Stanford Cline
Discrete Math
Jan 19, 2018

1. Let R be the set of positive real numbers and define addition, denoted by $\oplus$, and multiplication, denoted by $\otimes$, as follows. For every $a, b \in R$, $a \oplus b = ab$, and $a \otimes b = a^{\log b}$. Please prove or disprove $(R, \oplus, \otimes)$.

axiom -1: since $R \in R^+$, $\oplus$ is closed in R. There is no positive real number for which $a\dot{b} \in R^+$

axiom 0: since $R \in R^+$ and $\otimes$ is defined as $a \otimes b = a^{\log b}$, starting with the lowest possible value, $a, b = 1$, $1^{log 1}$ results in 1, which is included in R^+. as both values approach infinity the result also approaches infinity which is a positive real number.

axiom 1: addition is commutative, for all values a, b, $a \cdot b = b \cdot a$

axiom 2: addition is associative, for all values a, b, c, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

axiom 3: there exists an additive identity. since the $\otimes$ is defined as $a \oplus b = ab$, the additive identity is 1. $1 \cdot x = x \cdot 1 = x$.

axiom 4: There is an additive inverse. Since the additive identity is 1, there must be a value for every $x \in R^+$ where $x \cdot y = 1$. This would be the inverse of of the number x. For example: $x = 3$, $x \cdot x^{-1} = 3 \cdot 1/3 = 1$. Since both $x, x^{-1} \in R$ axiom 4 holds.

axiom 5: multiplication is commutative. $a \otimes bi = a^{\log b}$. This is commutative because if you take log_a of both sides it resutls in $logb = log_a b^{loga}$. Breaking apart the right side, $\frac{loga \cdot logb}{loga}$ in which both $loga$ cancel each other out and you're left with $logb = logb$. Since they are equal, $\otimes$ is commutative.

axiom 6: Multiplication is associative. Given a, b, c: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ on the left, the resultant is $(a^{logb}) \otimes c = a^{logb \cdot logc}$, and on the right side, $a \otimes b^{logc} = a^{logb \cdot logc}$. Since these are equal, multiplication is associative.

axiom 7: there exists a multiplicatave identity. This axiom holds true for the value 10. $(a^{log10} = 10^{loga} = a)$

axiom 8: there exists a multiplicatave inverse. since the multiplicatave identity is 10, the multiplicatave inverse will be a number where $a^{logb} = 10$ solving for be, $logb = log_a 10$, and solving for b, $b = 10^{log_a 10}$. Since $a \in R^+$, $log_a 10$ is positive for all values a. This means that 10^{R^+} and will result in a positive real number meaning that for all values a, there exists another number $b \in R^+$ for the multiplicatave inverse.

axiom 9: Multiplication distributes over addition. If $x, y, z \in R$, then $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = z \cdot x + z \cdot y$. With our redefined definition of both addition and multiplication, $x \cdot (y + z) = x \otimes (y \cdot z) = x^{logy \cdot logz}$ and on the right side of the equation $x \cdot y + x \cdot z = x^{logy} \cdot x^{logz} = x^{logy \cdot logz}$. Since they are equal on both sides of addition, axiom 9 holds.

axiom 10: the additive identity and multiplicatave identity are not equal. For our redefined $\oplus, \otimes$ the additive and multiplicatave identities were 1 and 10 respectively. since $1 \neq 10$ axiom 10 holds.

since all axioms hold, our redefined field $(R, \oplus, \otimes)$ is a field.

2. Denote the set 0,1,2,3 in Z_4 and define addition, denoted by $+$, and multiplication, denoted by $\cdot$ or juxtapostion, via the following tables:

axiom 8 states that for every number in Z_4 there is a multiplicative inverse where $a \cdot a^{-1} =$ multiplicatave inverse $= 1$. 2 from the range Z_4 has no multiplicatave inverse because there exists no other number, b, in Z_4 where $a \cdot b = 1$.

4. Please prove or disprove: if $n \in Z^+$, then $n^2 + n + 41$ is prime:

question 4 asserts that if $n \in Z^+$, then for all values n, $n^2 + n + 41$ will be prime. In order to disprove this, there must be some value for n where the output isn't prime. take $n = 40$, this results in $40^2 + 40 + 41 = 1681$. 1681 is not prime because 41 is a factor of 1681. Therefore $n^2 + n + 41$ is not prime for all $n \in Z^+$.