CS 3430: S19: SciComp with Py
Lecture 14
The Net Change of a Function, Reimann Sums and Consumer's Surplus

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Review

Definition of Antiderivative

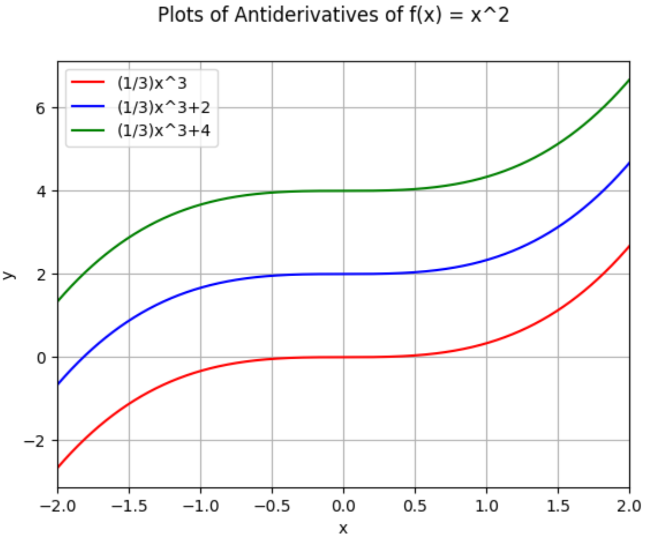
Suppose that $f(x)$ is a function and $F(x)$ is a function such that $F'(x) = f(x)$. Then $F(x)$ is an **antiderivative** of $f(x)$.

Antidifferentiation Theorem 1

Suppose that $f(x)$ is a continuous function on an internal I. If $F_1(x)$ and $F_2(x)$ are two antiderivatives of $f(x)$, then $|F_1(x) - F_2(x)| = C$, for $x \in I$.

Geometrically speaking, the graph of $F_2(x)$ can be obtained from $F_1(x)$ by a vertical shift (upward or downard) of $F_1(x)$.

Example of Antidifferentiation Theorem 1



Antidifferentiation Theorem 2

If $F'(x) = 0$ for all x in an interval I, there exists a constant C such that $F(x) = C$ for all x in I.

Verification of Theorem 1 with Theorem 2

Let $F_1(x)$ and $F_2(x)$ be two antiderivatives of $f(x)$.

Let $F(x) = F_2(x) - F_1(x)$. Then,

$$F'(x) = F_2'(x) - F_1'(x) = f(x) - f(x) = 0.$$

Thus, by Theorem 2, $F(x) = C$ and $F_2(x) = F_1(x) + C$.

Definition of Indefinite Integral

If $f(x)$ is a function whose antiderivatives are $F(x) + C$, then

$$\int f(x)dx = F(x) + C,$$

where $\int f(x)dx$ is called an **indefinite integral**.

The function $f(x)$ in $\int f(x)dx$ is called the **integrand**.

Antidifferentiation Rules for Powers and Exponentials

$$\int x^r dx = \frac{1}{r+1}x^{r+1} + C,\ r \neq -1.$$

$$\int e^{kx} dx = \frac{1}{k}e^{kx} + C,\ k \neq 0.$$

Antidifferentiation Rules for Logs

$$\int \frac{1}{x} dx = ln|x| + C, \; x \neq 0.$$

Antidifferentiation Rules for Sums and Constant Multiples

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx.$$

$$\int kf(x)dx = k \int f(x)dx, \; k \text{ is a constant.}$$

Net Change of a Function

Definition of Net Change

Suppose that $F' = f$. Then,

$$\int_a^b F'(x)dx = F(b) - F(a).$$

F' is the rate of change of F. The integral of the rate of change of F is the **net change** of F as x varies from a to b.

Example from Classical Mechanics

Suppose an object is moving in a straight line and let $s(t)$ be the object's position at time t as measured from a fixed referenced point.

Let $v(t)$ be the object's velocity. The net change in position of the object as t varies from a to b is

$$\int_a^b s'(x)dt = \int_a^b v(t)dt = s(b) - s(a).$$

Problem: Net Change in Position

The velocity at time t of an object moving in a straight line is $v(t) = 4t - 1$ m/s. What is the displacment of the object during the time interval $1 \leq t \leq 3$.

Solution

Let $s(t)$ be the object's position. Then, $s'(t) = v(t) = 4t - 1$ and

$$\int_1^3 s'(t)dt = \int_1^3 (4t-1)dt = 2t^2 - t\Big|_1^3 = (2(3^2)-3) - (2(1^2)-1) = 14.$$

The object moved 14 m to the right as t changed from 1 to 3.

Problem: Marginal Revenue Analysis

A company's marginal revenue function is $R'(x) = 0.03x^2 - 2x + 25$ dollars per unit, where x is the number of units produced in 1 day. Determine the net change in revenue if the production level is raised from $x = 20$ to $x = 25$ units.

Solution

Let $R(x)$ be the revenue function. Then, the net change in revenue is

$$\int_{20}^{25} R'(x)dx = \int_{20}^{25}(0.03x^2 - 2x + 25)dx = 0.01x^3 - x^2 + 25x|_{20}^{25} = -23.75.$$

So, the revenue will decrease by 23.75 dollars, if the company increases production from 20 to 25 units per day.

Problem: Net Increase in Federal Health Expenditure

During the late 1990s and early 2000s, the federal Medicare and Medicaid Services grew at an exponential rate with a growth constant of ≈ 0.12.

Let $R(t)$ denote the rate, in billions of dollars per year, of health expenditures at time t, where t is the number of years since 2000. Find the total amount of federal health expenditures from 2000 $(t = 0)$ to 2010 $(t = 10)$ if $R(t) = 380e^{0.12t}$.

Solution

Let $H(t)$ be the federal health expenditures from year 0 (2000) until year t. Then, $H'(t) = R(t)$ and the net change is

$$H(10) - H(0) = \int_0^{10} H'(t)dt = \int_0^{10} R(t)dt = \int_0^{10} 380e^{0.12t}dt = \frac{380}{0.12}e^{0.12t}\Big|_0^{10} \approx 7347 \text{ billion dollars.}$$

The Definite Integral and Area under a Graph

The Area under a Graph

Let $f(x)$ is a continuous non-negative function on some interval $[a, b]$. The area under the graph of $f(x)$ from a to b is the area bounded by the graph of $f(x)$ from above, the $x-$ axis, and the vertical lines $x = a$ and $x = b$.

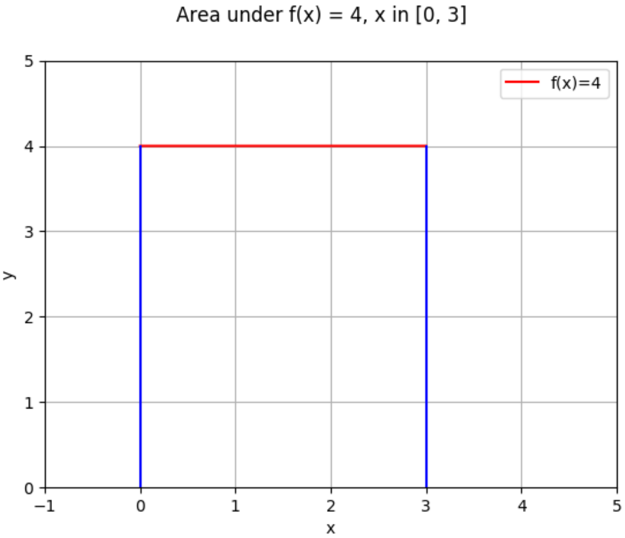
Theorem 1: A Fundamental Theorem of Integral Calculus

Let $f(x)$ is a continuous non-negative function on some interval $[a, b]$. The area under the graph of $f(x)$ from a to b bounded by the graph of $f(x)$ from above, the $x-$ axis, and the vertical lines $x = a$ and $x = b$ is

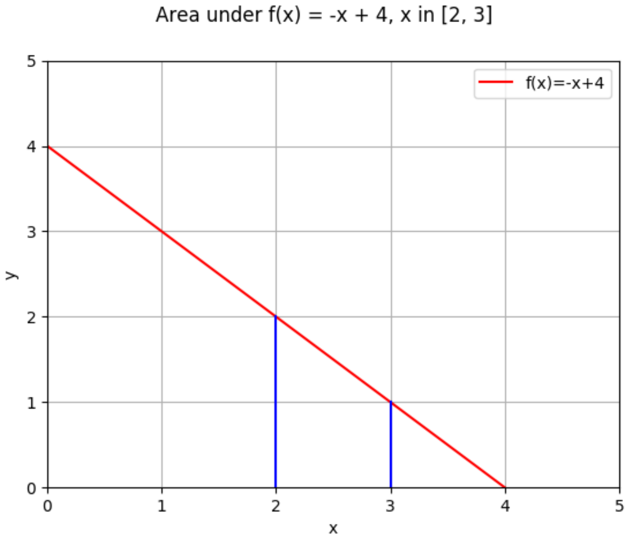
$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is an antiderivative of f.

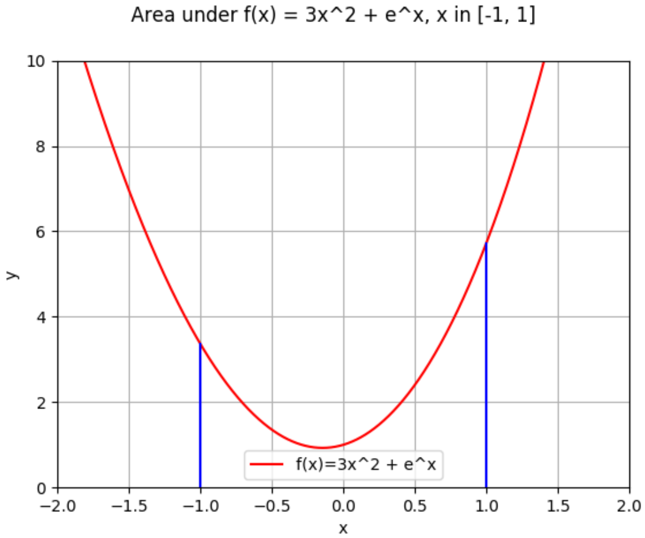
Example: $f(x) = 4, x \in [0, 3]$



Example: $f(x) = -x + 4, x \in [2, 3]$



Example: $f(x) = -x + 4, x \in [2, 3]$



Riemann Sums

Partition

Let $f(x)$ be a continuous non-negative function on the interval $a \leq x \leq b$. Let the x-axis interval be divided into $n > 0$ equal subintervals. This subdivision is called **partition**.

The width of each subinterval is $(b - a)/n$. Another way of saying it is

$$\Delta x = \frac{b-a}{n}.$$

Reimann Sum: Definition

Let's assume that we have a partition with n subinterval. Let's pick a point x_i in each subinterval so that x_1 is in the first subinterval, x_2 is in the second subinterval, etc.

Let Δx be the width of each subinterval. Then $f(x_1)\Delta x$ is the area of the rectangle above the first subinterval, $f(x_2)\Delta x$ is the area of the rectangle above the second subinterval, etc.

The Riemann sum is defined as

$$f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x = \\ [f(x_1) + f(x_2) + ... + f(x_n)]\Delta x.$$

Theorem 2: A Fundamental Theorem of Integral Calculus

Let $f(x)$ is a continuous non-negative function on some interval $[a, b]$. Then

$$\lim_{\Delta x \to 0}[f(x_1) + f(x_2) + ... + f(x_n)]\Delta x = \int_a^b f(x)dx = F(b) - F(a),$$

where F is an antiderivative of f.

Consumer's Surplus

The **consumer's surplus** for a commodity having demand curve $p = f(x)$ is

$$\int_0^A [f(x) - B]dx.$$

where A is the quantity demanded and $f(A) = B$ is the current price.

Problem

Let the demand curve be $p = 50 - 0.06x^2$ at the sales level $x = 20$. What is the consumer's surplus at this level of sales?

$$\int_0^A [f(x) - B] dx.$$

where A is the quantity demanded and $f(A) = B$ is the current price.

Solution

1) At the current level of sales the price $B = 50 - 0.06(20)^2 = 26$.

2) The consumer's surplus is

$$\int_0^{20}[(50 - 0.06x^2) - 26]dx = \int_0^{20}[24 - 0.06x^2]dx = 320.$$

References

1. L. Goldstein, D. Lay, D. Schneider, N. Asmar. *Calculus and its Applications*. Ch. 6, Pearson.
2. `www.python.org`.