1. Modelling
   a) Objective, minimize the integral of squared accelerate → 目标函数

$$J=\int_0^T g(x,u)dt=\int_0^T(1+u^TRu)dt=\int_0^T(1+a_x^2+a_y^2+a_z^2)dt$$

   b) State, Input and System equation

$$x=\begin{pmatrix}p_x\\p_y\\p_z\\v_x\\v_y\\v_z\end{pmatrix}, u=\begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix}, \dot{x}=f(x,u)=\begin{pmatrix}v_x\\v_y\\v_z\\a_x\\a_y\\a_z\end{pmatrix}$$

2. Solving
   a) Costate $\lambda=(\lambda_1\quad\lambda_2\quad\lambda_3\quad\lambda_4\quad\lambda_5\quad\lambda_6)^T$
   b) Define the Hamiltonian function

$$H(x,u,\lambda)=g(x,u)+\lambda^T f(x,u),$$
$$H(x,u,\lambda)=\left(1+a_x^2+a_y^2+a_z^2\right)+\lambda^T f(x,u) = (1+a_x^2+a_y^2+a_z^2) + \lambda_1 v_x + \lambda_2 v_y + \lambda_3 v_z + \lambda_4 a_x + \lambda_5 a_y + \lambda_6 a_z$$
$$\dot{\lambda}=-\nabla H(x^*,u^*,\lambda)=(0\quad 0\quad 0\quad -\lambda_1\quad -\lambda_2\quad -\lambda_3)^T$$

$$\dot{\lambda} = -\nabla_x H(x^*, u^*, \lambda)$$
$$= -\nabla_x\left(1+a_x^{*2}+a_y^{*2}+a_z^{*2}+\lambda_1 V_x+\lambda_2 V_y+\lambda_3 V_z+\lambda_4 a_x^*+\lambda_5 a_y^*+\lambda_6 a_z^*\right)$$
$$= (0,0,0,-\lambda_1,-\lambda_2,-\lambda_3)$$

   c) The costate is solved as

$$\lambda=\begin{pmatrix}2\alpha_1\\2\alpha_2\\2\alpha_3\\-2\alpha_1 t-2\beta_1\\-2\alpha_2 t-2\beta_2\\-2\alpha_3 t-2\beta_3\end{pmatrix}$$

$\lambda_1=\alpha_1$
$\lambda_2=\alpha_2$
$\lambda_3=\alpha_3$
$\dot{\lambda}_4=-\lambda_1=-\alpha_1$
$\lambda_4=-\alpha_1 t-\beta_1$
同理 $\lambda_5=-\alpha_2 t-\beta_2$
$\lambda_6=-\alpha_3 t-\beta_3$

   d) The optimal input is solved as

$$u^*=\arg\min_{a(t)} H(x^*(t),u(t),\lambda(t))=\begin{pmatrix}\alpha_1 t+\beta_1\\\alpha_2 t+\beta_2\\\alpha_3 t+\beta_3\end{pmatrix}$$

   e) The optimal state trajectory is solved as

$$x^*=\begin{pmatrix}\frac{1}{6}\alpha_1 t^3+\frac{1}{2}\beta_1 t^2+v_{x0}t+p_{x0}\\\frac{1}{6}\alpha_2 t^3+\frac{1}{2}\beta_2 t^2+v_{y0}t+p_{y0}\\\frac{1}{6}\alpha_3 t^3+\frac{1}{2}\beta_3 t^2+v_{z0}t+p_{z0}\\\frac{1}{2}\alpha_1 t^2+\beta_1 t+v_{x0}\\\frac{1}{2}\alpha_2 t^2+\beta_2 t+v_{y0}\\\frac{1}{2}\alpha_3 t^2+\beta_3 t+v_{z0}\end{pmatrix}, initial\ \ state: x(0)=\begin{pmatrix}p_{x0}\\p_{y0}\\p_{z0}\\v_{x0}\\v_{y0}\\v_{z0}\end{pmatrix}$$

代入 initial state

   f) α,β are solved as

$$u^* = \arg\min_{u(t)} H(x^*(t), u(t), \lambda(t))$$
$$= \arg\min_{a(t)}\left(1+a_x^2+a_y^2+a_z^2+\lambda_1 V_x^*+\lambda_2 V_y^*+\lambda_3 V_z^*+\lambda_4 a_x+\lambda_5 a_y+\lambda_6 a_z\right)$$
$$= \arg\min_{a(t)}\left(a_x^2+a_y^2+a_z^2+(-\alpha_1 t-\beta_1)a_x+(-\alpha_2 t-\beta_2)a_y+(-\alpha_3 t-\beta_3)a_z\right)$$

$$\Leftrightarrow \begin{bmatrix}2a_x-\alpha_1 t-\beta_1\\2a_y-\alpha_2 t-\beta_2\\2a_z-\alpha_3 t-\beta_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$a_x = -\frac{-\alpha_1 t-\beta_1}{2}\qquad a_y=-\frac{-\alpha_2 t-\beta_2}{2}\qquad a_z=-\frac{-\alpha_3 t-\beta_3}{2}$$

系数 ×2 就是文档里的答案

对 $a(t)$ 积分：$v(t) = \frac{1}{2}\alpha t^2 + \beta t + \gamma$ ($\gamma = V_0$)，$p(t) = \frac{1}{6}\alpha t^3 + \frac{1}{2}\beta t^2 + \gamma t + \theta$ ($\theta = P_0$)

代入 final state 求 α, β

$$\begin{pmatrix} \frac{1}{6}T^3 & 0 & 0 & \frac{1}{2}T^2 & 0 & 0 \\ 0 & \frac{1}{6}T^3 & 0 & 0 & \frac{1}{2}T^2 & 0 \\ 0 & 0 & \frac{1}{6}T^3 & 0 & 0 & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & 0 & 0 & T & 0 & 0 \\ 0 & \frac{1}{2}T^2 & 0 & 0 & T & 0 \\ 0 & 0 & \frac{1}{2}T^2 & 0 & 0 & T \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \Delta p_x \\ \Delta p_y \\ \Delta p_z \\ \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{pmatrix}$$

$P_f - P_0 = P(T) - P(0)$
$= \frac{1}{6}\alpha T^3 + \frac{1}{2}\beta T^2 + V_0 T$

$$\begin{pmatrix} \Delta p_x \\ \Delta p_y \\ \Delta p_z \\ \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{pmatrix} = \begin{pmatrix} p_{xf} - v_{x0}T - p_{x0} \\ p_{yf} - v_{y0}T - p_{y0} \\ p_{zf} - v_{x0}T - p_{z0} \\ v_{xf} - v_{x0} \\ v_{yf} - v_{y0} \\ v_{zf} - v_{z0} \end{pmatrix}$$

$V_f - V_0 = V(T) - V(0)$
$= \frac{1}{2}\alpha T^2 + \beta T$

定义 $\begin{cases} \Delta P = P_f - P_0 - V_0 T \\ \Delta V = V_f - V_0 \end{cases}$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} -\frac{12}{T^3} & 0 & 0 & \frac{6}{T^2} & 0 & 0 \\ 0 & -\frac{12}{T^3} & 0 & 0 & \frac{6}{T^2} & 0 \\ 0 & 0 & -\frac{12}{T^3} & 0 & 0 & \frac{6}{T^2} \\ \frac{6}{T^2} & 0 & 0 & -\frac{2}{T} & 0 & 0 \\ 0 & \frac{6}{T^2} & 0 & 0 & -\frac{2}{T} & 0 \\ 0 & 0 & \frac{6}{T^2} & 0 & 0 & -\frac{2}{T} \end{pmatrix} \begin{pmatrix} \Delta p_x \\ \Delta p_y \\ \Delta p_z \\ \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{pmatrix}$$

V_f 不确定时
需设计 $h(s(T))$
$\lambda = -\nabla h(s^*(T))$
$\downarrow$
$\lambda_{1,2,3} = -\frac{\partial h(s^*(T))}{\partial p_{1,2,3}}$
$\lambda_{4,5,6} = 0$
$\Downarrow$
$\alpha_{1,2,3} = -\frac{\beta_{1,2,3}}{T}$
$\alpha_{1,2,3}$ 已知
可求出 $\beta_{1,2,3}$

则

g) The cost

$$J = \int_0^T (1 + a_x^2 + a_y^2 + a_z^2)dt$$

$a^2 = \alpha^2 T^2 + \beta^2 + 2\alpha\beta T$
积分

$$J = T + \left(\frac{1}{3}{\alpha_1}^2T^3 + \alpha_1\beta_1T^2 + {\beta_1}^2T\right) + \left(\frac{1}{3}{\alpha_2}^2T^3 + \alpha_2\beta_2T^2 + {\beta_2}^2T\right)$$
$$+ \left(\frac{1}{3}{\alpha_3}^2T^3 + \alpha_3\beta_3T^2 + {\beta_3}^2T\right)$$

$\Delta P = \frac{1}{6}\alpha T^3 + \frac{1}{2}\beta T^2$
$\Delta V = \frac{1}{2}\alpha T^2 + \beta T$

h) J only depends on T, and the boundary states (known), so we can even get an optimal T!

i. You can use the Mathematica obtain the optimal analytic expression of T

ii. You can use Numerical calculation method obtain the optimal approximate solution of T

对任一维度

$\alpha = -\frac{12}{T^3}\cdot\Delta P + \frac{6}{T^2}\cdot\Delta V$

$\beta = -\frac{1}{2}\alpha T + \frac{\Delta V}{T}$

$\beta = \frac{6}{T^2}\Delta P - \frac{3}{T}\Delta V + \frac{\Delta V}{T}$

$\Delta P = \frac{1}{6}\alpha T^3 + \frac{1}{2}\left(-\frac{1}{2}\alpha T + \frac{\Delta V}{T}\right)T^2$

$= \frac{6}{T^2}\Delta P - \frac{2}{T}\Delta V$

$= \frac{1}{6}\alpha T^3 - \frac{1}{4}\alpha T^3 + \frac{1}{2}\Delta V\cdot T$
$= -\frac{T^3}{12}\alpha + \frac{1}{2}\Delta V\cdot T$

$$
\begin{aligned}
J = T + \Big[\; & \underbrace{\frac{1}{3}\left(-\frac{12}{T^3}\Delta P_x + \frac{6}{T^2}\Delta V_x\right)^2 T^3}_{=\frac{48}{T^3}\Delta P_x^2 + \frac{12}{T}\Delta V_x^2 - \frac{48}{T^2}\Delta P_x \Delta V_x} \\
& + \underbrace{\left(-\frac{12}{T^3}\Delta P_x + \frac{6}{T^2}\Delta V_x\right)\left(\frac{6}{T^2}\Delta P_x - \frac{2}{T}\Delta V_x\right) T^2}_{= -\frac{72}{T^3}\Delta P_x^2 - \frac{12}{T}\Delta V_x^2 + \frac{60}{T^2}\Delta P_x \Delta V_x} \\
& + \underbrace{\left(\frac{6}{T^2}\Delta P_x - \frac{2}{T}\Delta V_x\right)^2 T}_{= \frac{36}{T^3}\Delta P_x^2 + \frac{4}{T}\Delta V_x^2 - \frac{24}{T^2}\Delta P_x \Delta V_x} \;\Big]_x \\
& + \big[\cdots\big]_y + \big[\cdots\big]_z
\end{aligned}
$$

$$
\begin{aligned}
= T & + \frac{48-72+36}{T^3}\Delta P_x^2 + \frac{12-12+4}{T}\Delta V_x^2 + \frac{60-24-48}{T^2}\Delta P_x \Delta V_x \\
& + \cdots\cdots \Delta P_y^2 + \cdots\cdots \Delta V_y^2 + \cdots\cdots \Delta P_y \Delta V_y \\
& + \cdots\cdots \Delta P_z^2 + \cdots\cdots \Delta V_z^2 + \cdots\cdots \Delta P_z \Delta V_z
\end{aligned}
$$

$$
= T + \frac{12}{T^3}\underbrace{\left(\Delta P_x^2 + \Delta P_y^2 + \Delta P_z^2\right)}_{A} + \frac{4}{T}\underbrace{\left(\Delta V_x^2 + \Delta V_y^2 + \Delta V_z^2\right)}_{B} - \frac{12}{T^2}\underbrace{\left(\Delta P_x \Delta V_x + \Delta P_y \Delta V_y + \Delta P_z \Delta V_z\right)}_{C}
$$

$$
J' = 1 - \frac{36}{T^4}A - \frac{4}{T^2}B + \frac{24}{T^3}C
$$

令 $J' = 0 \iff \boxed{T^4 - 36A - 4T^2 B + 24TC = 0}$

由于 ΔP 中含 $V_0 T$

所以 $\downarrow$

$$\Delta P = P_f - V_0 T - P_0 \qquad \text{令} \quad d_{x,y,z} = P_{f\,x,y,z} - P_{0\,x,y,z}$$

$$\Delta P_{x,y,z} = d_{x,y,z} - V_{0\,x,y,z} T$$

$$J = T + \frac{12}{T^3}\left[dx^2 + dy^2 + dz^2 + T^2\left(V_{0x}^2 + V_{0y}^2 + V_{0z}^2\right) - 2T\left(V_{0x}dx + V_{0y}dy + V_{0z}dz\right)\right]$$

$$+ \frac{4}{T}\left(\Delta V_x^2 + \Delta V_y^2 + \Delta V_z^2\right)$$

$$- \frac{12}{T^2}\left(dx\,\Delta V_x + dy\,\Delta V_y + dz\,\Delta V_z - V_{0x}T\Delta V_x - V_{0y}T\Delta V_y - V_{0z}T\Delta V_z\right)$$

$$= \frac{12}{T^3}\underbrace{\left(dx^2 + dy^2 + dz^2\right)}_{A} + \frac{4}{T}\underbrace{\left(\Delta V_x^2 + \Delta V_y^2 + \Delta V_z^2\right)}_{B} - \frac{12}{T^2}\underbrace{\left(dx\,\Delta V_x + dy\,\Delta V_y + dz\,\Delta V_z\right)}_{C}$$

$$+ \frac{12}{T}\underbrace{\left(V_{0x}^2 + V_{0y}^2 + V_{0z}^2\right)}_{D} - \frac{24}{T^2}\underbrace{\left(V_{0x}dx + V_{0y}dy + V_{0z}dz\right)}_{E}$$

$$+ \frac{12}{T}\underbrace{\left(V_{0x}\Delta V_x + V_{0y}\Delta V_y + V_{0z}\Delta V_z\right)}_{F}$$

$$= T + \frac{12A}{T^3} - \frac{12C + 24E}{T^2} + \frac{4B + 12D + 12F}{T}$$

$$J' = 1 - \frac{36A}{T^4} + \frac{24C + 48E}{T^3} - \frac{4B + 12D + 12F}{T^2}$$

$$\Leftrightarrow T^4 - 36A + (24C + 48E)T - (4B + 12D + 12F)T^2 = 0$$

求根即可