac{D\eta}{Dt} = \frac{\partial \eta'}{\partial t} + U\frac{\partial \eta'}{\partial x}$$

and this hold for the velocities in the two layers with the common interface.

$$\frac{\partial \eta'}{\partial t} + U_1\frac{\partial \eta'}{\partial x} = \frac{\partial \psi_1'}{\partial x} \quad \Rightarrow \quad (U_1 - c)\tilde{\eta} = \tilde{\psi}_1$$

$$\frac{\partial \eta'}{\partial t} + U_2\frac{\partial \eta'}{\partial x} = \frac{\partial \psi_2'}{\partial x} \quad \Rightarrow \quad (U_2 - c)\tilde{\eta} = \tilde{\psi}_2$$

As a result

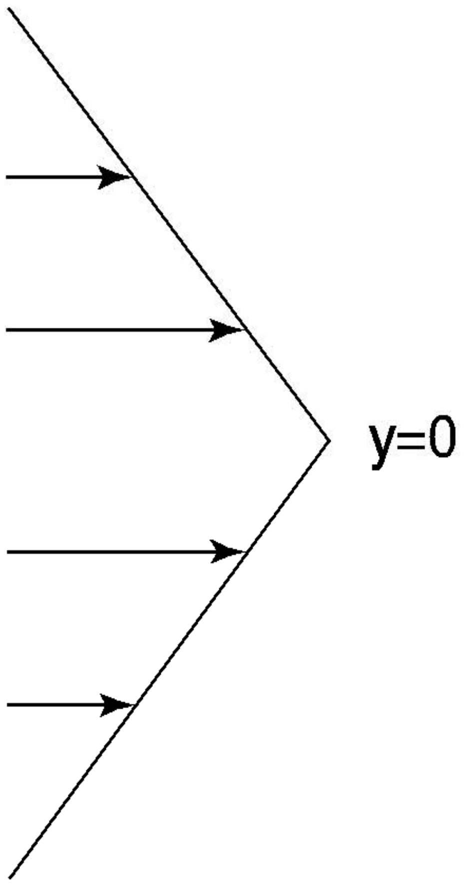
$$\Delta\left[\frac{\tilde{\psi}}{U - c}\right] = 0$$

A stable case: edge waves

Figure 6.3 Velocity profile of a stable jet. Although the vorticity is discontinuous, a small perturbation gives rise only to *edge waves* centered at $y = 0$.

from Vallis' book

Velocity is continuous. Vorticity is not!



The velocity profile has second order derivatives =0 (so U_{yy}=0)

The Rayleigh's equation is just

$$\left(U-c\right)\left(\tilde{\psi}_{yy}-k^2\tilde{\psi}\right)=0$$

If c ≠ U the solution is easy

$$\tilde{\psi}=\begin{cases}\phi_1 e^{-ky} & y>0\\ \phi_2 e^{ky} & y<0\end{cases}$$

Apply the jump condition for continuity of pressure and use the fact that the velocity U at the interface is the same for layer 1 and 2.

As a result the dispersion relation is

$$c = U_o + \frac{\partial_y U_1 - \partial_y U_2}{2k}$$

which is always a real number for any U: no instability! Only a bit of oscillatory motion at the interface that will die after a some time

(If edge waves interact together and have a sufficient cross stream c can be imaginary and an instability will take place for sufficiently long wavelengths).

necessary conditions for instability

- Rayleigh-Kuo or Rayleigh inflection point criteria

Let's start from the Rayleigh's equation written in a slightly different manner

$$\tilde{\psi}_{yy} - k^2\tilde{\psi} + \frac{\beta - U_{yy}}{(U-c)}\tilde{\psi} = 0$$

Multiply by the complex conjugate of $\tilde{\psi}$, $\tilde{\psi}^*$, and integrate over the domain of the flow →

$$\int_{y_1}^{y_2}\left(\left|\frac{\partial\tilde{\psi}}{\partial y}\right|^2 + k^2\left|\tilde{\psi}\right|^2\right)dy - \int_{y_1}^{y_2}\frac{\beta - U_{yy}}{U-c}\left|\tilde{\psi}\right|^2 dy = 0$$

The first integral does not contain c and it is always real. Taking the imaginary part of the second one and assuming that $\tilde{\psi}$ vanishes at the boundaries one obtains

$$c_i \int \frac{\beta - U_{yy}}{\left|U - c\right|^2} \left|\tilde{\psi}\right|^2 dy = 0$$

where c_i are the eigenvalues for each mode (and therefore have to be non-zero for the system to be of any interest).

The integral must vanishes and this happen if

$$\boxed{\beta - U_{yy} \qquad changes \qquad sign}$$

somewhere in the domain (i.e. it is zero somewhere in the domain).

(it is sufficient that it does not vanish for the system to be stable. It is necessary BUT not sufficient that vanishes for the system to be unstable)

- Fjortoft's criteria

The expression $$\boxed{\left(\beta - U_{yy}\right)\left(U - U_s\right)}$$

has to be positive somewhere in the domain.

U_s is the value of U(y) at which $\beta - U_{yy}$ vanishes

This criteria is satisfied if the vorticity has an extremum inside the domain (not at the boundary, not at infinity).

β can have both stabilizing or destabilizing effects.

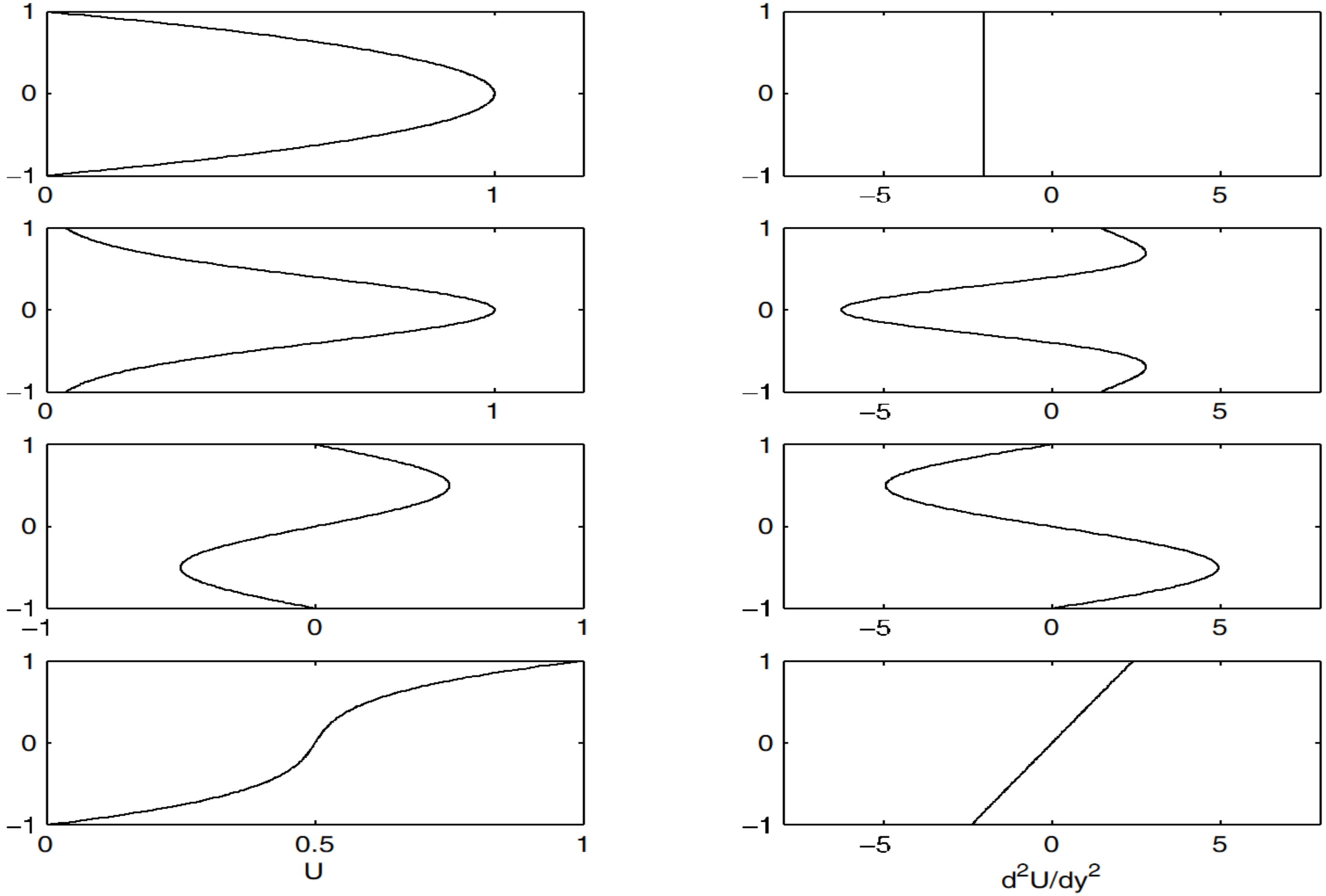


Fig. 6.8 Example parallel velocity profiles (left column) and their second derivatives (right column). From the top: Poiseuille flow ($u = 1 - y^2$); a Gaussian jet; a sinusoidal profile; a polynomial profile. By Rayleigh's criterion, the top profile is stable, whereas the lower three are potentially unstable. However, the bottom profile is stable by Fjørtoft's criterion (and note that the vorticity maxima are at the boundaries). If the β-effect were present and large enough it would stabilize the middle two profiles.

from Vallis' book

$(\frac{dz}{dx})_\phi < 0$

Baroclinic instability

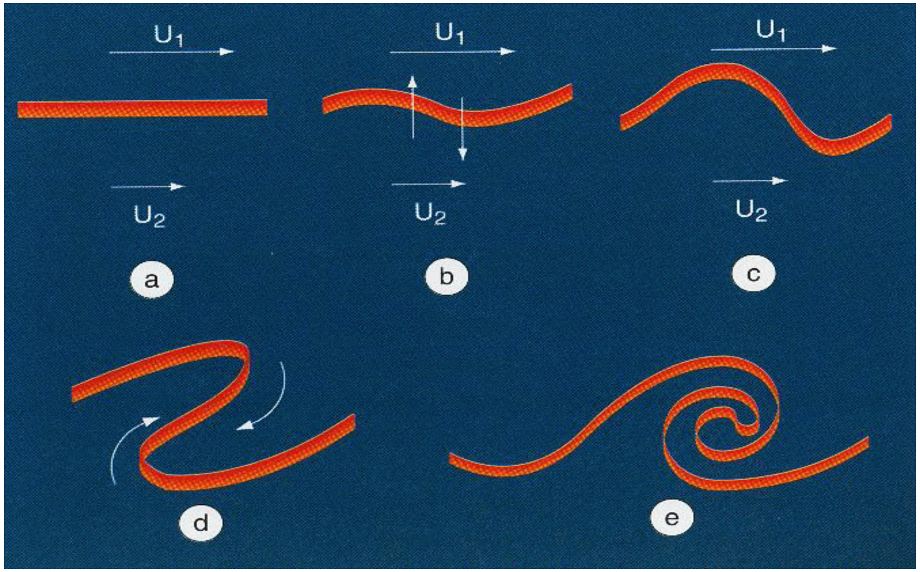
- Baroclinic instability produces wave motion due to vertical shear of the basic current in the presence of Coriolis and buoyancy forces. It is observed in oceanic and atmospheric flows. The structure of the baroclinic mode depends on the stratification.
- Available potential energy is the source of energy for baroclinic instability.
- For baroclinic conversion of energy, $P = \frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial z}\frac{\partial U_o}{\partial z}$ must be positive on the average, so the lines of constant ϕ must have a negative slope in regions of positive vertical shear to augment the production of perturbation of kinetic energy [Pedlosky, 1987].

Things to remember from the last week

- Instability theory
- GOAL: Demonstrate that the existence of fluctuations in the atmosphere and ocean can be attributed to the *instability* of the dynamical state without fluctuations to very small wavelike disturbances
- Initial value problem: identifying a “fluctuation-free state”
- **IS NOT the observed mean state.** In general the time-averaged state or an arbitrarily prescribe an initial state are used.

Barotropic instability

- Due to the shear
- examples: **Kelvin-Helmholtz instability**



- **Instability of parallel shear flows,** which brings us to the Rayleigh equation

$$\boxed{(U-c)\left(\tilde{\psi}_{yy}-k^2\tilde{\psi}\right)+\left(\beta-U_{yy}\right)\tilde{\psi}=0}$$

To be solved using the matching conditions (1) pressure must be continuous at the interface between two layers

$$\Delta\left[(U-c)\tilde{\psi}_y-U_y\tilde{\psi}\right]=0$$

(2) the normal velocity at the interface is given by the variation of the interface itself

$$\Delta\left[\frac{\tilde{\psi}}{U-c}\right]=0$$

Rayleigh-Kuo or Rayleigh inflection point criteria: necessary condition for instability

$$\boxed{\beta - U_{yy} \quad changes \quad sign}$$

somewhere in the domain

Fjortoft's criteria

$$\boxed{\left(\beta - U_{yy}\right)\left(U - U_{s}\right)}$$

has to be positive somewhere in the domain: i.e. the vorticity has an extremum inside the domain (not at the boundary, not at infinity).

week 8

- Baroclinic instability in QG flows
- Conditions for instability
- The Eady problem
- Two-layer baroclinic instability

Baroclinic instability

- Baroclinic instability produces wave motion due to vertical shear of the basic current in the presence of Coriolis and buoyancy forces. It is observed in oceanic and atmospheric flows. The structure of the baroclinic mode depends on the stratification.
- Available potential energy is the source of energy for baroclinic instability.
- For baroclinic conversion of energy, $P = \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial z} \frac{\partial U_o}{\partial z}$ must be positive on the average, so the lines of constant ϕ must have a negative slope in regions of positive vertical shear to augment the production of perturbation of kinetic energy [Pedlosky, 1987].

A physical description

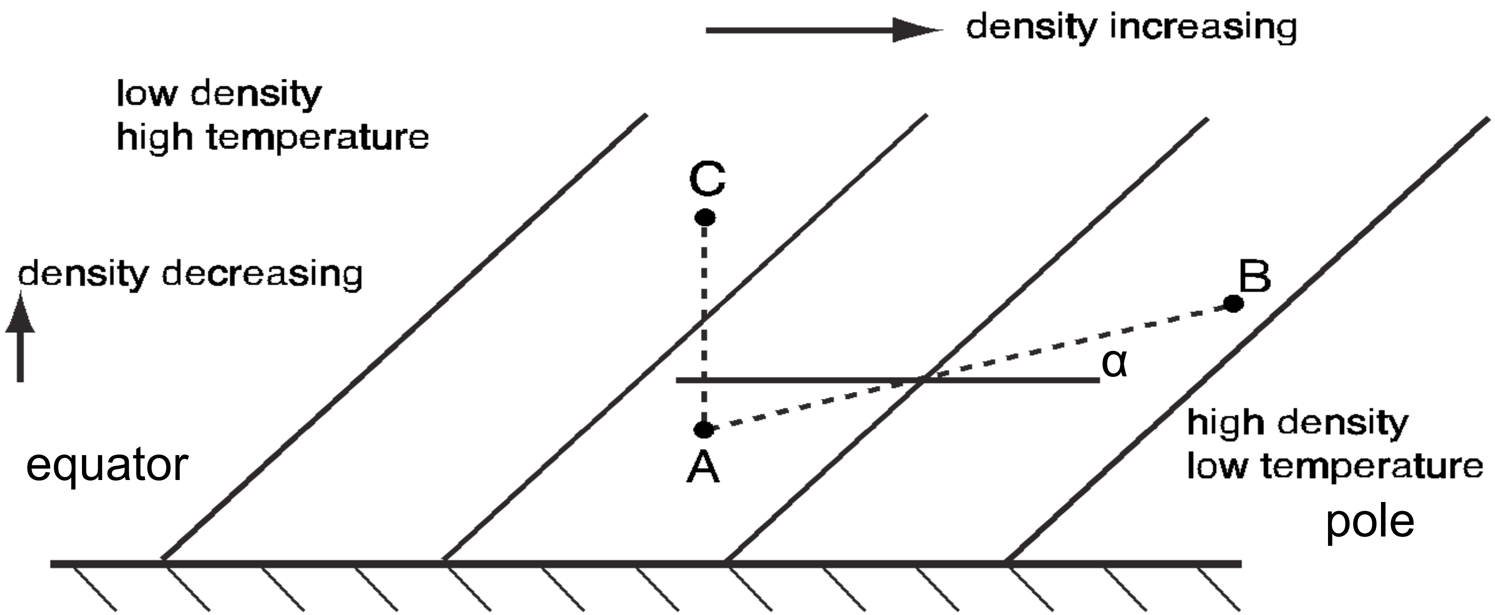


Fig. 6.9 A steady basic state giving rise to baroclinic instability. Potential density decreasing upwards and equatorwards, and the associated horizontal pressure gradient is balanced by the Coriolis force. Parcel 'A' is heavier than 'C', and so statically stable, but it is lighter than 'B'. Hence, if 'A' and 'B' are interchanged there is a release of potential energy.

If a parcel moves from A to C, such a perturbation is stable. But if parcels A and B exchange their positions, the perturbation responsible is creating an instability, because A finds itself surrounded by parcels of higher density, and it is therefore more buoyant. (figure in book have B and C exchanged)

B finds itself negatively buoyant and at lower latitude than where started. Overall the centre of gravity of the system is lowered, and so its potential energy. If potential energy is lost, kinetic must be gained.

The loss of potential energy $PE = \int \rho g dz$ is simply

$$\Delta PE = g(\rho_A z_A + \rho_B z_B - \rho_A z_B - \rho_B z_A) = g(z_A - z_B)(\rho_A - \rho_B) = g\Delta\rho\Delta z$$

If the content of both parenthesis is positive, energy is released and the state is unstable.

Let's call ϕ the slope of the isopycnals ($\phi = -\frac{\partial \rho}{\partial y} / \frac{\partial \rho}{\partial z}$) , α the slope of the displacement, and L the horizontal displacement, then for ϕ and α small

$$\Delta PE = g\Delta\rho\Delta z = g\left(L\frac{\partial \rho}{\partial y} + L\alpha\frac{\partial \rho}{\partial z} \right) L\alpha = gL^2\alpha\frac{\partial \rho}{\partial y}\left(1 - \frac{\alpha}{\phi}\right)$$

if $0 < \alpha < \phi$ then the energy is released by the perturbation

back to the eq. of motion: QG case

- Generic: QG PV equation for the interior + buoyancy or temperature eq. at two vertical boundaries

$$\frac{\partial q}{\partial t} + u \cdot \nabla q = 0, \qquad 0 < z < H$$

$$q = \nabla 2\psi + \beta y + \frac{\partial}{\partial z}\left(F \frac{\partial \psi}{\partial z} \right), \qquad F = f_o^2 / N^2$$

$$\frac{\partial b}{\partial t} + u \cdot \nabla b = 0, \qquad z = 0, H$$

$$b = f_o \frac{\partial \psi}{\partial z}$$

We assume : 1) purely zonal flow **u**=U(y,z)**i**;

2) a temperature field given by the thermal wind balance

The PV of the basic state is given by (on the beta-plane):

$$Q = \beta y - \frac{\partial U}{\partial y} + \frac{\partial}{\partial z} F \frac{\partial \psi}{\partial z} = \beta y + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial}{\partial z} F \frac{\partial \psi}{\partial z}$$

and the linearized eq for a perturbation on the basic state gives, for the interior and the top/bottom boundaries

$$\frac{\partial q'}{\partial t} + U \frac{\partial q'}{\partial x} + v' \frac{\partial Q}{\partial y} = 0, \quad 0 < z < H$$

$$\frac{\partial b'}{\partial t} + U \frac{\partial b'}{\partial x} + v' \frac{\partial B}{\partial y} = 0, \quad z = 0, H$$

Where $b' = f_o \partial_z \Psi'$, and $\partial_y B = \partial_y (f_o \partial_z \Psi) = - f_o \partial U / \partial z$

As usual, $\psi'(x,y,z,t) = \mathrm{Re}\,\tilde{\psi}(y,z)e^{ik(x-ct)}$

Substituting we obtain a set of two eq. very similar to the Rayleigh's one

$$\boxed{\begin{array}{ll} (U-c)\left(\tilde{\psi}_{yy} - k^2\tilde{\psi} + (F\tilde{\psi}_z)_z\right) + Q_y\tilde{\psi} = 0 & 0 < z < H \\ (U-c)\tilde{\psi}_z - U_z\tilde{\psi} = 0 & z = 0, H \end{array}}$$

Necessary conditions for instability

Again, it is necessary to consider the integral over the domain (in y: the flow is zonal) of the baroclinic equivalent of the Rayleigh's equation multiplied by the complex conjugate of the perturbation stream function ($\int_{y_1}^{y_2} \tilde{\psi}^*$)After quite a bit of algebra, the so-called *Charney-Stern-Pedlosky* necessary conditions for instability can be obtained.

For the instability to take place one of the following conditions must be satisfied (necessary but not sufficient, once more):

1. Q_y changes sign in the interior
2. Q_y is the opposite sign of U_z at the upper boundary, z=H
3. Q_y is of the same sign of U_z at the lower boundary, z=0
4. U_z has the same sign at the upper and lower boundaries (different from two before if Q_y=0)

The Eady problem

Together with the so-called Charney problem, this represents the first description of baroclinic instability. It's easier that the Charney' one (the mathematics is easier), even if less general. Four important simplifications:

1. $\beta = 0$
2. The stratification is constant (N^2=const)
3. The basic state has uniform shear ($U_o(z)=Uz/H=\Lambda z$) – not so true for the ocean-
4. The motion happens between two rigid lid at the top and bottom boundaries

The basic streamfunction for the Eady flow is simply Ψ= - Λzy, from which the basic state PV is therefore

$$Q = \nabla^2 \psi + \frac{H^2}{{L_d}^2}\frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z}\right) = 0$$

(which simplifies the calculations..)

The linearized eq. for a perturbation Ψ' is

$$\left(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x}\right)\left(\nabla^2 \psi' + \frac{H^2}{{L_d}^2}\frac{\partial^2 \psi'}{\partial z^2}\right) = 0$$

where again we are considering a periodic channel with

$$\psi'(x, y, z, t) = \mathrm{Re}\, \tilde{\psi}(y, z) e^{ik(x-ct)} \longrightarrow$$

$$\boxed{(\Lambda z - c)\left(\tilde{\psi}_{yy} - k^2 \tilde{\psi} + \frac{H^2}{{L_d}^2}\tilde{\psi}_{zz}\right) = 0}$$

- Boundary conditions

Horizontal: channel with walls or doubly-periodic (or infinite in one direction). Option 2 and 3 simplifies the analysis. otherwise analytically is problematic. $\longrightarrow$

Following option 3, Ψ = 0 at +L/2 and –L/2

$$\psi(y,z) = \phi(z)\sin ly \qquad \Rightarrow \qquad \psi'(x,y,z,t) = \mathrm{Re}\,\tilde{\phi}(z)\sin ly\, e^{ik(x-ct)}$$

Vertical (at z=0,H): w=0 $\longrightarrow$

$$\left(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x}\right)\frac{\partial \psi'}{\partial z} - \Lambda \frac{\partial \psi'}{\partial x} = 0, \qquad z = 0, H$$

Substituting the expression for Ψ' in the eqs for the interior

$$\left(\Lambda z - c\right)\left(-(k^2+l^2)\phi + \frac{H^2}{{L_d}^2}\phi_{zz}\right) = 0, \quad 0 < z < H$$

$$c\frac{d\phi}{dz} + \Lambda\phi = 0, \qquad z = 0$$

$$\left(c - \Lambda H\right)\frac{d\phi}{dz} + \Lambda\phi = 0, \qquad z = H$$

if Λz ≠ c, then the first equation can be written as

$$H^2\frac{d^2\phi}{dz^2} - \mu^2\phi = 0 \qquad with \qquad \mu^2 = {L_d}^2(k^2+l^2)$$

μ is the horizontal wavenumber scaled by the Rossby def. radius

and solution is

$$\phi(z) = A\cosh\mu\frac{z}{H} + B\sinh\mu\frac{z}{H}$$

and the boundaries

$$A\left[\Lambda H\right]+B\left[\mu c\right]=0$$

$$A\left[\left(c-\Lambda H\right)\mu\sinh\mu+\Lambda H\cosh\mu\right]+B\left[\left(c-\Lambda H\right)\mu\cosh\mu+\Lambda H\sinh\mu\right]=0$$

with A and B unknown. Non-trivial solutions exist if the determinant of the coefficients vanishes $\longrightarrow$

$$c^2-Uc+U^2\left(\mu^{-1}\coth\mu-\mu^{-2}\right)=0$$

Solution (finally) $c=\frac{U}{2}\pm\frac{U}{\mu}\left[\left(\frac{\mu}{2}-\coth\frac{\mu}{2}\right)\left(\frac{\mu}{2}-\tanh\frac{\mu}{2}\right)\right]^{1/2}$

It will grow if e^{-ikc} is positive, i.e. must exist the imaginary part of c $\longrightarrow$ $\left[\left(\frac{\mu}{2}-\coth\frac{\mu}{2}\right)\left(\frac{\mu}{2}-\tanh\frac{\mu}{2}\right)\right]^{1/2}$ must be negative

$\longrightarrow$ $\frac{\mu}{2}<\coth\frac{\mu}{2}$ $\Rightarrow$ $\mu<\mu_c=2.399$

The growth rate is given by kc_i - or ω_i if had chosen the notation $e^{i(kx-\omega t)}$ - and the 'natural' scales for non-dimensionalization are L_d, H and L_d/U $\longrightarrow$

$$kc_i = k\frac{U}{\mu}\left[\left(\coth\frac{\mu}{2}-\frac{\mu}{2}\right)\left(\frac{\mu}{2}-\tanh\frac{\mu}{2}\right)\right]^{1/2}$$

which is maximum for each given k when l=0 and $\mu=kL_d$.

The instability has a critical μ_c value above which the flow is stable $\longrightarrow$ there are a critical k and wavelength associated

$$k < k_c = \frac{\mu_c}{L_d}, \qquad \lambda > \lambda_c = \frac{2\pi L_d}{\mu_c} = 2.6L_d$$

It is also possible to find values for k and λ for which the growth rate is maximum.

Finally, given c one can use the BC to determine the vertical structure

Two-layer baroclinic instability

(see also Samelson and Pedlosky (JFM 1990) discussed in class. The derivation is the same)

Start from the vorticity equations

$$\frac{D}{Dt}\left[\varsigma_1 + \beta y + \frac{k_d^{\ 2}}{2}(\psi_2 - \psi_1)\right] = 0$$

$$\frac{D}{Dt}\left[\varsigma_2 + \beta y + \frac{k_d^{\ 2}}{2}(\psi_1 - \psi_2)\right] = 0$$

where $k_d = \dfrac{\sqrt{2^3}}{L_d}$

Assumptions: constant depth H and no topography $\longrightarrow$

The eq. are invariant under Galilean transformations (we are interested in translation), and without loss of generality we can set the basic state to

$$\psi_1 = -U_1 y, \qquad \psi_2 = -U_2 y = U_1 y$$

or $$Q_1 = \beta y + k_d{}^2 U_1 y, \qquad Q_2 = \beta y - k_d{}^2 U_1 y$$

The linearized vorticity equations are

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\nabla^2\psi_1{}' + \frac{k_d{}^2}{2}(\psi'_2 - \psi'_1)\right] + \frac{\partial \psi_1{}'}{\partial x}\left(\beta + k_d{}^2 U\right) = 0$$

$$\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)\left[\nabla^2\psi_2{}' + \frac{k_d{}^2}{2}(\psi'_1 - \psi'_2)\right] + \frac{\partial \psi_2{}'}{\partial x}\left(\beta - k_d{}^2 U\right) = 0$$

Doubly periodic domain (so that we can separate x and y variables and have a simple expression for the y component) $\longrightarrow$

$$\psi_1' = \mathrm{Re}\,\tilde{\psi}_1 e^{ik(x-ct)} e^{ily}, \qquad \psi_2' = \mathrm{Re}\,\tilde{\psi}_2 e^{ik(x-ct)} e^{ily}$$

substitute in the pv equations and the outcome are two equations in the form

$$[A]\tilde{\psi}_1 + [B]\tilde{\psi}_2 = 0 \qquad [C]\tilde{\psi}_1 + [D]\tilde{\psi}_2 = 0$$

and precisely

$$\left[(U-c)\left(k_d^{\,2}/2+K^2\right)-\left(\beta+k_d^{\,2}U\right)\right]\tilde{\psi}_1-\left[k_d^{\,2}(U-c)/2\right]\tilde{\psi}_1=0$$

$$-\left[k_d^{\,2}(U+c)/2\right]\tilde{\psi}_2+\left[(U+c)\left(k_d^{\,2}/2+K^2\right)+\left(\beta-k_d^{\,2}U\right)\right]\tilde{\psi}_2$$

which has solution

$$\boxed{c = -\frac{\beta}{K^2 + k_d^{\ 2}}\left\{1 + \frac{k_d^{\ 2}}{2K^2} \pm \frac{k_d^{\ 2}}{2K^2}\left[1 + \frac{4K^4\left(K^4 - k_d^{\ 4}\right)}{k_\beta^{\ 4} k_d^{\ 4}}\right]^{1/2}\right\}}$$

where $K^2=(k^2+l^2)$ and $k_\beta = \sqrt{\beta / U}$

Special (easier) cases:

1. zero shear, non zero β
2. zero β, non zero shear

General solution: for instability to occur the phase speed c must have an imaginary component, and therefore

$$\longrightarrow \quad {k_b}^4 {k_d}^4 + 4K^4 (K^4 - {k_d}^4) < 0$$

which implies that the critical wavenumber is

$${K_c}^4 = \frac{1}{2} {k_d}^4 \left(1 \pm \sqrt{1 - {k_\beta}^4 / {k_d}^4}\right)$$

(see fig in next slide)

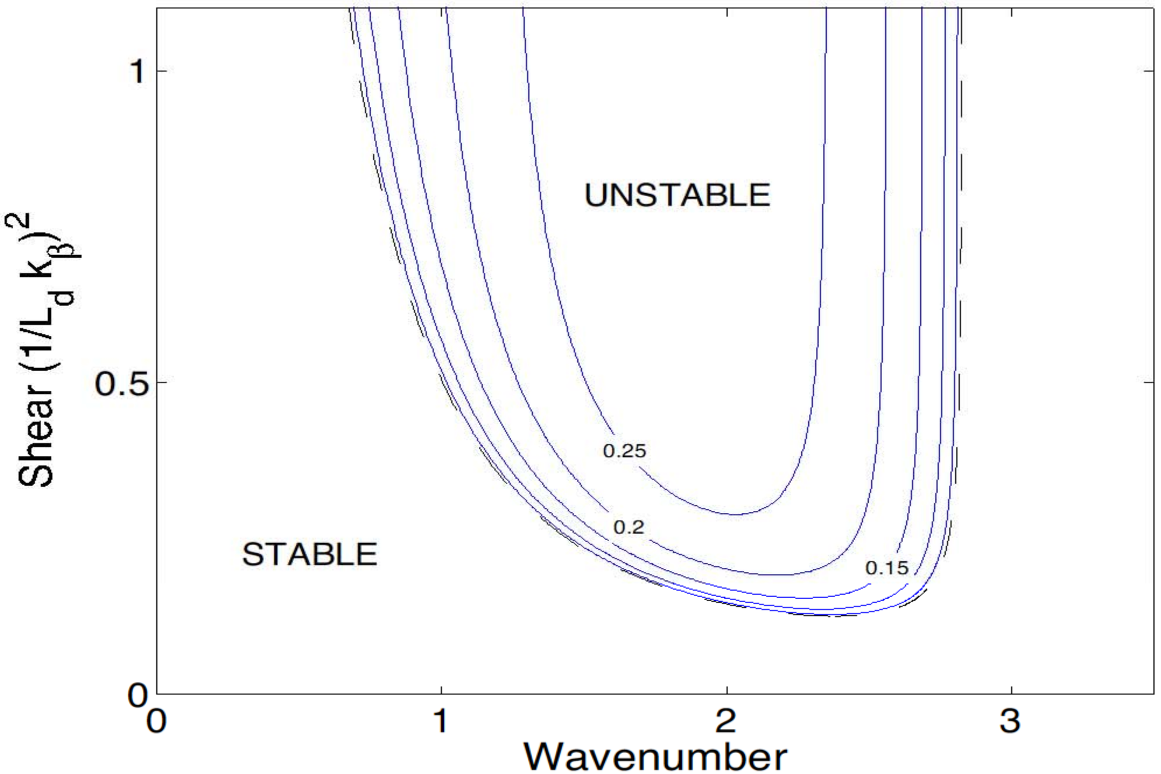


Fig. 6.15 Contours of growth rate in the two-layer baroclinic instability problem. The dashed line is the neutral stability curve obtained from (6.120), and the other curves are contours of growth rates obtained from (6.114). Outside of the dashed line, the flow is stable. The wavenumber is scaled by $1/L_d$ (i.e., by $k_d/\sqrt{8}$) and growth rates are scaled by the inverse of the Eady timescale (i.e., by U/L_d). Thus, for $L_d = 1000\,\mathrm{km}$ and $U = 10\,\mathrm{m\,s^{-1}}$, a nondimensional growth rate of 0.25 corresponds to a dimensional growth rate of $0.25 \times 10^{-5}\,\mathrm{s^{-1}} = 0.216\,\mathrm{day^{-1}}$.

more informally

consider the baroclinic and barotropic streamfunction

$$\tau = \frac{1}{2}\left(\psi_1 - \psi_2\right), \qquad \psi = \frac{1}{2}\left(\psi_1 + \psi_2\right)$$

β=0 and barotropic shear =0 to simplify the algebra $\longrightarrow$

$$\psi = 0 + \psi', \qquad \tau = -Uy + \tau'$$

linearize the eq. around the state above and obtain

$$\frac{\partial}{\partial t}\nabla^2\psi' = -U\frac{\partial}{\partial x}\nabla^2\tau'$$

$$\frac{\partial}{\partial t}\left(\nabla^2 - k_d{}^2\right)\tau' = -U\frac{\partial}{\partial x}\left(\nabla^2 + k_d{}^2\right)\psi'$$

solutions, as usual, in the form $\psi' = \mathrm{Re}\,\tilde{\psi} e^{ik(x-ct)}$

(same for τ')

simpler system:

$$c\tilde{\psi} - U\tilde{\tau} = 0$$

$$c\left(K^2 + k_d^{\,2}\right)\tilde{\tau} - U\left(K^2 - k_d^{\,2}\right)\tilde{\psi} = 0$$

solutions:

$$\boxed{c = \pm U \left(\frac{K^2 - k_d^{\,2}}{K^2 + k_d^{\,2}} \right)^{1/2}}$$

the flow is unstable if $K^2 < k_d^2$ and the wave speed has only imaginary component.

for unstable modes substituting in the eq. before

$$\tilde{\tau} = i\frac{c_i}{U}\tilde{\psi} = e^{i\pi/2}\frac{c_i}{U}\tilde{\psi}$$

the baroclinic streamfunction τ lags by 90° the barotropic one for a growing mode (c_i >0) and leads by 90° for a decaying one (c_i<0). τ is proportional to temperature in real flows and T is advected by the meridional velocity v=∂ψ/∂x and in Fourier components

$$\tilde{v} = \tilde{\tau}\frac{kU}{c_i}$$

in phase with temperature for growing modes, out of phase for decaying modes so for unstable modes positive velocities (northward) transport heat (high temperatures), while decaying modes transport low temperatures towards the pole (or heat equatorwards). Neutral waves do not transport heat.