ESCI 341 – Atmospheric Thermodynamics
Lesson 1 – Math Review

Partial Derivatives and Differentials

- The differential of a function of two variables, $f(x, y)$, is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{1}$$

  - Eq. (1) is true regardless of whether *x* and *y* are independent, or if they are both composite functions depending on a third variable, such as *t*.
- The terms like $\partial f/\partial x$ and $\partial f/\partial y$ are called partial derivatives, because they are taken assuming that all other variables besides that in the denominator are constant.
  - For example, $\partial f/\partial x$ describes how *f* changes as *x* changes (holding *y* constant), and $\partial f/\partial y$ describes how *f* changes as *y* changes (holding *x* constant).
- If *f* is a function of three variables, *x*, *y*, and *z*, then the differential of *f* is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\,. \tag{2}$$

- We often write the partial derivatives with subscripts indicating which variables are held constant,

$$df = \left(\frac{\partial f}{\partial x}\right)_{y,z} dx + \left(\frac{\partial f}{\partial y}\right)_{x,z} dy + \left(\frac{\partial f}{\partial z}\right)_{x,y} dz\,,$$

  though it is not absolutely necessary to do so.
- That partial and full derivatives are different can be illustrated by dividing Eq. (1) by the differential of *x* to get

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \tag{3}$$

- From Eq. (3) we see that the full derivative and the partial derivative are equivalent only if *x* and *y* are independent, so that dy/dx is zero.
- ***WARNING!*** Partial derivatives are not like fractions. The numerators and denominators cannot be pulled apart or separated arbitrarily. Partial derivatives must be treated as a complete entity. So, you should **NEVER** pull them apart as shown below

$$\frac{\partial f}{\partial t} = axt^2 \quad \Rightarrow \quad \partial f = axt^2 \partial t\,.$$ ***NEVER DO THIS!***

With a full derivative this is permissible, because is it composed of the ratio of two differentials. But there is no such thing as a *partial differential*, ∂f .

THE CHAIN RULE

- If x and y are not independent, but depend on a third variable such as s [i.e., $x(s)$ and $y(s)$], then the chain rule is

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}. \tag{4}$$

- If x and y depend on multiple variables such as s and t [i.e., $x(s,t)$ and $y(s,t)$], then the chain rule is

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\end{aligned}. \tag{5}$$

THE PRODUCT RULE AND THE QUOTIENT RULE

- The product and quotient rules also apply to partial derivatives:
  - o The *product rule*

$$\frac{\partial}{\partial x}(uv) \equiv u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}. \tag{6}$$

  - o The *quotient rule*

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) \equiv \frac{1}{v^2}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right). \tag{7}$$

PARTIAL DIFFERENTIATION IS COMMUTATIVE

- Another important property of partial derivatives is that it doesn't matter in which order you take them. In other words

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial^2 f}{\partial y \partial x}.$$

- Multiple partial derivatives taken with respect to different variables are known as *mixed* partial derivative.

OTHER IMPORTANT IDENTITIES

- The reciprocals of partial derivatives are:

$$\left(\frac{\partial f}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial f}\right)_y} \quad ; \quad \left(\frac{\partial f}{\partial y}\right)_x = \frac{1}{\left(\frac{\partial y}{\partial f}\right)_x}$$

- If a function of two variables is constant, such as $f(x, y) = c$, then its differential is equal to zero,

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy = 0 . \tag{8}$$

  - In this case, *x* and *y* must be dependent on each other, because in order for *f* to be a constant, as *x* change *y* must also change. For example, think of the function

$$f(x, y) = x^2 + y = c . \tag{9}$$

  - Eq. (8) can be rearranged to

$$\left(\frac{\partial f}{\partial x}\right)_y \frac{dx}{dy} + \left(\frac{\partial f}{\partial y}\right)_x = 0 . \tag{10}$$

    The derivative dx/dy in Eq. (10) is actually a partial derivative with *f* held constant , so we can write

$$\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_f + \left(\frac{\partial f}{\partial y}\right)_x = 0 ,$$

    which when rearranged leads to the identity

$$\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial y}{\partial f}\right)_x \left(\frac{\partial x}{\partial y}\right)_f = -1 . \tag{11}$$

  - Eq. (11) is only true if the function *f* is constant, so that $df = 0$.

INTEGRATION OF PARTIAL DERIVATIVES

- Integration is the opposite or inverse operation of differentiation.

$$
\begin{aligned}
\int_a^b \frac{\partial f(s,t)}{\partial s} ds &= f(b,t) - f(a,t) \\
\int_a^b \frac{\partial f(s,t)}{\partial t} dt &= f(s,b) - f(s,a)
\end{aligned} \tag{12}
$$

DIFFERENTIATING AN INTEGRAL

- If an integration with respect to one variable is then differentiated with respect to a separate variable, such as

$$\frac{\partial}{\partial t}\int_a^b f(s,t,u)ds$$

  the result depends on whether or not the limits of integration, *a* and *b*, depend on *t*.

- In general, if both *a* and *b*, depend on *t*, the result is

$$\frac{\partial}{\partial t}\int_{a(t,u)}^{b(t,u)} f(s,t,u)ds = \int_{a(t,u)}^{b(t,u)} \frac{\partial f(s,t,u)}{\partial t} ds + f(b,t,u)\frac{\partial b}{\partial t} - f(a,t,u)\frac{\partial a}{\partial t}\,. \tag{13}$$

  - If *a* does not depend on *t* then the term in Eq. (13) that involves $\partial a/\partial t$ will disappear. Likewise, if *b* does not depend on *t*, then the term containing $\partial b/\partial t$ will be zero.

ESCI 342 – Atmospheric Dynamics I
Lesson 1 – Vectors and Vector Calculus

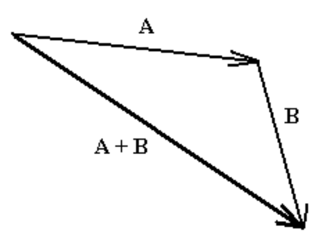
Reference: *Schaum's Outline Series: Mathematical Handbook of Formulas and Tables*
Suggested Reading: Martin, Section 1.2

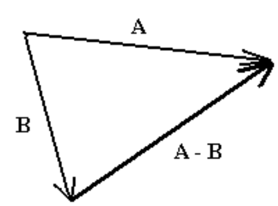
COORDINATE SYSTEMS

- An ***orthonormal*** coordinate system is one in which the basis vectors are mutually perpendicular and are of unit length.
- ***Cartesian*** coordinates are an orthonormal coordinate system where the coordinate lines are not curved.
- We will be using orthonormal coordinates in this class (either Cartesian or spherical coordinates).
- The unit basis vectors for our coordinate system are $\hat{i}$, $\hat{j}$, and $\hat{k}$, which point toward the East, North, and up, respectively.

VECTORS

- Vectors have both a magnitude (length) and a direction.
- Vectors are denoted by either writing them in boldface (**A**), or by placing an arrow over the top ($\vec{A}$).
- The magnitude of a vector is denoted either by A or $\left|\vec{A}\right|$.
- Vectors are added by placing them head to tail.





- Vector addition is
  - commutative: $\vec{A}+\vec{B}=\vec{B}+\vec{A}$
  - associative: $\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$
- Vectors can be multiplied by scalars. Scalar multiplication is
  - associative: $m(n\vec{A}) = n(m\vec{A}) = mn\vec{A}$
  - distributive: $(m+n)\vec{A}=m\vec{A}+n\vec{A}$; $m(\vec{A} + \vec{B}) = m\vec{A}+m\vec{B}$

COMPONENTS

- A vector can be written in terms of components along the coordinate system axes.

$$\vec{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

- $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the *unit vectors* (magnitude = 1) along the *x*, *y*, and *z* axes respectively.
  - The components of a vector may have negative values. For example, a velocity vector given by $\vec{V} = -2\,\mathrm{m\,s^{-1}}\,\hat{i} + 3\mathrm{m\,s^{-1}}\,\hat{j}$ has component vectors
$$\begin{aligned}\vec{V}_x &= -2\,\mathrm{m\,s^{-1}}\,\hat{i} \\ \vec{V}_y &= 3\mathrm{m\,s^{-1}}\,\hat{j}\end{aligned}.$$
  - We would think of $\vec{V}_x$ as having a magnitude of 2 m s−1 and a direction opposite to $\hat{i}$, while $\vec{V}_y$ has a magnitude of 3 m s−1 and in the direction of $\hat{j}$.
  - Another example is gravity, $\vec{g}$, which in component form is
$$\vec{g} = -g\hat{k}.$$
    We say that the magnitude of gravity is *g* and its direction is opposite to $\hat{k}$.
- In component form, vector addition is accomplished by adding the components.
$$\vec{A} + \vec{B} = (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j} + (a_z + b_z)\hat{k}$$
- In component form, multiplication by a scalar is
$$m\vec{A} = ma_x\hat{i} + ma_y\hat{j} + ma_z\hat{k}.$$
- The magnitude of a vector is found from its Cartesian components (axes are normal to one another) using the Pythagorean formula
$$A = \sqrt{a_x^2 + a_y^2 + a_z^2}.$$

DOT PRODUCT

- The dot (or scalar) product of two vectors is defined as $\vec{A} \bullet \vec{B} = AB\cos\theta$, where θ is the angle between the two vectors.
- ***The result of the dot product is a scalar, not a vector!***
- In component form the dot product is
$$\vec{A} \bullet \vec{B} = a_x b_x + a_y b_y + a_z b_z.$$
- If two vectors are normal, their dot product is zero.
- The dot product is
  - commutative: $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$
  - distributive: $\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$

CROSS PRODUCT

- The cross product is defined as $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{u}$, where θ is the angle between the two vectors and $\hat{u}$ is the unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ in the direction consistent with the right-hand rule.
  - Note: In European texts the cross product is often denoted using a “^” rather than a “×”.

- ***The result of the cross product is a vector!***
- If two vectors are parallel, their cross product is zero.
- In orthonormal coordinates, the component form the cross product is found by finding the determinant of a matrix whose first row is the unit vectors along the axes, and the second and third rows are the components of the vectors.

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=\left(a_y b_z-b_y a_z\right)\hat{\imath}-\left(a_x b_z-b_x a_z\right)\hat{\jmath}+\left(a_x b_y-b_x a_y\right)\hat{k}\,.$$

- ***The cross product is <u>not</u> commutative:*** $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$
- The cross product is distributive: $\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$

DERIVATIVES OF VECTORS

- A vector function is a vector whose magnitude and direction depends upon other scalars (for example, time).
- The derivative of a vector function is written in component form as

$$\frac{d\vec{A}}{ds}=\frac{da_x}{ds}\hat{\imath}+\frac{da_y}{ds}\hat{\jmath}+\frac{da_z}{ds}\hat{k}+a_x\frac{d\hat{\imath}}{ds}+a_y\frac{d\hat{\jmath}}{ds}+a_z\frac{d\hat{k}}{ds}\,\boldsymbol{.}$$

- The rules for differentiating dot and cross products are analogous to the product rule for scalar differentiation

$$\frac{d}{ds}\left(\vec{A}\bullet\vec{B}\right)=\vec{A}\bullet\frac{d\vec{B}}{ds}+\frac{d\overline{A}}{ds}\bullet\vec{B}$$

$$\frac{d}{ds}\left(\vec{A}\times\vec{B}\right)=\vec{A}\times\frac{d\vec{B}}{ds}+\frac{d\vec{A}}{ds}\times\vec{B}$$

THE GRADIENT

- The *del* operator in Cartesian coordinates is defined as $\nabla\equiv\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$.
- The del operator applied to a scalar yields a vector that points in the direction of steepest ascent (i.e., a vector that is normal to the contours and pointing toward higher values).

$$\nabla a\equiv\hat{\imath}\frac{\partial a}{\partial x}+\hat{\jmath}\frac{\partial a}{\partial y}+\hat{k}\frac{\partial a}{\partial z}\,.$$

  - When applying the del operator to a scalar field, the order of the unit vectors and the partial derivatives doesn't matter. This is why you will often see

$$\nabla a\equiv\frac{\partial a}{\partial x}\hat{\imath}+\frac{\partial a}{\partial y}\hat{\jmath}+\frac{\partial a}{\partial z}\hat{k}\,.$$

- ∇a is called the *gradient* of a.

- ***Worth repeating: The gradient is a vector that is normal to the contours and points toward higher values!***
- If the scalar is uniform in space (i.e., has the same value everywhere) then the gradient is zero.

- The del operator can also be applied to a vector (contrary to what some textbooks state), the result of which is a second-order tensor. When applying the del operator to a vector, it is important to write the unit coordinate vectors before each term rather than after it,

$$\nabla\vec{A} \equiv \hat{i}\frac{\partial\vec{A}}{\partial x} + \hat{j}\frac{\partial\vec{A}}{\partial y} + \hat{k}\frac{\partial\vec{A}}{\partial z}.$$

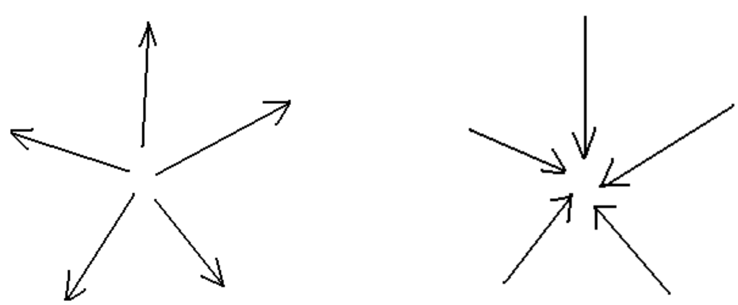
  This is because the order in which vectors are directly multiplied is not commutative (i.e., $\hat{i}\,\hat{j} \neq \hat{j}\hat{i}$).

DIVERGENCE

- The divergence of a vector field is defined as $\nabla \bullet \vec{A}$.
  - In Cartesian coordinates $\nabla \bullet \vec{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$
- ***The divergence is a scalar!***
- When meteorologists speak of divergence, they are referring to the divergence of the velocity vector ($\vec{V} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k}$), and so we usually see divergence written as

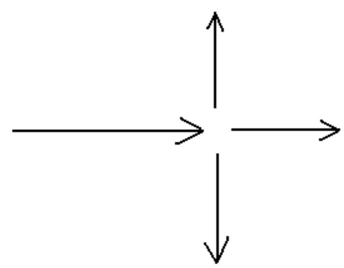
$$\nabla \bullet \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- Negative divergence is called convergence.
  - If $\nabla \bullet \vec{V} > 0$ there is *divergence*.
  - If $\nabla \bullet \vec{V} < 0$ there is *convergence*.
- The physical meaning of divergence can be illustrated as follows. If the vector field is pointing away from a point, the divergence at that point is positive. If the vector field is pointing into a point, the divergence at that point is negative.



- Direction alone cannot always be used to determine divergence or convergence. The vectors may be pointing in the same direction, and yet have divergence or convergence (see illustrations below).

- In many cases you cannot tell just by looking whether there is divergence or convergence. For example, the illustration below shows a case where you would have to perform the calculations to determine the divergence, since it is not obvious by examining the field.



CURL

- The *curl* of a vector field is defined as $\nabla \times \vec{A}$.
- The curl is a vector whose components are found by finding the cross product of the del operator with the vector. Iin Cartesian coordinates, the component form the curl is

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) \hat{i} - \left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z} \right) \hat{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) \hat{k} .$$

- The curl of the velocity vector, $\nabla \times \vec{V}$, is called *vorticity*.

THE LAPLACIAN

- The Laplacian operator is defined as $\nabla^2 \equiv \nabla \bullet \nabla$.
  - o In Cartesian coordinates, $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$
- For a scalar in Cartesian coordinates the Laplacian is $\nabla^2 a \equiv \dfrac{\partial^2 a}{\partial x^2} + \dfrac{\partial^2 a}{\partial y^2} + \dfrac{\partial^2 a}{\partial z^2}$.
- For a vector in Cartesian coordinates the Laplacian is $\nabla^2 \vec{A} \equiv \dfrac{\partial^2 \vec{A}}{\partial x^2} + \dfrac{\partial^2 \vec{A}}{\partial y^2} + \dfrac{\partial^2 \vec{A}}{\partial z^2}$.

THE DEL OPERATOR IS LINEAR

- $\nabla(m+n) = \nabla m + \nabla n$
- $\nabla \bullet \left(\vec{A}+\vec{B}\right) = \nabla \bullet \vec{A} + \nabla \bullet \vec{B}$
- $\nabla \times \left(\vec{A}+\vec{B}\right) = \nabla \times \vec{A} + \nabla \times \vec{B}$

SPHERICAL COORDINATES

- On the Earth it makes sense to use spherical coordinates rather than Cartesian coordinates.
- In spherical coordinates for the Earth, the position of a point is given by its distance from the center of the Earth, r; the latitude, ϕ; and the longitude, λ.
    - o The unit vectors in spherical coordinates are the same as in Cartesian coordinates, with $\hat{i}$, $\hat{j}$, and $\hat{k}$ pointing toward the East, North, and up, respectively.
- The relationships between the coordinates (x, y, z) and (r, ϕ, λ) are

$$\begin{aligned} dx &= r\cos\phi d\lambda \\ dy &= r d\phi \\ dz &= dr \end{aligned} \tag{1}$$

- The del operator in Cartesian coordinates is

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \tag{2}$$

- The del operator expressed in spherical coordinates is derived from that for Cartesian coordinates as follows:

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \hat{i}\frac{\partial}{\partial \lambda}\frac{d\lambda}{dx} + \hat{j}\frac{\partial}{\partial \phi}\frac{d\phi}{dy} + \hat{k}\frac{\partial}{\partial r}\frac{dr}{dz}$$

which using the relationships in (1) gives

$$\nabla = \hat{i}\left(\frac{1}{r\cos\phi}\frac{\partial}{\partial \lambda}\right) + \hat{j}\left(\frac{1}{r}\frac{\partial}{\partial \phi}\right) + \hat{k}\frac{\partial}{\partial r}. \tag{3}$$

- The choice of which coordinate system to use (Cartesian or spherical) is strictly up to us. They both have advantages and disadvantages. Spherical coordinates are attractive because the Earth is spherical. However, the del operator is more cumbersome in spherical coordinates, and also, giving position in (r, ϕ, λ) is more cumbersome and less intuitive than using (x, y, z). Let's face it – Cartesian coordinates are simpler and easier to use and think about! Plus, locally the Earth appears flat to us.
- Fortunately, we can still use spherical coordinates, but express position in terms of x, y, and z rather than r, ϕ, λ. This is going to make our equations look very similar to Cartesian coordinates, although it will introduce some additional terms, called

curvature terms, into some of our equations. We will discuss these terms as they arise in our studies.

- When we use spherical coordinates, but use x, y, and z in place of λ, ϕ, and r, we will call this ***parameterized spherical coordinates***.

EXERICISES

Three vectors are given by

$$\vec{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$
$$\vec{B} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$$
$$\vec{C} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k}$$

1. Show that each of the following is true:

   a. $\vec{A} \bullet \left(\vec{B} \times \vec{C}\right) = \left(\vec{A} \times \vec{B}\right) \bullet \vec{C}$

   b. $\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(\vec{A} \bullet \vec{C}\right)\vec{B} - \left(\vec{A} \bullet \vec{B}\right)\vec{C}$

   c. $\left(\vec{A} \times \vec{B}\right) \times \vec{C} = \left(\vec{A} \bullet \vec{C}\right)\vec{B} - \left(\vec{B} \bullet \vec{C}\right)\vec{A}$

2. Show that $\vec{A} \bullet \dfrac{d\vec{A}}{dt} = A\dfrac{dA}{dt}$ (remember that A is the magnitude of $\vec{A}$)

3. A vector is a function of time given by $\vec{A}(t) = 2t\,\hat{i} - 3t^3\,\hat{j} + 5\ln t\,\hat{k}$. Find $\dfrac{d\vec{A}}{dt}$.

4. For the following scalar fields find the magnitude of the gradient at the point indicated.

   a. $a(x, y, z) = 2x^3y^2 - xz - z\ln y$; $(x, y, z) = (4, 2, 2)$
   b. $a(x, y, z) = \cos x \sin y - z$; $(x, y, z) = (0, \pi, 1)$
   c. $a(x, y) = x^2 + y^2 - 16$; $(x, y) = (2, -2)$

5. For the following vector fields find the divergence at the point indicated.

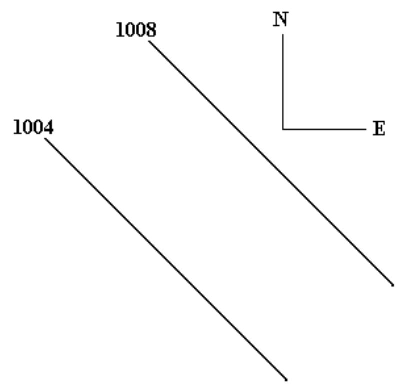
   a. $\vec{A}(x, y, z) = 3xy^2\hat{i} - xy^2z\,\hat{j} - 4x^2\ln y\,\hat{k}$; $(x, y, z) = (4, 2, 2)$
   b. $\vec{A}(x, y, z) = \cos x \sin y\,\hat{i} - z\,\hat{j}$; $(x, y, z) = (0, \pi, 1)$
   c. $\vec{A}(x, y) = \left(x^2 + y^2\right)\hat{i} + \left(x^2 + y^2\right)\hat{j}$; $(x, y) = (2, 1)$

6. The geostrophic wind is a vector field given by $\vec{V}_g = u_g \hat{i} + v_g \hat{j}$ where $u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$ and $v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}$.

   a. Show that if f and ρ are constant then the divergence of the geostrophic wind ($\nabla \bullet \vec{V}_g$) is zero.

   b. In reality, f is not constant, but instead increases as you go north. Show that in this case the divergence of the geostrophic wind is given by $\nabla \bullet \vec{V}_g = -v_g \frac{\partial(\ln f)}{\partial y}$.

   c. For the following isobar pattern, will the geostrophic wind be convergent or divergent?



7. For the following vector fields find the curl at the point indicated.

   a. $\vec{A}(x, y, z) = 3xy^2\hat{i} - xy^2z\ \hat{j} - 4x^2 \ln y\ \hat{k}$; $(x, y, z) = (4, 2, 2)$
   b. $\vec{A}(x, y, z) = \cos x \sin y\ \hat{i} - z\hat{j}$; $(x, y, z) = (0, \pi, 1)$
   c. $\vec{A}(x, y) = (x^2 + y^2)\hat{i} + (x^2 + y^2)\hat{j}$; $(x, y) = (2, 1)$

8. For the following scalar fields find the Laplacian at the point indicated.

   a. $a(x, y, z) = 2x^3y^2 - xz - z\ln y$; $(x, y, z) = (4, 2, 2)$
   b. $a(x, y, z) = \cos x \sin y - z$; $(x, y, z) = (0, \pi, 1)$
   c. $a(x, y) = x^2 + y^2 - 16$; $(x, y) = (2, -2)$

9. Show that the following identities involving the del operator are true. Do this by writing the operators and vectors in Cartesian coordinates and showing that the relationships hold. Because these identities are written in coordinate-free notation, if you prove them to be true in Cartesian coordinates, they are true in all coordinates.

$$\nabla \bullet \left(s\vec{A}\right) = \nabla s \bullet \vec{A} + s\left(\nabla \bullet \vec{A}\right)$$

$$\nabla \times \left(s\vec{A}\right) = \nabla s \times \vec{A} + s\left(\nabla \times \vec{A}\right)$$

$$\nabla \bullet \left(\vec{A} \times \vec{B}\right) = \vec{B} \bullet \left(\nabla \times \vec{A}\right) - \vec{A} \bullet \left(\nabla \times \vec{B}\right)$$

$$\nabla \times \left(\vec{A} \times \vec{B}\right) = \left(\vec{B} \bullet \nabla\right)\vec{A} - B\left(\nabla \bullet \vec{A}\right) - \left(\vec{A} \bullet \nabla\right)\vec{B} + \vec{A}\left(\nabla \bullet \vec{B}\right)$$

$$\nabla\left(\vec{A} \bullet \vec{B}\right) = \left(\vec{B} \bullet \nabla\right)\vec{A} + \left(\vec{A} \bullet \nabla\right)\vec{B} + \vec{B} \times \left(\nabla \times \vec{A}\right) + \vec{A} \times \left(\nabla \times \vec{B}\right)$$

$$\nabla \times \nabla s = 0$$

$$\nabla \bullet \left(\nabla \times \vec{A}\right) = 0$$

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \bullet \vec{A}\right) - \nabla^2 \vec{A}$$

ESCI 342 – Atmospheric Dynamics I
Lesson 2 – Fundamental Forces I

Suggested Reading: Martin, Section 2.1

UNITS

- A number is meaningless unless it is accompanied by a unit telling what the number represents.
- The standard unit system used internationally by scientists is known as the SI unit system. The basic units needed for a system of units are length, mass, and time. In the SI system, these are the meter (m), kilogram (kg), and second (s). Nearly every other unit can be derived from these three basic units. The SI unit system is sometimes referred to as the *m-k-s unit system* (as opposed to the c-g-s system, which uses centimeters, grams, and seconds as the basic units).
- Important units to remember are:

Phenomenon	Unit name	Basic units	Alternate units (non-SI)
Force	Newton (N)	kg m s^{-2}	dyne; pound
Energy	Joule (J)	N m	erg; foot-lb; calorie
Power	Watt (W)	J s^{-1}	Horsepower
Pressure	Pascal (Pa)	N m^{-2}	lb-in^{-2}; bar; torr; atmosphere; in-Hg
Temperature	Kelvin (K)	none	Celcius; Fahrenheit

- Prefixes for units:

Multiplier	Name	Abb.
10^9	giga	G
10^6	mega	M
10^3	kilo	k
10^2	hecta	h
10^1	deka	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n

- Though internationally meteorologists adhere to SI units, in the U.S. we continue to use some traditional units that differ from SI units. Some of these are
  - Pressure: millibar (mb) = 100 Pa = hecta-Pascal (hPa)
    atmosphere (atm) = 101325 Pa = 1013.25 mb
    inches of mercury (in-Hg) – 29.92 in-Hg = 1013.25 mb = 1 atm
  - Temperature: Celcius (°C) = K – 273.15
    Fahrenheit (°F) = (9/5)°C + 32
  - Length: statute mile (mi) = 1.16 km = 1760 yds
    nautical mile (M) = 1.1 mi = 2000 yds

- Speed: Knot (kt) = nautical mile per hour = 1.14 mph ≈ 2 × m-s^{-1}
- Energy: Calorie (cal) = 4.184 J

COORDINATES AND VELOCITY

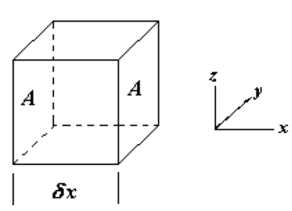
- In meteorology we use the following coordinate system:
  - The x-coordinate increases eastward
  - The y-coordinate increases northward
  - The z-coordinate increases upward
- The velocity components along each coordinate direction are defined as
  - $u \equiv dx/dt$; u is the speed in the eastward direction (*zonal* velocity)
  - $v \equiv dy/dt$; v is the speed in the northward direction (*meridional* velocity)
  - $w \equiv dz/dt$; w is the speed in the upward direction (*vertical* velocity)

FORCES

- The atmosphere obeys all the laws of physics, including Newton's second law of motion, $\vec{F} = m\vec{a}$.
- The forces acting on an air parcel are of one of two types:
  - *Body forces* – act on the center of mass, and are proportional to mass.
  - *Surface forces* – act on the surface of the parcel, and are independent of mass.
- ***In meteorology we often are using not the force, but the force per unit mass, F/m,*** which is really the acceleration.
- Meteorologists are often sloppy about whether they are talking about forces or accelerations.
  - Therefore, we often refer to the pressure gradient force when we really mean the pressure gradient acceleration.

THE PRESSURE GRADIENT FORCE

- The pressure gradient force is a surface force acting on the fluid parcel.
- For ease of derivation we often visualize the air parcel as being a cube-like shape with dimensions δx, δy, and δz.
- The pressure gradient force in the x-direction is derived as follows:



  - On the left side of the cube shown above, there will be force due to the pressure of $p_x A$ where p_x is the pressure on the left face, and A is the area.
  - On the right side of the cube the force due to the pressure is $-p_{x+\delta x}A$. Therefore, Newton's second law in the x-direction is

$$ma_{PGF\,x} = -\left(p_{x+\delta x} - p_x\right)A\,.$$

- o If the density of the fluid is ρ then the mass is $m = \rho A\delta x$, and Newton's law is written as

$$\left(\rho A\delta x\right)a_{PGF\,x} = -\left(p_{x+\delta x} - p_x\right)A$$

which after some rearranging becomes

$$a_{PGF\,x} = -\frac{1}{\rho}\frac{\left(p_{x+\delta x} - p_x\right)}{\delta x}.$$

- o In the limit as the volume becomes infinitesimally small, we have

$$a_{PGF\,x} = -\frac{1}{\rho}\lim_{\delta x \to 0}\frac{\left(p_{x+\delta x} - p_x\right)}{\delta x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

- The derivations for the pressure gradient force in the *y*- and *z*-directions proceed the same way. Therefore, the pressure gradient force (PGF, and more appropriately called the acceleration due to the pressure gradient force) is a vector having the components

$$\vec{a}_{PGF} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\hat{i} - \frac{1}{\rho}\frac{\partial p}{\partial y}\hat{j} - \frac{1}{\rho}\frac{\partial p}{\partial z}\hat{k}.$$

- Since $\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k} = \nabla p$, the pressure gradient force can also be written as

$$\vec{a}_{PGF} = -\frac{1}{\rho}\nabla p. \qquad \textbf{(1)}$$

  - o NOTE: Even though we derived (1) using Cartesian coordinates, this expression is valid in any coordinate system. It is what is called *geometric invariant*. It is only when it is expanded into components that the components will be different in different coordinate system.
- The pressure gradient force has the following properties:
  - o It always is directed in the opposite direction of the pressure gradient, ∇p.
  - o The stronger the pressure gradient, the stronger the pressure gradient force.
- The pressure gradient force can be estimated from maps of the isobars, as long as the distance between adjacent isobars is known, using the following approximation

$$\left|\nabla p\right| \cong \frac{\Delta p}{\Delta n}$$

where ∇p is the contour interval for the isobars and ∇n is the horizontal distance between the isobars.

THE GRAVITATIONAL FORCE

- The gravitational force between two objects of masses *m* and *M* is given by Newton's law of gravitation,

$$\vec{F} = -\frac{GmM}{r^2}\hat{r}$$

where *G* is the universal gravitational constant (6.673×10^{-11} N m^2 kg^{-2}), and *r* is the distance between the centers of mass of the objects ($\hat{r}$ is the unit vector along a line connecting the centers of mass of the two objects).

- The acceleration due to the gravitational force at the surface of the Earth ($r = a =$ 6378 km) is

$$\vec{g}_0{}^* = -\frac{GM}{a^2}\hat{r}\ .$$

- At some altitude z above the surface of the Earth the acceleration due to the gravitational force is

$$\vec{g}^* = -\frac{GM}{(a+z)^2}\hat{r}$$

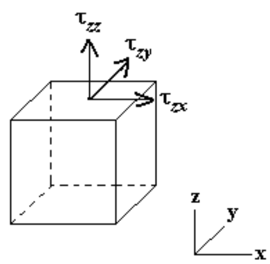
which can be written as

$$\vec{g}^* = \frac{\vec{g}_0{}^*}{(1+z/a)^2} \cong \vec{g}_0{}^* \quad \text{(as long as z << a).}$$

So, **<u>we can usually ignore changes in the gravitational acceleration with height</u>**.

VISCOUS FORCE

- Viscous force is due to friction caused by interactions of the molecules of a fluid.
- We will present the viscous force in Cartesian coordinates.
- Imagine a cubic fluid parcel in a non-uniform flow (see diagram below). There will be three forces along each face of the cube due to the frictional effects of the fluid flow around the cube.



- The force per unit area is called the stress (denoted by τ). The notation τ_{zx} denotes the stress acting on a face at z and directed along the x-axis.
- The acceleration in the x direction on the cube due to the frictional stresses on the upper and lower faces of the cube is given by[1]

$$a_{zx} = \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z}\ . \qquad \textbf{(2)}$$

- The accelerations due to the other components of the stress have a similar form. The following table summarizes the accelerations due to all the stresses.

acceleration due to stress in x-direction	$a_{xx} = \frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x}$	$a_{yx} = \frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial y}$	$a_{zx} = \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z}$
acceleration due to stress in y-direction	$a_{xy} = \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial x}$	$a_{yy} = \frac{1}{\rho}\frac{\partial \tau_{yy}}{\partial y}$	$a_{zy} = \frac{1}{\rho}\frac{\partial \tau_{zy}}{\partial z}$

[1] Equation (2) is presented without derivation. Details are beyond the scope of this course. Interested students may consult an advanced fluid dynamics text.

acceleration due to stress in *z*-direction	$a_{xz} = \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial x}$	$a_{yz} = \frac{1}{\rho}\frac{\partial \tau_{yz}}{\partial y}$	$a_{zz} = \frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z}$

- It turns out that the stresses themselves are dependent on the flow, and are given by

stresses in *x*-direction	$\tau_{xx} = \mu\frac{\partial u}{\partial x}$	$\tau_{yx} = \mu\frac{\partial u}{\partial y}$	$\tau_{zx} = \mu\frac{\partial u}{\partial z}$
stresses in *y*-direction	$\tau_{xy} = \mu\frac{\partial v}{\partial x}$	$\tau_{yy} = \mu\frac{\partial v}{\partial y}$	$\tau_{zy} = \mu\frac{\partial v}{\partial z}$
stresses in *z*-direction	$\tau_{xz} = \mu\frac{\partial w}{\partial x}$	$\tau_{yz} = \mu\frac{\partial w}{\partial y}$	$\tau_{zz} = \mu\frac{\partial w}{\partial z}$

where μ is the *dynamic viscosity coefficient*. Therefore, the accelerations can be written as

acceleration due to stress in *x*-direction	$a_x = \upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) = \upsilon\nabla^2 u$
acceleration due to stress in *y*-direction	$a_y = \upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) = \upsilon\nabla^2 v$
acceleration due to stress in *z*-direction	$a_z = \upsilon\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) = \upsilon\nabla^2 w$

where $\upsilon = \mu/\rho$ and is called the *kinematic viscosity coefficient*.

- The acceleration due to the viscous forces can be written in vector form as[2]

$$\vec{a}_{visc.} = \upsilon\nabla^2\vec{V} \qquad \textbf{(3)}$$

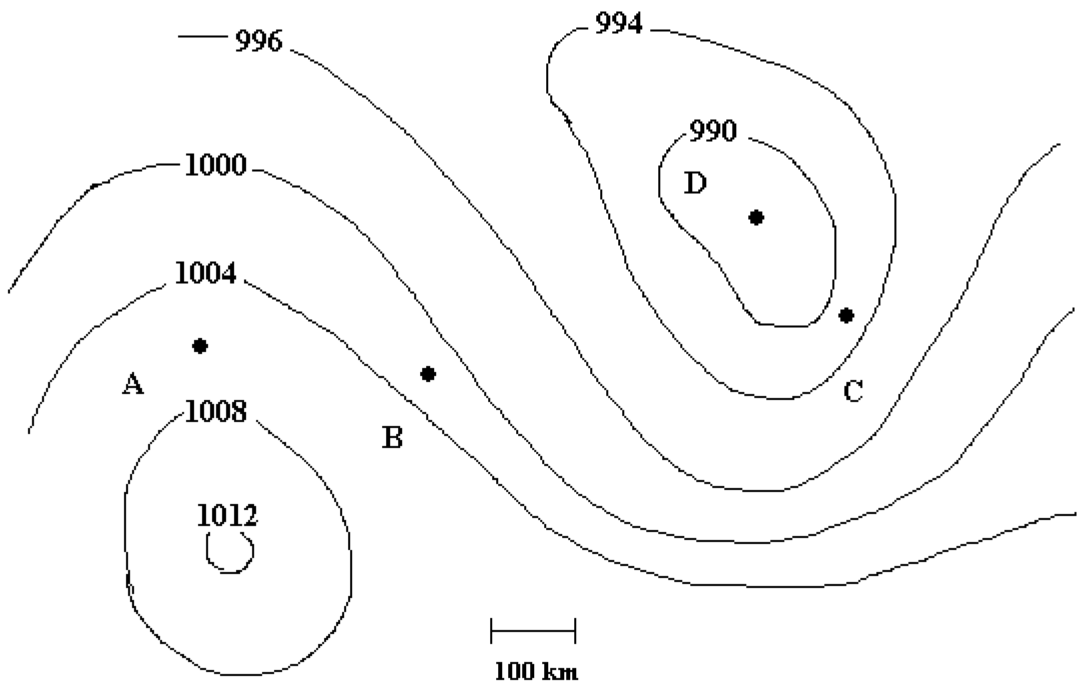
  - o NOTE: As with the pressure gradient acceleration, even though we derived (3) using Cartesian coordinates, this expression is a *geometric invariant* and is valid in any coordinate system. However, when expanded into component form, the components in Cartesian coordinates will look very different than those in spherical coordinates.

- The viscosity of the atmosphere is small, and under most circumstances we can ignore the viscous force in our meteorological equations.

[2] A more rigorous derivation of the viscous acceleration would result in an additional term in (3) of the form $\frac{1}{3}(\upsilon + 2\upsilon')\nabla\left(\nabla \bullet \vec{V}\right)$ where a second viscous coefficient, υ', appears. Since this additional term is very small, and is inconsequential for meteorological purposes, we choose to ignore it.

EXERCISES

1. At the four points shown in the picture below, estimate the magnitude of the acceleration due to the pressure gradient force. Assume a density of 1.23 kg/m^3. The isobars are labeled in mb.



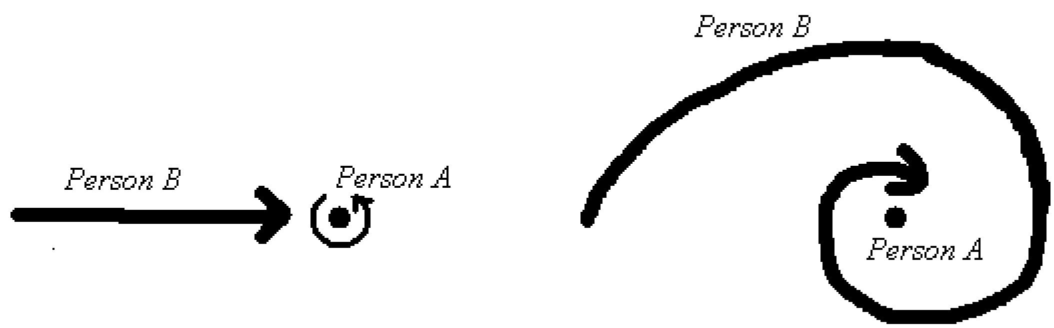
2. A man weighs 200 lb at the surface of the Earth. How much would the same man weigh at the summit of Mt. Everest (z = 8848 m)?

ESCI 342 – Atmospheric Dynamics I
Lesson 3 – Fundamental Forces II

Reading: Martin, Section 2.2

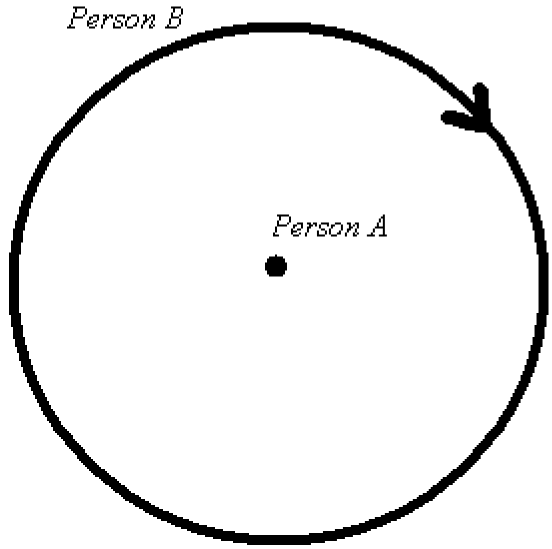
ROTATING REFERENCE FRAMES

- A reference frame in which an object with zero net force on it does not accelerate is known as an *inertial* reference frame.
- A reference frame attached to the Earth is a *noninertial* reference frame, since a net force is required to keep an object in one spot with respect to the reference frame.
- Use of a noninertial reference frame requires the introduction of *apparent* forces in order to use Newton's second law. This can be demonstrated as follows:
  - o Imagine that *Person A* is on a rotating turntable while *Person B* walks in a straight line at a constant speed toward them as pictured in the left illustration below.



  - o Since *Person B* is moving in a straight line at constant speed, there is no acceleration and therefore no net forces acting on *Person B*.
  - o From the standpoint of *Person A*, who is in a rotating reference frame, *Person B* is spiraling toward him as in the illustration on the right. Therefore, *Person A* assumes that there must be a force on *Person B*, since *Person B* is accelerating. This force is an apparent force, since it only appears in the non-inertial (rotating) reference frame.

- *If Person B* is not moving in the original reference frame there is still an apparent force in the noninertial frame since *Person B* will be moving in a circle around *Person A* (from *Person A*'s reference frame) as shown below.



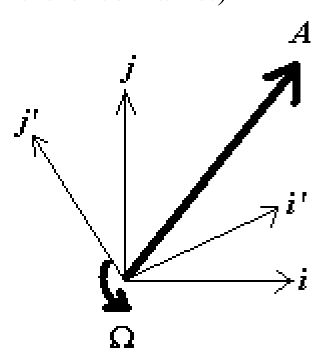
- Apparent forces are not real, though they appear to be real to someone in the noninertial reference frame.
- Apparent forces cannot do work on an object, nor can they change the speed of the object; they can only change the direction of motion of the object.

DERIVATIVES OF VECTORS IN ROTATING AND NONROTATING FRAMES

- The apparent forces can be developed mathematically as follows. Imagine two reference frames, one inertial and one rotating, sharing a common origin and a common *z*-axis. The rotating frame as an angular velocity of Ω (rad/s) about the *z*-axis.
- A vector $\vec{A}$ can be represented in component form in either frame as

$$\vec{A} = a_x\hat{i} + a_y\hat{j} = a'_x\hat{i}' + a'_y\hat{j}' . \qquad \textbf{(1)}$$

(Primes indicate the rotating reference frame.)



- The derivative of (1) with respect to time in the nonrotating frame is

$$\frac{d\vec{A}}{dt} = \frac{da_x}{dt}\hat{i} + \frac{da_y}{dt}\hat{j} = \frac{d_a\vec{A}}{dt} . \qquad \textbf{(2)}$$

The subscript *a* is just a way to remind us that this is the derivative from the perspective of an observer in the *absolute* (nonrotating) reference frame.

- In the rotating frame the derivative is

$$\frac{d\vec{A}}{dt} = \frac{da'_x}{dt}\hat{i}' + \frac{da'_y}{dt}\hat{j}' + a'_x\frac{d\hat{i}'}{dt} + a'_y\frac{d\hat{j}'}{dt}. \qquad \textbf{(3)}$$

  An observer in the rotating reference frame would perceive the derivative of $\vec{A}$ as simply

$$\frac{d_r\vec{A}}{dt} = \frac{da'_x}{dt}\hat{i}' + \frac{da'_y}{dt}\hat{j}'. \qquad \textbf{(4)}$$

  (they would be unaware of the additional terms involving the derivatives of the unit vectors). The subscript *r* indicates this is the derivative as perceived in the rotating frame.
  Using (4) in (3) we get

$$\frac{d\vec{A}}{dt} = \frac{d_r\vec{A}}{dt} + a'_x\frac{d\hat{i}'}{dt} + a'_y\frac{d\hat{j}'}{dt}. \qquad \textbf{(5)}$$

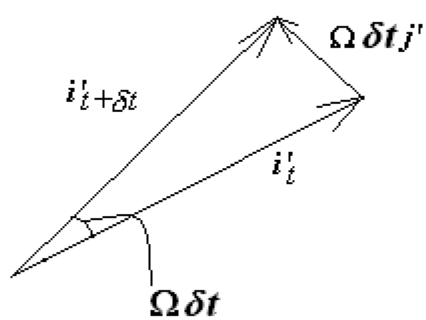
- The results from (3) and (5) must be identical, since $d\vec{A}/dt$ is a geometric invariant expression. We can therefore equate them to get

$$\frac{d_a\vec{A}}{dt} = \frac{d_r\vec{A}}{dt} + a'_x\frac{d\hat{i}'}{dt} + a'_y\frac{d\hat{j}'}{dt}. \qquad \textbf{(6)}$$

- We need to evaluate the derivatives of the unit vectors in the rotating frame. The term $d\hat{i}'/dt$ is evaluated as follows:

$$\frac{d\hat{i}'}{dt} = \lim_{\delta t\to 0}\frac{\hat{i}'_{t+\delta t} - \hat{i}'_t}{\delta t} = \lim_{\delta t\to 0}\frac{\Omega\delta t\,\hat{j}'}{\delta t} = \Omega\hat{j}'$$

  (refer to the diagram below).



  Similarly we can show that $d\hat{j}'/dt = -\Omega\,\hat{i}'$. Therefore we have

$$\frac{d_a\vec{A}}{dt} = \frac{d_r\vec{A}}{dt} + a'_x\Omega\,\hat{j}' - a'_y\Omega\hat{i}' \qquad \textbf{(7)}$$

  which is the same as

$$\frac{d_a\vec{A}}{dt} = \frac{d_r\vec{A}}{dt} + \vec{\Omega}\times\vec{A}. \qquad \textbf{(8)}$$

- Equation (8) shows how the derivative of a vector can be transformed between an inertial and a rotating reference frame.
  - o Even though the derivation was done for a vector in only two dimensions, it works regardless of the number of dimensions.

- o Equation (8) is also valid even if the *z*-axes of the origins of the two coordinate systems do not coincide and/or their *z*-axes aren't parallel.

ACCELERATIONS IN ROTATING VERSUS NONROTATING FRAMES

- Applying (8) to the position vector of a point in space yields

$$\frac{d_a\vec{r}}{dt} = \frac{d_r\vec{r}}{dt} + \vec{\Omega}\times\vec{r}$$

which is also

$$\vec{V}_a = \vec{V}_r + \vec{\Omega}\times\vec{r} \qquad \textbf{(9)}$$

where $\vec{V}_a$ is the velocity in the absolute frame and $\vec{V}_r$ is the velocity in the rotating frame.

- Applying (8) to the absolute velocity vector gives

$$\frac{d_a\vec{V}_a}{dt} = \frac{d_r\vec{V}_a}{dt} + \vec{\Omega}\times\vec{V}_a\,. \qquad \textbf{(10)}$$

Substituting $\vec{V}_a$ from (9) into the *right-hand side only* of (10) gives

$$\frac{d_a\vec{V}_a}{dt} = \frac{d\vec{V}_r}{dt} + \frac{d_r}{dt}\left(\vec{\Omega}\times\vec{r}\right) + \vec{\Omega}\times\vec{V}_r + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)$$

which simplifies to

$$\frac{d_a\vec{V}_a}{dt} = \frac{d_r\vec{V}_r}{dt} + 2\vec{\Omega}\times\vec{V}_r - \Omega^2\vec{r}\,. \qquad \textbf{(11)}$$

- Equation (11) shows the relationship between the acceleration observed in the absolute frame and the acceleration observed in the rotating frame. If the two reference frames did not share a common axis then equation (11) would be

$$\frac{d_a\vec{V}_a}{dt} = \frac{d_r\vec{V}_r}{dt} + 2\vec{\Omega}\times\vec{V}_r - \Omega^2\vec{R} \qquad \textbf{(12)}$$

where the vector $\vec{R}$ is a vector normal to the axis of rotation and pointing to the position of the object.

NEWTON'S SECOND LAW IN A ROTATING REFERENCE FRAME

- If we were writing Newton's second law of motion for an air parcel in the absolute reference frame we would only need to include the pressure gradient force, the gravitational force, and the viscous force, and would have

$$\frac{d_a\vec{V}_a}{dt} = -\frac{1}{\rho}\nabla p + \vec{g}\,* + \upsilon\nabla^2\vec{V}_a\,. \qquad \textbf{(13)}$$

However, from (11) this can be written as

$$\frac{d_r\vec{V}_r}{dt} + 2\vec{\Omega}\times\vec{V}_r - \Omega^2\vec{r} = -\frac{1}{\rho}\nabla p + \vec{g}\,* + \upsilon\nabla^2 V_a$$

and noting that $\nabla^2 V_a = \nabla^2\vec{V}_r$ (see exercises), we get

$$\frac{d_r\vec{V}_r}{dt} = -\frac{1}{\rho}\nabla p + \vec{g}\,* + \upsilon\nabla^2\vec{V}_r - 2\vec{\Omega}\times V_r + \Omega^2\vec{R} \qquad \textbf{(14)}$$

Equation (14) is Newton's second law for the rotating coordinate system. The two additional terms that appear in the rotating coordinate system version, but not in the absolute coordinate system version, equation (13), are the accelerations due to the apparent forces.

- The first new term is the ***Coriolis acceleration***, and the second is the ***centrifugal acceleration.***

THE CORIOLIS FORCE

- We've seen already that the acceleration due to the Coriolis force is given by

$$\vec{a}_{cor} = -2\vec{\Omega}\times\vec{V}_r . \qquad \textbf{(15)}$$

- Notice that the Coriolis force depends on the speed of the object relative to the rotating frame. If the object is at rest relative to the rotating frame then the Coriolis force is zero.
- Notice also that the Coriolis acts at right angles to the motion.
- Relative to the rotating reference frame at the surface of the Earth, $\vec{\Omega}$ can be written in component form as

$$\vec{\Omega} = \Omega\cos\phi\,\hat{j} + \Omega\sin\phi\,\hat{k} , \qquad \textbf{(16)}$$

where ϕ is latitude.
The velocity vector is

$$\vec{V}_r = u\hat{i} + v\hat{j} + w\hat{k} .$$

The Coriolis acceleration therefore has components of

$$\vec{a}_{cor} = \left(2\Omega v\sin\phi - 2\Omega w\cos\phi\right)\hat{i} - 2\Omega u\sin\phi\,\hat{j} + 2\Omega u\cos\phi\,\hat{k} . \qquad \textbf{(17)}$$

- An object moving toward the East ($u > 0$) will be a deflected upward and southward due to the Coriolis force.
- An object moving toward the West ($u < 0$) will be a deflected downward and northward due to the Coriolis force.
- An object moving toward the North ($v > 0$) will be deflected eastward due to the Coriolis force.
- An object moving toward the South ($v < 0$) will be deflected westward due to the Coriolis force.
- An object moving upward will be deflected westward due to the Coriolis force.
- An object moving downward will be deflected eastward due to the Coriolis force.

CENTRIFUGAL FORCE AND GRAVITY

- If an object is in circular motion in an absolute reference frame, it requires a ***centripetal*** acceleration of $-\Omega^2\vec{R}$, which is directed inward toward the axis of rotation.
- In a reference frame rotating with the object, the object is not accelerating, so there is an apparent force, the ***centrifugal*** force, that is invoked to balance the centripetal force and keep the object from accelerating.
- The term $\Omega^2\vec{R}$ in equation (4) represents the ***centrifugal*** acceleration due to the Earth's rotation.
- The centrifugal force is always directed away from the axis of rotation.

- On the Earth the centrifugal force appears to pull is away from the surface (except at the Poles) and to make us feel lighter.
    - This effect is most pronounced at the Equator.
- The centrifugal force is combined with the gravitational force to define a new force called *gravity*.
- The acceleration due to gravity force is defined as
$$\vec{g} = \vec{g}^* + \Omega^2 \vec{R} . \qquad \textbf{(18)}$$
  Equation (14) therefore becomes
$$\frac{d_r \vec{V}_r}{dt} = -\frac{1}{\rho} \nabla p - 2\vec{\Omega} \times \vec{V}_r + \vec{g} + \upsilon \nabla^2 \vec{V}_r . \qquad \textbf{(19)}$$
    - When you see gravity in an equation such as(19), keep in mind that it is a combination of the ***gravitational*** acceleration ***plus centrifugal*** acceleration.
- Note that gravity has a plus sign in (19), but keep in mind that if written in component form it lies solely in the negative $\hat{k}$ direction,
$$\vec{g} = -g\hat{k} . \qquad \textbf{(20)}$$
- Gravity is not directed exactly through the center of the Earth except at the Poles and at the Equator.
- The centrifugal force causes the Earth to not be a perfect sphere. Instead it is an *oblate spheroid*, with a larger radius at the Equator than at the Poles.

THE MOMENTUM EQUATON

- Equation (19) is known as the momentum equation.
- From now on we leave off the subscript *r* on the velocity, and just remember that it is the velocity relative to the Earth in our rotating coordinate system. So we will write it from now on as
$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho} \nabla p - 2\vec{\Omega} \times \vec{V} + \vec{g} + \upsilon \nabla^2 \vec{V} . \qquad \textbf{(21)}$$
- The terms in the momentum equation are:
    - acceleration due to the pressure gradient force: $-\frac{1}{\rho}\nabla p$
    - Coriolis acceleration: $-2\vec{\Omega} \times \vec{V}$
    - Gravity (includes centrifugal acceleration): $\vec{g}$
    - Friction (viscous acceleration): $\nu \nabla^2 \vec{V}$

COORDINATE REPRESENTATION OF THE MOMENTUM EQUATION

- Equation (21) is a vector equation. It is valid as is, without modification, in any coordinate system. However, for calculations and other purposes it is often necessary to write it out in component form. This is done by first choosing a coordinate system, and then expanding each term into its component form. Then, all the terms for a particular component are grouped together, resulting in three scalar equation, one for each coordinate.

- This is illustrated here using a ***local Cartesian coordinate***[1] system:
- Expanding each term of (21) into this coordinate system we get:
  - o For the acceleration term we have

$$\frac{d\vec{V}}{dt} = \frac{d}{dt}\left(u\hat{i} + v\hat{j} + w\hat{k}\right) = \frac{du}{dt}\hat{i} + \frac{dv}{dt}\hat{j} + \frac{dw}{dt}\hat{k} + \left(u\frac{d\hat{i}}{dt} + v\frac{d\hat{j}}{dt} + w\frac{d\hat{k}}{dt}\right), \quad \textbf{(22)}$$

    but in a Cartesian frame the unit vectors are constants in both time and space, so their derivative are zero. Therefore, in Cartesian coordinates

$$\frac{d\vec{V}}{dt} = \frac{du}{dt}\hat{i} + \frac{dv}{dt}\hat{j} + \frac{dw}{dt}\hat{k}, \quad \textbf{(23)}$$

  - o The pressure-gradient term expands as

$$-\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\frac{\partial p}{\partial x}\hat{i} - \frac{1}{\rho}\frac{\partial p}{\partial y}\hat{j} - \frac{1}{\rho}\frac{\partial p}{\partial z}\hat{k}. \quad \textbf{(24)}$$

  - o The Coriolis term expands as

$$-2\vec{\Omega}\times\vec{V} = \left(2\Omega v\sin\phi - 2\Omega w\cos\phi\right)\hat{i} - 2\Omega u\sin\phi\,\hat{j} + 2\Omega u\cos\phi\,\hat{k} \quad \textbf{(25)}$$

  - o The gravity term expands as

$$\vec{g} = -g\hat{k} \quad \textbf{(26)}$$

  - o The viscous term expands as

$$\upsilon\nabla^2\vec{V} = \upsilon\nabla^2 u\hat{i} + \upsilon\nabla^2 v\hat{j} + \upsilon\nabla^2 w\hat{k} + \left(\upsilon u\nabla^2\hat{i} + \upsilon v\nabla^2\hat{j} + \upsilon w\nabla^2\hat{k}\right), \quad \textbf{(27)}$$

    but in Cartesian coordinates the terms in parentheses are zero, because derivatives of the unit vectors are zero. Therefore, in a Cartesian frame we have

$$\upsilon\nabla^2\vec{V} = \upsilon\nabla^2 u\hat{i} + \upsilon\nabla^2 v\hat{j} + \upsilon\nabla^2 w\hat{k}\ . \quad \textbf{(28)}$$

- Collecting the like components ($\hat{i}$, $\hat{j}$, and $\hat{k}$) together we end up with the three Cartesian-component momentum equations,

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + \upsilon\nabla^2 u \quad \textbf{(29)}$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + \upsilon\nabla^2 v \quad \textbf{(30)}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\phi - g + \upsilon\nabla^2 w \quad \textbf{(31)}$$

- In parameterized spherical coordinates the derivatives of the unit vectors, the terms in parentheses in (22) and (27), are not zero, and these lead to additional terms, called ***curvature terms***, which involve the radius of the Earth (denoted by a). In parameterized spherical coordinates the component equations are

[1] Local Cartesian coordinates are coordinates based on a flat plane tangent to the Earth's surface at a particular latitude, ϕ.

$$\frac{du}{dt}-\left[\frac{uv\tan\phi}{a}-\frac{uw}{a}\right]=-\frac{1}{\rho}\frac{\partial p}{\partial x}+2\Omega v\sin\phi-2\Omega w\cos\phi$$
$$+\upsilon\left[\nabla^2 u-\frac{2}{a}\frac{\partial v}{\partial x}\tan\phi+\frac{1}{a}\frac{\partial w}{\partial x}-\frac{u}{a^2}\tan^2\phi\right] \tag{32}$$

$$\frac{dv}{dt}+\left[\frac{u^2\tan\phi}{a}+\frac{vw}{a}\right]=-\frac{1}{\rho}\frac{\partial p}{\partial y}-2\Omega u\sin\phi$$
$$+\upsilon\left[\nabla^2 v+\frac{2}{a}\frac{\partial u}{\partial x}\tan\phi-\frac{v}{a^2}\left(1+\tan^2\phi\right)+\frac{w}{a^2}\tan\phi+\frac{2}{a}\frac{\partial w}{\partial y}\right] \tag{33}$$

$$\frac{dw}{dt}-\left[\frac{u^2+v^2}{a}\right]=-\frac{1}{\rho}\frac{\partial p}{\partial z}+2\Omega u\cos\phi-g$$
$$+\upsilon\left[\nabla^2 w-\frac{2}{a}\frac{\partial u}{\partial x}+\frac{v}{a^2}\tan\phi-\frac{2w}{a^2}-\frac{u}{a^2}+\frac{1}{a}\frac{\partial w}{\partial x}-\frac{2}{a}\frac{\partial v}{\partial y}\right] \tag{34}$$

- The terms in square brackets in (32), (33), and (34) are the curvature terms, and are the only difference between the component representation in parameterized-spherical coordinates and Cartesian coordinates.

EXERCISES

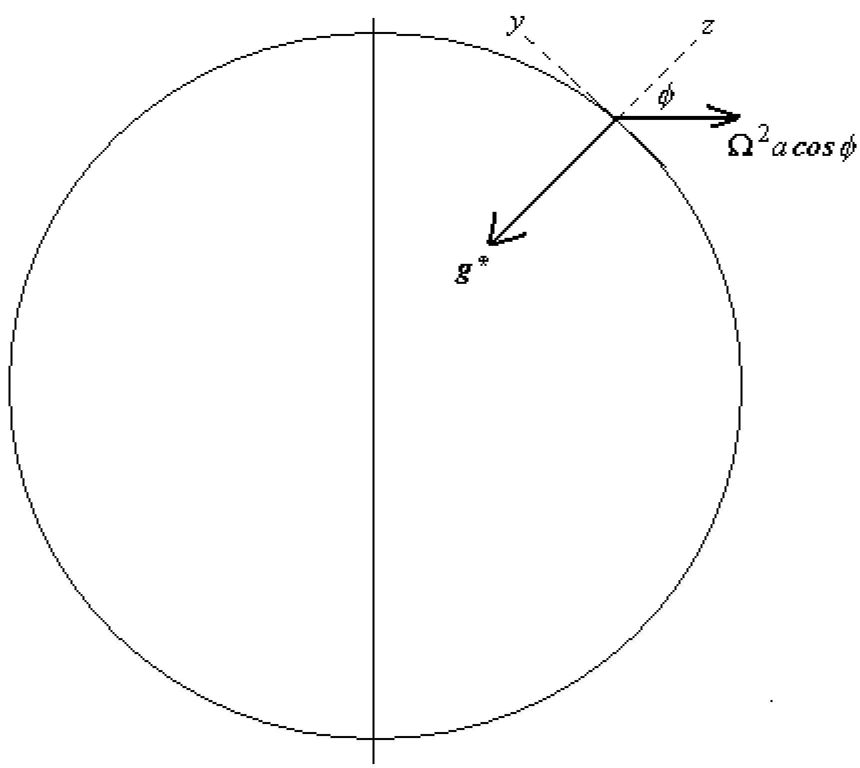
1. Show that if $\vec{\Omega} = \Omega\hat{k}'$ then $a'_x\Omega\,\hat{j}' - a'_y\Omega\hat{i}' = \vec{\Omega}\times\vec{A}$

2. Show that if $\vec{\Omega}$ and $\vec{r}$ are normal to each other then $\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) = -\Omega^2\vec{r}$.

3. a. Show that the magnitude of the gravity force at latitude ϕ is given by

$$g = \sqrt{\left(\Omega^2 a\cos^2\phi - g^*\right)^2 + \left(\Omega^2 a\cos\phi\sin\phi\right)^2}$$

where a is the radius of the Earth. (Hint: Write $\boldsymbol{g^*}$ and the centrifugal acceleration in component form and add them together. Then find the magnitude of the resultant vector. You may refer to the diagram below.)



b. Using the results from part a., find the magnitude of the gravity force at the North Pole, 45°N, and at the Equator (g^* = 9.81 m/s^2, the radius of the Earth is 6378 km, and $\Omega = 7.292\times10^{-5}$ rad/s).

c. Using your results for the gravity force from part a., find the geopotential height at an altitude of 5000 meters at the North Pole, 45°N, and at the Equator (assume that the gravity force is constant with height).

4. Assume that the gravity force at the surface is $g_0 = 9.80665$ m/s^2. Calculate the geopotential height at an altitude of 5,000 meters for the following two cases:

a. Gravity is constant with height.

b. Gravity decreases with height according to the following formula,

$$g = \frac{g_0}{(1 + z/a)^2}$$

where a is the radius of the Earth (6378 km).

c. From these results, do you think its important to include the variation of gravity with height?

5. Show that $-2\vec{\Omega} \times \vec{V} = (2\Omega v \sin\phi - 2\Omega w \cos\phi)\hat{i} - 2\Omega u \sin\phi\,\hat{j} + 2\Omega u \cos\phi\,\hat{k}$.

6. An ant is walking on a turntable that is rotating clockwise at 5 revolutions per minute (rpm). A coordinate system (x', y') is rotating with the turntable, the origin of which is the center of the turntable, with the $x'$and y'-axes pointing radially outward. At time $t = 0$, this coordinate system is perfectly aligned with a coordinate system fixed to the non-rotating room (x, y). The ant is initially at coordinates $x = x' = 0$, $y = y' = 1$ cm, and with respect to the turntable is traveling along the y'-axis at a constant speed of 0.5 cm/s.

a. What is the angular velocity of the turntable in rad/s?

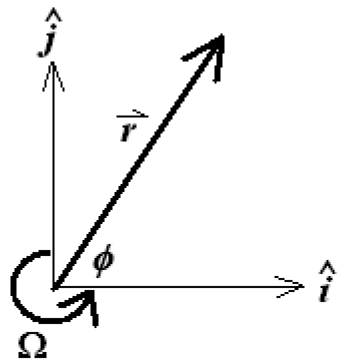
b. What are the components (in the rotating reference frame) of the ant's Coriolis acceleration at time $t = 0$?

c. What are the components (in the rotating reference frame) of the ant's centrifugal acceleration at time $t = 0$?

d. What are the components of the ant's velocity in the reference frame fixed to the room?

e. What are the components of the ant's acceleration in the reference frame fixed to the room?

7. A coordinate system is rotating counter-clockwise around it's $\hat{k}$ axis at an angular speed Ω. An object is located at position $\vec{r}$, and has a velocity such that the Coriolis acceleration is exactly balanced by the centrifugal acceleration.



a. What are the *u* and *v* components of the velocity of the object in the rotating coordinate system?

b. What are the *u* and *v* components of the velocity of the object in a non-rotating coordinate system?

c. What will the path of the object look like in the rotating coordinate system?

d. What will the path of the object look like in a non-rotating coordinate system?

12. Show that $\nabla^2 \vec{V}_a = \nabla^2 \vec{V}_r$. Hint: To do this, you will need to show that $\nabla^2 \left(\vec{\Omega} \times \vec{r} \right) = 0$. Use the identity $\nabla^2 \vec{A} = \nabla \left(\nabla \bullet \vec{A} \right) - \nabla \times \left(\nabla \times \vec{A} \right)$, and recognize that $\vec{\Omega} \times \vec{r}$ is the tangential velocity of solid-body rotation. Also, the velocity divergence in solid-body rotation is zero, and the vorticity in solid-body rotation is constant.

ESCI 342 – Atmospheric Dynamics I
Lesson 4 – Pressure

GEOPOTENTIAL

- The work required to raise a unit mass from the surface of the Earth to some height z is called the *geopotential*, defined as

$$\Phi = \int_0^z g\,dz\,. \qquad \textbf{(1)}$$

- The geopotential height is defined as

$$Z = \Phi / g_0 \qquad \textbf{(2)}$$

  where $g_0 = 9.80665$ m/s^2 and is called *standard gravity*.
- Since $g \cong g_0$, the geopotential height is approximately equal to the actual height ($z \cong Z$). However, for dynamic calculations involving the wind the geopotential height must be used for maximum accuracy, since even small deviations can lead to errors in the wind.
- Heights of pressure surfaces are reported in geopotential height rather than actual height.

THE HYDROSTATIC EQUATION

- The vertical momentum equation is[1]

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\varphi - g + \upsilon\nabla^2 w\,. \qquad \textbf{(3)}$$

- If the atmosphere is at rest then u, w, and dw/dt are all zero, so then this becomes

$$\frac{\partial p}{\partial z} = -\rho g\,, \qquad \textbf{(4)}$$

  which is known as the *hydrostatic equation*.
  - The hydrostatic equation is often written with a full derivative,

$$\frac{dp}{dz} = -\rho g\,.$$

    This is allowed because

$$\frac{dp}{dz} = \frac{\partial p}{\partial t}\frac{dt}{dz} + \frac{\partial p}{\partial x}\frac{dx}{dz} + \frac{\partial p}{\partial y}\frac{dy}{dz} + \frac{\partial p}{\partial z}$$

    and if the atmosphere is at rest then p is a function of height only so that

$$\frac{\partial p}{\partial t} = 0; \quad \frac{\partial p}{\partial x} = 0; \quad \frac{\partial p}{\partial y} = 0$$

    and therefore

$$\frac{\partial p}{\partial z} = \frac{dp}{dz}\,.$$

[1] Recall we are ignoring the curvature terms in the Laplacian. See the end of Lesson 3.

Just keep in mind that there will be occasions when we will use the hydrostatic equation even when the atmosphere is not at rest, and on those occasions it is more proper to use the form with the partial derivative.

- ***The hydrostatic equation states that in an atmosphere at rest the pressure gradient force is exactly balanced by gravity*.**
- In terms of geopotential height we can write

$$\frac{dp}{dz} = \frac{dp}{dZ}\frac{dZ}{dz} = \frac{g}{g_0}\frac{dp}{dZ}$$

so that the hydrostatic equation can be written as

$$\frac{dp}{dZ} = -\rho\, g_0 . \qquad \textbf{(5)}$$

- The hydrostatic equation can be used to find the vertical pressure profile of an atmosphere at rest as follows:
  - o Substitute for density from the ideal gas law to get

$$\frac{1}{p}\frac{dp}{dZ} = -\frac{g_0}{R_d T}.$$

  - o Integrating vertically from the surface to some geopotential height Z we get

$$p(Z) = p_0 \exp\left[-\frac{g_0}{R_d}\int_0^z \frac{1}{T}\,dZ\right]. \quad \textit{Pressure variation with height}$$

PRESSURE DECREASE IN AN ISOTHERMAL ATMOSPHERE

- Absolute temperature varies by only 20% or so through the troposphere, so we can get an idea how pressure changes with height by assuming a constant temperature (isothermal atmosphere). If this is done, the expression for the pressure profile becomes

$$p(Z) = p_o \exp\left(-\frac{g}{R_d T}Z\right) = p_0 \exp\left(-Z/H_P\right). \qquad \textbf{(6)}$$

- H_P is the *pressure scale height* of the atmosphere, and is a measure of how rapidly the pressure drops with height. A larger scale height means a slower rate of decrease with height.
  - o At $Z = H_p$ the pressure will have decreased to 37% of the surface value ($e^{-1} = 0.368$).
  - o The pressure scale height is the *e-folding* scale for pressure.

DENSITY PROFILE

- We can also use the hydrostatic equation and the equation of state to find how density changes with height. We first start by differentiating the ideal gas law with respect to geopotential height to get

$$\frac{dp}{dZ} = R_d\left[\rho\frac{dT}{dZ} + T\frac{d\rho}{dZ}\right].$$

From the hydrostatic equation we know that

$$\frac{dp}{dZ} = -\rho g_0 ,$$

so we can write

$$R_d\left[\rho\frac{dT}{dZ} + T\frac{d\rho}{dZ}\right] = -\rho g_0 .$$

Dividing through by $R_d T\rho$ gives

$$\frac{1}{T}\frac{dT}{dZ} + \frac{1}{\rho}\frac{d\rho}{dZ} = -\frac{g_0}{R_d T} .$$

If this is integrated from the surface to some level z we get

$$\rho(Z) = \rho_0 \frac{T_0}{T(Z)} \exp\left[-\frac{g_0}{R_d}\int_0^z \frac{1}{T} dZ\right].$$ ***Density variation with height***

- *Notice that the density and pressure profiles do not have the exact same functional dependence unless the atmosphere is isothermal* [$T(Z) = T_0$], in which case

$$\rho(Z) = \rho_0 \exp\left(-Z/H_P\right)$$
$$p(Z) = p_0 \exp\left(-Z/H_P\right).$$

THICKNESS AND THE HYPSOMETRIC EQUATION

- Equation (1) can be used to derive the *hypsometric equation* relating the average temperature in a layer to the thickness of the layer. We start with

$$\frac{1}{p}\frac{dp}{dZ} = -\frac{g_0}{R_d T}$$

which can be written as

$$1 = -\frac{R_d}{g_0}\frac{T}{p}\frac{dp}{dZ} .$$

Integrating this between two atmospheric levels

$$\int_{Z_1}^{Z_2} dZ = -\frac{R_d}{g_0}\int_{Z_1}^{Z_2}\frac{T}{p}\frac{dp}{dZ}dZ$$

gives

$$Z_2 - Z_1 = Z_\Delta = -\frac{R_d}{g_0}\int_{p_1}^{p_2} T\frac{dp}{p} ,$$

where $Z_\Delta \equiv Z_2 - Z_1$ and is called the *thickness* of the layer. Using the generalized mean-value theorem from calculus we can write this as

$$Z_\Delta = -\frac{R_d}{g_0}\bar{T}\int_{p_1}^{p_2}\frac{dp}{p}$$

where $\bar{T}$ is the layer-average temperature and is found by

$$\bar{T} = \int_{p_1}^{p_2} T\frac{dp}{p} \Bigg/ \int_{p_1}^{p_2}\frac{dp}{p} .$$

The equation for the thickness of the layer is then

$$Z_\Delta = -\frac{R_d}{g_0}\bar{T}\ln\frac{p_2}{p_1} = \frac{R_d}{g_0}\bar{T}\ln\frac{p_1}{p_2} \qquad \textbf{\textit{Hypsometric equation}}$$

- The hypsometric equation tells us that the thickness between two pressure levels is directly proportional to the average temperature within the layer.
- We can use thickness as a measure of the average temperature of a layer.
- Colder layers are thinner, warmer layers are thicker.
- We can use contours of thickness in a similar manner to how we use isotherms.
- ***The hypsometric equation is how the geopotential height of a pressure surface is determined from radiosonde observations***.

THE HORIZONTAL PRESSURE GRADIENT IN PRESSURE COORDINATES

- In meteorology it is often convenient to use pressure as the vertical coordinate in place of z or Z. This requires a slightly different representation for some of the terms in the momentum equation.
- On a constant pressure surface the differential of pressure is[2]

$$dp = \left(\frac{\partial p}{\partial x}\right)_{y,z,t} dx + \left(\frac{\partial p}{\partial y}\right)_{x,z,t} dy + \left(\frac{\partial p}{\partial z}\right)_{x,y,t} dz + \left(\frac{\partial p}{\partial t}\right)_{x,y,z} dt = 0\,. \qquad \textbf{(7)}$$

  - Dividing through by dx gives

$$\left(\frac{\partial p}{\partial x}\right)_{y,z,t} + \left(\frac{\partial p}{\partial y}\right)_{x,z,t}\frac{dy}{dx} + \left(\frac{\partial p}{\partial z}\right)_{x,y,t}\frac{dz}{dx} + \left(\frac{\partial p}{\partial t}\right)_{x,y,z}\frac{dt}{dx} = 0\,.$$

  - Since we are confined to the constant pressure surface (p is held constant) then we can write all the total derivatives as partial derivatives,

$$\left(\frac{\partial p}{\partial x}\right)_{y,z,t} + \left(\frac{\partial p}{\partial y}\right)_{x,z,t}\left(\frac{\partial y}{\partial x}\right)_p + \left(\frac{\partial p}{\partial z}\right)_{x,y,t}\left(\frac{\partial z}{\partial x}\right)_p + \left(\frac{\partial p}{\partial t}\right)_{x,y,z}\left(\frac{\partial t}{\partial x}\right)_p = 0$$

  which rearranges to

$$\left(\frac{\partial p}{\partial x}\right)_{y,z,t} = -\left(\frac{\partial p}{\partial y}\right)_{x,z,t}\left(\frac{\partial y}{\partial x}\right)_p - \left(\frac{\partial p}{\partial z}\right)_{x,y,t}\left(\frac{\partial z}{\partial x}\right)_p - \left(\frac{\partial p}{\partial t}\right)_{x,y,z}\left(\frac{\partial t}{\partial x}\right)_p\,. \qquad \textbf{(8)}$$

- Now, if we are constrained to remain on a constant pressure surface:
  1) We can still move in the x direction without changing the y coordinate, so x and y are independent. Likewise we can move in the y direction without changing the x coordinate.) This means that $(\partial x/\partial y)_p$ and $(\partial y/\partial x)_p$ are both zero.
  2) We can remain in a fixed horizontal location x, and y even if the pressure surface itself moves around. Therefore, x and t and also y and t are independent. Therefore, $(\partial t/\partial x)_p$ and $(\partial t/\partial y)_p$ are zero.
  3) We cannot arbitrarily move in the x or y directions without changing the z coordinate (unless the pressure surface is level), so there is a dependence between

[2] Notice that x, y, z, and t cannot all be independent in this case, since if dz is nonzero, the sum cannot equal zero unless either dx, dy, or dt is also nonzero.

z and x, and between z and y. This means that $\left(\partial z/\partial x\right)_p \neq 0$, and also $\left(\partial z/\partial y\right)_p \neq 0$.

- Eq. (8) is therefore

$$\left(\frac{\partial p}{\partial x}\right)_{y,z} = -\left(\frac{\partial p}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial x}\right)_p . \qquad \textbf{(9)}$$

- Substituting from the hydrostatic equation and rearranging yields

$$\left(\frac{\partial p}{\partial x}\right)_{y,z} = \rho\, g\left(\frac{\partial z}{\partial x}\right)_p = \rho\left(\frac{g\partial z}{\partial x}\right)_p = \rho\left(\frac{\partial \Phi}{\partial x}\right)_p . \qquad \textbf{(10)}$$

- From (10) we see that the horizontal acceleration due to the pressure gradient force can be written in terms of geopotential on a constant pressure surface

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \Phi}{\partial x}\right)_p ,$$

  or in terms of geopotential height of a constant pressure surface

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -g_0\left(\frac{\partial Z}{\partial x}\right)_p .$$

- A similar analysis can be done for the y-component of the pressure gradient force, and in vector form we have the following equivalences for the horizontal pressure gradient force.

$$-\frac{1}{\rho}\nabla_H p = -\nabla_{Hp}\Phi = -g_0\nabla_{Hp}Z \qquad \textbf{(11)}$$

  where

$$\nabla_H \equiv \left(\frac{\partial}{\partial x}\right)_z \hat{i} + \left(\frac{\partial}{\partial x}\right)_z \hat{j}$$

$$\nabla_{Hp} \equiv \left(\frac{\partial}{\partial x}\right)_p \hat{i} + \left(\frac{\partial}{\partial x}\right)_p \hat{j}$$

- It is important to note the following:

  1) In order for a horizontal pressure gradient to exist, a constant pressure surface must be tilted with respect to a surface of constant geopotential.

  2) The greater the tilt of a constant pressure surface, the greater the horizontal pressure gradient.

THE HYDROSTATIC EQUATION IN PRESSURE COORDINATES

- To find the hydrostatic equation in pressure coordinates, we start with the hydrostatic equation in height coordinates,

$$\frac{\partial p}{\partial z} = -\rho g , \qquad \textbf{(12)}$$

divide both sides by g to get

$$\frac{\partial p}{g\partial z} = -\rho \,. \qquad \textbf{(13)}$$

- Since we can ignore any dependence of gravity on altitude, (12) becomes

$$\frac{\partial p}{g\partial z} = \frac{\partial p}{\partial (gz)} = \frac{\partial p}{\partial \Phi} = -\rho \,. \qquad \textbf{(14)}$$

- Inverting (14) we get

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} \,,$$

or simply

$$\frac{\partial \Phi}{\partial p} = -\alpha \,. \qquad \textbf{(15)}$$

- Equation (15) is the hydrostatic equation in pressure coordinates.

EXERCISES

1. Show that $\dfrac{dZ}{dz} = \dfrac{g}{g_0}$.

2. If the atmosphere was incompressible (density constant at all altitudes), 100 km thick, and had a surface pressure of 1000 mb, at what altitude would the pressure be 250 mb? Sketch the graph of pressure vs. altitude for this case and discuss how it compares with the real atmosphere.

3. If the thickness of the 1000 – 500 mb layer is 5400 m, what is the layer average temperature (in °C)?

4. Find an expression for the vertical profile of pressure in an atmosphere that has a constant lapse rate of γ. [$T(z) = T_0 - \gamma z$]

5. a. Use a graphing calculator or other computer program to plot your result from problem 4. Use $\gamma = 6.5$°C/km, $T_0 = 288$K, and $p_0 = 1000$ mb.

b. On the same axis plot how pressure would change in an isothermal atmosphere having $T = 288$K and $p_0 = 1000$ mb.

c. Explain from a physical perspective the difference in the plots.

6. An atmosphere has a temperature profile as a function of pressure given by $T(p) = T_0 + a\ln(p/p_0)$ where T_0 is the temperature at sea level and p_0 is the pressure at sea level

a. For this atmosphere find a general expression for the layer-average temperature for a layer lying between pressures p_1 and p_2 ($p_2 > p_1$).

b. Use the expression found in part a. to find the geopotential height of the 500 mb pressure surface (use $p_0 = 1000$ mb, $T_0 = 288$ K, and $a = 36$ K.

7. If the temperature profile is linear in height ($\gamma = -\partial T/\partial z$ = constant), find an expression for temperature as a function of pressure. Hint: Start with the chain rule,

$$\frac{dT}{dp} = \frac{dT}{dz}\frac{dz}{dp}.$$

ESCI 342 – Atmospheric Dynamics I
Lesson 5 – The Total Derivative

THE TOTAL DERIVATIVE

- Meteorological variables such as p, T, $\vec{V}$ etc. can vary both in time and space. They are therefore functions of four independent variables, x, y, z and t.
- The differential of any of these variables (e.g., T) has the form

$$dT = \frac{\partial T}{\partial t}dt + \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz\,.$$

  - Dividing through by the differential of time gives

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}\,.$$

  - By definition

$$\frac{dx}{dt} \equiv u; \quad \frac{dy}{dt} \equiv v; \quad \frac{dz}{dt} \equiv w$$

    so that

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

    which can also be written as

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \vec{V} \bullet \nabla T\,.$$

- This shows that in general, ***the partial derivative is not equal to the full derivative***.
- We refer to the full derivative with respect to time as the ***total derivative*** or ***material derivative***, and give it the special notation of *D/Dt*, so that the total derivative operator is

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{V} \bullet \nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\,. \quad \textbf{(1)}$$

  - In the example using temperature we therefore have

$$\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \vec{V} \bullet \nabla T = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\,. \quad \textbf{(2)}$$

  - ***The total derivative* (*D/Dt*) *represents the change relative to*** **a reference frame attached to the air parcel and moving with it.**
    - This is referred to as a *Lagrangian derivative*.
- ***The term ∂/∂t represents the change from a coordinate system fixed x, y, and z coordinates***. This is called the ***local derivative***, or the *Eulerian derivative.*
- The term $\vec{V} \bullet \nabla$ is called the ***advection operator***,[1] and represents that part of the local change that is due to *advection* (transport of a property due to the mass movement of the fluid).
- In meteorology we usually measure the local derivative, since our instruments are usually fixed in space. Therefore, using temperature as an example, we write

---

[1] Do not confuse the advection operator, $\vec{V} \bullet \nabla$, with divergence $\nabla \bullet \vec{V}$!

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \vec{V} \bullet \nabla T .$$

- It is important to understand that the change we measure with our instruments may be due to either a change within the fluid itself (represented by the *DT/Dt* term), or due the movement of fluid with a different property over our instrument, represented by the $-\vec{V} \bullet \nabla T$ term.
    - o Example: The temperature at our station has been decreasing. This may be due the entire air mass losing heat due to radiation or conduction (*DT/Dt*) or due to the wind blowing colder air into our area, $-\vec{V} \bullet \nabla T$.

ADVECTION OF A SCALAR VERSUS A VECTOR

- Vector quantities can also be advected. The advection of a vector field looks like $-\vec{V} \bullet \nabla \vec{A}$.
- At first this may look odd, because we are used to the concept of the gradient operator operating on a scalar, not on a vector. But, $\nabla \vec{A}$ is indeed a defined and valid operation (contrary to what some textbooks may state), and is in fact a second-order tensor. The dot product of a vector with a second-order tensor results in a vector, so the result of $\vec{V} \bullet \nabla \vec{A}$ is just another vector.
- Since we may not want to involve ourselves with the concept of tensors, the advection of a vector is often written as $-\left(\vec{V} \bullet \nabla\right)\vec{A}$. In this form, the operator $\vec{V} \bullet \nabla$ is a scalar operator having the form

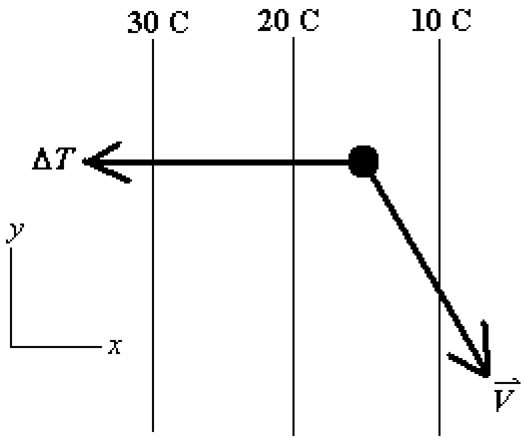
$$\vec{V} \bullet \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} .$$

- Keep in mind that whether or not we write the advection operator with parentheses, the two forms are equivalent,

$$\left(\vec{V} \bullet \nabla\right)\vec{A} = \vec{V} \bullet \nabla\vec{A} . \qquad \textbf{(3)}$$

MORE ON ADVECTION

- The advection term for a scalar involves the dot product of the velocity vector and the gradient vector. It is therefore readily evaluated.
    - o Example: The wind is from 330° at 25 m/s. The isotherms are oriented north-south as shown in the picture below, and are 100 km apart.



In component form the two vectors are

$$\vec{V} = (12.5\,m/s)\,\hat{\imath} - (21.6\,m/s)\,\hat{\jmath}$$

$$\nabla T = (-0.0001\,°C/m)\,\hat{\imath}$$

The advection is therefore

$$-\vec{V} \bullet \nabla T = -\left[(12.5\,m/s)\,\hat{\imath} - (21.6\,m/s)\,\hat{\jmath}\right] \bullet (-0.0001\,°C/m)\,\hat{\imath} = 0.00125\,°C/s\,.$$

The advection would cause the temperature at a fixed point to increase by 4.5°C in one hour, independent of any other temperature increase or decrease due to radiation or conduction.

- o Another way to solve this problem would be to find the angle between the two vectors (in this case is it 120°) and use the formula that

$$-\vec{V} \bullet \nabla T = -V\left|\nabla T\right|\cos\theta = -(25\,m/s)(0.0001°C/m)\cos 120° = 0.00125\,°C/s$$

- Advection itself is defined as $-\vec{V} \bullet \nabla s$ where s is any scalar property (e.g., u, v, w, T)
  - o The minus sign ensures that if the velocity and the gradient are opposite, then the advection is positive, since the property would be increasing with time.

ORIGIN OF THE CURVATURE TERMS

- In Lesson 3 we introduced the vector-form of the momentum equation, stating that it is frame-invariant.

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega}\times\vec{V} + \vec{g} + \upsilon\nabla^2\vec{V}\,. \quad \textbf{(4)}$$

- It is valid no matter what coordinate system (Cartesian, spherical, cylindrical) is chosen. It is only when we write the vector equation in component form that the choice of coordinate system becomes relevant.
- Expanding the total derivative yields

$$\frac{\partial\vec{V}}{\partial t} + \vec{V}\bullet\nabla\vec{V} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega}\times\vec{V} + \vec{g} + \upsilon\nabla^2\vec{V}\,. \quad \textbf{(5)}$$

- There are two terms in (5) that involve spatial derivatives of the velocity vector. These are the $\vec{V}\bullet\nabla\vec{V}$ and $\nabla^2\vec{V}$ terms. It is these terms that give rise to curvature terms, because when expanded out into coordinates, spatial derivative of the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ appear.

- In Cartesian coordinates the spatial derivatives of the unit vectors are zero, because both the direction and magnitude of the unit vectors is constant in space and time.
- In other coordinate systems the unit vectors have different directions depending on the location, so their spatial derivatives are not constant.
- The origin of the curvature terms is illustrated with he term $\vec{V}\bullet\nabla\vec{V}$ from the momentum equation. This term represents the advection of momentum by the wind itself. Expanded out it has the form

$$\vec{V}\bullet\nabla\vec{V} = u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}. \qquad \textbf{(6)}$$

The derivatives on the right-hand-side of (6) expand as follows

$$\frac{\partial\vec{V}}{\partial x} = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial v}{\partial x}\hat{j} + \frac{\partial w}{\partial x}\hat{k} + u\frac{\partial\hat{i}}{\partial x} + v\frac{\partial\hat{j}}{\partial x} + w\frac{\partial\hat{k}}{\partial x}$$

$$\frac{\partial\vec{V}}{\partial y} = \frac{\partial u}{\partial y}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial w}{\partial y}\hat{k} + u\frac{\partial\hat{i}}{\partial y} + v\frac{\partial\hat{j}}{\partial y} + w\frac{\partial\hat{k}}{\partial y}$$

$$\frac{\partial\vec{V}}{\partial z} = \frac{\partial u}{\partial z}\hat{i} + \frac{\partial v}{\partial z}\hat{j} + \frac{\partial w}{\partial z}\hat{k} + u\frac{\partial\hat{i}}{\partial z} + v\frac{\partial\hat{j}}{\partial z} + w\frac{\partial\hat{k}}{\partial z}$$

In Cartesian coordinates the terms involving derivatives of the unit vectors would all be zero. However, in spherical coordinates the directions of the unit vectors change with position. We therefore need to evaluate all of the following derivatives:

$$\frac{\partial\hat{i}}{\partial x};\quad \frac{\partial\hat{j}}{\partial x};\quad \frac{\partial\hat{k}}{\partial x};\quad \frac{\partial\hat{i}}{\partial y};\quad \frac{\partial\hat{j}}{\partial y};\quad \frac{\partial\hat{k}}{\partial y};\quad \frac{\partial\hat{i}}{\partial z};\quad \frac{\partial\hat{j}}{\partial z};\quad \frac{\partial\hat{k}}{\partial z}.$$

This is tedious, but not difficult, with the following results:

$\frac{\partial\hat{i}}{\partial x} = \frac{\tan\phi}{a}\hat{j} - \frac{1}{a}\hat{k}$	$\frac{\partial\hat{j}}{\partial x} = -\frac{\tan\phi}{a}\hat{i}$	$\frac{\partial\hat{k}}{\partial x} = \frac{1}{a}\hat{i}$
$\frac{\partial\hat{i}}{\partial y} = 0$	$\frac{\partial\hat{j}}{\partial y} = -\frac{1}{a}\hat{k}$	$\frac{\partial\hat{k}}{\partial y} = \frac{1}{a}\hat{j}$
$\frac{\partial\hat{i}}{\partial z} = 0$	$\frac{\partial\hat{j}}{\partial z} = 0$	$\frac{\partial\hat{k}}{\partial z} = 0$

so that we have

$$\frac{\partial\vec{V}}{\partial x} = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial v}{\partial x}\hat{j} + \frac{\partial w}{\partial x}\hat{k} + u\left(\frac{\tan\phi}{a}\hat{j} - \frac{1}{a}\hat{k}\right) + -v\frac{\tan\phi}{a}\hat{i} + \frac{w}{a}\hat{i}$$

$$\frac{\partial\vec{V}}{\partial y} = \frac{\partial u}{\partial y}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial w}{\partial y}\hat{k} - \frac{v}{a}\hat{k} + \frac{w}{a}\hat{j}$$

$$\frac{\partial\vec{V}}{\partial z} = \frac{\partial u}{\partial z}\hat{i} + \frac{\partial v}{\partial z}\hat{j} + \frac{\partial w}{\partial z}\hat{k}$$

and putting these into equation (6) gives the advection of momentum in component form,

$$\vec{V}\bullet\nabla\vec{V}=\left(\vec{V}\bullet\nabla u-\frac{uv\tan\phi}{a}+\frac{uw}{a}\right)\hat{i}$$
$$+\left(\vec{V}\bullet\nabla v+\frac{u^2\tan\phi}{a}+\frac{vw}{a}\right)\hat{j}+\left(\vec{V}\bullet\nabla w-\frac{\left(u^2+v^2\right)}{a}\right)\hat{k}. \tag{7}$$

- Through a similar, though more tedious analysis, we can show that in parameterized spherical coordinates the viscous term expands as

$$\upsilon\nabla^2\vec{V}=\upsilon\left[\nabla^2 u-\frac{u}{a^2}\left(\tan^2\phi+1\right)-\frac{2}{a}\frac{\partial v}{\partial x}\tan\phi+\frac{2}{a}\frac{\partial w}{\partial x}\right]\hat{i}$$
$$+\upsilon\left[\nabla^2 v-\frac{v}{a^2}\left(\tan^2\phi+1\right)+\frac{2}{a}\frac{\partial u}{\partial x}\tan\phi+\frac{w}{a^2}\tan\phi+\frac{2}{a}\frac{\partial w}{\partial y}\right]\hat{j} \tag{8}$$
$$+\upsilon\left[\nabla^2 w+\frac{v}{a^2}\tan\phi-\frac{2w}{a^2}-\frac{2}{a}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]\hat{k}$$

- The terms in (7) and (8) that involve the radius of the Earth, *a*, are called the **curvature terms**.
- Curvature terms will appear anytime we take spatial derivatives of a vector and expand it into components in spherical coordinates.

THE FULL MOMENTUM EQUATIONS

- The full momentum equations in component form in parameterized spherical coordinates are

$$\frac{\partial u}{\partial t}+\vec{V}\bullet\nabla u-\frac{uv\tan\phi}{a}+\frac{uw}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+2\Omega v\sin\phi-2\Omega w\cos\phi$$
$$+\upsilon\left[\nabla^2 u-\frac{u}{a^2}\left(\tan^2\phi+1\right)-\frac{2}{a}\frac{\partial v}{\partial x}\tan\phi+\frac{2}{a}\frac{\partial w}{\partial x}\right] \tag{9}$$

$$\frac{\partial v}{\partial t}+\vec{V}\bullet\nabla v+\frac{u^2\tan\phi}{a}+\frac{vw}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-2\Omega u\sin\phi$$
$$+\upsilon\left[\nabla^2 v-\frac{v}{a^2}\left(\tan^2\phi+1\right)+\frac{2}{a}\frac{\partial u}{\partial x}\tan\phi+\frac{w}{a^2}\tan\phi+\frac{2}{a}\frac{\partial w}{\partial y}\right] \tag{10}$$

$$\frac{\partial w}{\partial t}+\vec{V}\bullet\nabla w-\frac{\left(u^2+v^2\right)}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+2\Omega u\cos\phi-g$$
$$+\upsilon\left[\nabla^2 w-\frac{2w}{a^2}+\frac{v}{a^2}\tan\phi-\frac{2}{a}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right] \tag{11}$$

- The table below summarizes the various terms in the three momentum equations

local der.	advective terms	advection curvature terms	pressure gradient terms	Coriolis terms	gravity term	viscous terms	viscous curvature terms
$\frac{\partial u}{\partial t}$	$\vec{V}\bullet\nabla u$	$-\frac{uv\tan\phi}{a}+\frac{uw}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial x}$	$2\Omega v\sin\varphi$ $-2\Omega w\cos\varphi$		$\upsilon\nabla^2 u$	$\upsilon\begin{bmatrix}\nabla^2 u \\ -\frac{u}{a^2}\left(\tan^2\phi+1\right) \\ -\frac{2}{a}\frac{\partial v}{\partial x}\tan\phi \\ +\frac{2}{a}\frac{\partial w}{\partial x}\end{bmatrix}$
$\frac{\partial v}{\partial t}$	$\vec{V}\bullet\nabla v$	$\frac{u^2\tan\phi}{a}+\frac{vw}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial y}$	$-2\Omega u\sin\phi$		$\upsilon\nabla^2 v$	$\upsilon\begin{bmatrix}\nabla^2 v \\ -\frac{v}{a^2}\left(\tan^2\phi+1\right) \\ +\frac{2}{a}\frac{\partial u}{\partial x}\tan\phi \\ +\frac{w}{a^2}\tan\phi \\ +\frac{2}{a}\frac{\partial w}{\partial y}\end{bmatrix}$
$\frac{\partial w}{\partial t}$	$\vec{V}\bullet\nabla w$	$-\frac{\left(u^2+v^2\right)}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial z}$	$2\Omega u\cos\phi$	$-g$	$\upsilon\nabla^2 w$	$\upsilon\begin{bmatrix}\nabla^2 w \\ -\frac{2w}{a^2} \\ +\frac{v}{a^2}\tan\phi \\ -\frac{2}{a}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\end{bmatrix}$

VECTOR FORM VS. COMPONENT FORM OF MOMENTUM EQUATION

- Notice that even in spherical coordinates the momentum equation has the simple form of

$$\frac{D\vec{V}}{Dt}=-\frac{1}{\rho}\nabla p-2\vec{\Omega}\times\vec{V}+\vec{g}+\upsilon\nabla^2\vec{V} \qquad \textbf{(12)}$$

or

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \bullet \nabla \vec{V} = -\frac{1}{\rho} \nabla p - 2\vec{\Omega} \times \vec{V} + \vec{g} + \upsilon \nabla^2 \vec{V} \ , \quad \textbf{(13)}$$

when written in vector form.

- The curvature terms do not appear until we start writing the momentum equations in component form.
- ***<u>It is much easier to memorize the momentum equation in vector form</u>*.**
  - o If you know it's vector form, you can always then expand it into components using your knowledge of vector calculus.

EXERCISES

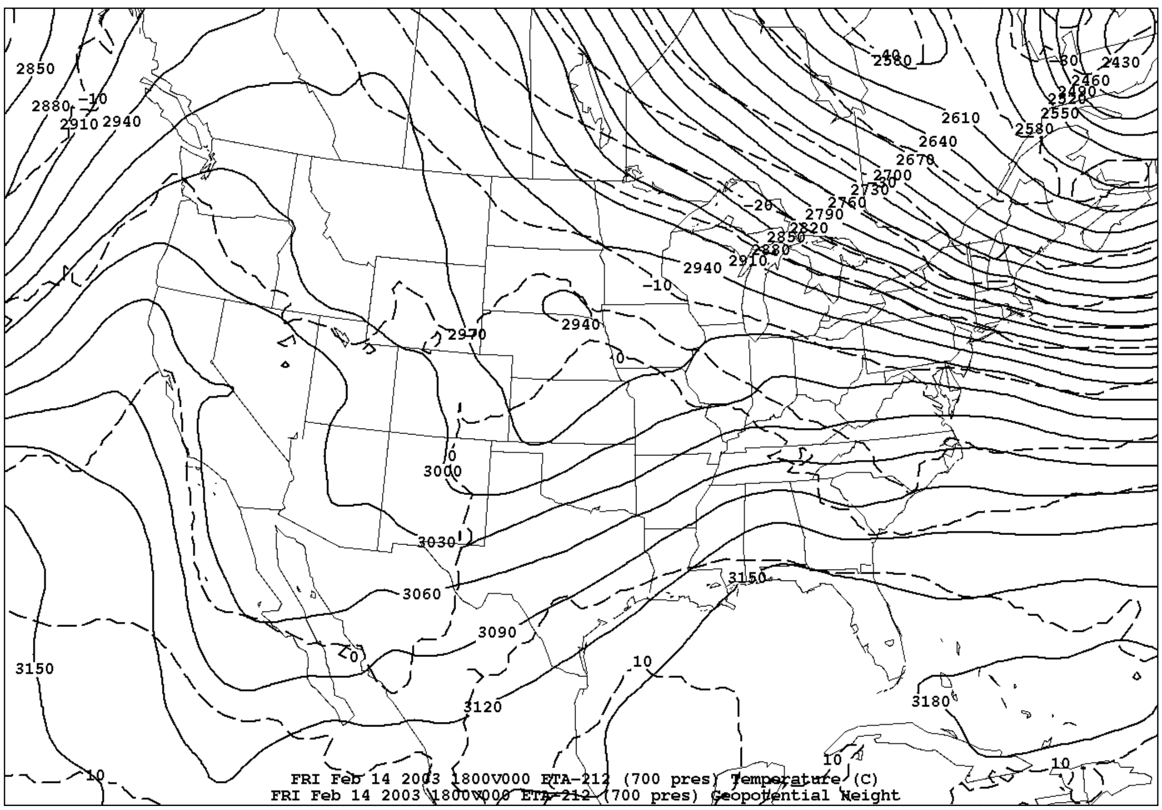
1. Show that if the wind is blowing parallel to the isotherms that the temperature advection is zero.

2. On the map below, indicate an area where:

a. The temperature would be increasing due to advection.

b. The temperature would be decreasing due to advection.

c. An area where temperature advection is very weak.



3. Show that in pure Cartesian coordinates

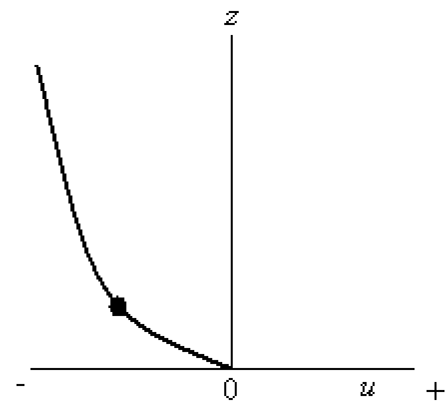
$$(\vec{V} \bullet \nabla \vec{A}) = (\vec{V} \bullet \nabla a_x)\hat{i} + (\vec{V} \bullet \nabla a_y)\hat{j} + (\vec{V} \bullet \nabla a_z)\hat{k}\,.$$

4. Show that in pure Cartesian coordinates

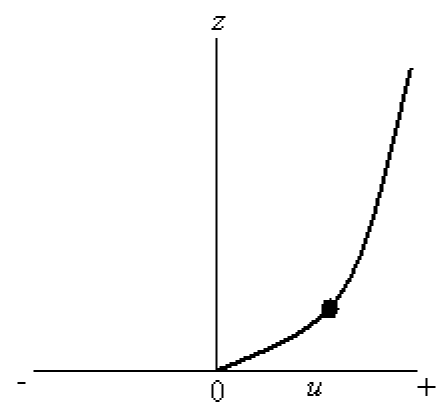
$$\frac{D\vec{V}}{Dt} = \left(\frac{\partial u}{\partial t} + \vec{V} \bullet \nabla u\right)\hat{i} + \left(\frac{\partial v}{\partial t} + \vec{V} \bullet \nabla v\right)\hat{j} + \left(\frac{\partial w}{\partial t} + \vec{V} \bullet \nabla w\right)\hat{k}$$

5. a. For the following profile of u, explain whether a downdraft would cause an increase or decrease in u at the location of the dot. Assume that u is constant in x and y [$u = u(t, z)$]

Hint: $\dfrac{\partial u}{\partial t} = -\vec{V} \bullet \nabla u = -u\dfrac{\partial u}{\partial x} - v\dfrac{\partial u}{\partial y} - w\dfrac{\partial u}{\partial z}$



b. Do the same as in 6.a., only for the following profile



6. Prove the following identities for parameterized spherical coordinates:

$\dfrac{\partial \hat{i}}{\partial x} = \dfrac{\tan\phi}{a}\hat{j} - \dfrac{1}{a}\hat{k}$	$\dfrac{\partial \hat{j}}{\partial x} = -\dfrac{\tan\phi}{a}\hat{i}$	$\dfrac{\partial \hat{k}}{\partial x} = \dfrac{1}{a}\hat{i}$
$\dfrac{\partial \hat{i}}{\partial y} = 0$	$\dfrac{\partial \hat{j}}{\partial y} = -\dfrac{1}{a}\hat{k}$	$\dfrac{\partial \hat{k}}{\partial y} = \dfrac{1}{a}\hat{j}$
$\dfrac{\partial \hat{i}}{\partial z} = 0$	$\dfrac{\partial \hat{j}}{\partial z} = 0$	$\dfrac{\partial \hat{k}}{\partial z} = 0$

7. Show that in parameterized spherical coordinates

$$\vec{V}\bullet\nabla\vec{V}=\left(\vec{V}\bullet\nabla u-\frac{uv\tan\phi}{a}+\frac{uw}{a}\right)\hat{i}+\left(\vec{V}\bullet\nabla v+\frac{u^2\tan\phi}{a}+\frac{vw}{a}\right)\hat{j}+\left(\vec{V}\bullet\nabla w-\frac{\left(u^2+v^2\right)}{a}\right)\hat{k}$$

8. Show that $\nabla^2\vec{V}=\nabla^2(\hat{i}u)+\nabla^2(\hat{j}v)+\nabla^2(\hat{k}w)$

9. Prove the identity $\nabla^2(ab)=a\nabla^2 b+2\nabla a\bullet\nabla b+b\nabla^2 a$

10. Derive the following expressions for parameterized spherical coordinates:

$$\nabla^2(\hat{i}u)=\hat{i}\left[\nabla^2 u-\frac{u}{a^2}\left(\tan^2\phi+1\right)\right]+\hat{j}\frac{2}{a}\frac{\partial u}{\partial x}\tan\phi-\hat{k}\frac{2}{a}\frac{\partial u}{\partial x}$$

$$\nabla^2(\hat{j}v)=-\hat{i}\frac{2}{a}\frac{\partial v}{\partial x}\tan\phi+\hat{j}\left[\nabla^2 v-\frac{v}{a^2}\left(\tan^2\phi+1\right)\right]+\hat{k}\left(\frac{v}{a^2}\tan\phi-\frac{2}{a}\frac{\partial v}{\partial y}\right)$$

$$\nabla^2(\hat{k}w)=\hat{i}\frac{2}{a}\frac{\partial w}{\partial x}+\hat{j}\left(\frac{w}{a^2}\tan\phi+\frac{2}{a}\frac{\partial w}{\partial y}\right)+\hat{k}\left(\nabla^2 w-\frac{2w}{a^2}\right)$$

11. Use the results of the previous problems to show that

$$\begin{aligned}\upsilon\nabla^2\vec{V}&=\upsilon\left[\nabla^2 u-\frac{u}{a^2}\left(\tan^2\phi+1\right)-\frac{2}{a}\frac{\partial v}{\partial x}\tan\phi+\frac{2}{a}\frac{\partial w}{\partial x}\right]\hat{i}\\&+\upsilon\left[\nabla^2 v-\frac{v}{a^2}\left(\tan^2\phi+1\right)+\frac{2}{a}\frac{\partial u}{\partial x}\tan\phi+\frac{w}{a^2}\tan\phi+\frac{2}{a}\frac{\partial w}{\partial y}\right]\hat{j}\\&+\upsilon\left[\nabla^2 w+\frac{v}{a^2}\tan\phi-\frac{w}{a^2}-\frac{2}{a}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]\hat{k}\end{aligned}$$

ESCI 342 – Atmospheric Dynamics I
Lesson 6 – Scale Analysis

SCALE ANALYSIS OF THE MOMENTUM EQUATIONS

- Not all of the terms in the momentum equations are significant. If a term is much smaller than the others then it is reasonable to ignore it under certain circumstances.
- To assess which terms can be neglected, we assign an "order of magnitude" to all the variables and parameters in the equations.
    - For scale analysis we often don't assign exact numbers…just orders of magnitude.
- The orders of magnitude are assigned for specific scales of motion. For instance, they would be quite different for the study of tornadoes than they would be for the study of hurricanes.
- The parameters that need to be scaled are shown in the table below:

Name	Symbol
Horizontal velocity	U
Vertical velocity	W
Horizontal length scale	L
Vertical length scale	H
Pressure change	δP
Density	ρ
Time[1]	$\tau = L/U$

- Appropriate scales for each are determined as follows:
    - Horizontal velocity, U: For most atmospheric circulations the u and v components are of similar magnitude, and so we use a single scale parameter, U, to represent both.
    - Vertical velocity, W.
    - Horizontal length scale, L: The horizontal length scale can be defined in a few ways.
        - For wavelike features in the atmosphere it is usually taken to be one-fourth of the total wavelength, $L \sim \lambda/4$.[2] This is because the scale of the spatial velocity derivatives such as $\partial u/\partial x$ are of the order of U/L (see figure below).
        - For a vortex, L is taken to be the radius R (not diameter), since the spatial derivative of velocity in a vortex will be of the order of U/R.

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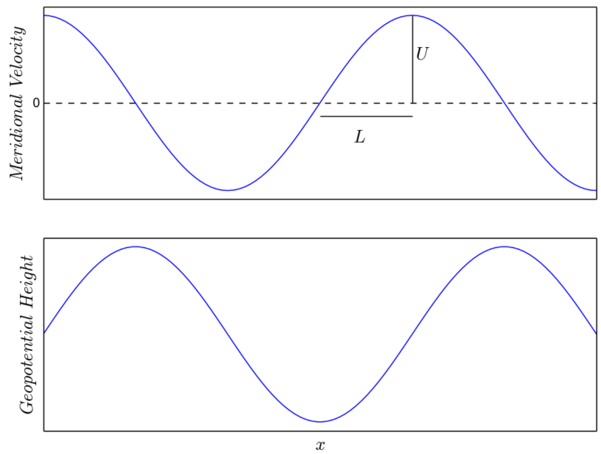
[1] The time scale $\tau = L/U$ is called the *advective* time scale. It is the time it would take for a parcel of fluid to travel the entire horizontal length of the flow.

[2] In very early work (Charney, J.G: 'On the scale of atmospheric motions', *Geof. Publ.*, **17**, 3-17; Burger, A.P., 1957: 'Scale consideration of planetary motions of the atmosphere', *Tellus*, **10**, 195-205) a horizontal scale of $\lambda/2$ was postulated. This was altered to $\lambda/4$ in Phillips, N.A., 1963: 'Geostrophic motion', *Rev. Geophys.*, **1**, 123-175



- o Vertical length scale, H: The vertical length scale is the height of the circulation or disturbance.
- o Pressure change, δP: The pressure change is needed for terms involving derivatives of the pressure.
  - In the horizontal, this will be the range between the maximum and minimum pressures found moving horizontally across the circulation.
  - In the vertical, it will be the maximum and minimum pressures found moving vertically through the circulation.
  - There may be very large differences for δp in the horizontal versus in the vertical.
- o Time, τ: For the time scale we use the *advective time scale*, defined as $\tau = L/U$. This is the time it would take for a parcel of fluid traveling at speed U to travel the distance L.

SYNOPTIC SCALE ANALYSIS OF THE HORIZONTAL MOMENTUM EQUATIONS

- For synoptic scales the following orders of magnitude are appropriate:

Name	Symbol	Order of magnitude
Horizontal velocity	U	10 m s^{-1}
Vertical velocity	W	0.01 m s^{-1}
Horizontal distance	L	1000 km (10^6 m)
Vertical distance	H	10 km (10^4 m)
Pressure change	δP	*Horizontal*: 10 mb (10^3 Pa) *Vertical*: 1000 mb (10^5 Pa)
Time	$\tau = L/U$	1 day (10^5 s)

o The following parameters are also used:

density	ρ	$1\ \text{kg m}^{-3}$
kinematic viscosity	υ	$1.46\times10^{-5}\ \text{m}^2\ \text{s}^{-1}$
omega	Ω	$7.292\times10^{-5}\ \text{rad s}^{-1}$
latitude	ϕ	$45°$
radius of Earth	a	$6.378\times10^{6}\ \text{m}$

- Using these scales and parameters, the terms in the u-momentum equation have the following orders of magnitude

$\frac{\partial u}{\partial t}$	$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}$	$w\frac{\partial u}{\partial z}$	$-\frac{uv\tan\phi}{a}$	$\frac{uw}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial x}$	$2\Omega v\sin\phi$	$-2\Omega w\cos\phi$
U^2/L	U^2/L	WU/H	U^2/a	UW/a	$\delta P/(\rho L)$	$2\Omega U \sin 45$	$2\Omega W \cos 45$
10^{-4}	10^{-4}	10^{-5}	10^{-5}	10^{-8}	10^{-3}	10^{-3}	10^{-6}

$\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$	$\upsilon\frac{\partial^2 u}{\partial z^2}$	$-\upsilon\frac{u}{a^2}(\tan^2\phi+1)$	$-\upsilon\frac{2}{a}\frac{\partial v}{\partial x}\tan\phi$	$\upsilon\frac{2}{a}\frac{\partial w}{\partial x}$
$\upsilon U/L^2$	$\upsilon U/H^2$	$\upsilon U/a^2$	$\upsilon U/La$	$\upsilon W/La$
10^{-16}	10^{-12}	10^{-18}	10^{-18}	10^{-20}

- A similar analysis for the v-momentum equation is

$\frac{\partial v}{\partial t}$	$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}$	$w\frac{\partial v}{\partial z}$	$\frac{u^2\tan\varphi}{a}$	$\frac{vw}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial y}$	$-2\Omega u\sin\phi$
U^2/L	U^2/L	WU/H	U^2/a	UW/a	$\delta P/(\rho L)$	$2\Omega U \sin 45$
10^{-4}	10^{-4}	10^{-5}	10^{-5}	10^{-8}	10^{-3}	10^{-3}

$\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$	$\upsilon\frac{\partial^2 v}{\partial z^2}$	$-\upsilon\frac{v}{a^2}\left(\tan^2\phi+1\right)$	$\upsilon\frac{2}{a}\frac{\partial u}{\partial x}\tan\phi$	$\upsilon\frac{w}{a^2}\tan\phi$	$\upsilon\frac{2}{a}\frac{\partial w}{\partial y}$
vU/L^2	vU/H^2	$\upsilon U/a^2$	$\upsilon U/La$	$\upsilon W/a^2$	$\upsilon W/La$
10^{-16}	10^{-12}	10^{-18}	10^{-18}	10^{-21}	10^{-20}

- Many of the terms are very small compared to others, and can be ignored without significant loss of accuracy. We can therefore ignore the curvature terms, the viscous terms, and the Coriolis term that involves the vertical velocity.
- Ignoring these terms yields a much simpler version of the horizontal equations of motions:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+2\Omega v\sin\varphi \qquad \textbf{(1)}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-2\Omega u\sin\varphi \qquad \textbf{(2)}$$

Note: We could have also ignored the vertical advection terms, but it is not too much of an inconvenience to keep them.

- By defining the *Coriolis parameter* as

$$f = 2\Omega \sin\phi \qquad \textbf{(3)}$$

the horizontal momentum equations assume the form

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv \qquad \textbf{(4)}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \qquad \textbf{(5)}$$

- In vector form the horizontal momentum equation is

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p - \hat{k}\times f\,\vec{V}\,. \qquad \textbf{(6)}$$

In (6), all derivatives and vectors are horizontal,

$$\vec{V} = u\,\hat{i} + v\,\hat{j}$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} \quad .$$

- The total derivative terms in (4), (5), and (6) are known as the ***inertial terms***. The terms on the right-hand-side are the ***pressure gradient*** and ***Coriolis terms*** respectively.

THE ROSSBY NUMBER

- Dividing the horizontal momentum equation, (6), through by $f\,V$ we get

$$\frac{1}{f\,V}\frac{D\vec{V}}{Dt} = -\frac{1}{\rho f\,V}\nabla p - \hat{k}\times\frac{f\,\vec{V}}{f\,V}\,.$$

- Using the representative scales the order of magnitude of these terms are

$$\frac{U}{f\,L} \quad = \quad \frac{\delta P}{\rho\, f\, U} \quad + \quad 1$$

- The dimensionless combination *U/f L* is defined as the *Rossby number* (named for Gustav Rossby),

$$Ro \equiv U/f\,L \qquad \textbf{(7)}$$

GEOSTROPHIC BALANCE (VERY SMALL ROSSBY NUMBER)

- When the Rossby number is much less than unity ($Ro \ll 1$), then the acceleration (inertial) term can be ignored and the only two terms left are the pressure gradient term and the Coriolis term, which must be nearly in balance.
  - o This is known as *geostrophic balance*, and the velocity in this case is known as the *geostrophic wind*.

- The momentum equation in this case reduces to

$$\hat{k} \times f\vec{V}_g = -\frac{1}{\rho}\nabla p$$

which is solved for the geostrophic wind to yield

$$\vec{V}_g = \frac{1}{f\rho}\hat{k} \times \nabla p\,, \qquad \textbf{(8)}$$

with wind speed (magnitude) equal to

$$V_g = \frac{|\nabla p|}{f\rho}\,. \qquad \textbf{(9)}$$

CYCLOSTROPHIC BALANCE (VERY LARGE ROSSBY NUMBER)

- When the Rossby number is much greater than unity ($Ro >> 1$) then the Coriolis term can be ignored. In this instance the only terms that are left are the acceleration and the pressure gradient terms, and so the acceleration is a direct result of the pressure gradient force

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p\,. \qquad \textbf{(10)}$$

  - This type of balance is called *cyclostrophic*.
  - In cyclostrophic balance the pressure gradient acceleration is exactly that required for the centripetal acceleration, and so we have

$$\frac{V_c^2}{r} = \frac{|\nabla p|}{\rho}$$

or

$$V_c = \sqrt{\frac{r|\nabla p|}{\rho}} \qquad \textbf{(11)}$$

where r is the radius of curvature of the flow.

GRADIENT BALANCE (ROSSBY NUMBER NEAR UNITY)

- If the Rossby number is of the order of unity ($Ro \sim 1$), then all three terms must be retained. This is known as *gradient balance*, and the wind in this case is known as the *gradient wind*. Details of gradient wind will be discussed in a future lesson.
- The following table summarized these results

Ro	**Terms**	**Balance**
<< 1	pressure gradient and Coriolis	*geostrophic*
~1	acceleration, pressure gradient, and Coriolis	*gradient*
>> 1	acceleration and pressure gradient	*cyclostrophic*

- For large-scale (synoptic scale) motion, the Rossby number is of the order

$$Ro \sim \frac{(10m/s)}{(10^{-4}s^{-1})(10^6 m)} = 0.1\,,$$

which shows that on these scales the atmosphere is close to being in geostrophic balance. Hence, the actual wind should be close to the geostrophic wind.

INERTIAL FLOW (ROSSBY NUMBER EXACTLY EQUAL TO ONE)

- If the pressure gradient is exactly zero, then the inertial terms must exactly balance the Coriolis term.[3]
- The balance in this case is called *inertial balance*, with the speed given by

$$V_{in} = -f\,r\,. \tag{12}$$

- In inertial balance the flow is circular, with radius of $R = -V_{in}/f$.
  - Since by definition V is always positive, then R must be negative and so inertial flow is anticyclonic.
- The period of the inertial flow is found by dividing the circumference of the inertial circle by the speed,

$$\tau = \frac{-2\pi r}{V_{in}} = \frac{2\pi}{f}\,. \tag{13}$$

  - The inertial period is shorter at higher latitudes, and is infinity at the Equator.

MORE ON THE GEOSTROPHIC WIND

- ***The geostrophic wind is a definition! On the synoptic scale the actual wind should be close to the geostrophic wind (because Ro << 1), but will rarely be exactly equal to the geostrophic wind.***
- The components of the geostrophic wind are

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y} \tag{14}$$

$$v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x} \tag{15}$$

- ***The geostrophic wind is parallel to the isobars with lower pressure to the left (in the Northern Hemisphere).***
- ***The geostrophic wind speed is directly proportional to the pressure gradient.***
- In pressure coordinates, the geostrophic wind and components are

$$\vec{V}_g = \frac{1}{f}\hat{k}\times\nabla\Phi = \frac{g_0}{f}\hat{k}\times\nabla Z \tag{16}$$

$$u_g = -\frac{1}{f}\frac{\partial\Phi}{\partial y} = -\frac{g_0}{f}\frac{\partial Z}{\partial y} \tag{17}$$

$$v_g = \frac{1}{f}\frac{\partial\Phi}{\partial x} = \frac{g_0}{f}\frac{\partial Z}{\partial x} \tag{18}$$

- Therefore, on a constant pressure surface
  - ***The geostrophic wind is parallel to the isohypses with lower heights to the left (in the Northern Hemisphere).***

[3] In this instance the Rossby number would be exactly equal to one. However, a Rossby number of one does not automatically imply inertial balance, because when calculating Rossby number we use characteristic orders of magnitude for the flow parameters, not exact values. Only if the pressure gradient is exactly zero do we have inertial balance. Pure inertial flow rarely if ever occurs. However, there often is an inertial component to the flow. Inertial balance plays a role in some atmospheric phenomena such as nocturnal low-level jets.

- *o* ***The geostrophic wind speed is directly proportional to the geopotential height gradient.***

- Another important feature of the geostrophic wind is that it is non-divergent ($\nabla \bullet \vec{V}_g = 0$) if f is constant.
- The geostrophic wind is sometimes written in terms of the *streamfunction*, ψ, defined as $\psi \equiv p/f\rho$ (in pressure coordinates $\psi = \Phi/f$).
  where f and ρ are constant. In this case, the geostrophic wind is

$$\vec{V}_g = \hat{k} \times \nabla \psi \tag{19}$$

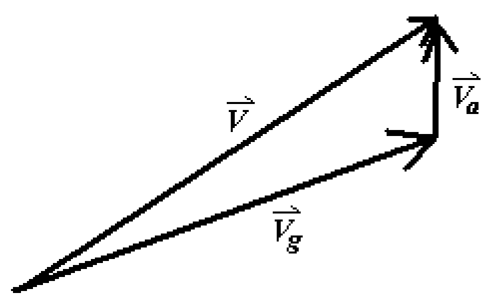
$$u_g = -\frac{\partial \psi}{\partial y} \tag{20}$$

$$v_g = \frac{\partial \psi}{\partial x} \tag{21}$$

THE AGEOSTROPHIC WIND

- The difference between the actual wind and the geostrophic wind is called the *ageostrophic wind*.

$$\vec{V}_a \equiv \vec{V} - \vec{V}_g \tag{22}$$



- Since the atmosphere is usually close to geostrophic balance, the ageostrophic wind is typically small in comparison to the geostrophic wind.
- ***Horizontal divergence is a very important mechanism for rising and sinking motions in the atmosphere.***
  - o Since the geostrophic wind is non-divergent, any divergence must be due to the ageostrophic wind.
  - o Therefore, ***even thought the ageostrophic wind is small, it is very important!***

SCALE ANALYSIS OF THE VERTICAL MOMENTUM EQUATION

- Scale analysis of the vertical momentum equation proceeds as follows (note that in this case δP is the vertical variation in pressure, which is ~ 1000 mb or 10^5 Pa).

$\frac{\partial w}{\partial t}$	$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}$	$w\frac{\partial w}{\partial z}$	$-\frac{u^2+v^2}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial z}$	$2\Omega u\cos\phi$	$-g$
UW/L	UW/L	W^2/H	U^2/a	$\delta P/\rho H$	$2\Omega U\cos 45$	g
10^{-7}	10^{-7}	10^{-8}	10^{-5}	10	10^{-3}	10

$\upsilon\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$	$\upsilon\frac{\partial^2 w}{\partial z^2}$	$-\upsilon\frac{w}{a^2}$	$\upsilon\frac{v}{a^2}\tan\phi$	$-\upsilon\frac{2}{a}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$
$\upsilon W/L^2$	$\upsilon W/H^2$	$\upsilon W/a^2$	$\upsilon U/a^2$	$\upsilon U/La$
10^{-19}	10^{-15}	10^{-20}	10^{-18}	10^{-17}

- This analysis shows that the pressure gradient and gravity terms are dominant.
  - Therefore, on the synoptic scale, the atmosphere can be assumed to be in hydrostatic balance, and the vertical momentum equation simplifies to

$$\frac{\partial p}{\partial z}=-\rho g\,.$$

  - A more rigorous analysis (see *Mesoscale Meteorological Modeling* by Pielke) shows that ***the hydrostatic relation is appropriate if the vertical length scale is much smaller than the horizontal length scale, H << L.*** This condition certainly applies on the synoptic scale.

EXERCISES

1. Perform a scale analysis of the horizontal momentum equations (in component form) for the whirlpool formed as your bathroom sink drains. Which terms are important in this case? Water has a density of 1000 kg/m^3 and a kinematic viscosity of $\nu =$ 1.8×10^{-6} m^2 s^{-1}. The horizontal pressure difference across the whirlpool is ~ 10 Pa. (Use a reasonable estimate for the horizontal velocity based on your own experiences.)

2. What is the Rossby number for a tornado? Does the Coriolis force effect a tornado?

3. Expand the horizontal momentum equation

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p - \hat{k} \times f\,\vec{V}$$

to show that in pure Cartesian-component form it is

$$\left(\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv\right)\hat{i} + \left(\frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu\right)\hat{j} = 0,$$

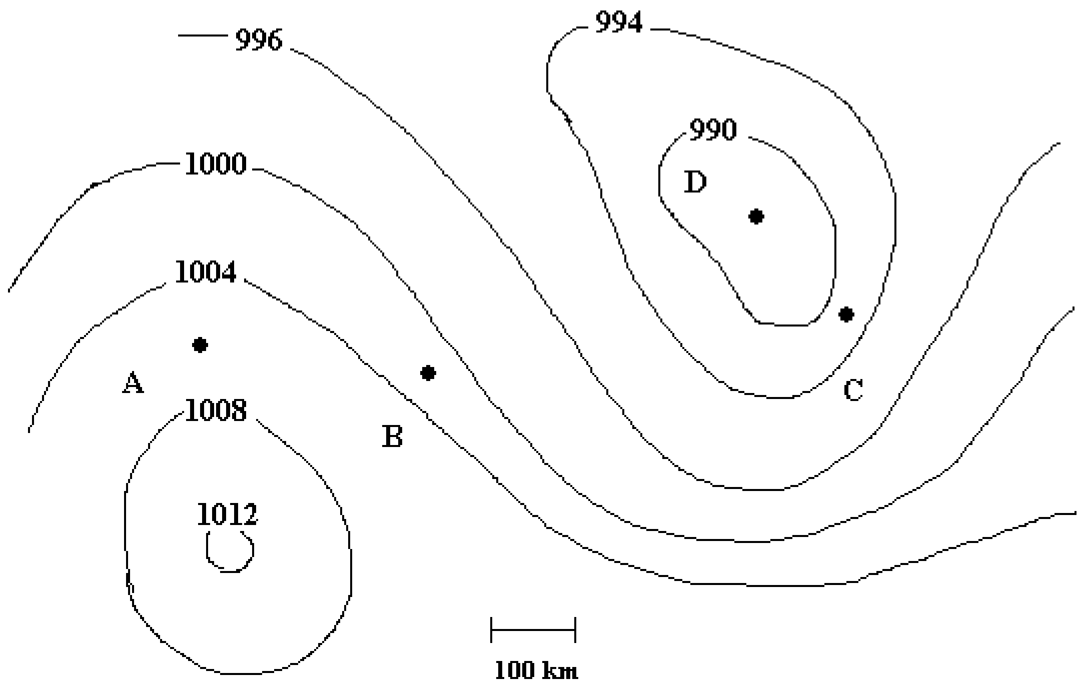
and therefore yields the two component equations

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu$$

4. Show that $\hat{k}\times(\hat{k}\times\vec{V}_g) = -\vec{V}_g$.

5. At the four points shown in the picture below, estimate the magnitude of the geostrophic wind. Assume a density of 1.23 kg/m^3 and a latitude of 45°. The isobars are labeled in mb.



6. Perform a scale analysis of the vertical momentum equation for a midlatitude thunderstorm to find out what terms can be ignored.

ESCI 342 – Atmospheric Dynamics I
Lesson 7 – The Continuity and Additional Equations

Suggested Reading: Martin, Chapter 3

THE SYSTEM OF EQUATIONS IS INCOMPLETE

- The momentum equations in component form comprise a system of three equations with 4 unknown quantities (u, v, p, and ρ).

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv \quad \textbf{(1)}$$

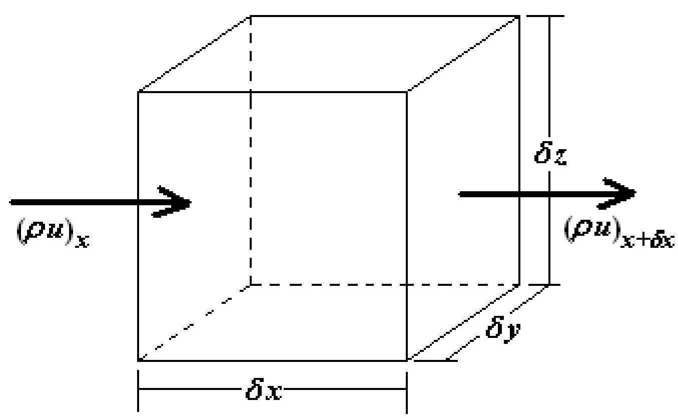
$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \quad \textbf{(2)}$$

$$\frac{\partial p}{\partial z} = -\rho g \quad \textbf{(3)}$$

  - o They are not a closed set, because there are four dependent variables (u, v, p, and ρ), but only three equations. We need to come up with some more equations in order to close the set.

DERIVATION OF THE CONTINUITY EQUATION

- Another principle on which we can derive a new equation is the *conservation of mass*. The equation derived from this principle is called the *mass continuity equation*, or simply the *continuity equation*.
- Imagine a cube at a fixed point in space. The net change in mass contained within the cube is found by adding up the mass fluxes entering and leaving through each face of the cube.[1]



- The mass flux across a face of the cube normal to the x-axis is given by ρu. Referring to the picture below, these fluxes will lead to a rate of change in mass within the cube given by

$$\frac{\partial m}{\partial t} = (\rho u)_x \delta y \delta z - (\rho u)_{x+\delta x} \delta y \delta z \quad \textbf{(4)}$$

---

[1] A flux is a quantity per unit area per unit time. Mass flux is therefore the rate at which mass moves across a unit area, and would have units of kg s^{-1} m^{-2}.

- The mass in the cube can be written in terms of the density as $m = \rho\delta x\delta y\delta z$ so that

$$\frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial t}\delta x\delta y\delta z\,. \qquad \textbf{(5)}$$

  Equating (4) and (5) gives

$$\frac{\partial \rho}{\partial t} = \frac{(\rho u)_x - (\rho u)_{x+\delta x}}{\delta x} = -\left[\frac{(\rho u)_{x+\delta x} - (\rho u)_x}{\delta x}\right]$$

  which becomes

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$$

  as $\delta x \to 0$.
- Similar analysis can be done for the fluxes across the other four faces to yield the continuity equation,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z},$$

  which can also be written in vector form as

$$\frac{\partial \rho}{\partial t} = -\nabla \bullet \left(\rho\vec{V}\right). \qquad \textbf{(6)}$$

- Equation (6) is the *Eulerian form* of the continuity equation.
- A physical interpretation of (6) is that the change in density at a fixed point in space is dependent upon the divergence of the mass flux.
  - o If there is divergence of the mass flux then $\nabla \bullet \left(\rho\vec{V}\right) > 0$ and density will decrease.
  - o If there is convergence of the mass flux then $\nabla \bullet \left(\rho\vec{V}\right) < 0$ and density will increase.
- Using the vector identity

$$\nabla \bullet \left(\rho\vec{V}\right) = \rho\nabla \bullet \vec{V} + \vec{V} \bullet \nabla\rho \qquad \textbf{(7)}$$

  equation (6) can also be written in *Lagrangian form* as

$$\frac{D\rho}{Dt} = -\rho\nabla \bullet \vec{V}\,. \qquad \textbf{(8)}$$

- Equations (6) and (8) are identical! You should be familiar with both forms of the continuity equation.
- A physical interpretation of (8) is that the change in density following a fluid parcel is dependent upon the velocity divergence.
  - o If there is velocity divergence then $\nabla \bullet \vec{V} > 0$ and density will decrease.
  - o If there is velocity convergence then $\nabla \bullet \vec{V} < 0$ and density will increase.

THE INCOMPRESSIBLE CONTINUITY EQUATION

- The Lagrangian form of the continuity equation is

$$\frac{D\rho}{Dt} = -\rho\nabla \bullet \vec{V}\ . \qquad \textbf{(9)}$$

- Under certain conditions the total derivative of pressure on the left-hand-side of the equation is much smaller than the right-hand-side, and we can ignore the time derivative. In this case the continuity equations is simply

$$\nabla \bullet \vec{V} = 0\,. \qquad \textbf{(10)}$$

  This is known as the ***incompressible*** continuity equation, because it is the form of the continuity equations obeyed by an incompressible fluid.
  - Physically, ***incompressibility means that the density of an air parcel doesn't change.***
- The conditions that must be met in order to use the incompressible continuity equation are (derivations are in the Appendix at the end of the lesson):
  - **Condition A:** $U^2 << c^2$ where U is the flow speed and c is the speed of sound.[2]
  - **Condition B:** $H << H_p$ where H is the vertical length scale of the flow, and H_p is the pressure scale height of the atmoshere.
- **Condition A** states that the flow speed must be much less than the speed of sound.
- **Condition B** states that the flow must be shallow compared to the scale height of the atmosphere.
- ***Both Conditions A and B must hold in order for use of the incompressible continuity equation to be valid.***
- Though **Condition A** is met in the atmosphere, **Condition B** is not (except for very shallow circulations). Therefore, the incompressible continuity equation is not appropriate under most circumstances, because as an air parcel moves up and down in the atmosphere its density will change.

THE ANELASTIC CONTINUITY EQUATION

- If **Conditions A** is met, but not **Condition B**, we can still come up with a simplified continuity equation for the atmosphere if we write the density in terms of a reference density that only changes with height, and a perturbation density that can change in any direction and with time such that

$$\rho(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t)$$

  where $\rho' << \rho$ (as is true in synoptic scale motion).
- Substituting this into the continuity equation gives

$$\frac{\partial\rho'}{\partial t} + \nabla \bullet \left(\rho_0\vec{V}\right) + \nabla \bullet \left(\rho'\vec{V}\right) = 0$$

  which scales as

$$\frac{\rho' U}{L} + \rho_0\left(\frac{U}{L} + \frac{W}{H}\right) + \rho'\left(\frac{U}{L} + \frac{W}{H}\right) = 0\,.$$

Since $\rho' << \rho$ we can ignore the terms involving ρ', so that we get the ***anelastic*** continuity equation

$$\nabla \bullet \left(\rho_0\vec{V}\right) = 0\,. \qquad \textbf{(11)}$$

---

[2] Laboratory experiments indicate that $U/C < \sim 0.5$ is sufficient.

- The anelastic continuity equation allows density changes due to vertical motion only.
- The anelastic continuity equation is the appropriate form of the continuity equation to use for the real atmosphere on the synoptic scale. However, for simplicity we will often make use the incompressible continuity equation instead (without introducing significant error for our purposes).

THERMODYNAMIC ENERGY EQUATION

- With the addition of the continuity equation we are up to 4 equations, but now we have 5 unknowns (u, v, w, p, and ρ). We need another equation. This is supplied by the thermodynamic energy equation.
- The thermodynamic energy equation comes from the 1st Law of Thermodynamics (conservation of energy). This equation is derived in detail in other courses, so it won't be derived here.

$$\underbrace{c_p \frac{DT}{Dt}}_{A} - \underbrace{\frac{1}{\rho}\frac{Dp}{Dt}}_{B} = \underbrace{J}_{C} . \qquad \textbf{(12)}$$

- This equation relates changes in temperature following an air parcel to adiabatic compression and expansion of the air parcel and to diabatic heating and cooling.
  - o *Term A* is the temperature change following an air parcel.
  - o *Term B* is the adiabatic heating and cooling term due to vertical motion.
  - o *Term C* is the diabatic heating rate due to radiation, condensation, etc.
- If the motion is adiabatic, then $J = 0$.

THE EQUATION OF STATE

- We are now up to 5 equations, but the thermodynamic energy equation has introduced yet another variable, T. So we ***still*** need another equation. This is supplied by the equation of state (ideal gas law).
- The ideal gas law for air is

$$p = \rho R_d T\left(1 + 0.61q\right) \qquad \textbf{(13)}$$

  where q is the specific humidity.

THE WATER-MASS CONTINUITY EQUATION

- Equations (1), (2), (3), (8), (12), and (13) are six equations, but there are now seven undknowns! So, we still need another equation that hopefully doesn't introduce a new unknown.
- This new equation is the *water-mass continuity equation*,

$$\frac{\partial\left(\rho q\right)}{\partial t} + \nabla \bullet \left(\rho q \vec{V}\right) = S \qquad \textbf{(14)}$$

  where the term S takes into account the sources and sinks of water to the atmosphere.

THE GOVERNING EQUATIONS

- We've now derived the set of equations that govern the atmosphere on the synoptic scale. They are:

$$\frac{\partial u}{\partial t}+\vec{V}\bullet\nabla u=-\frac{1}{\rho}\frac{\partial p}{\partial x}+fv$$

$$\frac{\partial v}{\partial t}+\vec{V}\bullet\nabla v=-\frac{1}{\rho}\frac{\partial p}{\partial y}-fu$$

$$\frac{\partial p}{\partial z}=-\rho g$$

$$\frac{D\rho}{Dt}+\rho\nabla\bullet\vec{V}=0 \quad or \quad \nabla\bullet\left(\rho\vec{V}\right)=0 \quad or \quad \nabla\bullet\vec{V}=0$$

$$c_p\frac{DT}{Dt}-\frac{1}{\rho}\frac{Dp}{Dt}=J$$

$$\frac{\partial\left(\rho q\right)}{\partial t}+\nabla\bullet\left(\rho q\vec{V}\right)=Sources\ and\ Sinks$$

$$p=\rho R_d T\left(1+0.61q\right)$$

- ***These constitute a closed set of 7 equations in 7 unknowns*** (u, v, w, p, T, q, and ρ).
- This set of equations is known as the governing equations. Theoretically, they can be solved to predict or diagnose the future values of the 7 variables.

APPENDIX – This section shows the derivations of conditions A and B. It is included for completeness, but will not be included on examinations or quizzes.

- The speed of sound in a fluid is given by the partial derivative of pressure with respect to density at constant entropy (potential temperature),
$$c^2=\left(\frac{\partial p}{\partial\rho}\right)_\theta .$$
- For an ideal gas, $c=\sqrt{\gamma R'T}$ where $\gamma=c_p/c_v$.
- The thermodynamic variables commonly used are T, ρ, θ, and p. We only need to specify two of them, and any others can be deduced from these two.
- We can therefore write density as a function of pressure and potential temperature, $\rho=f\left(p,\theta\right)$.
- The potential temperature (θ) is conserved under adiabatic conditions. Therefore, under adiabatic conditions $\rho=f\left(p\right)$, and
$$\frac{D\rho}{Dt}=\left(\frac{\partial\rho}{\partial p}\right)_\theta\frac{Dp}{Dt}=\frac{1}{c^2}\frac{Dp}{Dt} .$$
This means the continuity equation can be written as
$$\frac{1}{\rho c^2}\frac{Dp}{Dt}=-\nabla\bullet\vec{V}$$
which can be expanded to
$$\frac{1}{\rho c^2}\left(\frac{\partial p}{\partial t}+\vec{V}\bullet\nabla p\right)=-\nabla\bullet\vec{V} . \qquad \textbf{(15)}$$
  - o From the momentum equation

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p - \hat{k} \times f\vec{V} + \vec{g}$$

we can solve for ∇p to get

$$\nabla p = \rho\frac{D\vec{V}}{Dt} - \rho\hat{k} \times f\vec{V} + \rho\vec{g}\ .$$

Substituting this into (15) gives

$$\frac{1}{\rho c^2}\left[\frac{\partial p}{\partial t} + \vec{V} \bullet \left(\rho\frac{D\vec{V}}{Dt} - \rho\hat{k} \times f\vec{V} + \rho\vec{g}\right)\right] = -\nabla \bullet \vec{V}$$

which reduces to

$$\frac{1}{c^2}\left(\frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{1}{2}\frac{D|V|^2}{Dt} - gw\right) = -\nabla \bullet \vec{V} \qquad \textbf{(16)}$$

- In terms of order of magnitude, this equation is

$$\frac{\delta PU}{\rho c^2 L} + \frac{U^3}{c^2 L} + \frac{gW}{c^2} = \frac{U}{L}\ .$$

- If the pressure suddenly changed at a point in the fluid by an amount δP, you would expect that the change in velocity would be given by

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla(\delta p)$$

and so δP would be of the order of ρU^2. Therefore, the orders of magnitude of equation (16) become

$$\frac{U^3}{c^2 L} + \frac{U^3}{c^2 L} + \frac{gW}{c^2} = \frac{U}{L}\ .$$

- The first two terms of equation (16) can be ignored only if

$$\frac{U^3}{c^2 L} << \frac{U}{L}\ ,$$

which becomes

$$\frac{U^2}{c^2} << 1\ . \qquad \textbf{(A)}$$

- The third term can be ignored if

$$\frac{gW}{c^2} << \frac{U}{L}\ ,$$

which can also be written as

$$\frac{gH}{c^2}\frac{W}{H} << \frac{U}{L}\ .$$

Since

$$\frac{W}{H} \le \frac{U}{L}$$

the condition becomes

$$\frac{gH}{c^2} << 1\ .$$

This can be written as

$$\frac{gH}{\gamma R'T} \cong \frac{1}{\gamma}\frac{H}{H_p} << 1 \qquad \textbf{(B)}$$

where H_p is the pressure scale height.

EXERCISES

1. Show that ρu has the units of mass flux.

2. Use the vector identity $\nabla \bullet s\vec{A} = s\nabla \bullet \vec{A} + \vec{A} \bullet \nabla s$ to show that the two forms of the continuity equation we derived are equivalent.

3. You are studying the land-sea breeze circulation. This circulation has a typical depth of 1000 meters or so. Is it appropriate to use the incompressible continuity equation in this case?

4. Show that ρq is equal to absolute humidity, ρ_v.

ESCI 342 – Atmospheric Dynamics I
Lesson 8 – Geostrophic and Gradient Balance

Suggested Reading: Martin, Chapter 4 (except Section 4.3)

PRESSURE GRADIENT ACCELERATION IN HEIGHT AND PRESSURE COORDINATES

- In a prior lesson we've already established that the pressure gradient acceleration, which in height (z) coordinates is

$$PGA = -\frac{1}{\rho}\nabla p\,, \qquad \textbf{(1)}$$

is in pressure coordinates given as

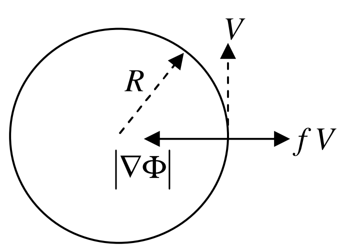
$$PGA = -\nabla\Phi\,, \qquad \textbf{(2)}$$

where Φ is geopotential.

- Since (2) is simpler and easier to write, we will use pressure coordinates in this lesson.
- Keep in mind that the equations we develop in this lesson can be converted to height coordinates simply by replacing $\nabla\Phi$ or spatial derivatives of Φ $\nabla p/\rho$ or its spatial derivatives.

GRADIANT WIND AROUND A LOW PRESSURE

- The diagram below shows the directions of the Coriolis and pressure-gradient accelerations for normal flow around a low. The radius of the flow is *R*, and the speed is *V*.



- The balance of acceleration in the radial direction is

$$\frac{V^2}{R} = |\nabla\Phi| - fV\,, \qquad \textbf{(3)}$$

which can be rearranged to

$$V^2 + fRV - R|\nabla\Phi| = 0\,. \qquad \textbf{(4)}$$

- Equation (4) is quadratic in *V*, and using the quadratic formula is solved for *V* as

$$V = -\frac{fR}{2} \pm \frac{1}{2}\sqrt{f^2R^2 + 4R|\nabla\Phi|}\;. \qquad \textbf{(5)}$$

- Equation (5) gives the gradient wind speed around a low pressure. What is interesting is that there are two solutions, corresponding to the two roots of the radical.
  - o The first solution, with the + sign, is

$$V = -\frac{fR}{2} + \frac{1}{2}\sqrt{f^2R^2 + 4R\left|\nabla\Phi\right|}\ , \qquad \textbf{(6)}$$

and is the gradient wind around a *regular low*.

- o The second solution, with the − sign, is

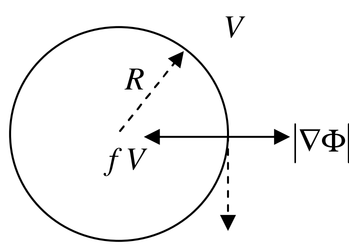
$$V = -\frac{fR}{2} - \frac{1}{2}\sqrt{f^2R^2 + 4R\left|\nabla\Phi\right|}\ , \qquad \textbf{(7)}$$

and is the gradient wind around an *anomalous low*.

- o The winds for the anomalous low are very large, and actually in the opposite direction of the regular low (since the speed is negative).
- o In the anomalous low, the pressure-gradient and Coriolis accelerations both point toward the center and contribute to the centripetal acceleration.

GRADIANT WIND AROUND A HIGH PRESSURE

- The diagram below shows the directions of the Coriolis and pressure-gradient accelerations for normal flow around a high.



- In order to be balanced, the pressure gradient acceleration and the Coriolis must sum up to equal the centripetal accelerations, which is V^2/R . Therefore, the following equation must be true,

$$V^2/R = f\,V - \left|\nabla\Phi\right|,$$

which can be rearranged to

$$V^2 - f\,RV + R\left|\nabla\Phi\right| = 0. \qquad \textbf{(8)}$$

- Solving (8) for *V* yields

$$V = \frac{fR}{2} \pm \frac{1}{2}\sqrt{f^2R^2 - 4R\left|\nabla\Phi\right|}\ . \qquad \textbf{(9)}$$

- Equation (9) gives the gradient wind speed around a high pressure. As with the low, there are two solutions, corresponding to the two roots of the radical.
  - o The first solution, with the + sign, is

$$V = \frac{fR}{2} + \frac{1}{2}\sqrt{f^2R^2 - 4R\left|\nabla\Phi\right|}\ , \qquad \textbf{(10)}$$

and is the gradient wind around an *anomalous high*.

  - o The second solution, with the − sign, is

$$V = \frac{fR}{2} - \frac{1}{2}\sqrt{f^2R^2 - 4R\left|\nabla\Phi\right|}\ , \qquad \textbf{(11)}$$

and is the gradient wind around a *regular high*.

  - o The winds for both the anomalous and regular high rotate in the normal sense, but the speed for the anomalous high is very large.

- For highs, there is a restriction on how strong the pressure gradient may be.
    - Notice in (10) and (11) that in order for the wind speed to be "real" and not have an "imaginary" component, the following condition must hold
$$f^2R^2 - 4R\left|\nabla\Phi\right| \geq 0\,. \tag{12}$$
    - This requires that
$$\left|\nabla\Phi\right| \leq \frac{f^2}{4}\left|R\right|. \tag{13}$$
    - This explains why, on a synoptic scale weather map, we often see tightly wound lows with large pressure gradients right down to the center, while we never see large pressure gradients in the center of high pressures. *For the high pressure case, there is a physical limit as to how large the pressure gradient can be near the center.*

MORE ON GRADIENT WIND

- Under most large-scale flow regimes the atmosphere is close to being in gradient balance, and the gradient wind equations, (6) and (11), are appropriate to use for the low and high respectively.
- The anomalous low and high are not readily observed in the atmosphere, primarily because anomalous low and high both have negative absolute vorticity, and it is difficult to imagine a situation where a large scale circulation can develop having negative absolute vorticity.
- It may be possible to observe anomalous gradient flows on the small scale. However, such flows are actually likely to have large Rossby numbers, and be closer to cyclostrophic balance rather than in gradient balance.
- For more information on the anomalous solutions refer to:
    - Fultz, 1991: "Quantitative nondimensional properties of the gradient wind", *J. Atmos. Sci.*, **48**, 869-875
    - Chew, F. and M.H. Bushnesll, 1990, "The half-inertial flow in the Eastern Equatorial Pacific: A case study", *J. Phys. Ocean.*, **20**, 1124-1133
    - Mogil, H.M. and R.L. Holle, 1972: "Anomalous gradient winds: Existence and implications", *Mon. Wea. Rev.*, **100**, 709-716
    - Alaka, M.A., 1961: "The occurrence of anomalous winds and their significance", *Mon. Wea. Rev.*, **89**, 482-494
- Note that the anomalous high exhibits the curious behavior that the wind speed actually increases as the pressure gradient force decreases!
- The anomalous high and anomalous low both become inertial flow as the pressure gradient goes to zero.

SIMPLIFIED EXPRESSION OF THE GRADIENT WIND EQUATION

- The gradient wind equation can also be written in terms of the geostrophic wind speed. Since, by definition,
$$V_g = \left|\nabla\Phi\right|/f\,, \tag{14}$$
  the gradient wind for the regular cyclone and anticyclone can be expressed as

Cyclone: $$V_{gr} = \frac{fR}{2}\left(\sqrt{1+\frac{4V_g}{fR}} - 1\right) \qquad \textbf{(15)}$$

Anticyclone: $$V_{gr} = \frac{fR}{2}\left(1 - \sqrt{1-\frac{4V_g}{fR}}\right) \qquad \textbf{(16)}$$

TRAJECTORIES VS. STREAMLINES

- A *trajectory* is a curve tracing the successive points of the particles position in time.
- A *streamline* is a line that is tangent to the velocity at a point, at a given instant.
- Trajectories and streamlines only coincide if the fluid motion is steady.
- The local rate of change of wind direction is

$$\frac{\partial \beta}{\partial t} = V\left(\frac{1}{R_T} - \frac{1}{R_S}\right)$$

where R_T and R_S are the radii of curvature for the trajectory and the streamline respectively.

EXERCISES

1. For the same values of pressure gradient, Coriolis parameter, and radius of curvature, the flow around a regular low is slower than the flow around a regular high. Give a physical explanation of this fact. Drawing force diagrams should be very helpful.

2. The geostrophic wind speed is given as

$$V_g = \frac{|\nabla \Phi|}{f}.$$

Use this to derive (15) and (16).

ESCI 342 – Atmospheric Dynamics I
Lesson 9 – Thermal Wind

Suggested Reading: Martin, Section 4.3

THERMAL WIND

- The geostrophic wind in pressure coordinates is

$$\vec{V}_g = \hat{k}\times\frac{g_0}{f}\nabla_p Z \quad \textbf{(1)}$$

- The difference in geostrophic wind between two levels is

$$\vec{V}_{g2}-\vec{V}_{g1}=\frac{g_0}{f}\hat{k}\times\nabla_p Z_2-\frac{g_0}{f}\hat{k}\times\nabla_p Z_1=\frac{g_0}{f}\hat{k}\times\nabla_p\left(Z_2-Z_1\right)$$

or

$$\vec{V}_{g2}-\vec{V}_{g1}=\frac{g_0}{f}\hat{k}\times\nabla_p Z_\Delta, \quad \textbf{(2)}$$

where

$$Z_\Delta = Z_2 - Z_1 \quad \textbf{(3)}$$

is the geopotential thickness between layers 2 and 1.
- This shows that the difference between the geostrophic wind at two layer is parallel to the contours of thickness.
- Substituting the hypsometric equation

$$Z_\Delta=\frac{R_d}{g_0}\ln\left(\frac{p_1}{p_2}\right)\overline{T} \quad \textbf{(4)}$$

into (2) shows that the difference in geostrophic wind is parallel to the contours of layer average temperature,

$$\vec{V}_{g2}-\vec{V}_{g1}=\frac{R_d}{f}\ln\left(\frac{p_1}{p_2}\right)\hat{k}\times\nabla_p\overline{T}\,. \quad \textbf{(5)}$$

- Since the difference in wind is parallel to the layer-mean isotherms, it is commonly referred to as the *thermal wind*, and denoted as $\vec{V}_T$ so that we have two equivalent expressions for the thermal wind,

$$\vec{V}_T=\frac{g_0}{f}\hat{k}\times\nabla_p Z_\Delta \quad \textbf{(6)}$$

or

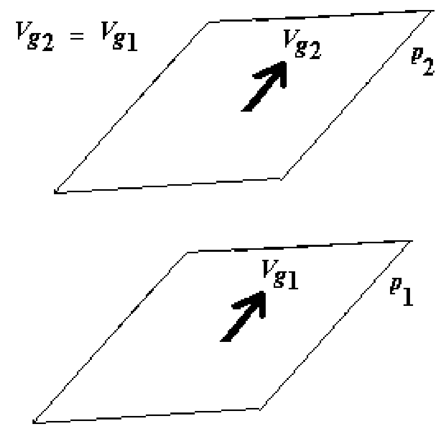
$$\vec{V}_T=\frac{R_d}{f}\ln\left(\frac{p_1}{p_2}\right)\hat{k}\times\nabla_p\overline{T}\,. \quad \textbf{(7)}$$

- Rules for the thermal wind
  - o The thermal wind is parallel to the thickness lines with low thickness to the left.
  - o The stronger the thickness gradient, the stronger the thermal wind.
- The rules for the thermal wind are analogous to those for the geostrophic wind, except that thickness is substituted for geopotential height.
- If you add the thermal wind to the geostrophic wind at the lower layer, you will get the geostrophic wind at the upper layer.
- Like the geostrophic wind, ***the thermal wind is a definition.***

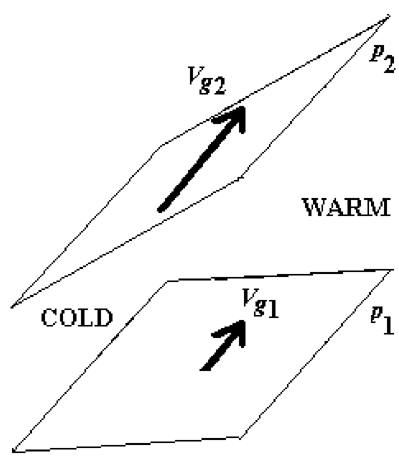
- o The actual difference between the wind at two levels will equal the thermal wind, only if the actual winds at the two levels are geostrophic. However, since the atmosphere is usually close to geostrophic balance, the thermal wind is a good approximation to the actual difference in wind between two levels.

PHYSICAL EXPLANATION OF THERMAL WIND

- The physical basis for the thermal wind can be explained as follows.
- First, remember that on a constant pressure surface the geostrophic wind is normal to the height gradient, and the speed is proportional to the slope of the pressure surface.
- Second, remember that the thickness between two pressure surfaces is proportional to the average temperature in the layer.
- If there is no thermal gradient in the layer, an upper level-pressure surface will be sloped the same as the lower-level pressure surface, and so the geostrophic wind on each surface will be identical.

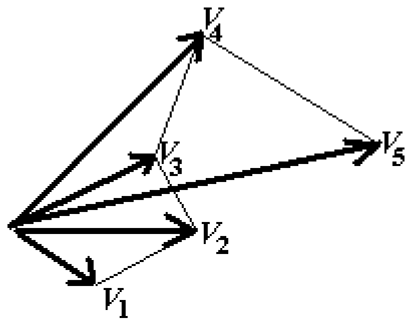


- If there is a thermal gradient in the layer, the upper-level surface will have a different slope than the lower-level surface, and therefore a different geostrophic wind.

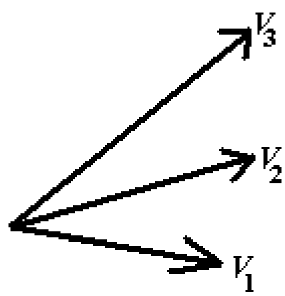


BACKING AND VEERING WINDS

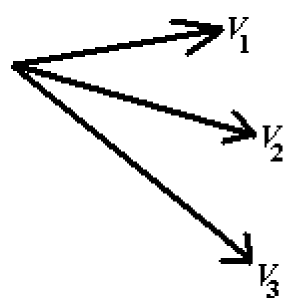
- A *hodograph* is a graph made by placing the tails of the wind vectors at different levels together, and then drawing a line that sequentially connects their heads in ascending order (see example below)



- *Backing winds* are winds whose vectors rotate counter-clockwise (either with time or with height).



- *Veering winds* are winds whose vectors rotate clockwise (either with time or with height).

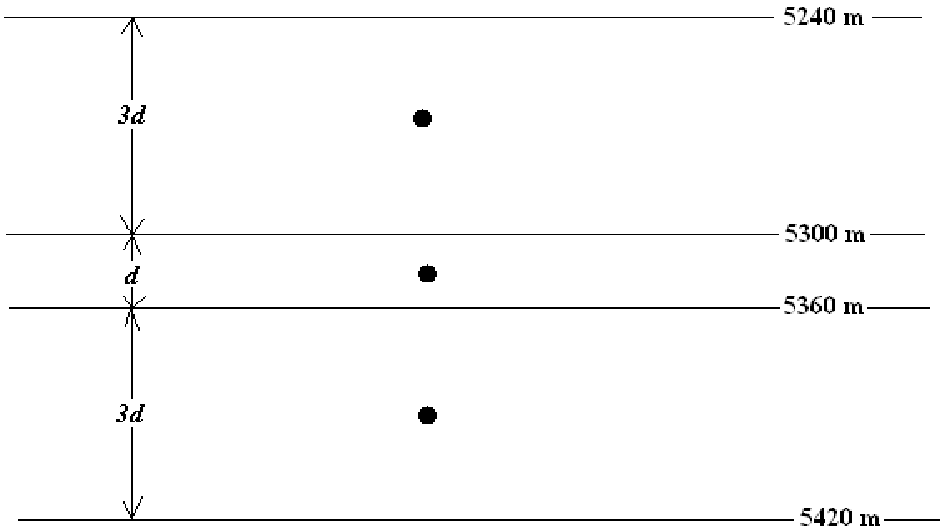


- The thermal wind leads to the following relations between the winds on a hodograph and temperature advection.
  - o Veering winds indicate warm-air advection
  - o Backing winds indicate cold-air advection

EXERCISES

1. The geostrophic wind is $\vec{V}_g = \hat{k} \times \frac{g_0}{f} \nabla_p Z$. Take the partial derivative of this with respect to p and show that $\frac{\partial \vec{V}_g}{\partial p} = -\frac{R_d}{f\,p} \hat{k} \times \nabla_p T$.

2. The diagram below shows contours of 1000 – 500 mb thickness.



a. Assume the 1000 mb geostrophic wind is SW at 5 m/s. At the three black dots draw wind barbs representing the geostrophic wind direction and speed at 500 mb. Use a latitude of 45 N, and d = 175 km.

b. Explain why the position of the jet stream seems linked to the position of the polar front.

ESCI 342 – Atmospheric Dynamics I
Lesson 10 – Vertical Motion, Pressure Coordinates

Reading: Martin, Section 4.1

PRESSURE COORDINATES

- Pressure is often a convenient vertical coordinate to use in place of altitude.
- If the hydrostatic approximation is used, the relationship between pressure and altitude is given by the hydrostatic equation,

$$\frac{\partial p}{\partial z} = -\rho g\,. \tag{1}$$

- In height coordinates the vertical velocity is defined as $w \equiv Dz/Dt$. In pressure coordinates the vertical velocity is defined as

$$\omega \equiv \frac{Dp}{Dt}, \tag{2}$$

  and is commonly called simply *omega*.
  - The units of ω are Pa/s (often microbars per second, μb/s, is also used).
  - ***Since pressure decreases upward, a negative omega means rising motion, while a positive omega means subsiding motion.***
- w and ω are related as follows:

$$\omega = \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\,. \tag{3}$$

- On the synoptic scale we can assume hydrostatic vertical balance, so that (3) becomes

$$\omega = \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} - \rho g w \cong -\rho g w\,, \tag{4}$$

  since the local pressure tendencies and horizontal pressure advection terms are much, much smaller in magnitude than the vertical pressure advection terms.
- The total derivative in pressure coordinates is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}\,.^{1} \tag{5}$$

- The conversion of a height derivative to a pressure derivative is accomplished using the chain rule as follows

$$\frac{\partial}{\partial p} = \frac{\partial z}{\partial p}\frac{\partial}{\partial z} = -\frac{\alpha}{g}\frac{\partial}{\partial z}\,. \tag{6}$$

- In pressure coordinates, the directions of the unit vectors ($\hat{i}$, $\hat{j}$, and $\hat{k}$) are the same as in height coordinates. The x and y axes are still horizontal, and <u>*not*</u> oriented along the constant pressure surface.[2] The vertical axis is still vertical (perpendicular to x and y.)

---

[1] If you take the total derivative of pressure you end up with the seeming absurdity that $\omega = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \omega$. However, the three partial-derivative terms are actually zero since they are taken holding pressure (the vertical coordinate) constant.

[2] See "The quasi-static equations of motion with pressure as independent variable", A. Eliassen, *Geof. Publ.*, **17**, 1949

- o The u and v components of the wind are the same in both height and pressure coordinates.

MOMENTUM EQUATIONS IN PRESSURE COORDINATES

- In pressure coordinates the horizontal momentum equation is

$$\frac{D\vec{V}_H}{Dt} = -\nabla_p \Phi - \hat{k} \times f\vec{V}_H \qquad \textbf{(7)}$$

- The hydrostatic equation in pressure coordinates is

$$\frac{\partial \Phi}{\partial p} = -\alpha\,. \qquad \textbf{(8)}$$

CONTINUITY EQUATION IN PRESSURE COORDINATES

- The continuity equation is pressure coordinates is derived by writing the conservation of mass, m, for a parcel as follows:

$$\frac{Dm}{Dt} = \frac{D}{Dt}\left(\rho \delta x \delta y \delta z\right) = 0\,. \qquad \textbf{(9)}$$

If the atmosphere is in hydrostatic balance, then $\rho\delta z = -\delta p/g$, so (9) becomes

$$\frac{D}{Dt}\left(\delta x \delta y \delta p\right) = 0\,. \qquad \textbf{(10)}$$

Equation (10) expands out as

$$\frac{D}{Dt}\left(\delta x \delta y \delta p\right) = \delta y \delta p \frac{D}{Dt}\left(\delta x\right) + \delta x \delta p \frac{D}{Dt}\left(\delta y\right) + \delta x \delta y \frac{D}{Dt}\left(\delta p\right) = 0$$

which can also be written as

$$\frac{1}{\delta x}\frac{D}{Dt}\left(\delta x\right) + \frac{1}{\delta y}\frac{D}{Dt}\left(\delta y\right) + \frac{1}{\delta p}\frac{D}{Dt}\left(\delta p\right) = 0\,. \qquad \textbf{(11)}$$

From the fact that

$$\frac{D}{Dt}\left(\delta x\right) = \delta u; \quad \frac{D}{Dt}\left(\delta y\right) = \delta v; \quad \frac{D}{Dt}\left(\delta p\right) = \delta \omega$$

equation (11) becomes

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} = 0$$

which in the limit as the parcel becomes infinitesimally small is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0\,. \qquad \textbf{(12)}$$

- Equation (12) is the ***full continuity equation*** in pressure coordinates. It contains no assumptions about incompressibility.
- ***The full continuity equation in pressure coordinates looks very much like the incompressible continuity equation. This is one of the advantages of using pressure coordinates.***

THERMODYNAMIC ENERGY EQUATION IN PRESSURE COORDINATES

- The thermodynamic energy equation in pressure coordinates is

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J\ , \tag{13}$$

which expanded out, and using the definition of ω, becomes

$$\underset{A}{\frac{\partial T}{\partial t}} = \underset{B}{-\vec{V} \bullet \nabla_p T} - \left(\underset{C}{\frac{\partial T}{\partial p}} - \underset{D}{\frac{\alpha}{c_p}} \right) \omega + \underset{E}{\frac{J}{c_p}} \quad . \tag{14}$$

In this form, the terms represent:

Term A – Local temperature tendency
Term B – Horizontal thermal advection
Term C – Vertical thermal advection
Term D – Adiabatic expansion/compression due to vertical motion
Term E – Diabatic heating (radiation, latent heat, etc.)

- Terms C and D can be combined and written as

$$\frac{\partial T}{\partial p} - \frac{\alpha}{c_p} = \left(\frac{\alpha}{\theta} \frac{\partial \theta}{\partial p} \right) \frac{p}{R_d}\ ,$$

  and defining the *static-stability parameter*, σ, as

$$\sigma \equiv -\frac{\alpha}{\theta} \frac{\partial \theta}{\partial p}\ , \tag{15}$$

  we get the following form of the thermodynamic energy equation in pressure coordinates.

$$\underset{A}{\frac{\partial T}{\partial t}} = \underset{B}{-\vec{V} \bullet \nabla_p T} + \underset{C}{\frac{\sigma p}{R_d} \omega} + \underset{D}{\frac{J}{c_p}}\ . \tag{16}$$

  - In this form of the equation, the vertical advection and adiabatic expansion/compression are combined into one term, Term C.
- The static stability parameter is a positive number for a stable atmosphere, and a negative number for an unstable atmosphere.

VERTICAL MOTION

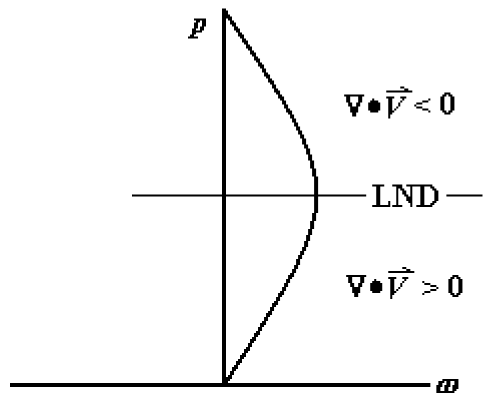
- Vertical motion is very important for forming clouds, and also effects the stability of the atmosphere; however, it is not routinely measured. Therefore, it must be inferred from other measured quantities.
- ***Kinematic method for calculating vertical motion***
  - One method of calculating the vertical motion is the *kinematic* method, which integrates the continuity equation between the surface and some pressure level above the surface to get

$$\omega(p) = \omega(p_s) + \int_p^{p_s} \nabla_p \bullet \vec{V}\, dp\ . \tag{17}$$

  - If the surface is flat and the surface pressure tendency is zero, then $\omega(p_s) = 0$ and (17) becomes

$$\omega(p) = \int_{p}^{p_s} \nabla_p \bullet \vec{V}\, dp\,. \qquad \textbf{(18)}$$

- o This gives the expected result that integrated convergence gives upward motion and integrated divergence gives downward motion.
- o The kinematic method has some major flaws. It is only the ageostrophic part of the wind field that can be divergent, and this is very small compared to the actual wind. In fact, the ageostrophic wind is of the order of the errors in the wind observations themselves. This means that divergences calculated from the observed winds may have large errors.
- o Though it isn't much use for calculating actual values of vertical motion, the kinematic method is good for illustrating some general points about divergence and its relation to vertical motion.
- o Since the vertical motion must disappear at the ground, and is also usually quite small at the top of the troposphere, a graph of the vertical motion with height would look something like that shown below



- o Since $\partial\omega/\partial p$ must disappear at some level, the divergence also disappears at that level. This leads to the conclusion that
  - ▪ ***There is some level in the atmosphere at which there is no horizontal divergence. This level is known as the level of non-divergence, or LND.***
- o Observations indicate that the level of non-divergence usually occurs at around 600 mb. However, since 600 mb is not a standard pressure level for reporting, traditionally meteorologists consider 500 mb to be the level of non-divergence.

- ***Adiabatic method for calculating vertical motion***
  - o Another method for calculating vertical motion uses the thermodynamic energy equation, (16), solved for ω and assuming adiabatic conditions, to get

$$\omega = \frac{R_d}{\sigma p}\left(\frac{\partial T}{\partial t} + \vec{V} \bullet \nabla_p T\right). \qquad \textbf{(19)}$$

  - o A drawback to this method is that temperature tendency must also be known.

PRESSURE TENDENCY EQUATION

- An equation for the change in pressure at a fixed point in the atmosphere can be derived as follows:
  - o Differentiate the hydrostatic equation with respect to time to get

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial z}\right) = -g\,\frac{\partial \rho}{\partial t}\,. \qquad \textbf{(20)}$$

Substituting for $\partial\rho/\partial t$ from the continuity equation gives

$$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}\right) = g\nabla \bullet\left(\rho\vec{V}\right) = g\nabla_H \bullet\left(\rho\vec{V}\right) + g\,\frac{\partial}{\partial z}\left(\rho w\right)$$

which when integrated from some level *z* to the top of the atmosphere yields

$$\left.\frac{\partial p}{\partial t}\right|_{\infty} - \left.\frac{\partial p}{\partial t}\right|_{z} = g\int_z^{\infty}\nabla_H \bullet\left(\rho\vec{V}\right)dz - g\rho w\,. \qquad \textbf{(21)}$$

Since $\partial p/\partial t$ at the top of the atmosphere is zero, equation (21) for the pressure tendency at level *z* is

$$\frac{\partial p}{\partial t} = -g\int_z^{\infty}\nabla_H \bullet\left(\rho\vec{V}\right)dz + g\rho w$$

This equation can be expanded as

$$\frac{\partial p}{\partial t} = -g\int_z^{\infty}\rho\nabla_H \bullet\vec{V}\,dz - g\int_z^{\infty}\vec{V}\bullet\nabla_H\rho\,dz + g\rho w\,. \qquad \textbf{(22)}$$

Using the ideal gas law we can show that

$$\nabla_H\rho = R_d^{-1}\nabla_H\left(p/T\right) = R_d^{-1}T^{-2}\left[T\nabla_H p - p\nabla_H T\right]$$

so (22) becomes

$$\underset{A}{\frac{\partial p}{\partial t}} = \underset{B}{-g\int_z^{\infty}\rho\nabla_H \bullet\vec{V}dz} - \underset{C}{\frac{g}{R_d}\int_z^{\infty}\frac{1}{T}\left(\vec{V}\bullet\nabla_H p\right)dz} + \underset{D}{\frac{g}{R_d}\int_z^{\infty}\frac{p}{T^2}\left(\vec{V}\bullet\nabla_H T\right)dz} + \underset{E}{g\rho w}\,. \qquad \textbf{(23)}$$

- The physical interpretation of the pressure tendency equation is as follows:
  - ***Term A*** represents the local pressure tendency
  - ***Term B*** represents the vertically integrated divergence above the level of interest.
    - Integrated divergence above the layer leads to lower pressure.
    - Integrated convergence above the layer leads to higher pressure.
  - ***Term C*** represents integrated advection of pressure.
    - If the winds are in geostrophic or gradient balance, this term will be zero.
  - ***Term D*** represents the integrated temperature advection.
    - Advection of warm air lowers the pressure.
  - ***Term E*** represents advection of mass across the layer.
    - Upward vertical velocity leads to increased pressure, as the mass is moved above the level (since it is the mass above the level that determines the pressure in a hydrostatic atmosphere).
    - At the surface of the Earth, if the surface if level, then ***Term E*** would be zero.

EXERCISES

1. Show that the hydrostatic equation in pressure coordinates is $\partial\Phi/\partial p = -\alpha$. Hint: Start with $\partial p/\partial z = -\rho g$ and use the chain rule and the definition of geopotential.

2. Show that

$$\frac{\partial T}{\partial p} - \frac{\alpha}{c_p} = \left(\frac{\alpha}{\theta}\frac{\partial \theta}{\partial p}\right)\frac{p}{R_d}.$$

Hint: Take $\partial/\partial p$ of $T = \theta(p/p_0)^{\kappa}$.

3. If horizontal advection and diabatic heating are negligible, then the local temperature tendency from the thermodynamic energy equation is

$$\frac{\partial T}{\partial t} = \frac{\sigma p}{R_d}\omega.$$

This equation says that if the atmosphere is stable then downward motion will result in an increase in temperature at a fixed level, while if the atmosphere is unstable then downward motion will result in a decrease in temperature at a fixed level. Give a ***physical*** explanation as to why this occurs.

4. Use the adiabatic method to estimate the 500 mb vertical velocity (ω) for the following situation. The temperature tendency is zero. The temperature at 600 mb is −13°C, at 500 mb it is −19°C. The wind at 500 mb is from the SW at 20 m/s, and the temperature at 500 mb increases toward the West at 1°C/100 km.

5. For a typical tropical cyclone (which is a warm-core circulation in gradient balance), explain whether each term in the pressure tendency equation contributes to surface development (lower pressures at the surface) or to weakening (higher pressures at the surface). Which terms do you think are most important for surface development?

ESCI 343 - Atmospheric Dynamics
Lesson 11 - Circulation
Dr. DeCaria

References:

An Introduction to Atmospheric Dynamics, 4th ed., Holton
An Informal Introduction to Theoretical Fluid Mechanics, Lighthill

Barotropic and Baroclinic Fluids

- A *barotropic* fluid is one in which surfaces of constant pressure and constant density are parallel (see Fig. 1).

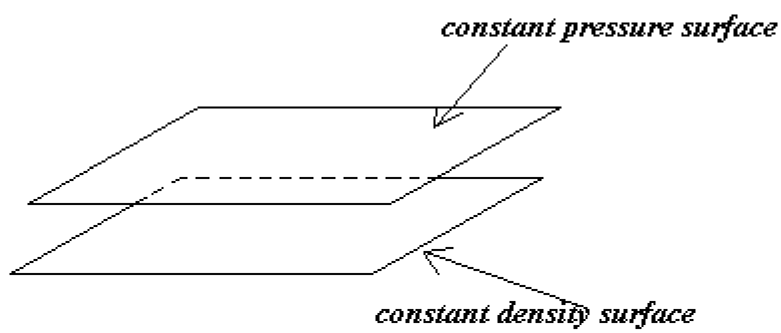


Figure 1: Pressure and density surfaces are parallel in a barotropic fluid.

- In a barotropic fluid the density is constant along a constant pressure surface.
  - From the ideal gas law this implies that, for a barotropic atmosphere, temperature is constant on a constant pressure surface, since $p/\rho = R_d T$, and since both p and ρ are constant on a pressure surface, then so would T.
  - If the atmosphere were truly barotropic there would be no isotherms on a constant pressure map.
- In a barotropic fluid the thermal wind is zero. Therefore, the flow is the same at all levels. ***There is no vertical wind shear in a barotropic atmosphere***.
- If a fluid is not barotropic it is *baroclinic*. In baroclinic fluids the pressure and density surfaces intersect as shown in Fig. 2.
  - In a baroclinic atmosphere there will be a temperature gradient on a constant pressure surface.
  - In a baroclinic atmosphere the flow will be different at different levels.

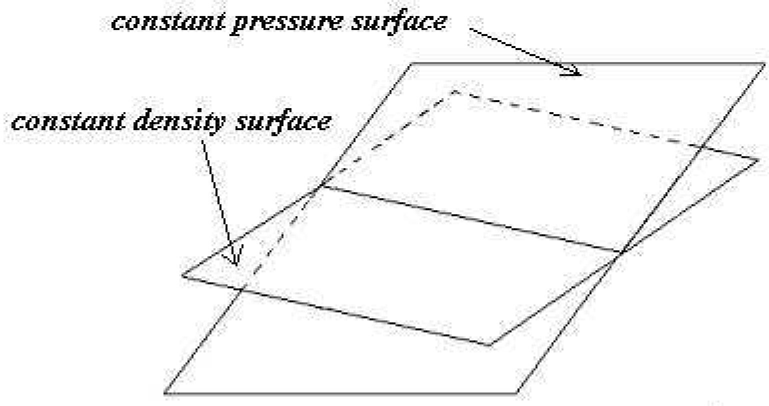


Figure 2: Pressure and density surfaces intersect in a baroclinic fluid.

- In the atmosphere the isotherms are sometimes parallel to the height contours (see Fig. 3).
  - In this case the wind changes speed with height, but is always in the same direction. Though technically baroclinic, this situation is typically referred to as *equivalent barotropic*.
  - **BEWARE!** Often times meteorologists say barotropic when they really mean equivalent barotropic.

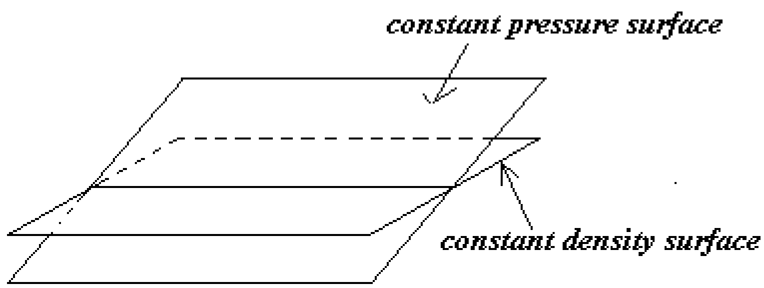


Figure 3: Pressure and density surfaces intersect along height contours in an equivalent barotropic fluid.

- The baroclinicity/barotropicity of the atmosphere can change with altitude.
  - Just because a particular pressure level is equivalent barotropic, does not mean the entire atmosphere above and below it is equivalent barotropic.
- To see if a particular level or region is barotropic, equivalent barotropic, or baroclinic we only have to look at a constant pressure surface and see the orientation of the isotherms (or thickness lines).

  - If the isotherms are very widely spaced then the region or level is close to barotropic.
  - If the isotherms are parallel to the height contours then the region or level is equivalent barotropic.
  - If the isotherms cross the height contours the region or level is baroclinic.

- In Fig. 4 the solid lines are the 850 mb height contours and the dashed lines are the 500-1000 mb thickness contours.

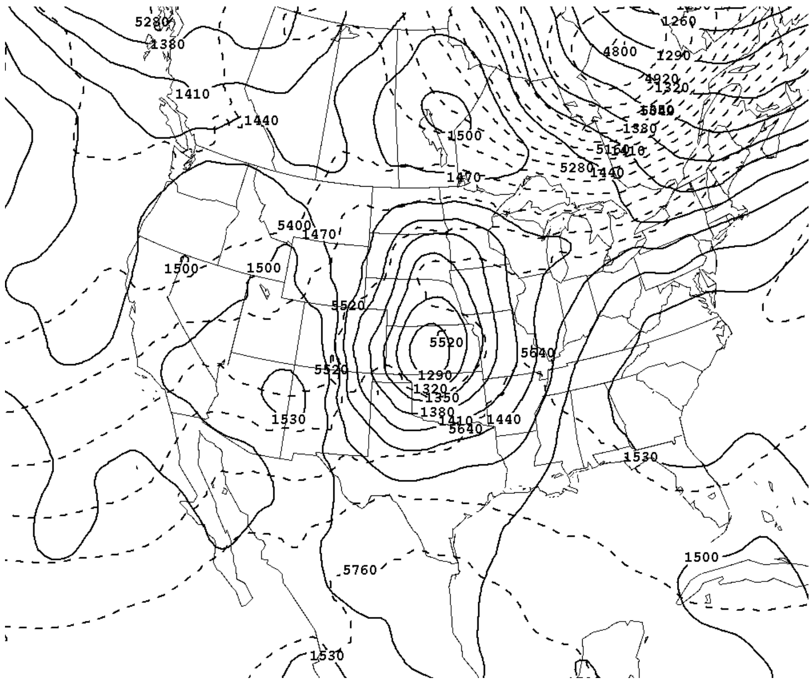


Figure 4: Example of 850 mb heights and 1000-500 mb thicknesses.

  - The region near New England at 850 mb is highly baroclinic, since the height and thickness lines cross at large angles.
  - The low over Kansas at 850 mb is nearly equivalent barotropic since the height and thickness lines are nearly parallel.

- A map of the tropics (Fig. 5) shows that the height contours and thickness contours are spread very far apart (see example below). This is characteristic of the tropical atmosphere away from cyclones, etc. The tropics therefore tend to be barotropic (this is easy to remember since tropic appears in barotropic).

  - Notice the equivalent barotropic low near the Philippines.

- A fluid that starts out barotropic can become baroclinic, except if the density is constant.

- A fluid with constant density is called *autobarotropic*, because it is always barotropic (density is always constant on a constant pressure surface).

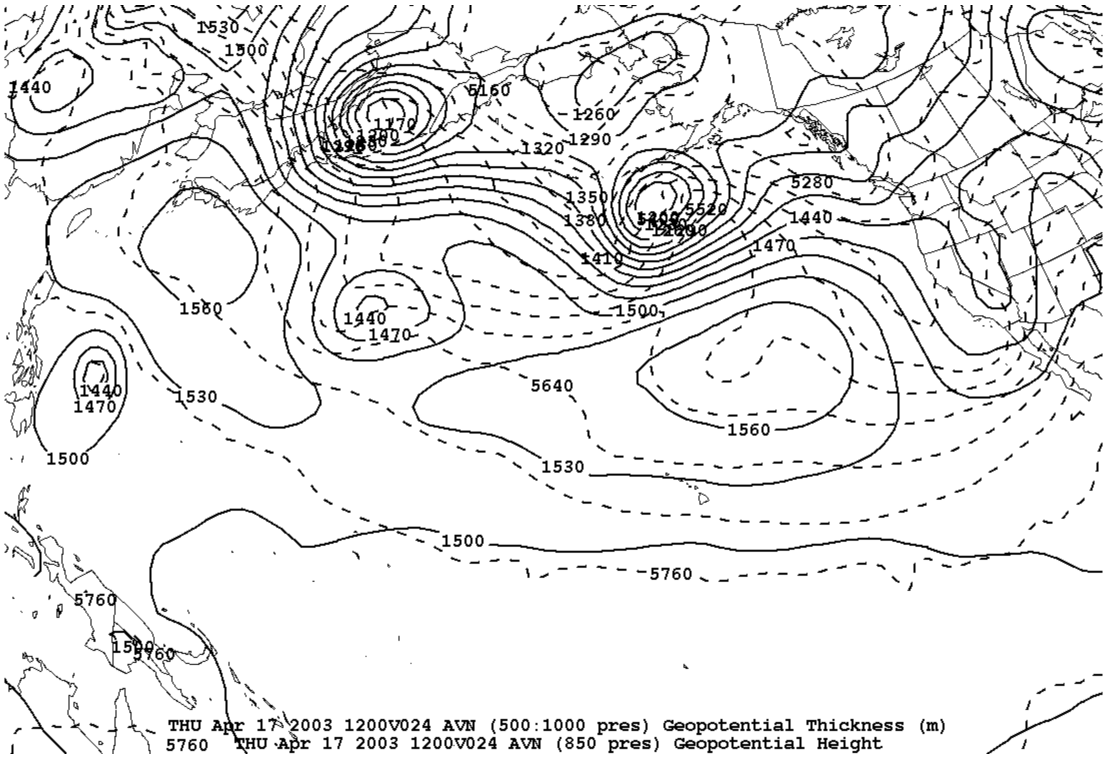


Figure 5: Example of 850 mb heights and 1000-500 mb thicknesses over the Tropical Pacific.

Mathematical Preliminaries

Here are a few mathematical preliminaries that will aid or upcoming discussion.

- ***Stokes Theorem***, which states that a line integral of the tangent-component of a vector along a closed path can be expressed as the normal-component of the curl integrated over the area within the closed path,

$$\oint \vec{F} \cdot d\vec{l} = \int\int_A \nabla \times \vec{F} \cdot d\vec{A}. \tag{1}$$

  - $d\vec{l}$ is a tangent vector along the closed path, and is positive in the counter-clockwise direction.
  - The direction of $d\vec{A}$ is perpendicular to the surface bounded by the closed path, and is positive in the right-hand sense.
  - The components of $d\vec{l}$ in Cartesian coordinates are

$$d\vec{l} = \hat{i}dx + \hat{j}dy + \hat{k}dz. \tag{2}$$

- The identity

$$df = \nabla f \cdot d\vec{l}, \tag{3}$$

  where f is any scalar function.

- The identity
$$\nabla \times \nabla f = 0. \tag{4}$$

- ***Exact differentials***, which are differentials whose integral around a closed path is zero,
$$\oint df = 0. \tag{5}$$

  – If a function f of two variables x and y has a differential that is written as
  $$df = Mdx + Ndy$$
  then it is an exact differential if
  $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{6}$$

  – If a diffential is function of only a single variable and is of the form, $df = M(x)dx$, then it is an exact differential as long as M is integrable (f is differentiable).

- ***Conservative vector fields***, which are vector fields $\vec{H}$ that can be written in terms of the gradient of a scalar, such as
$$\vec{H} = \nabla s. \tag{7}$$

  – A conservative vector field has the property
  $$\oint \vec{H} \cdot d\vec{l} = 0, \tag{8}$$
  which is shown as follows:
  $$\oint \vec{H} \cdot d\vec{l} = \oint \nabla s \cdot d\vec{l} = \int\int_A \nabla \times \nabla s \cdot d\vec{A} = 0. \tag{9}$$

  – So, if $\vec{H}$ is a conservative vector field, then $\vec{H} \cdot d\vec{l}$ is an exact differential.

  – Another important result from this is that for any scalar function, s, we have
  $$\oint \nabla s \cdot d\vec{l} = 0. \tag{10}$$

Circulation

- Circulation is a measure of the rotation in a fluid.

- Circulation is defined as the line integral around a closed path of the dot-product of velocity and the vector tangent to the path,
$$C = \oint \vec{V} \cdot d\vec{l}. \tag{11}$$

- By convention, integrating around the path in a counter-clockwise direction is positive.

- Circulation will be most useful to us if we develop a formula for how it changes with time,
$$\frac{DC}{Dt} = \frac{D}{Dt}\oint \vec{V}\cdot d\vec{l} = \oint \frac{D}{Dt}\left(\vec{V}\cdot d\vec{l}\right) = \oint \frac{D\vec{V}}{Dt}\cdot d\vec{l} + \oint \vec{V}\cdot \frac{D}{Dt}d\vec{l}. \tag{12}$$

- The term
$$\oint \vec{V}\cdot\frac{D}{Dt}d\vec{l}$$
is evaluated as follows. We first note that
$$\frac{D}{Dt}d\vec{l} = d\vec{V}$$
(this result is not conceptually straight-forward, but is explained in the Appendix). So now we have
$$\oint \vec{V}\cdot\frac{D}{Dt}d\vec{l} = \oint \vec{V}\cdot d\vec{V} = \oint udu + \oint vdv + \oint wdw = \oint \frac{1}{2}du^2 + \oint \frac{1}{2}dv^2 + \oint \frac{1}{2}dw^2. \tag{13}$$
From (3) each of these terms can be written in the form of the integral of a gradient,
$$\oint \frac{1}{2}du^2 = \oint d\left(\frac{u^2}{2}\right) = \oint \nabla\left(\frac{u^2}{2}\right)\cdot d\vec{l}.$$
And then from (10) each of these terms is seen to be zero! Therefore,
$$\frac{DC}{Dt} = \oint \frac{D\vec{V}}{Dt}\cdot d\vec{l}. \tag{14}$$

Change in Circulation in an Absolute Reference Frame

- Before looking at the concept of circulation applied to the atmosphere, lets apply it to fluid in an absolute (non-rotating) coordinate system . In this case the momentum equation is
$$\frac{D\vec{V}}{Dt} = -\alpha\nabla p + \vec{g^*}, \tag{15}$$
where $\vec{g^*}$ is the Newtonian gravitational acceleration, which can also be written as the gradient of a gravitational potential, $\vec{g^*} = -\nabla\Phi^*$.

- Inserting (15) into (14) yields,
$$\frac{DC}{Dt} = -\oint \alpha\nabla p\cdot d\vec{l} - \oint \nabla\Phi^*\cdot d\vec{l}. \tag{16}$$

- The last term in (16) has the form of (10), and is therefore zero, and so (16) becomes

$$\frac{DC}{Dt} = -\oint \alpha \nabla p \cdot d\vec{l} = -\oint \alpha dp \tag{17}$$

(remember that $\nabla p \cdot d\vec{l} = dp$).

- Equation (17) is the ***Bjerknes Circulation Theorem***.

- Before exploring the significance of Bjerknes theorem for a baroclinic fluid, lets look at its application to a barotropic fluid.

Circulation in a Barotropic Fluid

- In a barotropic fluid the density is a function of pressure only. In this case we have

$$\oint \alpha dp = \oint f(p) dp = 0,$$

since $f(p)dp$ only involves a single variable, and is therefore an exact differential. So, for a barotropic fluid the circulation theorem, (17), becomes

$$\frac{DC}{Dt} = 0. \tag{18}$$

- This result is Kelvins circulation theorem.

- Kelvins circulation theorem states that the circulation around a closed curve moving with a frictionless, barotropic fluid is constant!

- Kelvins theorem is very powerful in that it expresses the dynamics of the fluid flow in one compact conservation law (conservation of circulation).

  - In general, physics problems are easier to solve if they can be written in terms of conservation laws. For example, it is easier to solve for the velocity of an object sliding down a frictionless ramp by using the conservation of energy rather than Newtons second law.

- Kelvins theorem (and Bjerknes for that matter) only apply to frictionless fluids.

  - Circulation can be created or dissipated in a boundary layer, due to friction at the surfaces, which creates velocity shear. This is not incorporated into either Kelvins or Bjerknes circulation theorem.

Solenoids

- We can write Bjerknes' circulation theorem as

$$\frac{DC}{Dt} = -\oint \alpha dp = -\oint \alpha \nabla p \cdot d\vec{l} = -\int\int_A \nabla \times (\alpha \nabla p) \cdot d\vec{A}, \tag{19}$$

  where Stokes' Theorem was used to get to the last step.

- The term $\nabla \times (\alpha \nabla p)$ evaluates as

$$\nabla \times (\alpha \nabla p) = \nabla \alpha \times \nabla p + \alpha \nabla \times \nabla p, \tag{20}$$

  and by (4) the last term is zero. So, (19) becomes

$$\frac{DC}{Dt} = -\int\int_A (\nabla \alpha \times \nabla p) \cdot d\vec{A}. \tag{21}$$

- The physical meaning Bjerknes Circulation Theorem is best illustrated if the path of integration in (21) lies in a plane, as shown in Fig. 6. In this case

$$\frac{DC}{Dt} = |\nabla \alpha||\nabla p| \cos \beta, \tag{22}$$

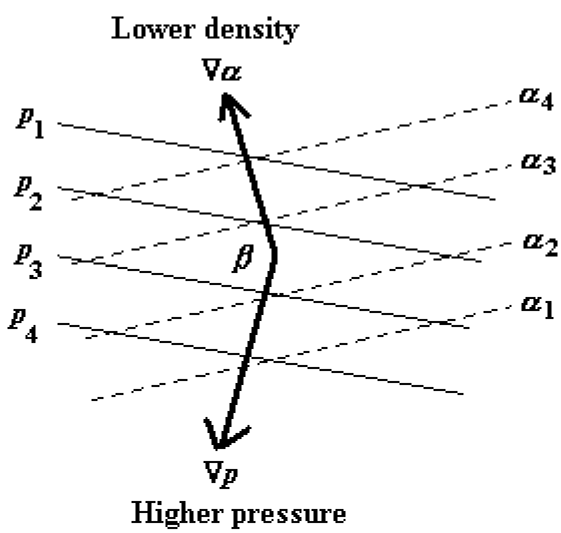


Figure 6: Solenoids of specific volume and pressure.

  where β is the angle between the gradients of specific volume and pressure.

  - In this example $DC/Dt < 0$, so a clockwise circulation would develop as shown in Fig. 7.

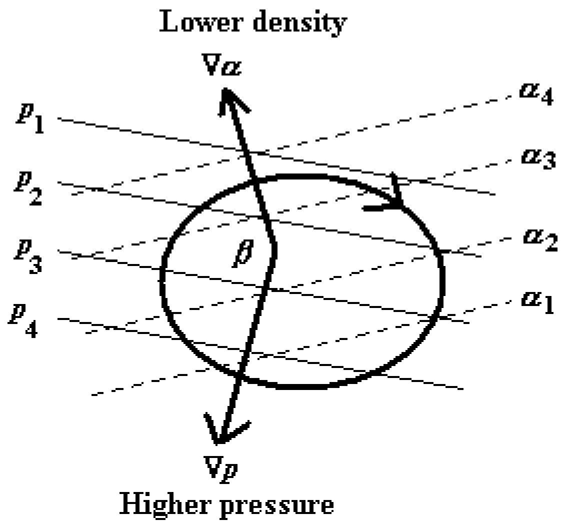


Figure 7: Solenoids resulting in a negative circulatioin tendency.

- In general the circulation that develops would be such that the density and pressure surfaces would become parallel.
- The diamond-shaped cells occurring between the isobars and isopycnals are called solenoids.
- For the atmosphere, which is an ideal gas, the solenoidal term can be written in terms of the temperature and pressure gradients as (see exercises)

$$\frac{DC}{Dt} = -R_d \int \int_A \nabla T \times \nabla(\ln p) \cdot d\vec{A}, \tag{23}$$

  which is easier and more relevant to apply to atmospheric processes.

Circulation Theorem in a Rotating Reference Frame

- Up to now we have limited our discussion of circulation to an absolute reference frame (or to scales small enough that the rotation of the reference frame is negligible).
- To see why rotation of the frame makes a difference, imagine a ring (or chain) of fluid at rest with respect to the absolute frame of reference (see Fig. 8).
  - In a rotating reference frame there would appear to be a circulation oriented opposite to that of the rotation of the reference frame.
  - Alternatively, if the fluid were circulating in the absolute reference frame, but was at rest with respect to the rotating frame, the circulation with respect to the rotating frame would be zero.

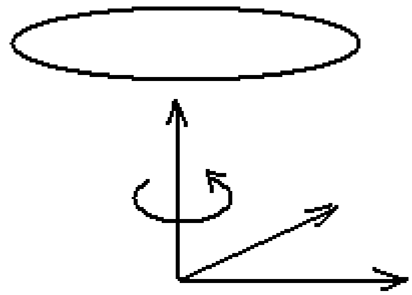


Figure 8: A ring or chain of fluid parcels in a rotating reference frame.

- To find the circulation in the rotating frame we need to use the momentum equation that includes both the Coriolis term and the centrifugal term. In this case the momentum equation is

$$\frac{D\vec{V}}{Dt} = -\alpha\nabla p - 2\vec{\Omega} \times \vec{V} + \vec{g}, \tag{24}$$

where $\vec{g}$ is gravity (apparent gravity)

$$\vec{g} = \vec{g^*} + |\vec{\Omega}|^2\vec{R},$$

and includes the centrifugal acceleration. Using (24) in (14) results in

$$\frac{DC}{Dt} = -\int\int_A (\nabla\alpha \times \nabla p)\cdot d\vec{A} - \oint (2\vec{\Omega} \times \vec{V})\cdot d\vec{l} + \oint \vec{g}\cdot d\vec{l}, \tag{25}$$

where the steps to get the solenoidal term in (25) are the same as those used to derive the solenoidal term in (21).

  - Since apparent gravity can be expressed in terms of the gradient of relative geopotential, $\vec{g} = -\nabla\Phi$, then by the same argument that we used to eliminate the gravitational (Newtonian gravity) acceleration in (16) we can eliminate the gravity (apparent gravity) acceleration term in (25), so we are left with

$$\frac{DC}{Dt} = -\int\int_A (\nabla\alpha \times \nabla p)\cdot d\vec{A} - 2\oint (\vec{\Omega} \times \vec{V})\cdot d\vec{l} \tag{26}$$

  - From Stokes' Theorem

$$\oint (\vec{\Omega} \times \vec{V})\cdot d\vec{l} = \int\int_A \nabla \times (\vec{\Omega} \times \vec{V})\cdot d\vec{A}. \tag{27}$$

- The circulation theorem applied to the rotating frame is therefore

$$\frac{DC}{Dt} = \int\int_A (\nabla\alpha \times \nabla p) \cdot d\vec{A} - 2\int\int_A \nabla \times (\vec{\Omega} \times \vec{V}) \cdot d\vec{A}. \tag{28}$$

Horizontal Circulations on the Earth

- For synoptic scale circulations we are primarily concerned with circulations around a vertical axis, so that $d\vec{A} = \hat{k}dA$. This greatly simplifies (28), which becomes

$$\frac{DC_z}{Dt} = \int\int_A \hat{k} \cdot (\nabla_H\alpha \times \nabla_H p)dA - 2\int\int_A \hat{k} \cdot \nabla \times (\vec{\Omega} \times \vec{V})dA. \tag{29}$$

  - The subscript z on C_z is used to denote horizontal circulation (circulation around the vertical axis).
  - The subscript H on the gradients in the solenoidal term indicate that we are only using the horizontal gradients of α and p.

- On the synoptic scale we can use the horizontal velocity $\vec{V} = u\hat{\imath} + v\hat{\jmath}$. The angular velocity of the Earth in local Cartesian coordinates is $\vec{\Omega} = 0\hat{\imath} + \Omega\cos\varphi\hat{\jmath} + \Omega\sin\varphi\hat{k}$. Using these vector components we can show that

$$2\hat{k} \cdot \nabla \times (\vec{\Omega} \times \vec{V}) = f\nabla_H \cdot \vec{V} + \frac{\partial f}{\partial y}v,$$

  where $f = 2\Omega\sin\varphi$.

- Therefore, the equation for the change in circulation in the horizontal plane on a rotating Earth is

$$\frac{DC_z}{Dt} = -\underbrace{\int\int_A \hat{k} \cdot (\nabla_H\alpha \times \nabla_H p)dA}_{A} - \underbrace{\int\int_A f(\nabla_H \cdot \vec{V})dA}_{B} - \underbrace{\int\int_A \beta v dA}_{C}, \tag{30}$$

  where $\beta = \partial f/\partial y$.

- The terms of this equation represent:

  Term A: This is just the solenoidal term that we have seen before.

  Term B: This is the divergence term.

  - Divergence leads to anticyclonic circulation.
  - Convergence leads to cyclonic circulation.

  Term C: This is the β-effect term.

  - Moving the chain of fluid parcels northward generates anticyclonic circulation.

– Moving the chain of fluid parcels southward generates cyclonic circulation.

- The solenoidal term can also be written in terms of the temperature and pressure gradients as (see exercises)

$$\frac{DC_z}{Dt} = -R_d \int\int_A \hat{k} \cdot (\nabla_H T \times \nabla_H \ln p) dA - \int\int_A f(\nabla \cdot \vec{V}) dA - \int\int_A \beta v dA. \quad (31)$$

- In this form, the circulation equations shows that if the isotherms are parallel to the isobars, then the solenoidal term is zero.
- This means that in an equivalent barotropic region of the atmosphere, the only way to develop a horizontal circulation is through divergence or north-south motion.

Appendix

- Figure 9 shows a segment of the closed path at two different times, t and $t + \Delta t$.

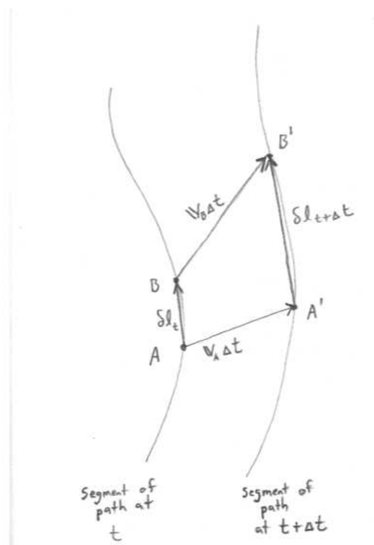


Figure 9: A segement of the fluid chain being advected.

- Point A is advected to Point A', and Point B is advected to Point B'.
- The vectors $d\vec{l}_t$ and $d\vec{l}_{t+\Delta t}$ are very small, but finite.
- From the diagram we have

$$\vec{V}_A \Delta t + d\vec{l}_{t+\Delta t} = \vec{V}_B \Delta t + d\vec{l}_t,$$

which rearranges to

$$\frac{d\vec{l}_{t+\Delta t} - d\vec{l}_t}{\Delta t} = \vec{V}_B - \vec{V}_A = \delta \vec{V}.$$

- Taking the limit as $\Delta t \to 0$ results in

$$\frac{D}{Dt}\delta\vec{l} = \delta\vec{V}.$$

- If we allow the distance between Points A and B to approach zero, then $\delta\vec{l} \to d\vec{l}$ and $\delta\vec{V} \to d\vec{V}$, resulting in the desired result that

$$\frac{D}{Dt}d\vec{l} = d\vec{V}.$$

Exercises

1. (a) Show that for an ideal gas that the solenoidal term of the circulation theorem can be written as

$$\frac{DC}{Dt} = -\oint \alpha dp = -R'\oint Td(\ln p).$$

   (b) Use identity (9) and Stokes Theorem to show that for an ideal gas the solenoidal term can be written as

$$\frac{DC}{Dt} = -R'\int\int_A \nabla T \times \nabla(\ln p)\cdot d\vec{A}.$$

2. (a) A chain of fluid parcels in your bathtub lies in the horizontal plane. The chain is circular with a radius of 5 cm. The chain is rotating clockwise (as viewed from above) with a tangential velocity of 0.5 cm/s. As the chain moves over the drain, horizontal convergence causes the radius of the chain to shrink to 1 cm. What is the new tangential velocity? (Hint: Your bathwater is barotropic.)

   (b) Repeat part (a), only assume that initially the chain is rotating counter-clockwise.

   (c) What do you think determines the direction of the whirlpool that forms over your bathtub drain?

3. Show that

$$\beta = \frac{2\Omega\cos\varphi}{a}$$

   where a is the radius of the Earth.

4. A circular chain of fluid parcels with a radius of 300 km is centered at latitude 30N. Its circulation is initially zero. The entire chain of fluid parcels begins moving northward at 5 m/s.

   (a) What is the rate of change of the circulation?

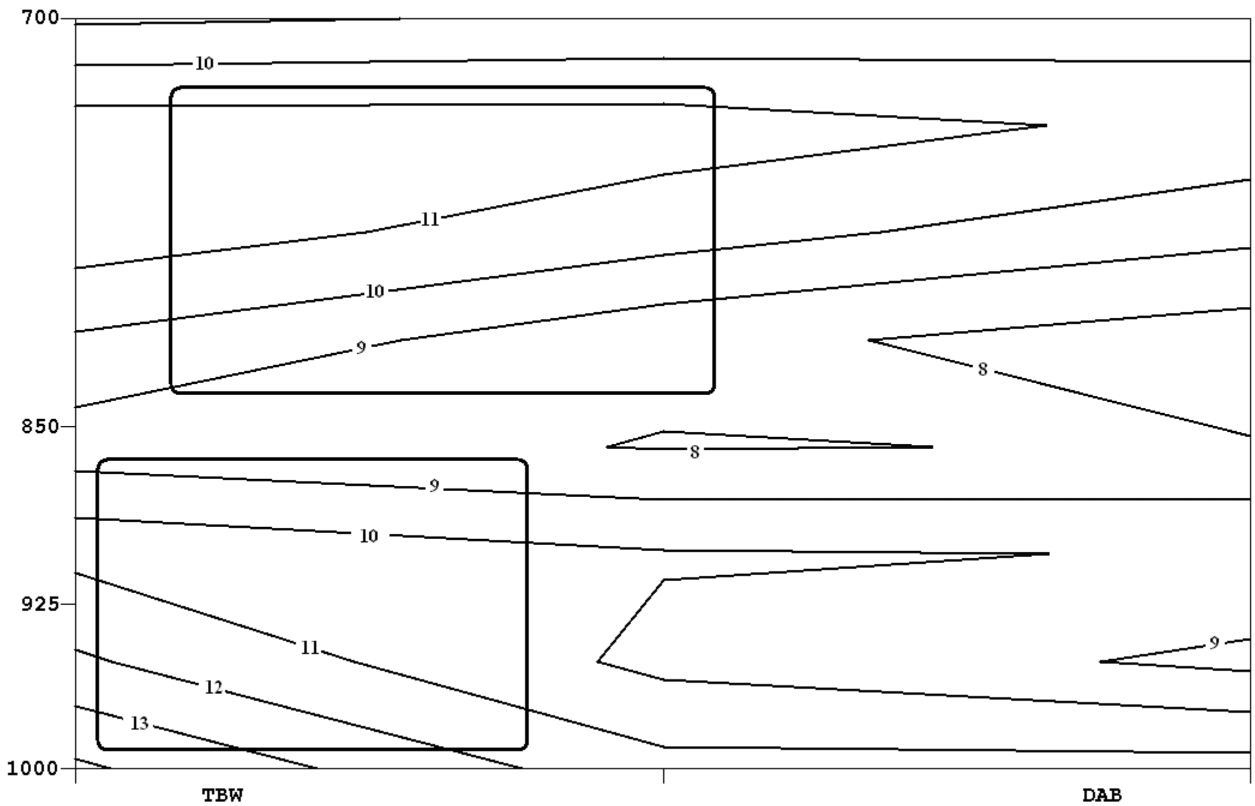
   (b) Assuming β remains nearly constant with small changes in latitude, what will the circulation be after 24 hours?

   (c) How strong will the tangential winds be after 24 hours?

5. According to the circulation equation on a rotating Earth, a circular chain of fluid parcels initially at rest will generate an anticyclonic circulation as it expands, and a cyclonic circulation as it contracts. Give a physical explanation for why this occurs.

6. Show that
$$2\hat{k} \cdot \nabla \times (\vec{\Omega} \times \vec{V}) = f \nabla_H \cdot \vec{V} + \frac{\partial f}{\partial y} v.$$

7. The figure shows a vertical cross-section of temperature along a line across central Florida between Tampa Bay and Daytona Beach, Florida.



(a) Draw arrows on the rectangles showing the orientation of the circulation that would develop due to the solenoidal term.

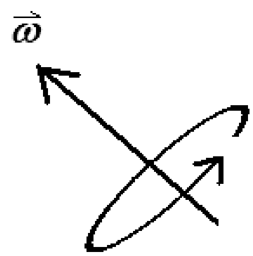
(b) Would the circulation on the lower rectangle be a land breeze, or a sea breeze?

ESCI 342 – Atmospheric Dynamics I
Lesson 12 – Vorticity

Reference: *An Introduction to Dynamic Meteorology* (4rd edition), Holton
An Informal Introduction to Theoretical Fluid Mechanics, Lighthill
Reading: Martin, Section 5.2

VORTICITY

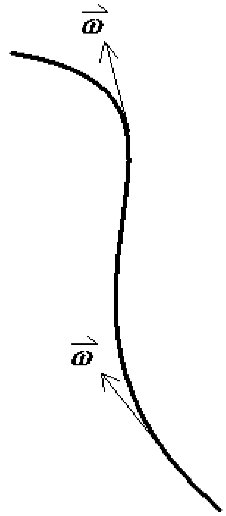
- The circulation of a fluid is defined as
$$C \equiv \oint \vec{V} \bullet d\vec{l}\ . \qquad \textbf{(1)}$$
- From Stokes's theorem this is the same as
$$C \equiv \int_A \left(\nabla \times \vec{V}\right) \bullet d\vec{A}\,. \qquad \textbf{(2)}$$
The quantity $\nabla \times \vec{V}$ is therefore also a measure of the rotation of the fluid, and is called the ***vorticity***.
    - o Vorticity is defined as
$$\vec{\omega} \equiv \nabla \times \vec{V}\ . \qquad \textbf{(3)}$$
- ***Vorticity is a vector***. The rotation of the fluid follows the right-hand-rule with respect to the vorticity vector.



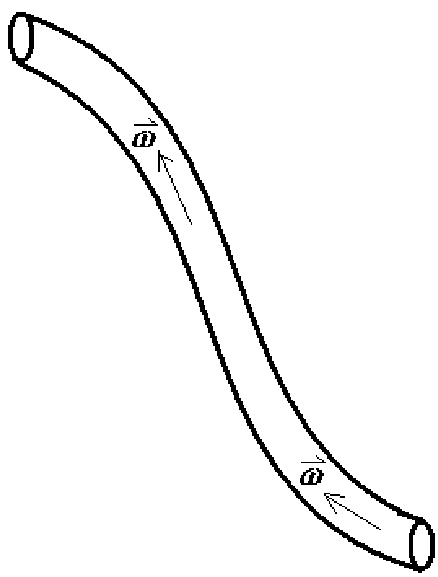
- Circulation and vorticity are closely related.
    - o For a flat surface we can write (2) using the generalized mean-value theorem as
$$C \equiv \overline{\nabla \times \vec{V}} \bullet \int_A d\vec{A} = \overline{\nabla \times \vec{V}} \bullet \vec{A} \qquad \textbf{(4)}$$
which is interpreted as meaning that the circulation around a planar surface is just the area-averaged vorticity normal to the surface multiplied by the area.
- The components of the vorticity vector are
$$\vec{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}\ . \qquad \textbf{(5)}$$
- ***In meteorology we are primarily concerned with circulations in the horizontal plane, so we are most interested in the vertical component of vorticity. From now on, when we speak of vorticity, we will usually be referring only to the vertical component.***
- The vertical component of vorticity is given the symbol ζ, and the following definition holds.
$$\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,. \qquad \textbf{(6)}$$

VORTEX STRETCHING

- A *vortex line* is a line that is everywhere parallel to the vorticity vector.



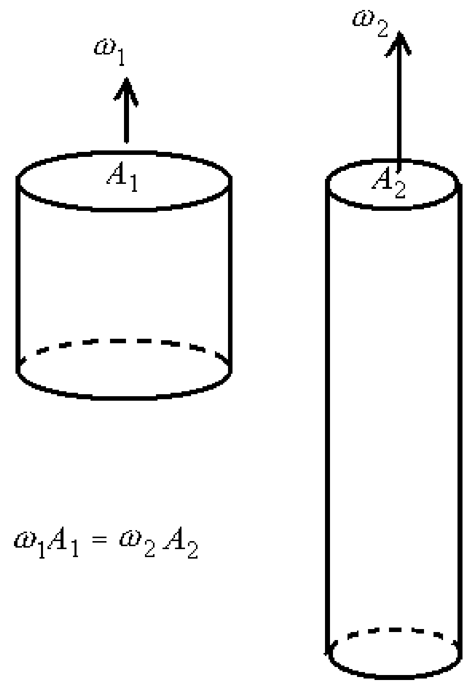
  - Vortex lines cannot begin or end in the interior of the fluid. They must terminate at a boundary of some sort.
  - Vortex lines move with the fluid.
- A *vortex tube* is a collection of vortex lines.



  - Vortex tubes move with the fluid and always consists of the same fluid parcels.
- The circulation taken around a disk that is perpendicular to the vortex tube is equal to the average vorticity of the tube times the area of the tube

$$C = \oint \vec{V} \bullet d\vec{l} = \int_A \left(\nabla \times \vec{V}\right) \bullet dA = \int_A \vec{\omega} \bullet d\vec{A} \equiv \bar{\omega} A \qquad \textbf{(7)}$$

- In a barotropic fluid, if a vortex tube is stretched the circulation doesn't change. However, the cross-sectional area of the tube will decrease, which means that the vorticity must increase!
  - Stretching a vortex tube causes it to spin faster.



- This phenomenon is known as *vortex stretching*.
  - Vortex stretching explains why a whirlpool forms over your bathtub drain. As the vortex tube moves over the drain it becomes stretched, causing it to spin more rapidly.
  - The sense of the rotation is determined by the original rotation of the vortex tube before it moved over the drain.
  - Vortex stretching also helps explain the formation of mesocyclones and tornadoes, as vertically oriented vortex tubes are stretched in the thunderstorm updraft.

RELATIVE VERSUS ABSOLUTE VORTICITY

- As with circulation, vorticity also depends on whether it is measured in an absolute reference frame or in a rotating frame.
  - The vorticity measured in the absolute reference frame is called *absolute vorticity*, and is given the symbol η.
  - The vorticity measured relative to the Earth is called *relative vorticity*, and is given the symbol ζ.
  - The vorticity of the surface of the Earth is called planetary vorticity. It is equal to the Coriolis parameter, f (see exercise 1).
- Absolute, relative, and planetary vorticity are related via

$$\eta = \zeta + f\ . \qquad \textbf{(8)}$$

- In the atmosphere, relative vorticity is usually much less than the planetary vorticity. Therefore, the absolute vorticity is usually a positive value.
  - ***Interesting aside:*** When we looked at the gradient wind back in Lesson 10 we saw that there were two anomalous cases of balanced flow: an anomalous high

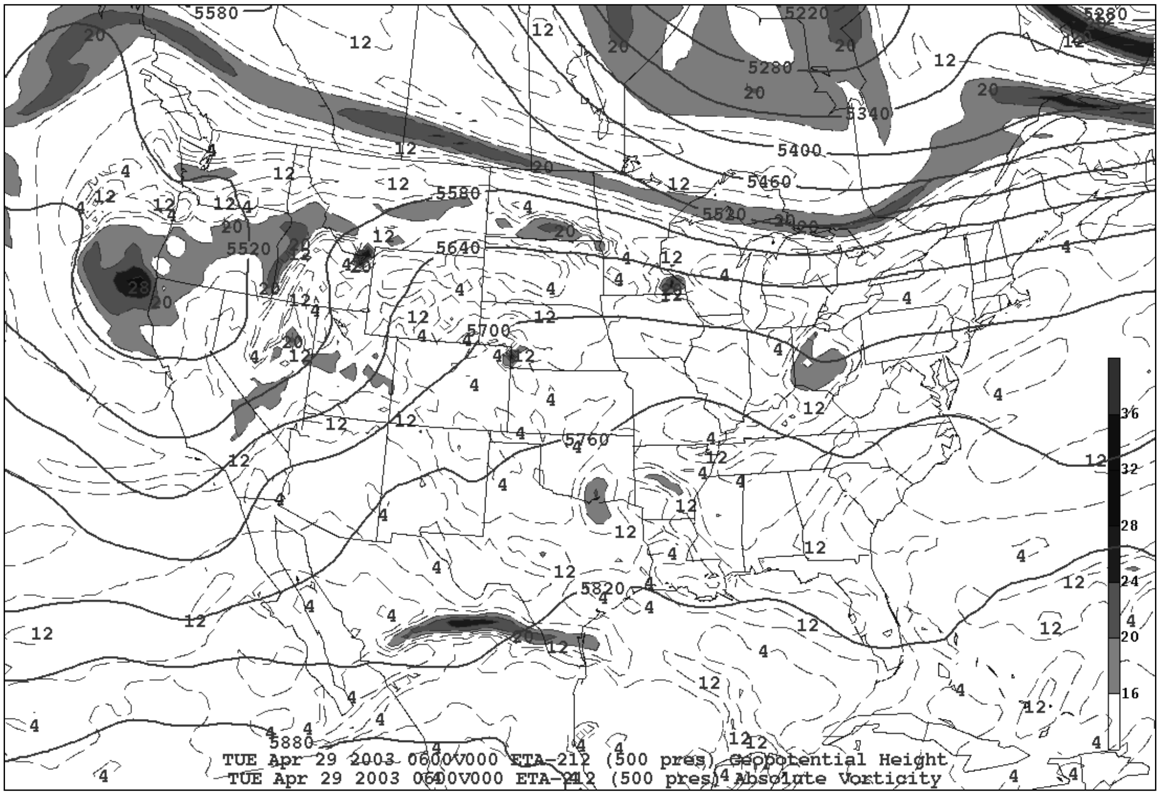
and an anomalous low. It turns out that both of these solutions have negative absolute vorticity, which may be one reason why they aren't observed frequently on the synoptic scale, since it is difficult to think of a process in the atmosphere that would generate negative absolute vorticity over a large area.

CURVATURE VERSUS SHEAR

- Vorticity may be visualized by imagining a paddle wheel moving with the fluid flow.
  - If the paddle wheel is rotating clockwise then there is negative (or anticyclonic) relative vorticity.
  - If the paddle wheel is rotating counter-clockwise then there is positive (or cyclonic) relative vorticity.
- The relative vorticity may be due either to
  - Curvature
  - Shear
- This is best visualized in natural coordinates, where the vorticity can be written as

$$\zeta = \underbrace{-\frac{\partial V}{\partial n}}_{A} + \underbrace{\frac{V}{R}}_{B} . \qquad \textbf{(9)}$$

  Note that n is always toward the left of the velocity vector. *Term A* is the vorticity due to ***shear***, and *Term B* is vorticity due to ***curvature***
  - You cannot necessarily tell the sense of vorticity by just looking at the streamlines. It is possible to have cyclonic curvature with anticyclonic shear, or vice-versa. In most cases, $\nabla \times \vec{V}$ must be calculated to find the sign of the relative vorticity.
  - Relative vorticity in the atmosphere is usually on the order of 10^{-5} to10^{-4} s^{-1}.
- The picture below shows the 500-mb geopotential height, and the 500-mb absolute vorticity (units are s^{-1} × 10^5).



GEOSTROPHIC VORTICITY

- The vorticity due to the geostrophic wind is called the *geostrophic vorticity*, ζ_g.
- Since the geostrophic wind can be given in terms of a streamfunction,

$$u_g = -\frac{\partial \psi}{\partial y}; \qquad v_g = \frac{\partial \psi}{\partial x},$$

  the geostrophic vorticity is equal to the LaPlacian of the streamfunction,

$$\zeta_g = \nabla^2 \psi . \qquad \textbf{(10)}$$

  - Since the stream function is related to the geopotential field via

$$\psi = f_0^{-1}\Phi = f_0^{-1} g_0 Z$$

    (where $f = f_0$ = constant) then the geostrophic vorticity can be calculated directly from the geopotential heights.
- On the synoptic scale we often approximate the actual wind by the geostrophic wind. In the same vein, we often approximate the actual relative vorticity by the geostrophic vorticity.
  - This is convenient, since we can calculate vorticity directly from the geopotential heights, and don't need the actual wind observations.
- Remember…the geostrophic vorticity, like the geostrophic wind, is a definition. It is close to, but not necessarily equal to, the actual vorticity.

$$\zeta = \zeta_g + \zeta_a \cong \zeta_g .$$

EXERCISES

1. a. Show that the circulation of a flat disk of radius r in solid-body rotation is $C = 2\pi P r^2$ where P is the component of angular velocity perpendicular to the disk.

b. Show that the component of vorticity perpendicular to the disk is just $2P$.

c. Use your result to show that the vorticity of a point on the Earth's surface is $2\Omega \sin\phi$ and is therefore equal to the Coriolis parameter.

2. A vertically oriented vortex tube is in your bathtub. The tube is circular with a radius of 5 cm. The tube is rotating clockwise (as viewed from above) with a tangential velocity of 0.5 cm/s.

a. Calculate the average vorticity of the tube.

b. As the tube moves over the drain it is stretched, and its radius shrinks to 1 cm. What is the new average vorticity?

3. Calculate the vorticity of the following flows at point $(x,y) = (1\text{m}, 2\text{m})$.

a. $\begin{aligned} u &= u_0 xy \\ v &= v_0 y \end{aligned}$ $\quad u_0 = 2\ \text{m}^{-1}\ \text{s}^{-1}$, $v_0 = 1\ \text{s}^{-1}$

b. $\begin{aligned} u &= u_0 y \\ v &= v_0 x \end{aligned}$ $\quad u_0 = 2\ \text{s}^{-1}$, $v_0 = 1\ \text{s}^{-1}$

c. $\begin{aligned} u &= u_0 \\ v &= v_0 x^2 \end{aligned}$ $\quad u_0 = 2\ \text{m s}^{-1}$, $v_0 = 1\ \text{m}^{-1}\ \text{s}^{-1}$

d. $\begin{aligned} u &= u_0 \\ v &= v_0 \cos kx\ \sin ly \end{aligned}$ $\quad u_0 = 2\ \text{m s}^{-1}$, $v_0 = 1\ \text{m s}^{-1}$, $k = 2.1\ \text{m}^{-1}$, $l = 0.9\ \text{m}^{-1}$

4. Show that if f and ρ are assumed constant, the geostrophic vorticity on a constant altitude surface is

$$\zeta_g = \frac{1}{f\rho}\nabla^2 p\,.$$

5. a. Show that if f is allowed to vary with latitude, that the geostrophic vorticity on a constant pressure surface is

$$\zeta_g = \frac{1}{f}\nabla^2\Phi - \frac{\beta}{f^2}\frac{\partial \Phi}{\partial y}\,.$$

b. Use scale analysis arguments to see if it is reasonable to ignore the second term in the above expression on the synoptic scale so that we can still use

$$\zeta_g \cong \frac{1}{f}\nabla^2\Phi$$

even though f is not constant with latitude. Hint: $f^{-1}\,\partial\Phi/\partial y$ is of the same order of magnitude as the geostrophic wind.

ESCI 342 – Atmospheric Dynamics I
Lesson 13 – The Vorticity Equation

Reference: *An Introduction to Dynamic Meteorology* (4[rd] edition), Holton
Atmosphere-Ocean Dynamics, Gill
Reading: Martin, Section 5.3

DERIVATION OF THE VORTICITY EQUATION

- An equation for the change in vorticity with time can be derived from the horizontal momentum equations,

$$\frac{\partial u}{\partial t}+\vec{V}\bullet\nabla u=-\alpha\frac{\partial p}{\partial x}+fv \qquad \textbf{(1)}$$

$$\frac{\partial v}{\partial t}+\vec{V}\bullet\nabla v=-\alpha\frac{\partial p}{\partial y}-fu\,. \qquad \textbf{(2)}$$

- Taking $\partial/\partial x$ of (2) and subtracting $\partial/\partial y$ of (1) gives

$$\frac{\partial}{\partial x}\frac{\partial v}{\partial t}-\frac{\partial}{\partial y}\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\vec{V}\bullet\nabla v\right)-\frac{\partial}{\partial y}\left(\vec{V}\bullet\nabla u\right)=-\frac{\partial}{\partial x}\left(\alpha\frac{\partial p}{\partial y}\right)+\frac{\partial}{\partial y}\left(\alpha\frac{\partial p}{\partial x}\right)-\frac{\partial}{\partial x}(fu)-\frac{\partial}{\partial y}(fv)$$

which can be expanded and rearranged to

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial x}\left(\vec{V}\bullet\nabla v\right)-\frac{\partial}{\partial y}\left(\vec{V}\bullet\nabla u\right)=-\left(\frac{\partial\alpha}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial p}{\partial x}\right)-f\nabla_H\bullet\vec{V}-\beta v$$

which then rearranges as

$$\frac{\partial\varsigma}{\partial t}+\frac{\partial}{\partial x}\left(\vec{V}\bullet\nabla v\right)-\frac{\partial}{\partial y}\left(\vec{V}\bullet\nabla u\right)=-\left(\frac{\partial\alpha}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial p}{\partial x}\right)-f\nabla_H\bullet\vec{V}-\beta v\,.$$

- It turns out that

$$\frac{\partial}{\partial x}\left(\vec{V}\bullet\nabla v\right)-\frac{\partial}{\partial y}\left(\vec{V}\bullet\nabla u\right)=\vec{V}\bullet\nabla\zeta+\zeta\nabla_H\bullet\vec{V}+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$

so that the equation for vorticity can be written as

$$\frac{\partial\zeta}{\partial t}=-\vec{V}\bullet\nabla\zeta-\beta v-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)-\left(\frac{\partial\alpha}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial p}{\partial x}\right)-(\zeta+f)\nabla_H\bullet\vec{V}\,.$$

or as

$$\underset{A}{\frac{\partial\zeta}{\partial t}}=\underset{B}{-\vec{V}\bullet\nabla\zeta}\;\underset{C}{-\beta v}\;\underset{D}{+\hat{k}\bullet\frac{\partial\vec{V}}{\partial z}\times\nabla w}\;\underset{E}{-\hat{k}\bullet\nabla\alpha\times\nabla p}\;\underset{F}{-\eta\nabla_H\bullet\vec{V}}\,. \qquad \textbf{(3)}$$

PHYSICAL MEANING OF THE TERMS IN THE VORTICITY EQUATION

- ***Term A:*** The local relative vorticity tendency
- ***Term B:*** Advection of relative vorticity
- ***Term C:*** Advection of planetary vorticity
  - o Accounts for generation of relative vorticity due to movement of the air poleward.
  - o Imagine the air at rest with respect to the Earth so there is zero relative vorticity. As the air moves toward the south, the local rotation of the Earth has decreased, so the air appears to have taken on a cyclonic circulation, even though from space,

its rotation hasn't changed. Thus, southward motion leads to an increase in relative vorticity, while northward motion leads to a decrease.
  - Since $-\beta v = -\vec{V} \bullet \nabla f$, this term appears as an advection of planetary vorticity, and that's what it is commonly called. It is also often referred to as simply the "beta term".
- ***Term D:*** Twisting/tilting term
  - Accounts for the tilting of horizontal relative vorticity into the vertical.
- ***Term E:*** Solenoidal term
  - Accounts for the generation of relative vorticity due to baroclinicity.
- ***Term F:*** Divergence term
  - Accounts for the generation of relative vorticity due to convergence.
  - The physical explanation of this term is simply the conservation of absolute angular momentum (angular momentum as viewed from space).
  - If a circulation has positive absolute angular momentum (and therefore positive absolute vorticity), if it converges, it must spin faster in the positive direction, and will gain cyclonic absolute vorticity.
  - If a circulation has negative absolute angular momentum (and therefore negative absolute vorticity), if it converges, it must spin faster in the negative direction and will gain anticyclonic absolute vorticity.

THE VORTICITY EQUATION ON THE SYNOPTIC SCALE

- On the synoptic scale, scale analysis shows that the vertical advection term, the solenoidal term, and the twisting/tilting term are all an order of magnitude less than the next largest terms. Therefore, we can ignore these terms and write the vorticity equation (***<u>for the synoptic scale only</u>***) as

$$\frac{\partial \zeta}{\partial t} = -\vec{V} \bullet \nabla_H \zeta - \beta v - \eta \nabla_H \bullet \vec{V} \qquad \textbf{(4)}$$

- Note: On the synoptic scale $\zeta << f$, so that $\eta \cong f$. Therefore, most authors write the vorticity equation with only planetary vorticity, rather than absolute vorticity, in the divergence term, as

$$\frac{\partial \zeta}{\partial t} = -\vec{V} \bullet \nabla_H \zeta - \beta v - f \nabla_H \bullet \vec{V} \ .$$

  - This is the form of the vorticity equation we will usually use, though you should always keep in mind that it is the absolute vorticity, not the planetary vorticity, that is really in this term.
- Because f is only dependent on y, equation (4) can be written as

$$\frac{D}{Dt}(\zeta + f) = -f \nabla_H \bullet \vec{V}$$

or

$$\frac{D\eta}{Dt} = -f \nabla_H \bullet \vec{V} \ . \qquad \textbf{(5)}$$

- ***<u>Equation (5) is one of the most important equations in all of meteorology. I expect you to memorize it, and to understand what it means.</u>***
- Equation

- (5) states that ***on the synoptic scale the absolute vorticity of a fluid parcel changes mainly in response to divergence or convergence.***
  - Divergence leads to a decrease in absolute vorticity.
  - Convergence leads to an increase in absolute vorticity.
- ***Caution***: By only including the planetary vorticity in the divergence term, we've constrained convergence to always produce cyclonic vorticity. This is okay on the synoptic scale, because negative absolute vorticity rarely if ever occurs at that scale.
  - Keep in mind though, that on the mesoscale or smaller, the absolute vorticity must be used in the divergence term (we can't ignore ζ), and so it is possible on smaller scales for convergence to lead to creation of negative vorticity.

THE VORTICITY EQUATION AND THE ROLE OF CONSERVATION OF ANGULAR MOMENTUM

- We've already mentioned that the convergence/divergence term is based on conservation of absolute angular momentum. This may not be completely clear from our derivation of the vorticity equation. The question may be asked, "Can the vorticity equation be derived directly from conservation of absolute angular momentum?" The answer is "yes it can." If you are interested in seeing this derivation, go to *www.atmos.millersville.edu/~adecaria/DERIVATIONS/Vorticity.pdf*.

THE QUASI-GEOSTROPHIC APPROXIMATION

- Another assumption that can be made on the synoptic scale is that the actual wind can be approximated by the geostrophic wind $\left(\vec{V} \cong \vec{V}_g\right)$ in every term except the divergence term.
  - This is the *quasi-geostrophic* approximation.
  - The rational for this approximation is that the ageostrophic wind is usually much smaller than the geostrophic wind and can therefore be ignored in the advection terms.

$$\vec{V} = \vec{V}_g + \vec{V}_a$$
$$\vec{V}_a << \vec{V}_g$$

  - However, since the ageostrophic wind is the only component of the wind that can be divergent, it must be retained in the divergence term.
- Using the quasi-geostrophic approximation , equation (4) becomes

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \bullet \nabla_H \zeta_g - \beta v_g - f \nabla_H \bullet \vec{V}_a . \qquad \textbf{(6)}$$

  - ***Equation* (6) *is the quasi-geostrophic vorticity equation.***
  - ζ_g is the geostrophic vorticity (the vorticity of the geostrophic wind). It can be written as

$$\zeta_g = \nabla^2 \psi . \qquad \textbf{(7)}$$

- If there is no divergence, then (5) says that the absolute vorticity must be conserved following a fluid parcel,

$$\frac{D\eta}{Dt} = 0 . \qquad \textbf{(8)}$$

- Thus, ***at the level-of-nondivergence, vorticity changes only due to advection, and absolute vorticity is conserved following the fluid parcel.***

VORTICITY ADVECTION AND THE MOVEMENT OF SYNOPTIC DISTURBANCES

- Focusing on the advection terms of the Q-G vorticity equation, and ignoring the effects of convergence and divergence, we get

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \bullet \nabla_H \zeta_g - \beta v_g . \qquad \textbf{(9)}$$

- The first term on the RHS represents advection of relative vorticity, and the second term represents planetary vorticity.
- For a typical midlatitude disturbance these terms tend to have opposite effects.
  - The relative vorticity advections favors eastward motion
  - The planetary vorticity advection term favors westward (retrograde) motion.
- Which term "wins" depends on the size, or wavelength of the disturbance.
  - For very large disturbances the planetary term will dominate, and the wave will propagate westward.
  - For smaller disturbances the relative vorticity advection is more significant, and the disturbance will move eastward.

EXERCISES

1. Expand $\dfrac{D}{Dt}(\zeta + f)$ to show that it equals $\dfrac{\partial \zeta}{\partial t} + \vec{V} \bullet \nabla \zeta + \beta v$.

2. a. Show that $-\left(\dfrac{\partial w}{\partial x}\dfrac{\partial v}{\partial z} - \dfrac{\partial w}{\partial y}\dfrac{\partial u}{\partial z}\right)$ is the same as $\hat{k} \bullet \dfrac{\partial \vec{V}}{\partial z} \times \nabla w$.

b. Show that $-\left(\dfrac{\partial \alpha}{\partial x}\dfrac{\partial p}{\partial y} - \dfrac{\partial \alpha}{\partial y}\dfrac{\partial p}{\partial x}\right)$ is the same as $-\hat{k} \bullet \nabla \alpha \times \nabla p$.

3. a. Starting with the momentum equations in isobaric coordinates

$$\frac{Du}{Dt} = -\frac{\partial \Phi}{\partial x} + fv \quad (a)$$

$$\frac{Dv}{Dt} = -\frac{\partial \Phi}{\partial y} - fu \quad (b)$$

take $\partial/\partial x$ of (b) and subtract $\partial/\partial y$ of (a) to derive the vorticity equation in isobaric coordinates,

$$\frac{\partial \zeta}{\partial t} = -\vec{V} \bullet \nabla \zeta - \beta v + \hat{k} \bullet \frac{\partial \vec{V}}{\partial p} \times \nabla \omega - (\zeta + f)\nabla_p \bullet \vec{V}.$$

b. Why is there no solenoidal term?

4. a. Remembering that the geostrophic wind can be written in terms of the streamfunction, ψ, as

$$\vec{V}_g = \hat{k} \times \nabla \psi$$

show that $\zeta_g = \nabla^2 \psi$.

b. Show that $\vec{V}_g \bullet \nabla \zeta_g = J(\psi, \nabla^2 \psi)$ where $J(a,b) = \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x}$. ($J(a,b)$ is called the Jacobian of a and b.)

c. Show that the quasi-geostrophic vorticity equation can be written as

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi\right) + J(\psi, \nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = -f \nabla \bullet \vec{V}_a$$

5. Show that $\dfrac{\partial}{\partial x}\dfrac{D}{Dt} \neq \dfrac{D}{Dt}\dfrac{\partial}{\partial x}$ (i.e., material derivatives don’t commute with partial derivatives.

ESCI 342 – Atmospheric Dynamics I
Lesson 14 –Potential Vorticity

Reference: *An Introduction to Dynamic Meteorology* (4^{rd} edition), Holton
Atmosphere-Ocean Dynamics, Gill
Reading: Holton, Sections 5.3 and 5.4

THE BAROTROPIC QUASI-GEOSTROPHIC VORTICITY EQUATION

- Using the incompressible continuity equation we can write the quasi-geostrophic vorticity equation as
$$\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \bullet \nabla_H \zeta_g + \beta v_g = \eta \frac{\partial w}{\partial z}. \qquad \textbf{(1)}$$
- If we integrate this equation from the surface to the top of the atmosphere we have
$$\int_{z_0}^{z_T} \left(\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \bullet \nabla_H \zeta_g + \beta v_g \right) dz = \int_{z_0}^{z_T} (\zeta + f) \frac{\partial w}{\partial z} dz = \eta \left[w(z_T) - w(z_0) \right] \qquad \textbf{(2)}$$
- In the case of a barotropic fluid the velocity (and hence, vorticity) is independent of height, so we get
$$\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \bullet \nabla_H \zeta_g + \beta v_g = \frac{f}{h}(w_T - w_0) = \frac{\eta}{h}\left(\frac{Dz_T}{Dt} - \frac{Dz_0}{Dt} \right). \qquad \textbf{(3)}$$
- The right-hand-side can be rewritten as
$$\frac{Dz_T}{Dt} - \frac{Dz_0}{Dt} = \frac{D}{Dt}(z_T - z_0) = \frac{Dh}{Dt} \qquad \textbf{(4)}$$
where h is the total depth of the fluid.
- Equation (3) can therefore be written as
$$\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \bullet \nabla_H \zeta_g + \beta v_g = \frac{\eta}{h}\frac{Dh}{Dt}. \qquad \textbf{(5)}$$

BAROTROPIC POTENTIAL VORTICITY

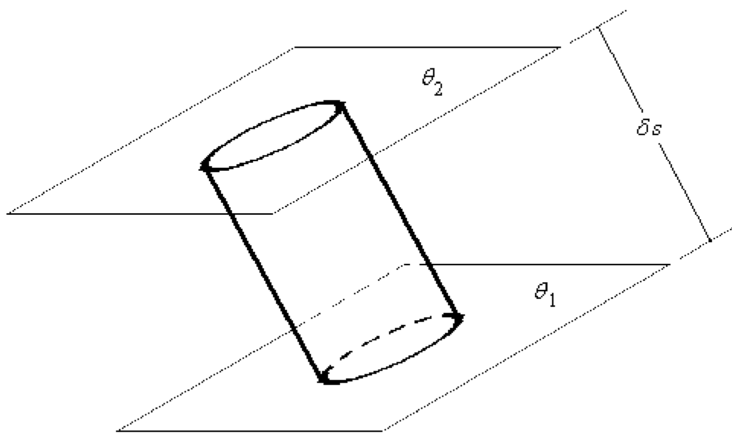
- Equation (5) can be rearranged to
$$\frac{D}{Dt} \ln(\zeta_g + f) = \frac{D}{Dt} \ln h$$
or
$$\frac{D}{Dt}\left(\frac{\zeta_g + f}{h} \right) = 0.$$
- The quantity $(\zeta_g + f)/h$ is conserved following the fluid parcel in a barotropic fluid.
  - o This quantity is called the *barotropic potential vorticity*.

POTENTIAL VORTICITY IN A BAROCLINIC FLUID

- The circulation theorem for a baroclinic fluid is
$$\frac{DC}{Dt} = -\oint \alpha dp = -\oint \frac{dp}{\rho}. \qquad \textbf{(6)}$$
- The equation of state for the atmosphere can be written as

$$\rho = \frac{p}{R_d T} = \frac{p}{R_d \theta \left(\frac{p}{p_0} \right)^{\kappa}} = \frac{p_0^{\kappa}}{R_d \theta} p^{1-\kappa} \quad \textbf{(7)}$$

- On a surface of constant potential temperature (7) shows us that the density is a function of pressure only.[1] This makes the right-hand-side of (6) equal to zero. Thus:
  - ***<u>On an isentropic surface the circulation is constant</u>.***
- Imagine a section of a stream tube of cross-sectional area δA lying between two isentropic surfaces.



- The circulation on the isentropic surface is given by

  $$C = \omega \delta A.$$

  The mass contained within the stream tube is

  $$m = \rho \delta A \, \delta s$$

  so that the circulation per unit mass is

  $$\frac{C}{m} = \frac{\omega}{\rho \delta s}.$$

  The length of the stream tube is

  $$\delta s = \frac{\partial s}{\partial \theta} \delta \theta$$

  so that the circulation per unit mass is

  $$\frac{C}{m} = \frac{\omega}{\delta \theta \rho} \frac{\partial \theta}{\partial s}.$$

  - If the flow is adiabatic then the ends of the stream tube will always lie on the isentropic surfaces. Thus, m is constant (as is $\delta\theta$). Since the circulation (C) is also constant, this means that

    $$\frac{\omega}{\rho} \frac{\partial \theta}{\partial s} = const.$$

- On the synoptic scale we can assume that isentropic surfaces are nearly horizontal, so that the stream tube is oriented along the z-axis. Therefore we can write

[1] This result is actually true for any fluid (not just ideal gases) for which density can be written as a function of only p and θ.

$$\frac{\eta_\theta}{\rho}\frac{\partial\theta}{\partial z} = const.,$$

and in terms of pressure this is

$$\frac{\eta_\theta}{\rho}\frac{\partial\theta}{\partial p}\frac{\partial p}{\partial z} = const.$$

- If the atmosphere is hydrostatic then this becomes

$$-g\eta_\theta\frac{\partial\theta}{\partial p} = const.$$

or

$$-g\left(\zeta_g + f\right)\frac{\partial\theta}{\partial p} = const.$$

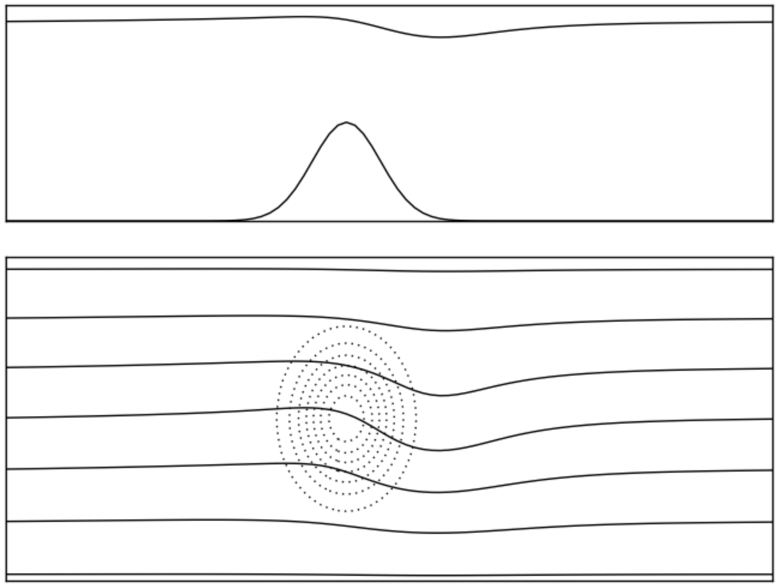
- The quantity

$$P = -g\left(\zeta_g + f\right)\frac{\partial\theta}{\partial p} \qquad \textbf{(8)}$$

is called *Ertel's potential vorticity*, and is the form of potential vorticity appropriate to the atmosphere.
  - o P is conserved following an air parcel in adiabatic flow, and is therefore a good tracer of air parcels under conditions where diabatic heating (latent heat of condensation, radiation, etc.) can be neglected.

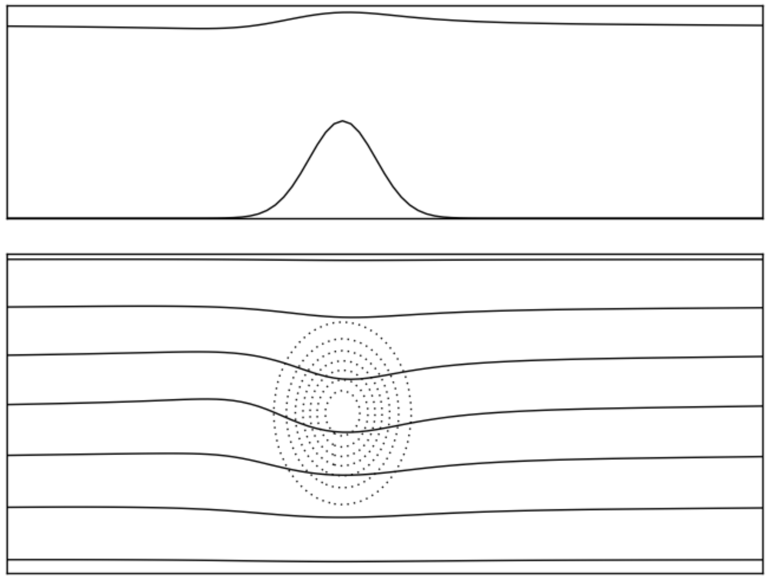
BAROTROPIC POTENTIAL VORTICITY AND LEE-SIDE TROUGHS

- Conservation of potential vorticity can be used to explain the formation of lee-side troughs in westerly flow over a mountain barrier.
- The diagram below shows a model simulation for a barotropic fluid. The model was initialized with a purely westerly flow over a Gaussian shaped hill.
  - o The upper diagram is a cross section along the x-axis for the middle of the domain, and shows the upper surface of the fluid and the model terrain
  - o The lower diagram is a plot of the streamfunction.
- We can qualitatively explain the features we see as follows:



- As the parcel starts to ascend the hill its depth decreases. This requires the absolute vorticity to also decrease. The atmosphere accomplishes this by creating both anticyclonic shear and curvature vorticity on the upstream side of the hill.
- As the parcel crests the hill and starts moving back down slope, the depth starts to increase. The absolute vorticity now must increase, and so the atmosphere adjusts by having both cyclonic shear and curvature on the downstream side of the hill.
- The result is a trough downstream (on the lee side) of the hill.

- For easterly flow a very different set of events occurs. This is shown in the diagram below.
- We explain the results qualitatively as:



- As the parcel ascends the East side of the hill, again the absolute vorticity must decrease. This is accomplished by the parcel heading South, to where the planetary vorticity is less. However, the curvature cannot be too great here, because the curvature is cyclonic and would work against a decrease in absolute vorticity.
- Once the parcel crests the hill and stretches as it descends, the absolute vorticity must once again increase, so the parcel curves gently back to the North.
- In easterly flow, instead of a lee-side trough, a ridge is formed directly over the obstacle.

EXERCISES

1. Show that $w(z_T) - w(z_0) = Dh/Dt$

2. Expand $\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0$ to show that it gives $\frac{\partial \zeta}{\partial t} + \vec{V} \bullet \nabla_h \zeta + \beta v = \frac{(\zeta + f)}{h}\frac{Dh}{Dt}$.

3. Use the conservation of Ertel's potential vorticity to demonstrate the formation of a lee-side trough for westerly flow, and a ridge over the hill for easterly flow.

ESCI 343 – Atmospheric Dynamics II
Lesson 1 – Ageostrophic Wind

References: *An Introduction to Dynamic Meteorology* (3rd edition), J.R. Holton

THE QG MOMENTUM EQUATIONS

- The QG momentum equations are derived as follows:
  - Start with the momentum equation in vector form,

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \bullet \nabla \vec{V} = -\nabla \Phi - \hat{k} \times f \vec{V} . \tag{1}$$

  - Split the horizontal wind into geostrophic and ageostrophic components,

$$\vec{V} = \vec{V}_g + \vec{V}_a . \tag{2}$$

and substitute into (1), ignoring the ageostrophic wind in the advection term. This gives

$$\frac{\partial \vec{V}_g}{\partial t} + \vec{V}_g \bullet \nabla \vec{V}_g = -\nabla \Phi - \hat{k} \times f \left(\vec{V}_g + \vec{V}_a \right) . \tag{3}$$

  - Next, we replace the Coriolis parameter by $f = f_0 + \beta y$, so that the momentum equations become

$$\frac{\partial \vec{V}_g}{\partial t} + \vec{V}_g \bullet \nabla \vec{V}_g = -\nabla \Phi - \hat{k} \times \left(f_0 + \beta y \right) \left(\vec{V}_g + \vec{V}_a \right) . \tag{4}$$

  - Expand (4) to get

$$\frac{\partial \vec{V}_g}{\partial t} + \vec{V}_g \bullet \nabla \vec{V}_g = -\nabla \Phi - \hat{k} \times f_0 \vec{V}_g - \hat{k} \times f_0 \vec{V}_a - \hat{k} \times \beta y \vec{V}_g - \hat{k} \times \beta y \vec{V}_a . \tag{5}$$

  - The last term in (5) is very small, and can be ignored, so we now have

$$\frac{\partial \vec{V}_g}{\partial t} + \vec{V}_g \bullet \nabla \vec{V}_g = -\nabla \Phi - \hat{k} \times f_0 \vec{V}_g - \hat{k} \times f_0 \vec{V}_a - \hat{k} \times \beta y \vec{V}_g . \tag{6}$$

  - By the definition of the geostrophic wind,

$$f_0 \vec{V}_g = \hat{k} \times \nabla \Phi \tag{7}$$

so that the first two terms on the right-hand side of (6) cancel, resulting in

$$\frac{D_g \vec{V}_g}{Dt} = -\hat{k} \times f_0 \vec{V}_a - \hat{k} \times \beta y \vec{V}_g \tag{8}$$

where

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \vec{V}_g \bullet \nabla \,. \tag{9}$$

THE AGEOSTROPHIC WIND EQUATION

- We've been stressing the fact that though the atmosphere is close to being in geostrophic balance, the unbalanced component of the wind (the ageostrophic wind) is very important for the dynamics of synoptic disturbances.
- The ageostrophic circulation is included in QG analysis through the divergence terms.
- We are now going to derive an equation for the ageostrophic wind itself
  - We start with the QG momentum equation (8) on the *f*-plane, so that the beta term is zero,

$$\frac{D_g \vec{V}_g}{Dt} = -\hat{k} \times f_0 \vec{V}_a \,.$$

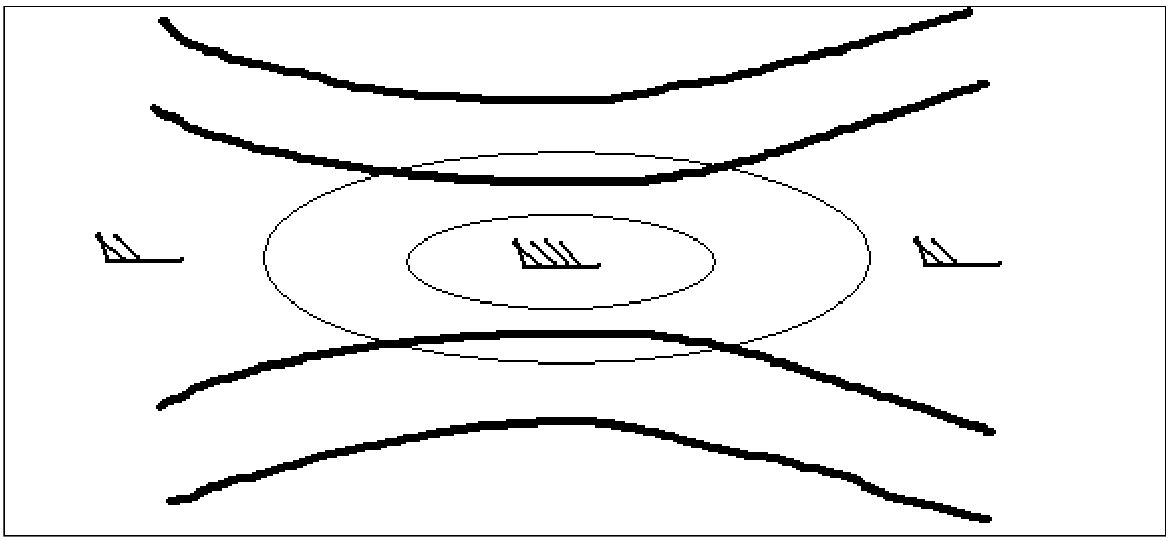
  - Dividing both sides by f_0 and taking $\hat{k} \times$ of both sides results in

$$\vec{V}_a = \frac{1}{f_0} \hat{k} \times \frac{D_g \vec{V}_g}{Dt} \,. \tag{10}$$

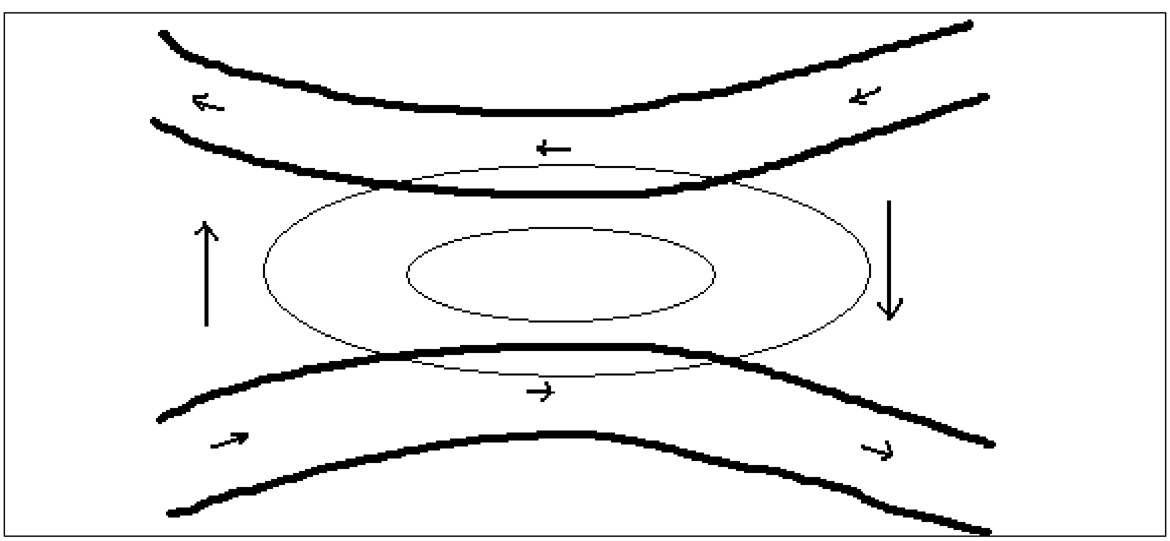
- The first thing to notice is that the ageostrophic wind always points to the left of the geostrophic acceleration (in the Northern Hemisphere).
  - This makes physical sense because if the wind flow is from an area of loose pressure gradient to tight pressure gradient, the wind must accelerate in order to get closer to geostrophic balance. The ageostrophic wind accomplishes this by blowing to the left (toward lower pressure).
  - If the flow is from tight gradient to loose gradient, the wind must decelerate, so the ageostrophic wind will blow toward higher pressure (and still left of the acceleration, which is now pointing upstream).

AGEOSTROPHIC CIRCULATION IN JET STREAKS

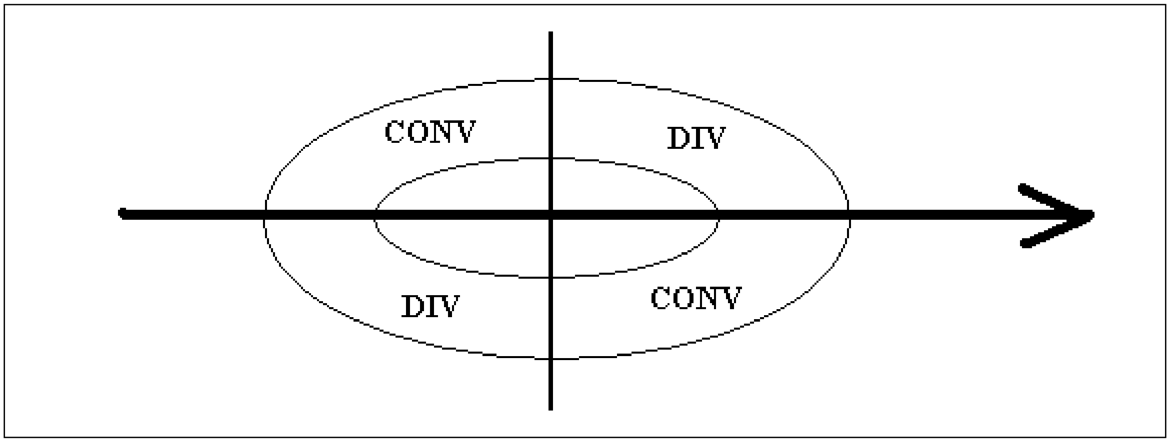
- Equation (10) can be used to explain why enhanced upward motion occurs in the right-entrance and left-exit region of jet streaks (maxima in the jet stream wind speeds).
- The diagram below shows height contours (bold) and isotachs (thin) for a typical jet streak. The wind barbs show the geostrophic wind.



- Remembering that the ageostrophic wind is to the left of the geostrophic acceleration, the ageostrophic wind will be oriented as shown below.



- From this diagram it is seen that the ageostrophic wind is divergent in the right-entrance and left-exit regions of the jet streak, and convergent in the other regions (see diagram below). This leads to upward motion in the right-entrance and left-exit regions of the jet streak**.**



THE ISALLOBARIC WIND

- The total derivative in (10) can be split into a local and an advective derivative, which results in

$$\vec{V}_a = \frac{1}{f_0}\hat{k}\times\frac{\partial \vec{V}_g}{\partial t} + \frac{1}{f_0}\hat{k}\times\left(\vec{V}_g \bullet \nabla \vec{V}_g\right). \qquad (11)$$

- The first term on the RHS of (11) is the *isallobaric wind*, because it can be written as

$$\vec{V}_{isall} = \frac{1}{f_0}\hat{k}\times\frac{\partial \vec{V}_g}{\partial t} = -\frac{1}{f_0^2}\nabla\left(\frac{\partial \Phi}{\partial t}\right) = -\frac{1}{f_0^2}\nabla\chi \qquad (12)$$

and thus blows perpendicular to the *isallobars* (lines of constant geopotential tendency, denoted as) and toward falling heights.

- The divergence of the isallobaric wind is

$$\nabla \bullet \vec{V}_{isall} = -\frac{1}{f_0^2}\nabla^2\chi\,. \qquad (13)$$

- When heights are falling the isallobaric wind is convergent.
- When heights are rising the isallobaric wind is divergent.

THE ADVECTIVE WIND

- The second term on the RHS of (11) is the *advective wind*,

$$\vec{V}_{adv} = \frac{1}{f_0}\hat{k}\times\left(\vec{V}_g \bullet \nabla \vec{V}_g\right). \tag{14}$$

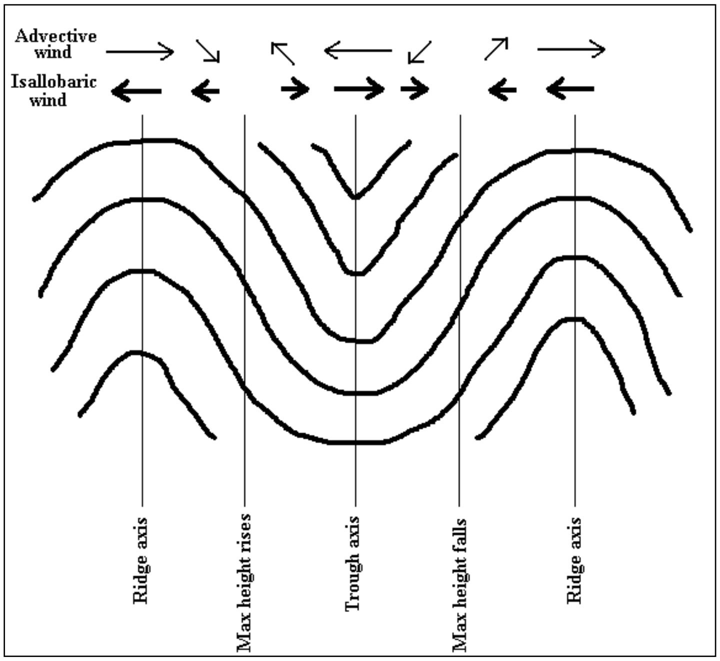
- The divergence of the advective wind is

$$\nabla \bullet \vec{V}_{adv} = -\frac{1}{f_0}\vec{V}_g \bullet \nabla \zeta_g\ . \tag{15}$$

- The advective wind is divergent when there is positive vorticity advection (PVA).
- The advective wind is convergent when there is negative vorticity advection (NVA).

DIVERGENCE OF THE AGEOSTROPHIC WIND IN TROUGHS/RIDGES

- We have established that the advective wind is divergent in regions of PVA, and so
  - Downstream of a trough the advective wind is divergent
  - Upstream of trough the advective wind is convergent.
- However, if a trough is propagating in the direction of the flow then downstream of a trough the heights are falling, while upstream the heights are rising. This means that
  - Downstream of a propagating trough the isallobaric wind is convergent
  - Upstream of a propagating trough the isallobaric wind is divergent.
- The net divergence or convergence downstream of the trough depends on which is more dominant: the advective or the isallobaric wind.
  - For ***upper-level troughs*** the advective wind dominates (due to the higher wind speeds and large vorticity advection found aloft), resulting in net divergence ahead of the upper-level trough, and convergence behind it.
  - For ***lower-level troughs***, the isallobaric wind tends to either cancel or be larger than the advective wind. Therefore, there is usually net convergence ahead of a low-level trough, and net divergence behind it.
- The diagram below shows the orientation of the advective and isallobaric wind vectors for a trough-ridge system.



EXERCISES

1. Show that $\frac{1}{f_0}\hat{k}\times\frac{\partial\vec{V}_g}{\partial t}=-\frac{1}{f_0^2}\nabla\chi$.

2. Show that the divergence of the isallobaric and advective winds are given by

$$\nabla\bullet\vec{V}_{iso}=-\frac{1}{f_0^2}\nabla^2\chi$$

$$\nabla\bullet\vec{V}_{adv}=-\frac{1}{f_0}\vec{V}_g\bullet\nabla\zeta_g$$

ESCI 343 – Atmospheric Dynamics II
Lesson 2 – Q-G Height-tendency Equation

Reference: *An Introduction to Dynamic Meteorology* (3rd edition), J.R. Holton
Synoptic-dynamic Meteorology in Midlatitudes, Vol 1, H.B. Bluestein

THE QUASIGEOSTROPHIC THERMODYNAMIC ENERGY EQUATION

- The thermodynamic energy equation in pressure coordinates is

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J\ , \tag{1}$$

which when expanded out, and using the definition of ω, becomes

$$\underset{A}{\frac{\partial T}{\partial t}} = \underset{B}{-\vec{V} \bullet \nabla_p T} - \left(\underset{C}{\frac{\partial T}{\partial p}} - \underset{D}{\frac{\alpha}{c_p}} \right) \omega + \underset{E}{\frac{J}{c_p}} \quad . \tag{2}$$

In this form, the terms represent:

Term A – Local temperature tendency

Term B – Horizontal thermal advection

Term C – Vertical thermal advection

Term D – Adiabatic expansion/compression due to vertical motion

Term E – Diabatic heating (radiation, latent heat, etc.)

- Terms C and D can be combined and written as

$$\frac{\partial T}{\partial p} - \frac{\alpha}{c_p} = \left(\frac{\alpha}{\theta} \frac{\partial \theta}{\partial p} \right) \frac{p}{R_d}\,,$$

and defining the *static-stability parameter*, σ, as

$$\sigma \equiv -\frac{\alpha}{\theta} \frac{\partial \theta}{\partial p}\,, \tag{3}$$

we get the following form of the thermodynamic energy equation in pressure coordinates.

$$\underset{A}{\frac{\partial T}{\partial t}} = \underset{B}{-\vec{V} \bullet \nabla_p T} + \underset{C}{\frac{\sigma p}{R_d} \omega} + \underset{D}{\frac{J}{c_p}}\ . \tag{4}$$

- In this form of the equation, the vertical advection and adiabatic expansion/compression are combined into one term, Term C.

- The static stability parameter is a positive number for a stable atmosphere, and a negative number for an unstable atmosphere.
- The quasigeostrophic form of the thermodynamic energy equation in pressure coordinates is

$$\left(\frac{\partial}{\partial t}+\vec{V}_g \bullet \nabla_p\right)T-\left(\frac{\sigma p}{R_d}\right)\omega=\frac{J}{c_p} \tag{5}$$

where we have simply substituted the geostrophic wind for the actual wind in the advection term.

THE HYDROSTATIC EQUATION IN PRESSURE COORDINATES

- The hydrostatic equation in pressure coordinates is derived as follows:

In height coordinates we have

$$\frac{\partial p}{\partial z}=-\rho g\ . \tag{6}$$

Using the chain rule

$$\frac{\partial p}{\partial z}=\frac{\partial p}{\partial \Phi}\frac{\partial \Phi}{\partial z}=g\frac{\partial p}{\partial \Phi},$$

and so from (6)

$$g\frac{\partial p}{\partial \Phi}=-\rho g$$

or

$$\frac{\partial \Phi}{\partial p}=-\alpha\ . \tag{7}$$

- Equation (7) is the hydrostatic equation in pressure coordinates.

AN ALTERNATE FORM OF THE QG THERMODYNAMIC EQUATION

- From the ideal gas law we can write equation (7) as

$$\frac{\partial \Phi}{\partial p}=-\alpha=-\frac{R_d T}{p}, \tag{8}$$

which when solved for T gives

$$T = -\frac{p}{R_d}\frac{\partial \Phi}{\partial p}. \tag{9}$$

- Substituting (9) for the temperature in the QG thermodynamic energy equation, (5) gives

$$\left(\frac{\partial}{\partial t} + \vec{V}_g \bullet \nabla_p\right)\frac{\partial \Phi}{\partial p} + \sigma\omega = -\frac{J}{c_p}\frac{R_d}{p}, \tag{10}$$

which rearranged yields

$$\frac{\partial}{\partial p}\frac{\partial \Phi}{\partial t} = -\vec{V}_g \bullet \nabla_p\left(\frac{\partial \Phi}{\partial p}\right) - \sigma\omega - \frac{R_d J}{p c_p}. \tag{11}$$

- Geopotential tendency is defined as

$$\chi = \frac{\partial \Phi}{\partial t}, \tag{12}$$

so the QG thermodynamic energy equation becomes

$$\frac{\partial \chi}{\partial p} = -\vec{V}_g \bullet \nabla_p\left(\frac{\partial \Phi}{\partial p}\right) - \sigma\omega - \frac{R_d J}{p c_p}. \tag{13}$$

- **NOTE! Equations (5) and (13) are identical! They are just written in different forms.**

THE QG VORTICITY EQUATION REVISITED

- The QG vorticity equation in pressure coordinates is

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \bullet \nabla_p\left(\zeta_g + f\right) + f_0\frac{\partial \omega}{\partial p}. \tag{14}$$

- The geostrophic vorticity in terms of the geopotential is

$$\zeta_g = \frac{1}{f_0}\nabla_p^2\Phi. \tag{15}$$

- Substituting (15) into (14) give another form of the QG vorticity equation

$$\nabla^2\chi = -f_0\vec{V}_g \bullet \nabla_p\left(\frac{1}{f_0}\nabla_p^2\Phi + f\right) + f_0^2\frac{\partial \omega}{\partial p}. \tag{16}$$

- **NOTE! Equations (14) and (16) are identical! They are just written in different forms.**

- Equations (13) and (16) are two equations with two dependent variables, χ and ω.
  - If we know the what the geopotential field (Φ) is, then these equations form a complete system which can be solved for either χ or ω.

THE GEOPOTENTIAL TENDENCY EQUATION

- The first equation we will derive is the geopotential tendency equation, found by eliminating ω between (13) and (16).
- The idea behind this is simple, but the individual mathematical steps become complicated.
  - Differentiate (13) with respect to pressure, then multiply it by f_0^2/σ.
  - Add the result to (16).
- The result is (note that for ease of notation the subscript p is no longer written on the del operator, but is implied).

$$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\chi = -f_0\vec{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - \frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] - \frac{f_0^2 R_d}{\sigma c_p}\frac{\partial}{\partial p}\left(\frac{J}{p}\right) - \frac{f_0^2}{\sigma}\omega\frac{\partial \sigma}{\partial p} \quad (17)$$

- The static stability parameter normally increases with height; however, analysis of the Q-G tendency equation is slightly easier if we assume that σ is constant so that the last term disappears. In this case, the equation becomes

$$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\chi = -f_0\vec{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - \frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] - \frac{f_0^2 R_d}{\sigma c_p}\frac{\partial}{\partial p}\left(\frac{J}{p}\right). \quad (18)$$

Q-G geopotential tendency equation

- Though ugly, this equation has a sort of inner beauty. We first try to see this beauty by analyzing the terms of the equation in a *qualitative* fashion.
- We do this by imaging that the horizontal structure of disturbances in the atmosphere can be approximated by sinusoidal functions such as

$$\chi(x, y, p, t) = \mathrm{X}(p,t)\exp[i(kx + ly)]. \quad (19)$$

- If we ignore the pressure derivatives on the left hand side (LHS) of the tendency equation, then the LHS becomes

$$\nabla^2\left[\chi(x,y,p,t)\right]=\nabla^2\left\{\mathrm{X}(p,t)\exp[i(kx+ly)]\right\}$$
$$=-(k^2+l^2)\mathrm{X}(p,t)\exp[i(kx+ly)]=-(k^2+l^2)\chi(x,y,p,t)$$

or more simply,

$$\nabla^2\chi\propto-\chi\,. \tag{20}$$

- What this means is that ***for a sinusoidal disturbance having a zero mean value, the horizontal Laplacian of a field is proportional to the negative of the field***.
- So, we can qualitatively think of the LHS of the equation as being nothing more than $-\chi$, so that ***if the LHS is negative it means that the geopotential tendency is positive.***
- It may help to write the equation using the 'proportional to' symbol ($\propto$), in the following manner,

$$-\chi\quad\propto\quad -f_0\vec{V}_g\bullet\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right)-\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g\bullet\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]-\frac{f_0^2R_d}{\sigma c_p}\frac{\partial}{\partial p}\left(\frac{J}{p}\right). \tag{21}$$

- If we can find the signs of the terms on the right hand side (RHS) of the equation we will be able to tell whether heights are going to rise or fall.

THE VORTICITY ADVECTION TERM

- Though the terms on the RHS look intimidating, they really aren't. The first term on the RHS is nothing more than absolute vorticity advection,

$$\textit{Absolute vorticity advection}=-\vec{V}_g\bullet\nabla\left(\zeta_g+f\right)=-\vec{V}_g\bullet\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right). \tag{22}$$

- If vorticity advection is positive, this means that the geopotential tendency, χ, is negative (falling heights).

THE DIFFERENTIAL THERMAL ADVECTION TERM

- Remember that we earlier found that

$$T=-\frac{p}{R}\frac{\partial\Phi}{\partial p}\,. \tag{23}$$

- This means that the second term on the RHS is proportional to the vertical derivative of temperature advection.
- If temperature advection decreases with height (increases with pressure) then

$$\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] > 0$$

and so χ will be positive, so heights will be rising.

- Two important points to note:
  - *Remember that we are using pressure coordinates, so if something is increasing with height, it is decreasing with pressure, and therefore* $\partial/\partial p < 0$.
  - *It is the vertical derivative of the advection that matters.* Strong cold advection over weak cold advection has the same effect as weak warm advection over strong warm advection, because in both cases the derivative has the same value.

THE DIFFERENTIAL DIABATIC HEATING TERM

- The differential heating term (third term on RHS) behaves similarly to the differential thermal advection term.
  - If the heating decreases with height, or cooling increases with height, then heights will rise.
- Another useful way of writing the essence of the Q-G tendency equation is

$\partial\Phi/\partial t \propto$ −absolute vorticity advection +$\partial/\partial p$(thermal advection) +$\partial/\partial p$(heating)
or
$\partial\Phi/\partial t \propto$ −absolute vorticity advection −$\partial/\partial z$(thermal advection) −$\partial/\partial z$(heating)

- The previous analysis leads us to a very important conclusion. ***In quasi-geostrophic theory, there are only three ways for heights to fall…either through positive vorticity advection, through warm advection that increases with height, or through diabatic heating that increases with height*!**

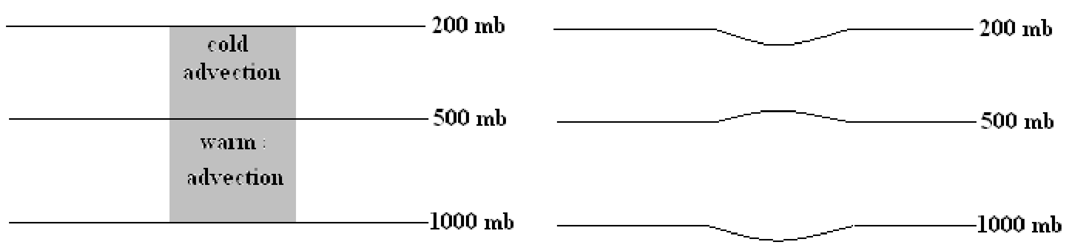
A PHYSICAL INTERPRETATION OF THE TENDENCY EQUATION

- The effects of the terms on the RHS of the tendency equation can be explained physically, as well as mathematically.
- *Vorticity advection*: We know that on the synoptic scale there is a direct relationship between vorticity and geopotential, via

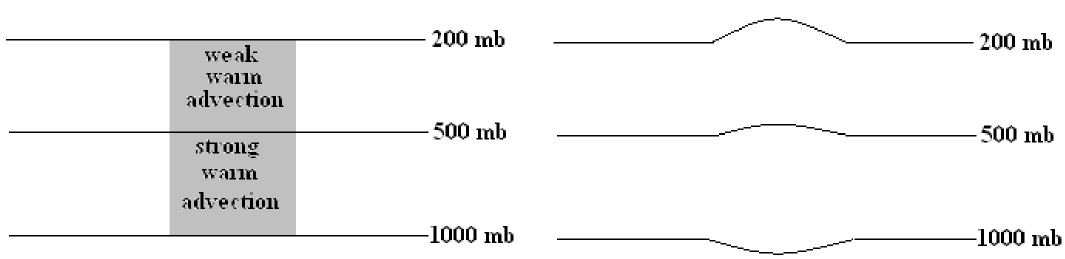
$$\zeta_g = \frac{1}{f_0}\nabla^2\Phi \,. \tag{24}$$

- PVA leads to increasing values of vorticity, which if the atmosphere is going to remain in near geostrophic balance requires $\nabla^2\Phi$ must also increase, which means that Φ itself must adjust to a lower value (high values of $\nabla^2\Phi$ imply low values of Φ itself).
- In essence, the height anomaly is advected with the mean flow.

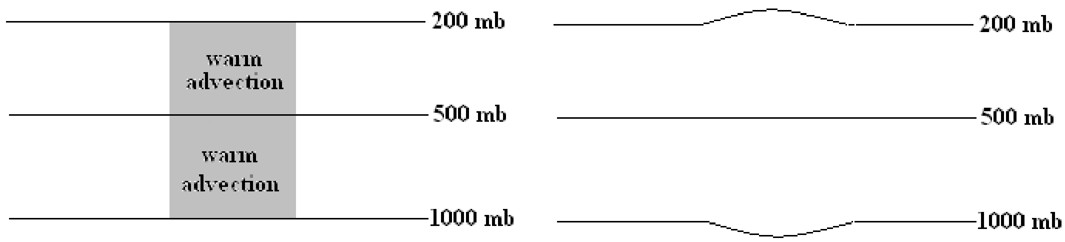
- *Differential thermal advection*: The effects of differential thermal advection can be thought of as follows:
  - Imagine a scenario where there is net warm advection in the lower levels (below 500 mb), and net cold advection in the upper levels (above 500 mb).
  - Since the thickness between two pressure surfaces is proportional to temperature, the low-level warm advection will lead to increased thickness of the 1000 – 500 mb layer, while the upper-level cold advection will lead to decreased thickness of the 500 – 200 mb layer.
  - The net result is height rises at 500 mb (see diagram below).



- The same result will occur with weak warm advection aloft and stronger warm advection in the low-levels (see diagram below).



- If the advection is the same strength both aloft and below, then there is no change in height at 500 mb (see diagram below).



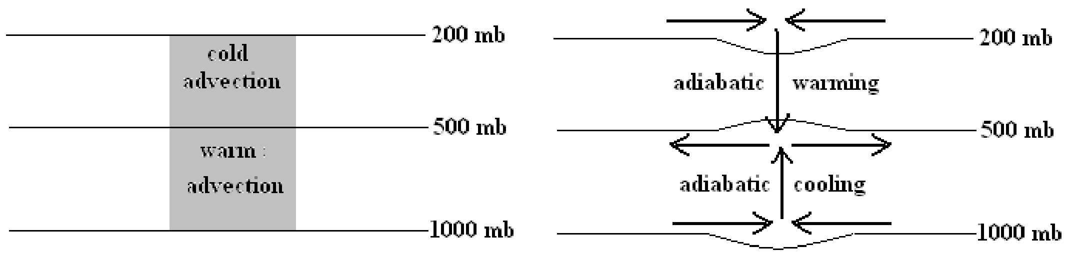
- Typically, thermal advection is very small in the upper troposphere (above 500 mb) compared to that in the lower-levels, so it is really the low level advection that determines the 500 mb geopotential tendency.
    - Cold advection in the lower levels will decrease the thickness of the 1000 –500 mb layer, and lower the heights at 500 mb, as would be expected (since *cold advection decreasing with height is the same as warm advection increasing with height*).
    - Warm advection in the lower levels will increase the thickness of the 1000 –500 mb layer, and result in height rises at 500 mb.
- *Differential diabatic heating*. The physical interpretation for the differential diabatic heating term is similar to that for differential advection.
    - If there is more heating above a level than below it, the heights at that level will fall.
        - Phrased another way, we can say that above the level of maximum heating (J/p) the heights will rise, and below the level of maximum heating heights will fall.

Le CHATELIER'S PRINCIPLE

- Le Chatelier's Principle, named for Henry Louis Le Chatelier, states that many natural systems will resist changes, and if forced to change, will react with process that try to restore the original state.
- Though Le Chatelier's Principle isn't as rigorous and general as often thought to be[1], we can see Le Chatelier's principle at work in the differential thermal advection and diabatic heating terms of the Q-G tendency equation.

[1] see J. de Heer, *J. Chem. Educ.*, **34**, 375 (1957)

- For example, cold advection (or diabatic cooling) over warm advection (or diabatic heating) forces height rises at 500 mb, as well as height falls at 200 and 1000 mb (as per the diagram below.)



- However, these height rises and falls indicate that there must be a change in the vorticity at these levels (increased vorticity where there are height falls, and decreased vorticity where there are height rises.)
- To accomplish this vorticity change in a quasi-geostrophic framework, there must be convergence where there are height falls, and divergence where there are height rises.
- This convergence/divergence pattern is the result of the isallobaric wind.
- The convergence/divergence pattern leads to upward motion and adiabatic cooling in the lower levels, and subsidence and adiabatic warming in the upper levels.
- The adiabatic heating/cooling opposes the original temperature change due to advection.
- LeChatelier's Principle doesn't mean that the effects of the differential heating (advection) will be completely cancelled by the adiabatic heating/cooling from the secondary circulation, but does illustrate that the atmosphere will resist the changes imposed by the thermal forcing, and will respond with a secondary circulation.

● LeChatelier's Principle can also be seen in the vorticity advection term of the QG height tendency equation.
  - PVA leads to falling heights. But, the advective wind is divergent in regions of PVA, and since divergence decreases vorticity this effect is counter to the vorticity increase due to the PVA.

EFFECTS OF STATIC STABILITY

- The static stability of the troposphere on the synoptic scale is rarely negative.
- Since static stability appears in the denominator of the heating terms, an increase in static stability will cause the height rises or falls from these terms to be of a less magnitude than in a less stable atmosphere.
- The vertical change of static stability is a little more complex. The static stability term (which we've previously neglected) has the following effect on the height tendency:

$$\chi \propto \frac{f_0^2}{\sigma}\omega\frac{\partial\sigma}{\partial p}. \tag{25}$$

- Since static stability usually increases with height, this implies that downward vertical motion will lead to a lowering of the geopotential heights, while upward motion will lead to raising of geopotential heights. To understand this physically, recall the from the thermodynamic energy equation that vertical motion affects temperature tendency via

$$\frac{\partial T}{\partial t} \propto \sigma\omega. \tag{26}$$

- The impact of the vertical motion is enhanced as the static stability increases. Therefore, if the motion is downward, there will be more heating at higher altitudes (where σ is larger) than at lower altitudes (where σ is smaller). Thus, there will be heating increasing with height, which we have already seen leads to height falls.
- For upward motion, there will be more cooling aloft than below, which will lead to height rises.
- If static stability decreases with height, then the effect of vertical motion is opposite from that just described, since there will be larger heating (or cooling) below, rather than aloft.

AN ADVANCED TREATMENT OF THE TENDENCY EQUATION

- Since we've previously assumed that disturbances in the atmosphere have a sinusoidal horizontal structure, we will also assume that the forcing (terms on the RHS of the tendency equation) also have a sinusoidal structure. So, we assume

$$\chi(x,y,p,t) = \mathrm{X}(p,t)\exp[-(kx+ly)]$$
$$\textit{vorticity advection term} = F_v(p)\exp[-(kx+ly)]$$
$$\textit{thermal advection term} = -\frac{dF_T(p)}{dp}\exp[-(kx+ly)]$$
$$\textit{diabatic heating term} = -\frac{dF_J(p)}{dp}\exp[-(kx+ly)]$$

and put these into the tendency equation. This gives an ordinary differential equation,

$$\frac{d^2\mathrm{X}}{dp^2} - \alpha^2\mathrm{X} = \frac{\sigma}{f_0^2}\left(F_v - \frac{dF_T}{dp} - \frac{dF_J}{dp}\right) \tag{27}$$

where

$$\alpha^2 = \frac{K^2\sigma}{f_0^2} \tag{28}$$

(we've assumed the static stability parameter, σ, is constant with height).

- The solutions to (27) are hyperbolic sines and cosines with a characteristic vertical length scale of

$$D = \frac{2\pi}{\alpha} = \frac{2\pi f_0}{K\sqrt{\sigma}} = \frac{f_0 L}{\sqrt{\sigma}} \tag{29}$$

where L is the wavelength of the disturbance.

- So, *the longer the wavelength of a disturbance, the deeper the effects of its forcing terms are felt in the atmosphere*.

QUASI-GEOSTROPHIC POTENTIAL VORTICITY

- The tendency equation (ignoring the diabatic heating term, J) can be written as

$$\frac{D_g}{Dt}\left[\frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right)\right] = 0\,.$$

- The quantity in brackets is called the *quasi-geostrophic potential vorticity*,

$$q \equiv \frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)$$

and is conserved following a fluid parcel in adiabatic motion.

EXERCISES

1. Derive the Q-G tendency equation, showing all steps.

2. Is the vertical extent of the forcing terms in the Q-G tendency equation larger or smaller in the tropics as compared to the middle latitudes?

ESCI 343 – Atmospheric Dynamics II
Lesson 3 – QG Omega Equation

Reference: *An Introduction to Dynamic Meteorology* (3rd edition), J.R. Holton
Synoptic-dynamic Meteorology in Midlatitudes, Vol 1, H.B. Bluestein

THE QG OMEGA EQUATION

- We previously derived the QG geopotential tendency equation by eliminating omega from Eqns. (1) and (2)

$$\frac{\partial \chi}{\partial p} = -\vec{V}_g \bullet \nabla\left(\frac{\partial \Phi}{\partial p}\right) - \sigma\omega - \frac{R_d J}{p c_p}. \tag{1}$$

$$\nabla^2 \chi = -f_0 \vec{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + f_0^2 \frac{\partial \omega}{\partial p}. \tag{2}$$

- To get the QG omega equation we instead eliminate the geopotential tendency from these two equations.
- This is accomplished by differentiating (2) with respect to pressure and then subtracting the horizontal Laplacian of (1).
- The result is the QG Omega Equation,

$$\begin{aligned}\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = &-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] \\ &-\frac{1}{\sigma}\nabla^2\left[-\vec{V}_g \bullet \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] - \frac{R_d}{\sigma c_p}\nabla^2\left(\frac{J}{p}\right)\end{aligned} . \tag{3}$$

QG omega equation

- Like the geopotential tendency equation, we can make sense of the omega equation in a qualitative fashion my assuming that atmospheric disturbances are sinusoidal. The LHS of the equation is then proportional to the negative of omega, or

$$\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto -\omega. \tag{4}$$

- Also, we know

$$-\nabla^2\left[-\vec{V}_g \bullet \nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right] \propto -\vec{V}_g \bullet \nabla\left(-\frac{\partial\Phi}{\partial p}\right)$$

$$-\nabla^2\left(\frac{J}{p}\right) \propto \frac{J}{p}$$

which allows us to write

$$\omega \propto \frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] - \left[-\vec{V}_g \bullet \nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right] - \left(\frac{J}{p}\right). \qquad (5)$$

or

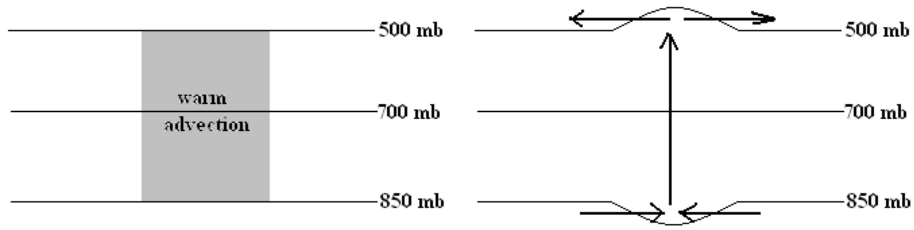
$\omega \propto \partial/\partial p$ (absolute vorticity advection) – thermal advection – heating

or

$w \propto \partial/\partial z$ (absolute vorticity advection) + thermal advection + heating

A PHYSICAL INTERPRETATION OF THE OMEGA EQUATION

- As with the tendency equation, the terms on the RHS of the omega equation can be explained physically.
- *Differential vorticity advection*:
  - If PVA increases with height, then this implies that there will be increasing divergence of the advective wind with height.
  - This increasing divergence with height will result in upward vertical motion.
- *Thermal advection*:
  - Warm advection will increase the thickness of a layer and result in higher heights aloft compared to below.
  - This results in divergence of the isallobaric wind aloft, and convergence of the isallobaric wind below.
  - This convergence/divergence pattern leads to upward motion.



Thickness increase due to warm advection at 700 mb raises 500 mb heights and lowers 850 mb heights. The isallobaric wind diverges at 500 mb and converges at 850 mb, resulting in upward vertical motion at 700 mb.

- Note that in this case, Le Chatelier's principle is at work, because the upward motion will lead to adiabatic cooling, which opposes the temperature change forced by the advection.
- *Diabatic Heating*:
    - The diabatic heating term has essentially the same physical explanation as the advection term. Warming leads to upward motion, regardless of the cause of the warming (warm advection or diabatic heating.)

THE Q-VECTOR

- The differential vorticity advection term and the thermal advection term in the omega equation represent different physical processes.
- They may add or cancel each other, and so analysis of the net result is difficult.
- Another useful way of looking at diagnosing vertical motion is to derive an alternate form of the omega equation, called the Q-vector form of the equation.
- For the *f*-plane (constant Coriolis parameter) under adiabatic conditions the Q-vector form of the QG omega equation is

$$\sigma\nabla^2\omega + f_0^2\frac{\partial^2\omega}{\partial p^2} = -2\nabla\bullet\vec{Q}$$

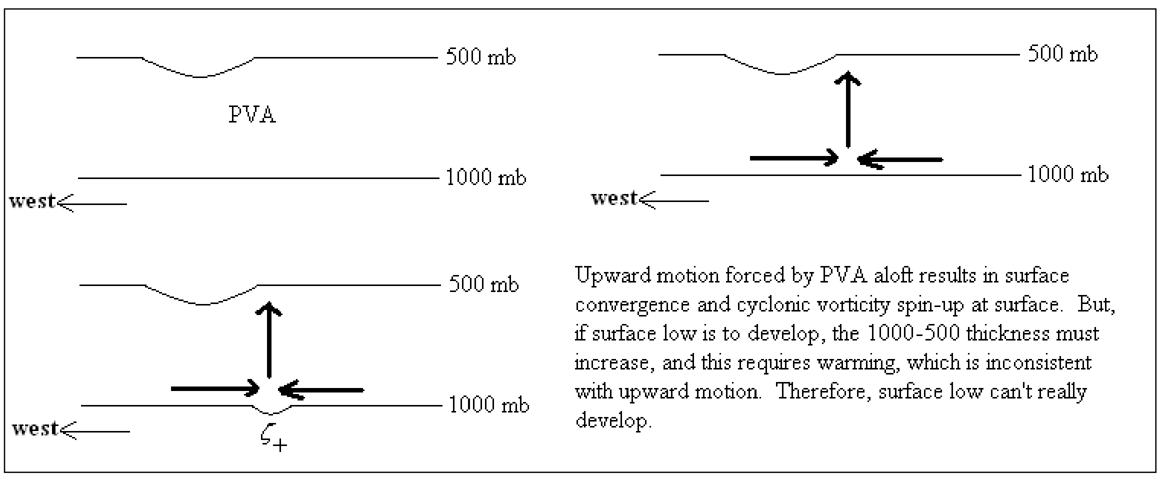
$$\vec{Q} \equiv \left(-\frac{R_d}{p}\frac{\partial\vec{V}_g}{\partial x}\bullet\nabla T, -\frac{R_d}{p}\frac{\partial\vec{V}_g}{\partial y}\bullet\nabla T\right)$$

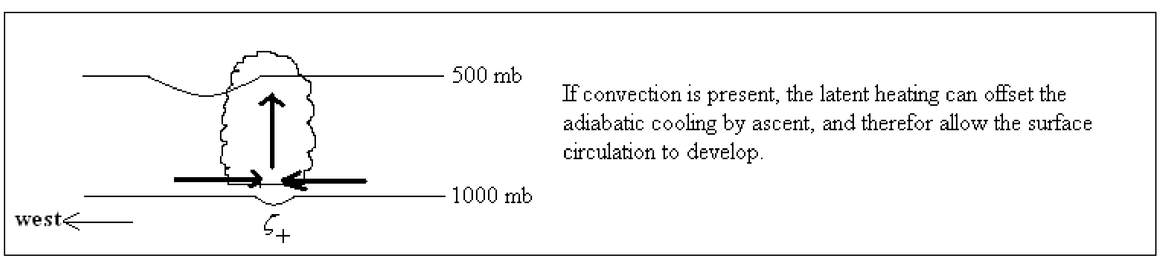
Q-vector form of omega equation

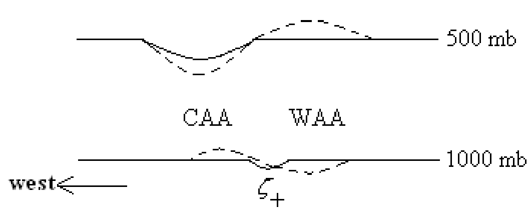
- In this form the vertical motion is only a function of the divergence of the vector **Q**. This can be analyzed on weather maps (by computers) to diagnose the omega field. The rule to remember is
    - ***DIVERGENCE OF Q MEANS DOWNWARD MOTION, CONVERGENCE OF Q MEANS UPWARD MOTION!***

THE SELF-DEVELOPMENT PROCESS FOR EXTRATROPICAL CYCLONES

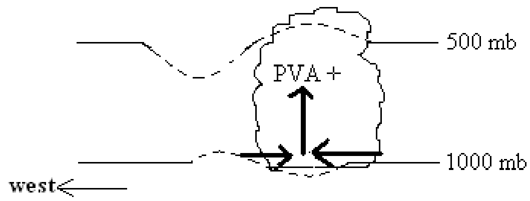
- Quasigeostrophic theory can be used to form a conceptual model of how an extratropical cyclone develops, through what is known as Pettersons's self-development process.
- This occurs when an upper-level trough approaches an old frontal boundary or baroclinic zone, and is explained below.







There will be warm-air advection in the southerly flow to the east of the surface cyclone, and cold-air advection in the northerly flow to the west of the surface cyclone. This will amplify the trough-ridge structure aloft, and also assist in propogation of the surface cyclone toward the East. The dashed lines show these forcings, but keep in mind that the upper-level low will also be propogating toward the East, so the surface low isn't going to outrun the upper-level trough.



The amplified trough-ridge structure leads to enhanced PVA over the surface low, leads to upward motion, more surface convergence, and a subsequent enhancement of the surface low. The process continues to repeat itself in a positive feedback loop.

- The diagrams above aren't meant to show exactly what the pressure surfaces will look like, but just to give an idea of how self-development operates.
- The development process will proceed as long as the upper-level trough is upstream of the surface cyclone, so that there is PVA over the surface cyclone.

EXERCISES

1. Derive the QG omega equation from the QG thermodynamic energy equation and the QG vorticity equation.

ESCI 343 – Atmospheric Dynamics II
Lesson 4 – Introduction to Waves

Reference: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
Waves in Fluids, J. Lighthill
Atmosphere-Ocean Dynamics, A.E. Gill

Reading: Holton, Section 7.2

GENERAL

The governing equations support many wavelike motions (waves are broadly defined as oscillations of the dependent variables.) Some of the waves supported by the equations are:

- External (surface) gravity waves
- Internal gravity waves
- Inertia-gravity waves
- Acoustic waves (including Lamb waves)
- Rossby waves
- Kelvin waves
- Kelvin-Helmholtz waves

Some of these waves are important for the dynamics of synoptic scale systems, while others are merely "noise." In order to understand dynamic meteorology, we must understand the waves that can occur in the atmosphere.

BASIC DEFINITIONS

- *amplitude* – half of the difference in height between a crest and a trough.
- *wavelength* (λ) – the distance between crests (or troughs)
- *wave number* (K) – $2\pi/\lambda$; the number of radians in a unit distance in the direction of wave propagation (sometimes the wave number is just defined as $1/\lambda$, in which case it is the number of wavelengths per unit distance.)
  - A higher wave number means a shorter wavelength.
  - Units are radians m^{-1}, or sometimes written as just m^{-1}.
  - We can also define wave numbers along each of the axes.
    - k is the wave number in the x-direction ($k = 2\pi/\lambda_x$).
    - l is the wave number in the y-direction ($l = 2\pi/\lambda_y$).
    - m is the wave number in the z-direction. ($m = 2\pi/\lambda_z$).
  - The wave number vector is given by
$$\vec{K} \equiv k\hat{i} + l\hat{j} + m\hat{k}$$
(don't confuse k and $\hat{k}$) and points in the direction of propagation of the wave.
- *angular frequency* (ω) – 2π times the number of crests passing a point in a unit of time.
  - Units are radians s^{-1}, sometimes just written as s^{-1}.
- *phase speed* (c) –the speed of an individual crest or trough.
  - For a wave traveling solely in the x-direction, $c = \omega/k$.

- For a wave traveling solely in the *y*-direction , $c = \omega / l$.
- For a wave traveling solely in the *z*-direction , $c = \omega / m$.
- For a wave traveling in an arbitrary direction, $c = \omega / \mathrm{K}$, where K is the *total wave number* given by $\mathrm{K}^2 = k^2 + l^2 + m^2$.
- For a wave traveling in an arbitrary direction, there is a phase speed along each axis, given by $c_x = \omega / k$, $c_y = \omega / l$, and $c_z = \omega / m$. ***Note that these are not the components of a vector!***

$$\vec{c} \neq c_x \hat{i} + c_y \hat{j} + c_z \hat{k}$$

  The phase velocity vector is actually given by

$$\vec{c} = \frac{\omega}{\mathrm{K}^2} \vec{\mathrm{K}} = \frac{\omega}{\mathrm{K}^2} (k \hat{i} + l \hat{j} + m \hat{k}) .$$

- The magnitude of the phase velocity (the phase speed) is given by

$$c = \frac{\omega}{K} \ .$$

- *group velocity* ($\boldsymbol{c}_g$) – the velocity at which the wave energy moves. Its components are given by

$$\vec{c}_g = \frac{\partial \omega}{\partial k} \hat{i} + \frac{\partial \omega}{\partial l} \hat{j} + \frac{\partial \omega}{\partial m} \hat{k}$$

  - The magnitude of the group velocity (the group speed) is given by

$$c_g = \frac{\partial \omega}{\partial K} \ .$$

- *dispersion relation* – an equation that gives the angular frequency of the wave as a function of wave number and physical parameters,

$$\omega = F\left(k, l, m, \ physical\ parameters\right).$$

  Each wave type has a unique dispersion relation. One of our main goals when studying waves is to determine the dispersion relation.

WAVE DISPERSION

- If the group velocity is the same as the phase speed of the individual waves making up the packet, then the waves are *non-dispersive*.
  - If waves are non-dispersive, then the shape of the wave packet never changes in time.
- If the group velocity is different than the phase speed on the waves making up the packet, then the waves are *dispersive*.
  - If the waves are dispersive, then the shape of the wave packet will change with time.
- Waves are dispersive if the phase velocity is not equal to the group velocity.
- Waves are non-dispersive if the phase velocity is equal to the group velocity.
- The two links below show animated GIF loops illustrating dispersive and non-dispersive wave behavior.
  - Dispersive wave animation: http://www.atmos.millersville.edu/~adecaria/ESCI343/dispersion-loop-DEEP.gif

- Non-dispersive wave animation: http://www.atmos.millersville.edu/~adecaria/ESCI343/dispersion-loop-SHALLOW.gif

THE EQUATION FOR A WAVE

The equation for a wave traveling in the positive *x* direction is

$$u(x,t) = A\sin(k\,x - \omega\,t) + B\cos(k\,x - \omega\,t)$$

An alternate way of writing this is

$$u(x,t) = A\sin k(x - ct) + B\cos k(x - ct)$$

For a wave traveling in the negative x direction, the equation is

$$u(x,t) = A\sin(k\,x + \omega\,t) + B\cos(k\,x + \omega\,t)$$

EULER'S FORMULA

Euler's formula states that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

From Euler's formula we have the following two identities:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = -i\,\frac{e^{i\theta} - e^{-i\theta}}{2}$$

Using Euler's formula a wave traveling in the positive *x*-direction can be written as

$$u(x,t) = Ae^{i(kx-\omega t)}$$

a wave traveling in the negative *x*-direction can be written as

$$u(x,t) = Ae^{i(kx+\omega t)},$$

where the amplitude *A* may itself be a complex number,

$$A = a_r + ia_i\,,$$

and gives information about the phase of the wave.

We will frequently use this complex notation for waves because it makes differentiation more straightforward because you don't have to remember whether or not to change the sign (as you do when differentiating sine and cosine functions).

The complex amplitude, *A*, gives information about the phase of the wave. In this form we have the following phase relations between two waves (*u* and *v*), given by

$$u = Ae^{i(kx-\omega t)}$$

$$v = Be^{i(kx-\omega t)}$$

$$\begin{aligned} &u \propto v && \text{in phase} \\ &u \propto -iv && 90^\circ \text{ out of phase} \\ &u \propto -v && 180^\circ \text{ out of phase} \\ &u \propto iv && 270^\circ \text{ out of phase} \end{aligned}$$

SPECTRAL ANALYSIS

It is rare to find a wave of a single wavelength in the atmosphere. Instead, there are many waves of different wavelengths superimposed on one another. However, we can use the concept of spectral analysis to isolate and study individual waves, recognizing that we can later sum them up if need be. So, keep in mind that real atmospheric disturbances are a collection of many individual waves of differing wavelengths.

Fourier Series – Applies to Continuous, Periodic Functions

Most continuous periodic functions (period = L) can be represented by an infinite sum of sine and cosine functions as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{L}\right)$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{L}\int_{-L/2}^{L/2} f(x)dx$$

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2\pi nx}{L}\right)dx$$

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2\pi nx}{L}\right)dx .$$

The Fourier coefficients give the amplitudes of the various sine and cosine waves needed to replicate the original function.

- The coefficient a_0 is just the average of the function.
- The coefficients a_n are the coefficients of the cosine waves (the even part of the function).
- The coefficients b_n are the coefficients of the sine waves (the odd part of the function).

For a completely even function, the b_n's would all be zero, while for a completely odd function, the a_n's would be zero.

Fourier series can also be represented using complex notation, and in this notation

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n \exp\left[\frac{i2\pi nx}{L}\right]$$

where the coefficients α_n are complex numbers, with the real part representing the amplitudes of the cosine waves, and the imaginary part representing the amplitudes of the sine waves,

$$\alpha_n = \frac{1}{2}\left(a_n - ib_n\right) = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\exp\left[\frac{-i2\pi nx}{L}\right]dx .$$

Each of the Fourier coefficients, α_n, are associated with a sinusoidal wave of a certain wavelength. If the original function contained one pure wave, then there would only be two Fourier coefficients (a_1 and b_1). The more sinusoids (more wave numbers) needed to represent the function, the more Fourier coefficients are necessary.

In general:

- Smoother functions require fewer waves to recreate, and have fewer higher frequency components.
- Sharper functions require more waves to recreate, and have more higher frequency components.
- Broad functions require fewer waves to recreate, and have fewer higher frequency components.
- Narrow functions require more waves to recreate, and have more higher frequency components.

Fourier Transforms – Applies to Continuous, Aperiodic Functions

Fourier analysis can be extended to functions that are continuous, but not periodic (aperiodic functions). This is done by representing the function as an infinite integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\exp\left[ikx\right]dk \qquad \textbf{(1)}$$

where the Fourier coefficients are represented by F(k), which is a complex number given by

$$F(k) = \int_{-\infty}^{\infty} f(x)\exp\left[-ikx\right]dx . \qquad \textbf{(2)}$$

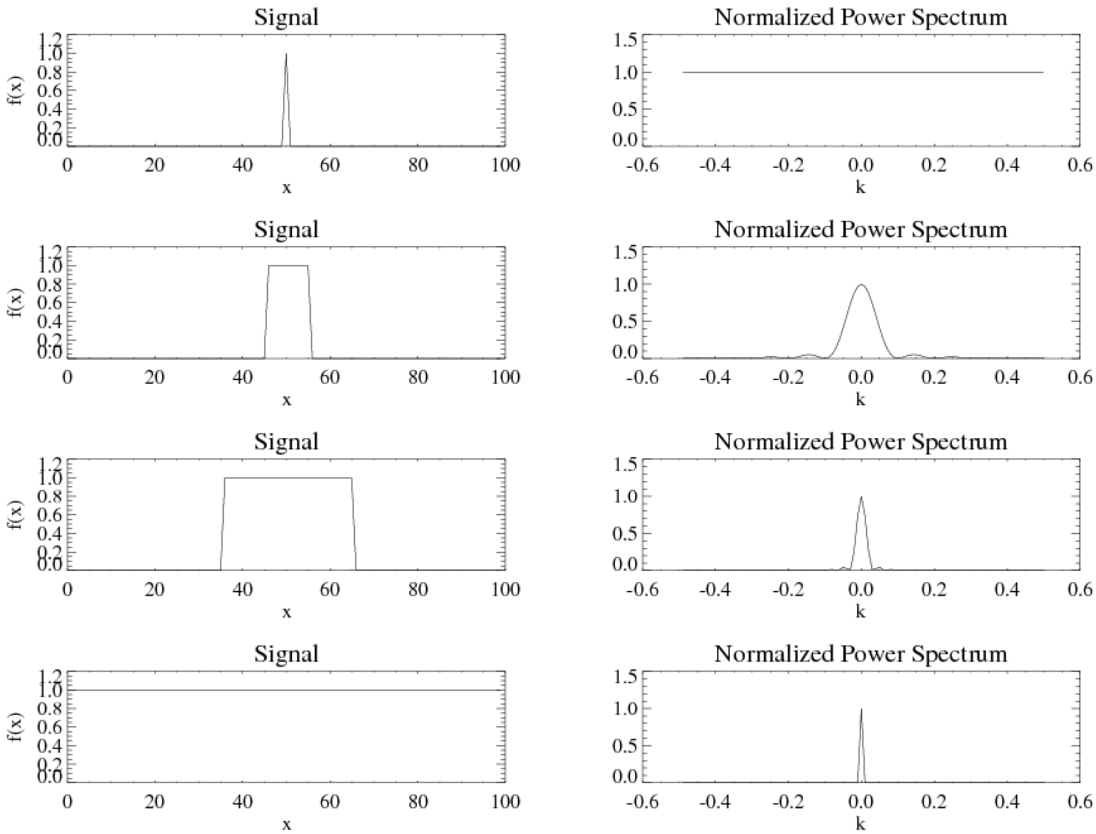
Equations (1) and (1) are called the Fourier transform pairs. Equation (1) is the representation of the function in “physical” space. Equation (2) is the representation of the function in “frequency” or “wave number” space. As with Fourier series, the real part of the Fourier coefficient, Re[$F(k)$], represents the cosine, or even part of the function, while the imaginary part, Im[$F(k)$], represents the sine, or odd part of the function.

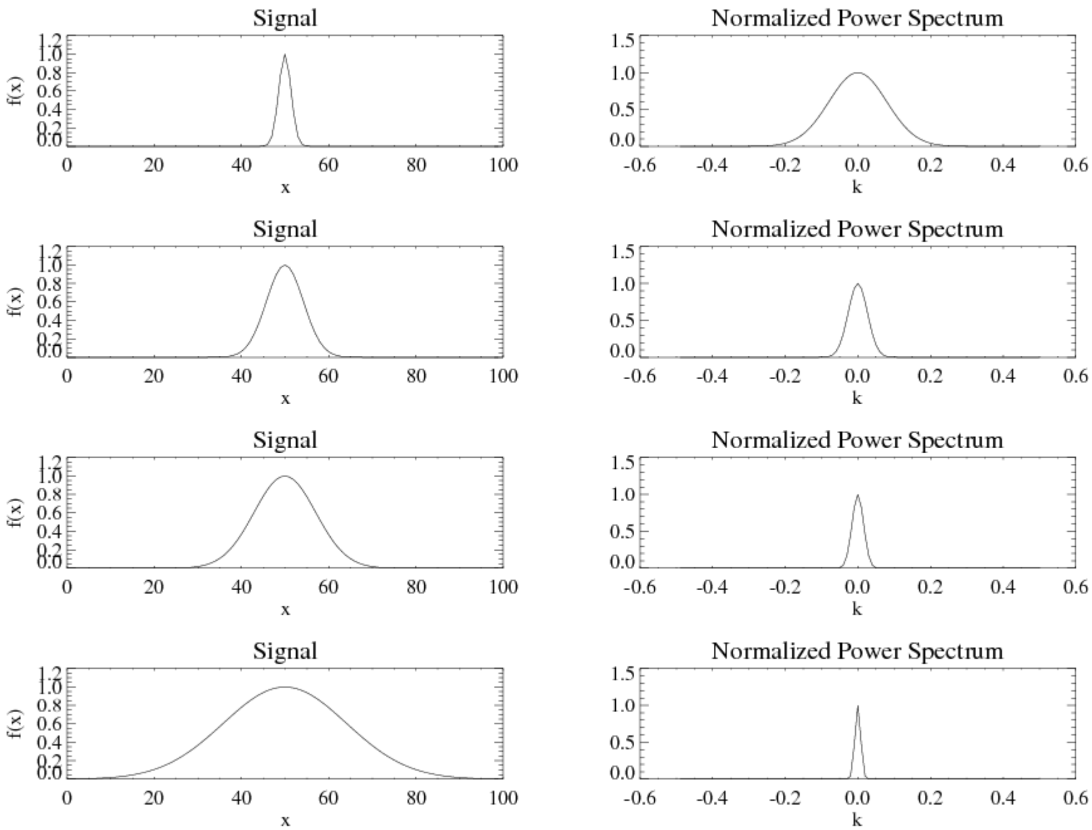
FOURIER SPECTRA OF SOME EXAMPLE FUNCTIONS

As mentioned previously, sharp, narrow functions have more and higher frequency waves in their Fourier spectra then do smooth, broad functions. The figures below shows some example functions and their associated Fourier spectra. The first four figures show box functions of various width, while the second four pictures show Gaussian curves of various width. Things to note:

- In general, the narrower the function, the broader the spectrum, and vice versa.
- The power series of a Gaussian curve is also a Gaussian curve.

- An impulse function has an infinitely broad power spectrum, while an infinitely broad function has a single spike for its power spectrum.




Signal
Normalized Power Spectrum
f(x)
x
k
Signal
Normalized Power Spectrum
f(x)
x
k
Signal
Normalized Power Spectrum
f(x)
x
k
Signal
Normalized Power Spectrum
f(x)
x
k

EXERCISES

1. Show the following to be true:

$$\cos(kx-\omega t)=\mathrm{Re}\left[e^{i(kx-\omega t)}\right]$$
$$-\cos(kx-\omega t)=\mathrm{Re}\left[-e^{i(kx-\omega t)}\right]$$
$$\sin(kx-\omega t)=\mathrm{Re}\left[-ie^{i(kx-\omega t)}\right]$$
$$-\sin(kx-\omega t)=\mathrm{Re}\left[ie^{i(kx-\omega t)}\right]$$

2. Show the following to be true:

$$\frac{\partial}{\partial x}e^{i(kx-\omega t)}=ike^{i(kx-\omega t)}$$
$$\frac{\partial^2}{\partial x^2}e^{i(kx-\omega t)}=-k^2e^{i(kx-\omega t)}$$
$$\frac{\partial}{\partial t}e^{i(kx-\omega t)}=-i\omega\, e^{i(kx-\omega t)}$$
$$\frac{\partial^2}{\partial t^2}e^{i(kx-\omega t)}=-\omega^2\, e^{i(kx-\omega t)}$$

3. A wave is represented in complex notation as

$$u(x,t)=Ae^{i(kx-\omega t)}$$

where $A=2-3i$. Show that this is equivalent to representing the wave as

$$u(x,t)=2\cos(kx-\omega t)+3\sin(kx-\omega t).$$

4. Find the phase difference between the following two waves,

$$u(x,t)=Ae^{i(kx-\omega t)}$$
$$v(x,t)=Be^{i(kx-\omega t)}$$

for the following values of A and B.

a. $A=2+3i;\quad B=-3+2i$

b. $A=2+3i;\quad B=-2-3i$

c. $A=2+3i;\quad B=3-2i$

d. $A=2+3i;\quad B=4+6i$

e. $A=2+3i;\quad B=9-6i$

5. a. Let a wave be represented by

$$u(x) = e^{ikx}.$$

Show that u and du/dx are 270° out of phase.

b. Let a wave be represented by

$$u(x) = \cos kx.$$

Show that u and du/dx are 270° out of phase, which shows the consistency of representing sinusoids using complex notation.

6. A wave traveling in two dimensions is represented as

$$u(x, y, t) = Ae^{i(kx+ly-\omega t)}.$$

Show that

$$\nabla^2 u = -\left(k^2 + l^2\right)u,$$

demonstrating the Laplacian of a sinusoidal function is proportional to the negative of the original function.

7. What is the physical meaning of a complex frequency? In other words, if ω has an imaginary part, what does this imply? Hint: Put $\omega = \omega_r + i\omega_i$ into

$$u = e^{i(kx-\omega t)}$$

and see what you get.

8. Start with the definition of group velocity, $\vec{c}_g = \frac{\partial \omega}{\partial k}\hat{i} + \frac{\partial \omega}{\partial l}\hat{j} + \frac{\partial \omega}{\partial m}\hat{k}$, and show that the magnitude of the group velocity (the group speed) is given by $c_g = \frac{\partial \omega}{\partial K}$.

ESCI 343 – Atmospheric Dynamics II
Lesson 5 – Linear Waves

Reference: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
Waves in Fluids, J. Lighthill
Atmosphere-Ocean Dynamics, A.E. Gill

Reading: Holton, Section 7.2

LINEAR VERSUS NONLINEAR WAVES

A linear set of equations does not contain products of the dependent variables. A non-linear set of equation contains products of dependent variables. The momentum equations are definitely nonlinear, since the advective terms look like

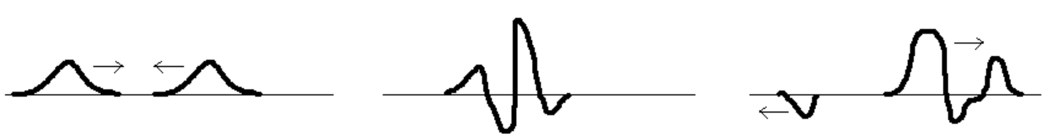
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$$

which has products of the dependent variables u, v, and w.

A linear set of equations support linear waves, while a nonlinear set of equations supports nonlinear waves. ***Linear waves and nonlinear waves behave very differently!*** The primary difference is that ***linear waves do not interact with one another, and can't exchange energy!*** Two linear waves will pass right through one another. Any interference between the two waves is strictly linear, meaning at a given point, the effect of the wave is just the sum of the effects of the two waves. ***Nonlinear waves interact and may exchange energy!*** The diagram below illustrates this.

Nonlinear waves are much more complex, and more difficult to study, than are linear waves. Unfortunately, the governing equations are highly non-linear (due to the advective terms), and therefore, atmospheric waves are nonlinear.

Linear waves pass right through each other without tranferring energy.
The effect of the waves is just their sum.



Nonlinear waves interact with one another, and may exchange energy.
The effect of the waves is not just their sum.

THE PERTURBATION METHOD

The governing equations are nonlinear. In order to study the properties of atmospheric waves we "linearize" the governing equations, and then study the linear waves supported by these equations. By studying these linear waves we hope to learn some information about the waves and their relevance.

In order to linearize the equations we use the *perturbation method*. We start by dividing all the dependent variables into two parts. The first part is known as the *basic state*, and is assumed to be either constant, or only a function of the spatial coordinates.

The second part is the *perturbation*, and is allowed to vary with time, and in all three space directions. For example,

$$
\begin{aligned}
u(x,y,z,t) &= \overline{u} + u'(x,y,z,t) \\
v(x,y,z,t) &= \overline{v} + v'(x,y,z,t) \\
w(x,y,z,t) &= w'(x,y,z,t); \quad assume\ \overline{w} = 0 \\
p(x,y,z,t) &= \overline{p}(x,y,z) + p'(x,y,z,t) \\
\rho(x,y,z,t) &= \overline{\rho}(x,y,z) + \rho'(x,y,z,t)
\end{aligned}
$$

Another assumption is that the basic state must satisfy the equations of motion when the perturbations are zero.

A third critical assumption for the perturbation method is that the perturbations much be small so that products of perturbations can be neglected. (Do not confuse this procedure with Reynolds averaging. Although the two procedures may look similar, they are really very different.)

We then take the divided dependent variables, substitute them into the equations, and multiply everything out. Since we can ignore terms that are the products of two perturbations, any such term can be crossed out.

THE PERTURBATION METHOD APPLIED TO THE *U*-MOMENTUM EQUATION

Applying the perturbation method to the *u*-momentum equation is illustrated below.

$$\frac{\partial(\overline{u}+u')}{\partial t} + (\overline{u}+u')\frac{\partial(\overline{u}+u')}{\partial x} + (\overline{v}+v')\frac{\partial(\overline{u}+u')}{\partial y} + (\overline{w}+w')\frac{\partial(\overline{u}+u')}{\partial z} = -\frac{1}{(\overline{\rho}+\rho')}\frac{\partial(\overline{p}+p')}{\partial x} + f(\overline{v}+v')$$

We can simplify this equation by recognizing that the basic state variables are independent of time, and that only the pressure and density basic state variables are functions of z. Also, we assumed that the base-state vertical velocity is zero. The equation then becomes

$$\frac{\partial u'}{\partial t} + (\overline{u}+u')\frac{\partial u'}{\partial x} + (\overline{v}+v')\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z} = -\frac{1}{(\overline{\rho}+\rho')}\frac{\partial(\overline{p}+p')}{\partial x} + f(\overline{v}+v')$$

Since the perturbation quantities are very small, we assume that we can ignore products of perturbation quantities. This further simplifies the equation to

$$\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \overline{v}\frac{\partial u'}{\partial y} = -\frac{1}{(\overline{\rho}+\rho')}\frac{\partial(\overline{p}+p')}{\partial x} + f(\overline{v}+v')$$

We also assume that we can ignore perturbations of density in the horizontal pressure gradient term (similar to the Boussinesq approximation), to get

$$\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \overline{v}\frac{\partial u'}{\partial y} = -\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x} - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x} + f\overline{v} + fv'$$

And finally, if we assume that the basic state is in geostrophic balance, then

$$-\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x}+f\bar{v}=0,$$

so that we are left with

$$\frac{\partial u'}{\partial t}+\bar{u}\frac{\partial u'}{\partial x}+\bar{v}\frac{\partial u'}{\partial y}=-\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}+fv'.$$

This is the linearized, or perturbation form of, the *u*-momentum equation. Linearization of the *v*-momentum equation proceeds in a similar manner.

LINEARIZING THE *W*-MOMENTUM EQUATION

The *w*-momentum equation is a bit trickier, because we can't ignore the density perturbation in the vertical pressure gradient term like we could in the horizontal pressure gradient term of the *u*-momentum equation. So, after substituting the basic state and perturbation variables into the *w*-momentum equation we get

$$\frac{\partial w'}{\partial t}+\bar{u}\frac{\partial w'}{\partial x}+\bar{v}\frac{\partial w'}{\partial y}=-\frac{1}{(\bar{\rho}+\rho')}\frac{\partial(\bar{p}+p')}{\partial z}-g.$$

A rule of algebra tells us that if $a << 1$, then

$$\frac{1}{1+a}\cong 1-a$$

Using this rule we can write

$$\frac{1}{\bar{\rho}+\rho'}=\frac{1}{\bar{\rho}(1+\rho'/\bar{\rho})}\cong\frac{1}{\bar{\rho}}\left(1-\frac{\rho'}{\bar{\rho}}\right).$$

Using this, the RHS of the *w*-momentum equation becomes

$$-\frac{1}{\bar{\rho}+\rho'}\frac{\partial}{\partial z}(\bar{p}+p')-g=\frac{1}{\bar{\rho}}\left(\frac{\rho'}{\bar{\rho}}-1\right)\frac{\partial}{\partial z}(\bar{p}+p')-g=\frac{1}{\bar{\rho}}\left(\frac{\rho'}{\bar{\rho}}\frac{\partial\bar{p}}{\partial z}+\frac{\rho'}{\bar{\rho}}\frac{\partial p'}{\partial z}-\frac{\partial\bar{p}}{\partial z}-\frac{\partial p'}{\partial z}\right)-g$$

and since we can ignore products of perturbation terms, this simplifies to

$$\frac{1}{\bar{\rho}}\left(\frac{\rho'}{\bar{\rho}}\frac{\partial\bar{p}}{\partial z}-\frac{\partial\bar{p}}{\partial z}-\frac{\partial p'}{\partial z}\right)-g.$$

If the basic state is in hydrostatic balance, then

$$\frac{\partial \overline{p}}{\partial z} = -\overline{\rho}\, g \; .$$

Substituting this into the equation above it gives

$$\frac{1}{\overline{\rho}}\left(\frac{\rho'}{\overline{\rho}}\left(-\overline{\rho}g\right) - \left(-\overline{\rho}g\right) - \frac{\partial p'}{\partial z}\right) - g = -\frac{\rho'}{\overline{\rho}}\, g - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z}$$

so that the linearized w-momentum equation is

$$\frac{\partial w'}{\partial t} + \overline{u}\frac{\partial w'}{\partial x} + \overline{v}\frac{\partial w'}{\partial y} = -\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} - \frac{\rho'}{\overline{\rho}}\, g \; .$$

Note that what we've done is to use the basic state density everywhere except in the buoyancy term (the term involving g), where we used the perturbation density. This is essentially the Boussinesq approximation, the difference being that the reference density is allowed to vary spatially, whereas in the Boussinesq approximation the reference density is assumed to be a true constant.

THE FINAL FORM OF THE PERTURBATION EQUATIONS

If we assume that the basic state is in geostrophic and hydrostatic balance, and that the base-state density is a function of z only, the linearized momentum and continuity equations are

$$\begin{aligned}
&\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \overline{v}\frac{\partial u'}{\partial y} = -\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x} + fv' \\
&\frac{\partial v'}{\partial t} + \overline{u}\frac{\partial v'}{\partial x} + \overline{v}\frac{\partial v'}{\partial y} = -\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial y} - fu' \\
&\frac{\partial w'}{\partial t} + \overline{u}\frac{\partial w'}{\partial x} + \overline{v}\frac{\partial w'}{\partial y} = -\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} - \frac{\rho'}{\overline{\rho}}\, g \\
&\frac{\partial \rho'}{\partial t} + \overline{u}\frac{\partial \rho'}{\partial x} + \overline{v}\frac{\partial \rho'}{\partial y} + w'\frac{d\overline{\rho}}{dz} = -\overline{\rho}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)
\end{aligned}$$

THE GENERAL METHOD FOR FINDING THE DISPERSION RELATION

There is a general method for finding the dispersion relation for the waves supported by a linearized set of equations. The method is best illustrated by example. We will use the linearized, one-dimensional shallow-water equations given by

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x}$$
$$\frac{\partial h'}{\partial t} = -H\frac{\partial u'}{\partial x}$$

as our example (here, g is the acceleration due to gravity, and the depth of the fluid, h is given by

$$h = H + h'.$$

These equations support shallow-water gravity waves.

1. Step 1 is to assume that all dependent variables have a sinusoidal form
$$u' = Ae^{i(kx-\omega t)}$$
$$h' = Be^{i(kx-\omega t)}$$
2. Step 2 is to plug the assumed form of the dependent variables into the linearized governing equations. In our case this yields two algebraic equations in A and B.
$$\omega A - kgB = 0$$
$$kHA - \omega B = 0$$
3. Step 3 is to write these equations in matrix form
$$\begin{pmatrix} \omega & -kg \\ kH & -\omega \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
In order for these two equations to be linearly independent (i.e., not have a trivial solution of $A = B = 0$), the determinant of the coefficient matrix must equal zero
$$\begin{vmatrix} \omega & -kg \\ kH & -\omega \end{vmatrix} = 0$$
4. Step 4 is to take the determinant of the coefficient matrix and solve for ω. In our case this becomes
$$-\omega^2 + k^2 gH = 0$$
$$\omega = \pm k\sqrt{gH}$$
with a phase speed of
$$c = \frac{\omega}{k} = \pm\sqrt{gH}$$
and a group velocity of
$$c_g = \frac{\partial \omega}{\partial k} = \pm\sqrt{gH}$$
(Note that since the phase speed does not depend on k then these waves are nondispersive , also evident because the phase speed and group velocity are identical). This is the standard method of determining the dispersion relation for a set of equations, and will be applied to more complex equations.

EXERCISES

1. Assume that all base-state variables are constant except for density, which is a function of height only. Also assume the base-state vertical velocity is zero. Show that the linearized continuity equation is

$$\frac{\partial \rho'}{\partial t} + \bar{u}\frac{\partial \rho'}{\partial x} + \bar{v}\frac{\partial \rho'}{\partial y} + w'\frac{d\bar{\rho}}{dz} = -\bar{\rho}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right).$$

2. Show that if a fluid is incompressible then the linearized continuity equation is simply

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

3. a. Find the dispersion relation for waves supported by the shallow-equations with a mean flow,

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} = -g\frac{\partial h'}{\partial x}$$
$$\frac{\partial h'}{\partial t} + \bar{u}\frac{\partial h'}{\partial x} = -H\frac{\partial u'}{\partial x}$$

and show that it is

$$\omega = k\bar{u} \pm k\sqrt{gH}$$

b. What are the phase speed and group velocity of these waves?

c. Are these waves dispersive?

d. How does the phase speed and group velocity of these waves compare to shallow-water gravity waves without a mean flow?

ESCI 343 – Atmospheric Dynamics II
Lesson 6 – Shallow-water Surface Gravity Waves

References: *An Introduction to Dynamic Meteorology* (3rd edition), J.R. Holton
Numerical Prediction and Dynamic Meteorology (2nd edition), G.J. Haltiner and R.T. Williams
Waves in Fluids, J. Lighthill

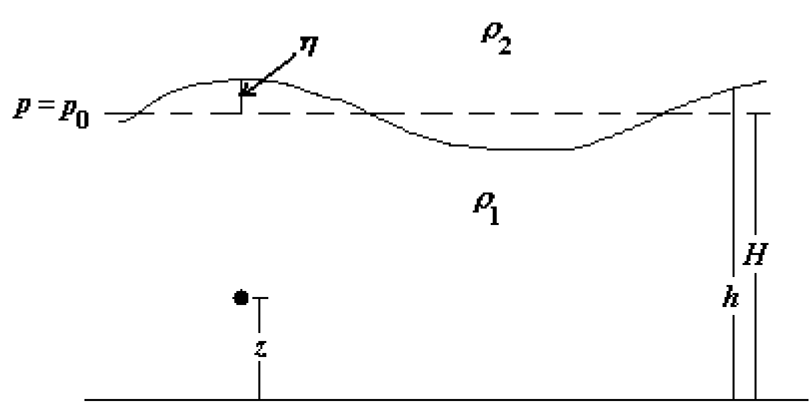
GENERAL

Surface gravity waves are waves on the surface of a liquid, the restoring force of which is gravity. These waves are familiar to all of us as the waves on the ocean or a lake. Though pure surface gravity waves do not appear in the atmosphere, their study is a useful prelude to the study of other gravity wave types in the atmosphere.

We will first limit our study to surface gravity waves on the free surface of a constant density fluid. We will also limit our study to waves in hydrostatic balance. This assumption implies that we are studying waves whose wavelength is much larger than the depth of the fluid (remember the condition from scale analysis for assuming hydrostatic balance is that the horizontal length scale be much larger than the vertical length scale). Thus we are limited to either very-long wavelengths, or very shallow water. This is alternately known as either the *shallow-water* approximation or the *long-wave* approximation.

THE SHALLOW-WATER MOMENTUM EQUATIONS

The diagram below shows the interface between two fluids of different, constant densities. The dashed line shows the position of the interface if the fluids are undisturbed. The solid line shows the interface displaced. The depth of the lower fluid is H.



If we assume that the upper fluid is very deep compared to the displacement of the interface, η (note that $H + \eta = h$), then we can assume that the pressure at the level of the undisturbed interface (the dashed line in the figure) remains constant at a value of p_0[1]. If

[1] We have to assume that the upper fluid is deep in order to assume that p_0 is constant. This is because a displacement of the interface upward results in either divergence or convergence in the upper fluid as it adjusts to the change in interface height. This means that there would be horizontal flow in the upper fluid, which would require a horizontal pressure gradient in the upper fluid. By constraining our

the lower fluid is in hydrostatic balance, then the pressure at any point in the lower fluid is proportional to the weight of the fluid above it. Therefore, at the point shown in the diagram the pressure will be

$$p = p_0 + \rho_1 g\left(H - z\right) + \rho_1 g\eta - \rho_2 g\eta \tag{1}$$

and the horizontal pressure gradient force will be

$$-\frac{1}{\rho_1}\frac{\partial p}{\partial x} = -\left(\frac{\rho_1 - \rho_2}{\rho_1}\right) g\frac{\partial \eta}{\partial x}\,. \tag{2}$$

Since

$$h = H + \eta \tag{3}$$

then

$$\frac{\partial h}{\partial x} = \frac{\partial \eta}{\partial x} \tag{4}$$

so

$$-\frac{1}{\rho_1}\frac{\partial p}{\partial x} = -\left(\frac{\rho_1 - \rho_2}{\rho_1}\right) g\frac{\partial h}{\partial x}\,.\text{[2]} \tag{5}$$

The quantity

$$g' = \left(\frac{\rho_1 - \rho_2}{\rho_1}\right) g \tag{6}$$

is called *reduced gravity*, and the momentum equations for the lower fluid are written as

$$\begin{aligned} \frac{Du}{Dt} &= -g'\frac{\partial h}{\partial x} + fv \\ \frac{Dv}{Dt} &= -g'\frac{\partial h}{\partial y} - fu\,. \end{aligned} \tag{7}$$

If the two fluids are greatly different in densities (such as air and water), then $\rho_1 - \rho_2 \cong \rho_1$, and $g' \cong g$ (note that the prime on g does not refer to a perturbation).

Since we've assumed that the lower fluid is in hydrostatic balance, we've constrained our analysis to motions whose horizontal scale is much greater than the vertical scale (the depth of the fluid). For this reason, we refer to equation set (7) as the *shallow-water* momentum equations.

Note that the pressure gradient force at any point in the lower fluid is independent of depth! This means that the fluid motion is also independent of depth. Therefore, the lower fluid is *barotropic*.

---

discussion to a very deep upper fluid, there is minimal convergence or divergence in the upper fluid (since the amount of mass replaced is minimal compared to the overall mass in the fluid column), and therefore, minimal horizontal flow in the upper fluid.

2 For convenience we derived this expression as though H were constant in x and y. However, if H is allowed to vary in x and y we would obtain the same result. The derivation would just be a little more cumbersome.

THE SHALLOW-WATER CONTINUITY EQUATION

The continuity equation in the lower fluid is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{8}$$

If we integrate the continuity equation from the bottom of the fluid to the interface we get

$$\int_0^h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz+\int_0^h\frac{\partial w}{\partial z}dz=0 \tag{9}$$

which becomes

$$w(h)-w(0)=-h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right). \tag{10}$$

The vertical velocity at the bottom of the fluid is zero. Also,

$$w(h)=\left.\frac{Dz}{Dt}\right|_h=\frac{Dh}{Dt} \tag{11}$$

so the shallow-water continuity equation is

$$\frac{Dh}{Dt}=-h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right). \tag{12}$$

Note: Equation (12) was derived for a flat bottom, but it actually applies to a fluid with a non-flat bottom. In this case, *h* is the total depth of the fluid from bottom to top.

LINEARIZED SHALLOW-WATER EQUATIONS

To linearize the shallow-water equations we use the following perturbation forms for the dependent variables

$$\begin{aligned} u &= \bar{u}+u' \\ v &= \bar{v}+v' \\ h &= H(x,y)+\eta-z_0. \end{aligned} \tag{13}$$

Note that H is the mean undisturbed height of the fluid surface, which can depend on x, and y. Thus, the undisturbed surface of the fluid can be sloped to support a geostrophically-balanced mean flow. The term z_0 is the elevation of the bottom topography, and for a flat bottom would be zero.

We will assume a flat bottom ($z_0=0$), and also assume that the base state is in geostrophic balance so that

$$\begin{aligned} \bar{u} &= -\frac{g}{f}\frac{\partial H}{\partial y} \\ \bar{v} &= \frac{g}{f}\frac{\partial H}{\partial x}, \end{aligned} \tag{14}$$

Putting (13) into equations (7) and (12), ignoring products of perturbations, and using (14), we get

$$
\begin{aligned}
&\frac{\partial u'}{\partial t}+\bar{u}\,\frac{\partial u'}{\partial x}+\bar{v}\,\frac{\partial u'}{dy}=-g'\frac{\partial \eta}{\partial x}+fv' \\
&\frac{\partial v'}{\partial t}+\bar{u}\,\frac{\partial v'}{\partial x}+\bar{v}\,\frac{\partial v'}{dy}=-g'\frac{\partial \eta}{\partial y}-fu' \\
&\frac{\partial \eta}{\partial t}+\bar{u}\left(\frac{\partial H}{\partial x}+\frac{\partial \eta}{\partial x}\right)+\bar{v}\left(\frac{\partial H}{\partial y}+\frac{\partial \eta}{dy}\right)+u'\frac{\partial H}{\partial x}+v'\frac{\partial H}{\partial y}=-H\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right).
\end{aligned}
\tag{15}
$$

Equations (15) are the linearized shallow-water equations.

Note: Usually we work with cases where $\frac{\partial H}{\partial x} << \frac{\partial \eta}{\partial x}$ and $\frac{\partial H}{\partial y} << \frac{\partial \eta}{\partial y}$. This allows the $\frac{\partial H}{\partial x}$ and $\frac{\partial H}{\partial y}$ terms to be omitted in the linearized continuity equation, which then simply becomes

$$
\frac{\partial \eta}{\partial t}+\bar{u}\,\frac{\partial \eta}{\partial x}+\bar{v}\,\frac{\partial \eta}{dy}=-H\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right). \tag{16}
$$

DISPERSION RELATION FOR SHALLOW-WATER GRAVITY WAVES

To find the dispersion relation for shallow-water gravity waves in the absence of a mean flow, with constant *H,* and ignoring Coriolis, we start with

$$
\begin{aligned}
&\frac{\partial u'}{\partial t}=-g'\frac{\partial \eta}{\partial x} \\
&\frac{\partial v'}{\partial t}=-g'\frac{\partial \eta}{\partial y} \\
&\frac{\partial \eta}{\partial t}=-H\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right).
\end{aligned}
\tag{17}
$$

Assuming a sinusoidal disturbance such as

$$
\begin{aligned}
u' &= Ae^{i(kx+ly-\omega t)} \\
v' &= Be^{i(kx+ly-\omega t)} \\
\eta &= Ce^{i(kx+ly-\omega t)}
\end{aligned}
\tag{18}
$$

and substituting into (17), we get the following matrix equation

$$
\begin{pmatrix} \omega & 0 & -g'k \\ 0 & \omega & -gl \\ kH & lH & -\omega \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{19}
$$

In order that *A*, *B*, and *C* not be zero, then

$$\begin{vmatrix} \omega & 0 & -g'k \\ 0 & \omega & -g'l \\ kH & lH & -\omega \end{vmatrix} = 0\,, \tag{20}$$

and solving this for ω gives

$$\omega = \sqrt{\left(k^2 + l^2\right) g'H}\ . \tag{21}$$

The total wave number in the direction of propagation is given by

$$K^2 = k^2 + l^2 \tag{22}$$

so we get the following dispersion relation and phase speed,

$$\begin{aligned} \omega &= K\sqrt{g'H} \\ c &= \frac{\omega}{K} = \sqrt{g'H}\ . \end{aligned} \tag{23}$$

These waves are nondispersive since the wavenumber does not appear on the right-hand side of (23).

EXERCISES

1. Find the dispersion relation for one-dimensional (x-direction) shallow-water gravity waves with a non-zero mean flow in the zonal direction (i.e., $\bar{u} \neq 0$, $\bar{v} = 0$).

2. a. Find the dispersion relation for two-dimensional (x and y-directions) shallow-water gravity waves with a zero mean flow, but including the Coriolis parameter (these are known as *shallow-water, inertial-gravity waves*).

b. Find the group velocity and phase speed of these waves. Are they dispersive?

3. The general dispersion relation for one-dimensional surface gravity waves (not restricted to shallow water) traveling in the x-direction is

$$\omega = \bar{u}k \pm \sqrt{gk \tanh kH} \, .$$

a. What is the phase speed for these waves?

b. Are these waves dispersive?

4. a. For very short waves, or for very deep water, ($kH >> 1$). Show that in this case the dispersion relation for surface gravity waves is

$$\omega = \bar{u}k \pm \sqrt{gk} \, .$$

(This is known as the *short-wave approximation*, or *deep-water approximation*). Note that for $x >> 1$, $\tanh x \cong 1$.

b. What is the group velocity for these waves?

c. Are these waves dispersive?

5. a. For very long waves, or for very shallow water, ($kH << 1$). Show that in this case the dispersion relation for surface gravity waves is

$$\omega = \bar{u}k \pm k\sqrt{gH} \, .$$

(This is known as the *long-wave approximation*, or *shallow-water approximation*). Note that for $x << 1$, $\tanh x \cong x$.

b. What is the group velocity for these waves?

c. Are these waves dispersive?

6. Calculate the speed of a shallow-water surface gravity wave for a fluid having a depth equal to the scale height of the atmosphere (~8.1 km) (assume zero mean flow). How does this compare with the speed of sound?

ESCI 343 – Atmospheric Dynamics II
Lesson 7 –Gravity Waves in a Two-layer Fluid

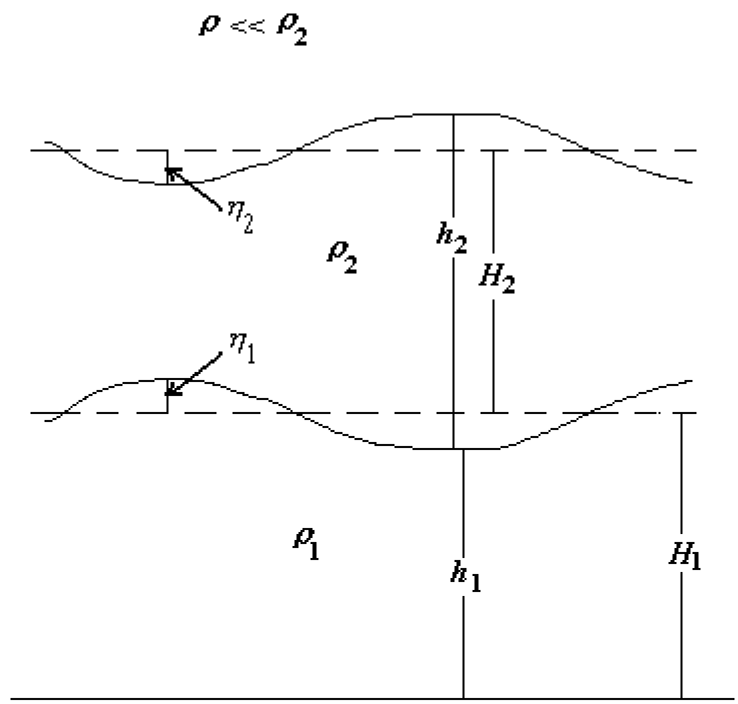
References: *Atmosphere-Ocean Dynamics*, A.E. Gill

GENERAL

In a two-layer fluid such as that shown in the diagram below, gravity waves can exist on either the interface at the top of the upper fluid, or on the interface between the two fluids. We assume that the space above the uppermost fluid is filled with a fluid of very low density and very large depth, so that at the upper interface we can use g instead of g', and that p_0 at the upper interface is zero (we will refer to this upper interface as the *free surface*). The gravity waves that form on the free surface and the interface are not independent of one another.

DERIVATION OF DISPERSION RELATION FOR TWO-LAYER FLUID

We can examine their dependence by applying the shallow-water equations to each layer of fluid.



Equations in the upper fluid. In the upper fluid the pressure gradient force is solely due to the slope of the free surface, and the shallow-water momentum equations (ignoring rotation) are

$$\begin{aligned}\frac{Du_2}{Dt} &= -g\frac{\partial \eta_2}{\partial x}\\ \frac{Dv_2}{Dt} &= -g\frac{\partial \eta_2}{\partial y}\,.\end{aligned} \tag{1}$$

The continuity equation in the upper fluid is

$$\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial w_2}{\partial z}=0 \tag{2}$$

which when integrated over the depth of the upper fluid (h_2) becomes

$$\frac{D\eta_2}{Dt}-\frac{D\eta_1}{Dt}=-h_2\left(\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}\right). \tag{3}$$

Equations in the lower fluid. In the lower fluid the pressure gradient is not only due to the slope of the interface, but also due to the slope of the free surface. This is seen by finding the pressure at a point at height z in the lower fluid, which is

$$p=p_0+\rho_2 g\eta_2+\rho_2 gH_2+\rho_1 g(H_1-z)+\rho_1 g\eta_1-\rho_2 g\eta_1 \tag{4}$$

so that

$$\frac{1}{\rho_1}\frac{\partial p}{\partial x}=\frac{\rho_2}{\rho_1}g\frac{\partial \eta_2}{\partial x}+g'\frac{\partial \eta_1}{\partial x}. \tag{5}$$

Therefore, the momentum equations in the lower fluid are

$$\begin{aligned}\frac{Du_1}{Dt}&=-\frac{\rho_2}{\rho_1}g\frac{\partial \eta_2}{\partial x}-g'\frac{\partial \eta_1}{\partial x}\\ \frac{Dv_1}{Dt}&=-\frac{\rho_2}{\rho_1}g\frac{\partial \eta_2}{\partial y}-g'\frac{\partial \eta_1}{\partial y}.\end{aligned} \tag{6}$$

The continuity equation in the lower layer is

$$\frac{D\eta_1}{Dt}=-h_1\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right). \tag{7}$$

DISPERSION RELATION FOR SHALLOW-WATER GRAVITY WAVES IN A TWO-LAYER FLUID

The linearized shallow-water equations for a two-layer fluid with zero mean flow, and for waves traveling in the x-direction only, are

$$\frac{\partial u_2'}{\partial t}=-g\frac{\partial \eta_2}{\partial x} \tag{8}$$

$$\frac{\partial \eta_2}{\partial t}-\frac{\partial \eta_1}{\partial t}=-H_2\frac{\partial u_2'}{\partial x} \tag{9}$$

$$\frac{\partial u_1'}{\partial t}=-\frac{\rho_2}{\rho_1}g\frac{\partial \eta_2}{\partial x}-g'\frac{\partial \eta_1}{\partial x} \tag{10}$$

$$\frac{\partial \eta_1}{\partial t}=-H_1\frac{\partial u_1'}{\partial x} \tag{11}$$

To find the dispersion relation we could proceed by assuming sinusoidal functions for the four dependent variables and substituting them into the governing equations to get a set of four algebraic equations. This would be very messy, since we would have to find the determinant of a 4×4 matrix. Instead, we will try to reduce the four equations and four unknowns into a system of two equations and two unknowns.

Eliminating the velocity component from equations (8) and (9) gives

$$\frac{\partial^2 \eta_1}{\partial t^2} = \frac{\partial^2 \eta_2}{\partial t^2} - gH_2 \frac{\partial^2 \eta_2}{\partial x^2}. \tag{12}$$

Eliminating the velocity component from equations (10) and (11) gives

$$\frac{\partial^2 \eta_1}{\partial t^2} - g'H_1 \frac{\partial^2 \eta_1}{\partial x^2} = \frac{\rho_2}{\rho_1} gH_1 \frac{\partial^2 \eta_2}{\partial x^2}. \tag{13}$$

Assuming sinusoidal forms for η_1 and η_2 of

$$\begin{aligned} \eta_1 &= Ae^{i(kx-\omega t)} \\ \eta_2 &= Be^{i(kx-\omega t)} \end{aligned} \tag{14}$$

into equations (12) and (13) yields the following algebraic set of equations,

$$\begin{aligned} &\omega^2 A - \left(\omega^2 - gH_2k^2\right)B = 0 \\ &\left(\omega^2 - g'H_1k^2\right)A - \left(\rho_2/\rho_1\right)gH_1k^2 B = 0 \end{aligned} \tag{15}$$

which gives a fourth-order polynomial for ω,

$$\omega^4 - gHk^2\omega^2 + g'gH_1H_2k^4 = 0 \tag{16}$$

where H is the total depth defined as

$$H \equiv H_1 + H_2. \tag{17}$$

In terms of phase speed, c, the fourth-order polynomial is

$$c^4 - gHc^2 + g'gH_1H_2 = 0 \tag{18}$$

which solved for c^2 is

$$c^2 = \frac{gH}{2} \pm \frac{1}{2}\sqrt{g^2H^2 - 4g'gH_1H_2} \tag{19}$$

or

$$c^2 = gH\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4g'H_1H_2}{gH^2}}\right). \tag{20}$$

Equation (20) has two roots for c^2. This indicates that the two-layer fluid supports two types (or modes) of wave motion, one associated with each root. The ratio of the disturbance amplitude on the free surface to that of the interior interface is found from equation (5.a) to be

$$\frac{B}{A} = \frac{\omega^2}{\omega^2 - gH_2k^2} = \frac{c^2}{c^2 - gH_2} \tag{21}$$

STRUCTURE OF THE TWO MODES

The speed of shallow-water gravity waves in a two-layer fluid is

$$c^2 = gH\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4g'H_1H_2}{gH^2}}\right). \tag{20}$$

Equation (20) has two roots for c. This indicates that the two-layer fluid supports two types (or modes) of wave motion, one associated with each root. The ratio of the disturbance amplitude on the free surface (B) to that of the interior interface (A) is

$$\frac{B}{A} = \frac{\omega^2}{\omega^2 - gH_2k^2} = \frac{c^2}{c^2 - gH_2} \tag{22}$$

The two-modes supported by the two-layer fluid are very different. We can best study them by assuming that the densities of the two fluids differ only slightly. This means that

$$\frac{g'}{g} = \frac{\rho_1 - \rho_2}{\rho_1} << 1$$

so that

$$\frac{4g'H_1H_2}{gH^2} << 1.$$

Through Taylor series expansion it can be shown for $x << 1$ that

$$\sqrt{1-x} \cong 1 - \frac{1}{2}x.$$

This allows us to write the phase speed of the wave (from 6) as

$$c^2 \cong gH\left[\frac{1}{2} \pm \frac{1}{2}\left(1 - \frac{2g'H_1H_2}{gH^2}\right)\right] \tag{23}$$

Structure of mode with positive root. For the mode with the positive root, the phase speed is

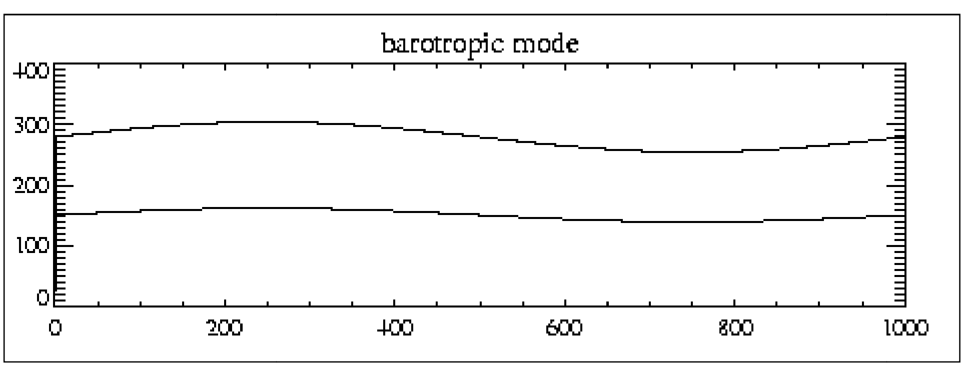
$$c^2 = gH, \tag{24}$$

which is identical to that of an external gravity wave on the surface of a homogeneous fluid of depth H. The disturbances on the two interfaces are seen to be in phase, as is shown from substituting (24) into (22) to get

$$\frac{B}{A} = \frac{H}{H_1}.$$

This also shows that the disturbance on the free surface is larger in amplitude than the disturbance on the interior interface (since the total depth of the fluid, H, is greater than the depth of the lower layer, H_1).

For this mode, the u-components of velocity are also in phase and are nearly equal, leading to the general name for this mode as the *barotropic mode*. The structure of this mode is shown below.



Structure of mode with negative root. For the mode with the negative root, the phase speed is

$$c^2 = g' \frac{H_1 H_2}{H}, \tag{25}$$

The ratio of the disturbance amplitudes on the two interfaces are found by substituting (25) into (22) to get

$$\frac{B}{A} = \frac{g' H_1 H_2}{g' H_1 H_2 - g H H_2},$$

which can be written as

$$\frac{B}{A} = \frac{\dfrac{g' H_1 H_2}{g H H_2}}{\dfrac{g' H_1 H_2}{g H H_2} - 1} = \frac{\dfrac{g' H_1}{g H}}{\dfrac{g' H_1}{g H} - 1};$$

since

$$\frac{g' H_1}{g H} << 1,$$

and for small x,

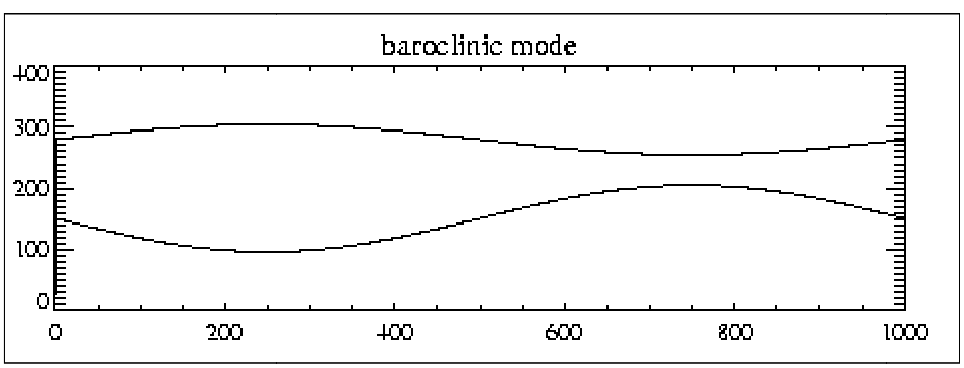
$$\frac{x}{x-1} \cong -x,$$

we have that for the negative root,

$$\frac{B}{A} \cong -\frac{g'}{g} \frac{H_1}{H}. \tag{26}$$

This also shows that the disturbance on the free surface is much smaller in amplitude than the disturbance on the interior interface, and is of opposite sign, so the two interfaces are 180° out of phase.

For this mode, the u-components of velocity are also 180° out of phase, leading to the general name for this mode as the *baroclinic mode*. The structure of this mode is shown below.



MULTIPLE LAYERED FLUIDS

We've that for a two-layer, shallow-water fluid there are two distinct modes, a barotropic and a baroclinic mode. In a three layer fluid it turns out there are three modes: a barotropic mode and two baroclinic modes. In general, an n-layer fluid will have n modes: a barotropic mode, and $n-1$ baroclinic modes. A fluid with continuous stratification can therefore be expected to have an infinite number of baroclinic modes.

EQUIVALENT DEPTH

One concept from the discussion of two-layer fluids will be important to us later when discussing internal waves in either a multi-layered, or continuously stratified fluid. That is the concept of *equivalent depth*. From the phase speed for the baroclinic mode in a two-layer fluid,

$$c^2 = g'\frac{H_1 H_2}{H},$$

we see that the speed of this mode is equal to that of an external gravity wave in a uniform fluid having a depth equal to

$$c^2 = gH_e \tag{27}$$

where

$$H_e \equiv \frac{g'}{g}\frac{H_1 H_2}{H}. \tag{28}$$

H_e is called the *equivalent depth*, and is the depth that the fluid would have to have in order for an external gravity wave to have the same speed as the baroclinic mode. The equivalent depth is important, because if we can calculate the equivalent depth for the fluid we know that the phase speed of the baroclinic mode is given by (27).

For a fluid with multiple layers, there are multiple baroclinic modes. However, each mode will have an equivalent depth, and once we know that, we know the dispersion relation! In a fluid with continuous stratification there would be an infinite number of baroclinic modes; however, not all of the modes will be important (some will be very weak), and if we can identify the most important baroclinic modes (maybe as few as two or three) and can find the equivalent depths for these modes, then we know the dispersion relations for these modes! This type of analysis is often done in both the atmosphere and the ocean, and is known as *normal mode* analysis.

WHY WE STUDY SHALLOW-WATER GRAVITY WAVE THEORY

As meteorologists, why do we bother studying shallow-water gravity waves? After all, the atmosphere is neither shallow, nor water! The answer is, ***"because of the concept of equivalent depth."***

The atmosphere is a multi-layer fluid, and supports many baroclinic modes of oscillation. For the scales of motion studied in synoptic-scale dynamics the hydrostatic assumption is valid. Recall that shallow-water wave theory also assumes hydrostatic balance. If we can find the equivalent depths for each of the baroclinic modes of oscillation, then we can use the shallow-water gravity wave dispersion relation,

$$\begin{aligned} \omega^2 &= gH_e k^2 \\ c^2 &= gH_e \end{aligned} \tag{29}$$

to study the linear gravity waves that are supported by the atmosphere.

ESCI 343 – Atmospheric Dynamics II
Lesson 8 – Sound Waves

References: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
Waves in Fluids, J. Lighthill

SOUND WAVES

We will limit our analysis to sound waves traveling only along the *x*-axis, but keep in mind that we could easily extend this to waves traveling in an arbitrary direction. We start with the linearized equations of motion for the case of zero mean flow, and for which the reference density is constant with height. These are

$$\begin{aligned}\frac{\partial u'}{\partial t} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}\\ \frac{\partial \rho'}{\partial t} &= -\bar{\rho}\frac{\partial u'}{\partial x}\end{aligned} \qquad \textbf{(1)}$$

We also have an equation of state that relates any three of the thermodynamic variables. We will use ρ, p, and θ as our thermodynamic variables, so our equation of state can be written as

$$p = p(\rho, \theta).$$

The equation of state can be written in differential form as

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_\theta d\rho + \left(\frac{\partial p}{\partial \theta}\right)_\rho d\theta. \qquad \textbf{(2)}$$

Sound waves are adiabatic, so that θ is constant. Therefore, we can write

$$\frac{dp}{dt} = \left(\frac{\partial p}{\partial \rho}\right)_\theta \frac{d\rho}{dt}$$

or in linearized form

$$\frac{\partial p'}{\partial t} = \left(\frac{\partial p}{\partial \rho}\right)_\theta \frac{\partial \rho'}{\partial t}. \qquad \textbf{(3)}$$

If we substitute this into the continuity equation, the linearized set of equations for one-dimensional sound waves are then

$$\begin{aligned}\frac{\partial u'}{\partial t} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}\\ \frac{\partial p'}{\partial t} &= -\bar{\rho}\left(\frac{\partial p}{\partial \rho}\right)_\theta \frac{\partial u'}{\partial x}\end{aligned} \qquad \textbf{(4)}$$

.

To find the dispersion relation for sound waves we assume sinusoidal solutions of

$$u' = Ae^{i(kx-\omega t)}$$
$$p' = Be^{i(kx-\omega t)} \tag{5}$$

and substitute them into the two prior equations to find that these are nondispersive waves travelling at a phase speed of

$$c = \pm\sqrt{\left(\frac{\partial p}{\partial \rho}\right)_\theta} . \tag{6}$$

Since we know that these are sound waves, we have shown that for a general fluid the speed of sound is given by

$$c_s^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_\theta . \tag{7}$$

This shows us that the speed of sound is fundamental property of a fluid. It also shows us that sound waves are non-dispersive.[1]

THE CONTINUITY EQUATION WRITTEN WITH THE SPEED OF SOUND

Using Eqs. (2) and (7) we can write

$$\frac{D\rho}{Dt} = \frac{1}{c_s^2}\frac{Dp}{Dt} \tag{8}$$

which allows us to express the fully compressible continuity equation in terms of the material derivative of pressure,

$$\frac{1}{c_s^2}\frac{Dp}{Dt} = \rho\nabla \bullet \vec{V} . \tag{9}$$

This is a common way of writing the continuity equation.

THE SPEED OF SOUND IN AN IDEAL GAS

In an ideal gas, the equation of state has the form

$$p = \rho R'T . \tag{10}$$

In terms of potential temperature, this can be written as

$$p = \rho R'\theta\left(\frac{p}{p_0}\right)^{R'/c_p} . \tag{11}$$

[1] Actually, we've only shown that linear sound waves are nondispersive. We haven't, and won't discuss any of the effects of non-linear sound waves.

Taking the partial derivative of this with respect to density at constant potential temperature, and making use of the fact that for an ideal gas $c_p = c_v + R'$, after some patience you can find that for an ideal gas, the speed of sound is given by

$$c_s = \sqrt{\gamma R' T} \tag{12}$$

$$\gamma = \frac{c_p}{c_v}. \tag{13}$$

SOUND WAVES WITH A NON-ZERO MEAN FLOW

So far we've ignored the mean flow. If there is a basic state mean flow then the analysis is slightly more complex. Our linearized equations of motion become

$$\begin{aligned} &\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \\ &\frac{1}{c_s^2}\left(\frac{\partial p'}{\partial t} + \bar{u}\frac{\partial p'}{\partial x}\right) = -\bar{\rho}\frac{\partial u'}{\partial x} \end{aligned} \tag{14}$$

Remember that our goal is to find the dispersion relation for the waves. We do so by assuming a sinusoidal form for both dependent variables, of the form of Eqs. (5) and substitute these directly into Eqs. (14). The result is the following dispersion relation,

$$\begin{aligned} &\omega = k\left(\bar{u} \pm c_s\right) \\ &c = \bar{u} \pm c_s \end{aligned} \tag{15}$$

Note that the effect of the mean flow is additive. This is a property of linear waves. ***For linear waves, the phase speed with mean flow is just the phase speed without the mean flow plus the mean flow itself.***

VERTICALLY PROPOGATING SOUND WAVES

For sound waves propagating in all three dimensions the linearized governing equations are

$$\begin{aligned} &\frac{\partial u'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \\ &\frac{\partial v'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} \\ &\frac{\partial w'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} - \frac{\rho'}{\bar{\rho}}g \\ &\frac{1}{c_s^2}\left(\frac{\partial p'}{\partial t} + w'\frac{\partial \bar{p}}{\partial z}\right) = -\bar{\rho}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right). \end{aligned} \tag{16}$$

For sufficiently large wave number (sufficiently small wavelengths) it turns out that we can ignore the effects of buoyancy (gravity) and the vertical gradient of the reference pressure. In practical terms this means as long as the waves are sufficiently short such that the wavelength is small compared to the scale over which pressure and density change with height, or

$$\mathrm{K} >> -\frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial z} \tag{17}$$

(details can be found in Lighthill, Section 4.2). Condition (17) can be expressed as

$$\lambda << 2\pi H_\rho \tag{18}$$

where H_ρ is the ***density scale height*** of the atmosphere.[2] So, as long as we limit ourselves to sound waves in the normal range of human hearing we can ignore the effects of gravity on sound waves. Our equations are then

$$\begin{aligned}
\frac{\partial u'}{\partial t} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \\
\frac{\partial v'}{\partial t} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} \\
\frac{\partial w'}{\partial t} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} \\
\frac{1}{c_s^2}\frac{\partial p'}{\partial t} &= -\bar{\rho}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right).
\end{aligned} \tag{19}$$

Substituting sinusoidal forms for u', v', w', and ρ' of

$$\begin{aligned}
u' &= Ae^{i(kx+ly+mz-\omega t)} \\
v' &= Be^{i(kx+ly+mz-\omega t)} \\
w' &= Ce^{i(kx+ly+mz-\omega t)} \\
p' &= De^{i(kx+ly+mz-\omega t)}
\end{aligned} \tag{20}$$

yields the following dispersion relation

$$\omega^2 = c_s^2\left(k^2+l^2+m^2\right) = c_s^2 K^2 . \tag{21}$$

[2] Density scale height is the *e*-folding scale for density, i.e., the altitude at which density is 37% of the surface value.

EXERCISES

1. The linearized governing equations for one-dimensional sound waves with zero mean flow are

$$\frac{\partial u'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}$$

$$\frac{1}{c_s^2}\frac{\partial p'}{\partial t} = -\bar{\rho}\frac{\partial u'}{\partial x}$$

Substitute the assumed solutions

$$u' = Ae^{i(kx-\omega t)}$$

$$p' = Be^{i(kx-\omega t)}$$

into these equations to derive the dispersion relation for sound waves.

2. The linearized governing equations for three-dimensional sound waves with zero mean flow are

$$\frac{\partial u'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y}$$

$$\frac{\partial w'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z}$$

$$\frac{1}{c_s^2}\frac{\partial p'}{\partial t} = -\bar{\rho}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)$$

Derive the dispersion relation

$$\omega^2 = c_s^2\left(k^2 + l^2 + m^2\right) = c_s^2 \mathrm{K}^2$$

for these waves.

3. Show that the speed of sound in an ideal gas is

$$c_s = \pm\sqrt{\gamma R' T}\ .$$

4. Show that for an isothermal atmosphere that Condition (17) becomes Condition (18).

5. a. Find the scale height and speed of sound for an isothermal atmosphere with a temperature of 255K.

b. For this atmosphere, find out how large the wavelength of an acoustic wave would need to be before we started concerning ourselves with the effects of gravity and buoyancy on these waves.

ESCI 343 – Atmospheric Dynamics II
Lesson 9 – Internal Gravity Waves

References: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
A*tmosphere-Ocean Dynamics*, A.E. Gill
Waves in Fluids, J. Lighthill

Reading: Holton, 7.4.1

THE BRUNT-VÄISÄLÄ FREQUENCY

Before progressing with an analysis of internal waves we should review the important concept of the Brunt-Väisälä frequency. The vertical acceleration on an air parcel is

$$\frac{D\tilde{w}}{Dt} = -\frac{1}{\tilde{\rho}}\frac{\partial \overline{p}}{\partial z} - g \tag{1}$$

where quantities with a '~' character are properties of the air parcel, while those with an overbar are for the surrounding environment. Assuming that the atmosphere is in hydrostatic balance we can write

$$\frac{\partial \overline{p}}{\partial z} = -\overline{\rho} g \tag{2}$$

and so

$$\frac{D\tilde{w}}{Dt} = -\frac{1}{\tilde{\rho}}\left(-\overline{\rho} g\right) - g = -\left(\frac{\tilde{\rho}-\overline{\rho}}{\tilde{\rho}}\right) g\ . \tag{3}$$

Defining the perturbation density as the difference in density between the parcel and its surrounding air at the same level,

$$\rho' = \tilde{\rho} - \overline{\rho}\,, \tag{4}$$

then (3) can be written as

$$\frac{D\tilde{w}}{Dt} = -\frac{\rho'}{\tilde{\rho}} g\ . \tag{5}$$

If the parcel starts out at level z_0 and has the same density as its environment,

$$\tilde{\rho}(z_0) = \overline{\rho}(z_0) \tag{6}$$

and is displaced adiabatically a small vertical distance *z*, then its new density will can be expressed as a Taylor series expansion

$$\tilde{\rho}(z_0 + z) \cong \overline{\rho}(z_0) + \frac{\partial \tilde{\rho}}{\partial z} z = \overline{\rho}(z_0) + \frac{\partial \tilde{\rho}}{\partial \tilde{p}}\frac{\partial \tilde{p}}{\partial z} z\,. \tag{7}$$

As parcels rise and expand their pressure instantaneously adjusts to be equal to that of the surrounding environment, so that

$$\tilde{p} = \bar{p} \tag{8}$$

and therefore

$$\frac{\partial \tilde{p}}{\partial z} = \frac{\partial \bar{p}}{\partial z} = -\bar{\rho} g \ . \tag{9}$$

Furthermore, we know that

$$\left(\frac{\partial \tilde{\rho}}{\partial \tilde{p}}\right)_\theta = \frac{1}{c_s^2} \ . \tag{10}$$

Using (9) and (10), (7) becomes

$$\tilde{\rho}(z_0 + z) \cong \bar{\rho}(z_0) - \frac{\bar{\rho} g}{c_s^2} z \ . \tag{11}$$

The density of the environment can also be expanded using Taylor series,

$$\bar{\rho}(z_0 + z) \cong \bar{\rho}(z_0) + \frac{\partial \bar{\rho}}{\partial z} z \ . \tag{12}$$

From (11) and (12) the perturbation density, (4), becomes

$$\rho' = -\left(\frac{\partial \bar{\rho}}{\partial z} + \frac{\bar{\rho} g}{c_s^2}\right) z \tag{13}$$

and so (5) becomes

$$\frac{D\tilde{w}}{Dt} = \frac{\left(g\frac{\partial \bar{\rho}}{\partial z} + \frac{\bar{\rho} g^2}{c_s^2}\right) z}{\tilde{\rho}} \ . \tag{14}$$

For small displacements the denominator can be approximated as

$$\tilde{\rho} \cong \bar{\rho} \tag{15}$$

and so (14) is now

$$\frac{D\tilde{w}}{Dt} = \left(\frac{g}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial z} + \frac{g^2}{c_s^2}\right) z \ . \tag{16}$$

Equation (16) has the form of

$$\frac{D^2 z}{Dt^2} + N^2 z = 0 \tag{17}$$

where

$$N^2 \equiv -\left(\frac{g}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial z} + \frac{g^2}{c_s^2}\right) . \tag{18}$$

Equation (17) is a homogeneous 2nd-order differential equation. Although N^2 is a function of z, in those cases were is a constant then (17) has solutions given by

$$z(t) = Ae^{\sqrt{-N^2}t} + Be^{-\sqrt{-N^2}t} \ . \tag{19}$$

These solutions are fundamentally different depending on whether N^2 is positive or negative.

If N^2 is positive, N itself if real, and solutions to (19) are

$$z(t) = Ae^{iNt} + Be^{-iNt} \tag{20}$$

which are oscillations having an angular frequency of N. N is therefore a fundamental frequency of the oscillation, and is referred to as the *Brunt-Väisälä frequency* (or buoyancy frequency).

If N^2 is negative, then N itself is imaginary, and solutions to (19) are

$$z(t) = Ae^{|N|t} + Be^{-|N|t}\,. \tag{21}$$

These solutions are exponential with time, and are not oscillatory.

The Brunt-Väisälä frequency frequency is directly related to the static stability of the atmosphere.

N^2	***N***	***Solutions for z(t)***	***Static Stability***
positive	real	oscillations	stable
negative	imaginary	exponential growth	unstable

For the atmosphere

$$\frac{g}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial z} >> \frac{g^2}{c_s^2} \tag{22}$$

and so we can get away with defining N^2 as

$$N^2 \cong -\frac{g}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial z}\,. \tag{23}$$

Also, for an ideal gas, the Brunt-Väisälä frequency can be written in terms of potential temperature as

$$N^2 = \frac{g}{\bar{\theta}}\frac{\partial\bar{\theta}}{\partial z}\,. \tag{24}$$

Equation (24) is valid for an ideal gas only, whereas (18) is true for any fluid. For an ideal gas, (18) and (24) are equivalent (see exercises).

An additional result that will be of use in the next section is that from (13) and (18) we can derive a direct relationship between the Brunt-Väisälä frequency and the perturbation density,

$$\rho' = -\frac{\bar{\rho}N^2}{g}z\,. \tag{25}$$

DISPERSION RELATION FOR PURE INTERNAL WAVES

For the present discussion we will ignore changes in density due to local compression or expansion, which is a valid assumption as long as the waves are short compared to the scale at which the density changes with height (large values of wave number). We will therefore use the linearized, anelastic continuity equation, so that the governing equations are

$$\frac{\partial u'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \tag{26}$$

$$\frac{\partial v'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} \tag{27}$$

$$\frac{\partial w'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} - \frac{\rho'}{\bar{\rho}} g \tag{28}$$

$$w'\frac{d\bar{\rho}}{dz} = -\bar{\rho}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) \tag{29}$$

which when written in *flux form* are

$$\frac{\partial}{\partial t}\left(\bar{\rho}\, u'\right) = -\frac{\partial p'}{\partial x} \tag{30}$$

$$\frac{\partial}{\partial t}\left(\bar{\rho}\, v'\right) = -\frac{\partial p'}{\partial y} \tag{31}$$

$$\frac{\partial}{\partial t}\left(\bar{\rho}\, w'\right) = -\frac{\partial p'}{\partial z} - \rho' g \tag{32}$$

$$\frac{\partial}{\partial x}\left(\bar{\rho}\, u'\right) + \frac{\partial}{\partial y}\left(\bar{\rho} v'\right) + \frac{\partial}{\partial z}\left(\bar{\rho}\, w'\right) = 0\,. \tag{33}$$

Equations (30) thru (33) are four equation in five unknowns (u', v', w', p', and ρ'). The fifth equation is found by taking $\partial^2/\partial t^2$ of (25) to get

$$\frac{\partial^2 \rho'}{\partial t^2} = \frac{\bar{\rho}}{g} N^2 \frac{\partial w'}{\partial t} = \frac{N^2}{g}\frac{\partial}{\partial t}\left(\bar{\rho} w'\right) \tag{34}$$

Though we could write sinusoidal solutions for the five dependent variables and solve a 5×5 determinant to get the dispersion relation, this would be tedious. We can eliminate u', v', and w' from the equations and reduce the number of equations to two as follows:

- Take $\partial/\partial t$ of (33), and combining it with $\partial/\partial x$ of (30), $\partial/\partial y$ of (31), and $\partial/\partial z$ of (32) to get

$$\frac{\partial \rho'}{\partial z} = -\frac{1}{g}\nabla^2 p'\,. \tag{35}$$

- Eliminate w' between (32) and (34) to get

$$\frac{\partial^2 \rho'}{\partial t^2} + N^2 \rho' = -\frac{N^2}{g}\frac{\partial p'}{\partial z}\,. \tag{36}$$

Equations (35) and (36) are two equations in two unknowns. Substituting the sinusoidal solutions

$$p' = Ae^{i(kx++ly+mz-\omega t)}$$
$$\rho' = Be^{i(kx+ly+mz-\omega t)} \tag{37}$$

into equation set (35) and (36) yields the following dispersion relation for internal gravity waves of

$$\omega = \pm \frac{\sqrt{k^2+l^2}N}{\sqrt{k^2+l^2+m^2}} = \pm \frac{\mathrm{K}_H N}{\mathrm{K}} \tag{38}$$

where $\mathrm{K}_H = \sqrt{k^2+l^2}$ is the horizontal wave number, and $\mathrm{K} = \sqrt{k^2+l^2+m^2}$ is the total wave number.

ANALYSIS OF INTERNAL WAVE DISPERSION

The dispersion relation for internal waves (38) shows that for purely horizontal waves ($\mathrm{K} = \mathrm{K}_H$) the frequency is equal to the Brunt-Väisälä frequency. For non-horizontally traveling waves the frequency is less than the Brunt-Väisälä frequency. Therefore, the Brunt-Väisälä frequency is an upper-limiting frequency for internal waves. In other words, for internal waves $\omega^2 \le N^2$.

The phase speed for internal waves is given by

$$c = \frac{\omega}{\mathrm{K}} = \pm \frac{\mathrm{K}_H N}{\mathrm{K}^2}. \tag{39}$$

The phase velocity is given by

$$\vec{c} = \frac{\omega}{\mathrm{K}^2}\vec{\mathrm{K}} = \pm \frac{\mathrm{K}_H N}{\mathrm{K}^3}\left(k\hat{i} + l\hat{j} + m\hat{k}\right). \tag{40}$$

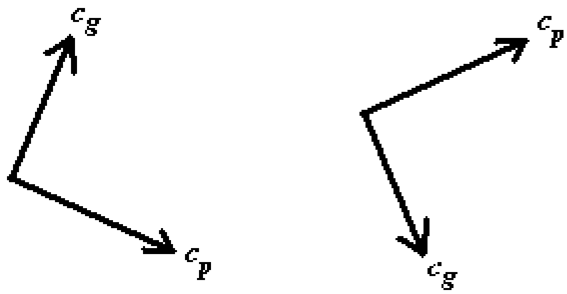
The group velocity is

$$\vec{c}_g = \frac{\partial \omega}{\partial k}\hat{i} + \frac{\partial \omega}{\partial l}\hat{j} + \frac{\partial \omega}{\partial m}\hat{k} = \pm \frac{m^2 N}{\mathrm{K}_H \mathrm{K}^3}\left(k\hat{i} + l\hat{j} - \frac{\mathrm{K}_H^2}{m}\hat{k}\right). \tag{41}$$

Inspection of (40) and (41) shows a curious fact that for internal waves, if there is a downward component to the phase velocity, then there is an upward component to the group velocity, and vice-versa. In fact, by taking the dot product of (40) and (41) we find that

$$\vec{c} \bullet \vec{c}_g = 0. \tag{42}$$

which shows that ***the group velocity and phase velocity are actually oriented at 90° to each other in the vertical plane!*** This is illustrated in the diagram below.



The link below contains an animated GIF loop showing internal wave dispersion for a wave number pointing toward the upper right. Note that individual crests propagate toward the upper-right corner, while the groups of waves propagate toward the lower-right corner. http://www.atmos.millersville.edu/~adecaria/ESCI343/internal-wave-loop.gif.

EXERCISES

1. Show that for an ideal gas

$$\frac{g}{\bar{\theta}}\frac{d\bar{\theta}}{dz} \equiv -\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz} - \frac{g^2}{c_s^2}\ .$$

2. Substitute sinusoidal solutions into equations (35) and (36) to derive the dispersion relation for internal gravity waves,

$$\omega^2 = \frac{\mathrm{K}_H^2 N^2}{\mathrm{K}^2}.$$

3. a. Show that the group velocity for internal waves is

$$\vec{c}_g = \pm\frac{m^2 N}{\mathrm{K}_H \mathrm{K}^3}\left(k\hat{i} + l\hat{j} - \frac{\mathrm{K}_H^2}{m}\hat{k}\right).$$

b. What is the magnitude of the group velocity for purely vertically propagating waves?

c. What is the magnitude of the group velocity for purely horizontally propagating waves?

4. Use equations (40) and (41) to show that for internal waves, $\vec{c} \bullet \vec{c}_g = 0$.

5. For an ideal, incompressible gas the linearized governing equations in the *x*-*z* plane can be written as

$$\frac{\partial u'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \qquad \textbf{(a)}$$

$$\frac{\partial w'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} + \frac{\theta'}{\bar{\theta}}g \qquad \textbf{(b)}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \qquad \textbf{(c)}$$

$$\theta' = -\frac{\partial \bar{\theta}}{\partial z}\Delta z \qquad \textbf{(d)}$$

a. Substitute (d) into (b) and then take $\partial/\partial t$ of the resulting equation to get

$$\frac{\partial^2 w'}{\partial t^2} = -\frac{1}{\bar{\rho}}\frac{\partial^2 p'}{\partial t \partial z} - N^2 w' \qquad \textbf{(e)}$$

b. Substitute sinusoidal solutions into (a), (c), and (e) to find the dispersion relation and phase speed.

c. What kind of waves are these?

6. Show that for the atmosphere, $\frac{g}{\rho}\frac{d\rho}{dz} >> \frac{g^2}{c_s^2}$.

ESCI 343 – Atmospheric Dynamics II
Lesson 10 - Topographic Waves

Reference: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton

Reading: Holton, Section 7.4.2

STATIONARY WAVES

Waves will appear to be stationary if their phase speed is equal and opposite to the mean flow,

$$c = -\bar{u}\,. \tag{1}$$

Stationary waves will have a frequency of zero, since they do not oscillate in time, only in space.

For dispersive waves, the wavelength of the stationary wave will correspond to that wavelength which has a phase speed equal and opposite to the mean flow. None of the other wavelengths will be stationary.

An example of stationary waves is the stationary wave pattern that forms in a river when it flows over a submerged rock or obstacle. The flow of the obstacle generates many different waves of various wavelengths, but only the ones whose phase speed is equal and opposite to the flow will remain stationary. If the flow speeds up or slows down, the wavelength of the stationary wave will change.

DISPERSION RELATION FOR STATIONARY INTERNAL GRAVITY WAVES

In the atmosphere, flow of a stably stratified fluid over a mountain barrier can also generate standing waves. If we consider the flow over a sinusoidal pattern of ridges that are perpendicular with the *x*-axis, and with a horizontal wave number of *k*, then we can analyze the structure of these waves. Since these waves are internal gravity waves in the presence of a mean flow their phase speed and dispersion relation is simply

$$\begin{aligned} c &= \bar{u} \pm \frac{N}{\sqrt{k^2 + m^2}} \\ \omega &= \bar{u}k \pm \frac{kN}{\sqrt{k^2 + m^2}} \end{aligned} \tag{2}$$

(remember that since these are linear waves, the mean flow simply adds a term $\bar{u}k$ to the frequency). But, we are only interested in the standing waves generated by the topography, for which the phase speed (and therefore, frequency) is zero. For standing waves then, we have

$$\bar{u} \pm \frac{N}{\sqrt{k^2 + m^2}} = 0\,. \tag{3}$$

The horizontal wave number, *k*, is determined by the wave number of the terrain; $\bar{u}$ and *N* are properties of the atmosphere. Since these quantities are predetermined, then there is only one value of vertical wave number, *m*, which can satisfy the dispersion relation. Thus, *m* is given by

$$m^2 = \frac{N^2}{\bar{u}^2} - k^2\,. \tag{4}$$

Equation (4) is the dispersion relation for stationary internal gravity waves.

VERTICALLY PROPAGATING VERSUS VERTICALLY DECAYING WAVES

The vertical structure of the standing waves is determined by how N, k, and $\overline{u}$ relate to one another.

Vertically propagating waves. If $k < N/\overline{u}$ then the right hand side is positive, and m is real. This corresponds to waves that propagate vertically, since the sinusoidal form for the dependent variables is

$$e^{i(kx+mz)}. \tag{5}$$

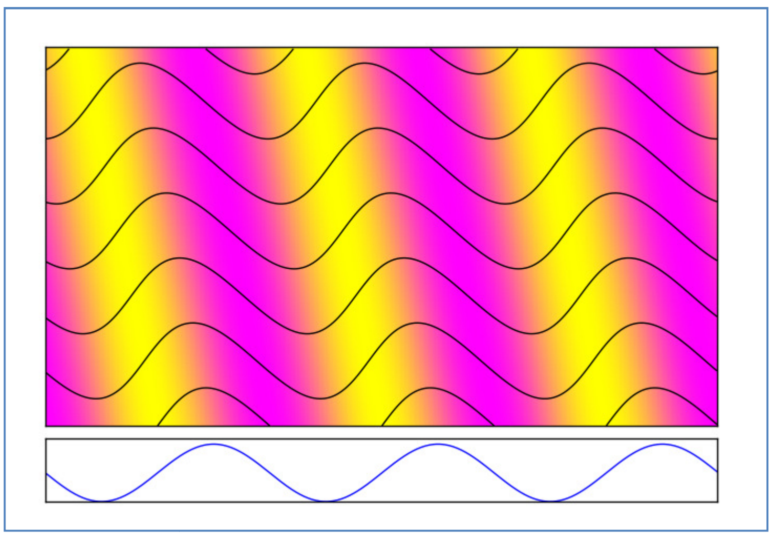
Along lines of constant phase ($kx + mz$) the following relation holds

$$kx + mz = b \tag{6}$$

where b is a constant. This means that phase lines obey the following equation,

$$z = -\frac{k}{m}x + \frac{b}{m}. \tag{7}$$

This tells us that the lines of constant phase tilt upwind with height, since they have a negative slope (see figure below). Since the wave is propagating upstream (it has to be, since it is a standing wave), the individual wave crests have a downward component of phase speed. This means that the group velocity is upward, so that topographically forced waves propagate energy upward.

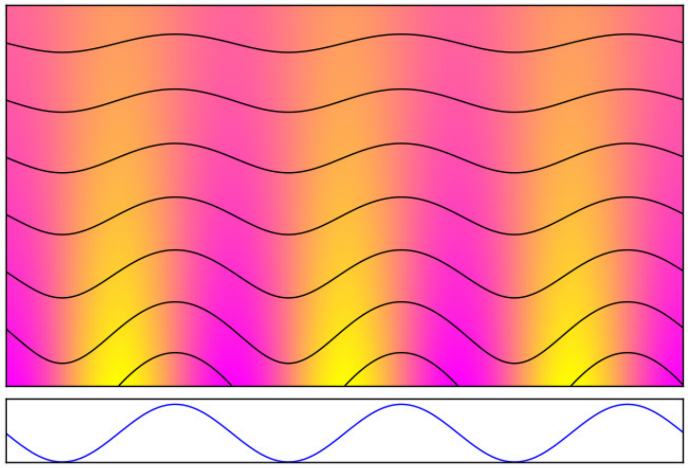


Vertically propagating topographically-forced stationary waves. The wind speed is from left-to-right. This is for $\overline{u} < N/k$, so the value of m^2 is positive and m is real. Solid lines are streamfunction; colors are for vertical velocity.

Vertically decaying waves. If $k > N/\overline{u}$ then the right-hand-side negative. In this case the sinusoidal form for the dependent variables is

$$e^{ikx}e^{-mz},$$

and the waves decay with height (such waves are also known as *evanescent*). In this case, lines of constant phase are vertical. This is illustrated in the figure below.



Vertically decaying topographically-forced stationary waves. This is for $\bar{u} > N/k$, so the value of m^2 is negative and m is imaginary. Solid lines are streamfunction; colors are for vertical velocity.

WAVES GENERATED BY AN ISOLATED MOUNTAIN RIDGE

In the real world, mountains are not pure sinusoids. However, through Fourier analysis, we can approximate the real topography by its Fourier components, with the component of wave number k having an amplitude of $H(k)$. $H(k)$ and $h(x)$ are the *Fourier transforms* of each other, and are defined by the following equations

$$
\begin{aligned}
h(x) &= \int_{-\infty}^{\infty} H(k)\, e^{-ikx}\, dk \\
H(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} h(x)\, e^{ikx}\, dx
\end{aligned}
\tag{8}
$$

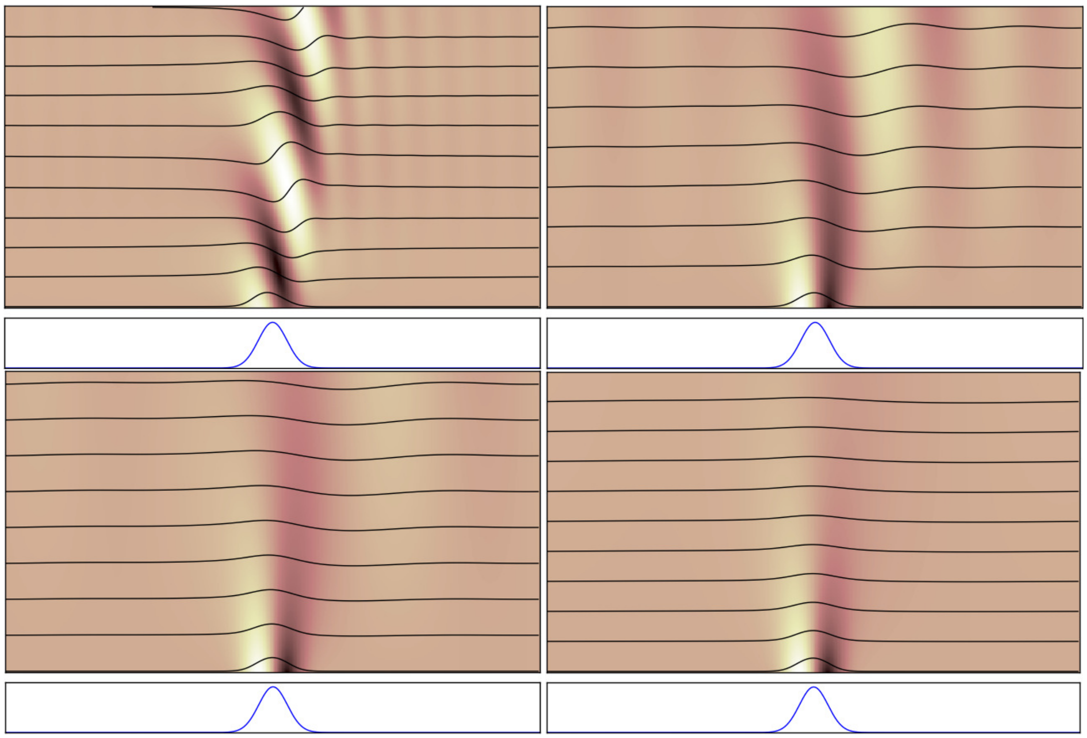
A very sharp, or discontinuous function has a greater number of high frequency (high wave number) components in its transform. A broad function has few high frequency components, and is mostly made up of low frequency (low wave number) components.

Flow over a mountain will generate a whole spectrum of gravity waves. Each wave component generated will either propagate vertically, or decay vertically, depending on whether its vertical wave number (m) determined from

$$m^2 = \frac{N^2}{\bar{u}^2} - k^2 \tag{9}$$

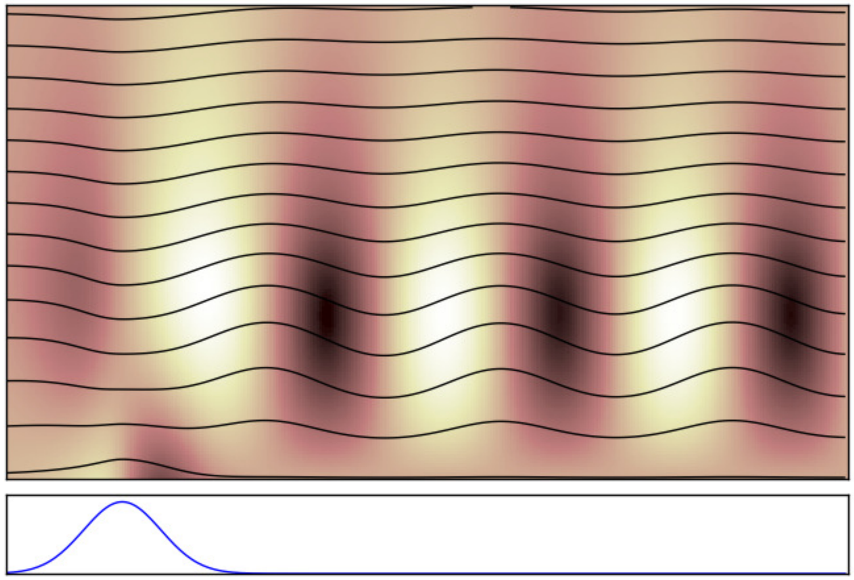
is real or imaginary. Based on the previous discussion of Fourier transforms, we expect that a narrow mountain will generate a lot of high wave number gravity waves, while a broad mountain will generate more low wave-number gravity waves.

Whether or not flow over an isolated mountain will generate vertically propagating waves, or vertically decaying waves, depends on the wind speed and the width of the mountain. Vertically propagating waves are more likely if the wind speed is slow, or the mountain is wide. Faster wind speeds and narrower mountains are more likely to result in vertically decaying waves. The plots below are for various wind speeds over a Gaussian-shaped hill. The wind speed in each plot is constant, but is faster in each successive plot.



TRAPPED (LEE) WAVES

In the previous discussion on mountain waves, we've assumed that the mean flow does not have vertical shear, and that the static stability is constant with height. Since wind speed normally increases with height it is possible that (9) will yield vertically propagating waves in the lower layer, but have vertically decaying waves in the upper part of the atmosphere. This is illustrated in the figure below. In this figure there are two distinct vertical layers. In each layer the quantity $N/\bar{u}$ (called the *Scorer parameter*) is constant (though N and $\bar{u}$ themselves may vary within each layer). If the upper layer has a larger wind speed and/or lower stability, then the scorer parameter is larger in the lower layer than in the upper layer. This results in waves that are 'trapped' in the lower layer downwind of the mountain.



EXERCISES

1. For an isothermal, compressible atmosphere show the following (remember that $H = R'T/g$):

a. $c_s^2 = \gamma g H$

b. $N^2 = (g/H)(1-\gamma^{-1})$

2. Assume that the Allegheny Mountains can be approximated as a parallel series of ridges approximately 25 km apart. Also, assume an isothermal, compressible atmosphere with a scale height of 8100 m. Calculate the critical wind speed below which topographically forced waves will propagate vertically, and above which they will decay with height.

ESCI 343 – Atmospheric Dynamics II
Lesson 11 - Rossby Waves

Reference: *An Introduction to Dynamic Meteorology (4rd edition)*, J.R. Holton
Atmosphere-Ocean Dynamics, A.E. Gill
Fundamentals of Atmospheric Physics, M.L. Salby

Reading: Holton, 7.7 and 12.3

BAROTROPIC ROSSBY WAVES

Rossby waves owe their existence to the principle of conservation of potential vorticity. We first start with a barotropic fluid, for which the principle of conservation of barotropic potential vorticity states that

$$\frac{D}{Dt}\left(\frac{\zeta+f}{h}\right)=0\,.$$

Expanding this out yields the barotropic vorticity equation

$$\frac{\partial\zeta}{\partial t}+\vec{V}\bullet\nabla_h\zeta+\beta v=\frac{f}{h}\frac{Dh}{Dt}\,.$$

The linearized form of this equation with zonal mean flow only ($\bar{u}\neq 0;\ \bar{v}=0$) is

$$\frac{\partial\zeta'}{\partial t}+\bar{u}\frac{\partial\zeta'}{\partial x}+\beta v'=\frac{f_0}{H}\left(\frac{\partial\eta}{\partial t}+\bar{u}\frac{\partial\eta}{\partial x}+\bar{u}\frac{\partial H}{\partial x}+u'\frac{\partial H}{\partial x}+v'\frac{\partial H}{\partial y}\right). \qquad \textbf{(1)}$$

If we assume that the mean depth of the fluid (*H*) is constant then equation (1) becomes

$$\frac{\partial\zeta'}{\partial t}+\bar{u}\frac{\partial\zeta'}{\partial x}+\beta v'=\frac{f_0}{H}\left(\frac{\partial\eta}{\partial t}+\bar{u}\frac{\partial\eta}{\partial x}\right). \qquad \textbf{(2)}$$

Further assuming geostrophic balance we can write them in terms of the perturbation height as follows,

$$\zeta'=\frac{g}{f_0}\nabla^2\eta$$

$$v'=\frac{g}{f_0}\frac{\partial\eta}{\partial x}$$

so that equation (2) becomes

$$\frac{\partial}{\partial t}\left(\nabla^2\eta-\frac{f_0^2}{c^2}\eta\right)+\bar{u}\frac{\partial}{\partial x}\left(\nabla^2\eta-\frac{f_0^2}{c^2}\eta\right)+\beta\frac{\partial\eta}{\partial x}=0 \qquad \textbf{(3)}$$

where *c* is the phase speed of a shallow-water gravity wave. The waves supported by equation (3) are called *Rossby waves*.

To find the dispersion relation for the waves supported by equation (3) we assume a perturbation of the form

$$\eta=Ae^{i(kx+ly-\omega t)}$$

and substitute it into (3). This yields the following dispersion relation for Rossby waves,

$$\omega=\bar{u}k-\frac{\beta k}{K^2+f_0^2/c^2}\,. \qquad \textbf{(4)}$$

For waves that are short compared to the Rossby radius of deformation (given by c/f_0), the wave number will be large compared to f_0/c. In this case the dispersion relation becomes

$$\omega = \bar{u}k - \frac{\beta k}{K^2},^{1} \qquad \textbf{(4')}$$

a result known as the ***shortwave approximation***.

DISPERSION PROPERTIES OF ROSSBY WAVES

The phase velocity of Rossby waves with zero mean flow ($\bar{u} = 0$) is

$$\vec{c} = \frac{\omega}{K^2}\vec{K} = \frac{-k\beta}{K^2\left(K^2 + f_0^2/c^2\right)}\left(k\hat{i} + l\hat{j}\right), \qquad \textbf{(5)}$$

and the group velocity is

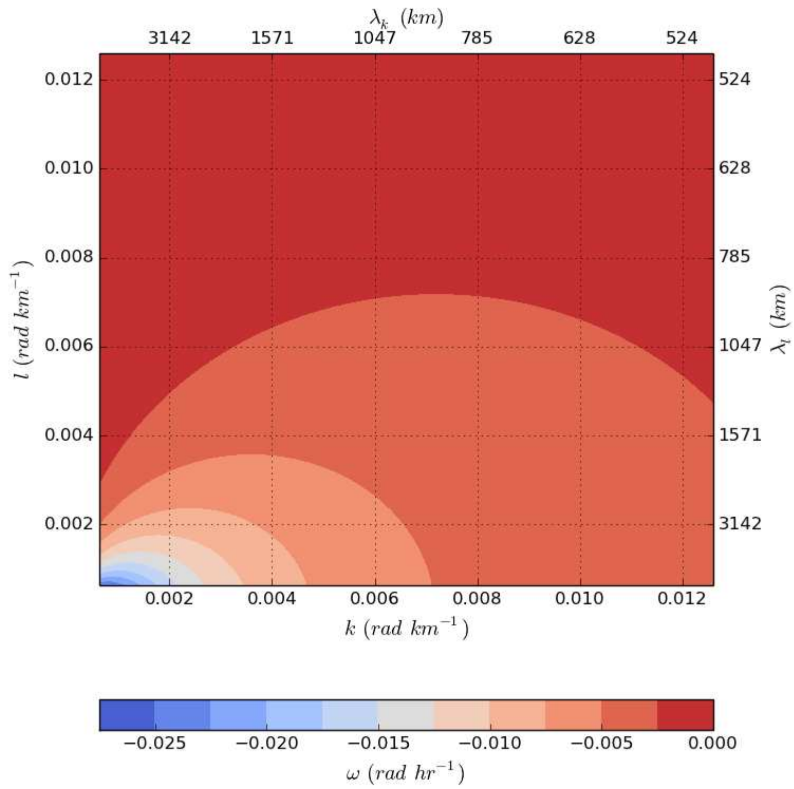
$$\vec{c}_g = \frac{\beta\left(k^2 - l^2 - f_0^2/c^2\right)}{\left(K^2 + f_0^2/c^2\right)^2}\hat{i} + \frac{2\beta kl}{\left(K^2 + f_0^2/c^2\right)^2}\hat{j}. \qquad \textbf{(6)}$$

The plots on the next page show the dispersion properties for Rossby waves with zero mean flow, using a value of $\beta = 10^{-11}$ m^{-1} s^{-1} and Rossby radius of deformation of 3000 km. Some things to note:

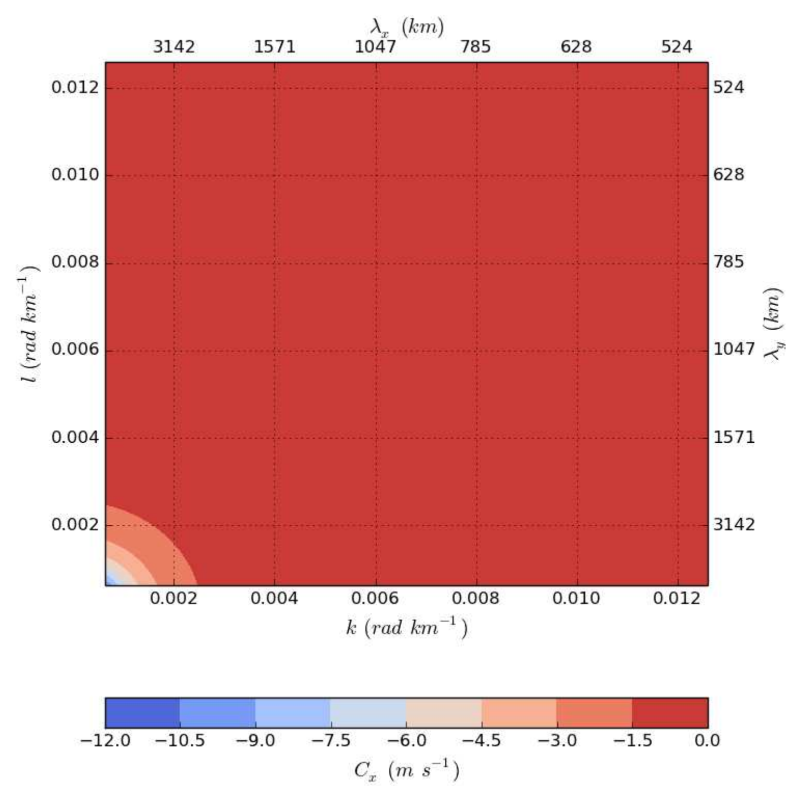
- The frequency is always negative, and becomes larger in magnitude for the longer wavelengths (smaller wave numbers).
- The zonal phase speed is always negative in the absence of mean flow.
- The zonal group speed may be either positive or negative, depending on the horizontal wave number.
  - Long waves propagate energy westward in the same direction as the phase speed.
  - Shortwaves propagate energy eastward, opposite to the phase speed.
- The meridional phase speed is negative, but the meridional group speed is positive.
  - The meridional energy propagation is opposite to the phase speed.

If there were a non-zero mean flow it would simply be added to the phase speeds and group speeds.

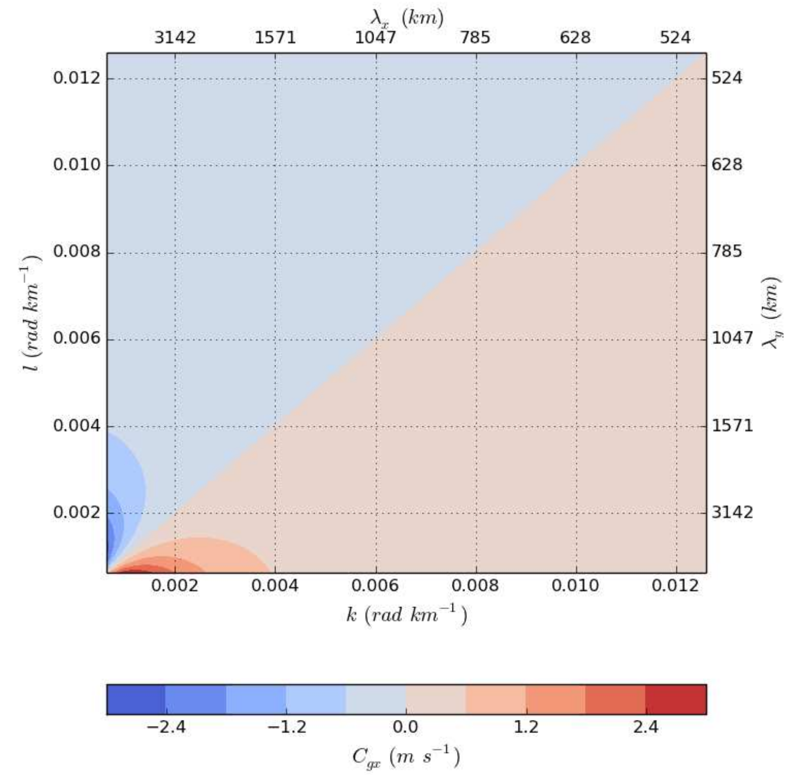
[1] In most meteorological textbooks equation (4') is the dispersion relation that is given for Rossby waves, and is derived directly by ignoring the right-hand-side of equation (2). Physically this implies ignoring the vertical stretching of the fluid column. Equation (4) is the more general form of the dispersion relation.



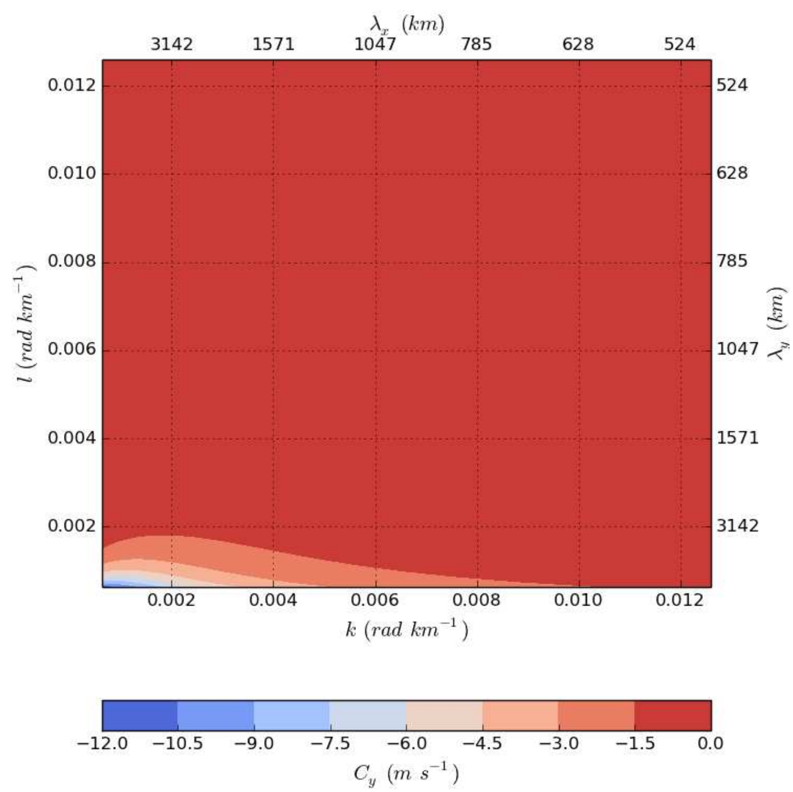
Angular frequency.



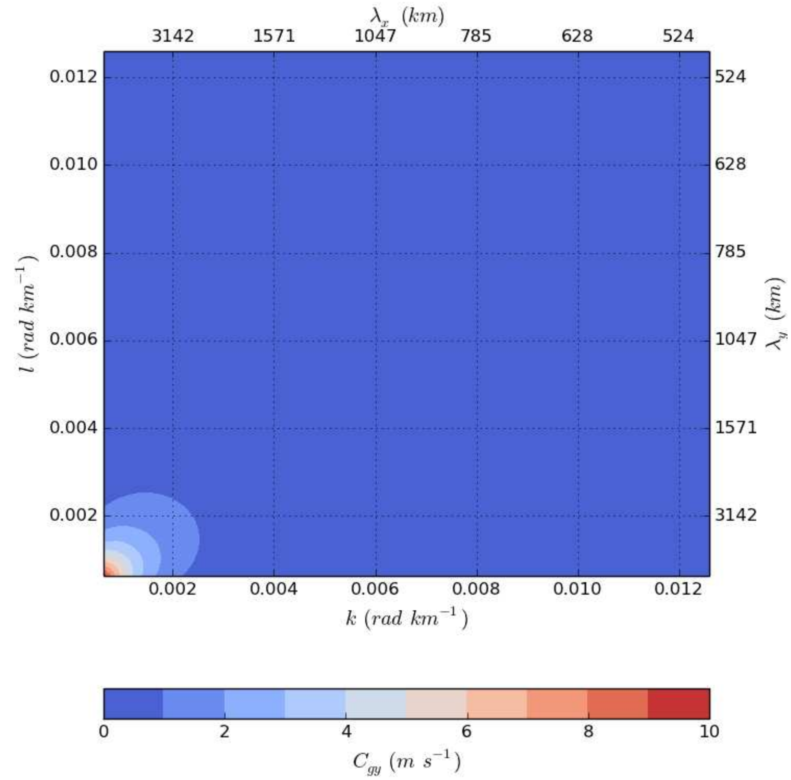
Zonal phase speed.



Zonal group speed.



Meridional phase speed.



Meridional group speed.

VERTICALLY PROPAGATING ROSSBY WAVES

In a stratified fluid it is possible to have Rossby waves that have a vertical component of propagation. To study these waves we have to use the concept of quasi-geostrophic potential vorticity (in place of the barotropic potential vorticity used in the previous discussions). In the absence of diabatic heating, quasi-geostrophic potential vorticity is conserved and therefore the following equation holds (see Lesson 4)

$$\frac{D_g}{Dt}\left[\frac{1}{f_0}\nabla^2\Phi + f + f_0\frac{\partial}{\partial p}\left(\frac{1}{\sigma}\frac{\partial\Phi}{\partial p}\right)\right] = 0$$

where the static-stability parameter is given as

$$\sigma = -\frac{\alpha}{\theta}\frac{\partial\theta}{dp}.$$

Converting this to height coordinates yields

$$\frac{D_g}{Dt}\left[\frac{1}{f_0}\nabla^2\Phi + f + \frac{f_0}{\bar{\rho}N^2}\frac{\partial}{\partial z}\left(\bar{\rho}\frac{\partial\Phi}{\partial z}\right)\right] = 0.$$

We can write this in terms of the streamfunction,

$$\psi \equiv \Phi/f_0,$$

so that

$$\frac{D_g}{Dt}\left[\nabla^2\psi + f + \frac{f_0^2}{\bar{\rho}N^2}\frac{\partial}{\partial z}\left(\bar{\rho}\frac{\partial\psi}{\partial z}\right)\right] = 0. \quad \textbf{(7)}$$

The linearized form of equation (7), with mean flow in the zonal direction only, is

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left[\nabla^2\psi' + f + \frac{f_0^2}{\bar{\rho}N^2}\frac{\partial}{\partial z}\left(\bar{\rho}\frac{\partial\psi'}{\partial z}\right)\right] + \beta\frac{\partial\psi'}{\partial x} = 0. \quad \textbf{(8)}$$

Since we are interested in waves which may propagate vertically great distances, we need to use a slightly modified form of the sinusoidal solution for the streamfunction,

$$\psi' = \frac{A}{\sqrt{\bar{\rho}}}e^{i(kx+ly+mz-\omega t)}, \quad \textbf{(9)}$$

where $\bar{\rho}$ is a function of height. To simplify things we also assume and isothermal, anelastic atmosphere, so that

$$N^2 = -\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz} = \frac{g}{H}.$$

Putting equation (9) into equation (8) yields the dispersion relation

$$\omega = \bar{u}k - \frac{\beta k}{\mathrm{K}_H^2 + \left(\frac{f_0^2}{N^2}\right)\left(m^2 + \frac{N^4}{4g^2}\right)}. \quad \textbf{(10)}$$

Rossby waves are often forced by topography. To find out when such stationary waves (with frequency zero) can propagate vertically we rearrange equation (10) for vertical wave number to get

$$m^2 = \left(\frac{N^2}{f_0^2}\right)\left[\frac{\beta}{\bar{u}} - \mathrm{K}_H^2 - \frac{N^2 f_0^2}{4g^2}\right]. \quad \textbf{(11)}$$

Vertical propagation is only possible if the term in brackets is positive. This means that for vertical propagation the mean zonal wind speed must be in the range

$$0 < \bar{u} \le \frac{\beta}{\mathrm{K}_H^2 + N^2 f_0^2/4g^2}.$$

Therefore, ***Rossby waves cannot propagate vertically if the mean zonal winds are easterly, or if they are westerly and exceed a certain speed.***

This has important implications for the dynamics of the middle atmosphere (defined as the stratosphere and mesosphere). In the summertime the zonal winds in the middle atmosphere are easterly, and so energy from topographically forced Rossby waves cannot reach the middle atmosphere. In the wintertime, however, the zonal winds in the middle atmosphere are westerly, allowing Rossby waves to reach the middle atmosphere and deposit energy. This explains the sudden stratospheric warming episodes (as much as 40-50 K within a few days) observed in the Northern Hemisphere wintertime. This phenomenon is not as pronounced in the Southern Hemisphere because there are not as many topographical features in that hemisphere to generate topographically forced Rossby waves.

EXERCISES

1. For a mean zonal flow of 30 m/s, at what wavelength will a Rossby wave be stationary? Use β for 45°N, and assume $l = 0$.

2. Assume a barotropic fluid with a mean-depth, H, that varies in the y-direction only. Also, assume that $\beta = 0$, and that $\bar{v} = 0$.

a. Show that equation (1) then supports waves whose dispersion relation (using the short-wave approximation) is

$$\omega = \bar{u}k + \frac{f_0}{H}\frac{\partial H}{\partial y}\frac{k}{\mathrm{K}^2}.$$

These are Rossby waves that owe their existence to the bottom topography rather than to β, and can occur in the ocean along coastlines.

b. What is the phase speed and group velocity for these waves (assume $l = 0$)?

c. Assume that the bottom topography has an exponential shape such as $H(y) = A\exp(-\alpha y)$. What value should α be in order that these waves travel at the same speed as a regular Rossby wave at latitude 45°N?

3. a. Show for an isothermal atmosphere that

$$\bar{\rho}^{-1/2} = \rho_0^{-1/2} e^{z/(2H)}.$$

b. Use this result to show that equation (9) can be written as

$$\psi' = Ae^{i(kx+ly+[m+1/(2H)]z-\omega t)}.$$

c. Substitute this assumed solution into (8) to get the following dispersion relation for vertically propagating Rossby waves

$$\omega = \bar{u}k - \frac{\beta k}{\mathrm{K}_H^2 + \left(\frac{f_0^2}{N^2}\right)\left(m^2 + \frac{1}{4H^2}\right)}.$$

d. Show that this dispersion relation is identical to equation (10).

ESCI 343 – Atmospheric Dynamics II
Lesson 12 - Inertial-gravity Waves

Reference: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
Atmosphere-Ocean Dynamics, A.E. Gill

Reading: Holton, Section 7.5

INERTIAL-GRAVITY WAVES

Inertial-gravity waves occur when a statically stable flow is also inertially stable. They are essentially gravity waves that have a large enough wavelength to be affected by the earth's rotation. To study inertial-gravity waves we need to include the Coriolis terms in the governing equations. For simplicity, we will use the incompressible continuity equation. Therefore, the linearized governing equations are

$$\frac{\partial u'}{\partial t} = -\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x} + fv' \tag{1}$$

$$\frac{\partial v'}{\partial t} = -\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial y} - fu' \tag{2}$$

$$\frac{\partial w'}{\partial t} = -\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} - \frac{\rho'}{\overline{\rho}}g \tag{3}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{4}$$

and

$$\rho' = \frac{\overline{\rho}}{g}N^2 \Delta z\,. \tag{5}$$

Combining (3) and (5) to eliminate ρ' gives

$$\frac{\partial^2 w'}{\partial t^2} = -\frac{1}{\overline{\rho}}\frac{\partial^2 p'}{\partial z \partial t} - N^2 w'\,. \tag{6}$$

Equations (1), (2), (4), and (6) are the governing equations for inertial-gravity waves. Assuming the usual sinusoidal solutions

$$u' = Ae^{i(kx+ly+mz-\omega t)}$$

$$v' = Be^{i(kx+ly+mz-\omega t)}$$

$$w' = Ce^{i(kx+ly+mz-\omega t)}$$

$$p' = De^{i(kx+ly+mz-\omega t)}$$

and substituting into (1), (2), (4), and (6) results in the following algebraic equations

$$i\omega A + fB - \left(ik/\overline{\rho}\right)D = 0 \tag{7}$$

$$fA - i\omega B + \left(il/\overline{\rho}\right)D = 0 \tag{8}$$

$$kA + lB + mC = 0 \tag{9}$$

$$\left(\omega^2 - N^2\right)C - \left(m\omega/\overline{\rho}\right)D = 0 \tag{10}$$

and the resulting dispersion relation for inertial-gravity waves

$$\omega^2 = \left(f^2 m^2 + N^2 \mathrm{K}_H^2 \right) \Big/ \mathrm{K}^2 \ . \tag{11}$$

Notice that if the effects of rotation are ignored ($f = 0$) then the dispersion relation becomes that for pure internal waves.

The dispersion relation can be written in terms of the propagation angle of the waves. The wave number vector makes an angle φ with the horizontal plane, and this angle is given by

$$\varphi = \arctan\left(m/K_H \right). \tag{12}$$

Equation (11) can then be written as

$$\omega^2 = f^2 \sin^2 \varphi + N^2 \cos^2 \varphi . \tag{13}$$

From this we see that the frequency of inertial-gravity waves is constrained to always lie between f and N,

$$f \le \omega \le N \ . \tag{14}$$

Purely horizontally propagating waves have a frequency of N, while purely vertically propagating waves have a frequency of f.

Since inertial-gravity waves have long enough wavelengths to be effected by the earth's rotation, we can assume that they are in hydrostatic balance. This implies that $m >> \mathrm{K}_H$. Therefore, we sometimes write the dispersion relation for inertial-gravity waves as

$$\omega^2 \cong f^2 + \frac{N^2 \mathrm{K}_H^2}{m^2} . \tag{15}$$

DISPERSION AND STRUCTURE OF INERTIAL-GRAVITY WAVES

The phase velocity of inertial gravity waves is

$$\vec{c} = \pm \sqrt{f^2 + \frac{N^2 \mathrm{K}_H^2}{m^2}} \frac{\vec{\mathrm{K}}}{\mathrm{K}^2} , \tag{16}$$

while the group velocity is

$$\vec{c}_g = \pm \frac{N^2}{m^2 \sqrt{f^2 + N^2 \mathrm{K}_H^2 / m^2}} \left(k\hat{i} + l\hat{j} - \frac{\mathrm{K}_H^2}{m} \hat{k} \right). \tag{17}$$

Thus, for inertial-gravity waves the group velocity and phase velocity are orthogonal, and upward propagating waves transport energy downward, while downward propagating waves transport energy upward.

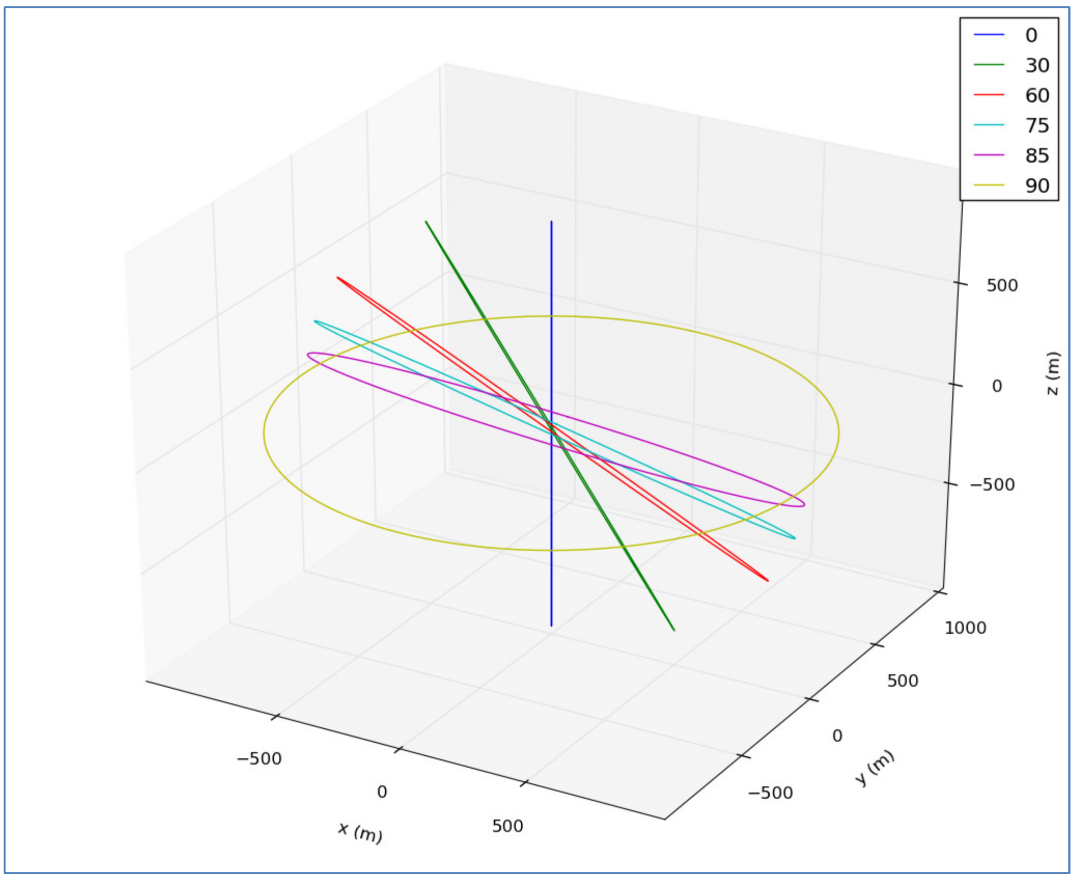
The parcel trajectories for inertial-gravity waves are ellipses. From (9) we can deduce that

$$\vec{K} \bullet \vec{V} = 0$$

(see exercises). This means that the velocity is always perpendicular to the wave number vector; i.e., there is no component of particle motion along the direction of the phase propagation. Therefore, the ellipses are always at 90° to the direction of phase propagation. The particles move along these ellipses in an anticyclonic fashion (in the

Northern Hemisphere), regardless of whether the wave propagation is up or down. This makes sense, since the Coriolis force must be directed toward the inside of the ellipse. The figure below shows the direction of the particle trajectories for upward and downward propagating waves.

The figures below show the 3-dimensional trajectories for waves traveling in the positive x and z directions for various propagation angles. Notice that the trajectory is a vertical lines for a propagation angle of zero (when $\omega = N$), as would be expected for a pure internal wave. As the propagation angle increases, ω decreases and the trajectories begin to slant, and also open up into ellipses. At the lowest frequency possible ($\omega = f$) the trajectory is a circle that lies completely in the horizontal plane.



EXERCISES

1. Show that if $m >> K_H$ then (11) becomes (15).

2. Show that the group velocity for inertial-gravity waves is given by equation (11)

3. Show that for inertial-gravity waves, $\vec{c} \bullet \vec{c}_g = 0$.

4. Use equations (7), (8), (9), and (10) to show the following phase relations between u', v', and w' for a wave traveling in the x-z plane ($l = 0$).

a. $u' = (i\omega/f)v'$

b. $u' = -(m/k)w'$

c. $v' = (ifm/k\omega)w'$

d. If $w' = \cos(kx + mz - \omega t)$, what are u' and v'?

5. a. Use the results from 4.a to determine whether the horizontal velocity vector will rotate cyclonically or anticyclonically with time. Will this change if the wave is propagating upward versus downward?

b. Use the results from 4.a to determine whether the horizontal velocity vector will rotate cyclonically or anticyclonically with height. Will this change if the wave is propagating upward versus downward?

6. Show that $\vec{K} \bullet \vec{V} = 0$ is the same as equation (9).

ESCI 343 – Atmospheric Dynamics II
Lesson 13 – Geostrophic/gradient Adjustment

Reference: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
Atmosphere-Ocean Dynamics, A.E. Gill

Reading: Holton, Section 7.6

GEOSTROPHIC ADJUSTMENT OF A BAROTROPIC FLUID

The atmosphere is nearly always close to geostrophic and hydrostatic balance. If this balance is disturbed through such processes as heating or cooling, the atmosphere adjusts itself to get back into balance. This process is called *geostrophic adjustment*, although it may more accurately be referred to as gradient adjustment, since in curved flow the atmosphere tends toward gradient balance. One method of studying geostrophic adjustment is to first study adjustment in a barotropic fluid using the shallow-water equations. Once we understand adjustment in a barotropic fluid we can easily extend our results to a baroclinic fluid by use of the concept of equivalent depth, studied in a previous lesson.

For this we can use the linearized shallow-water equations with zero mean flow,

$$\frac{\partial u'}{\partial t} - fv' = -g\frac{\partial h'}{\partial x} \tag{1}$$

$$\frac{\partial v'}{\partial t} + fu' = -g\frac{\partial h'}{\partial y} \tag{2}$$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \tag{3}$$

If we take $\partial/\partial x$ of (1) and add it to $\partial/\partial y$ (2) we get

$$\frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) - f\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right) = -g\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right). \tag{4}$$

Rearranging (3) we get

$$\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = -\frac{1}{H}\frac{\partial h'}{\partial t}. \tag{5}$$

From the definition of vorticity we know that

$$\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \equiv \zeta'. \tag{6}$$

Putting (5) and (6) into (4) we get

$$\frac{\partial^2 h'}{\partial t^2} - gH\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + fH\zeta' = 0\,. \tag{7}$$

We need one more equation that relates h' and ζ'. This is the shallow-water vorticity equation, found by taking $\partial/\partial x$ of (2) and subtracting $\partial/\partial y$ (1) to get

$$\frac{\partial \zeta'}{\partial t} = -f\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right), \tag{8}$$

which, using (5), can be written (after some rearranging) as

$$\frac{\partial}{\partial t}\left(\frac{\zeta'}{f} - \frac{h'}{H}\right) = 0\,. \tag{9}$$

Integrating (9) with respect to time gives

$$\frac{\zeta'}{f} - \frac{h'}{H} = \frac{\zeta'_0}{f} - \frac{h'_0}{H}\,, \tag{10}$$

where ζ'_0 and h'_0 refers to the initial values of relative vorticity and height perturbation. Using this in (7) results in

$$\frac{\partial^2 h'}{\partial t^2} - gH\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f^2 h' = -f\left(H\zeta'_0 - fh'_0\right). \tag{11}$$

Since the quantity gH is the square of the speed of a gravity wave in this fluid, we can denote it by c^2 and write this equation as

$$\frac{\partial^2 h'}{\partial t^2} - c^2\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f^2 h' = -f\left(H\zeta'_0 - fh'_0\right). \tag{12}$$

Equation (12) governs the geostrophic adjustment process in a barotropic fluid.

THE STEADY-STATE SOLUTION

Lets simplify things somewhat by assuming the initial state is at rest, and has an abrupt step in the surface height given by

$$h'_0 = -\hat{h}\,\mathrm{sgn}(x)\,.^1 \tag{13}$$

We also assume that there is no dependence in the y-direction. Equation (12) then becomes

$$\frac{\partial^2 h'}{\partial t^2} - c^2\frac{\partial^2 h'}{\partial x^2} + f^2 h' = -f^2\hat{h}\,\mathrm{sgn}(x)\,, \tag{14}$$

[1] The sgn(x) function is defined to be +1 for $x \geq 0$, and −1 for $x < 0$.

a second order, non-homogeneous partial differential equation. The homogeneous form of this equation supports shallow-water inertial-gravity waves (see exercises). After these waves have subsided, there will remain a steady-state solution which obeys the steady state equation

$$\frac{d^2h'}{dx^2} - \left(\frac{f}{c}\right)^2 h' = \left(\frac{f}{c}\right)^2 \hat{h}\,\mathrm{sgn}(x)\,. \tag{15}$$

Equation (15) is a second-order, non-homogeneous ordinary differential equation with constant coefficients (assuming f and c are constant). The solution to (15) consists of a complementary solution (the general solution to the homogeneous equation) plus a particular solution,

$$h'(x) = h'_c(x) + h'_p(x)\,. \tag{16}$$

The complementary solution is found from the characteristic equation for the homogeneous form of (15), which is

$$r^2 - (f/c)^2 = 0\,. \tag{17}$$

Therefore, the complementary solution is then

$$h'_c(x) = Ae^{\alpha x} + Be^{-\alpha x} \tag{18}$$

where

$$\alpha \equiv f/c\,. \tag{19}$$

For the particular solution we use the ***method of undetermined coefficients***, guessing that the particular solution will have the form

$$h'_p(x) = C\,\mathrm{sgn}(x)\,. \tag{20}$$

Putting (20) into (15) shows that $C = -\hat{h}$, so that the particular solution is

$$h'_p(x) = -\hat{h}\,\mathrm{sgn}(x)\,. \tag{21}$$

Therefore, the general solution of (15) is

$$h'(x) = Ae^{\alpha x} + Be^{-\alpha x} - \hat{h}\,\mathrm{sgn}(x)\,, \tag{22}$$

or

$$h'(x) = \left\{\begin{matrix} Ae^{\alpha x} + Be^{-\alpha x} - \hat{h} & x \ge 0 \\ Ae^{\alpha x} + Be^{-\alpha x} + \hat{h} & x < 0 \end{matrix}\right\}. \tag{23}$$

All that remains is to apply the boundary conditions, which require that:

- $h'(x)$ remain bounded as $x \to \pm\infty$: This requires that $A = 0$ for positive x and $B = 0$ for negative x, so that

$$h'(x) = \left\{\begin{matrix} Be^{-\alpha x} - \hat{h} & x \ge 0 \\ Ae^{\alpha x} + \hat{h} & x < 0 \end{matrix}\right\} \tag{24}$$

- $h'(x)$ be continuous at $x = 0$: This means that

$$B - \hat{h} = A + \hat{h} \tag{25}$$

- The first derivative of $h'(x)$ be continuous at $x = 0$: This means that

$$-\alpha B = \alpha A \tag{26}$$

Solving (25) and (26) for A and B yields

$$A = -\hat{h}$$

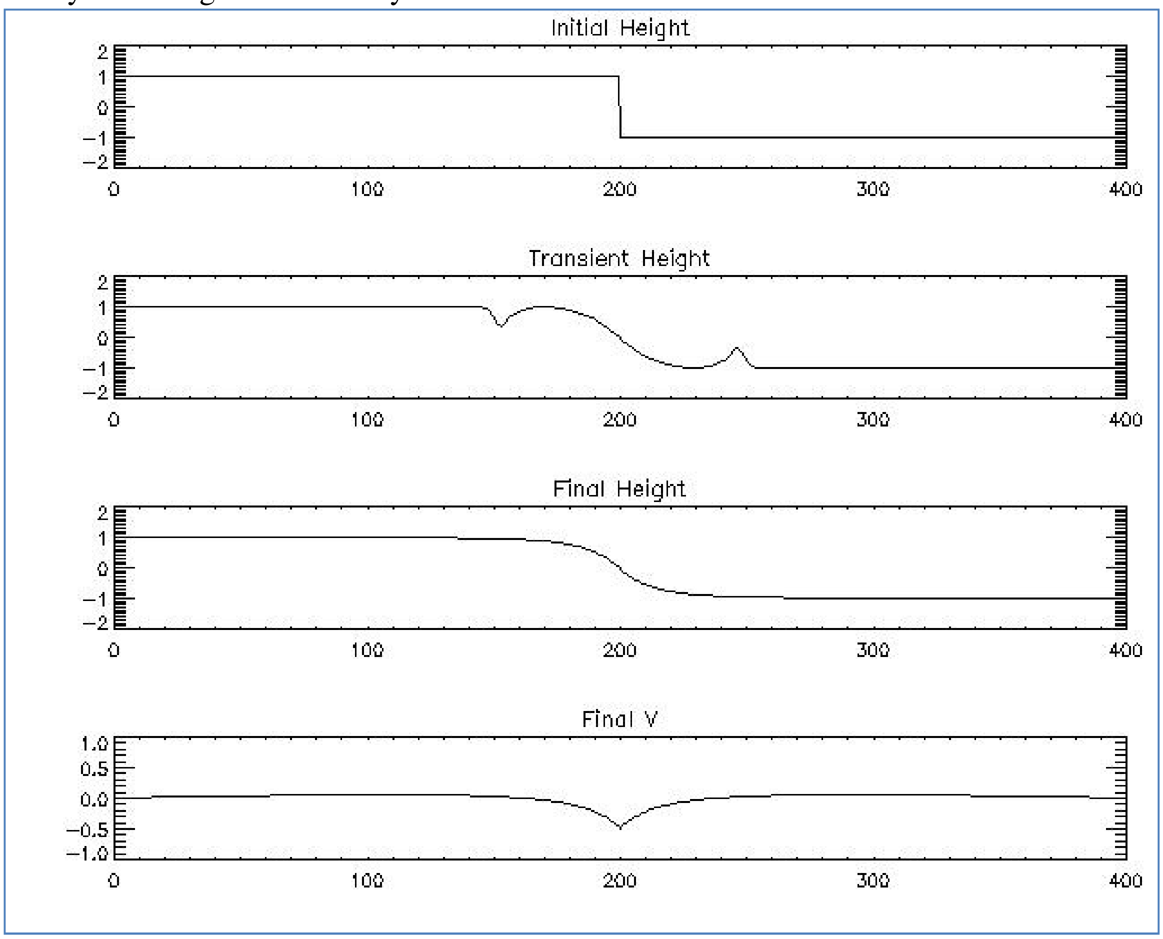
$$B = \hat{h}$$

so that the steady-state solution is

$$h'(x) = \hat{h}\begin{Bmatrix} e^{-\alpha x} - 1 & x \geq 0 \\ 1 - e^{\alpha x} & x < 0 \end{Bmatrix}. \tag{27}$$

ANALYSIS OF THE SOLUTION

The figures below show the initial height field, the transient height field, and the steady-state height and velocity fields taken from a 1-D shallow-water numerical model.



The transient solution consists of the shallow-water inertial gravity waves. The final height solution is the steady state solution from (27). The figures are striking in that, though the step that was in the initial conditions is smoothed out, there is still a region near the center of the domain with a horizontal pressure gradient, and therefore, with a geostrophic flow out of the page. The initial height field adjusted under the influence of gravity, and set up a flow that is in geostrophic balance with the final height field. The excess mass and potential energy were removed by the inertial-gravity waves which propagated away as part of the non-steady state solution.

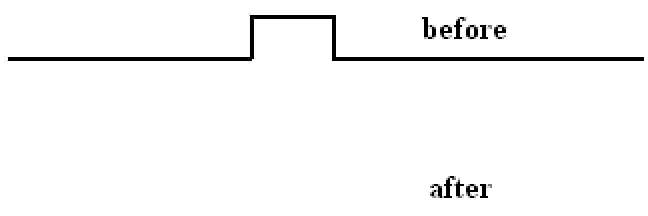
The region in which there is a remaining height gradient is characterized by an *e*-folding scale of $1/\alpha$. This length scale is of fundamental importance. It measures the scale over which the influence of the earth's rotation affects the flow, and is called the ***Rossby radius of deformation***. It is defined as

$$\lambda_R \equiv c/f \,. \tag{28}$$

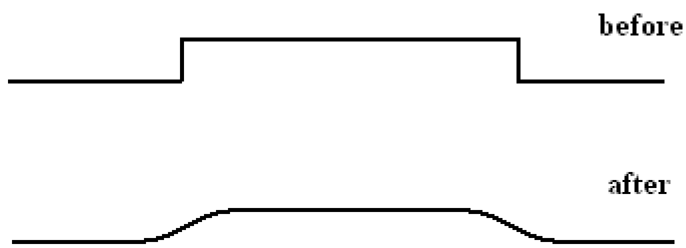
The radius of deformation is given by the group velocity of a gravity wave divided by the Coriolis parameter. The physical essence of the radius of deformation can be seen by recalling that the *inertial period* (the time scale for which rotational effects are important) is $2\pi f^{-1}$, so that $2\pi\lambda_R$ can be interpreted as the distance traveled by a gravity wave during one inertial period. For dispersive gravity wave modes the group velocity, c_g, should be used rather than the phase speed.

ROSSBY RADIUS OF DEFORMATION

The Rossby radius of deformation is a fundamental physical parameter of a fluid on a rotating reference frame. It gives a length scale that can be used as a measure of how large a disturbance has to be in order for rotational effects to be important. The physical concept of the radius of deformation is better illustrated using the following example. Imagine that you immerse a tumbler into a lake, turn it upside down, and lift it up to the point just before the lip breaks the surface of the water (a quick calculation will show that the radius of the tumbler is much smaller than the radius of deformation.) Right as you lift the lip of the tumbler completely from the water the initial height field would look like that pictured below. As soon as you lift the tumbler from the water, the water surface begins to adjust by generating gravity waves which propagate away from the initial disturbance. In the steady state all that remains after the water calms is a flat surface, as shown.

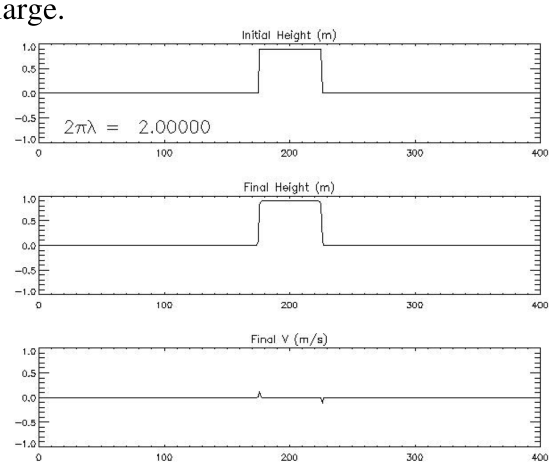


Now, imagine the same experiment performed in the ocean, only using an extremely wide tumbler that has a radius much greater than the radius of deformation. Performing the same experiment will result in a steady state solution pictured below, with a hump of water remaining, around which an anticyclonic geostrophic (actually, gradient) circulation has developed!

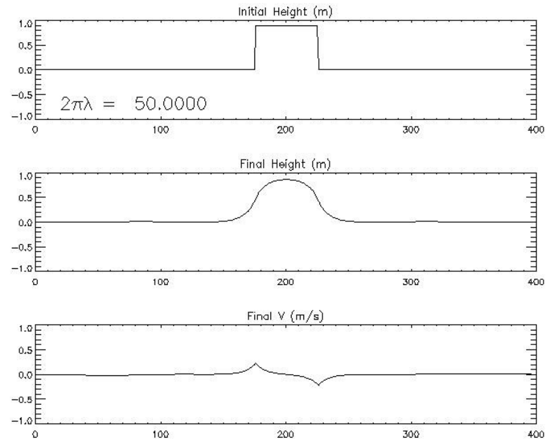


What is the difference between the two experiments? It is how the horizontal scale of the initial disturbance compares with the Rossby radius of deformation!

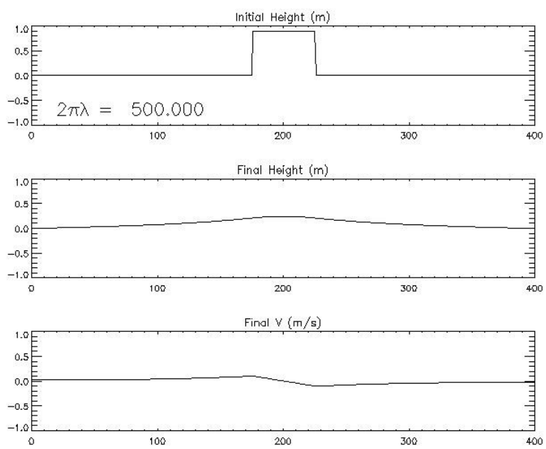
We can illustrate this using a 1-D shallow-water model for disturbances ranging in size from very small compared to the Rossby radius of deformation to those that are very large.



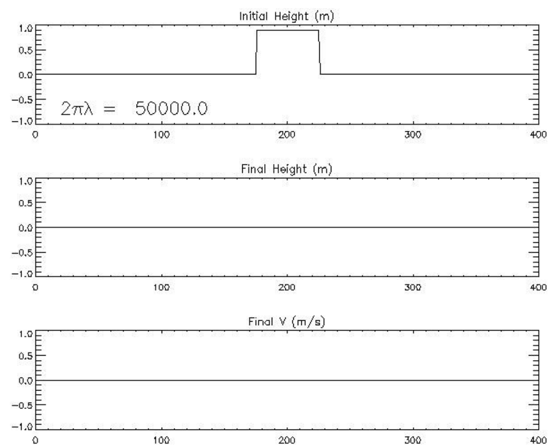
(a)



(b)



(c)



(d)

In these figures, the ratio of the horizontal length of the disturbance, L divided by $2\pi\lambda_R$ is (a) 25; (b) 1.0; (c) 0.1; and (d) 0.001 (in the model the physical size of the disturbances is the same in each case, but the Coriolis acceleration is varied to achieve a variable radius of deformation). In all cases the final height and velocity fields are in geostrophic balance. However, for very small disturbances the height field adjusted to the initial velocity field, while for very large disturbances the velocity field adjusted to the initial height field.

The following rules apply in all cases, and are suggested for commitment to memory:

If the size of the disturbance is much greater than 2π times the Rossby radius of deformation ($L \gg 2\pi\lambda_R$), then the velocity field adjusts to the initial height (mass) field.[2]

[2] The terms "height" field and "mass" field are synonymous.

If the disturbance is much less than 2π times the Rossby radius of deformation ($L << 2\pi\lambda_R$), then the height field adjusts to the initial velocity field.

If the disturbance is of the same order as 2π times the Rossby radius of deformation ($L \sim 2\pi\lambda_R$), then the height and velocity fields undergo mutual adjustment.

In all cases, the final height and velocity fields are in geostrophic/gradient balance!

ADJUSTMENT IN A VORTEX

Our derivation of the Rossby radius of deformation was for a 1-D fluid, so there was no curvature to the flow. The general form of the Rossby radius of deformation for a vortex is

$$\lambda_R = \frac{c}{\sqrt{\eta\left(f_0 + 2v/r\right)}} \tag{29}$$

where η is the absolute vorticity, v is the tangential velocity, and r is the radius of curvature of the flow (see the notes for Tropical Meteorology, Lesson 9, if you are interested in a derivation for a vortex). For flows whose absolute vorticity is primarily due to the earths vorticity (i.e., flows where $\zeta << f$) this becomes

$$\lambda_R = c/f\ ,$$

which is the same as what we derived here.

GEOSTROPHIC ADJUSTMENT IN A MULTI-LAYER FLUID

The principles of geostrophic adjustment in a multi-layer, hydrostatic fluid are identical to that in a barotropic fluid, except that there are several modes of inertial-gravity waves generated: one for the barotropic mode, and one for each baroclinic mode. Each mode has its own unique radius of deformation, given by

$$\begin{aligned} \lambda_0 &= c_0/f \\ \lambda_1 &= c_1/f \\ &\vdots \\ \lambda_n &= c_n/f \end{aligned} \tag{30}$$

where the subscript 0 refers to the barotropic mode, and the subscript n refers to the n^{th} baroclinic mode. The barotropic mode is often called the *external mode*, and its radius of deformation the *external radius of deformation*. In a two-layer fluid the baroclinic mode is often called the *internal mode*, and its radius of deformation the *internal radius of deformation*. In a continuously stratified fluid (such as the ocean or atmosphere) there are in theory an infinite number of baroclinic modes possible; however, most of the energy is confined to a few of the lower baroclinic modes, so application of geostrophic adjustment is greatly simplified, as we can concern ourselves with a smaller, finite number of modes.

In a continuously stratified fluid the group velocity of the modes of oscillation can be approximated as

$$c_n \cong NH/n\pi; \quad n = 0, 1, 2, \cdots \tag{31}$$

where N is the Brunt-Vaisala frequency and H is the scale height. In this case the Rossby radii of deformation are

$$\lambda_n \cong \frac{NH}{nf\pi}; \quad n = 0, 1, 2, \cdots. \tag{32}$$

Since the baroclinic modes have a much smaller wave speed than the barotropic mode, the baroclinic radius of deformation is much smaller than that of the barotropic radius of deformation (see exercises).

SUMMARY AND FURTHER DISCUSSION

On the synoptic scale, the atmosphere is close to geostrophic and hydrostatic balance. Radiational heating and cooling, latent heat release, and other factors push the atmosphere from geostrophic balance. The atmosphere adjusts back into geostrophic balance by generating inertial-gravity waves which propagate energy away from the disturbance. The nature of the adjustment depends on how the horizontal scale of the disturbance compares with the Rossby radius of deformation. Since the atmosphere is stratified, it is usually the baroclinic radii of deformation that are important.

For small-scale phenomena, such as individual thunderstorms, the disturbance is much smaller than the radius of deformation. Therefore, the mass field adjusts to the initial velocity field, and no residual synoptic scale circulations are generated. However, larger-scale phenomena can be of the order of the baroclinic radius of deformation, and can therefore leave a synoptic scale circulation as the velocity field adjusts to the mass field.

The radius of deformation also provides us with a length scale by which to gauge whether phenomena are effected by the earth's rotation. Phenomenon whose horizontal length scales are much smaller than the radius of deformation are unlikely to be effected by the earth's rotation (unless they persist for a time scale on the order of the inertial period, $2\pi f^{-1}$. Therefore, thunderstorms, tornadoes, and toilets are not affected by the earth's rotation.

EXERCISES

1. Show that the homogeneous form of equation (12),

$$\frac{\partial^2 h'}{\partial t^2} - c^2 \frac{\partial^2 h'}{\partial x^2} + f^2 h' = 0,$$

supports shallow-water inertial-gravity waves having a dispersion relation of

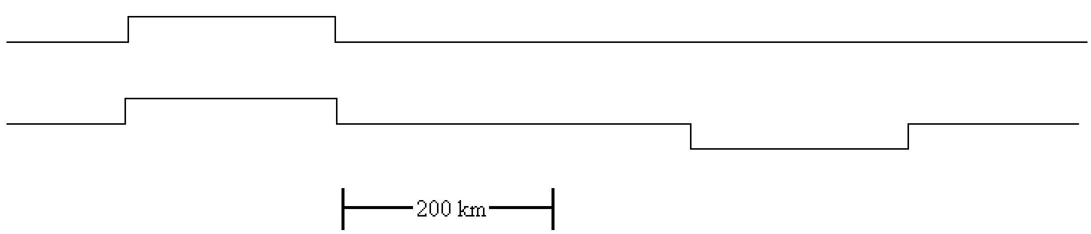
$$\omega^2 = f^2 + c^2 k^2.$$

2. The ocean is often represented as a two-layer fluid. Assume the upper layer has a depth of 700 m and a density of 1021 kg/m^3, while the lower layer has a depth of 3300 m and a density of 1023 kg/m^3.

a. Find the barotropic (external) radius of deformation at latitude 45°N.

b. Find the baroclinic (internal) radius of deformation at the same latitude.

c. For the disturbances in this ocean shown below, sketch the final position of the upper and lower surfaces. Assume the disturbance on the left only generates waves in the barotropic mode, while the disturbance on the right only generates waves in the baroclinic mode. The top line represents the external surface, while the bottom line represents the internal interface.



3. Calculate the radius of deformation for a typical bathtub. How large would a disturbance in the tub have to be in order for rotational effects to be important?

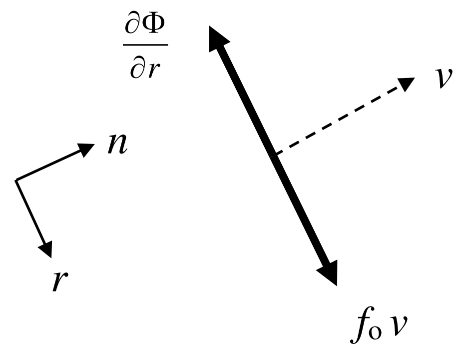
4. Does the radius of deformation increase or decrease with latitude?

ESCI 343 – Atmospheric Dynamics II
Lesson 14 – Inertial/slantwise Instability

Reference: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
Atmosphere-Ocean Dynamics, A.E. Gill
Mesoscale Meteorology in Midlatitudes, P. Markowski and Y. Richardson

INERTIAL INSTABILITY

Imagine an air parcel in geostrophic balance at speed v on the f-plane. The diagram below shows the balance of accelerations in the lateral direction. Note that the direction of v is completely arbitrary. The coordinate r is directed to the right of the wind, while the coordinate n is in the direction of the geostrophic wind. The wind component in the transverse direction (along r) is denoted as u.



The momentum equations in this coordinate system are

$$\frac{Du}{Dt} = -\frac{\partial \Phi}{\partial r} + f_0 v \tag{1}$$

$$\frac{Dv}{Dt} = -f_0 u . \tag{2}$$

Imagine that an air parcel starts in geostrophic balance. If the parcel is suddenly impelled laterally in the direction of r at a speed u, the balance of accelerations will change. Taking the time derivative of (1) gives us an equation for how the lateral acceleration changes with time,

$$\frac{D}{Dt}\left(\frac{Du}{Dt}\right) = -\frac{D}{Dt}\left(\frac{\partial \Phi}{\partial r}\right) + \frac{D}{Dt}\left(f_0 v\right) . \tag{3}$$

The terms on the right-hand side of (3) are evaluated as follows:

$$\frac{D}{Dt}\left(\frac{\partial \Phi}{\partial r}\right) = \frac{\partial \Phi}{\partial t} + u\frac{\partial}{\partial r}\frac{\partial \Phi}{\partial r} + v\frac{\partial}{\partial n}\frac{\partial \Phi}{\partial r} = u\frac{\partial^2 \Phi}{\partial r^2} \tag{4}$$

and

$$\frac{D}{Dt}\left(f_0 v\right) = f_0\frac{Dv}{Dt} = f_0\left(-f_0 u\right) = -f_0^2 u , \tag{5}$$

so that (3) becomes

$$\frac{D^2 u}{Dt^2} = -u\frac{\partial^2 \Phi}{\partial r^2} - f_0^2 u \quad \textbf{(6)}$$

or

$$\frac{D^2 u}{Dt^2} + \left(\frac{\partial^2 \Phi}{\partial r^2} + f_0^2\right)u = 0 \; . \quad \textbf{(7)}$$

The solutions to (7) will be oscillatory provided that

$$\frac{\partial^2 \Phi}{\partial r^2} + f_0^2 > 0 \, . \quad \textbf{(8)}$$

In this case, the parcel will oscillate around its original line of motion, and the flow is *inertially stable*. The angular frequency of the oscillations is

$$\omega^2 = \frac{\partial^2 \Phi}{\partial r^2} + f_0^2 \, . \quad \textbf{(9)}$$

If instead,

$$\frac{\partial^2 \Phi}{\partial r^2} + f_0^2 < 0 \, , \quad \textbf{(10)}$$

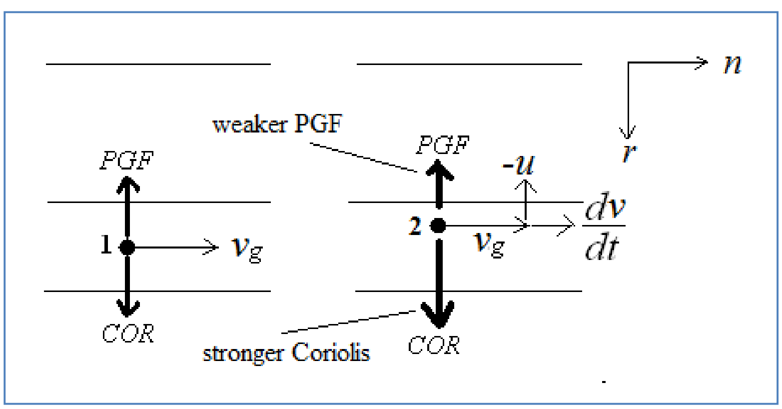
then the transverse velocity will grow exponentially with time and the parcel will accelerate away from its original line of motion.

PHYSICAL INTERPRETATION OF INERTIAL STABILITY

The physical interpretation of inertial stability/instability is directly linked to how the pressure gradient tightens or loosens in the direction of r. The figure below shows the case of the pressure gradient becoming tighter with increasing r, which implies that

$$\frac{\partial^2 \Phi}{\partial r^2} > 0 \, . \quad \textbf{(11)}$$

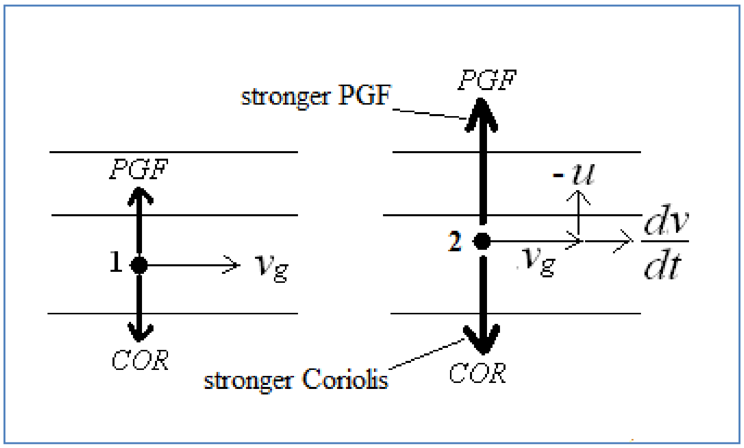
The stability criteria (8) tells us that this case is inertially stable, so that a parcel displaced latitudinally will return to its base latitude. To see why this occurs, refer to the diagram below.



Imagine parcel in geostrophic balance at Point 1. If the parcel is perturbed in the direction of $-r$, then there will also be a positive acceleration in the direction of n due to Coriolis. This will increase the v component of the wind and thus increase the component of the Coriolis acceleration in the direction of r. Since the parcel is also moving into an area of weaker pressure gradient, there is a net acceleration on the parcel toward positive r. Thus, a lateral perturbation will result in a restoring acceleration back toward the original line of motion.

For the case where the pressure gradient decreases in the r direction the physical interpretation is a little more complex. The diagram below shows this case, where

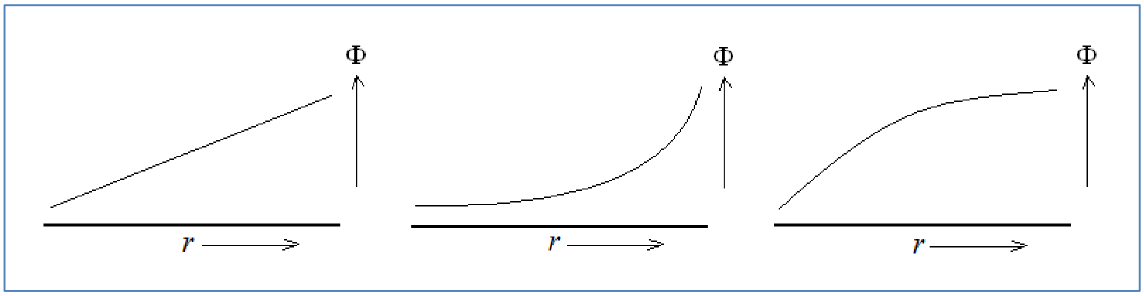
$$\frac{\partial^2 \Phi}{\partial r^2} < 0. \qquad \textbf{(12)}$$



As before the parcel is perturbed in the negative r direction at velocity u. There is still an acceleration due to Coriolis in the n direction, which will increase the component of the Coriolis acceleration in the r direction. However, the pressure gradient acceleration is also increasing as the parcel moves to Point 2. If the pressure gradient acceleration is larger than the Coriolis acceleration, the parcel will accelerate toward the negative r direction. If, however, the increase in Coriolis acceleration outweighs the increase in the pressure gradient acceleration, the parcel will accelerate back toward its original line of motion. Thus, a decreasing pressure gradient with increasing r is not sufficient to produce inertial instability. In order for instability to occur in this case, (8) shows us that

$$\frac{\partial^2 \Phi}{\partial r^2} > -f_0^2. \qquad \textbf{(13)}$$

Plots of geopotential versus r for a constant pressure surface are shown in the diagrams below. For the first two diagrams the atmosphere is inertially stable, because the second derivative of Φ is either positive or zero. For the third diagram the second derivative of Φ is negative, but instability would depend on just how negative the second derivative is.



ABSOLUTE MOMENTUM AND INERTIAL STABILITY

The stability criteria (8) can be written in alternate forms as follows:

$$f_0^2 + \frac{\partial^2 \Phi}{\partial r^2} = f_0^2 + \frac{\partial}{\partial r}\left(\frac{\partial \Phi}{\partial r}\right) = f_0^2 + \frac{\partial}{\partial r}\left(f_0 v_g\right) > 0$$

or

$$f_0 + \frac{\partial v_g}{\partial r} > 0. \qquad \textbf{(14)}$$

For ease of notation we can rewrite (14) as

$$\frac{\partial}{\partial r}\left(f_0 r + v_g\right) > 0,$$

and defining a quantity called the *absolute momentum*[1] as

$$M \equiv f_0 r + v_g \qquad \textbf{(15)}$$

the condition for inertial stability/instability can be written as

$$\begin{aligned} \frac{\partial M}{\partial r} &> 0: \quad \textit{Inertially stable} \\ \frac{\partial M}{\partial r} &= 0: \quad \textit{Inertially neutral} \\ \frac{\partial M}{\partial r} &< 0: \quad \textit{Inertially unstable} \end{aligned} \qquad \textbf{(16)}$$

Even though M is called absolute momentum, it is not exactly equal to the momentum as viewed from space, but is equal to it within some function of r. The reason it is defined this way is so that its r-derivative is equal to the absolute vorticity via

$$\eta = \frac{\partial M}{\partial r} \quad . \qquad \textbf{(17)}$$

Thus, we can view inertial instability of the flow as occurring if the absolute vorticity of is negative. This partially explains why we don't see negative absolute vorticity occurring on the synoptic scale, because if it does occur, inertial instability will occur.

---

[1] Our derivation of inertially instability and absolute momentum was done in coordinates that are completely arbitrary, with no preferred direction for the geostrophic wind. Many basic treatments of this topic do the derivation for a purely zonal geostrophic flow. In this case, absolute momentum is defined instead as $M = f_0 y - u_g$, and the derivatives in (16) are taken with respect to y instead of r.

INERTIAL STABILITY OF A VORTEX

The concept of inertial instability can be extended to curved flow (for a detailed derivation see Lesson 10 of the Tropical Meteorology class notes). In this case the background flow is assumed to be in gradient wind balance. Also, it is the radial gradient of the absolute *angular* momentum that is important, rather than the gradient of the absolute momentum. The absolute angular momentum is given by

$$M_a = vr + f_0 r^2/2 \tag{18}$$

where v is the tangential velocity of the vortex, and r is the distance from the vortex center. The condition for inertial stability/instability of a vortex is

$$\frac{\partial M_a^2}{\partial r} > 0: \quad \textit{Inertially stable}$$

$$\frac{\partial M_a^2}{\partial r} = 0: \quad \textit{Inertially neutral}$$

$$\frac{\partial M_a^2}{\partial r} < 0: \quad \textit{Inertially unstable}$$

The relationship between absolute angular momentum and absolute vorticity is

$$\eta = \frac{1}{r}\frac{\partial M_a}{\partial r}. \tag{19}$$

Most treatments of inertial stability mention that negative absolute vorticity is inertially unstable. While this is true for weak or straight-line flow, in strong vortexes it is possible to have negative absolute vorticity and still be inertially stable.

SLANTWISE/SYMMETRIC INSTABILITY

Static stability refers to an air parcel's resistance to vertical displacement, whereas inertial instability refers to its resistance to transverse-horizontal displacement. It turns out that it is possible for a parcel to be both statically (vertically) and inertially (horizontally) stable, and yet be unstable with respect to diagonal displacement. Such instability is called slantwise instability, or symmetric instability, and may be an important instability mechanism near fronts or other baroclinic zones.

Mathematically, the condition for static instability is

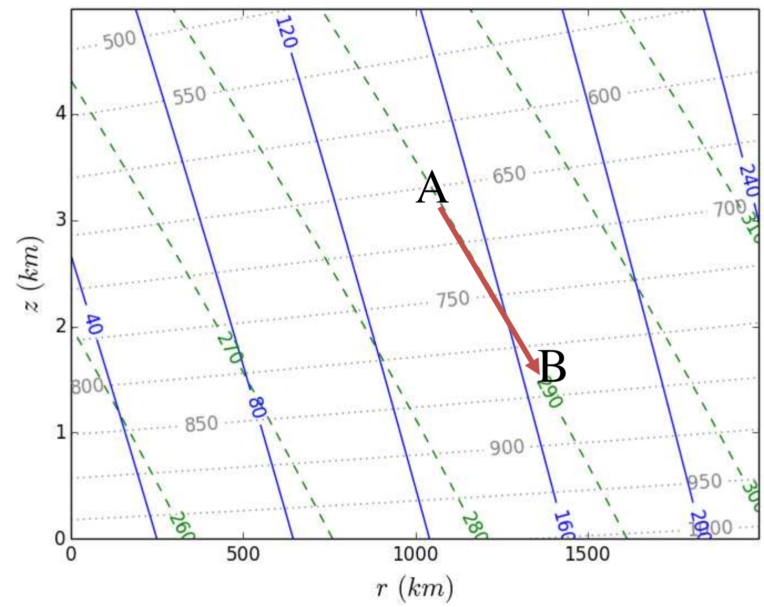
$$\left(\frac{\partial M}{\partial y}\right)_\theta > 0: \quad \textit{Slantwise stable}$$

$$\left(\frac{\partial M}{\partial y}\right)_\theta = 0: \quad \textit{Slantwise neutral}$$

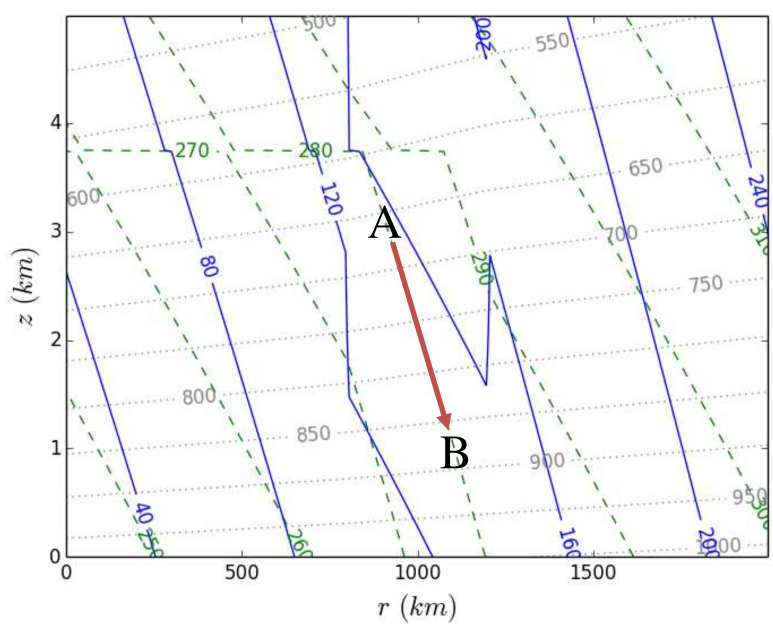
$$\left(\frac{\partial M}{\partial y}\right)_\theta < 0: \quad \textit{Slantwise unstable}$$

This looks just like the condition for inertial stability except the derivative is taken along an adiabatic surface, rather than on a horizontal surface.

The figure below shows lines of potential temperature (dashed) and the lines of absolute momentum (solid). The pressure surfaces are also shown with dotted lines. In this configuration the atmosphere is stable with respect to horizontal motion (inertial stability), vertical motion (static stability) and diagonal motion along an adiabat (slantwise stability). A parcel moving adiabatically from Point A to Point B would be moving into a region of higher absolute momentum, so that $\left(\partial M / \partial r\right)_\theta > 0$ and the atmosphere is slantwise stable.



The next figure (below) shows what can happen near a baroclinic zone (front). In this case there are regions where the adiabats are more steeply sloped than the absolute momentum lines. In this region a parcel moving in the positive *r* direction on an adiabat from Point A to Point B would be moving into a region of lower absolute momentum, so that $\left(\partial M / \partial r\right)_\theta < 0$ and the atmosphere is slantwise unstable.



ESCI 343 – Atmospheric Dynamics II
Lesson 15 – Barotropic and Baroclinic Instability

Reference: *Numerical Prediction and Dynamic Meteorology (2nd edition)*, G.J. Haltiner and R.T. Williams
An Introduction to Dynamic Meteorology (3rd edition), J.R. Holton
Dynamics of the Atmosphere: A Course in Theoretical Meteorology, W. Zdunkowski and A. Bott

HYDRODYNAMIC INSTABILITY

A flow is hydrodynamically unstable if a small perturbation in the flow grows spontaneously. Examples of hydrodynamic instability that we've already studied are buoyant instability and inertial instability. In both these cases an air parcel moved from its original position will continue to accelerate away from where it started, instead of oscillating around its original position.

One method of assessing whether or not a flow is stable or unstable is by assuming that the perturbation has a sinusoidal waveform such as

$$\psi' = Ae^{i(kx-\omega t)}$$

and determining under what circumstances the frequency will have an imaginary component. If the dispersion relation has an imaginary component such as

$$\omega = \omega_r + i\omega_i ,$$

then the perturbation will have the form

$$\psi' = Ae^{i(kx-\omega t)} = Ae^{i(kx-\omega_r t)}e^{k\omega_i t}$$

which grows exponentially in time (and is therefore unstable) if $\omega_i > 0$.

An example of hydrodynamic instability: Internal gravity waves with imaginary Brunt -Vaisala frequency

Recall that the phase speed for internal gravity waves is

$$\omega = \pm\frac{N\mathrm{K}_H}{\mathrm{K}} ,$$

where N is the Brunt-Vaisala frequency given by

$$N^2 = -\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz} - \frac{g^2}{c_s^2} .$$

If N is real then the fluid is stable, and a parcel disturbed vertically from rest would oscillate about its original position. However, if N is imaginary then we know a parcel will be unstable, and if perturbed from rest it will accelerate away from its original position. This can also be seen from the dispersion relation, since N will be imaginary, and hence ω will have an imaginary component.

BAROTROPIC INSTABILITY

One form of hydrodynamic instability that can occur in the atmosphere is barotropic instability. The derivation of the condition for barotropic instability is beyond the scope of this course. But, the condition for barotropic instability involves the horizontal shear of the mean wind. The necessary condition for barotropic instability to occur is that, somewhere within the flow, the following condition must be true:

$$\frac{d^2\overline{u}}{dy^2} - \beta = 0\,. \tag{1}$$

This means that for barotropic instability to occur that the second derivative of the mean zonal wind must be equal to β somewhere in the flow. Condition (1) can also be written as

$$\frac{d}{dy}\left(\frac{d\overline{u}}{dy} - f\right) = 0 \tag{2}$$

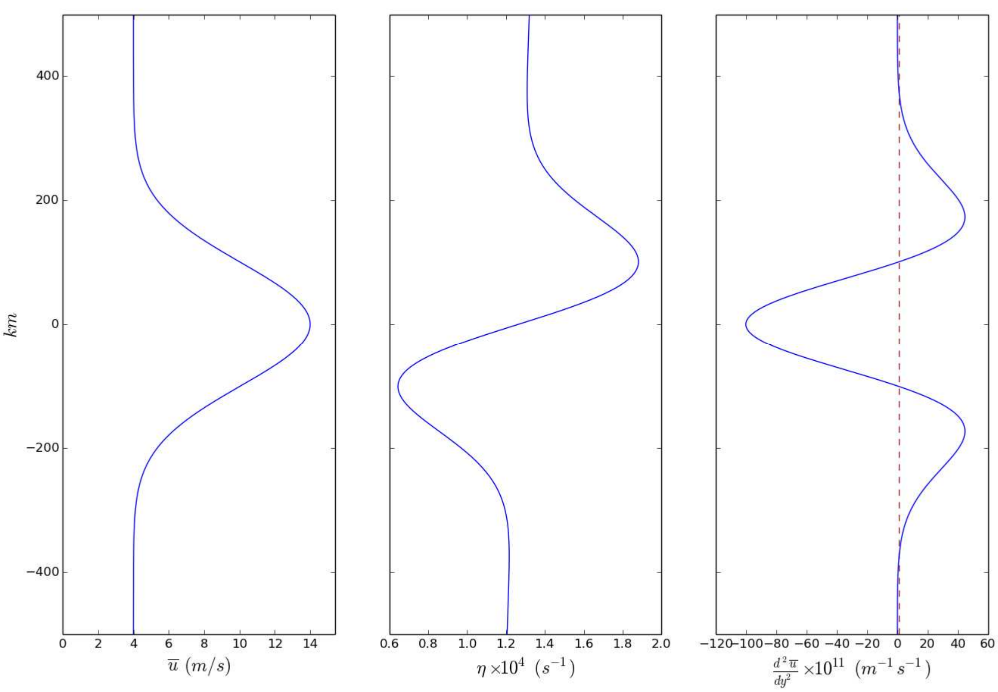
or

$$\frac{d\eta}{dy} = 0\,. \tag{3}$$

We can interpret this to mean that the absolute vorticity must have a minimum or maximum value somewhere in the flow in order for barotropic instability to occur.

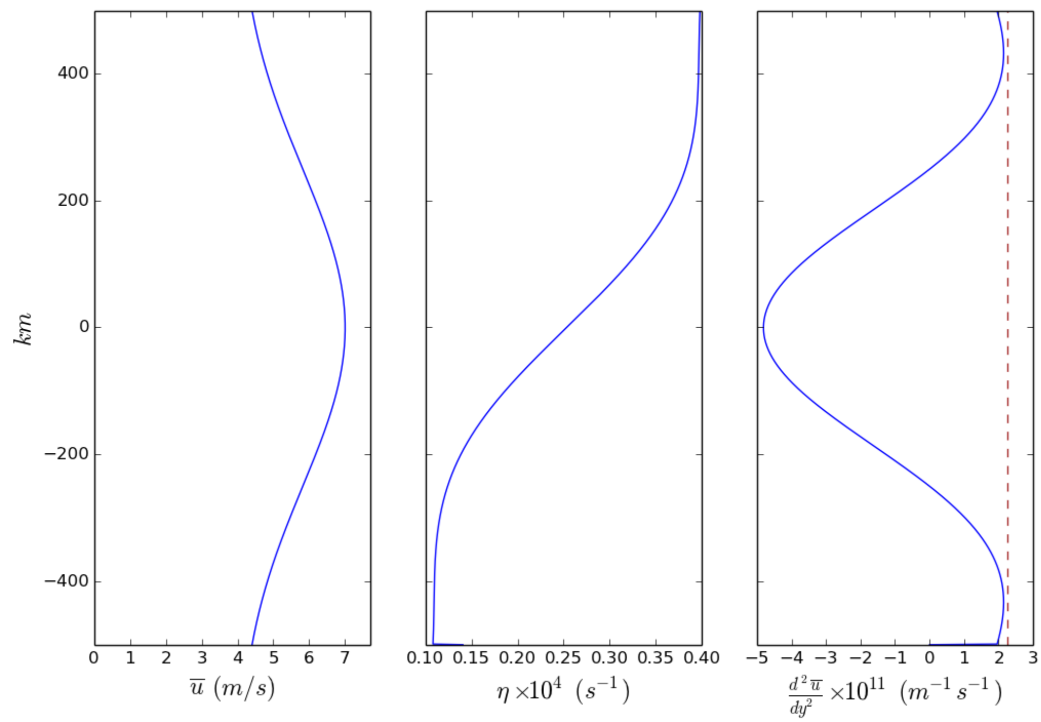
BAROTROPIC INSTABILITY IN A WESTERLY JET STREAM

Barotropic instability is dependent upon horizontal shear of the mean flow. To examine if barotropic instability is possible, the horizontal profile of the absolute vorticity must be examined. The plot below shows the zonal velocity, absolute vorticity, and the second derivative of the velocity for an idealized westerly jet stream on the beta plane. The dashed line on the third diagram is the value of beta.



There are absolute vorticity minima and maxima on both flanks of the jet, near the locations of the inflection points in the velocity profile. Thus, the condition for barotropic instability is met in these two regions.

However, the presence of an inflection point does not automatically mean that there is a minimum or maximum in the absolute vorticity. If beta is large compared to the second derivative of the velocity, such as for a broad, weak jet at low latitudes, as shown below, then there will not be any maxima or minima in vorticity, even though there are inflection points in the velocity profile. Thus, beta acts as stabilizing influence against barotropic instability.



ENERGETICS OF BAROTROPIC DISTURBANCES

Barotropic disturbances derive their energy from the mean flow. Energy considerations show that ***for a barotropic disturbance to grow it must tilt opposite to*** $d\overline{u}/dy$.[1] Since midlatitude disturbances tend to tilt in the <u>same</u> direction as $d\overline{u}/dy$, they actually lose energy back to the mean flow due to barotropic instability. Thus, barotropic instability is not a viable way for midlatitude disturbances to form and grow. However, interestingly enough, ***since midlatitude disturbance decay due to barotropic instability, they give up energy to the mean flow and help maintain the mean flow against friction***. Thus, barotropic instability is somewhat important for the maintenance of the mean flow in the midlatitudes.

BAROCLINIC INSTABILITY IN A TWO-LAYER FLUID

[1] See Haltiner and Williams, pp.74-75.

Since barotropic instability is not a viable option for the formation of midlatitude cyclones, then another mechanism must be invoked. This mechanism is ***baroclinic instability***. For baroclinic instability it is the ***vertical shear***, rather than the horizontal shear, that is important.

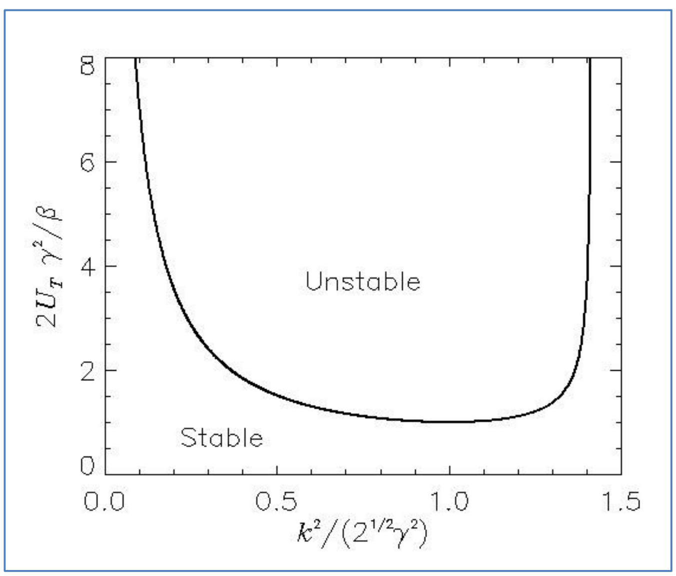
Baroclinic instability is often studied for the simple case of a two-layer fluid, for which waves are unstable if the following condition is true:

$$\delta = \frac{\beta^2\gamma^4}{k^4\left(k^2+2\gamma^2\right)^2} - \frac{U_T^2\left(2\gamma^2-k^2\right)}{\left(k^2+2\gamma^2\right)} < 0\,. \qquad \textbf{(4)}$$

In this expression, U_T is the vertical wind shear parameter (equal to half the difference in U between the two layers, and γ is inversely proportional to static stability (large stability means small γ). This expression points out the importance of vertical wind shear on baroclinic instability. There is also a wavelength dependence for baroclinic instability, which is best illustrated by writing Eq. (4) in terms of the dimensionless variables

$$\tilde{U} = \frac{2U_T\gamma^2}{\beta} \qquad \tilde{K} = \frac{k^2}{\sqrt{2}\gamma^2}\,,$$

where $\tilde{U}$ is proportional to vertical shear and $\tilde{K}$ is proportional to wavenumber, and plotting the curve of $\tilde{U}$ versus $\tilde{K}$ for the neutrally stable case of δ = 0. This plot is shown below, and demonstrates that for small values of shear the flow is stable, but as shear increases, instability will set in when $\tilde{U}$ is greater than unity. The plot also shows that there will be a certain wave number ($\tilde{K}$ = 1.0) at which the instability will first occur.



BAROCLINIC INSTABILITY IN THE ATMOSPHERE

Analysis of baroclinic instability in the real (continuously stratified) atmosphere is much more complicated than for the two-layer fluid. Qualitatively the results are similar, with instability depending on the vertical shear. However, in the real atmosphere there is always an unstable wave number, so barotropic instability is pervasive in the middle latitudes. The amount of instability, and growth rates, increase with the amount of wind shear and other factors. Baroclinic instability in the regions of strong horizontal thermal gradients is the mechanism by which midlatitude cyclones form. Baroclinic instability (combined with barotropic instability) is also important for the formation of African Easterly waves.