CS 3430: Sprint 2019: SciComp with Py
Lecture 7
Differentiation of Exponential and Logarithmic Functions

Vladimir Kulyukin
Department of Computer Science
Utah State University

Review

Chain Rule

Chain rule allows us to differentiate function compositions.

To differentiate $f(g(x))$, differentiate first the outside function $f(x)$, substitute $g(x)$ for x in the result, and multiply by the derivative of the inside function $g(x)$.

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

Alternative Notation for the Chain Rule

Let $y = f(g(x))$ and $u = g(x)$. Let $\frac{dy}{du} = f'(u) = f'(g(x))$ and $\frac{du}{dx} = g'(x)$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Here is a quick example to understand this notation: if y varies 4 times as fast as u and u varies 3 times as fast as x, then y varies 12 times as x. In other words,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 4 \cdot 3 = 12.$$

Implicit Differentiation

In some applications, the variables are related by an equation rather than a function, e.g., $x^2 + y^2 = 4$ or $x^2y^3 = 10$.

In such cases, the technique that allows us to determine the rate of change of one varible with respect to the other is called **implicit differentiation**.

Think of y as some function of x *implicitly* defined by the equation and use the chain rule to differentiate.

Related Rates

In implicit differentiation, y is treated as an unknown function of x.

In some applications, y and x are related via an equation but both are functions of a third variable t, which typically refers to time.

When we differentiate such an equation with respect to t, we end up with a new equation that relates $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

These quantities, $\frac{dy}{dt}$ and $\frac{dx}{dt}$, are called **related rates**.

Exponential Functions

Introduction

When a bacteria culture grows in a laboratory dish, the rate of growth of the culture at any moment is proportional to the total number of bacteria in the dish. In science, this is called **exponential growth**.

Radioactive materials typically decay at rates proportional to their present amounts. This is called **exponential decay**.

These models are used in many areas of science such as biology, archeology, nuclear physics, economics, and public health.

Laws of Exponents

1) $b^x \cdot b^y = b^{x+y}$;

2) $b^{-x} = \frac{1}{b^x}$;

3) $\frac{b^x}{b^y} = b^x \cdot b^{-y}$;

4) $\frac{b^x}{b^y} = b^x \cdot b^{-y} = b^{x-y}$;

5) $(b^x)^y = b^{xy}$;

6) $a^x b^x = (ab)^x$;

7) $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$.

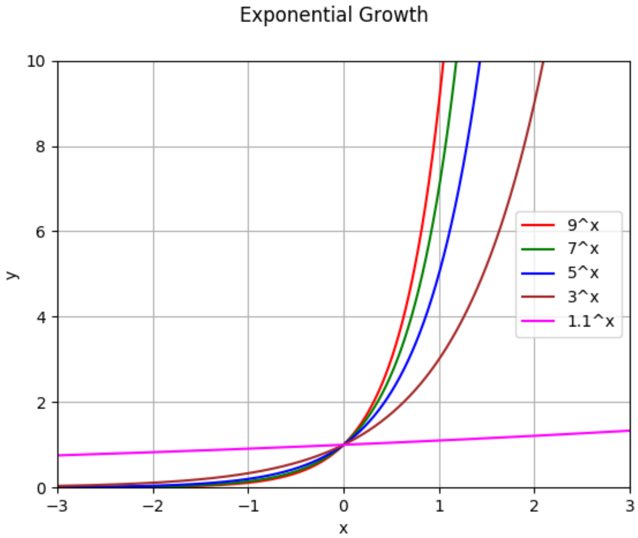
Exponential Function: Definition

Let $b \in \mathbb{R}^+$. Then the function

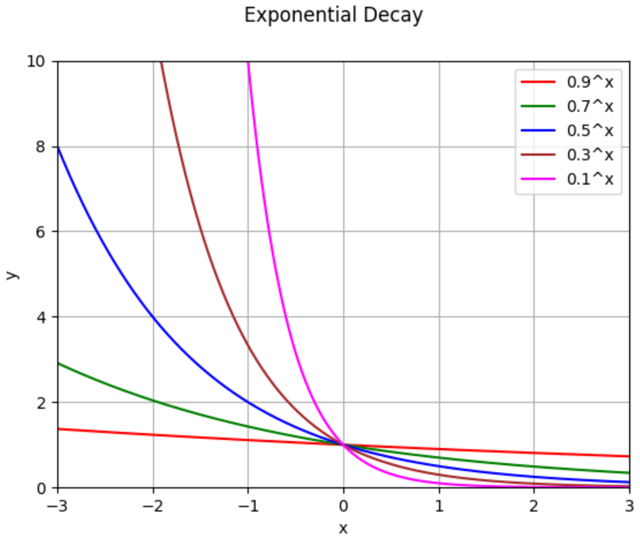
$$f(x) = b^x,$$

is called an **exponential function**.

Exponential Growth



Exponential Decay



The Exponential Function e^x

All exponential growth and decay functions pass through $(0, 1)$.

The exponential functions have different slopes at this point (e.g., 2^x has a slope of 0.693 and 3^x has a slope of 1.1).

There must be an exponential function whose slope is exactly 1.

This function is known as $f(x) = e^x$, where $e = 2.718281828$.

Approximating Slopes of b^x at $(0, 1)$

We can use the secant line method to approximate the slope of b^x at $(0, 1)$ for various values of b.

Let the slope be m. If $f(x) = b^x$, then at $(0, 1)$, m is

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{b^h - 1}{h},$$

$h = 0.1, 0.01, 0.001, \ldots.$

Approximating Slopes of 2^x at $(0, 1)$

We can define two simple functions to approximate the slope of 2^x at $(0, 1)$.

```
def slope_of_b_to_x(b):
  for d in range(1, 11):
    h = 1.0/(10.0**d)
    print((b**h-1.0)/h)

def slope_of_2_to_x():
  print('\nApproximating f\'(x) = 2^x, at (0, 1):')
  slope_of_b_to_x(2.0)
  print('********')
```

```
Approximating f'(x) = 2^x, at (0, 1):
0.717734625363
0.695555005672
0.693387462581
...
```

Approximating Slopes of 3^x at $(0, 1)$

We can define two simple functions to approximate the slope of 3^x at $(0, 1)$.

```
def slope_of_b_to_x(b):
  for d in range(1, 11):
    h = 1.0/(10.0**d)
    print((b**h-1.0)/h)

def slope_of_3_to_x():
  print('\nApproximating f\'(x) = 3^x, at (0, 1):')
  slope_of_b_to_x(3.0)
  print('********')
```

```
Approximating f'(x) = 3^x, at (0, 1):
1.16123174034
1.10466919379
1.099215984
...
```

Approximating Slopes of 4^x at $(0, 1)$

We can define two simple functions to approximate the slope of 4^x at $(0, 1)$.

```
def slope_of_b_to_x(b):
  for d in range(1, 11):
    h = 1.0/(10.0**d)
    print((b**h-1.0)/h)

def slope_of_4_to_x():
  print('\nApproximating f\'(x) = 4^x, at (0, 1):')
  slope_of_b_to_x(4.0)
  print('********')
```

```
Approximating f'(x) = 4^x, at (0, 1):
1.48698354997
1.395947979
1.38725571133
...
```

Derivative of $y = b^x$

$$\frac{d}{dx}(b^x) = mb^x,$$

where $m = \frac{d}{dx}(b^x)$ at $x = 0$.

Derivative of $y = e^x$

$$\frac{d}{dx}(e^x) = m \cdot e^x = 1 \cdot e^x = e^x,$$

because $m = \frac{d}{dx}(e^x) = 1$ at $x = 0$.

Approximating Slopes of e^x at $(0, 1)$

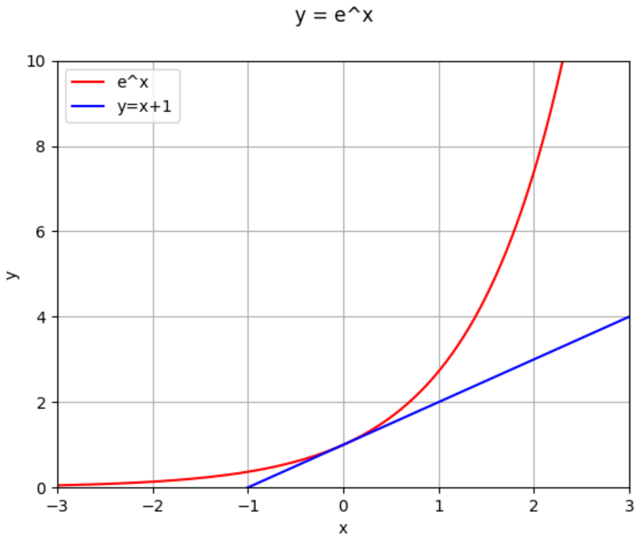
We can define two simple functions to approximate the slope of e^x at $(0, 1)$.

```
def slope_of_b_to_x(b):
  for d in range(1, 11):
    h = 1.0/(10.0**d)
    print((b**h-1.0)/h)

def slope_of_e_to_x():
  print('\nApproximating f\'(x) = e^x, at (0, 1):')
  slope_of_b_to_x(math.e)
  print('********')

Approximating f'(x) = e^x, at (0, 1):
1.05170918076
1.00501670842
1.00050016671
...
```

Graph of $y = e^x$ and Tangent Line at $(0, 1)$



Example

Find the tangent line to the graph of e^x when $x = 1$.

When $x = 1$, $y = f(1) = e$. Then $\frac{d}{dx}e^x = e^x$. So, the slope of the tangent line at $(1, e)$ is e.

Using the point-slope formula, we have

$$y - e = e(x - 1) \text{ or } y = ex.$$

Problem 1

Differentiate $(1+x^2)e^x$ and $\frac{1+e^x}{2x}$.

Solution

1. $\frac{d}{dx}(1+x^2)e^x = e^x(x^2+2x+1) = e^x(x+1)^2$;

2. $\frac{d}{dx}\left(\frac{1+e^x}{2x}\right) = \frac{xe^x - e^x - 1}{2x^2}$.

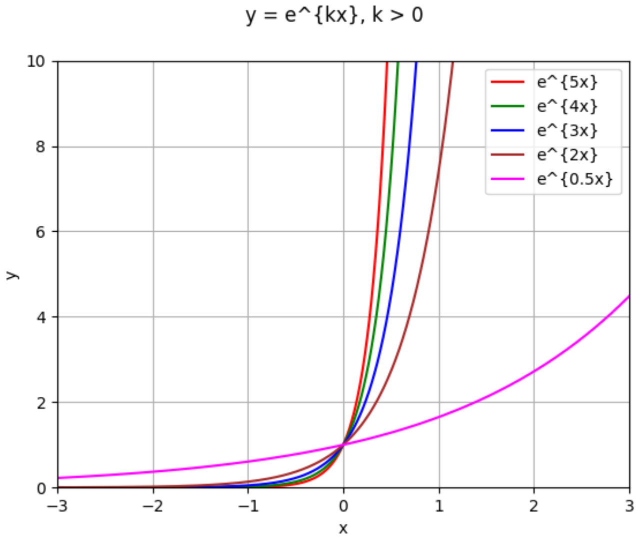
Problem 2

Compute $\frac{d}{dx}e^{-x}$.

Solution

$$\frac{d}{dx}e^{-x} = \frac{d}{dx}(e^x)^{-1} = (-1)(e^x)^{-2}\frac{d}{dx}e^x = -\frac{e^x}{(e^x)^2} = -\frac{1}{e^x} = -e^{-x}.$$

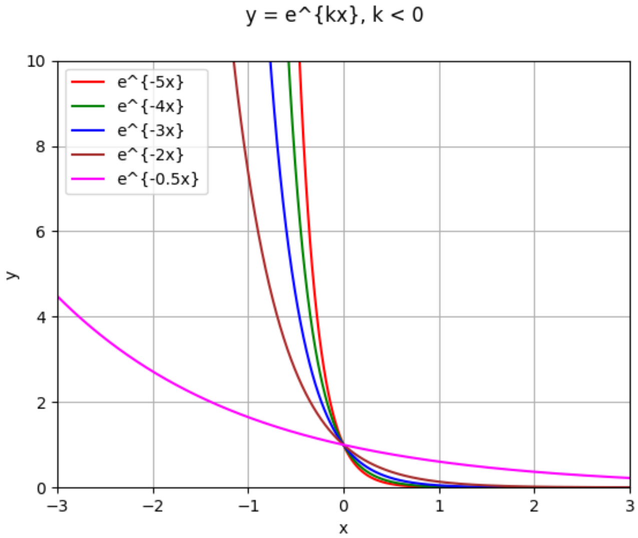
Graph of $y = e^{kx}$, $k > 0$



Properties of $y = e^{kx}$, $k > 0$

1. $(0, 1)$ is on the graph;
2. The graph of e^x is strictly above the x-axis;
3. The x-axis is an asymptote as x becomes negative;
4. The graph is always increasing and concave up.

Graph of $y = e^{kx}$, $k < 0$



Properties of $y = e^{kx}$, $k < 0$

1. $(0, 1)$ is on the graph;
2. The graph of e^x is strictly above the x-axis;
3. The x-axis is an asymptote as x becomes positive;
4. The graph is always decreasing and concave up.

Functions $f(x) = b^x, b > 0$ and $f(x) = e^x$

Let $b \in \mathbb{R}^+$. Then there is some value of $x = k$ such that $e^k = b$.

$f(x) = b^x = (e^k)^x = e^{kx}$.

Conclusion: any function $f(x) = b^x$ can be written as $f(x) = e^{kx}$, for some value of k. Hence, it is conceptually and implementationally advantageous to focus on using functions $f(x) = e^{kx}$ instead of studying 2^x, 3^x, 0.5^x, etc.

Chain Rule for Exponential Functions

Let $g(x)$ be any differentiable function. Then

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}\frac{d}{dx}g(x).$$

If we write $u = g(x)$, then the above equation can be written as

$$\frac{d}{du}\left(e^{u}\right) = e^{u}\frac{d}{dx}u.$$

Problem 3

Compute

1. $y = e^{5x}$;
2. $y = e^{x^2-1}$;
3. $y = e^{x-\frac{1}{x}}$.

Solution

1. $\frac{d}{dx}e^{5x} = e^{5x}\frac{d}{dx}5x = 5e^{5x}$;
2. $\frac{d}{dx}e^{x^2-1} = e^{x^2-1}\frac{d}{dx}(x^2 - 1) = 2xe^{x^2-1}$;
3. $\frac{d}{dx}e^{x-\frac{1}{x}} = e^{x-\frac{1}{x}}\frac{d}{dx}\left(x - \frac{1}{x}\right) = e^{x-\frac{1}{x}}(1 + \frac{1}{x^2})$.

Useful Differentiation Formula

Let C and k be arbitrary constants. Then

$$\frac{d}{dx}\left(Ce^{kx}\right) = kCe^{kx}.$$

Problem 4

Compute $\frac{d}{dx}\frac{3e^{2x}}{1+x^2}$.

Solution

$$\frac{d}{dx}\left(\frac{3e^{2x}}{1+x^2}\right)=\frac{(1+x^2)\frac{d}{dx}(3e^{2x})-(3e^{2x})\frac{d}{dx}(1+x^2)}{(1+x^2)^2}=$$

$$6e^{2x}\left(\frac{x^2-2x+1}{(1+x^2)^2}\right).$$

Problem 5

The highest price ever paid for an artwork at auction was for Pablo Picasso's 1955 painting *Les Femmes d'Algier*. The artwork fetched \$179.4 million in a Christie's auction in 2015. Prior to this sale, the painting was last sold in 1997 for \$31.9 million. If the painting keeps appreciating at its current rate, then a model for its value can be approximated by a model $f(t) \approx 31.87e^{0.096t}$, where $f(t)$ is in millions of dollars and t is the number of years since 1997.

1) At what rate was the painting appreciating in 2015?

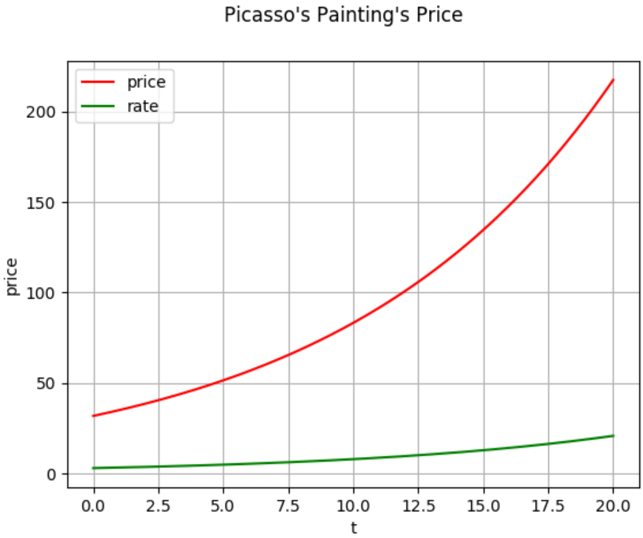
2) What will be the price of the painting in 2020 and the rate at which it will be appreciating?

Solution

1) $f'(t) = 31.87 \cdot 0.096 \cdot e^{0.096t}$. Then $f'(2015 - 1997) = f(18) = 31.87 \cdot 0.096 \cdot e^{0.096 \cdot 18} \approx 17.22$. So, in 2015, the painting was appreciating at $\approx$ \$17,220,000 a year.

2) $f(2020 - 1997) = f(23) = 31.87 \cdot e^{0.096 \cdot 23} = 289.94$. So, the price of the painting in 2020 is $\approx$ \$289,940,000.
$f'(2020 - 1997) = f(23) = 31.87 \cdot 0.096 \cdot e^{0.096 \cdot 23} \approx 27.83$. So, in 2020, the painting will be appreciating at $\approx$ \$27,830,000 per year.

Picasso's Painting Price

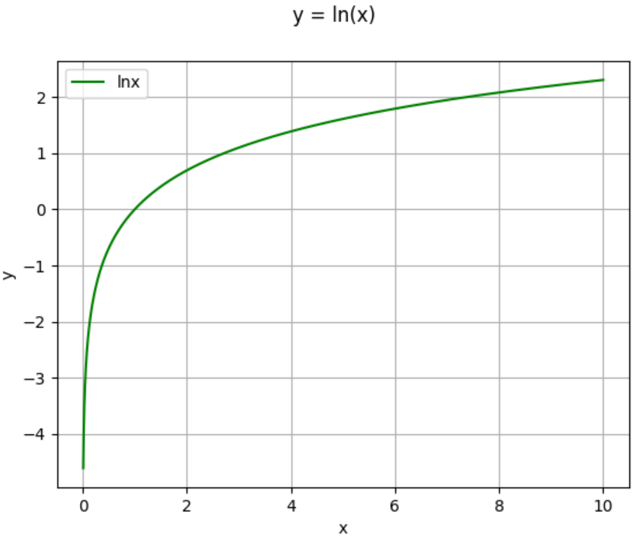


The Natural Logartihm Function

Definition

For $x > 0$, $y = lnx$ if and only if $x = e^y$.

The Graph of $y = lnx$



Problem 6

Solve $5e^{x-3} = 4$.

Solution

$$5e^{x-3} = 4;$$

$$e^{x-3} = 0.8;$$

$$x - 3 = ln(0.8);$$

$$x = ln(0.8) + 3;$$

References

1. L. Goldstein, D. Lay, D. Schneider, N. Asmar. *Calculus and its Applications*. Ch. 4, Pearson.
2. `www.python.org`.