# CS 3430: S19: SciComp with Py
# Lecture 12
# Antidifferentiation

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# Background

Two fundamental problems of numerical analysis are: 1) finding the slope of a curve at a point and 2) to find the area of a region under a curve.

These problems are straightforward when the curve is a straight line; even when the curve consists of several straight lines, the curve can be segemented into line segments, each of which can be processed separately.

More sophisticated methods are needed when the curves are not straight lines.

The area problem is connected with the notion of **antiderivative** (aka **integral**), which is the reverse of the derivative of a function. This connection forms another fundamental theorem of calculus.

# Antidifferentiation

# Why Antidifferentiation is Needed

We have developed a few standard techniques of differentiation used in scientific computing and have seen several classes of problems from various areas of science where differentiation is productively used.

In many applications, however, it is necessary to find the original function $F(x)$ given its derivative $F'(x)$.

# Example

A rocket is fired vertically. Its velocity at $t$ seconds after liftoff is $v(t) = 6t + 0.5$ meters per second. The top of the rocket is 8 meters above the launch pad. What is the height of the rocket (measured from the top of the rocket to the launch pad) at time $t$.

If $s(t)$ is the height of the rocket at time $t$, then $s'(t)$ is the rate at which the height is changing. In other words, $s'(t) = v(t)$.

Although we do not yet know the formula for $s(t)$, we know the formula for $s'(t)$. Thus, the problem is how to find $s(t)$ from $s'(t)$.

We'll get back to this example at the end of this lecture.

# Definition of Antiderivative

Suppose that $f(x)$ is a function and $F(x)$ is a function such that $F'(x) = f(x)$. Then $F(x)$ is an **antiderivative** of $f(x)$.

# Problem

Find an antiderivative of $f(x) = x^2$.

# Solution

We need to find $F(x)$ such that $F'(x) = f(x)$. We know that the derivative of $x^3$ is $3x^2$.

All we need to do is to multiply $x^3$ by $1/3$ to get $F(x)$. Thus.

$$F(x) = \frac{1}{3}x^3.$$

Check:

$$\frac{d}{dx}F(x) = \frac{1}{3}3x^2 = x^2 = f(x).$$

# Solution

Observe that if we take $F(x) = \frac{1}{3}x^3 + 5$. Then, $F'(x) = x^2 = f(x)$.

We can actually generalize the above observation to state that the antiderivative of $f(x) + x^3$ is

$$F(x) = \frac{1}{3}x^3 + C,$$

where $C$ is an arbitrary constant.

# Problem

Find an antiderivative of $f(x) = e^{-2x}$.

## Solution

We know that the derivative of $e^{cx}$ is $ce^{cx}$. Thus, the antiderivative of $e^{-2x}$ is of the form $F(x) = ke^{-2x} + C$.

Let's differentiate $F(x)$:

$$\frac{d}{dx}\left(ke^{-2x} + C\right) = k\frac{d}{dx}e^{-2x} + 0 = -2ke^{-2x},$$

Since $F'(x) = f(x)$, we want $-2ke^{-2x} = e^{-2x}$, so that $k = -\frac{1}{2}$.

Check:

$$\frac{d}{dx}\left(-\frac{1}{2}\right)e^{-2x} = \left(-\frac{1}{2}\right)(-2)e^{-2x} = e^{-2x}.$$

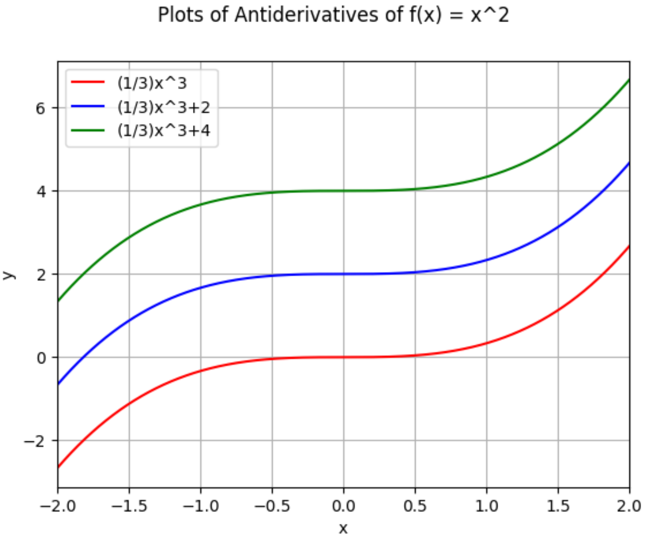
So, $F(x) = -\frac{1}{2}e^{-2x} + C$.

# Antidifferentiation Theorem 1

Suppose that $f(x)$ is a continuous function on an internal $I$. If $F_1(x)$ and $F_2(x)$ are two antiderivatives of $f(x)$, then $|F_1(x) - F_2(x)| = C$, for $x \in I$.

Geometrically speaking, the graph of $F_2(x)$ can be obtained from $F_1(x)$ by a vertical shift (upward or downard) of $F_1(x)$.

# Example of Antidifferentiation Theorem 1



# Antidifferentiation Theorem 2

If $F'(x) = 0$ for all $x$ in an interval $I$, there exists a constant $C$ such that $F(x) = C$ for all x in $I$.

# Problem: Verification of Theorem 1

Use Theorem 2 to verify Theorem 1.

# Solution

Let $F_1(x)$ and $F_2(x)$ be two antiderivatives of $f(x)$.

Let $F(x) = F_2(x) - F_1(x)$. Then,

$F'(x) = F_2'(x) - F_1'(x) = f(x) - f(x) = 0.$

Thus, by Theorem 2, $F(x) = C$ and $F_2(x) = F_1(x) + C$.

# Definition of Indefinite Integral

If $f(x)$ is a function whose antiderivatives are $F(x) + C$, then

$$\int f(x)dx = F(x) + C,$$

where $\int f(x)dx$ is called an **indefinite integral**.

The function $f(x)$ in $\int f(x)dx$ is called the **integrand**.

# Antidifferentiation Formulas

# Antidifferentiation Rules for Powers and Exponentials

$$\int x^r dx = \frac{1}{r+1} x^{r+1} + C,\ r \neq -1.$$

$$\int e^{kx} dx = \frac{1}{k} e^{kx} + C,\ k \neq 0.$$

# Problem

Compute $\int \sqrt{x} dx$.

# Solution

$$\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} = \frac{2}{3}x^{\frac{3}{2}} + C.$$

# Problem

Compute $\int \frac{1}{x^2} dx$.

# Solution

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{1}{-2+1} x^{-2+1} = -x^{-1} + C = -\frac{1}{x} + C.$$

Technically, since $x \neq 0$, the solution is

$$\int \frac{1}{x^2} dx = \begin{cases} -\frac{1}{x} + C_1 & \text{if } x > 0, \\ -\frac{1}{x} + C_2 & \text{if } x < 0. \end{cases}$$

# Antidifferentiation Rules for Logs

$$\int \frac{1}{x} dx = ln|x| + C,\ x \neq 0.$$

Technically, since $x \neq 0$,

$$\int \frac{1}{x} dx = \begin{cases} lnx + C_1 & \text{if } x > 0, \\ ln(-x) + C_2 & \text{if } x < 0. \end{cases}$$

# Antidifferentiation Rules for Sums and Constant Multiples

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx.$$

$$\int kf(x)dx = k \int f(x)dx, \ k \text{ is a constant.}$$

# Problem

Compute $\int \left(x^{-3} + 7e^{5x} + \frac{4}{x}\right) dx$.

# Solution

$$\int \left(x^{-3} + 7e^{5x} + \frac{4}{x}\right) dx = \int x^{-3} dx + \int 7e^{5x} dx + \int \frac{4}{x} dx = -\frac{1}{2}x^{-2} + \frac{7}{5}e^{5x} + 4 ln|x| + C.$$

# Problem

Solve the differential equation $y' = x^2 - 2$, $y(1) = \frac{4}{3}$.

# Solution

1) Compute

$$\int (x^2 - 2)dx = \int x^2 dx - \int 2dx = \frac{x^3}{3} - 2x + C,\ C \text{ is a constant.}$$

2) Solve for $C$

$$y(1) = \frac{1}{3}1^3 - 2 + C = \frac{4}{3},\ C = \frac{4}{3} + \frac{5}{3} = 3.$$

# Problem

Let's get back to a motivation example at the beginning of the lecture and solve it as a problem.

A rocket is fired vertically. Its velocity at $t$ seconds after liftoff is $v(t) = 6t + 0.5$ meters per second. The top of the rocket is 8 meters above the launch pad. What is the height of the rocket (measured from the top of the rocket to the launch pad) at time $t$.

# Solution

1) The position function $s(t)$ is an antiderivative of $v(t)$. So,

$$s(t) = \int v(t)dx = \int (6t + 0.5)dt = 3t^2 + 0.5t + C,\ C \text{ is a constant.}$$

2) When $t = 0$, the rocket's height is 8 meters. In other words, $s(0) = 8$ and

$$8 = s(0) = 3 \cdot 0^2 + 0.5 \cdot 0 + C = C.$$

3) Thus,

$$s(t) = 3t^2 + 0.5t + 8.$$

# Problem

A company's marginal cost is $0.015x^2 - 2x + 80$ dollars, where $x$ is the number of units produced in 1 day. The company has a fixed cost of \$1000 per day.

What is the cost of producing $x$ units per day?

How high will the cost rise if the production level is raised from 30 units per day to 60 units per day?

# Solution

1) Use antidifferentiation to compute the cost $C(x)$

$$C(x) = \int \left(0.015x^2 - 2x + 80\right)dx = \int 0.005x^3 - x^2 + 80x + K,\ K$$
is a constant.

2) Since \$1000 is a fixed cost,

$$C(0) = 0.005 \cdot 0^3 - 0^3 + 80 \cdot 0 + K = 1000.\text{ So, } K = 1000.$$

3) Thus,

$$C(x) = 0.005x^3 - x^2 + 80x + 1000.$$

4) We compute

$$C(60) - C(30) = 3280 - 2635 = \$645.$$

# References

1. L. Goldstein, D. Lay, D. Schneider, N. Asmar. *Calculus and its Applications*. Ch. 4, Pearson.
2. www.python.org.