CS 3430: Lecture 2
Differentiation Rules

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Review

The Point-Slope Formula

If we have a slope m and a point (x_1, y_2), we can write the equation of the line with the slope m passing through that point as

$$y - y_1 = m(x - x_1).$$

Derivative

The slope formula is called the **derivative** of $f(x)$ and is written as $f'(x)$.

For each value of x, $f'(x)$ gives the slope of $y = f(x)$ at $(x, f(x))$.

The process of computing $f'(x)$ is called **differentiation**.

Geometric Meaning of the Derivative

$f'(a) =$ slope of the graph of $f(x)$ at $(a, f(a)) =$ slope of the tangent line to $f(x)$ at $(a, f(a))$.

Equation of the Tangent Line: $y - f(a) = f'(a)(x - a)$.

Common Derivatives

Derivative of Linear Function: If $f(x) = mx + b$, then $f'(x) = m$.

Derivative of Constant Function: If $f(x) = c, c \in \mathbb{R}$, the $f'(x) = 0$.

Power Rule: If $f(x) = x^r, r \in \mathbb{R}$, then $f'(x) = rx^{r-1}$.

Function Representation with Python

We need to represent functions in terms of data structures.

Specifically, we need to represent:

1. variables;
2. constants;
3. powers;
4. sums;
5. products;
6. quotients.

Function Representation with Python

```
class const(object):
    def __init__(self, val=0.0):
        self.__val__ = val

    def get_val(self):
        return self.__val__

    def __str__(self):
        return str(self.__val__)
```

Function Representation with Python

```
class const(object):
    @staticmethod
    def add(c1, c2):
        assert isinstance(c1, const)
        assert isinstance(c2, const)
        v1, v2 = c1.get_val(), c2.get_val()
        return const(val=(v1 + v2))

    @staticmethod
    def mult(c1, c2):
        assert isinstance(c1, const)
        assert isinstance(c2, const)
        v1, v2 = c1.get_val(), c2.get_val()
        return const(val=(v1 * v2))
```

Sums

```
class plus(object):
    def __init__(self, elt1=None, elt2=None):
        self.__elt1__ = elt1
        self.__elt2__ = elt2

    def get_elt1(self):
        return self.__elt1__

    def get_elt2(self):
        return self.__elt2__

    def __str__(self):
        return '(' + str(self.__elt1__) + '+' \
               + str(self.__elt2__) + ')'
```

Function Representation with Python: Example

Let's represent $y = x^3 + 5x$.

```
>>> fex1 = make_pwr('x', 3.0)
>>> fex2 = prod(mult1=make_const(5.0),
                mult2=make_pwr('x', 1.0))
>>> y = plus(elt1=fex1, elt2=fex2)
>>> print y
((x^3.0)+(5.0*(x^1.0)))
```

Differentiation Rules

Another Derivative Notation

Let $f(x) : \mathbb{R} \mapsto \mathbb{R}$ be a function. Then

$$f'(x) \equiv \tfrac{d}{dx} f(x).$$

Examples:

If $f(x) = x^2 + 3x - 10$, then $f'(x) \equiv \frac{d}{dx} f(x)$.

If $f(t) = \frac{t^2 - 7t + 10}{t - 5}$, then $f'(x) \equiv \frac{d}{dt} f(t)$.

If $f(u) = \sqrt{u^2 + \pi u}$, then $f'(x) \equiv \frac{d}{du} f(u)$.

Differentiation Rules

1. $\frac{d}{dx}C = 0$, for any $C \in \mathbb{R}$;
2. $\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}f(x), k \in \mathbb{R}$;
3. $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$;
4. $\frac{d}{dx}[g(x)]^r = r \cdot g(x)^{r-1} \cdot \frac{d}{dx}g(x)$.

Examples: Constant-Multiple Rule

1. $\frac{d}{dx}\left(2x^5\right)$;
2. $\frac{d}{dx}\left(\frac{x^3}{4}\right)$;
3. $\frac{d}{dx}\left(-\frac{3}{x}\right)$;
4. $\frac{d}{dx}\left(5\sqrt{x}\right)$.

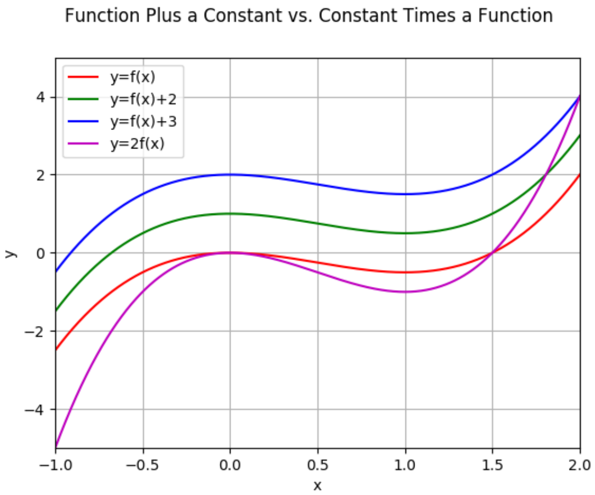
Examples: Sum Rule

1. $\frac{d}{dx}(x^3 + 5x)$;
2. $\frac{d}{dx}\left(x^4 - \frac{3}{x^2}\right)$;
3. $\frac{d}{dx}(2x^7 - x^5 + 8)$.

Examples: General Power Rule

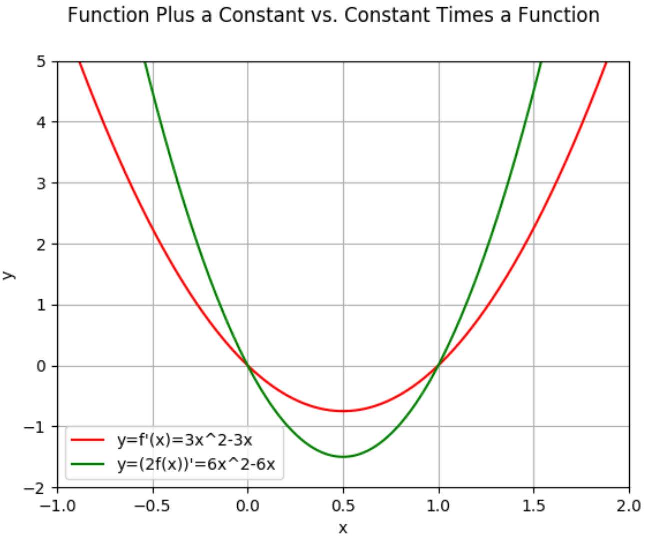
1. $\frac{d}{dx}(x^3 + 5)^2$;
2. $\frac{d}{dx}\sqrt{1 - x^2}$;
3. $\frac{d}{dx}\left(\frac{1}{x^3+4x}\right)$;
4. $\frac{d}{dx}5\sqrt[3]{1 + x^3}$.

Function Plus a Constant vs. Constant Times a Function



$$f(x) = x^3 - \frac{3}{2}x^2$$

Function Plus a Constant vs. Constant Times a Function



Problem: Tangent Line to a Curve at a Point

Find the equation of the tangent line to $y = x^3 + 3x - 8$ at (2, 6).

Second Derivative

Let $f(x)$ is a function. If $f(x)$ is differentiated, we obtain $f'(x)$, which is a function that computes the slope of the curve $y = f(x)$. If $f'(x)$ is differentiated, then we obtain $f''(x)$, which is a function that computes the slope of the slope of $y = f(x)$.

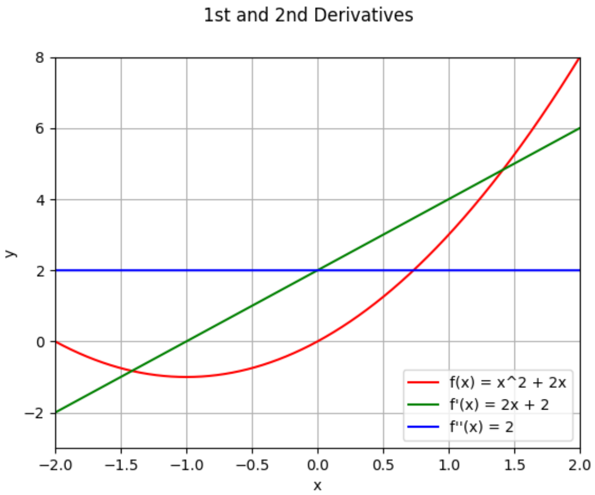
Second Derivative Notation:

$$f''(x) \equiv \frac{d^2}{dx^2} f(x).$$

Examples: Second Derivative

1. $f(x) = x^3 + \frac{1}{x}$;
2. $f(x) = 2x + 1$;
3. $f(t) = t^{1/2} + t^{-1/2}$.

1st and 2nd Derivatives



$$f(x) = x^2 + 2x;\ \frac{d}{dx}f(x) = 2x + 2;\ \frac{d^2}{dx^2}f(x) = \frac{d}{dx}(2x + 2) = 2.$$

Derivative as Approximate Rate of Change

Derivative as Approximate Rate of Change for a Unit Increase

$$f(a+1) - f(a) \approx f'(a)$$

$$f(a+1) \approx f'(a) + f(a)$$

Marginal Concept in Economics

Let $C(x)$ be a cost function that computes the cost of producing x units of some commodity.

Economists are interested in approximating the quantity $C(a+1) - C(a)$, which is called the **additional cost** incurred if the production level is increased by 1 unit, i.e., from a to $a+1$.

The first derivative of $C(x)$, i.e., $C'(x)$, is called the **marginal cost**.

The quantity $C(a+1) - C(a)$ is also the cost of producing the $(a+1)^{st}$ unit.

Marginal Cost

The marginal cost at production level a is

$$C'(a) \approx C(a+1) - C(a).$$

Marginal Cost: Example

Suppose the cost function of a small company that produces chairs is $C(x) = -0.005x^3 - 0.5x^2 + 28x + 300$ (measured in dollars). The current daily production is 50 chairs. What is the cost of increasing daily production from 50 to 51? What is the marginal cost when $x = 50$?

Marginal Cost: Example

What is the cost of increasing daily production from 50 to 51?

$C(51) - C(50) = \$15.76.$

What is the marginal cost when x = 50?

$C'(x) = 0.015x^2 - x + 28.$ $C'(50) = \$15.5.$

$C(51) - C(50) = \$15.76 \approx \$15.5 = C'(50).$

Marginal Revenue and Marginal Profit

If $R(x)$ is the revenue generated from the production of x units of some commodity, then $R'(x)$ is called **marginal revenue**.

If $P(x)$ is the profit generated from the production of x units of some commodity (i.e., $P(x) = R(x) - C(x)$), then $P'(x)$ is called **marginal profit**.

$$R'(a) \approx R(a+1) - R(a).$$

$$P'(a) \approx P(a+1) - P(a).$$

Marginal Revenue and Marginal Profit: Example

Let $R(x)$ be the revenue (in thousands of dollars) generated by a company from the production of x units of some commodity. Let $R(4) = 7$ and $R'(4) = -0.5$.

1. Estimate the additional revenue that results from increasing the production level by 1 unit from $x = 4$ to $x = 5$.
2. Estimate the revenue generated from the production of 5 units.
3. Is it profitable to raise the production to 5 units if the cost function is $C(x) = x + \frac{4}{x+1}$ (in thousands of dollars)?

Marginal Revenue and Marginal Profit: Example

1. Estimate the additional revenue that results from increasing the production level by 1 unit from $x = 4$ to $x = 5$.

   Use $R'(a) \approx R(a+1) - R(a)$.
   $R'(4) \approx R(5) - R(4) \approx -0.5 = -500$ dollars.

2. Estimate the revenue that results from increasing the production level by 1 unit from $x = 4$ to $x = 5$.

   Use $R'(a) \approx R(a+1) - R(a)$ or $R(a+1) \approx R'(a) + R(a)$.
   $R(5) \approx R'(4) + R(4) = 6.5 = \$6,500$.

3. Is it profitable to raise the production to 5 units if the cost function is $C(x) = x + \frac{4}{x+1}$ (in thousands of dollars)?

   Use $P(x) = R(x) - C(x)$. At production $x = 5$, the cost is
   $C(5) = 5 + \frac{4}{5+1} = \frac{17}{3} \approx 5.667 \approx \$5,667$.
   $P(5) = R(5) - C(5) = 6,500 - 5,667 = \$833$.

References

1. L. Goldstein, D. Lay, D. Schneider, N. Asmar. *Calculus and its Applications*. Ch. 1, Pearson.
2. `www.python.org`.