CS 3430: S19: SciComp with Py
Lecture 11
Further Exponential Models in Physics, Sociology, and Medicine

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Terminal Velocity

Terminal Velocity: Definition

Suppose a skydiver jumps out of an airplane. The skydiver falls at an increasing rate.

As the time goes on, the wind creates an upward force that begins to counterbalance the downward force of gravity.

The air fraction eventually becomes so great that the skydiver's downward velocity reaches a limiting speed called the **terminal velocity**.

Formula

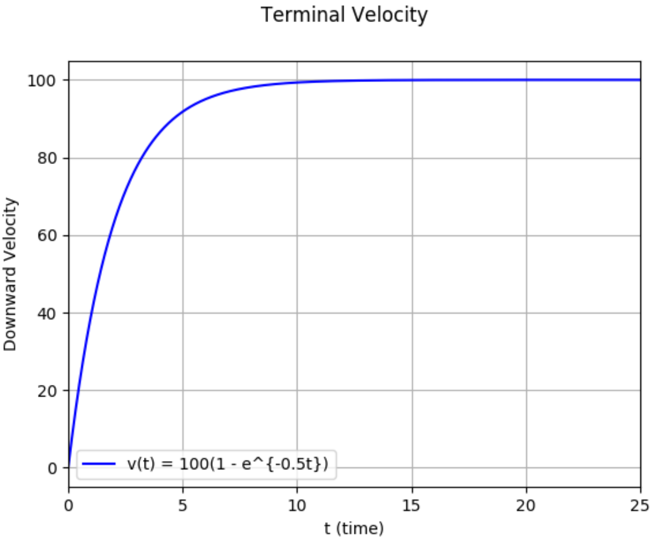
Let $v(t)$ be the downward velocity of the skydriver after t seconds of free fall. A frequently used model for $v(t)$ is

$$v(t) = M(1 - e^{-kt}),$$

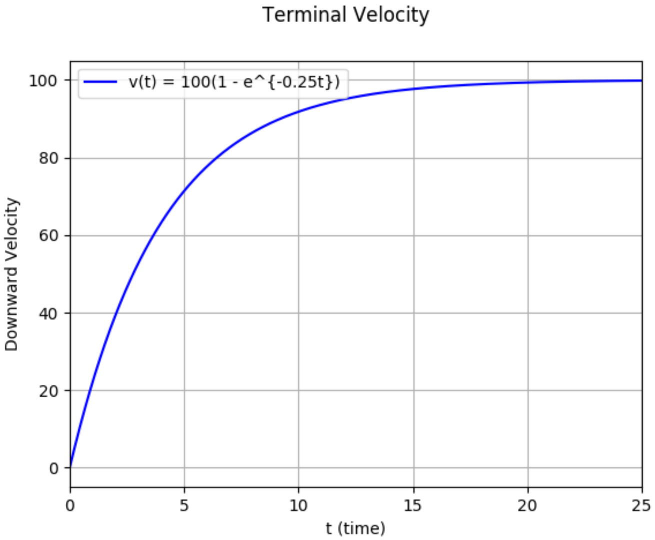
where M is the terminal velocity and k is a positive constant.

As $t \to 0$, $e^{-kt} \to 1$ and $v(t) \to 0$. As t increases, $e^{-kt} \to 0$ and $v(t) \to M$.

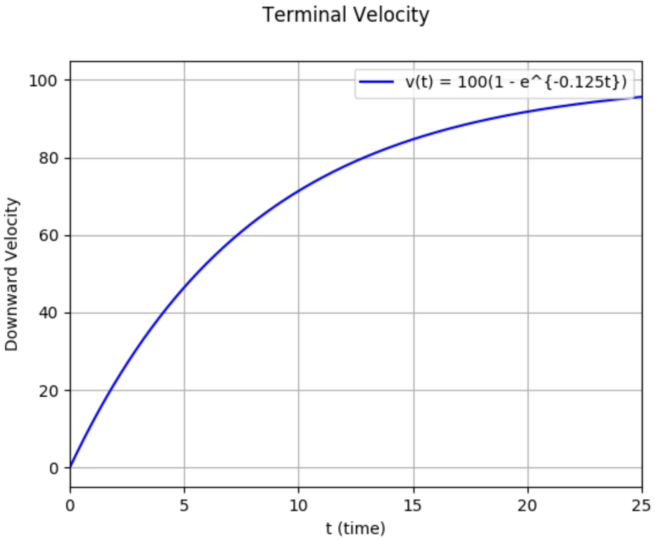
Time vs. Downward Velocity: Plot 1



Time vs. Downward Velocity: Plot 2



Time vs. Downward Velocity: Plot 3



Skydiver's Velocity

Let's compute $v'(t)$. We know that $v(t) = M(1 - e^{-kt}) = M - Me^{-kt}$.

So, $\frac{d}{dt}v(t) = -M(e^{-kt})'(-kt)' = -M(e^{-kt})(-k) = kMe^{-kt}$.

But, $k[M - v(t)] = k[M - (M - Me^{-kt})] = kMe^{-kt}$ and $v(0) = M - Me^0 = 0$.

Conclusion: $\frac{d}{dt}v(t) = k[M - v(t)], v(0) = 0$.

Theorem

The unique solution of the differential equation and initial condition

$$y'(t) = k(M - y(t)), y(0) = 0$$

is

$$y(t) = M(1 - e^{-kt}).$$

The Learning Curve

How Humans Learn

Psychologists have discovered that in many learning situations a person's learning rate is rapid at first and then slows down.

As the task is mastered, the person's level of performance reaches a level above which it is almost impossible to rise.

Within reasonable limits, each person seems to have a certain capacity for memorizing a list of nonsense syllables.

Example: A person can memorize M syllables in a row if given sufficient time (e.g., 1 hour) to study the list but cannot memorize $M + 1$ syllables even if allowed several hours of study.

How Humans Learn

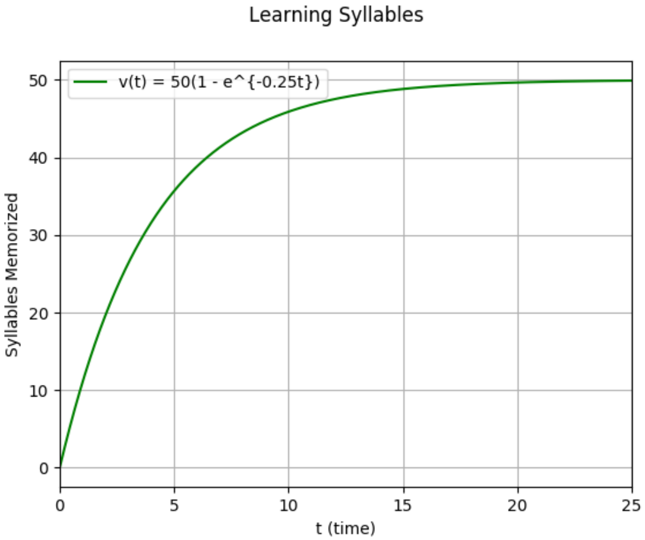
In multiple experiments, psychologists give the same subject different lists of syllables and varying lengths of time to study the lists and then attempt to determine an empirical relationship between the number of syllables memorized accurately and the amount of study time.

A remarkable fact is that many psychologists independently discovered that a reasonable model of this situation is

$$f(t) = M(1 - e^{-kt}),$$

where $f(t)$ is the number of syllables learned in t minutes, $k > 0$, and M is the limit of syllables the subject can master.

A Plot of Human Learning Curve



Learning Rate

The slope of the learning curve at time t is approximately the number of additional syllables the subject can memorize if the subject is given 1 more minute of study time.

The slope of the learning curve is a measure of the rate of learning.

The differential equation satisfied by $y = f(t)$ is

$$y' = k(M - y), f(0) = 0.$$

What does this equation say? If the subject is given a list of M nonsense syllables, the rate of memorization is proportional to the number of syllables remaining to be memorized.

Diffusion of Information by Mass Media

Interesting Quote

Affirmation, pure and simple, kept free of all reasoning and all proof, is one of the surest means of making an idea enter the mind of crowds. The conciser an affirmation is, the more destitute of every appearance of proof and demonstration, the more weight it carries... Statesmen called upon to defend a political cause, and commercial men pushing the sale of their products by means of advertising are acquainted with the value of affirmation... At the end of a certain time we have forgotten who is the author of the repeated assertion, and we finish by believing in it. To this circumstance is due the astonishing power of avertisements. When we have read a hundred, a thousand, times that X's chocoloate is the best, we imagine we have heard it said in many quarters, and we end by acquiring the certitude that such is the fact.

Gustave Le Bon. "The Crowd: A Study of the Popular Mind," 1896.

How Information is Spread

Sociologists have discovered that the equation $y'(t) = k[M - y(t)], y(0) = 0$ is a reasonable model of how information is spread through a population when information is constantly being pushed by mass media (e.g., print, television, online).

Given a population, P, let $f(t)$ be the number of people who have already heard a certain piece of news by time t. Then $P - f(t)$ is the number of people who are yet to hear the information by time t. The is the number of the "uninformed."

Rate of Information Diffusion

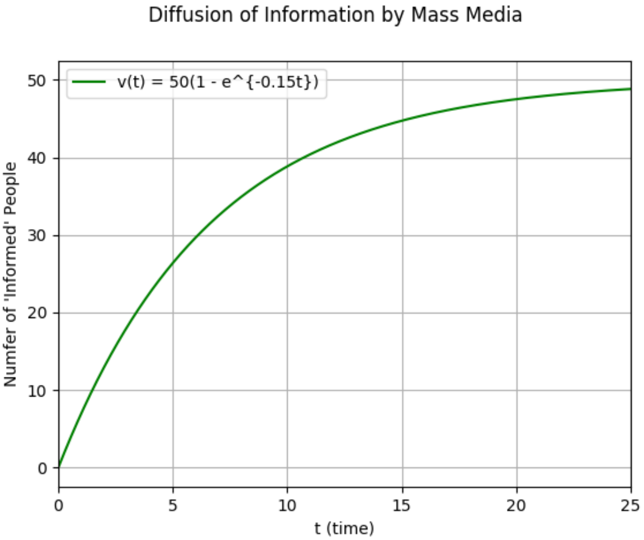
The derivative of $f(t)$, $f'(t)$, is the rate of information diffusion. If one assumes that the number of the newly informed people per unit of time is proportional to the number of people who have not heard the news, then

$$f'(t) = k[P - f(t)].$$

If $f(0) = 0$, then the solution to the above differential equation is

$$f(t) = P[1 - e^{-kt}].$$

A Plot of Diffusion of Information by Mass Media



Problem

The news of a public official's resignation is broadcast frequently by internet media and TV stations. Approximately $1/2$ of the residents of the city of X have heard the news within 4 hours of its initial release. Estimate the time (in hours) when $\approx 90\%$ of the residents will have heard the news.

Solution

Our model is $f(t) = P(1 - e^{-kt})$. We know from the problem that within $t = 4$ hours half the population heard the news, which means

$$P(1 - e^{-k \cdot 4}) = \frac{1}{2}P.$$

We solve the above equation for $k = -\frac{1}{4} ln(0.5) \approx 1.73$. So, the model is

$$f(t) = P(1 - e^{-1.73 \cdot t}).$$

Now we can solve for t in $f(t) = 0.9P$, i.e., $0.9P = P(1 - e^{-1.73 \cdot t})$ to get $t \approx 13.3$ hours.

Intravenous Diffusion of Glucose

Production and Use of Glucose

The human body produces and consumes glucose (aka blood sugar). In a normally functioning body, there is a balance between these two processes.

If a body cannot produce glucose, the equilibrium is broken, and the body must get glucose externally. Sometimes this is done through intravenous injections.

Suppose that $A(t)$ is the amount of glucose, in milligrams, above the equilibrium level (excess glucose). Then the body starts consuming excess glucose at a rate proportional its amount, i.e.,

$$A'(t) = -\lambda A(t),$$

where λ is called the **velocity constant of elimination**.

Amount of Excess Glucose

Suppose a patient receives a continuous intravenous infusion of glucose.

The amount of excess glucose in the blood is influenced by two factors: the amount glucose being continuously injected and the amount of glucose consumed by the body (see previous slide).

If r is the rate of infusion, then the amount of excess glucose is increasing at $A'(t) = r$. Then,

$$A'(t) = r - \lambda A(t).$$

Amount of Excess Glucose

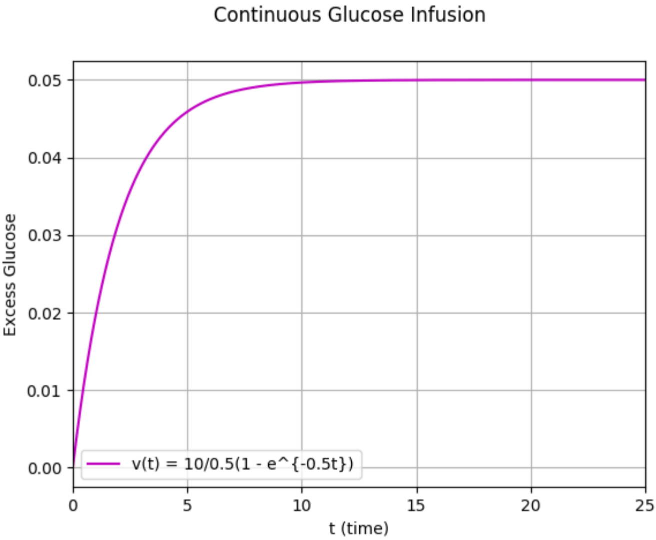
Let M be $\frac{r}{\lambda}$. Then,

$$A'(t) = r - \lambda A(t) = \lambda(\tfrac{r}{\lambda} - A(t)) = \lambda(M - A(t)),$$
$$A(0) = 0.$$

The solution to the above equation is

$$A(t) = M(1 - e^{-\lambda t}) = \tfrac{r}{\lambda}(1 - e^{-\lambda t}).$$

A Plot of Continuous Infusion of Glucose



Logistic Growth

Limits on Growth

A population cannot grow exponentially forver. Eventually the environment begins to inhibit its growth.

The logistic growth model is an exponential model that takes into account some effects of the environment on a population.

For small values of time t, logistic growth models have the same basic shape as exponential growth curves. But, then the enivornmental factors (e.g., lack of food, overcrowding, etc.) begin to slow down the population's growth.

Logistic Growth Equation

$$y = \frac{M}{1+Be^{-Mkt}},$$

where B, M, k are positive constants.

The above, $f(t)$, satisfies the following partial differential equation:

$$y' = ky(M - y).$$

In this equation, the growth rate, y', depends on y, the size of the population, but also on how close y is to the maximum, i.e., $M - y$.

Problem

A lake is stocked with 100 fish. There are 250 fish in the lake 3 months later. A ecological study of the lake predicts that the lake can support ≈ 1000 fish. What is the formula, $P(t)$, for the number of fish in the lake t months later?

Solution

The limiting population, M, is 1000. Thus,

$$P(t) = \frac{1000}{1+Be^{-1000kt}}.$$

At $t = 0$, there are 100 fish. Thus,

$$P(0) = 100 = \frac{1000}{1+Be^{0}} = \frac{1000}{1+B}. \text{ Thus, } B = 9.$$

We know that $P(3) = 250$. Thus,

$$250 = \frac{1000}{1+9e^{-3\cdot 1000\cdot k}}. \text{ So, } k \approx 0.00037.$$

Spread of Epidemics

Spread of Highly Contagious Diseases

Mathematical models of highly contagious diseases are based on the following assumptions:

1. The population is a fixed number P, and each member of P is susceptible to the disease.
2. The duration of the disease is long so that no cures occur during the time period under study.
3. All infected individuals are contagious and circulate freely among the population.
4. During each time period, each infected person makes c contacts, and each contact with an uninfected person results in transmission.

Spread of Highly Contagious Diseases

A frequently used mathematical model of spread of epidemics is

$$f(t) = \frac{P}{1+Be^{-ct}},$$

where B and c are numerical characteristics of an epidemic and P is the population.

Problem

In a city of 500,000 people, the city's department of public health monitors the spread of a strain of flu. At the beginning of the first week of monitoring, 200 infections are reported. After the first week, 300 new cases are reported. Estimate the number of infected individuals after 6 weeks.

Solution

$P = 500,000$. Thus,

$$f(t) = \frac{P}{1+Be^{-ct}} = \frac{500{,}000}{1+Be^{-ct}}.$$

At $t = 0$, $f(0) = 200$. Thus,

$$f(0) = 200 = \frac{500{,}000}{1+Be^{0}} = \frac{500{,}000}{1+B}.\ \text{Thus, } B = 2,499.$$

We know that $f(1) = 200 + 300 = 500$. Thus,

$$f(1) = 500 = \frac{500{,}000}{1+2499e^{-c}}.\ \text{So, } c \approx 0.92.$$

Now we know B and c and can estimate $f(6)$.

References

1. G. Le Bon. *The Crowd: A Study of the Popular Mind*, 1896.
2. P. Bonacich and P. Lu. *Introduction to Mathematical Sociology*, Princeton University Press.
3. L. Goldstein, D. Lay, D. Schneider, N. Asmar. *Calculus and its Applications*. Ch. 4, Pearson.
4. `www.python.org`.