CS 3430: Sprint 2019: SciComp with Py
Lecture 9
Exponential Growth and Decay Models

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Review

Question

How many exponential and logarithmic functions are sufficient for most scientific computing purposes?

Two Functions: $y = e^{kx}$ and $y = lnx$

We need one exponential function, $f(x) = e^{kx}$, because any other exponential function $f(x) = b^x$ can be expressed in terms of it.

We need one logarithm function, $ln(x)$, because any logarithm with a different base can be expressed in terms of it.

Question

What is $\frac{d}{dx} ln(x)$?

Derivative of $ln(x)$

$$\frac{d}{dx} ln(x) = \frac{1}{x}, x > 0.$$

Question

State the chain rule for $ln(x)$.

Chain Rule for $ln(x)$

Let $g(x)$ be differentiable.

$$\frac{d}{dx}[ln(g(x))] = \frac{1}{g(x)} \cdot \frac{d}{dx}g(x).$$

If $u = g(x)$, then

$$\frac{d}{du}[ln(u)] = \frac{1}{u} \cdot \frac{du}{dx}.$$

Question

What is $\frac{d}{dx} ln|x|$?

Derivative of $ln|x|$

If $x > 0$, then $|x| = x$, then $\frac{d}{dx} ln|x| = \frac{d}{dx} ln(x) = \frac{1}{x}$.

If $x < 0$, then $|x| = -x$, then
$\frac{d}{dx} ln|x| = \frac{d}{dx} ln(-x) = \frac{1}{-x} \frac{d}{dx}(-x) = \frac{1}{x}$.

Question

What is logarithmic differentiation used for?

Logarithmic Differentiation

In scientific computing, logarithmic differentiation is used to simplify the task of differentiating long products.

Example:

$$\frac{d}{dx}\left(\frac{(x+1)(2x+1)(3x+1)}{\sqrt{4x+1}}\right).$$

Question

State the logarithmic differentiation rule.

Derivation of Logarithmic Differentiation Rule

Let $f(x)$ be a differentiable function. Then, by the chain rule,

$$\frac{d}{dx} ln[f(x)] = \frac{1}{f(x)} \cdot \frac{d}{dx} f(x).$$

So,

$$\frac{d}{dx} f(x) = f(x) \cdot \frac{d}{dx} ln[f(x)].$$

Exponential Growth and Decay

Motivation

The functions $y = e^x$ and $y = lnx$ are used in many scientific problems.

Many insect colonies grow in size at rates proportional to their current sizes.

Many radioactive materials decay at rates proportional to their current sizes.

Exponential Growth

Proportional Rate of Growth

Let $P(t)$ denote the number of units in some population at time t.

Let $P'(t)$ be a rate of change of P with respect to t, i.e., $\frac{d}{dt}P(t)$.

If the rate of change of the population is proportional to its current size, **then**

$$P'(t) = kP(t), \text{ for some constant } k.$$

Differential Equations

If the rate of change of the population is proportional to its current size, then

$$P'(t) = kP(t), \text{ for some constant } k.$$

Let $y = P(t)$. Then

$$y' = ky, \text{ for some constant } k.$$

Note that the above equation expresses a relationship between y and y'. These types of equations are called **differential equations**.

Solving Differential Equations

A solution to $y' = ky$ is any *function* whose derivative is a constant multiple of itself.

Let C and k be arbitrary constants. Let $y = Ce^{kt}$. Then $y' = kCe^{kt}$, i.e., $y' = ky$.

So, $y = Ce^{kt}$ is a solution to $y' = ky$.

Theorem 1

The function $y = Ce^{kt}$ satisfies the differential equation $y' = ky$. Conversely, if $y = f(t)$ satisfies $y' = ky$, then $y = Ce^{kt}$, for some constant C.

In biology, chemistry, economics, etc., k is called the **growth constant**.

Problem 1

Determine all functions $y = f(t)$ such that $y' = 0.3y$.

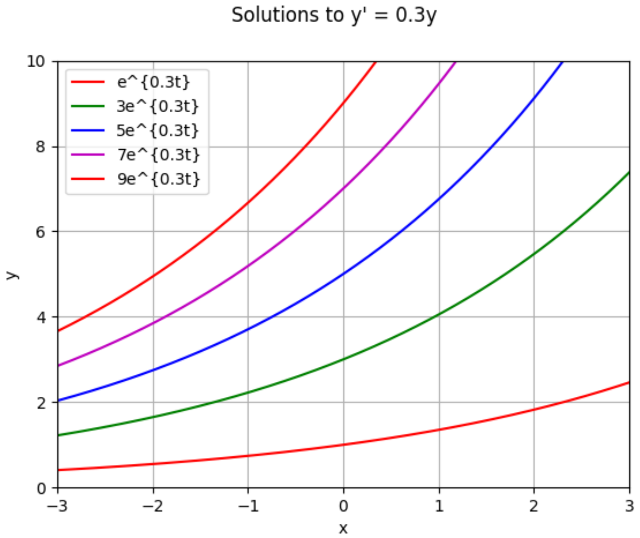
Solution

1) The equation $y' = 0.3y$ has the form $y' = ky$ with $k = 0.3$.

2) By theorem 1, any solution to this equation has the form $y = Ce^{0.3t}$, where C is an arbitrary constant.

Note that the equation $y' = 0.3y$ has infinitely many solutions, e.g., $y = e^{0.3t}$, $y = 3e^{0.3t}$, $y = 5e^{0.3t}$, $y = 7e^{0.3t}$, etc. We can choose a solution that models a given situation best.

Solutions to $y' = 0.3y$



Problem 2

Let $y = P(t)$ be the number of fruit flies in a laboratory t days after the first observation is made. It is known that this particular species of flies exhibits exponential growth with growth constant $k = 0.3$. Suppose that the initial count of fruit flies is 6. What is $P(t)$? What is the number of fruit flies in the lab after 7 days?

Solution

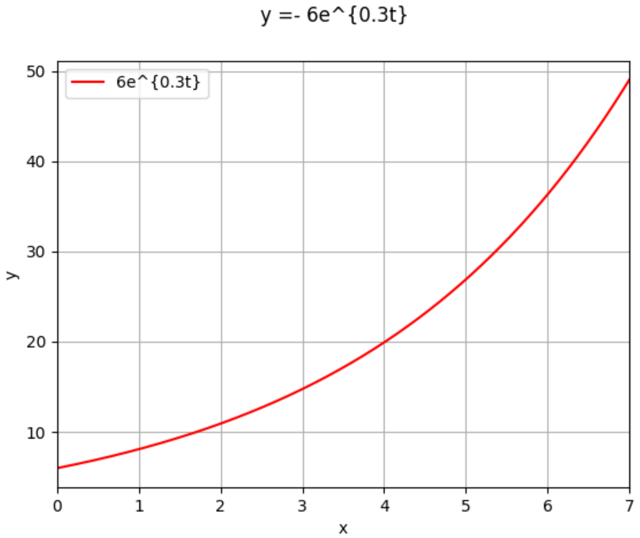
1) Since $y = P(t)$ exhibits exponential growth with $k = 0.3$, it satisfies the differential equation $y' = 0.3y$. Thus, $P(t) = Ce^{0.3t}$.

2) We know that the initial count of fruit flies is 6, i.e., $P(0) = Ce^{0.3\cdot 0} = 6$. So, $C = 6$. So, $P(t) = 6e^{0.3t}$.

3) After 7 days the number of fruit flies in the lab is

$$P(7) = 6e^{0.3\cdot 7} = 6e^{2.1} \approx 49.$$

Exponential Growth Model for Fruit Flies



Theorem 2

The *unique* solution to the differential equation $y' = ky$ with the initial condition $y(0) = P_0$ is

$$y = P(t) = y(0)e^{kt} = P_0 e^{kt}.$$

Problem 3

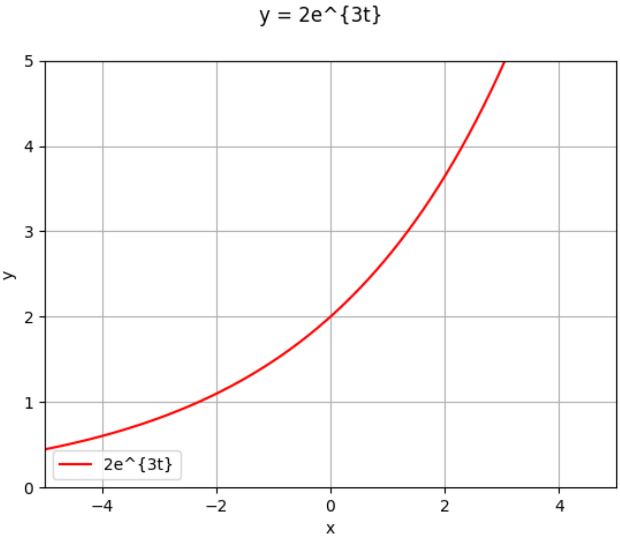
Solve $y' = 3y$, $y(0) = 2$.

Solution

$k = 3$, $y(0) = P_0 = 2$. So, the unique solution is

$$y(t) = y(0)e^{3t} = 2e^{3t}.$$

Solution



Problem 4

A colony of fruit flies grows at a rate proportional to its size. At time $t = 0$, ≈ 20 fruit flies are present. In 5 days there are 400 fruit flies. Determine a function that expresses the size of the colony as a function of time measured in days.

Solution

1) Since, by assumption, the colony grows at a rate proportional to its size, the size of the colony, $P(t)$, satisfies a differential equation of the form $y' = ky$. Thus, it has the form $P(t) = P_0 e^{kt}$.

2) To solve the problem, we must determine both P_0 and k.

3) We know that $P(0) = 20$. Thus, $P(t) = 20e^{kt}$.

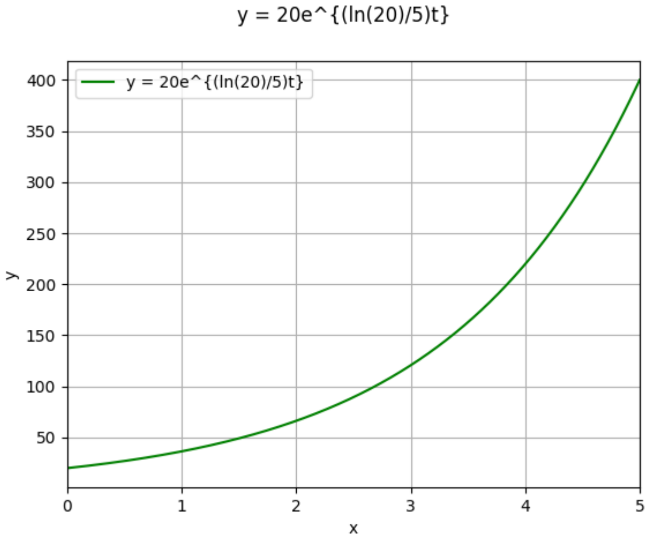
3) We know that $P(5) = 400$. Thus,

$P(5) = 20e^{k5} = 400 \Rightarrow e^{k5} = 20 \Rightarrow 5k = ln(20) \Rightarrow k = \frac{ln(20)}{5} \approx 0.6$.

So, the exponential growth model that we seek is

$$P(t) = 20e^{\frac{ln(20)}{5}t}.$$

Solution



y = 20e^{(ln(20)/5)t}
y = 20e^{(ln(20)/5)t}
400
350
300
250
200
150
100
50
y
0
1
2
3
4
5
x

Problem 5

A colony of insects is growing according to $P(t) = P_0 e^{kt}$. The size of the colony doubles in 9 days. Determine the growth constant k.

Solution

1) We know that the size of the colony doubles in 9 days. Thus, $P(9) = 2P(0)$. In other words, $P(9) = P_0 e^{k9} = 2P_0$.

2) $P_0 e^{k9} = 2P_0 \Rightarrow e^{9k} = 2 \Rightarrow 9k = ln(2) \Rightarrow k = \frac{ln(2)}{9} \approx 0.077$.

Note that the growth constant k does not depend on us knowing the initial size of the population.

Problem 6

Suppose that a population of insects is modeled by $P(t) = P_0 e^{0.077t}$. Let the initial population of the colony be 100.

1. How large will the colony be after 41 days?
2. How fast will the colony be growing at that time?
3. At what time will the colony contain 800 insects?
4. How large is the colony when it is growing at a rate of 200 insects per day?

Solution

1) Since $P(0) = 100$, $P(t) = 100e^{0.077t}$. So, after 41 days, $P(41) = 100e^{0.077 \cdot 41} \approx 2350$ insects.

2) Since $P(t)$ satisfies $y' = 0.077y$, $P'(t) = 0.077P(t)$. Thus, when $t = 41$, $P'(41) = 0.077P(41) = (0.077)(2350) \approx 181$, i.e., the colony is growing at 181 insects per day.

3) $100e^{0.077t} = 800 \Rightarrow t = \frac{ln 8}{0.077} \approx 27$ days. In other words, the colony will include 800 insects in 27 days.

4) When the colony is growing at a rate of 200 insects per day, $P'(t) = 200$. We have $200 = 0.077P(t)$. Thus, $P(t) = \frac{200}{0.077} \approx 2597$. Thus, when the colony is growing at 200 insects per day, the size of the colony is 2597 insects.

Exponential Decay

Decay Constant

We can solve $y' = ky$ for $k < 0$ by using Theorem 1 to obtain the solution $y = Ce^{kt}$, where C is an arbitrary constant.

When $k < 0$, we are dealing with **exponential decay**.

To emphasize that that fact that k is negative, $P(t) = kP(t)$ is often written as $P(t) = -\lambda P(t)$, where $\lambda > 0$.

Then, $P(t)$ satisfies the differential equation $P'(t) = -\lambda P(t)$. By theorem 1, the solution has the form $P(t) = P_0 e^{-\lambda t}$, for some positive number P_0. The constant λ is called the **decay constant**.

Problem 7

The decay constant of Strontium 90, a highly radioactive element, is $\lambda = 0.0244$ with t measured in years. How long will it take for a quantity P_0 of Strontium 90 to decay to $1/2$ of its original size?

Solution

1) Since $\lambda = 0.0244$, we have $P(t) = P_0 e^{-0.0244t}$.

2) The problem asks us to determine when $P(t) = \frac{1}{2} P_0$.

3) $P_0 e^{-0.0244t} = \frac{1}{2} P_0 \Rightarrow t = \frac{ln \frac{1}{2}}{-0.0244} \approx 28$ years.

Half-Life of Radioactive Elements

The **half-life** of a radioactive element is the length of time required for a given quantity of the element to decay to $1/2$ of its original size.

In problem 7, we found out that the half-life of Strontium 90 is 28 years.

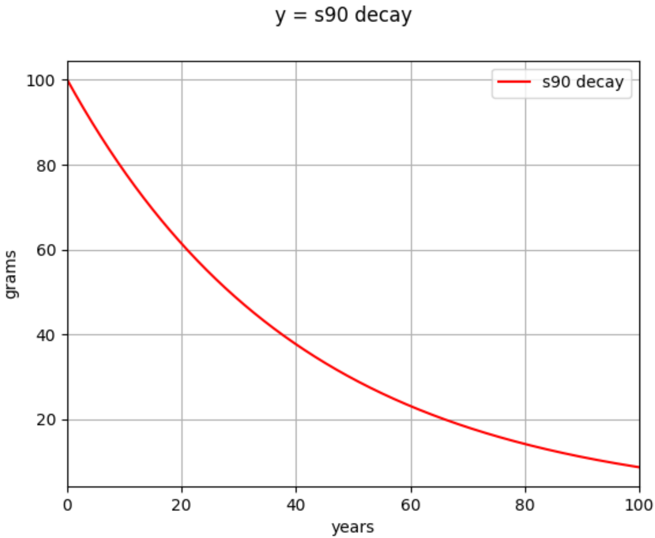
In other words, it takes 28 years for the original mass to become $1/2$ of itself, another 28 years for the original mass to become $1/4$ of itself, then another 28 years for the original mass to become $1/8$ of itself, etc.

Note that the half-life of an element does not depend on its original mass.

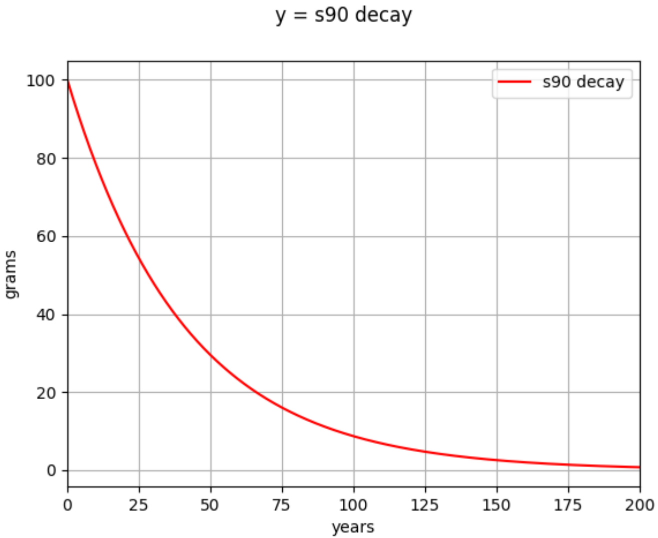
Question

How long, do you think, it will take 100g of Strontium 90 to decay completely?

How Fast Will 100g of Strontium 90 Decay?



How Fast Will 100g of Strontium 90 Decay?



References

1. E. Batschelet. *Introduction to Mathematics for Life Scientists*, Springer.
2. L. Goldstein, D. Lay, D. Schneider, N. Asmar. *Calculus and its Applications*. Ch. 4, Pearson.
3. `www.python.org`.