CS 3430: S19: SciComp with Py
Lecture 19

Function Approximation with Taylor Polynomials

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Review

Introduction

Whenever we need the value of such functions as e^x, $ln(x)$, $sin(x)$, $cos(x)$, and $tan(x)$ for a particular value of x (e.g., $e^{0.023}$, $ln(5.8)$, etc.), we use a computer or a calculator and the answer magically shows up on the screen.

Since computers and calculators compute these values, there must exist corresponding algorithms.

Taylor polynomials furnish us one tool to implement such algorithms.

Example: Polynomial Approximation

Consider $f(x) = e^x$ and the tangent line through $(0, f(0)) = (0, 1)$. The slope of the tangent line is $f'(0) = e^0 = 1$.

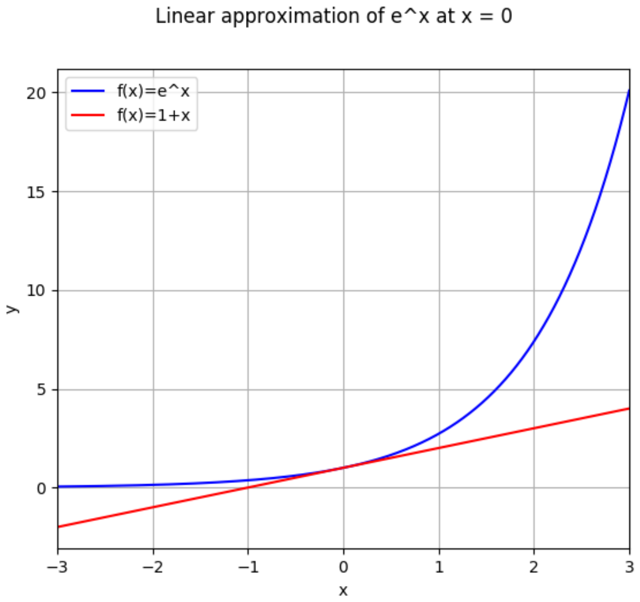
The equation of the tangent line is

$$y - f(0) = f'(0)(x - 0).$$

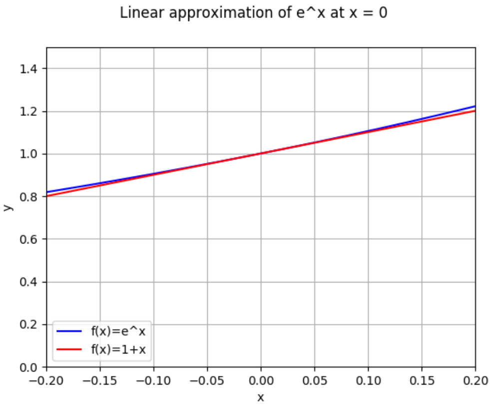
$$y = f(0) + f'(0)x.$$

$$y = 1 + x.$$

Example: Polynomial Approximation



Example: Polynomial Approximation



Let $p_1(x) = 1 + x$. Then $p_1(x)$ are close to $f(x) = e^x$ for x near 0.

Taylor Polynomials

1st Taylor Polynomial

A given function $f(x)$ can be approximated for values of x near 0 by the polynomial

$$p_1(x) = f(0) + f'(0)x.$$

The above polynomial is called the **first Taylor polynomial of** $f(x)$ at $x = 0$. The graph of $p_1(x)$ is the tangent line to $y = f(x)$ at $x = 0$.

1st Taylor Polynomial

The first Taylor polynomial resembles $f(x)$ near $x = 0$, because

1. Both graphs go through the same point at $x = 0$, i.e., $p_1(0) = f(0)$;
2. Both graphs have the same slope at $x = 0$, i.e., $p_1'(0) = f'(0)$.

The two above observations suggest that, to approximate $f(x)$ more closely at $x = 0$, we should look for a polynomial that coincides with $f(x)$ in its value at $x = 0$ and the values of its first and second derivatives at $x = 0$.

Problem

Given a function $f(x)$, suppose that $f(0) = 1$, $f'(0) = -2$, $f''(0) = 7$, and $f'''(0) = -5$. Find a polynomial approximation of degree 3, i.e., $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, such that $p(x)$ coincides with $f(x)$ up to the third derivative at $x = 0$.

Solution

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \ p(0) = a_0;$$

$$p'(x) = 0 + a_1 + 2a_2 x + 3a_3 x^2, \ p'(0) = a_1;$$

$$p''(x) = 0 + 0 + 2a_2 + 2 \cdot 3a_3 x, \ p''(0) = 2a_2;$$

$$p'''(x) = 0 + 0 + 0 + 2 \cdot 3a_3, \ p'''(0) = 2 \cdot 3a_3;$$

Since we want $p(x)$ and its derivatives to coincide with the given values of $f(x)$ and its derivatives, we have $a_0 = 1$, $a_1 = -2$, $a_2 = 7/2$, and $a_3 = -5/6$.

Nth Taylor Polynomial

Given a function $f(x)$, the **nth Taylor polynomial of $f(x)$ at $x = 0$** is the polynomial $p_n(x)$ defined by

$$p_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n.$$

This polynomial coincides with $f(x)$ up to the n-th derivative at $x = 0$ in the sense that $p_n(0) = f(0)$, $p_n'(0) = f'(0)$, ..., $p_n^{(n)}(0) = f^{(n)}(0)$.

Problem

Determine the first three Taylor polynomials of $f(x) = e^x$ at $x = 0$.

Solution

$$f(0) = f'(0) = f''(0) = f'''(0) = e^0 = 1.$$

So, the first three Taylor polynomials are

1. $p_1(x) = 1 + \frac{1}{1!}x = 1 + x;$
2. $p_2(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 = 1 + x + \frac{1}{2}x^2;$
3. $p_3(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3.$

Taylor Polynomial at $x = a$

Given a function $f(x)$, the **nth Taylor polynomial of $f(x)$ at $x = a$** is the polynomial $p_n(x)$ defined by

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This polynomial coincides with $f(x)$ up to the n-th derivative at $x = a$ in the sense that $p_n(a) = f(a)$, $p_n'(a) = f'(a)$, ..., $p_n^{(n)}(a) = f^{(n)}(a)$.

Problem

Calculate the 2nd Taylor polynomial of $f(x) = \sqrt{x}$ at $x = 1$ and use this polynomial to estimate $\sqrt{1.02}$.

Solution

In this case, $a = 1$. Let's calculate the values of $f(x)$ and its first two derivatives at $x = 1$.

$f(x) = x^{1/2}$, $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$.

$f(1) = 1$, $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{4}$.

The desired polynomial is

$p_2(x) = 1 + \frac{1/2}{1!}(x-1) + \frac{-1/4}{2!}(x-1)^2 = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$.

Solution

Since 1.02 is pretty close to 1, $p_2(1.02)$ approximates $f(1.02) = \sqrt{1.02}$.

$$p_2(1.02) = 1 + \frac{1}{2}(1.02 - 1) - \frac{1}{8}(1.02 - 1)^2 = 1.00995.$$

References

1. L. Goldstein, D. Lay, D. Schneider, N. Asmar. *Calculus and its Applications*, Chapter 11. Pearson.
2. `www.python.org`.