CS 3430: Lecture 3
Derivative as Approximate Rate of Change

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Review

Another Derivative Notation

Let $f(x) : \mathbb{R} \mapsto \mathbb{R}$ be a function. Then

$$f'(x) \equiv \tfrac{d}{dx} f(x).$$

Examples:

If $f(x) = x^2 + 3x - 10$, then $f'(x) \equiv \frac{d}{dx} f(x)$.

If $f(t) = \frac{t^2 - 7t + 10}{t-5}$, then $f'(x) \equiv \frac{d}{dt} f(t)$.

If $f(u) = \sqrt{u^2 + \pi u}$, then $f'(x) \equiv \frac{d}{du} f(u)$.

Differentiation Rules

1. $\frac{d}{dx}C = 0$, for any $C \in \mathbb{R}$;
2. $\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}f(x), k \in \mathbb{R}$;
3. $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$;
4. $\frac{d}{dx}[g(x)]^r = r \cdot g(x)^{r-1} \cdot \frac{d}{dx}g(x)$.

Second Derivative

Let $f(x)$ is a function. If $f(x)$ is differentiated, we obtain $f'(x)$, which is a function that computes the slope of the curve $y = f(x)$. If $f'(x)$ is differentiated, then we obtain $f''(x)$, which is a function that computes the slope of the slope of $y = f(x)$.

Second Derivative Notation:

$$f''(x) \equiv \frac{d^2}{dx^2} f(x).$$

Derivative as Approximate Rate of Change for a Unit Increase

$$f(a+1) - f(a) \approx f'(a)$$

$$f(a+1) \approx f'(a) + f(a)$$

Marginal Cost

The marginal cost at production level a is

$$C'(a) \approx C(a+1) - C(a).$$

Marginal Revenue and Marginal Profit

If $R(x)$ is the revenue generated from the production of x units of some commodity, then $R'(x)$ is called **marginal revenue**.

If $P(x)$ is the profit generated from the production of x units of some commodity (i.e., $P(x) = R(x) - C(x)$), then $P'(x)$ is called **marginal profit**.

$$R'(a) \approx R(a+1) - R(a).$$

$$P'(a) \approx P(a+1) - P(a).$$

Derivative as Approximate Rate of Change

Approximating Change with Derivative

If h is small, then

$$\frac{f(a+h)-f(a)}{h} \approx f'(a);$$

$$f(a+h) - f(a) \approx f'(a) \cdot h.$$

Example

Let the production function $p(x)$ give the number of units of goods produced when x units of labor are employed. If 5000 units of labor are employed, $p(5000) = 300$ and $p'(5000) = 2$.

(a) What does $p(5000) = 300$ mean? When 5000 units of labor are employed, 300 units of goods are produced.

(b) What does $p'(5000) = 2$ mean? If we add one unit of labor, the productivity is expected to increase at a rate of 2 units of goods for each additional unit of labor.

(c) How many additional units of goods will be produced if x (i.e., units of labor) is increased from 5000 to 5000.5? We use the formula $f'(a) \cdot h$, i.e., $p'(5000) \cdot 0.5 \approx 1$.

(d) How many additional units of goods will be produced if x (i.e., units of labor) is decreased from 5000 to 4999? We use the formula $f'(a) \cdot h$, i.e., $p'(5000) \cdot -1 \approx -2$.

Average Rate of Change

Average rate of change of $f(x)$ over the interval $a \leq x \leq b$ is

$$\frac{f(b)-f(a)}{b-a}.$$

Instantaneous Rate of Change

The derivative $f'(a)$ measures the instantaneous rate of change of $f(x)$ at $x = a$.

Problem 1

Let $f(x) = x^2$. Then $f'(x) = 2x$. The instantaneous rate of change of $f(x)$ at 1 is $f'(1) = 2$. Compute the average rates of change of $f(x)$ over the following intervals:

a) $1 \leq x \leq 2$;

b) $1 \leq x \leq 1.1$;

c) $1 \leq x \leq 1.01$.

Solution

a) $1 \leq x \leq 2$; $\frac{f(2)-f(1)}{2-1} = \frac{2^2-1^2}{2-1} = 3$;

b) $1 \leq x \leq 1.1$; $\frac{f(1.1)-f(1)}{1.1-1} = \frac{1.1^2-1^2}{1.1-1} = 2.1$;

c) $1 \leq x \leq 1.01$; $\frac{f(1.01)-f(1)}{1.01-1} = \frac{1.01^2-1^2}{1.01-1} = 2.01$.

Observation: Average rates of change approach the instantaneous rate of change at 1 as the intervals starting at 1 become smaller.

Py Solution

```
intervals = ((1.0, 2.0), (1.0, 1.5), (1.0, 1.25),
             (1.0, 1.1), (1.0, 1.01), (1.0, 1.001),
             (1.0, 1.0001))

def f1(x): return x**2
def df1(x): return 2*x

def avrg_rate(f, l, u):
  return (f(u)-f(l))/(u - l)

def test_avrg_rates(f, df, intervals):
  for l, u in intervals:
    ar = avrg_rate(f, l, u)
    print ar, abs(df(l) - ar)

if __name__ == '__main__':
  test_avrg_rates(f1, df1, intervals)
```

Py Solution

Python code output is below.

```
3.0 1.0
2.5 0.5
2.25 0.25
2.1 0.1
2.01 0.01
2.001 0.000999999999918
2.0001 9.99999993923e-05
```

Average rates of change approach the instantaneous rate of change at 1 as the intervals starting at 1 become smaller.

Analysis of Functions

Increasing and Decreasing Functions

$f(x)$ is **increasing over an interval** if f's graph continuously rises as x goes from left to right through the interval. In other words, whenever x_1 and x_2 are in the interval and $x_1 < x_2$, then $f(x_1) < f(x_2)$.

$f(x)$ is **increasing at** $x = c$ if $f(x)$ is increasing at some open interval on the x-axis that contains c.

$f(x)$ is **decreasing over an interval** if f's graph continuously falls as x goes from left to right through the interval. In other words, whenever x_1 and x_2 are in the interval and $x_1 < x_2$, then $f(x_1) > f(x_2)$.

$f(x)$ is **decreasing at** $x = c$ if $f(x)$ is decreasing over some open interval on the x-axis that contains c.

Relative Maxima and Minima

A **relative extreme point** or an **extremum** of a function f is a point at which its graph changes from increasing to decreasing or from decreasing to increasing.

An extreme point is a **relative maximum** of a function f if f's graph changes from increasing to decreasing.

An extreme point is a **relative minimum** of a function f if f's graph changes from decreasing to increasing.

The term *relative* means that a point is maximal or minimal relative only to nearby points of the graph.

Absolute Maxima and Minima

The absolute maximum of a function is the largest value the function assumes on its domain.

The absolute minimum of a function is the smallest value the function assumes on its domain.

Sometimes the absolute maxima/minima occur at the end points of the domain.

Concavity

A function is **concave up** at $x = a$ if there is an open interval on the x-axis containing a through which the graph of f lies above all tangent lines to it.

Equivalently, $f(x)$ is concave up at $x = a$ if the slope of $f(x)$ increases as we move from left to right through $(a, f(a))$.

A function is **concave down** at $x = a$ if there is an open interval on the x-axis containing a through which the graph of f lies below all tangent lines to it.

Equivalently, $f(x)$ is concave down at $x = a$ if the slope of $f(x)$ decreases as we move from left to right through $(a, f(a))$.

First- and Second-Derivative Tests

First-Derivative Test

If $f'(a) > 0$, then $f(x)$ is increasing at $x = a$.

If $f'(a) < 0$, then $f(x)$ is decreasing at $x = a$.

If $f'(a) = 0$, then the first-derivative test is inconclusive.

First-Derivative Test for Local Extrema

Suppose that $f'(a) = 0$.

(a) If f' changes from positive to negative at $x = a$, then f has a local maximum at $x = a$.

(b) If f' changes from negative to positive at $x = a$, then f has a local minimum at $x = a$.

(c) If f' does not change sign at $x = a$, then f has no extreme point at $x = a$.

Problem 2

Find the local extreme points of $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 1$ analytically and graphically.

Analytical Solution

(1) $\frac{d}{dx}f(x) = x^2 - 4x + 3 = (x-1)(x-3)$; in implementation, it is easier to use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

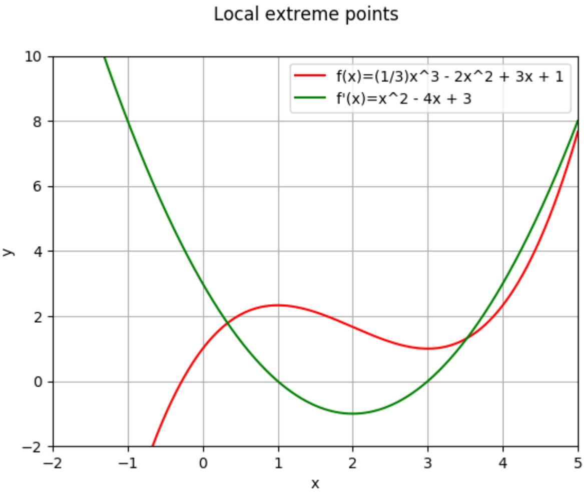
(2) $\frac{d}{dx}f(x) = 0$ if $x = 1$ and $x = 3$.

(3) The critical points are $(1, \frac{7}{3})$ and $(3, 1)$.

(4) The intervals are $x < 1$, $1 < x < 3$, and $3 < x$: $f'(x) > 0$ on $x < 1$; $f'(x) < 0$ on $1 < x < 3$; $f'(x) > 0$ on $3 < x$.

(5) $(1, \frac{7}{3})$ is a local maximum; $(3, 1)$ is a local minimum.

Graphic Solution



Second-Derivative Test for Local Extrema

(a) If $f'(a) = 0$ and $f''(a) < 0$, then $f(x)$ has a local maximum at $x = a$.

(b) If $f'(a) = 0$ and $f''(a) > 0$, then $f(x)$ has a local minimum at $x = a$.

Problem 3

Find one relative extreme point of $f(x) = \frac{1}{4}x^2 - x + 2$.

Solution

(a) Compute the 1st and 2nd derivatives: $\frac{d}{dx}f(x) = \frac{1}{2}x - 1$; $\frac{d^2}{dx^2}f(x) = \frac{1}{2}$.

(b) Solve $\frac{d}{dx}f(x) = 0$ to get $x = 2$. Thus, the critical point is $(2, f(2)) = (2, 1)$.

(c) Since $\frac{d^2}{dx^2}f(2) = \frac{1}{2}$, $(2, 1)$ is the local minimum.

Second-Derivative Test

The second-derivative test also gives useful information about concavity of the graph of $f(x)$.

Suppose that $f''(a)$ is negative. This means that $f'(x)$ has a negative derivative at $x = a$, i.e., the slope of the graph $f'(x)$ is decreasing near the point $(a, f(a))$. In other words, the graph is concave down.

Suppose that $f''(a)$ is positive. This means that $f'(x)$ has a positive derivative at $x = a$, i.e., the slope of the graph $f'(x)$ is increasing near the point $(a, f(a))$. In other words, the graph is concave up.

Second-Derivative Test

If $f''(a) > 0$, then $f(x)$ is concave up at $x = a$.

If $f''(a) < 0$, then $f(x)$ is concave down at $x = a$.

If $f''(a) = 0$, then the second-derivative test is inconclusive.

Problem 4

Locate all relative extreme points of $f(x) = x^3 - 3x^2 + 5$ and determine the concavity of the function at these points analytically and graphically.

Analytical Solution

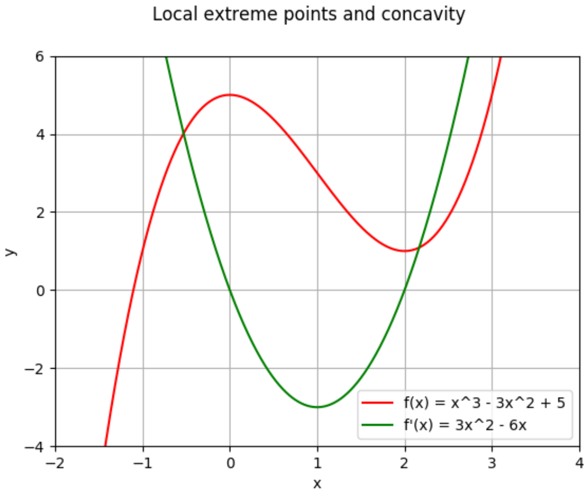
(a) Compute the 1st and 2nd derivatives: $\frac{d}{dx} f(x) = 3x^2 - 6x$; $\frac{d^2}{dx^2} f(x) = 6x - 6$.

(b) Solve $\frac{d}{dx} f(x) = 3x^2 - 6x = 3x(x-2) = 0$ to get $x = 0$ and $x = 2$.

(c) Compute the critical points $(0, f(0)) = (0, 5)$ and $(2, f(2)) = (2, 1)$.

(d) Compute the 2nd derivative values $\frac{d^2}{dx^2} f(0) = -6 < 0$, which makes $(0, 5)$ is a local maximum; $\frac{d^2}{dx^2} f(2) = 6 > 0$, which makes $(2, 1)$ a local minimum.

Graphic Solution



Combining First- and Second Derivatives

Possible combinations of increasing, decreasing, concave up, and concave down.

Derivatives	Behavior of $f(x)$ at $x = a$
$f'(a) > 0$; $f''(a) > 0$	$f(x)$ is increasing; $f(x)$ is conc. up
$f'(a) > 0$; $f''(a) < 0$	$f(x)$ is increasing; $f(x)$ is conc. down
$f'(a) < 0$; $f''(a) > 0$	$f(x)$ is decreasing; $f(x)$ is conc. up
$f'(a) < 0$; $f''(a) < 0$	$f(x)$ is decreasing; $f(x)$ is conc. down

Inflection Points

An **inflection point** is a point on the graph of a function $f(x)$ at which the function is continuous and at which the graph changes from concave up to concave down or from concave down to concave up.

An inflection point of a function $f(x)$ can occur at a value of x for which $f''(x) = 0$.

Problem 5

Find the inflection points of $f(x) = x^3 - 3x^2 + 5$.

Solution

(a) Take the 2nd derivative: $f''(x) = 6(x - 1)$.

(b) Solve $f''(x) = 0$ to obtain $x = 1$. Thus, the only possible inflection point is $(1, f(1)) = (1, 3)$.

(c) Test the sign of $f''(x)$ over $x < 1$ and $x > 1$. $f''(0) = -6 < 0$ and $f''(2) = 6 > 0$. Thus, the concavity of $f(x)$ changes from down to up.

References

1. L. Goldstein, D. Lay, D. Schneider, N. Asmar. *Calculus and its Applications*. Ch. 1, Pearson.
2. `www.python.org`.